Universitat Autònoma de Barcelona

Gravity, Holography and Applications to Condensed Matter

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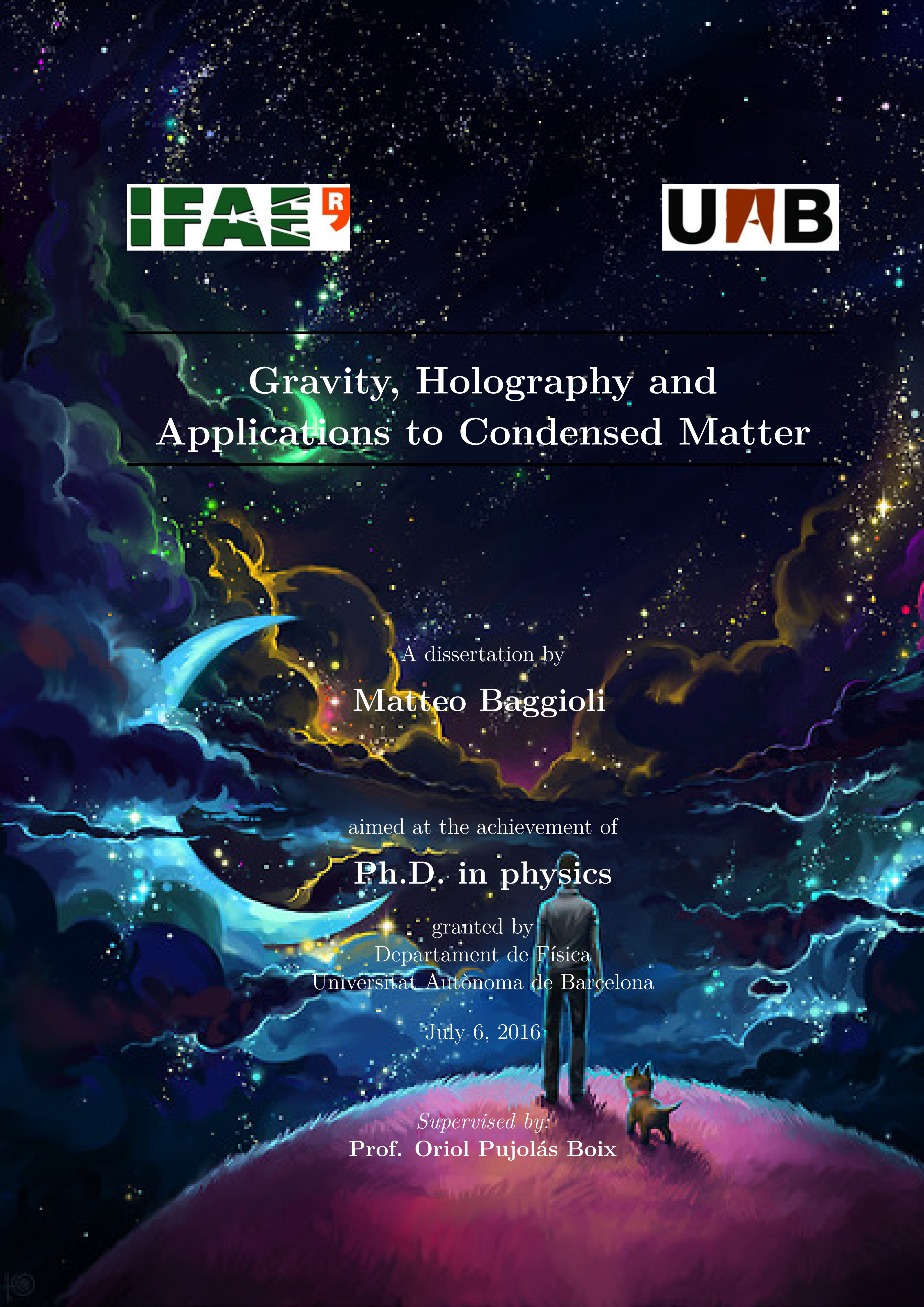

Gravity, Holography and Applications to Condensed Matter

A dissertation by

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Acknowledgements

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I have been planning for a long time to make a huge and very funny list of acknowledgements to start with a laugh the reading of my thesis and with the convictions that physicists could be social and amusing animals but I suddenly realized that maybe it is a good occasion to be serious (once in life) and maybe... we are really the nerds people think we are.
That said, the first thank goes, without any doubt, to my advisor Oriol Pujolàs for bearing with me all these 3 years. It must have been a hard task!!! Thanks for sharing your time and your knowledge with me and for teaching me how to behave in this scientific world.
A sincere thank goes also to all my collaborators and colleagues (too many to list them all), who enlightened my lazy brain with deep discussions and intriguing problems. I owe you much of what I know.
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Everybody of you gave me something (and someone of you, the lucky ones lol, are still giving it) and most importantly taught me something, which I will bring with me forever and everywhere together with the tons of ink I have on my skin.
Part of these 200 pages of "very useful" scientific developments are yours and more than that my survival after 3 years of intense theoretical physics activity has been possible just because all of you.
Thanks...

Non ce la farai mai ...

La Dodo

Non importa chi sei ma chi vuoi diventare ...

Big Z

DEDICATION

Dedicated to my sister who always told me to do something better remunerated. . .

Dedicated to my mom, now you know what I was thinking about when I was not replying you about choosing pasta or pizza. . .

Dedicated to my dad, because you gave me the locura I need for this mad world. . .

Dedicated to all the eyes I crossed during these years and all the people who made me smile when a smile was the last of my thoughts. . .

Dedicated to all the girls (and sometimes boys, huh!) who stole my time from physics, you could have done it way more, who cares about physics c'mon . . .

Dedicated to all the people I did not listen to, it looks like I did well. . .

Abstract

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Strongly coupled physical systems along with their corresponding, and usually exotic, features are elusive and not suitable to be described by conventional and perturbative approaches, which in those cases are not able to provide a controllable and robust tool for computations. Nevertheless non perturbative effects and strongly correlated frameworks are ubiquitous in nature, expecially in Condensed Matter physics. The AdS/CFT correspondence, born from the excitement of ideas and efforts employed in finding out a possible description of Quantum Gravity, lead to a flurry of fresh air into the subject, introducing an unexpected and brandnew perspective for dealing with strongly coupled field theories. In its more general formulation, known as Gauge-Gravity duality, this setup accounts for an effective and efficient weapon to tackle those kind of problems using a dual gravitational description which turns out to be way easier than the original one. In the last years, a huge number of developments have been achieved in applying the duality towards modern and hot condensed matter misteries, such as the Strange Metals nature or the mechanism underlying the High-Tc Superconductivity.
Momentum relaxation is an ever-present and unavoidable ingredient of any realistic Condensed Matter system. In real-world materials the presence of a lattice, impurities or disorder forces momentum to dissipate and leads to relevant physical effects such as the finiteness of the DC transport properties, i.e. conductivities. Several open questions are connected to those quantities expecially in the limit of strong momentum relaxation where novel insulating states appear and unexpected quantum phase transitions between the latter and metallic states (MIT) arise.

The main purpose of this thesis is the introduction of momentum dissipation and its consequent effects into the framework of AdS/CMT, namely the applications of the Gauge-Gravity duality to Condensed Matter.
A convenient and effective way of breaking translational symmetry of the the dual quantum field theory is provided by Massive Gravity (MG) theories, which constitues a tractable and easy tool to adress several interesting questions in strongly coupled systems with momentum dissipation. Born to solve cosmological puzzles, MG can now be reconsidered under a completely new perspective and could become a useful framework for ”Real-world” phenomena and ”low energy” applications. We consider generic massive gravity models embedded into asymptotically Anti de Sitter spacetime and we analyze them using holographic techniques.

- We study in detail the definition, the meaning, the consistency and the stability of such massive gravity models.
- We concentrate on the transport properties of MG theories with particular attention on the electric conductivity. We focus on the features of strongly coupled insulating states and we study in detail the possible existence of holographic metal-insulator transitions. Moreover

we initiate the study of the elastic response of these gravitational configurations dual to strongly coupled solids.

- We discuss the existence of universal bounds on physical quantities in the context of strongly coupled materials with momentum dissipation. We consider the famous viscosity/entropy ratio and possible lower bound on conductivities.
- We introduce momentum dissipation into the framework of holographic superconductors and we attempt to build and discuss the actual phase diagram of High-Tc superconductors.

This work represents a step further towards the definition of effective holographic models for Condensed matter able to reproduce (and maybe one day to predict) non trivial features of realistic systems.

List of works

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i. **Electron-Phonon Interactions, Metal-Insulator Transitions, and Holographic Massive Gravity**

   M. Baggioli, O. Pujolas

   Published in Phys.Rev.Lett. 114 (2015) no.25, 251602,
   arXiv:1411.1003 [hep-th,cond-mat.str-el]

ii. **Phases of holographic superconductors with broken translational symmetry**

   M. Baggioli, M. Goykhman

   Published in JHEP 1507 (2015) 035 ,
   arXiv: 1504.05561[hep-th,cond-mat.str-el,cond-mat.supr-con]

iii. **Drag Phenomena from Holographic Massive Gravity**

   M. Baggioli, D. K. Brattan

   arXiv: 1504.07635[hep-th]

iv. **Under The Dome: Doped holographic superconductors with broken translational symmetry**

   M. Baggioli, M. Goykhman

   Published in JHEP 1601 (2016) 011 ,
   arXiv: 1510.06363[hep-th, cond-mat.str-el, cond-mat.supr-con]

v. **Solid Holography and Massive Gravity**

   L. Alberte, M. Baggioli, A. Khmelnitsky, O. Pujolas

   Published in JHEP 1602 (2016) 114,
   arXiv: 1510.09089[hep-th, cond-mat.str-el]

vi. **Viscosity bound violation in holographic solids and the viscoelastic response**

   L. Alberte, M. Baggioli, O. Pujolas
   arXiv: 1601.03384[hep-th]

vii. **On holographic disorder-driven metal-insulator transitions**

   M. Baggioli, O. Pujolas
   arXiv: 1601.07897[hep-th, cond-mat.str-el]

viii. **Chasing the cuprates with dilatonic dyons**

A. Amoretti, M. Baggioli, N. Magnoli, D. Musso

Published in JHEP06(2016)113,
arXiv: 1603.03029[hep-th, cond-mat.str-el]

ix. **On effective holographic Mott insulators**

M. Baggioli, O. Pujolas
arXiv: 1604.08915[hep-th, cond-mat.str-el]

Contents

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List of abbreviations

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QCD Quantum Cromodynamics

QGP QUark Gluon Plasma

FD Fermi-Dirac

MB Maxwell-Boltzmann

BH Black Hole

GSL Generalized second law

SC Superconductor

RN Reissner Nordstrom

BF Breitenlohner-Freedman

MIT Metal Insulator transition

KSS Kovtun, Son and Starinets

NG Nambu Goto

MG Massive Gravity

NED Non Linear Electrodynamics

DBI Dirac Born Infield

EMD Einstein Maxwell Dilaton

GPKW Gubser, Polyakov, Klebanov, Witten

GR General Relativity

AdS Anti De Sitter

CFT Conformal Field Theory

GGD Gauge Gravity Duality

HMG Holographic Massive Gravity

dRGT De Rham Gabadadze Tolley

BB Black Brane

CM Condensed Matter

LVMG Lorentz Violating Massive Gravity

UV Ultraviolet

IR Infrared

DC Direct Current

AC Alternate Current

VEV Vacuum Expectation Value

QNM Quasinormal Mode

GH Gibbons Hawking

RG Renormalization Group

TB Translation Breaking

DDMIT Disorder Driven Metal Insulator Transition

CCS Charge Conjugation Symmetric

FP Fierz-Pauli

vDVZ van Dam-Veltman-Zakharov

EH Einstein-Hilbert

ADM Arnowitt, Deser and Misner

BD Boulware-Deser

Part I

Introduction

Duality in mathematics is not a theorem, but a ”principle”

Michael Atiyah

The idea of **duality** is ubiquitous in fundamental sciences. It is very powerful and useful, and has a long history going back hundreds of years. Over time it has been adapted and modified and it has finally taken the stage in the modern scientific scenario. It appears in many subjects in mathematics (geometry, algebra, analysis) and in physics. *Fundamentally, duality gives two different points of view of looking at the same object.* In theoretical physics one often says that a non-trivial equivalence between two models is a duality and that two very different looking physical systems can nevertheless be identical. The first example of such an idea in the context of physics goes back to the early history and it refers to the nature of light. Aristotle was one of the first to publicly hypothesize about the nature of light, proposing that light is a disturbance in the element aether (that is, it is a wave-like phenomenon). Democritus -the original atomist- argued that all things in the universe, including light, are composed of indivisible sub-components (light being some form of solar atom). This dicotomy formalized later on through the work of Max Planck, Albert Einstein, Louis de Broglie, Arthur Compton, Niels Bohr, and many others, takes the name of *Wave-Particle duality* and it is nowadays phrased as: all particles also have a wave nature (and vice versa).
The idea that a particular problem can have more than one description and that depending on the situation one is more convenient than the other spread into several fields of physics and becomes a strong and robust computational tool. Early prototypes are the *Electro-Magnetic duality* and the *Kramers-Wannier duality*, which allows for example to solve the 2-dimensional Ising model exactly. Along with the formulation of Supersymmetry and String Theory a huge class of dualities has been discovered and analyzed: S-Duality, T-Duality, U-Duality, Mirror Symmetry, Montonen-Olive duality, etc…
The astonishing results following this program are that entities concerning the theoretical description of a system such as:

- the nature of the fundamental degrees of freedom;
- the number of spacetime dimensions;
- the spacetime's size and topology;

- the couplings' strengths;

are not *duality invariant* concepts and that despite the physics of a particular system is one and only its description can be absurdly different in different *duality frames*.
All these ideas along with the brandnew openminded attitude lead to the birth of the so called **AdS-CFT correspondence**, first formulated by Juan Maldacena in 1997, which represents not only the single most important result in string theory in the last twenty years but also the most shining and deeply surprising example of duality. The original example of AdS/CFT linked two very special theories. The gravitational side involved a particular extension of gravity (type IIB supergravity) on a particular geometry ($AdS_5 \times S_5$) whereas the QFT was the unique theory with the largest possible amount of supersymmetry ($\mathcal{N} = 4\, SYM$). There is a specific dictionary that translates between the theories. This relationship has no formal mathematical proof. However a very large number of checks have been performed. These checks involve two calculations, using different techniques and methods, of quantities related by the dictionary. Continual agreement of these calculations constitutes strong evidence for the correspondence. The first example has by now been extended to many other cases, and AdS/CFT is more generally referred to as the **Gauge-Gravity duality** (GGD). Formally this is the statement that gravitational theories in (N+1) dimensions can be entirely and completely equivalent to non-gravitational quantum field theories in N dimensions. The AdS/CFT correspondence has a very useful property. When the gravitational theory is hard to solve, the QFT is easy to solve, and vice-versa! This opens the door to previously intractable problems in QFT through simple calculations in gravity theories. Moreover AdS/CFT allows a conceptual reworking of the classic problems of QFT. Indeed if a QFT can be equivalent to gravitational theory, then neither one is deeper than the other. Maybe, the non-perturbative definition of a QFT is not a QFT anymore but it takes the form of a gravitational one. Physicists can therefore use it to develop new intuitions for both QFT and Quantum Gravity in a symbiotic fashion.
The main feature of the GGD is that it qualifies as a *Strong-Weak duality* in the sense that it relates a theory with a coupling constant g to an equivalent theory with coupling constant $1/g$. More in details, the *dual* of a strongly coupled quantum field theory is represented by a weakly coupled and classical theory of gravity, i.e. General Relativity. Therefore, exploiting this connection GGD has become a very efficient (and sometimes the only one available) tool to attack strongly coupled problems in the context of:

(a) QCD and Quark Gluon Plasma (QGP) Physics

(b) Condensed Matter and Quantum Phase Transitions

(c) Non Equilibrium Physics

(d) Information Theory

In this thesis we focus our attention on the applications of the Gauge-Gravity Duality towards the Condensed Matter world, which are usually referred as **AdS-CMT**, making use of the *motto*:

Strongly Coupled/Correlated and Quantum Condensed Matter Systems

$$\updownarrow$$

Weakly Coupled and Classical Gravitational Theories (GR)

It is inter-disciplinarity at its best: suitably interpreted, the equations of string theory can be a powerful tool for analysing some exotic states of matter, ranging from super-hot balls of quarks and gluons to ultracold atoms. Condensed Matter is a boiling pot of interesting questions and open problems which seems to conflict the old-known and well established paradigms of the field itself. The access and the study of strongly coupled and strongly correlated materials opened a completly new and misterious scenario where the single particle approximation and the perturbative methods are proved of no help. Despite sceptics still question whether this strange alliance will actually lead to new insights, or whether it is just a marriage of convenience, for the time being, the advantage to both partners is clear. String theory, long criticized for having lost touch with reality, gets experimental credibility. And condensed-matter physics, never the media darling that string theory has been, gets a new mathematical tool and a chance to bask in new-found glamour.

Through the chapters of this thesis we will encounter the hottest open problems in CM such as:

(a) the nature of the *Strange Metals*

(b) the role of the *High-Tc Superconductors*

(c) the existence of *Metal-Insulator transitions* (MIT)

(d) the role of *disorder* in CM systems and the appearance of *Anderson Localization*

and we will attack them using the new tool given us by the GGD.

The novelty and the crucial point of the present work is the introduction of momentum dissipation effects into the GGD setup. The (explicit/spontaneous) breaking of translational symmetry is a mandatory ingredient to describe condensed matter system where the presence of *lattice, impurities, disorder, etc...* is at the order of the day. In the spirit of *effective field theories* (EFT) we mimick such a mechanism considering **Massive Gravity** (MG) theories where diffeomorphism invariance is (partially) broken. Such a modification of the usual GR picture will allow us to consider "duals" of metallic and insulating configurations and eventual transitions between them. This represents a step forward in realizing **holographic effective field theories** for condensed matter and in sharpening the GGD tool towards its concrete application to "real world" systems.

