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ORIGINAL ARTICLE

A conformable fractional calculus on arbitrary time scales

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Abstract A conformable time-scale fractional calculus of order $\alpha \in]0,1]$ is introduced. The basic tools for fractional differentiation and fractional integration are then developed. The Hilger time-scale calculus is obtained as a particular case, by choosing $\alpha = 1$.

1. Introduction

Fractional calculus is nowadays one of the most intensively developing areas of mathematical analysis (Jahanshahi et al., 2015; Machado et al., 2011; Tarasov, 2015), including several definitions of fractional operators like Riemann–Liouville, Caputo, and Grünwald–Letnikov. Operators for fractional differentiation and integration have been used in various fields, such as signal processing, hydraulics of dams, temperature field problem in oil strata, diffusion problems, and waves in liquids and gases (Benkhettou et al., 2015; Boyadjiev and Scherer, 2004; Schneider and Wyss, 1989). Here we introduce the notion of conformable fractional derivative on a time scale $\mathbb{T}$. The notion of conformable fractional derivative in $\mathbb{T} = [0,\infty)$ is a recent one: it was introduced in Khalil et al. (2014), then developed in Abdeljawad (2015), and is currently under intensive investigations (Batarfi et al., 2015). In all these works, however, only the case $\mathbb{T} = [0,\infty)$ is treated, providing a natural extension of the usual derivative. In contrast, here we introduce the conformable natural extension of the time-scale derivative. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. It serves as a model of time. The calculus on time scales was initiated by Aulbach and Hilger (1990), in order to unify and generalize continuous and discrete analysis (Hilger, 1990, 1997). It has a tremendous potential for applications and has recently received much attention (Agarwal et al., 2002). The reader interested on the subject of time scales is referred to the books (Bohner and Peterson, 2001, 2003).

The paper is organized as follows. In Section 2, the conformable fractional derivative for functions defined on arbitrary time scales is introduced, and the respective conformable fractional differential calculus developed. Then, in Section 3, we introduce the notion of conformable fractional integral on time scales (the α-fractional integral) and investigate some of its basic properties. We end with Section 4 of conclusion.

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2. Conformable fractional differentiation

Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, and $\delta > 0$. We define the δ-neighborhood of t as $\mathcal{V}_t :=]t-\delta,\ t+\delta[\cap\mathbb{T}$. We begin by introducing a new notion: the conformable fractional derivative of order $\alpha \in]0,1]$ for functions defined on arbitrary time scales.

Definition 1. Let $f: \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in]0,1]$. For $t > 0$, we define $T_\alpha(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ of t, $\delta > 0$, such that $\left|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)[\sigma(t) - s]\right| \leqslant \epsilon|\sigma(t) - s|$ for all $s \in \mathcal{V}_t$. We call $T_\alpha(f)(t)$ the conformable fractional derivative of f of order α at t, and we define the conformable fractional derivative at 0 as $T_\alpha(f)(0) = \lim_{t\to 0^+} T_\alpha(f)(t)$.

Remark 2. If $\alpha = 1$, then we obtain from Definition 1 the delta derivative of time scales. The conformable fractional derivative of order zero is defined by the identity operator: $T_0(f) := f$.

Remark 3. Along the work, we also use the notation $(f(t))^{(\alpha)} = T_\alpha(f)(t)$.

The next theorem provides some useful relationships concerning the conformable fractional derivative on time scales introduced in Definition 1.

Theorem 4. *Let $\alpha \in]0,1]$ and $\mathbb{T}$ be a time scale. Assume $f: \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The following properties hold.*

(i) *If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with*

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}. \tag{1}$$

(iii) *If t is right-dense, then f is conformable fractional differentiable of order α at t if, and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,*

$$T_\alpha(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}. \tag{2}$$

(iv) *If f is fractional differentiable of order α at t, then $f(\sigma(t)) = f(t) + (\mu(t))t^{\alpha-1}T_\alpha(f)(t)$.*

Proof.