Organization of the thesis:

Part I is devoted to the theoretical background necessary in order to get through this work.

In chapter 1 we present the Condensed Matter world in a "particle physicist fashion" analyzing the existing open issues and the reason why the standard approaches fail in giving any explanation.

In chapter 2 we introduce the tool we will be using all the time, i.e. the GGD. We discuss all its main features and we give heuristic reasonings based on the *Holographic Principle* and the *Renormalization Group Flow* to motivate the duality. We then present the formal dictionary of the mapping and give some explicit easy examples of its application.

In chapter 3 we consider another main character of our tale, namely Massive Gravity (MG) theories. We outline the history and the formal developments of this framework. We then finally connect it with the AdS-CMT picture and we describe its role in describing holographic condensed matter systems.

In part II we present the original results of this thesis which contribute to the development of the AdS-CMT field and its ”Real-World” applications.

In part III we conclude with some final remarks, a brief summary and a list of ideas and homeworks for the future.

The present thesis is based on the following papers:

i. ”Electron-Phonon Interactions, Metal-Insulator Transitions, and Holographic Massive Gravity”

ii. ”Phases of holographic superconductors with broken translational symmetry”

iii. ”Under The Dome: Doped holographic superconductors with broken translational symmetry”

iv. ”Solid Holography and Massive Gravity”

v. ”Viscosity bound violation in holographic solids and the viscoelastic response”

vi. ”On holographic disorder-driven metal-insulator transitions”

Chapter 1

Condensed Matter Crash Course

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Contents

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Bad times have a scientific value. These are occasions a good learner would not miss.

Ralph Waldo Emerson

In this first part of the thesis we provide in a simple and coincise way, and from a particle physicist perspective, the condensed matter background necessary for the rest of the work. We will start our journey from the dawn of *Solid State Physics* and we will focus our description on the electric transport properties of the materials and their interpretation. We will wander among the various historical attempts of explaining such features up to the modern definition of the distinction between standard metals and insulators.

Once fixed a solid background, we will face the modern challenges that Condensed Matter present us getting into the modern mysteries of novel materials. We will provide a description of these

new exotic phases of matter finishing our itinerary in the fascinating and unexplained world of High-Tc Superconductors.
All of this will constitute our "Real-World" motivations for the rest of the work with the awareness that Holography could provide a new and unexpected tool to tackle such a problems.

Disclaimer: This is not meant to be a detailed and complete solid state physics essay for which we refer to standard condensed matter textbooks [1–3].

1.1 Solid state physics for dummies

The Drude Model

91 of the 118 elements of the Periodic Table are **metals**. Metals are widespread in nature, excellent conductors of heat and electricity, ductile, malleable and shiny. The definition of a metallic material and the challenge of accounting for these features go back in the days and gave rise to the birth of *Solid State Physics*. Soon after the discovery of the electron by J.J.Thomson in 1897, finding a simple model which, at least qualitatively, explains the distinction between insulators and metals and the transport properties of the latter has been a pressing issue.
The first attempt appeared already in 1900, just three years later, when Paul Drude proposed his model. Despite the simplicity and the *classical* nature of the **Drude Model** and its failure to account for some of the features, its success was considerable high and it still represents a practical and quick way to form a sketchy picture of what is really happening in a metal. Drude simply applied the *kinetik theory* of gases to a metal, imagining it as a gas of electrons. He wrongly assumed that the electrons could be modelled by a dilute gas whereas the usual electron density in a metal, around $n \sim 10^{28}/\text{cm}^3$, is approximately 1000 times bigger than the one of a classical gas at room temperature. Moreover, it is pretty clear, nowadays, that electrons do not constitute an ideal gas because they follow Fermi-Dirac distribution and they have appreciable interactions. Despite this naive assumption the results were pretty accurate and the reason why was found, years later (1957), by Landau. At that time he prooved that a gas of interacting particles can be equivalently described by a system of almost non interacting "*quasi-particles*" that can be indeed modelled accordingly to the Drude theory. Starting from the assumption

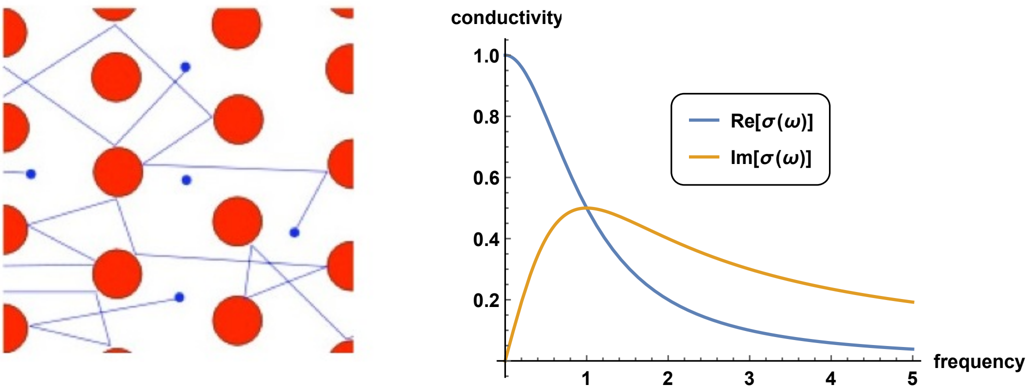

Figure 1.1: Drude model. **Left:** Sketch of the pinball ballistic picture where the blue dots are the electrons and the red ones the immobile ions. **Right:** Optical conductivity $\sigma(\omega)$ with $\sigma_0 = \tau = 1$.

that materials are globally neutral ($Q = 0$) the Drude Model considers the metal to be formed of a collection of heavy and positively-charged ions from which a number of "free electrons", *i.e.* the *conduction electrons*, were detached. The ions are considered to be immobile objects while electrons are free of moving around the "lattice" and scatter on the ions (see fig.1.1).
Several assumptions are then made:

i. All the events between one collision and the following are neglected. Electrons are therefore treated as free and they consequently move in straight lines in the middle of the scattering events. This assumption assumes that there are no appreciable $e^- - e^-$ interactions (*indipendent e^- approximation*) and that *e^--ions* interactions are irrelevant too (*free e^- approximation*). Despite the first approximation is surprisingly good[1], the second one is very bad and must be abandoned in order to account for the features observed in metals. All in all the only possible interactions between the free electrons and the environment is via collisions.

ii. The collisions are considered as instantaneous events and theiy result in a change in the electron velocity . The important point is the existence of *some* scattering mechanism without the need of inquiring too close what it is. Surprisingly enough, the Drude Model does not need any microscopic details of such a collisions to achieve its results and conclusions.

iii. The probability of a collision is defined as $1/\tau$ where τ is the *collisions time* or *relaxation time* and it has a fundamental role in the Drude model. This time scale is indipendent of the velocity and the position and of all the previous events; it is therefore defined as some "averaged time" which effectively describes the scattering events without any microscopic appeal.

iv. The electrons in the metal achieve thermal equilibrium only through the collisions.

In order to compute the transport properties within the Drude model we have to introduce an external electric field E which drives the electrons motion. Electrons will be accelerated in the opposite direction by the average electric field at their location. With each collision, though, the electron is deflected in a random direction with a velocity that is much larger than the velocity gained by the electric field. The net result is that electrons take a zigzag path due to the collisions, but generally drift in a direction opposing the electric field. Let us consider a density n of electrons whose correspondent current takes the form:

$$\vec{J} = -n\,e\,\vec{v} \tag{1.1}$$

The net current, parallel to the charge flow in the material, will be the average of the previous quantity which depends indeed on the average velocity of the electrons $\bar{v}$. In absence of any electric field the average velocity would be zero because the electrons would move in random directions colliding with random impurities and/or lattice imperfections in the crystal arising from thermal motion of ions about their equilibrium positions. At every collision the velocity can be written down as:

$$v = v_0 - \frac{e\,E\,t}{m} \tag{1.2}$$

[1]Electron-electron interactions are usually not taken into account because their interaction energy is commonly smaller than their kinetik energy. This just happens not to be true in a class of novel materials labelled as *Mott Insulator* where electron-electron interactions have to be considered and are the main responsables for the observed features.

where m is the electrons mass.
Averaging out on this quantity we recover the expression:

$$< v >= \underbrace{< v_0 >}_{=0} - \frac{e\,E\, < t >}{m} = -\frac{e\,E\,\tau}{m} \tag{1.3}$$

where $\tau =< t >$ is precisely the average time of collision. We can then express the net current as:

$$J = \left(\frac{n\,e^2\,\tau}{m}\right) E \tag{1.4}$$

As a theoretical description of the generalized Ohm's Law, $J = \sigma E$, one can finally extract the electric DC conductivity within the Drude Model, which reads:

$$\boxed{\sigma_{DC} = \frac{n\,e^2\,\tau}{m}} \tag{1.5}$$

The same type of computation leads also to the definition of the *mobility* $\mu = \frac{|v|}{E}$ of the charge carriers which in the Drude model appears to be:

$$\mu = \frac{e\,\tau}{m} \tag{1.6}$$

The net result of all this maths is a reasonable approximation of the conductivity of a number of monovalent metals. At room temperature, by using the kinetic theory of gases to estimate the drift velocity, the Drude model gives $\sigma \sim 10^{-6}\,\Omega^{-1}\,m^{-1}$. This is about the right order of magnitude for many monovalent metals, such as sodium ($\sigma \sim 2.13\,10^{-5}\,\Omega^{-1}\,m^{-1}$). If we substitute the room-temperature value of σ for a typical metal along with a typical n into the Drude equation, a value of $\tau \sim 1/10$ fs emerges. In Drude's picture, the electrons are the particles of a classical gas, so that they will possess a mean kinetic energy $\frac{1}{2}m_e < v^2 >= \frac{3}{2}k_B T$. Using this expression to derive a typical classical room temperature electron speed, we arrive at a mean free path $v\tau \sim 0.1/1$ nm. This is roughly the same as the interatomic distances in metals, a result consistent with the Drude picture of electrons colliding with the ionic cores.
One can do more and describe the dynamics of the electrons under an external applied generic force f as:

$$\frac{d\,\vec{p}(t)}{dt} = -\frac{\vec{p}(t)}{\tau} + \vec{f}(t) \tag{1.7}$$

where the individual electron collisions are incorporated in a frictional damping item which relates to τ. This description allows to compute several additional transport quantities such as the Hall Conductivity and the Magnetoresistance[2] and to extend the computations to the finite frequency regime.
Defining $f = -e\,E$ and going to Fourier space the momentum equation can be written as:

$$-\,\imath\,\omega\,p(\omega) = -\frac{p(\omega)}{\tau} - e\,E(\omega) \tag{1.8}$$

and the current can be therefore defined as:

$$J(\omega) = -\frac{n\,e\,p(\omega)}{m} = \frac{(n\,e^2\,/m)}{1/\tau - \imath\,\omega}\,E(\omega) \tag{1.9}$$

[2]One in this case has to define the external force to be $\vec{f} = -\,e\,(\vec{E} + \frac{p \times B}{m})$.

All in all the optical conductivity $\sigma(\omega)$ (see fig.1.1) derived using the Drude theory reads:

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - \imath \omega \tau}\,, \qquad \sigma_0 = \frac{n\,e^2\,\tau}{m}\,.} \tag{1.10}$$

Using similar reasonings the Drude model, joined with the Maxwell-Boltzmann distribution of kinetik theory, is able to provide also results for the Hall conductivity, the Thermo-Electric conductivities, and many others quantities.

Drude Model successes
First theoretical proof of Ohm's law
Predicts the Hall effect
Predicts the presence of a Plasma frequency
Predicts electric and thermal conductivities to a very good accuracy
Weidemann-Franz law
Drude Model failures
Presence of materials which are insulators and semiconductors (i.e. not metals)
Temperature dependence of the electric conductivity
Temperature dependence of the thermal conductivity
Temperature dependence of the specific heat
Overestimating the electronic heat capacity
Too long mean free path $l \sim v_f \tau$

In conclusion the Drude model gives a good enough classical description of electrical conduction in metals which leads to Ohm's law and shows that resistivity in a metal can be explained by the motion of its free electrons. We summarize its successes and failures in table 1.1. The biggest mistery that the Drude Model leaves us is the answer to the question:

Why are some materials metals and other insulators?

In order to improve the description of the transport properties in a metal and to answer this question we need to take into account *Quantum* effects and to relax some of the assumptions of the Drude Model such as the *free electron approximation*. The two main resolutions, which take us to a more realistic scenario, are the following:

i. We relax the assumption that between the collision the ions do not affect at all the electron's motion. We therefore let the electrons move in a static (and periodic) potential due to the ions rather than in free space.

ii. We relax the assumptions that the ions are static and immobile because heavy. We let the ions vibrate around their equilibrium position due to thermal fluctuations.

The Sommerfeld Model

The first step towards a more complete description is the promotion of classical mechanics to its quantum version, which will lead us to the introduction of the so-called **Sommerfeld model**.

Treating the electrons like quantum particles (i.e. fermions) rather than molecules of a classical gas represents the first main improvement to the Drude model. Pauli exclusion principle replaces the Maxwell-Boltzmann distribution with the Fermi-Dirac one and at the temperatures of interest ($T < 10^3$ K) those two can be amazingly different (see fig.1.2). We start considering the quantum-

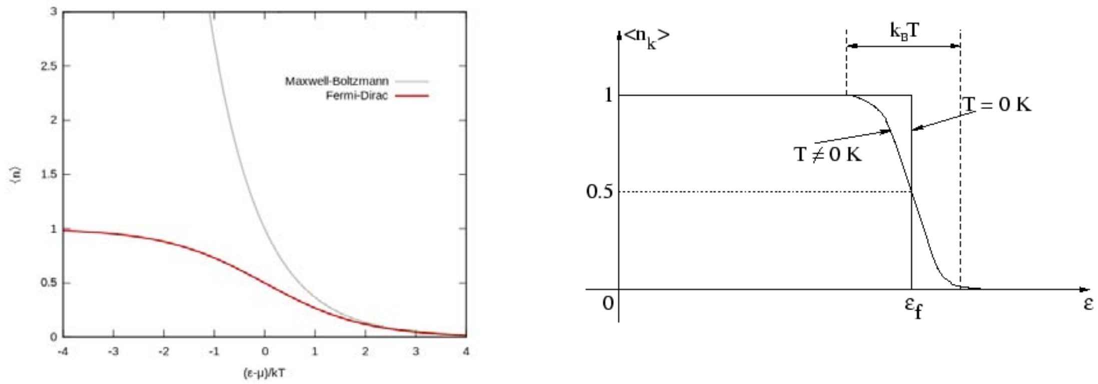

Figure 1.2: **Left:** Comparison between the Maxwell-Boltzmann and the Fermi-Dirac distributions. **Right:** Fermi-Dirac distribution at zero and finite temperatures.

mechanical problem of a single electron living in a volume V and satisfying the Schrodinger equation:

$$- \frac{\hbar}{2\,m} \nabla^2 \Psi(r) \;=\; \epsilon\, \Psi(r) \tag{1.11}$$

A solution of the former equation is the plane wave:

$$\Psi_k(r) \;=\; \frac{1}{\sqrt{V}}\, e^{\imath\, k\, r} \tag{1.12}$$

which implies the following energetic spectrum:

$$\epsilon(k) \;=\; \frac{\hbar^2\, k^2}{2\, m}\,, \qquad p \;=\; \hbar\, k\,, \qquad v \;=\; \frac{\hbar\, k}{m}\,. \tag{1.13}$$

Because of the volume restrictions, with the appropriate boundary conditions, the momentum of the electron gets quantized into:

$$k_i \;=\; \frac{2\,\pi\, n_i}{L}\,, \qquad i\,:\,x\,,\,y\,,\,z \tag{1.14}$$

and in a k-space region of volume Ω there therefore exist $\frac{\Omega\, V}{8\,\pi^3}$ allowed values for k.
Now assuming that the electrons are non interacting we can build the ground state ($T = 0$) of N electron states placing the electrons in the one-electron levels we just found. If N is large enough the k-space volume occupied by piling up the electrons will have the topology of a Sphere (*i.e.* the Fermi Sphere) with radius k_F and whose surface takes the name of the **Fermi Surface** (see fig.1.3). This is a direct application and consequence of the Pauli Exclusion principle and the density distribution f_k of the electrons in the case of $T = 0$ (*i.e.* the ground state) takes just the form of a step function centered at ϵ_F:

$$\begin{cases} f_k \;=\; 1 & \text{if } \epsilon_k < \epsilon_f \\ f_k \;=\; 0 & \text{if } \epsilon_k > \epsilon_f \end{cases} \tag{1.15}$$

With some easy computations is easy to show that the total number of electrons and the relative electron density are function of the Fermi momentum k_F as following:

$$N = \frac{k_F^3}{3\,\pi^2}\,V \quad \rightarrow \quad n = \frac{k_F^3}{3\,\pi^2}\,. \tag{1.16}$$

One can also define a Fermi velocity v_F which plays the role of the thermal velocity $v_K = (3K_BT/m)^{1/2}$ in a classical gas, used in the Drude model. Substituting a typical electrons density n we get a Fermi Energy ϵ_F in the range $\sim 1.5 - 15$ eV (*i.e.* $\sim$ atomic energies). and a Fermi velocity $v_F \sim 0.01c$. In conclusion the total energy of the N electrons ground state is given by summing the single state energies up to the fermi momentum k_F (and taking into account the spin degeneracy) as:

$$E = 2 \sum_{k<k_F} \frac{\hbar^2\,k^2}{2\,m} \tag{1.17}$$

Introducing some temperature T and a chemical potential μ, the distribution of the states gets

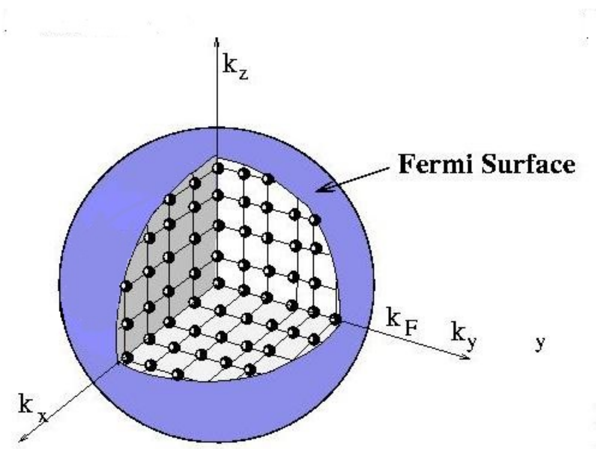

Figure 1.3: Fermi Sphere and Fermi Surface.

smoothed out and takes the form of the famous Fermi Dirac distribution:

$$f_i = \frac{1}{e^{(\epsilon_i-\mu)/k_BT} + 1} \tag{1.18}$$

which defines the mean number of electrons in the i level $i : \{k, s\}$, labelled by momentum k and spin s, whose energy takes the value $\epsilon(k) = \frac{\hbar^2 k^2}{2m}$. The total number of electrons is just given by summing up what just found as $N = \sum_i f_i$. Taking the $T \rightarrow 0$ limit of the Fermi-Dirac distribution we get the condition:

$$\lim_{T\rightarrow 0} \mu = \epsilon_F \tag{1.19}$$

which turns out to be true to an high degree of precision. The main implications of the Fermi-Dirac distribution are that:

- For $k_BT \ll \epsilon_F \rightarrow \mu \simeq \epsilon_F$, condition realized in metals for all the accessible T (until they melt):

$$\mu \simeq \epsilon_F \simeq \mu(T = 0) \tag{1.20}$$

- Only the electrons with energy ϵ within $k_B T$ and μ contribute to thermal processes, transport, etc...

Sommerfeld reexamined the Drude model with the Fermi-Dirac (FD) distribution instead of the Maxwell-Boltzmann(MB) one[3]. The use of the FD distribution affects only the predictions that require some knowledge of the electrons distribution and generates the following improvements to the Drude model results:

i. The mean free path gets smaller ($l \sim 100$ Angstrom) ;

ii. The specific heat is smaller by a factor $k_B T/\epsilon_F$ because $v_F^2 = \frac{2\,\epsilon_F}{m} \gg v_{kinetik}^2$;

iii. The overestimate of the thermopower gets corrected.

The Sommerfeld model does not modify any predictions concerning the DC and AC conductivities, the Hall coefficient and the magnetoresistance as far as the relaxation time τ is kept energy indipendent.

Sommerfeld Model successes
Temperature dependence of the electronic specific heat
Approximate T dependence of thermo-electric conductivities
T indipendence of the electronic magnetic susceptibility
Weidemann-Franz law
Sommerfeld Model failures
Presence of materials which are insulators and semiconductors (i.e. not metals)
Hall coefficient of many metals
Magnetoresistance
Different shapes of the Fermi Surfaces

Actually, shortly after Drude built his model, Lorentz introduced into the Drude model a energy dependent relaxation time $\tau(\epsilon)$ and noticed that in this way the DC and AC transport coefficients get dependent on the temperature T as in realistic situations. Anyway, once corrected the Drude model with the FD distribution, using a energy indipendent $\tau = \tau(\epsilon_F)$ or a energy dependent one $\tau(\epsilon)$ does not make any difference. This happens because the physical quantities are determined almost entirely by the scattering event and the dynamics around the fermi surface $\epsilon = \epsilon_F$.
Despite the several improvements given by the Sommerfeld theory (summarized with the still existing failures in table 1.1), the theoretical description is still lacking of an answer for the simple and fundamental question:
Why some materials are insulators and semiconductors (and not metals) ?
Technically speaking, it is intellectually unsatisfying to completely disregard the interactions between the electrons and the ionic cores, except as a source of instantaneous collisions. To get

[3]Note that despite he used a quantum distribution the model still belong to classical mechanics. The classical description becomes impossible if one has to consider electrons localized to within atomic distances. However the conduction electrons are not bond to particular ions, but can wander freely. There is thus a wide range of phenomena in which the system is well described by classical mechanics.

rid of the failures of the Sommerfeld model, and account also for insulating states, we must add interactions between these two elements; in other words, we have to take into account the periodic potential due to the lattice.

Band theory for solids

The free electrons assumption accounts for a wide range of metallic features but has to abandoned to reach more efficient descriptions for solids. The deficiency of the Drude model was due mainly by the use of statistical classical mechanics which lead for example to an estimate of the heat capacity hundreds of time too large but was obscured somehow by the fortuit success in determining the Wiedemann Franz ratio. The FD application of the Sommerfeld model eliminates this class of problems still retaining the free electrons approximation but continues to predict results in contradiction with experiments:

i. The Hall Coefficient can be derived to be $R_H = -1/nec$ where no T, τ, B dependence appears at all whereas in real experiments this is not the case and for example the B dependence is often dramatic.

ii. It predicts a null Magnetoresistance with the resistivity ρ indipendent of the magnetic field B.

iii. The Temperature dependence of the various quantities (DC and AC) can just be inserted by hand in the definition of the relaxation time τ; there is nothing in the free electrons models which accounts for a T dependence.

iv. Why are some elements non metals? For example why Boron is an insulator while its neighbor, aluminium, is an excellent metal? Why is Carbon an insulator in the form of a diamond and a conductor when in the form of graphite?

Despite all the previous oversimplifications must be abandoned to achieve an accurate model for solids, a remarkable amount of progress can be made by first just abandoning the free electrons approximation (without modifying the single electron approximation or the relaxation time approximation).
Once aware of this fact, waving the free electrons assumptions proceed in two stages:

- The electrons do not move in empty space but inside a static potential created by the ionic structure of the metal.

- The ionic cores are not immobile anymore and the dynamics of the ionic position has to be taken into account.

For the moment we stick to the first stage, which will be already able to provide the wanted distinction between a metal and an insulator.
Therefore the main point is to include in the single electrons dynamics a potential term due to the ionic cores $V(r)$ which modifies the Schrodinger equation into:

$$\left(-\frac{\hbar^2\,\nabla^2}{2\,m} + V(r)\right)\,\Psi(r) = \epsilon\,\Psi(r) \tag{1.21}$$

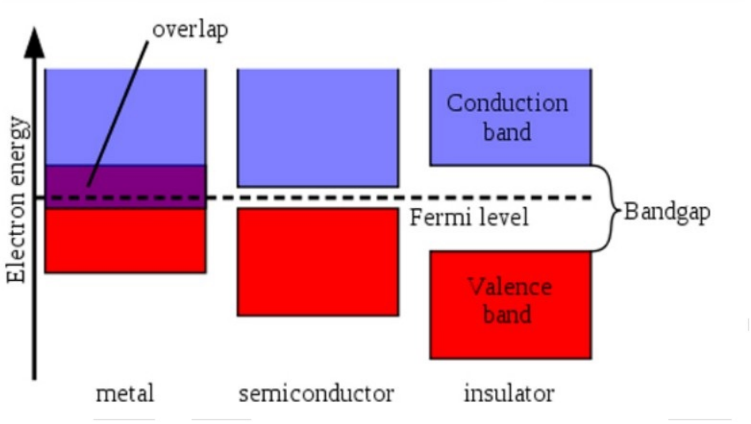

Figure 1.4: A sketchy explanation of how Band Theory works and distinguishes metals from semiconductors and insulators.

where the potential V is defined to be periodic $V(r+R) = V(r)$ with periodicity R, defined as the *Bravais lattice vector*[4].
Within this scenario one can prove, and it goes under the name of *Bloch's theorem*, that the eigenstates of Ψ take the form:

$$\Psi_{n,k}(r) \;=\; e^{\imath\, k\, r}\, u_{n,k}(r) \tag{1.22}$$

with $u_k(r)$ functions which are periodic in R. Note this implies that $\Psi(r+R) = e^{\imath kR}\,\Psi(r)$ with Ψ the N electrons wave-function.
If we insert now the Bloch's ansatz into the Schrodinger equation we get:

$$\underbrace{\left(\frac{\hbar^2}{2\,m^2}\left(\frac{1}{\imath}\,\nabla\,+\,k\right)^2\,+\,V(r)\right)}_{H_k}\,u_{n,k}(r)\;=\;\epsilon_k\,u_{n,k}(r) \tag{1.23}$$

equipped with the boundary condition $u_{n,k}(r) = u_{n,k}(r+R)$.
The wave vector k appears only as a parameter in the Hamiltonian H_k such that the energy levels $\epsilon_{n,k}$, for given k, vary continuosly as k varies. The descriptions of electrons in a periodic potential is therefore given in term of the continuous functions $\epsilon_n(k)$. The information contained in those function is referred to the **Band structure** and the electrons level specified by $\epsilon_n(k)$ is called an *energy band*. Note already as the functions $\epsilon_n(k)$ have to be bounded because continuous and periodic functions.
If the free electrons approximation predicted a discrete set of allowed energies, now with the introduction of a periodic potential the available energy states form bands which are somehow the results of the overlap of atomic orbitals.
The first striking conclusion is that electrons in a band are specified by a non vanishing velocity v:

$$v_n(k) \;=\; \frac{1}{\hbar}\,\nabla_k\,\epsilon_n(k) \tag{1.24}$$

This means that despite the interactions, there are stationary levels for the electrons in a periodic lattice such that they can move forever with a velocity $v_n(k)$ without any dissipation of momentum. This is totally different from the Drude picture we analyzed previously.

[4]Note that the scale of periodicity is usually $R \sim 10^{-8}$ cm ; therefore the use of quantum mechanics is mandatory.

Moreover, the concept of *Fermi Surface* is still the same as before with the only difference that now the single electron states are labelled by two quantum numbers n and k.
Now the crucial point for conduction is the position of the Fermi Surface within this electronic band structure. Two possible situations can arise:

- A certain number of bands are completely filled with all the others remaining empty. The difference in energy between the highest occupied one and the lowest unoccupied one defines the *band gap* ϵ_{GAP}. If $\epsilon_{GAP} \gg k_B T$ then we are in presence of an insulator, whereas if $\epsilon_{GAP} \approx k_B T$ we are speaking of a semiconductor. In the second case the gap is not big and thermal or other fluctuations can bridge it.
- A specific band is partially filled and the Fermi energy ϵ_F lies within the energy range of that band. In this case we have a metal.

Let's rephrase this concept in a different way. We can define a delocalized band of energy levels in a crystalline solid which is partly filled with electrons as a *conduction band*. The electrons present in the conduction band are vacant, they have great mobility and are responsible for electrical conductivity. On the other way the highest range of electron energies in which electrons are normally present at absolute zero temperature is called *valence band*. The position of the fermi level respect to the conduction band is a crucial fact in determining the electric transport properties of a material. If the Fermi level lies on top of the conduction band, which overlaps with the valence one, then the material is a metal[5]; if, on the contrary, there is a big gap between the two bands and the fermi level turns out to be just on top of this gap, the corresponding material will be an insulator (see fig.1.4).
At this stage, we are finally able to distinguish the materials accordingly to their conductivity properties into metals and insulators and to provide a simple but quite often accurate description of the observed physics. We will see in the next section that this is sometimes not enough and that the idea of a static lattice and eventually the single electron approximation have to be abandoned too.

Phonons

So far we have considered the ions as a fixed, immobile and rigid array. This is of course an approximation since the ions are not infinitely massive. In a classical theory this is true just at $T = 0$; in a quantum theory even at $T = 0$ this statement is false because of the indetermination principle $\Delta x \Delta p \geq \hbar$. This oversimplified assumption resulted to be impressively succesful whenever the physical property considered is dominated by the conduction electrons. To understand in a complete fashion the features of the metals (for example the temperature dependence of the DC transport coefficients) and expecially to achieve a more accurate description that a rudimentary theory of insulators we must go beyond. One point which is already clear is that under the assumption that the lattice is a static object, in insulators, where the electrons are quiescent, there are no degrees of freedom left to account for their varied features. That said let us analyze more extensively the failures of the static lattice model:

- The specific heat attributed to the electronic degrees of freedom is $\sim T$; this is true in real material just at low T (order $T > 10$K) whereas at high T it goes like $\sim T^3$ and at higher T

[5]If the overlap is small we are in presence of a *semimetal* with pretty peculiar features. We do not discuss this case here.

it reaches even a constant. These additional contributions are entirely due to the neglected *d.o.f.* of the ionic lattice (**phonons**).

- The static lattice model predicts a null specific heat for insulators, whereas in real situations it is not null and usually scales like $\sim T^3$.
- The temperature dependence of the resistivity can't be explained.
- The explanation of the Wiedemann-Franz law at intermediate temperature needs the introduction of of electrons-phonons scatterings.
- Thermal conduction in electric insulators is absent for the static lattice model.
- There is no sound propagation in insulators.
- The presence of Superconducting instabilities and SC phases are not explainable.

Once the lattice is not static anymore we can consider the normal modes of vibrations of the crystal as a whole and the dynamics of the small displacements around the equilibrium configuration. The corresponding standing waves, if longitudinally polarized, are called *sound waves* and the quanta of the lattice vibrational field are referred to as **phonons** [4]. The easiest possible picture is given by replacing the lattice by a volume formed by a gas of phonons carrying energy and momentum and considering the relative normal modes in the so-called harmonic approximation[6]. We will not enter in details the full quantum description of phonons theory which can be find in any ordinary CM textbook; we restrict ourselves just in collecting the major results and conclusions.
If one proceeds with the quantum mechanical description of phonons through the idealization into an elastic spring of atoms, one finds out that the solutions of the problem are just harmonic oscillators with dispersion relation:

$$\omega_k = 2\,\frac{K}{m}\left|\sin\frac{k\,a}{2}\right| \tag{1.25}$$

and energy $E_n = \left(\frac{1}{2}+n\right)\hbar\omega_k$ where a is the lattice spacing, m the atom mass and K is related to the spring constant of the atomic chain.
One can of course complicate the situation to a 1-D array of two species with masses $m_{1,2}$ and get the following result:

$$\omega_\pm^2 = k\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \pm k\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2 - \frac{4\sin^2(k\,a/2)}{m_1\,m_2}} \tag{1.26}$$

The two branches (see fig.1.5) are called *acoustic* and *optical* phonons. The one which at small momentum k takes the linear form $\omega = v_s k$ is the acoustic one and it is related with the sound propagation, while the other one encounters for the optical features of the material. Clearly realistic materials go beyond this simplistic approximations and the number of modes and their dispersion relations get more and more complicated (see fig.1.5). The theory of phonons gives rise in its continuum description to the elastic property of materials and it is much wider than what discussed here. This allows for example to distinguish clearly solids and fluids by the fact that fluids support just longitudinal waves and their rigidity is null. Additionally we can explore the

[6] Of course there could be and there are anharmonic terms resulting in interactions.

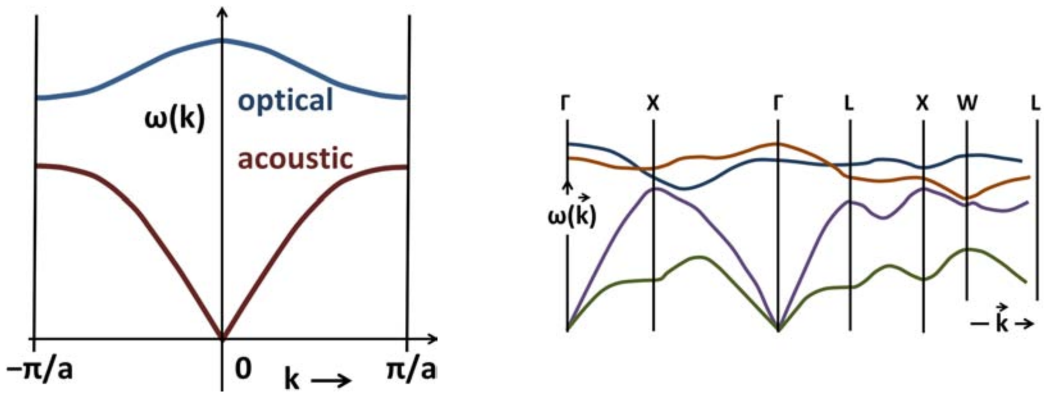

Figure 1.5: **Left:** Dispersion curves in linear diatomic chain. **Right:** Dispersion relation $\omega(k)$ for phonons in GeAs.

thermodynamical properties of phonons considering them as a gas and applying the Bose-Einstein statistic:

$$n(\omega_{k,s}) \;=\; \frac{1}{e^{\hbar\omega/k_B T} - 1} \tag{1.27}$$

and constructing the so-called Debye theory which for example predicts that the energy U takes the form of:

$$U \;=\; 3\,k_B\,T\,N\left(\frac{k_B\,T}{\hbar\Omega_D}\right)\int_0^{\beta\,\hbar\,\Omega_D}\frac{x^3}{e^x-1}\,dx \tag{1.28}$$

which turns out to be very successful in explaining the thermodynamical features of metallic and insulating materials.
Instead of focusing on the explicit computations, let's underline the various features that the phonons theory can explain with success:

- Specific Heat. The T^3 contributions and the constant behaviour at high T are entirely due to phonons physics and well explained by the Debye theory.
- Thermo-electric conductivities. Phonons explain the deviation of thermal conductivity from the electric one in non-metals materials and additional properties about transport.
- Superconductivity. The mechanism underling the ordinary superconductors (as we shall see later) is indeed due to pairing with phonons.
- If there are no phonons, all materials would be acoustic insulators and this is certainly not the case.

In conclusion, starting from the classical Drude Model, inserting the effects of quantum mechanics, relaxing the free electron approximation and finally introducing the dynamics of the ionic cores we reached a good description of lots of phenomena which real solids show off.

We end here our quick journey through the basics of solid state physics. This is of course not meant to be a complete, precise and detailed discussion of the topics followed but just a small appetizer for the reader. We end up with a successful, even if simple and approximated, description of several features of metallic and insulating states. A considerable percentual of realistic materials are well enough described by the frameworks we presented and just in recent years we had to face new challenges linked with novel exotic phases. These new situations, which do not fit in what already explained, are direct consequence of **strong coupling** and **strong correlation** and will force us to take a new perspective and rely on new tools.