(i) Assume that f is conformable fractional differentiable at t. Then, there exists a neighborhood $\mathcal{V}_t$ of t such that $|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)[\sigma(t) - s]| \leqslant \epsilon|\sigma(t) - s|$ for $s \in \mathcal{V}_t$. Therefore, $|f(t) - f(s)| \leqslant |[f(\sigma(t) - f(s)] - T_\alpha(f)(t)[\sigma(t) - s]t^{\alpha-1}| + |[f(\sigma(t)) - f(t)]| + |f^{(\alpha)}(t)||[\sigma(t) - s]||t^\alpha - 1|$ for all $s \in \mathcal{V}_t \cap]t-\epsilon, t+\epsilon[$ and, since t is a right-dense point,

$$\begin{aligned}|f(t) - f(s)| &\leqslant \left|[f^\sigma(t) - f(s)] - f^{(\alpha)}(t)[\sigma(t) - s]^\alpha\right| + \left|f^{(\alpha)}(t)[t - s]^\alpha\right| \\ &\leqslant \epsilon\delta + \left|t^{\alpha-1}\right||T_\alpha(f)(t)|\delta.\end{aligned}$$

Since $\delta \to 0$ when $s \to t$, and $t > 0$, it follows the continuity of f at t.

(ii) Assume that f is continuous at t and t is right-scattered. By continuity,

$$\lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} t^{1-\alpha} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} t^{1-\alpha} = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}.$$

Hence, given $\epsilon > 0$ and $\alpha \in]0,1]$, there is a neighborhood $\mathcal{V}_t$ of t such that

$$\left|\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} t^{1-\alpha} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}\right| \leqslant \epsilon$$

for all $s \in \mathcal{V}_t$. It follows that

$$\left|[f(\sigma(t)) - f(s)]t^{1-\alpha} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}(\sigma(t) - s)\right| \leqslant \epsilon|\sigma(t) - s|$$

for all $s \in \mathcal{V}_t$. The desired equality (1) follows from Definition 1.

(iii) Assume that f is conformable fractional differentiable of order α at t and t is right-dense. Let $\epsilon > 0$ be given. Since f is conformable fractional differentiable of order α at t, there is a neighborhood $\mathcal{V}_t$ of t such that $|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|$ for all $s \in \mathcal{V}_t$. Because $\sigma(t) = t$,

$$\left|\frac{f(t) - f(s)}{t - s} t^{1-\alpha} - T_\alpha(f)(t)\right| \leqslant \epsilon$$

for all $s \in \mathcal{V}_t, s \neq t$. Therefore, we get the desired result (2). Now, assume that the limit on the right-hand side of (2) exists and is equal to L, and t is right-dense. Then, there exists $\mathcal{V}_t$ such that $|(f(t) - f(s))t^{1-\alpha} - L(t - s)| \leqslant \epsilon|t - s|$ for all $s \in \mathcal{V}_t$. Because t is right-dense,

$$\left|(f(\sigma(t)) - f(s))t^{1-\alpha} - L(\sigma(t) - s)\right| \leqslant \epsilon|\sigma(t) - s|,$$

which leads us to the conclusion that f is conformable fractional differentiable of order α at t and $T_\alpha(f)(t) = L$.

(iv) If t is right-dense, i.e., $\sigma(t) = t$, then $\mu(t) = 0$ and $f(\sigma(t)) = f(t) = f(t) + \mu(t)T_\alpha(f)(t)t^{1-\alpha}$. On the other hand, if t is right-scattered, i.e., $\sigma(t) > t$, then by (iii)

$$f(\sigma(t)) = f(t) + \mu(t)t^{\alpha-1} \cdot \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha} = f(t) + (\mu(t))^{\alpha-1}T_\alpha(f)(t),$$

and the proof is complete. $\square$

Remark 5. In a time scale $\mathbb{T}$, due to the inherited topology of the real numbers, a function f is always continuous at any isolated point $t \in \mathbb{T}$.

Example 6. Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$. Then $\sigma(t) = t + h$ and $\mu(t) = h$ for all $t \in \mathbb{T}$. For function $f: t \in \mathbb{T} \mapsto t^2 \in \mathbb{R}$ we have $T_\alpha(f)(t) = (t^2)^{(\alpha)} = (2t + h)t^{1-\alpha}$.

Example 7. Let $q > 1$ and $\mathbb{T} = \overline{q^\mathbb{Z}} := q^\mathbb{Z} \cup \{0\}$ with $q^\mathbb{Z} := \{q^k : k \in \mathbb{Z}\}$. In this time scale

$$\sigma(t) = \begin{cases} qt & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and} \quad \mu(t) = \begin{cases} (q-1)t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Here 0 is a right-dense minimum and every other point in $\mathbb{T}$ is isolated. Now consider the square function f of Example 6. It follows that

$$T_\alpha(f)(t) = \left(t^2\right)^{(\alpha)} = \begin{cases} (q+1)t^{2-\alpha} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Example 8. Let $q > 1$ and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$. For all $t \in \mathbb{T}$ we have $\sigma(t) = qt$ and $\mu(t) = (q-1)t$. Let $f : t \in \mathbb{T} \mapsto \log(t) \in \mathbb{R}$. Then $T_\alpha(f)(t) = (\log(t))^{(\alpha)} = \frac{\log(q)}{(q-1)t^\alpha}$ for all $t \in \mathbb{T}$.

Proposition 9. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then $T_\alpha(f)(t) = (c)^{(\alpha)} = 0$.*

Proof. If t is right-scattered, then by Theorem 4 (ii) one has $T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha} = 0$. Otherwise, t is right-dense and, by Theorem 4 (iii), $T_\alpha(f)(t) = \lim_{s\to t} \frac{c-c}{t-s} t^{1-\alpha} = 0$. □

Proposition 10. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then*

$$T_\alpha(f)(t) = (t)^{(\alpha)} = \begin{cases} t^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

Proof. From Theorem 4 (iv), it follows that $\sigma(t) = t + \mu(t)t^{\alpha-1}T_\alpha(f)(t)$, $\mu(t) = \mu(t)t^{\alpha-1}T_\alpha(f)(t)$. If $\mu(t) \neq 0$, then $T_\alpha(f)(t) = t^{1-\alpha}$ and the desired relation is proved. Assume now that $\mu(t) = 0$, i.e., $\sigma(t) = t$. In this case t is right-dense and, by Theorem 4 (iii), $T_\alpha(f)(t) = \lim_{s\to t} \frac{t-s}{t-s} t^{1-\alpha} = t^{1-\alpha}$. Therefore, if $\alpha = 1$, then $T_\alpha(f)(t) = 1$; if $0 < \alpha < 1$, then $T_\alpha(f)(t) = t^{1-\alpha}$. □

Now, let us consider the two classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, $h > 0$.

Corollary 11. *Function $f : \mathbb{R} \to \mathbb{R}$ is conformable fractional differentiable of order α at point $t \in \mathbb{R}$ if, and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s} t^{1-\alpha}$ exists as a finite number. In this case,*

$$T_\alpha(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}. \tag{3}$$

Proof. Here $\mathbb{T} = \mathbb{R}$, so all points are right-dense. The result follows from Theorem 4 (iii). □

Remark 12. The identity (3) corresponds to the conformable derivative introduced in Khalil et al. (2014) and further studied in Abdeljawad (2015).

Corollary 13. *Let $h > 0$. If $f : h\mathbb{Z} \to \mathbb{R}$, then f is conformable fractional differentiable of order α at $t \in h\mathbb{Z}$ with*

$$T_\alpha(f)(t) = \frac{f(t+h) - f(t)}{h} t^{1-\alpha}.$$

Proof. Here $\mathbb{T} = h\mathbb{Z}$ and all points are right-scattered. The result follows from Theorem 4 (ii). □

Now we give an example using the time scale $\mathbb{T} = \mathbb{P}_{a,b}$, which is a time scale with interesting applications in Biology (Fenchel and Christiansen, 1977).

Example 14. Let $a, b > 0$ and consider the time scale $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty}[k(a+b),\ k(a+b)+a]$. Then

$$\sigma(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^{\infty}[k(a+b),\ k(a+b)+a), \\ t+b & \text{if } t \in \bigcup_{k=0}^{\infty}\{k(a+b)+a\} \end{cases}$$

and

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty}[k(a+b),\ k(a+b)+a), \\ b & \text{if } t \in \bigcup_{k=0}^{\infty}\{k(a+b)+a\}. \end{cases}$$