1.2 Metal-insulator transitions and disordered electronic systems

Reamarkably simple theories have been prooved to be successful in describing noninteracting or weakly interacting electronic systems. The generic argument based on the filling of the electronic bands is able to provide a robust distinction between good insulators such as Silicon and Germanium from good metals such as Gold and Silver. Although the band picture is successful in many respects, de Boer and Verwey found out in 1937 that many transition transition-metal oxides with a partially filled d-electron band were nonetheless poor conductors and indeed often insulators. This raised the following questions:

How partially filled bands could be insulators?

How could an insulator become a metal as a continuous external parameter is varied?

The second phenomena go under the name of **metal-insulator transitions (MIT)** and take a particular role in the field of continuous phase transitions [5–9]. In the first place, they are not nearly so well understood, either experimentally and theoretically. In the second place, they usually belong to a particular subclass called **quantum phase transitions**, occuring at $T = 0$, where the critical behaviour is determined by quantum fluctuations rather then thermal ones (see fig.1.6). In contrast to simple situations, in systems close to a MIT physical features change

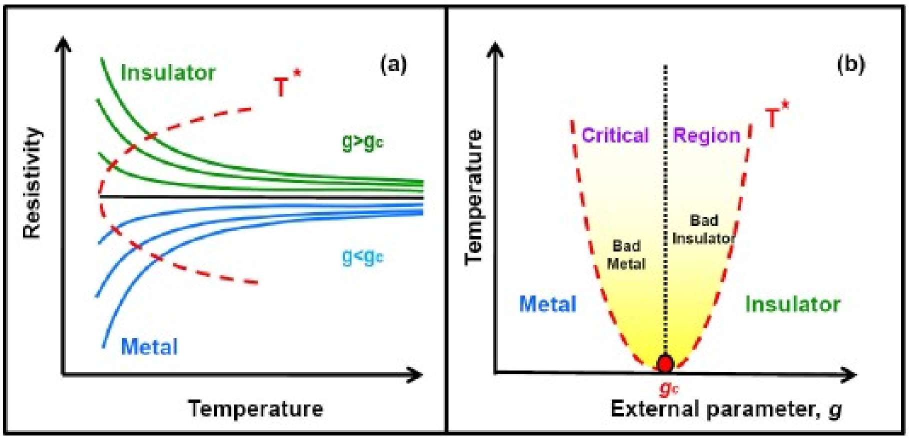

Figure 1.6: Metal insulator transitions.

dramatically upon varying an external parameter such as the carrier concentration, the pressure or the magnetic field. As a benchmark example, the electric resistivity can vary on several order of magnitudes along an MIT. As other quantum phase transitions (QPT), one expects the qualitative behaviour to display a certain degree of universality inside its critical nature, allowing an understanding based on simple yet fundamental physical pictures and concepts.

Despite these simple reasonings, MITs qualify as extremely difficult phenomena to be takled and described for two main reasons:

- The two limits, the one of good metals and the one of good insulators, are very different and stable systems characterized by completely different excitations. Metals are governed by fermionic quasiparticles excited above the Fermi sea while insulators by long-lived bosonic excitations, *i.e.* phonons. Therefore it looks extremely hard to connect continuously these two pictures together.
- No definite symmetry-breaking pattern is associated to any MIT. These are indeed better described by a dinamical transition where no obvious order parameter nor Laudau approach is available.

When band theory does fail: MITs!

The success of band theory was so impressive that already in 1930 Slater announced that solid state physics was a solved problem. (Un)-fortunately he was wrong. The band theory picture describes the motion of a single electron moving through a solid. This approximation is reliable only when its kinetik energy is consistently bigger that the typical energy scale of the system. One can quantify this statement with the so called r_s parameter, $r_s = E_c/E_F$, where E_c is the average Coulomb energy and E_F the Fermi energy. r_s is usually $\sim 3/5$ in good metals, thus one would naively think that band theory should never work. However one has to take into account the following points:

- Screening effects reduce the electron-electron and electron-impurity interactions significantly.
- The largest part of the Coulomb energy does not give rise to correlation and many body effects but just renormalizes the effective potential.
- Pauli exclusion principle restricts a lot the phase space for electron-electron scatterings.

As a result, a good description of the system is usually provided by a diluite collection of quasiparticles as defined in Landau **Fermi Liquid theory**. In a nutshell, Fermi Liquid is "protected" by a large kinetik scale of the electrons such that electronic correlations and impurities effects can be treated as small perturbations.
Sometimes this fails! Materials close to the MIT have small Fermi energy and quantum effects driven by Pauli exclusion start to weaken. Notable examples are:

i. Narrow band materials such as transition-metal oxide V_2O_3.

ii. Doped semiconductors

iii. Doped magnetic (Mott) insulators such as the famous high-Tc cuprate $La_{2-x}Sr_xCuO_4$.

In these scenarios the potential energy due to electron-electron interactions or disorder effects becomes comparable to the Fermi energy and the ground state undergoes a sudden and dramatic change: electrons become bound or "localized". The materials cease to conduct although band theory does not produce any gap at the Fermi surface. In the following we will briefly describe how that can take place and some generic features of such quantum phase transitions.
Historically, we can divide the MIT into two classes:

i. MIT triggered by electronic correlations (or electron-electron interactions): **Mott transitions**

ii. MIT triggered by disorder: **Anderson transitions**

Despite the increasing interests and progresses in understanding such an interesting mechanism like the Mott transition and the effects of the electron-electron self interactions through the rest of this work we will focus our attention just to the second case.

Disordered electronic systems

Impurities and defects are ubiquitous in real materials and they simply produce random scattering of mobile electrons. In ordinary metals the random potential due to impurities can be considered as a small perturbation whose only effect is giving additional contributions to the relaxation time appearing in the Drude formula accordingly to the *Matthiessen's rule*:

$$\tau^{-1} = \tau_{el}^{-1} + \tau_{ee}^{-1}(T) + \tau_{ep}^{-1}(T) \tag{1.29}$$

where τ_{el}^{-1} describes the elastic scattering rate by impurities and $\tau_{ee}^{-1}(T), \tau_{ep}^{-1}(T)$ the inelastic scattering processes by electrons and phonons. Note as in this picture the total resistivity becomes:

$$\rho(T) = \rho_0 + A\,T^n \tag{1.30}$$

where the residual resistivity ρ_0 is indeed a measure of impurity (elastic) scattering.
In contrast, in low carrier density systems, the impurity potential is comparable or larger then the Fermi energy, and the electrons can get trapped, *i.e.* "localized" by the impurities [10, 11]. Of course, this process generally leads to a sharp metal-insulator transition only at $T = 0$, since at finite temperature the electrons can overcome the impurity binding potential through thermal activation.
The possibility that true electronic bound states can be formed in presence of a random potential was first discussed by Anderson in 1958 [12]. Traditionally the quantum theory of electronic

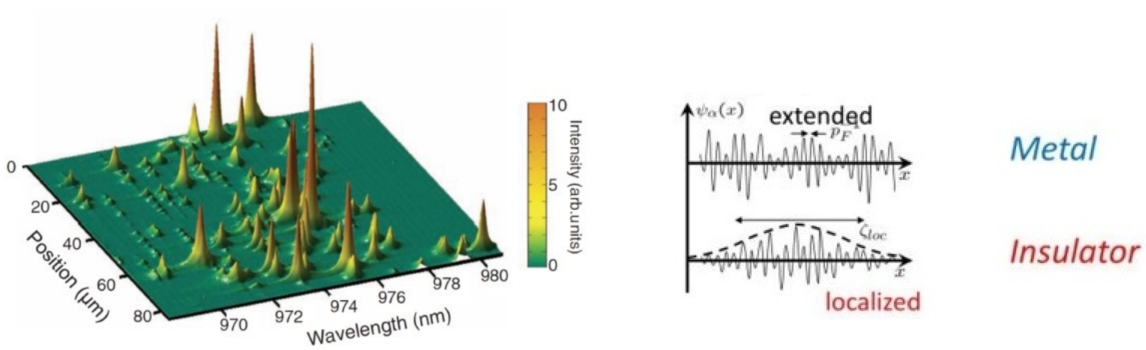

Figure 1.7: Anderson Localization. **Left:** Wavefunction amplitudes $|\psi_i^2|$ of strongly localized states. **Right:** Localized and extended states.

conductivity was built on the picture of an electron being multiply scattered by impurities and diffusing through the solid. A cardinal concept in the description of the diffusion of the electron is the mean free path, the average length the electron travels before it suffers a collision. The electron executes a zigzag motion and the mean free path is the average zigzag length. The appearance of strong multiple scattering correlates with a very short mean free path. Beyond a critical amount of impurity scattering the diffusive motion of the electron will come to a halt. There is not much

to wave anymore for a wave when its mean free path has become shorter than its De Broglie wave length. This stopping or localization has dramatic consequences for the conductivity - the material turns into an insulator. The phenomenon of **Anderson localization**, particularly that of weak localization, finds its origin in the wave interference between multiple-scattering paths. In the strong scattering limit, the severe interferences, due to random scatterings, can completely halt the waves inside the disordered medium.
Anderson, in his seminal work in 1958, considered the problem of a single electron in a dirty cristal: the quantum mechanical analogue of random walk in a random environment. While Einstein prooved in 1905 that classical random walk alway implies *diffusion*, namely:

$$< r^2 >= D\,t \tag{1.31}$$

with D the so called *diffusion constant*; Anderson showed that for a quantum particle this is not always the case and diffusion can be in certain limit (strong disorder) replaced by:

$$\lim_{t\to\infty} < r^2 >= const\,, \qquad \Longrightarrow D\,=\,0\,. \tag{1.32}$$

because of the consequences of *quantum intereference.*
The corresponding electric conductivity can be written like:

$$\sigma\,=\,e^2\,D\,v\,, \qquad v\,=\,\frac{dn}{d\mu} \tag{1.33}$$

where v is the electronic density of states. This means that in absence of diffusion the conductivity becomes null and the material presents insulating features. In these terms it seems that a possible good order parameter for the MIT is the value of the electric conductivity (at $T = 0$) itself.
The original formulation based on the absence of diffusion can be recasted in a quantum mechanical language using the Schrodinger equation:

$$\left(-\frac{\nabla^2}{2\,m}\,+U(r)\,-\epsilon_F\right)\,\Psi(r)\,=\,\epsilon\,\Psi(r) \tag{1.34}$$

The local density of states $\rho_{loc}(\epsilon)$ for an electron with energy ϵ is proportional to the wavefunction amplitude on site:

$$\rho_{loc}(\epsilon)\,\sim\,\left|\Psi_i^2(\epsilon)\right|^2 \tag{1.35}$$

The absence of diffusion and the phenomenon of Anderson Localization can be defined as the exponential localization of the single electron wave function:

$$\Psi(r)\,=\,e^{-|r|/\xi_{loc}} \tag{1.36}$$

where ξ_{loc} is the so called *localization length.* If the states localize only a small number of them have an appreciable overlap and therefore the conductivity drops down. Therefore in an Anderson Insulator, the local density of states (LDOS) will consist only of a few discrete δ-function peaks with appreciable weight (usually of order ϵ_F).
Localization effects are already present, but only in reduced dimensions, in the weak scattering limit where they take the name of "weak localization". Despite the numerous progresses in the field and the various proposals (phenomenological β function, scaling theories, random matrix models, DMFT) Anderson Localization still remains an open and intriguing question. Indeed, in

most realistic systems the disorder strength and electron-electron interactions have comparable magnitude and the number of particle playing in the game is usually very high. These two factors make the non-interacting single particle reasoning made by Anderson too naive and open the door towards:

- Anderson-Mott transitions, where both the effects of disorder and electrons self-interactions have to be both taken into account;
- Many-Body Localization (MBL) where the single particle wavefunction is not a reliable tool anymore.

What is the fate of Anderson Localization when the constituent particles interact between themselves?

What happens to Anderson Localization in a many-body problem where the single electron approximation is not valid anymore?

Strongly correlated systems, which cannot be effectively described in terms of indipendent and non-interacting entities, still constitute one of the most intriguing and misterious research field in modern solid state physics. The absence of a single particle approximation and a perturbative regime makes the theoretical description of such a systems very hard and call for new innovative tools. **Gauge gravity duality** could possibly be one of them.

1.3 High-Tc superconductivity

Conventional Superconductivity

Superconductivity is a state of matter characterized by a vanishing static electrical resistivity and an expulsion of the magnetic field from the interior of the sample [13, 14].
After H.K. Onnes had managed to liquify Helium, it became for the first time possible to reach temperatures low enough to achieve superconductivity in some chemical elements. In 1911, he found that the static resistivity of mercury abruptly fell to zero at a critical temperature T_c of about 4.1 K. In a normal metal, the resistivity decreases with decreasing temperature but saturates at a finite value for $T \to 0$. That was not the case and he immediately realized that he was standing in front of a new state of solid matter. Under a certain temperature, defined as the *critical temperature*, the system undergoes a phase transition into this novel phase where the resistivity drops down to 0 (see fig.1.8) which takes the name of **Superconducting state**. He also realized that a certain amount of magnetic field (*critical magnetic field*) $H_c(T)$ and a critical current $J_C(T)$ would destroy that state of matter and restore the usual metallic normal phase (see fig.1.8). A second striking feature is the so called *Meissner Effect*, namely the strong repulsion of the magnetic field from the SC sample. This somehow qualifies a superconductor as a perfect diamagnetic material with zero magnetic permeability $\mu = 0$[7].
Further experiments indicated that the critical temperature, at which the SC transition appears, $T_c \approx \Omega$ where Ω is the typical oscillation frequency of the ions in the materials. This constituted a strong indication that the SC mechanism is somehow linked to the oscillations of the ionic lattice, *i.e.* the phonons. Conventional superconductivity is indeed the physics of the **Cooper Pairs**,

[7] This can be better formalized using the so called *London equations* $\nabla^2 B = \frac{1}{\lambda_c} B$ and $J \sim A$ where λ_c is the penetration length and A the gauge field. We refer to a ordinary CM textbook for such details.

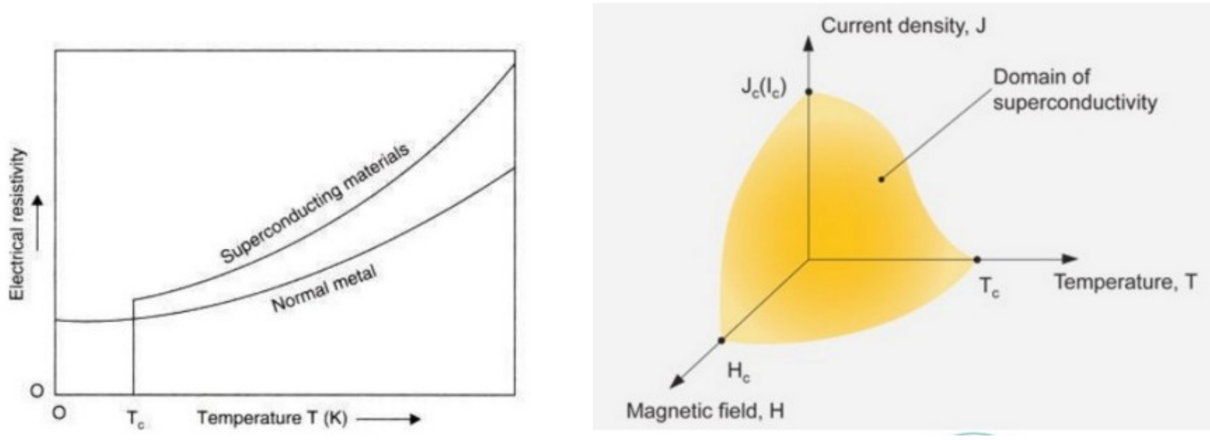

Figure 1.8: Superconducting materials. **Left:** Comparison of the electric resistivity in the normal metallic phase and in the SC one. **Right:** Sketch of the phase diagram for the SC state.

bound states of two electrons glued together by electron-phonon interactions. **BCS** (Bardeen, Cooper and Schrieffer) theory predicts that at sufficiently low temperatures, electrons near the Fermi surface become unstable against the formation of Cooper pairs. Cooper showed that such binding will occur in the presence of an attractive potential, no matter how weak. In conventional superconductors, an attraction is generally attributed to an electron-lattice interaction. The BCS theory, however, requires only the potential to be attractive, regardless of its origin.
Naively we can imagine the following picture: let us take an electron e_1 with defined momentum, energy and spin $e_1 = (k, \epsilon_k, \uparrow)$ and another one with same energy but opposite momentum and spin $e_2 = (-k, \epsilon_k, \downarrow)$. The Coulomb interaction between the first electron e_1 and the ions provokes a displacement in the ionic structure which takes the name of *polarization*; as a consequence the region around e_1 is now more positive charged than its equilibrium configuration. This account for an attractive potential U for the second electron e_2 which is now forced to follow and form a bond to the first one, creating indeed the so called *Cooper pair*. In conventional SC this is driven by electron-phonon interactions and can be explicitely computed in a diagrammatic fashion. As a result the corresponding critical temperature T_c is directly proportional to the coupling of the electron-phonon interactions:

$$T_c \sim g_{e-ph} \tag{1.37}$$

and because of this reason BCS theory predicts a maximum critical temperature of order $T_c \sim 30K$[8]. To increase the critical temperature the electron-phonon interactions should be stronger and would make the material unstable towards the formation of charge density waves.
Once the *Cooper pairs* are formed the electrons are not obliged anymore to follow the Fermi-Dirac statistic and the pairs themselves, now bosonic objects, can undergo *Bose-Einstein* condensation and create a macroscopic ground state which is energetically favoured and whose electric resistivity becomes null. In this regard, superconductivity can be strictly related to superfluidity and analyzed in the optic of *Landau Theory*.
The main idea is to identify an *order parameter*: a thermodynamical variable which is 0 on one side of the transition and not null on the other one. Let us assume that this order parameter ζ is constant in space and time and let us follow the so called *mean field theory*. In analogy with

[8]The highest BCS superconductor turns out to be Nb_3Ge with $T_c \approx 23K$.