Let $f : \mathbb{P}_{a,b} \to \mathbb{R}$ be continuous and $\alpha \in]0,1]$. It follows from Theorem 4 that the conformable fractional derivative of order α of a function f defined on $\mathbb{P}_{a,b}$ is given by

$$T_\alpha(f)(t) = \begin{cases} \lim_{s\to t} \frac{f(t)-f(s)}{(t-s)} t^{1-\alpha} & \text{if } t \in \bigcup_{k=0}^{\infty}[k(a+b),\ k(a+b)+a), \\ \frac{f(t+b)-f(t)}{b} t^{1-\alpha} & \text{if } t \in \bigcup_{k=0}^{\infty}\{k(a+b)+a\}. \end{cases}$$

For the conformable fractional derivative on time scales to be useful, we would like to know formulas for the derivatives of sums, products, and quotients of fractional differentiable functions. This is done according to the following theorem.

Theorem 15. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are conformable fractional differentiable of order α. Then,*

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with $T_\alpha(f+g) = T_\alpha(f) + T_\alpha(g)$;
(ii) for any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with $T_\alpha(\lambda f) = \lambda T_\alpha(f)$;
(iii) if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with $T_\alpha(fg) = T_\alpha(f)g + (f \circ \sigma)T_\alpha(g) = T_\alpha(f)(g \circ \sigma) + fT_\alpha(g)$;
(iv) if f is continuous, then $1/f$ is conformable fractional differentiable with

$$T_\alpha\left(\frac{1}{f}\right) = -\frac{T_\alpha(f)}{f(f \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^\kappa$ for which $f(t)f(\sigma(t)) \neq 0$;
(v) if f and g are continuous, then f/g is conformable fractional differentiable with

$$T_\alpha\left(\frac{f}{g}\right) = \frac{T_\alpha(f)g - fT_\alpha(g)}{g(g \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^\kappa$ for which $g(t)g(\sigma(t)) \neq 0$.

Proof. Let us consider that $\alpha \in]0,1]$, and let us assume that f and g are conformable fractional differentiable at $t \in \mathbb{T}^\kappa$.

(i) Let $\epsilon > 0$. Then there exist neighborhoods $\mathcal{V}_t$ and $\mathcal{U}_t$ of t for which

$$\left|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)(\sigma(t) - s)\right| \leqslant \frac{\epsilon}{2}|\sigma(t) - s| \quad \text{for all } s \in \mathcal{V}_t$$

and

$$\left|[g(\sigma(t)) - g(s)]t^{1-\alpha} - T_\alpha(g)(t)(\sigma(t) - s)\right| \leqslant \frac{\epsilon}{2}|\sigma(t) - s| \quad \text{for all } s \in \mathcal{U}_t.$$

Let $\mathcal{W}_t = \mathcal{V}_t \cap \mathcal{U}_t$. Then $|[(f+g)(\sigma(t)) - (f+g)(s)]t^{1-\alpha} - [T_\alpha(f)(t) + T_\alpha(g)(t)](\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|$ for all $s \in \mathcal{W}$. Thus, $f+g$ is conformable differentiable at t and $T_\alpha(f+g)(t) = T_\alpha(f)(t) + T_\alpha(g)(t)$.

(ii) Let $\epsilon > 0$. Then $|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|$ for all s in a neighborhood $\mathcal{V}_t$ of t. It follows that

$$\left|[(\lambda f)(\sigma(t)) - (\lambda f)(s)]t^{1-\alpha} - \lambda T_\alpha(f)(t)(\sigma(t) - s)\right| \leqslant \epsilon|\lambda|\,|\sigma(t) - s| \quad \text{for all } s \in \mathcal{V}_t.$$

Therefore, λf is conformable fractional differentiable at t and $T_\alpha(\lambda f) = \lambda T_\alpha(f)$ holds at t.

(iii) If t is right-dense, then

$$\begin{aligned} T_\alpha(fg)(t) &= \lim_{s\to t}\left[\frac{f(t) - f(s)}{t - s}t^{1-\alpha}\right]g(t) + \lim_{s\to t}\left[\frac{g(t) - g(s)}{t - s}t^{1-\alpha}\right]f(s) \\ &= T_\alpha(f)(t)g(t) + T_\alpha(g)(t)f(t) \\ &= T_\alpha(f)(t)g(\sigma(t)) + T_\alpha(g)(t)f(t). \end{aligned}$$

If t is right-scattered, then

$$\begin{aligned} T_\alpha(fg)(t) &= \left[\frac{f(\sigma(t)) - f(t)}{\mu(t)}t^{1-\alpha}\right]g(\sigma(t)) + \left[\frac{g(\sigma(t)) - g(t)}{\mu(t)}t^{1-\alpha}\right]f(t) \\ &= T_\alpha(f)(t)g(\sigma(t)) + f(t)T_\alpha(g)(t). \end{aligned}$$

The other product rule formula follows by interchanging the role of functions f and g.

(iv) We use the conformable fractional derivative of a constant (Proposition 9) and property (iii) of Theorem 15 (just proved): from Proposition 9 we know that $T_\alpha\left(f \cdot \frac{1}{f}\right)(t) = (1)^{(\alpha)} = 0$. Therefore, by (iii)

$$T_\alpha\left(\frac{1}{f}\right)(t)f(\sigma(t)) + T_\alpha(f)(t)\frac{1}{f(t)} = 0.$$

Since we are assuming $f(\sigma(t)) \neq 0$, $T_\alpha\left(\frac{1}{f}\right)(t) = -\frac{T_\alpha(f)(t)}{f(t)f(\sigma(t))}$.

(v) We use (ii) and (iv) to obtain

$$\begin{aligned} T_\alpha\left(\frac{f}{g}\right)(t) &= T_\alpha\left(f \cdot \frac{1}{g}\right)(t) = f(t)T_\alpha\left(\frac{1}{g}\right)(t) + T_\alpha(f)(t)\frac{1}{g(\sigma(t))} \\ &= \frac{T_\alpha(f)(t)g(t) - f(t)T_\alpha(g)(t)}{g(t)g(\sigma(t))}. \end{aligned}$$

This concludes the proof. □

Theorem 16. *Let c be a constant, $m \in \mathbb{N}$, and $\alpha \in]0,1]$.*

(i) If $f(t) = (t-c)^m$, then $T_\alpha(f)(t) = t^{1-\alpha}\sum_{p=0}^{m-1}(\sigma(t) - c)^{m-1-p}(t-c)^p$.

(ii) If $g(t) = \frac{1}{(t-c)^m}$ and $(t-c)(\sigma(t) - c) \neq 0$, then $T_\alpha(g)(t) = -t^{1-\alpha}\sum_{p=0}^{m-1}\frac{1}{(\sigma(t)-c)^{p+1}(t-c)^{m-p}}$.

Proof. We prove the first formula by induction. If $m = 1$, then $f(t) = t - c$ and $T_\alpha(f)(t) = t^{1-\alpha}$ holds from Propositions 9 and 10 and Theorem 15 (i). Now assume that

$$T_\alpha(f)(t) = t^{1-\alpha}\sum_{p=0}^{m-1}(\sigma(t) - c)^{m-1-p}(t-c)^p$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. We use Theorem 15 (iii) to obtain $(F(t))^{(\alpha)} = T_\alpha(t-c)f(\sigma(t)) + T_\alpha(f)(t)(t-c) = t^{1-\alpha}\sum_{p=0}^{m}(\sigma(t)-c)^{m-p}(t-c)^p$. Hence, by mathematical induction, part (i) holds. (ii) Let $g(t) = \frac{1}{(t-c)^m} = \frac{1}{f(t)}$. From (iv) of Theorem 15,

$$g^{(\alpha)}(t) = -\frac{T_\alpha(f)(t)}{f(t)f(\sigma(t))} = -t^{1-\alpha}\sum_{p=0}^{m-1}\frac{1}{(\sigma(t)-c)^{p+1}(t-c)^{m-p}},$$

provided $(t-c)(\sigma(t)-c) \neq 0$. □

We show some examples of application of Theorem 16.

Example 17. Let $\alpha \in]0,1]$ and $f(t) = t^m$. Then $T_\alpha(f)(t) = t^{1-\alpha}\sum_{p=0}^{m-1}\sigma(t)^{m-1-p}t^p$. Note that if t is right-dense, then $T_\alpha(f)(t) = mt^{m-\alpha}$. If we choose $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, then we obtain the usual derivative: $T_1(f)(t) = mt^{m-1} = f'(t)$.