superfluidity we can build up the free energy F as a function of the temperature T and the order parameter ζ and we can expand it as:

$$F = \alpha \zeta^2 + \frac{\beta}{2} \zeta^4 \tag{1.38}$$

In a superfluid:

$$\int d^3 r \left| \Psi_s(r) \right|^2 = n_s V \tag{1.39}$$

where n_s is the superfluid density and the wavefunction module $\left|\Psi_s\right|^2$ can indeed take the place of the order parameter such that $F = \alpha \left|\Psi\right|^2 + \frac{\beta}{2}\left|\Psi\right|^4$. The reasoning can follow, with some caveats, also for the SC scenario. Now if $\alpha > 0$ there is only a single minimum (see fig.1.9) at $\Psi = 0$ where the superfluid/superconductor density is 0. On the contrary for $\alpha < 0$ there is another minimum at $\Psi = \sqrt{-\frac{\alpha}{\beta}}$ where the density n_s is finite. If one then defines:

$$\alpha = \alpha' (T - T_c), \qquad \beta = const \tag{1.40}$$

the phase transition appears indeed at a critical temperature T_c and the order parameter scales like:

$$\Psi \sim \sqrt{T - T_c} \tag{1.41}$$

which is a characteristic result of mean field theory. BCS theory predicts that the correlations

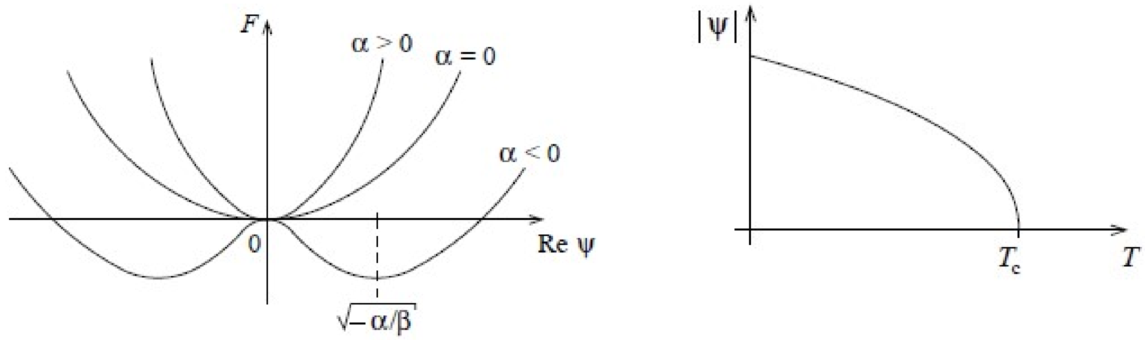

Figure 1.9: Landau theory for super-(fluids/conductors). **Left:** Free energy. **Right:** Order parameter.

between the electrons, mediated by phonons, can be broken with a certain amount of energy Δ_{gap} and their "binding energy" can indeed defined as $= 2\Delta_{gap}$. This quantity takes the name of the Superconducting gap. It represents the energy gain of the SC state and it is normally a function of the temperature (and eventually of momentum[9]). A SC material can therefore uniquely be defined by two parameters:

$$\boxed{\mathbf{SC} \longrightarrow \{T_c, \Delta_0\}} \tag{1.42}$$

BCS theory fixes in a universal way these two quantities to satisfy:

$$\frac{2\,\Delta_0}{k_B\,T_c} = 3.52 \tag{1.43}$$

[9]We will restrict ourselves to isotropic situations, namely *S-Wave SC* where the gap is constant and can be define as $\Delta(k = 0)$.

The physics of conventional SC materials is more intricated, complicated and wider than what we just described, for length constrictions, here. We refer to standard CM textbooks for a detailed analysis.

Beyond BCS theory

The BCS framework turned out to be very succesful and led Bardeen, Cooper and Schrieffer to the Nobel Prize in 1972 ”for their jointly developed theory of superconductivity”. Years later, in 1986, two IBM researchers G.Bednorz and K.A. Muller[10] found out that a particular material, whose electronic structure reads $La_{2-x}Ba_xCuO_4$, can undergo a superconducting transition at $T_c \sim 35K$ [15]. That represented a shocking result and opened the scenario for a large class

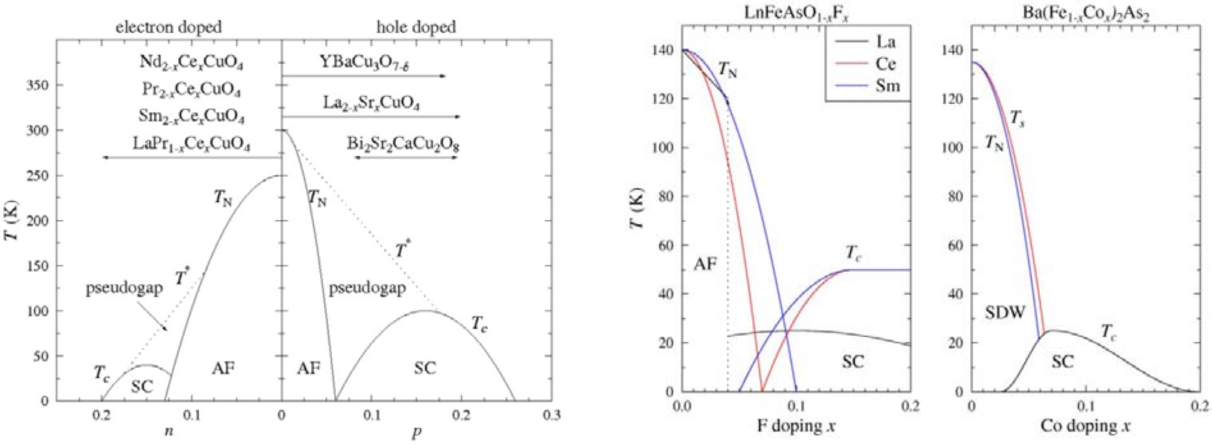

Figure 1.10: Phase diagrams for various high-Tc SC compunds. **Left:** Simplified doping dependent phase diagram of cuprate superconductors for both electron (n) and hole (p) doping. The phases shown are the antiferromagnetic (AF) phase close to zero doping, the superconducting phase around optimal doping, and the pseudogap phase. **Right:** Simplified doping dependent phase diagrams of iron-based superconductors for both Ln-1111 and Ba-122 materials. The phases shown are the antiferromagnetic/spin density wave (AF/SDW) phase close to zero doping and the superconducting phase around optimal doping..

of new materials, called **High-Tc Superconductors**, whose critical temperature is unusually high and in contrast with the conventional BCS predictions [16, 17]. Until 2008, only specific compounds of Copper and Oxygen, called ”Cuprates”, were thought to possess this unexplained feature but later on several other materials have been found such as the Iron-based compounds (”Pnictides”) [18]. Nowadays, the highest known critical temperature is about $T_c \sim 203K$ and it referes to sulfur hydride H_2S at extremly high pressure [19].
High-Tc superconductors provide extremely challenging questions and unexplained features:

- The extremely high critical temperature can’t be explained within BCS theory by electrons pairing through phonon interactions. If naively one pushes this further, realizes that such a high T_c will require an interaction with a very strong coupling which would make the full framework not reliable.

[10]Who were awarded the 1987 Nobel Prize in Physics ”for their important break-through in the discovery of superconductivity in ceramic materials” too.

- The normal phase of such High-Tc superconductors are not Fermi Liquids. They indeed shows off a peculiar linear in T resistivity $\rho \sim T$ which is in constrast with the Fermi Liquid prediction[11]. In these novel phases there is no clear Fermi Surface and therefore BCS fails just from the beginning. A fermi liquid instability requires a Fermi Surface! How do we get a SC from a non Fermi liquid?

- The coherence length (which measures the "extension" of the Cooper pairs) of the high-Tc superconductors is way smaller than BCS predicition because of the very small Fermi Energy ϵ_F.

- Within the phase diagram of these materials (see fig.1.10) there are several open questions (Antiferromagnetic ordering, Pseudogap phases, etc. . .) and the interplay between superconductivity and magnetism appears to be pretty relevant.

After two decades of intense experimental and theoretical research, with over 100000 published papers on the subject, several common features in the properties of high-temperature superconductors have been identified. As of 2016, no widely accepted theory explains their properties.

1.4 Quantum Criticality

Classical phase transitions occur at a finite temperature. A material that is tuned close to a classical phase transition senses the imminent change of state as the order parameter develops thermal fluctuations over larger and larger regions of the sample: such a state is known as a "critical state". **Quantum phase transitions** (QPT) are phase transitions at temperature $T = 0$ which occur upon varying a non-thermal control parameter (such as pressure, magnetic field, or chemical composition) [20]. A QPT implies non-analytic behavior of the ground-state energy as function of that control parameter. As there are no thermal fluctuations at zero temperature, a QPT is apparently driven by "quantum fluctuations". Today the phenomenon of quantum phase transitions has emerged as a major challenge to our understanding of condensed matter. As in classical phase transitions the coherence length ζ diverges at the critical point $r = r_c$. The wavefunction for the quantum state at $r = r_c$ is then a complicated superposition of an exponentially large set of configurations fluctuating at all length scales: in modern parlance, it has long-range quantum entanglement. The quantum critical point is defined by the ground state wavefunction, and so, strictly speaking, it is present only at the absolute zero of temperature. Thus, from an experimental perspective, it may seem that a quantum phase transition is an abstract theoretical idea. However, as will become amply clear below, the influence of the critical point extends over a wide regime in the $T > 0$ phase diagram: this is the regime of quantum criticality[12], which is crucial for interpreting a wide variety of experiments. Upon increasing temperature starting from the QCP, this critical continuum will be excited, resulting in power-law behavior of thermodynamic observables as function of temperature with non-trivial exponents. These power laws are the experimentally accessible signatures of quantum criticality; they signal the ground state at the QCP being a "novel state of matter".
As we have seen above, the critical point is characterized by a diverging correlation length:

[11] Fermi Liquid theory predicts the resistivity to be quadratic in temperature $\rho \sim T^2$; this result comes just from the T dependence of electron-eletron scatterings.

[12] Technically speaking this region extends wherever the thermal fluctuations are "small" compared to the quantum ones.

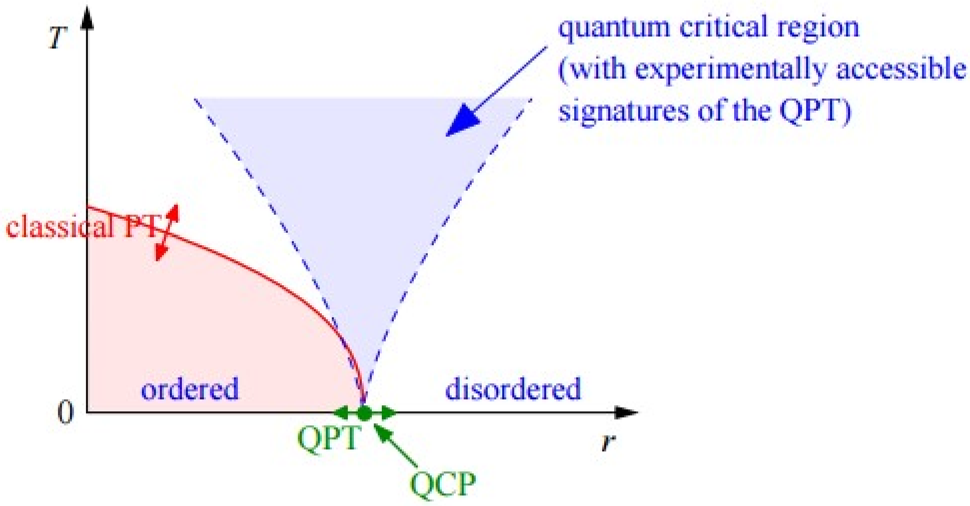

Figure 1.11: Generic phase diagram in the vicinity of a continuous quantum phase transition.

$\zeta \to \infty$. This implies that the order-parameter fluctuations do not display a characteristic length scale at criticality, hence fluctuations exist on all length scales. The system is said to be scale-invariant, i.e., looks ”similar” on all length scales. A consequence of scale invariance is that observables depend on parameters in the form of power laws, because power laws are the only scale-invariant dependencies. These power laws define critical exponents and are an important part of critical phenomena. Critical phenomena display a high degree of *universality*. This means that critical exponents are identical for classes of phase transitions, the so-called universality classes. Universality is rooted in the divergence of the correlation length: given that critical phenomena are determined by the physics at large length scales, microscopic details become unimportant.
The idea of universality and scale invariance call for an understanding of such quantum phenomena through **conformal field theories**. This qualifies the class of issues described in this section as suitable for being tackled by the so called **AdS-CFT** correspondence, which will be indeed the main tool presented and exploited along this thesis.
The development of a unified understanding of thermal phase transitions and classical criticality was a triumph of the 20th century. We are still far from a complete understanding of quantum phase transitions, but already, many suspect that the ultimate solution to this problem may be needed to understand and ultimately control phenomena such as high temperature superconductivity or metal-insulator transitions will depend on the development of a new theory of quantum phase transitions. In this sense Gauge Gravity duality is a promising direction to pursue.

Chapter 2

AdS-CF(M)T correspondence

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Contents

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The distance between insanity and genius
is measured only by success.

Bruce Feirstein

What electrons moving in a strongly correlated material and strings moving in a 11-dimensional spacetime have in common?

What Black Holes and Quantum Gravity can tell us about High-Tc Superconductivity and Disordered systems?

How can a String Theorist and a Solid state physicist eat now at the same table?

These are few of the questions we will adress in this chapter.
An unpredictable and astonishing connection between completely different branches of physics is out in the market providing a revolutionary point of view for lots of the long standing problems of modern physics.
The ”magic stick” goes under the name of **AdS-CFT correspondence** and it is one of the most important result of theoretical physics in the last decades. It is a powerful duality between a Quantum Field Theory in d dimensions (without dynamical gravity) and a theory of Quantum Gravity in $d+1$ dimensions with the following suprising characteristics:

- The number of dimensions of the two sides does not correspond! This is why the theory is denominated *holographic*.
- One side contains dynamical (and quantum) gravity while the other one is defined on a fixed background and it is defined by completely different degrees of freedom.
- When one side is strongly coupled the other one is weakly coupled and viceversa. For this reason AdS-CFT lies in the class of *Strong-Weak* dualities.

The number of new directions, perspectives and connections that this discovery has introduced in the field of physics (and not only, *i.e.* maths) is unbelievable and represented by the incredible amount of efforts and published papers in the last 20 years.
In this section we will drive our DeLorean DMC-12 back in time at the origin of such a discovery and we will revisit the major steps of its formulation. We will sketch the original stringy derivation of such a duality and we will give at least three motivations:

- Gauge Theory at Large N $\Longleftrightarrow$ String Theory.
- Gravitational d.o.f. $\sim$ area : *holographic principle*.
- RG is local in the energy scale: QFT has an extra ”emergent” dimension.

which hints towards its existence and the identification of Holography as the ”geometrization” of the QFT Renormalization Group (RG) flow.
We will then, in a *Bottom-Up* fashion, underline the main aspects and features before showing explicitely how the tool works with three benchmark examples:

i. The scalar field in AdS spacetime: an example of the AdS-CFT dictionary.

ii. The Reissner-Nordstrom Black Hole: the core of the AdS-CMT program.

iii. The holographic superconductors model: the first ”real-world” application.

The number of existing reviews on the Gauge Gravity duality is nowadays huge; we list here just some of the main ones we will be following with particular attention to the bottom-up setup and the applicative side [21–24].

2.1 The stringy tale: history class

The AdS-CFT correspondence was originally formulated in the context of String Theory [25–27]. Although some of the contents of this section are not indispensable to understand some of the subsequent chapters, they are important for building the reader's intuition about the duality and to sketch at least its historical origin.