Example 18. Let $\alpha \in]0,1]$ and $f(t) = \frac{1}{t^m}$. Then $T_\alpha(f)(t) = -t^{1-\alpha}\sum_{p=0}^{m-1}\frac{1}{t^{p-m}\sigma(t)^{p+1}}$. If t is right-dense, then $T_\alpha(f)(t) = -\frac{m}{t^{m+\alpha}}$. Moreover, if $\alpha = 1$, then we obtain $T_1(f)(t) = -\frac{m}{t^{m+1}}$.

Example 19. If $f(t) = (t-1)^2$, then $T_\alpha(f)(t) = t^{1-\alpha}\left[(\sigma(t)+1)^2 + (\sigma(t)+1)(t+1) + (t+1)^2\right]$ for all $\alpha \in]0,1]$.

The chain rule, as we know it from the classical differential calculus, does not hold for the conformable fractional derivative on times scales. This is well illustrated by the following example.

Example 20. Let $\alpha \in (0,\ 1); \mathbb{T} = \mathbb{N} = \{1, 2, \ldots\}$, for which $\sigma(t) = t + 1$ and $\mu(t) = 1$; and $f,\ g : \mathbb{T} \to \mathbb{T}$ be given by $f(t) = g(t) = t$. Then, $T_\alpha(f \circ g)(t) \neq T_\alpha(f)(g(t))T_\alpha(g)(t)$: $T_\alpha(f \circ g)(t) = t^{1-\alpha}$, while $T_\alpha(f)(g(t))T_\alpha(g)(t) = t^{2(1-\alpha)}$.

We can prove, however, the following result.

Theorem 21 (*Chain rule*). *Let $\alpha \in]0,1]$. Assume $g : \mathbb{T} \to \mathbb{R}$ is continuous and conformable fractional differentiable of order α at $t \in \mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$T_\alpha(f \circ g)(t) = f'(g(c))\,T_\alpha(g)(t). \tag{4}$$

Proof. Let $t \in \mathbb{T}^\kappa$. First we consider t to be right-scattered. In this case,

$$T_\alpha(f \circ g)(t) = \frac{f(g(\sigma(t))) - f(g(t))}{\mu(t)} t^{1-\alpha}.$$

If $g(\sigma(t)) = g(t)$, then we get $T_\alpha(f \circ g)(t) = 0$ and $T_\alpha(g)(t) = 0$. Therefore, (4) holds for any c in the real interval $[t, \sigma(t)]$. Now assume that $g(\sigma(t)) \neq g(t)$. By the mean value theorem we have

$$\begin{aligned} T_\alpha(f \circ g)(t) &= \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \cdot \frac{g(\sigma(t)) - g(t)}{\mu(t)} t^{1-\alpha} \\ &= f'(\xi) T_\alpha(g)(t), \end{aligned}$$

where $\xi \in]g(t),\ g(\sigma(t))[$. Since $g : \mathbb{T} \to \mathbb{R}$ is continuous, there is a $c \in [t, \sigma(t)]$ such that $g(c) = \xi$, which gives the desired result. Now let us consider the case when t is right-dense. In this case

$$T_\alpha(f \circ g)(t) = \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha}.$$

By the mean value theorem, there exists $\xi_s \in]g(t), g(\sigma(t))[$ such that

$$T_\alpha(f \circ g)(t) = \lim_{s \to t} \left\{ f'(\xi_s) \cdot \frac{g(t) - g(s)}{t - s} t^{1-\alpha} \right\}.$$

By the continuity of g, we get that $\lim_{s \to t} \xi_s = g(t)$. Then $T_\alpha(f \circ g)(t) = f'(g(t)) \cdot T_\alpha(g)(t)$. Since t is right-dense, we conclude that $c = t = \sigma(t)$, which gives the desired result. $\square$

Example 22. Let $\mathbb{T} = 2^{\mathbb{N}}$, for which $\sigma(t) = 2t$ and $\mu(t) = t$. (i) Choose $f(t) = t^2$ and $g(t) = t$. Theorem 21 guarantees that we can find a value c in the interval $[t, \sigma(t)] = [t, 2t]$, such that

$$T_\alpha(f \circ g)(t) = f'(g(c)) T_\alpha(g)(t). \tag{5}$$

Indeed, from Theorem 4 it follows that $T_\alpha(f \circ g)(t) = 3t^{1-\alpha}$, $T_\alpha(g)(t) = t^{1-\alpha}$, and $f'(g(c)) = 2c$. Equality (5) leads to $3t^{1-\alpha} = 2ct^{1-\alpha}$ and so $c = \frac{3}{2}t \in [t, 2t]$. (ii) Now let us take $f(t) = g(t) = t^2$ for all $t \in \mathbb{T}$. We obtain $15t^{4-\alpha} = T_\alpha(f \circ g)(t) = f'(g(c)) T_\alpha(g)(t) = 2c^2 3t^{2-\alpha}$. Therefore, $c = \sqrt{\frac{5}{2}} t \in [t, 2t]$.

To end Section 2, we consider conformable derivatives of higher-order. More precisely, we define the conformable fractional derivative T_α for $\alpha \in (n,\ n+1]$, where n is some natural number.

Definition 23. Let $\mathbb{T}$ be a time scale, $\alpha \in (n,\ n+1], n \in \mathbb{N}$, and let f be n times delta differentiable at $t \in \mathbb{T}^{\kappa^n}$. We define the conformable fractional derivative of f of order α as $T_\alpha(f)(t) := T_{\alpha-n}\left(f^{\Delta^n}\right)(t)$. As before, we also use the notation $(f(t))^{(\alpha)} = T_\alpha(f)(t)$.

Example 24. Let $\mathbb{T} = h\mathbb{Z}, h > 0, f(t) = t^3$, and $\alpha = 2.1$. Then, by Definition 23, we have $T_{2.1}(f) = T_{0.1}\left(f^{\Delta^2}\right)$. Since $\sigma(t) = t + h$ and $\mu(t) = h, T_{2.1}(f)(t) = (t^3)^{(2.1)} = (6t + 6h)^{(0.1)}$. By Proposition 9 and Theorem 15 (i) and (ii), we obtain that $T_{2.1}(f)(t) = 6(t)^{(0.1)}$. We conclude from Proposition 10 that $T_{2.1}(f)(t) = 6t^{0.9}$.

Theorem 25. *Let $\alpha \in (n,\ n+1], n \in \mathbb{N}$. The following relation holds:*

$$T_\alpha(f)(t) = t^{1+n-\alpha} f^{\Delta^{1+n}}(t). \tag{6}$$

Proof. Let f be a function n times delta-differentiable. For $\alpha \in (n, n+1]$, there exist $\beta \in (0, 1]$ such that $\alpha = n + \beta$. Using Definition 23, $T_\alpha(f) = T_\beta\left(f^{\Delta^n}\right)$. From the definition of (higher-order) delta derivative and Theorem 4 (ii) and (iii), it follows that $T_\alpha(f)(t) = t^{1-\beta}\left(f^{\Delta^n}\right)^\Delta(t)$. $\square$

3. Fractional integration

Now we introduce the α-conformable fractional integral (or α-fractional integral) on time scales.

Definition 26. Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the α-fractional integral of f, $0 < \alpha \leqslant 1$, is defined by $\int f(t)\Delta^\alpha t := \int f(t) t^{\alpha-1} \Delta t$.

Remark 27. For $\mathbb{T} = \mathbb{R}$ Definition 26 reduces to the conformable fractional integral given in Khalil et al. (2014); for $\alpha = 1$ Definition 26 reduces to the indefinite integral of time scales (Bohner and Peterson, 2001).

Definition 28. Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite α-fractional integral of f of order $\alpha, \alpha \in (0,\ 1]$, as follows: $F_\alpha(t) = \int f(t)\Delta^\alpha t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy α-fractional integral by $\int_a^b f(t)\Delta^\alpha t = F_\alpha(b) - F_\alpha(a)$.

Example 29. Let $\mathbb{T} = \mathbb{R}$, $\alpha = \frac{1}{2}$, and $f(t) = t$. Then $\int_1^{10^{2/3}} f(t)\Delta^\alpha t = 6$.