All you need to know about String Theory

We just give the reader the necessary String Theory ingredients in order to understand the original formulation of the AdS-CFT correspondence; for a complete description of String Theory several sources are available [28–33].
In 1961 G.Chew and S.Frautschi [34] recognized that the mesons, hadronic quark-antiquark states, follow the so-called *Regge trajectories*, namely their masses M turn out to be proportional to their spin J:

$$M^2 \sim J \tag{2.1}$$

Such a property of the hadronic resonances implies that the scattering of these particles should fall off exponentially quickly at large angles. Scattering of pointlike constituents leads to large angular deviations at high energies and therefore a theory of such a composite states following straight Regge trajectories was missing at that time. The first theory of this sort, the dual resonance model, was constructed by Gabriele Veneziano in 1968 [35], who noted that the Euler Beta function could be used to describe 4-particle scattering amplitude data for particles on Regge trajectories; it was soon later realized by Miguel Virasoro and Joel A. Shapiro [36, 37] that such a behaviour had to do with string-like objects. That was the born of String Theory (see more details in [38])! If one consider indeed an open string with tension T, it is easy to demonstrate that the relative mass obeys a relation of the type $M^2 \sim TJ$ which represents a Regge trajectory with slope T.
The main idea was to replace pointlike objects as fundamental degrees of freedom with extended objects, namely strings with length l_s. It is a non-local description which boils down to the common point particle representation in the limit $l_s \to 0$. The classical description of the string dynamics takes inspiration directly from the case of a point particle in special relativity. Let's consider a point particle of mass m moving in Minkoswki space $\eta_{\mu\nu}$; its motion could be described by a curve in spacetime $x^\mu(\tau)$ (*i.e.* wordline) where x^μ is the position of the point particle moving. The action for the point particle is proportional to the integral of the line element along the trajectory in spacetime, with the coefficient being given by the mass m of the particle:

$$\mathcal{S} = -m \int ds = -m \int_{\tau_0}^{\tau_1} d\tau \sqrt{-\eta_{\mu\nu}\, \dot{x}^\mu \, \dot{x}^\nu} \tag{2.2}$$

Let's push now the analogy to the case of the string, a one-dimensional object moving into the spacetime describing a surface, the so-called *worldsheet* 2.1. Its action is again constructed in terms of the area element dA of the worldsheet surface as:

$$\mathcal{S} = -T \int dA \tag{2.3}$$

where T is the tension of the stringy object which is usually taken to be $T = \frac{1}{2\pi\alpha'}$. The various parameters of the string are correlated between each other as:

$$l_s = \sqrt{\alpha'} = \frac{1}{M_s} \tag{2.4}$$

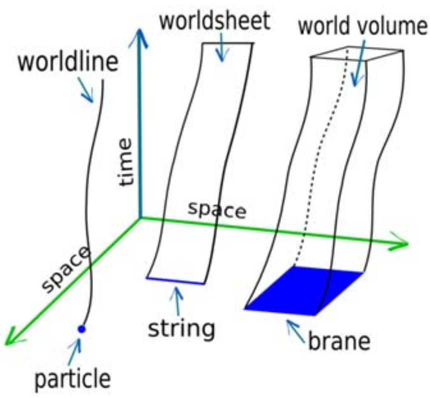

Figure 2.1: The dynamics of a point particle versus the dynamics of extended objects.

where l_s and M_s are the length and the mass of the string.
The 2-dimensional worldsheet Σ can be parametrized by a set of coordinates $\xi^a = (\tau, \sigma)$ and lives in a target space $\mathcal{M}$ with metric $G_{\mu\nu}$ such that there exists an embedding , *i.e.* a map $\Sigma \rightarrow \mathcal{M}$ described by $\xi^a \rightarrow X^\mu(\xi^a)$. With this in mind the action of the string, the *Nambu-Goto (NG) action*, can be written down in the following form:

$$\mathcal{S} = -T \int \sqrt{- det\tilde{G}_{ab}}\, d^2\xi \tag{2.5}$$

where $\tilde{G}_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ is the induced metric on the worldsheet. The classical equations of motion coming from the NG action can be solved in flat space $G_{\mu\nu} = \eta_{\mu\nu}$ for both Dirichlet and Neumann conditions. The correspondent solutions, *i.e.* the close and open strings, can be Fourier expanded as an infinite superposition of oscillation modes, much as in the string of a violin. This procedure, the quantization of the string, can be carried out in several ways from the simplest method of canonical quantization to the more advanced BRST techniques. Those vibration modes are particles from the point of view of the target spacetime such that the string represents an infinite tower of particles with growing masses and spins and with a mass gap defined by $\sim l_s$.
The quantization of the string leads to several suprising results. The first one is the presence of tachyonic modes with negative mass $m^2 < 0$ which are the sign of an instability. In order to avoid such a issue, fermionic d.o.f. have to be introduced in the theory such that Supersymmetry emerges in the target space. The final theory takes the name of *Superstring* and can be consistentely defined only in $D = 10$ dimensions[1]
The other very relevant point is that the string spectrum contains a massless spin 2 field which can be identified as the graviton. String theory is not a theory of Hadrons, it is a theory of **Quantum Gravity**! This problem is of particular importance in the landscape of modern theoretical physics because of the inconsistency between Quantum Mechanics and General Relativity. The search for such a theory of quantum gravity is one of the most interesting and pressing issue of the modern times and String Theory could probably provide an answer to that question. The String length l_s has therefore to be identified with the Planck scales l_P, the scale at which quantum corrections to Einstein's theory of General Relativity become important and mandatory.

[1]If not, the Lorentz group would aquire quantum anomalies leading to negative norm states. Note that for the Bosonic string something similar happens and the consistent number of dimensions has to be fixed to $D = 26$.

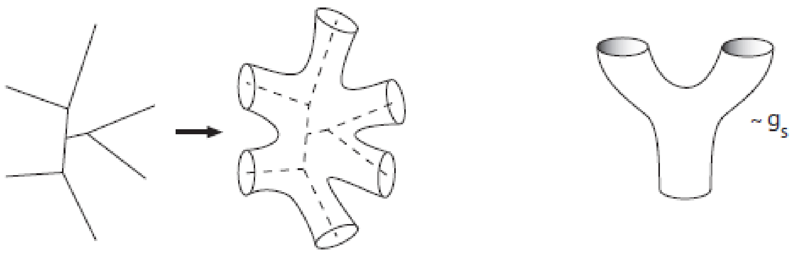

Figure 2.2: **Left:** The field theory vertex corresponds to a two-dimensional surfaces with boundaries. **Right:** The triple vertex for three closed strings is represented on the right as a pants surface along with its coupling constant g_s. Figure taken from [23].

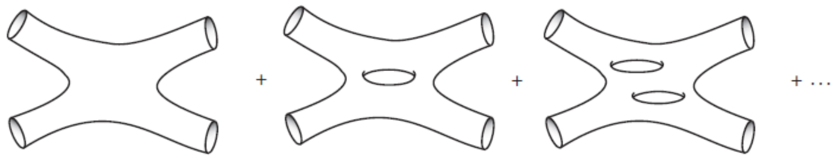

Figure 2.3: Perturbative expansion of the amplitude corresponding to four closed strings. Figure taken from [23].

In the low energy limit $l_s \to 0$ the massive modes decouple and only the massless ones are the relevant IR degrees of freedom. To be more precise, the quantization of the closed string leads to a massless metric field $g_{\mu\nu}$, with the opportune diffeomorphism invariance properties, while the quantization of the open string is responsible for the massless spin 1 field connected to the Gauge symmetry. There is also a collection of higher order forms which will be relevant in the following.
The amazing outcome is that at low energy $E \ll M_s$ the quantum consistency of the theory imposes the equations of motion for the $g_{\mu\nu}$ field to be:

$$\text{Einstein} + \ldots = 0 \,. \tag{2.6}$$

where the dots stand for correction to the Einstein equations, which are recovered in the limit of low energy.
The corrections can be organized in a double expansion: in powers of $\alpha' E^2$ from integrating out the massive modes and in powers of g_s from the string loops.
Strings indeed can interact between themselves. For example two close strings can join together and then split again into two. A vertex in field theory corresponds to a two-dimensional surface with boundaries which is associated to a coupling constant g_s 2.2 and a loop to a Riemann Surface with an hole. The higher order loops correspond to surfaces with more than a hole. The number of holes h of a Riemann surface, corresponding to the number of string loop, is called *genus* and the perturbative series of String Theory takes therefore the structure of a topological expansion in genus 2.3! In particular a generic string scattering amplitude can be written down as a genus

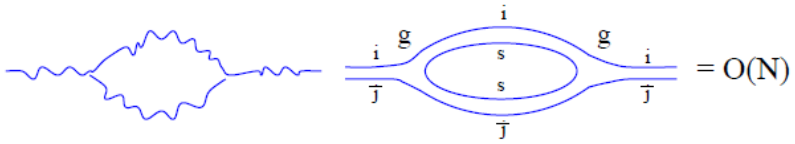

Figure 2.4: An explicit example: a graph contributing to the gluon self-energy. Figure taken from [23].

(number of loops) expansion[2]

$$\mathcal{A} = \sum g_s^{2h-2}\, \mathcal{F}_h(\alpha') \tag{2.7}$$

where the $\mathcal{F}$ part can be ulteriorly expanded in powers of α'.
We will see in a moment how this expansion is the first hint towards the point that a theory of gravity (String theory in this case) could be equivalent to a completely different quantum field theory.

Large N Quantum Field theories

Studies of the non-perturbative features of quantum field theories are at the forefront of theoretical physics research. Though remarkable progress has been achieved in recent years, still, some of the more fundamental questions have only a descriptive answer, whereas non-perturbative calculable schemes are seldom at hand. The absence of calculable dynamics in realistic models is often supplemented by simpler models in which the essence of the dynamics is revealed. Such a calculable framework for exploring theoretical ideas is given by large N quantum field theories [39]. It was realized indeed by t'Hooft in 1974 that a U(N) Gauge Theory[3] extremely simplifies when the number of colors N is taken to be large $N \to \infty$ [40]. A $1/N$ expansion can be performed and it turns out to be useful and efficients in various directions; for example some QCD models become solvable in the large N limit.
Let's consider a U(N) gauge theory defined by the action:

$$\mathcal{L} = Tr\left(F_{\mu\nu}^2 + \mathcal{L}_{matter}\right) \tag{2.8}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\, g_{YM}\,[A_\mu, A_\nu]$ the non-abelian field strength and $\mathcal{L}_{matter}$ the matter lagrangian, which generically includes fields in the fundamental and adjoint representations.
There is a convenient pictorial representation of Feynman graphs in terms of a double line notation. Fundamental and anti-fundamental fields can be written q_i and $q_{\bar{j}}$, respectively, where $i, \bar{i} = 1, \dots, N$ and the bar distinguishes indices transforming in the anti-fundamental representation. Adjoint fields of U(N) can be therefore written as hermitian matrices $A_{i\bar{j}}$ and thought as formal products of a fundamental and antifundamental representation. We shall use a Feynman graph notation where oriented lines are associated with indices i and $\bar{j}$ and not with fields. In this way, the propagator for an adjoint field can be then naturally written as a double line. An efficient way to understand this new language is by looking at explicit examples, such as the gluon self energy diagram of fig.2.4. The self-energy, pictured in the example of fig.2.4, diverges

[2]To be more precise the string coupling constant g_s does not represent an indipendent parameter but it is set by the expectation value of a scalar field contained in the string spectrum called *dilaton* : $g_s = e^{\langle\phi\rangle}$. It can therefore depends on the spacetime coordinate. The constant coupling we refer in the main text is defined be $g_s = e^{\langle\phi_\infty\rangle}$.
[3]This is actually true also for O(N) vector models.

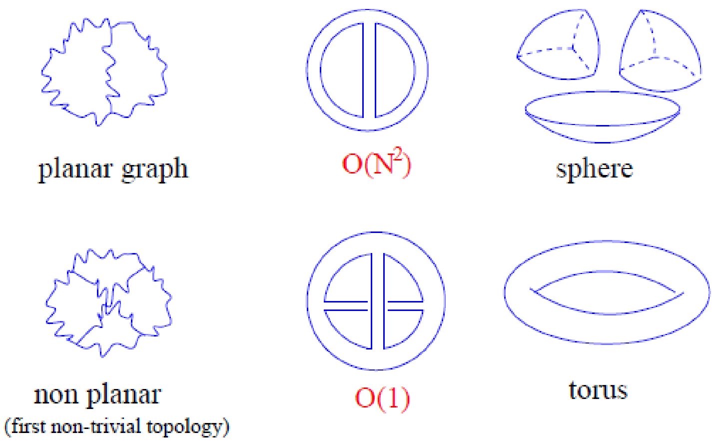

Figure 2.5: Planar and non-planar graphs and their relation with Riemann surfaces. Figure taken from [23].

as $\mathcal{O}(N)$ and lots of other diagrams do as well. It seems therefore that the large N limit is not a sensible limit to take. However if we analyze the self energy diagram in more details we realize that it is actually of order $\sim g_{YM}^2 N$. As a consequence if we take the so-called *t'Hooft limit*:

$$N \to \infty\,, \qquad g_{YM} \to 0\,, \qquad \lambda = g_{YM}^2\, N = \text{fixed} \tag{2.9}$$

where λ is called the *t'Hooft parameter* the self energy remains finite. The same happens to all other graphs. For more and more details see [41]. The *t'Hooft limit* is a sensible limit and gives rise to a consistent entire perturbative expansion.
It is better to redefine the fields and bring all the dependence on g_{YM} in front of the Lagrangian:

$$\Longrightarrow \mathcal{L} = \frac{N}{\lambda} Tr \left(F_{\mu\nu}^2 + \dots \right) \tag{2.10}$$

With this notation the propagator account for a λ/N factor while vertices provide a factor N/λ. Within this language one can see that graphs with different topology contribute with different powers of N. In particular there is a clear distinction between the *planar* diagrams and the *non planar* ones. Let's get into the details using the explicit example shown in fig.2.5. The first diagram is planar, meaning it can be drawn on a plane, or more technically it can be seen as a triangulation of a sphere. This is explicitely showed in the double line notation where no superposition of lines is evident. The correspondent Riemann surface has a topology with genus $h = 0$. The second graph is instead non planar, if we insist in drawing it on a plane some of its lines will interesect in points which are not vertices of the graph. The best we can do is to represent it on a Torus, a Riemann surface of genus $h = 1$.
In more generality, every graph can be drawn without interesecting lines on a Riemann surface whose Euler Characteristic is given by:

$$2 - 2\,h = F - V + E \tag{2.11}$$

where F is the number of faces of the graph, E is the number of edges and V the number of vertices and h is the genus, or the number of holes, of the Riemann surface.

Taking into account that we have a factor of N/λ for each propagator (E), a factor of λ/N for each vertex (V) and a factor of N for each loop (F), we can derive a general formula for every diagram:

$$\lambda^{V-E}\,N^{F-V+E} \;=\; \mathcal{O}(N^{2-2h}) \tag{2.12}$$

It is now clear that the t'Hooft expansion organizes graphs according to their topology. The expansion of the Free Energy of the theory in the large N limit becomes particulary simple:

$$F \;=\; \sum_{h=0}^{\infty} N^{2-2h}\, f_h(\lambda) \tag{2.13}$$

and shows amazing similarities with the perturbative string expansion (2.7)! The large N expansion considerably simplifies the perturbation theory. For $N \to \infty$ only the planar graphs survive. However the most striking result is the incredible similarity of the perturbative string expansion (2.7) and the gauge theory large N expansion (2.13) which seem to be equivalent under identifying:

$$N \;\sim\; g_s^{-1}\,, \qquad \alpha' \;\longleftrightarrow\; \lambda \tag{2.14}$$

This sketchy map will be made more precise in the following after introducing the original formulation of the AdS/CFT correspondence.

Maldacena's original argument

String theory is not only a theory of strings but it contains also extended solitonic objects called D_p branes [42]. They are $p+1$ dynamical hypersurfaces and their existence can be motivated via (at least) two arguments:

- These objects have to be necessary in the spectrum of the theory because one of its symmetries property known as *T-duality* (see for example [43]);
- They are naturally coupled to higher order p-forms A_p, coming from the quantization of the string, through their worldvolume $\mathcal{M}_{p+1}$:

  $$\sim \int_{\mathcal{M}_{p+1}} A_{\mu_1,\ldots,\mu_{p+1}}\, dx^{\mu_1}\,\ldots\, dx^{\mu_{p+1}} \tag{2.15}$$

  and they can be therefore defined to result charged under those higher rank gauge fields.

The action for such objects takes schematically the form of[4]:

$$\mathcal{S}_{D_p} \;=\; -\,T_{D_p} \int d^{p+1}x\;[\ldots] \tag{2.17}$$

where T_{D_p} is the brane tension and can be computed as:

$$T_{D_p} \;=\; \frac{1}{(2\,\pi)^p\, g_s\, l_s^{p+1}} \tag{2.18}$$

[4]To be more precise $[\ldots]$ is the famous DBI action:

$$\sim\; \sqrt{-\,det\;(g_{\mu\nu} \,+\, 2\,\pi\, l_s^2\, F_{\mu\nu})} \tag{2.16}$$

with g the induced metric and F the field strength of the worldvolume gauge field. For more details see [44].

From the latter it is evident that D-branes are non perturbative objects $\sim 1/g_s$ which can be anyway defined within perturbative string theory as hypersurfaces where open strings end.
If we take a collection of N D-branes then open strings can end on different branes such that they can be described by two indices i, j running from $i = 1, \ldots, N$. In other words, the low energy description of such extended objects can be done via a U(N) gauge theory.
In this section we will sketch the original proposal by Maldacena [25] stating the duality between type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}_5$ in 10 dimensions and $N = 4$ Super Yang Mills (SYM) in $3+1$ dimensions.
Let's start considering a stack of N D_3 branes in flat ten dimensional Minkowski space. In this setup there are two kind of excitations: the closed string, empty space excitations and the open string, which encode the excitations of the D_3 branes. In the low energy description $E \ll \frac{1}{l_s}$ only the massless states survive and contribute to the dynamics. The low energy closed string massless states, a gravity supermultiplet in 10 dimensions, are effectively described by type IIb supergravity (SUGRA). On the other side the low energy description of the open massless states, a $N = 4$ vector supermultiplet in $3 + 1$ dimensions, is econded in a $N = 4$ SYM theory.
The complete effective action for the massless modes will take the form of:

$$\mathcal{S} \, = \, \mathcal{S}_{bulk} \, + \, \mathcal{S}_{brane} \, + \, \mathcal{S}_{int} \tag{2.19}$$

where:

- $\mathcal{S}_{bulk}$ is the action of ten dimensional type IIB supergravity, plus some higher derivative corrections;
- $\mathcal{S}_{brane}$ is defined on the $3+1$ brane worldvolume and it containts the action of $N = 4$ SYM plus some higher derivative corrections, for example terms of the form $\alpha'^2 \, Tr(F^4)$;
- $\mathcal{S}_{int}$ describes the interaction between the bulk modes and the brane modes. The leading term can be obtained by covariantizing the brane action, introducing the background metric for the brane [45]. The correspondent coupling for such a tower of interactions reads $\kappa = g_s \alpha'^2$.

It is easy to proove that in the low energy limit all the interaction terms drop out; it is indeed a well known fact that gravity becomes free at long distances (low energies).
In order to see better what happens it is convenient to take the low energy limit keeping the energy fixed and shrinking the string length $l_s \to 0$ ($\alpha' \to 0$) mantaining all the dimensionless parameters, including g_s and N, finite. In this limit it is evident that the couling $\kappa \to 0$ and the bulk-brane system completely decouples. The supergravity theory in the bulk becomes free, while the higher derivative terms on the brane vanish leaving a pure $N = 4$ SYM theory, which is known to be a *conformal theory*. All in all we are left with a free gravity theory in the bulk and a pure conformal gauge theory on the brane which do not talk to each other (see fig.2.6). Let's now consider te system from a different point of view. D-branes are massive dynamical objects which act as gravitational source for the supergravity fields. It is possible to find a supergravity solution for a D_3 brane [46] of the form:

$$\begin{aligned} ds^2 \, &= \, f^{-1/2} \, \Big(- dt^2 \, + \, dx_1^2 \, + \, dx_2^2 \, + \, dx_3^2 \Big) \, + \, f^{1/2} \, \Big(dr^2 \, + \, r^2 \, d\Omega_5^2 \Big) \, , \\ f \, &= \, 1 \, + \, \frac{R^4}{r^4} \, , \qquad R^4 \, = \, 4 \, \pi \, g_s \, \alpha'^2 \, N \, . \end{aligned} \tag{2.20}$$

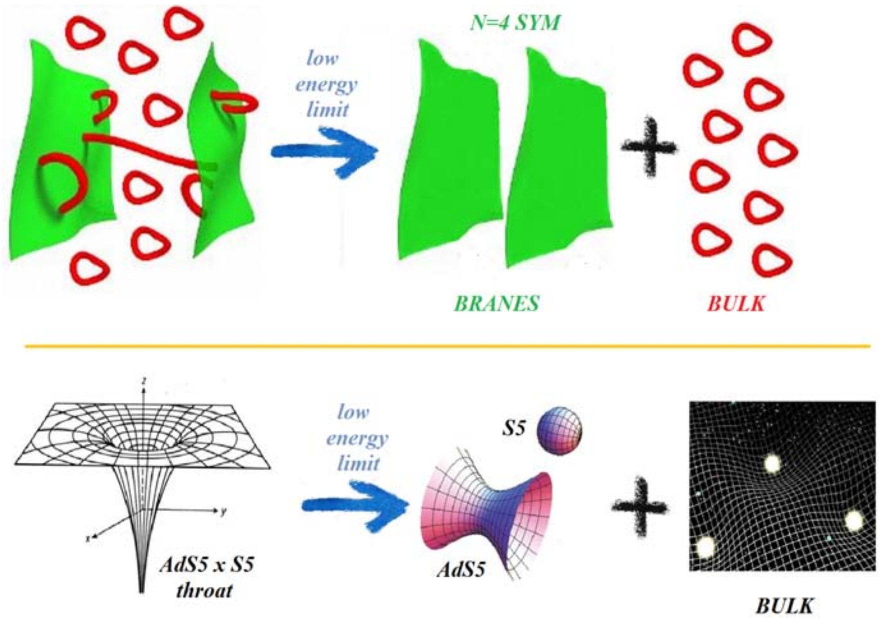

Figure 2.6: Low energy limit $E \ll \frac{1}{l_s}$ of the N D-branes system and bulk-brane decoupling procedure in the two pictures described in the main text.

plus a field strength F_5 corresponding to the gauge field living on the brane worldvolume, which will not be relevant for the present discussion.
Note that because of the non trivial g_{tt} component the energy E_p measured by an observer at a certain radius $r = r_p$ and the energy $E = E_\infty$ measured by an observer at infinity are related by a redshift factor via:

$$E = f^{-1/4} E_p . \tag{2.21}$$

Now let's proceed with the low energy limit also in this picture. There are two types of low energy excitations:

- massless particle propagating in the bulk region with very large wavelengths;
- any kind of excitation that we bring closer and closer to the $r = 0$ *near-horizon* region.

In the low energy limit these two types of modes completely decouple. In conclusion we are left with a free bulk supergravity theory and a *near-horizon* geometry which becomes (because for $r \ll R$ we have $f \sim R^4/r^4$) :

$$ds^2 = \frac{r^2}{R^2}\Big(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\Big) + R^2\frac{dr^2}{r^2} + R^2\, d\Omega_5^2 . \tag{2.22}$$

which is exactly the geometry of an $AdS_5 \times S_5$ spacetime.
We see that both from the point of view of open strings living on the brane and from the supergravity description we are left with two decoupled theories in the low energy limit (fig.2.6). In both cases one of the decoupled systems is supergravity in flat space. It is therefore natural to identify the other two sides, procedure which leads to state the original **AdS/CFT conjecture** [25] as:

$$\boxed{N = 4 \text{ U(N) SYM in } 3+1 \text{ dimensions}}$$

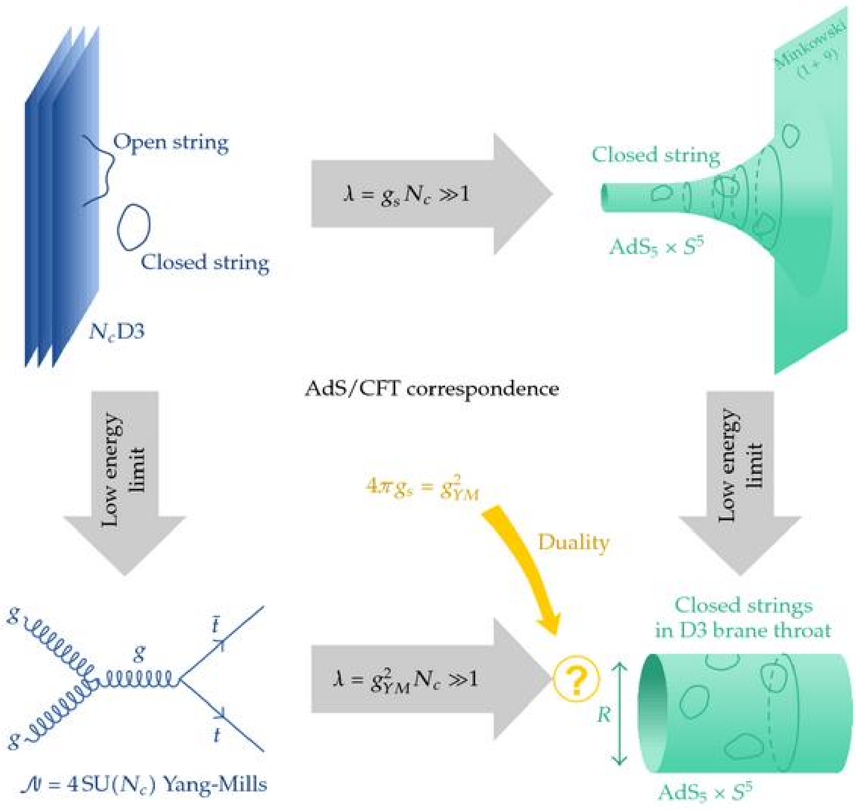

Figure 2.7: Sketch of how the original formulation of the duality [25] works.

$$\updownarrow$$

$$\boxed{\text{Type IIB supergravity in } AdS_5 \times S_5}$$

where the double arrow stands for **dual** (in a sense that will be more precise in the following). If we perform the sketched analysis in more details we discover that the various parameters of the U(N) gauge theory and the Supergravity theory are related in a one to one correspondence as:

$$\left(\frac{L}{l_s}\right)^4 = N\, g^2_{YM} = \lambda\,, \qquad \left(\frac{l_p}{L}\right)^8 = \frac{\pi^4}{2\,N^2} \tag{2.23}$$

This means that considering a classical description of gravity, where the quantum effects can be neglected because $\frac{l_p}{L} \ll 1$, corresponds to take the large N limit $N \gg 1$ of the U(N) "dual" gauge theory.
On the other side, in the limit $\frac{l_s}{L} \ll 1$, we can neglect all the stringy effects and this corresponds to take the strongly coupled limit $\lambda \gg 1$ of the "dual" gauge theory.
We finally discover that the classical gravity description maps to the strongly coupled and large N regime of the dual gauge theory. This weaker version of the duality takes the name of **Bottom-Up** and it will be what we will be using for the rest of this thesis. The idea is to forget about Strings and Branes and just consider and study classical theory of gravity which reduces to:

$$\text{General Relativity + bunch of fields on curved spacetime} \tag{2.24}$$

The point is that this limit is sensible and expecially interesting and useful because it corresponds to that regime of the QFT side which is less known and less tractable with standard and perturbative techniques.
Despite nowadays we have several hints and we know several examples, beyond the original $[N = 4]\, SYM \leftrightarrow [AdS_5 \times S_5]$ case, there is no non perturbative proof of the conjecture available yet.
Nevertheless people believe in what is called the strong conjecture which states the existence of such a duality between the gravitational theory and the quantum field theory for all g_s and N.
In full generality we now refer to the **Gauge-Gravity duality** as a generic duality between a specific theory of gravity and a generic quantum field theory. The question of searching such "duals" and the requirement of both sides in order to have a "dual" is still an open and active question we will not adress in this work.
We just simply accept the correspondence, in its *Bottom-Up* formulation, and we will use it as a tool to study Condensed Matter applications.

2.2 Three hints to motivate it

One of the most intriguing and at the same time surprising feature of the AdS/CFT correspondence is the fact that the two sides of the duality, meaning the two equivalent descriptions, live in spacetimes with different number of dimensions. This characteristic justifies the label of **Holography** which is usually associated to the correspondence. To be more precise, considering a QFT in d spacetime dimensions, the dual gravitational picture has to live in a $d+1$ dimensional bulk. Despite being quite shocking, there are at least three hints which point towards the confirmation that it must be the correct way.

Weinberg-Witten no-go theorem

The first consideration turns around the definition of a theory of Quantum Gravity, which can be defined as a quantum theory with a dynamical spin 2 massless particle, *i.e.* the graviton. A spin 2 graviton, composite object somehow made up from the gauge theory degrees of freedom, has to be in the spectrum of the theory. This seems naively in contrast with the famous Weinberg-Witten no-go theorem [47], which states: *massless particles (either composite or elementary) with spin $j > 1/2$ cannot carry a Lorentz-covariant current, while massless particles with spin $j > 1$ cannot carry a Lorentz-covariant stress-energy. The theorem is usually interpreted to mean that the graviton ($j = 2$) cannot be a composite particle in a relativistic quantum field theory* .
There are several ways of overcoming the no-go theorem. One of them is the possibility that gravity does not live in the same space of the QFT, which is somehow a prelude of the idea of holography.

Renormalization Group

The second hint comes directly from the modern analysis of Quantum Field theories with extensive degrees of freedom. The idea of considering quantum field theories at different scales/energies goes back to the 70's when, in the context of statistical mechanics, Wilson formulated the first version of the so called **RG** flow. Let's consider for example a spin system with hamiltonian:

$$S_i = \pm 1 , \qquad H = \sum_{<ij>} J_{ij}\, S_i\, S_j + \sum_{<<ij>>} K_{ij}\, S_i\, S_j + \dots \tag{2.25}$$

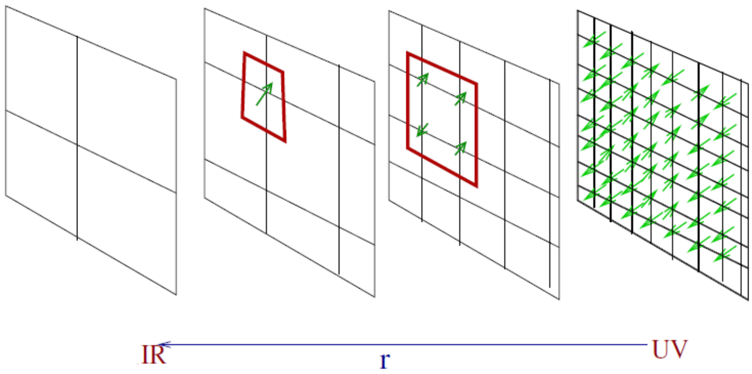

Figure 2.8: Sketchy representation of the Wilsonian RG flow idea with a spin system. Figure taken from J.McGreevy lectures.

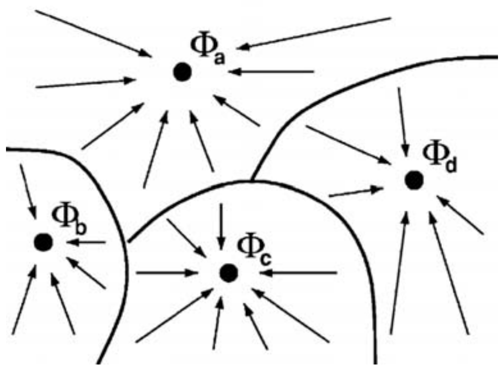

Figure 2.9: RG flow sketch: fixed points and universality classes.

The idea is to measure the system with coarser and coarser rulers (see fig.2.8). In other words, at each step we are going to "block" the spins and average the value of the spins inside the block . The so called Wilsonian effective action of the theory, and the Hamiltonian itself, will have the property of being invariant under these transformations of scale as far as the coupling constants of the theory g_i (J, K in the example) are taken to run with the scale of the system μ. Imposing such a invariance constraint fixes those couplings to obey particular differential equations, called *Beta function* equations:

$$\mu\, \partial_\mu\, g(\mu) \;=\; \beta_g(g(\mu))\,. \tag{2.26}$$

where indeed β_g is defined as the *Beta function* of the coupling g. This collection of equations defines the behaviour of the system under scale transformations and therefore at different energies, under the so called *Renormalization group flow* (RG).
In conclusion the idea is that a QFT has to be thought as sliced by scale as a family of trajectories of the RG governed by the previous equation. Those particular points in the phase space of the couplings g_i where the β function is vanishing, $\beta_g = 0$, are defined *fixed points* and they exploit the property of *scale invariance* or self-similarity, meaning that upon changing the resolution their "pictures" stay the same (as the famous cauliflower analogy). Often, but not necessarily[5], scale invariance implies *conformal invariance*, which is indeed what stands in the right side of AdS/CFT and it will aquire a fundamental role in the correspondence.

[5]See for example [48] for discussions about this issue.

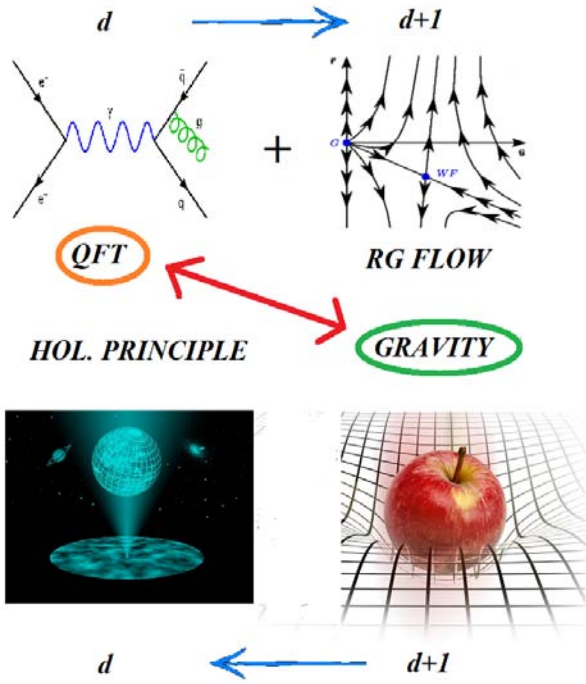

Figure 2.10: Sketchy relations between QFTs in d dimensions and theories of gravity in $d + 1$ dimensions hinting at the existence of a duality between the two pictures.

Many different microscopic theories can end up, under RG flow, in the same IR fixed point. This is the idea of *universality*: the behaviour of the IR theory is determined by a small number of relevant couplings and it can be shared by various microscopic UV theories which end up in the same IR fixed point and they are elements of the same "universality class", namely living inside the domain of attraction Φ_a of the same fixed point (see.fig 2.9).
One very important point is that the RG flow equations (2.26) turn out to be local in the scale μ. Therefore the energy scale μ can be assumed as an additional coordinate for the QFT, which can be imagined to be living in a dimension more $(d + 1)$ of the "usual" spacetime coordinates d. This constitutes a strong hint that a QFT in d dimensions can somehow be described by a different theory which lives in $d + 1$ dimensions and that it incorporates in its dynamics also the RG evolution of the QFT itself. The RG flow will just be encoded in the dynamical evolution along this new extra coordinate in a very geometrical way as we will see in the next sections. Holography can be indeed thougth as the *geometrization* of the QFT RG flow.

The holographic principle

So far, we focused on the QFT side showing that perhaps the idea of describing it with a theory with an additional extra dimension is not that surprising. Now we jump to the other side, showing on the contrary that a gravitational theory can be described by a theory of degrees of freedom living on a spacetime with one less dimension. With both these ingredients it will be natural to conjecture a correspondence between a QFT in d dimensions and a theory of gravity in $d + 1$ dimensions (see fig.2.10 which will be formalized in details in the formulation of the AdS/CFT duality. The idea that Einstein's equations of General Relativity contain singular solutions was realized immediately after its formulation, in 1916, by K.Schwarzschild [49]. These solutions, named **Black Holes**, are singular spacetime configurations provided by an *event horizon* outside which nothing can escape. Very dense mass configurations can collapse and form a Black Hole[6]. There is a strong and deep relation between these new objects and thermodynamics. It was indeed proven by Hawking and Bekenstein in a series of papers that black holes have an entropy and their associated entropy is directly proportional to the area $\mathcal{A}$ of their event horizon:

$$S_{BH} = \frac{\mathcal{A}}{4\, l_p^2} \tag{2.27}$$

where the Boltzmann constant is fixed to $k_B = 1$.
This opens already a series of questions about BH thermodynamics[7].
Does the BH satisfy the laws of thermodynamics we know ?
Can we write down for the BH a 1st law of the form $dM = T\, dS_{BH}$?
The answer is yes; directly from Einstein's equations we can prove that a BH satisfy a "generalized 1st law":

$$dM = \frac{\kappa}{8\,\pi}\, \mathcal{A} \tag{2.28}$$

where $\mathcal{A}$ is the area of the event horizon and κ the so called *surface gravity*. The latter represents the gravitational acceleration of an object at its surface and for the easiest BH solution, *Schwarzschild BH*, it reads $\kappa = \frac{1}{4\,M}$.
Once we relate the horizon area with the black hole entropy we get close to the definition of a 1st law for BH objects, but we are still missing a fundamental ingredient.
How and why does a BH possess a temperature T?
It was proven later on, through semiclassical computations performed by S.Hawking [52], that a BH emits thermal radiation. This black-body radiation, due to quantum effects at the horizon, is associated to a temperature T:

$$T = \frac{\kappa}{2\,\pi} \tag{2.29}$$

All in all we know now that BHs satisfy the 1st law of thermodynamics, but that's not all.