Theorem 30. *Let $\alpha \in (0,\ 1]$. Then, for any rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exists a function $F_\alpha : \mathbb{T} \to \mathbb{R}$ such that $T_\alpha(F_\alpha)(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Function F_α is said to be an α-antiderivative of f.*

Proof. The case $\alpha = 1$ is proved in Bohner and Peterson (2001). Let $\alpha \in (0,\ 1)$. Suppose f is rd-continuous. By Theorem 1.16 of Bohner and Peterson (2003), f is regulated. Then, $F_\alpha(t) = \int f(t)\Delta^\alpha t$ is conformable fractional differentiable on $\mathbb{T}^\kappa$. Using (6) and Definition 26, we obtain that $T_\alpha(F_\alpha)(t) = t^{1-\alpha}(F_\alpha(t))^\Delta = f(t),\ t \in \mathbb{T}^\kappa$. $\square$

Theorem 31. *Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and f, g be two rd-continuous functions. Then,*

(i) $\int_a^b [f(t) + g(t)]\Delta^\alpha t = \int_a^b f(t)\Delta^\alpha t + \int_a^b g(t)\Delta^\alpha t;$
(ii) $\int_a^b (\lambda f)(t)\Delta^\alpha t = \lambda \int_a^b f(t)\Delta^\alpha t;$
(iii) $\int_a^b f(t)\Delta^\alpha t = -\int_b^a f(t)\Delta^\alpha t;$
(iv) $\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t;$
(v) $\int_a^a f(t)\Delta^\alpha t = 0;$

(vi) *if there exist* $g : \mathbb{T} \to \mathbb{R}$ *with* $|f(t)| \leqslant g(t)$ *for all* $t \in [a, b]$, *then* $\left|\int_a^b f(t)\Delta^\alpha t\right| \leqslant \int_a^b g(t)\Delta^\alpha t$;

(vii) *if* $f(t) > 0$ *for all* $t \in [a, b]$, *then* $\int_a^b f(t)\Delta^\alpha t \geqslant 0$.

Proof. The relations follow from Definitions 26 and 28, analogous properties of the delta-integral, and the properties of Section 2 for the conformable fractional derivative on time scales. □

Theorem 32. *If* $f : \mathbb{T}^\kappa \to \mathbb{R}$ *is a rd-continuous function and* $t \in \mathbb{T}^\kappa$, *then*

$$\int_t^{\sigma(t)} f(s)\Delta^\alpha s = f(t)\mu(t)t^{\alpha-1}.$$

Proof. Let f be a rd-continuous function on $\mathbb{T}^\kappa$. Then f is a regulated function. By Definition 28 and Theorem 30, there exist an antiderivative F_α of f satisfying

$$\int_t^{\sigma(t)} f(s)\Delta^\alpha s = F_\alpha(\sigma(t)) - F_\alpha(t) = T_\alpha(F_\alpha)(t)\mu(t)t^{1-\alpha}$$
$$= f(t)\mu(t)t^{1-\alpha}.$$

This concludes the proof. □

Theorem 33. *Let* $\mathbb{T}$ *be a time scale,* $a, b \in \mathbb{T}$ *with* $a < b$. *If* $T_\alpha(f)(t) \geqslant 0$ *for all* $t \in [a, b] \cap \mathbb{T}$, *then* f *is an increasing function on* $[a, b] \cap \mathbb{T}$.

Proof. Assume $T_\alpha(f)$ exist on $[a, b] \cap \mathbb{T}$ and $T_\alpha(f)(t) \geqslant 0$ for all $t \in [a, b] \cap \mathbb{T}$. Then, by (i) of Theorem 4, $T_\alpha(f)$ is continuous on $[a, b] \cap \mathbb{T}$ and, therefore, by Theorem 31 (vii), $\int_s^t T_\alpha f(\xi)\Delta^\alpha \xi \geqslant 0$ for s, t such that $a \leqslant s \leqslant t \leqslant b$. From Definition 28, $f(t) = f(s) + \int_s^t T_\alpha f(\xi)\Delta^\alpha \xi \geqslant f(s)$. □

4. Conclusion

A fractional calculus, that is, a study of differentiation and integration of non-integer order, is here investigated via the recent and powerful calculus on time scales. Our new calculus includes, in a single theory, discrete, continuous, and hybrid fractional calculi. In particular, the new fractional calculus on time scales unifies and generalizes: the Hilger calculus (Bohner and Peterson, 2001; Hilger, 1990), obtained by choosing $\alpha = 1$; and the conformable fractional calculus (Abdeljawad, 2015; Khalil et al., 2014; Batarfi et al., 2015), obtained by choosing $\mathbb{T} = \mathbb{R}$.

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