In ordinary thermodynamics the second law requires that the entropy of a closed system shall never decrease, and shall typically increase as a consequence of generic transformations. While this law may hold good for a system including a black hole, it is not informative in its original form. For example, if an ordinary system falls into a black hole, the ordinary entropy becomes invisible to an exterior observer, so from her viewpoint, saying that ordinary entropy increases does not provide any insight: the ordinary second law is transcended. Including the black hole entropy in the entropy ledger gives a more useful law, the generalized second law of thermodynamics (GSL) (Bekenstein 1973 [53]):

$$\Delta S_0 + \Delta S_{BH} = \Delta S_{total} \geq 0\,. \tag{2.30}$$

where S_{BH} is the entropy of the black hole and S_0 the entropy of the "rest" of the system.
When matter entropy flows into a black hole, the GSL demands that the increase in black hole entropy shall more than compensate for the disappearance of ordinary entropy from sight.
The generalized 2nd law takes us immediately to the definition of the so-called **Holographic Principle** (see [54] for a review and [55, 56] for the original papers by t'Hooft and Susskind).
In an ordinary system with no gravity, the number of d.o.f. N_S is extensive and it relates to the

[6] This happens whenever the mass of the object overcomes the well known Chandrasekhar limit [50].

[7] One very relevant question we will not adress in this work is about the identification of the actual BH microstates which give rise to such an entropy accorging to Boltzmann's formula. This dilemma can be actually resolved in the context of String Theory [51].

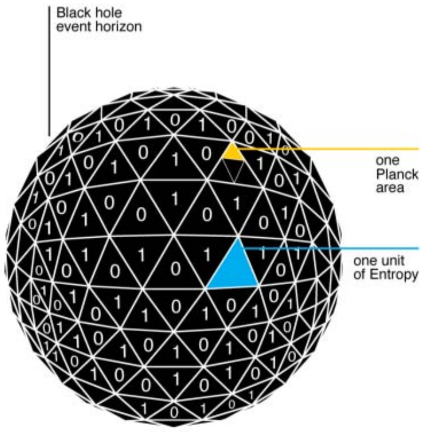

Figure 2.11: Black Hole entropy and Bekeinstein-Hawking idea.

volume as $N_S \sim e^V$. This means that the maximum entropy $S \sim lnN_S$ will be proportional to the volume $V \sim L^d$ of the system itself.
For gravity the story is different! For gravitational theories indeed the *Holographic principle* states that: the maximum entropy of a region of volume V is the entropy of the biggest black hole that fits.
This means that:

$$S_{max} = S_{BH} = \frac{1}{4\,\pi\,G_N} \times \text{horizon area} \tag{2.31}$$

The reason why the number of the degrees of freedom in theory with gravity scales like an area and not like a volume can be derived from the generalized 2nd law of thermodynamics we just introduced. Suppose for a moment that the entropy of the system is bigger than the entropy of the biggest BH fitting $S > S_{BH}$; now let's start throwing matter inside the system (increasing S and E). At a certain point the system will collapse into a BH state with entropy S_{BH} and if we compute the total entopy variation for such a process we discover that:

$$\Delta S < 0 \tag{2.32}$$

which violates the GSL!
The punchline is that gravity in $d+1$ dimensions shares the same number of d.o.f. of a QFT in fewer (d) dimensions!
This last remark, joined with the previous consideration about the nature of QFT and its RG flow, is sketched in figure 2.10 and it represents an handwaving hint towards the formulation of the AdS/CFT correspondence.
In the next sections we will better formalize such a duality starting to introduce in a separate way its two sides:

i. Anti de Sitter spacetime: AdS

ii. Conformal field theory: CFT

2.3 Conformal field theory & Anti de Sitter spacetime

In this section we will provide an accurate description of the two sides of the AdS/CFT correspondence: AdS which stands for Anti de Sitter Spacetime and CFT which stands for Conformal field theory. This will be enough to map finally the two sides and picture how the dictionary really works.
There are several good reviews about Conformal field theory, we just collect here some of the ones we found useful [57–59].
In regard to the second topic, the AdS spacetime, here there is a good reference as well [60]. That said, one can find introductory materials about the two subjects in every AdS-CFT review.

Conformal field theory

A field theory which does not contain any scale or dimensionful parameter enjoys classical scale invariance. This means that under a scale transformation the system remains invariant such that the physics does not depend on the scale itself. The easiest example of such a system is provided by a massless scalar with only quartic interaction:

$$\mathcal{S} \,=\, \int d^4x \left((\partial\phi)^2 \,+\, \frac{\lambda}{4!}\,\phi^4 \right) \tag{2.33}$$

Under a scale transformation, the coordinates and the field transform as:

$$\vec{x} \,\to\, \lambda\,\vec{x}\,, \qquad \phi(x) \,\to\, \lambda^{\Delta}\,\phi(\lambda\,x)\,. \tag{2.34}$$

where Δ is the *scaling dimension* of the field ϕ that in this case coincides with its canonical dimension $\Delta = 1$. It is straightforward to notice that in case we add a mass term $\sim m^2\,\phi^2$ such a symmetry gets spoiled. Of course, we are just talking about a classical realization of the symmetry and quantum effects usually breaks it.
Very often, a theory which enjoys scale invariance enjoys **conformal invariance** as well. It is "folk theorem" that scale invariance + Poincaré symmetry implies conformal invariance. This is not always true and there are various caveats (see [48] for details) and also some easy counterexamples like electrodynamics in $d \neq 4$ [61] and linear elasticity in $d = 2$ [62].We will avoid this discussion here.
We can think of a conformal transformation as a spacetime dependent dilatation:

$$\begin{aligned} &\text{scale:} \qquad x_\mu \,\to\, \lambda\,\tilde{x}_\mu \qquad ds^2 \,\to\, \lambda^2\,ds^2 \\ &\text{conformal:} \qquad x_\mu \,\to\, x'_\mu \qquad ds^2 \,\to\, ds'^2 \,=\, \Omega^2(x)\,ds^2 \end{aligned} \tag{2.35}$$

where in the limit $\Omega(x) = \lambda$ we recover the usual scale transformation. A conformal transformation rescales lengths but preserves angles between vectors (see fig.2.12). At an infinitesimal level we can write down:

$$x'_\mu \,=\, x_\mu \,+\, v_\mu(x)\,, \qquad \Omega(x) \,=\, 1 \,+\, \frac{\omega(x)}{2} \tag{2.36}$$

and we can derive the following equation:

$$\partial_\mu\,v_\nu \,+\, \partial_\nu\,v_\mu \,=\, \omega(x)\,\eta_{\mu\nu} \tag{2.37}$$

which after some easy manipulation leads to identify conformal transformations at linear level through:

$$\partial_\mu\,v_\nu \,+\, \partial_\nu\,v_\mu \,-\, \frac{2}{d}\,\left(\partial^\tau\,v_\tau\right)\eta_{\mu\nu} \,=\, 0 \tag{2.38}$$

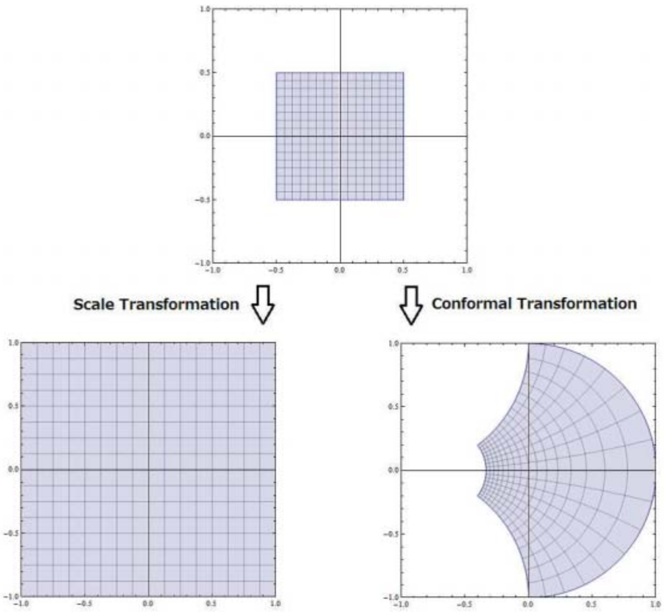

Figure 2.12: Scale transformations versus Conformal transformations. Figure taken from [48].

where d is the number of spacetime dimensions.
In two dimensions there are infinite solutions to this equation given, after Euclidean continuation, by all the holomorphic functions on a plane; the conformal group in $d = 2$ is indeed infinite-dimensional (Virasoro algebra).
In dimensions different from two $d \neq 2$ the number of solutions is smaller and it is given by at most quadratic function $v_\mu(x)$. The general solution contains:

- Translations: $\delta x_\mu = a_\mu$ whose generator P_μ is defined as $P_\mu = \partial_\mu$;
- Lorentz transformations: $\delta x_\mu = \omega_{\mu\nu} x^\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$; the generator $J_{\mu\nu}$ is defined by $J_{\mu\nu} = \frac{1}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)$;
- Dilatations: $\delta x_\mu = \lambda x_\mu$ whose generator D is a scalar defined by $D = x^\mu \partial_\mu$;
- Special conformal transformations: $\delta x_\mu = b_\mu x^2 - 2 x_\mu (b\,x)$ where the generator K_μ is defined by $K_\mu = x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu$. The corresponding finite transformations are:

$$x_\mu \to \frac{x_\mu + c_\mu x^2}{1 + 2\,cx + (cx)^2} \tag{2.39}$$

Altogether we have:

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)\,(d+2)}{2} \tag{2.40}$$

generators.
In fact one can prove that the group is isomorphic to SO(2,d). More precisely, there is an additional discrete symmetry that acts as a conformal transformation:

$$x_\mu \to \frac{x_\mu}{x^2} \qquad dx^2 \to x^2\,dx^2 \tag{2.41}$$

Adding it to the previous group we get the full conformal group O(2,d).
We can easily construct currents asssociated to the conformal transformations:

$$J_\mu \,=\, T_{\mu\nu}\,\delta x^\nu \tag{2.42}$$

which can be defined, with some subtleties, from Noether's theorem.
Conservation of the current corresponding to translations requires conservation of the stress energy tensor $\partial_\mu T^{\mu\nu} \,=\, 0$ and conservation of the current corresponding to Lorentz transformations is then automatic if $T_{\mu\nu}$ is symmetric. The current for dilation is now conserved if:

$$T^\nu_\nu \,=\, 0 \tag{2.43}$$

namely if the stress tensor results traceless; this is actually also the condition for scale invariance.
In the presence of supersymmetry, the conformal group is enhanced to a supergroup obtained by O(2,d) by adding the supercharges $\mathcal{Q}_a$ and the R-symmetry that rotates them. We also need to add the so-called conformal supercharges S_a. These are required to close the superconformal algebra $[K,Q] \sim S$. We shall not use explicitly the algebra of the superconformal group.
In a quantum theory, conformal invariance is broken by the introduction of a renormalization scale. The Renormalization Group (RG) and the Callan-Symanzick equation can be seen as anomalous Ward identity for dilatations. For example, in a pure Yang-Mills theory, which is classically scale invariant, the gauge coupling g runs with the energy scale, a dimensionful parameter Λ_{QCD} is introduced by dimensional transmutation, and the quantum stress energy tensor is not traceless anymore:

$$T^\mu_\mu \,\sim\, \beta(g)\,F^2_{\mu\nu} \tag{2.44}$$

The classical dimension Δ_0 of a field will be corrected by the anomalous dimension:

$$\Delta \,=\, \Delta_0 \,+\, \gamma(g)\,, \qquad \gamma \,=\, \frac{1}{2}\,\mu\,\frac{d}{d\mu}\,ln\,Z\,. \tag{2.45}$$

Nevertheless, conformal invariance can be present also at the Quantum level:

- if $\beta(g*) \,=\, 0$, we call the point in the phase space $g \,=\, g*$ a fixed point and there scale invariance and , under mild assumptions, conformal invariance are present;
- $\beta(g) \,=\, 0$: we say the theory is fully conformal also at the quantum level; there is no RG flow. This usually happens in Supersymmetric theories and the most famous example is $N = 4$ SYM.

In a conformal invariant theory we have an unitary action of the conformal group on the Hilbert space. The generators P,J,D,K will be represented by hermitian operators. It is a tedious exercise to check that the generators P,J,D,K close the following algebra:

$$\begin{aligned}
&[J_{\mu\nu}\,,\,J_{\rho\sigma}] \,=\, \eta_{\mu\rho}\,J_{\nu\sigma} \,+\, \text{permutations}\,,\\
&[J_{\mu\nu}\,,\,P_\rho] \,=\, i\,(\eta_{\mu\rho}\,P_\nu \,-\, \eta_{\nu\rho}\,P_\mu)\,,\\
&[J_{\mu\nu}\,,\,K_\rho] \,=\, i\,(\eta_{\mu\rho}\,K_\nu \,-\, \eta_{\nu\rho}\,K_\mu)\,,\\
&[J_{\mu\nu}\,,\,D] \,=\, 0\,,\\
&[D\,,\,P_\mu] \,=\, i\,P_\mu\,,\\
&[D\,,\,K_\mu] \,=\, -\,i\,P_\mu\,,\\
&[K_\mu\,,\,P_\nu] \,=\, -\,2\,i\,J_{\mu\nu} \,-\, 2\,i\,\eta_{\mu\nu}\,D\,.
\end{aligned} \tag{2.46}$$

The first line is the algebra of the Lorentz group SO(1,d-1), the next three lines state that D is a scalar whereas P_μ, K_μ are vectors. The next two lines P_μ and K_μ as ladder operators for D, which act increasing and decreasing its eigenvalue respectively. The last equation states that P and K close on a Lorentz transformation and a dilatation.
We can assemble all the generators in:

$$J_{MN} \,=\, \begin{pmatrix} J_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ \frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix}\,, \qquad M\,,\,N \,=\, 1\,,\,\ldots\,,\,d+2 \tag{2.47}$$

which turn out to be rotations in a $d+2$ dimensional space with signature $(2,d)$ (with metric $\eta_{MN} \,=\, diag\,(-1,1,\ldots,1,-1)$):

$$[J_{MN}\,,\,J_{RS}] \,=\, i\,\eta_{MR}\,J_{NS} \,\pm\, \text{permutations} \tag{2.48}$$

We then recover the group $SO(2,d)$ that we claimed before to be isomorphic to the conformal group in d dimensions.
Particles are usually identified by mass and Lorentz quantum numbers, corresponding to the Casimirs of the Poincaré group. Whenever conformal invariance is present, the mass operator $P_\mu P^\mu$ does not commute anymore with other generators, for example dilatations D. Mass and energy can be in fact rescaled by a conformal transformation. If a representation of the conformal group contains a state with given energy, it will contain states with arbitrary energy from zero to infinity obtained by applying dilatations. For this reason the entire formalism of S matrix does not make sense for conformal theories[8]. We need to find different ways of labeling states. A good way of doing it is to use the dilatations and the scaling of the fields under their action. The quantum version of a dilation acts on the fields as:

$$[D\,,\phi(x)] \,=\, i\,\left(\Delta \,+\, x_\mu\,\partial^\mu\right)\,\phi(x) \tag{2.49}$$

and identifies fields of conformal dimension Δ.
We can distinguish two type of operators in the infinite tower of a CFT:

- *Primary* operators: annihilated by the lowering operator K_μ;
- *Descendants*: all the other operators which can be built upon acting with P_μ and the other generators repeatedly.

Primary operators are classified according to the dimension Δ and the Lorentz quantum numbers. Conformal invariance is a very strong requirement and gives many constraints on a quantum field theory:

i. The Ward identities for the conformal group give constraints on the Green functions. One can always find a basis of primary operators $O_i(x)$, with fixed scale dimension Δ_i. The set of (O_i, Δ_i) gives the spectrum of the CFT. One-, two- and three-point functions are completely fixed by conformal invariance.
   For example, one-point functions are zero, while two-point functions equal:

$$\langle\, O_i(x)\,,\,O_j(y)\,\rangle \,=\, \frac{A\,\delta_{ij}}{|x-y|^{2\,\Delta_i}} \tag{2.50}$$

[8] This is sometimes stated as: there are no particles in a CFT.

ii. Unitarity of the theory gives bounds restricting the possible dimensions of primary fields. We have inequalities that depend on the Lorentz quantum number $\Delta \geq f(j_1, j_2)$ (see [63]). Important examples are:

- The dimension of a four-dimensional scalar field must be greater than one, $\Delta \geq 1$, and the saturation of the bound, $\Delta = 1$ ($(d-2)/2$ generically), implies that the operator obeys free field equations.
- For a vector field O_μ , $\Delta \geq 3$ (generically is $\Delta \geq (d-1)$) and the bound is saturated if and only if the operator is a conserved current $\partial^\mu O_\mu = 0$. Analogously, a spin 2 symmetric operator $O_{\mu\nu}$ (the stress tensor for example) has $\Delta \geq 4$ (generically is $\Delta \geq d$), and $\Delta = 4$ corresponds to conservation $\partial^\mu O_{\mu\nu} = 0$. In particular, conserved currents have canonical dimension and are not renormalized.

Checking these constrained results will constitute a first good test of the AdS/CFT correspondence.

Anti de Sitter spacetime

The most generic metric in $d+1$ dimensions consistent with Poincaré invariance can be written in the form:

$$ds^2 = \Xi[z] \left(-dt^2 + d\vec{x}^2 + dz^2\right) \tag{2.51}$$

where z is the extra "holographic" direction.
The dependence of the Ξ function just on that coordinate is imposed by the requirement of translational symmetries in the other coordinates $(t, \vec{x})$. At this stage not much can be said about the form of $\Xi(z)$. However if we consider a quantum field theory which is conformal such a constaint fixes univoquely that function. If we indeed perform a scale transformation:

$$(t\,, \vec{x}) \to C\,(t\,, \vec{x}) \tag{2.52}$$

where C is a constant, the gravity theory, formulated to describe the CFT, should enjoy this scaling simmetry. In order for that to happens we have to rescale at the same time the holographic coordinate z:

$$z \to C\,z \tag{2.53}$$

in the proper way to account that z represents the length scale of the boundary theory.
Asking the metric to be invariant under this full set of transformations, we need to impose that:

$$\Xi(z) \to \frac{1}{C}\,\Xi(z) \quad \text{under} \quad z \to C\,z \tag{2.54}$$

This uniquely determines:

$$\Xi(z) = \frac{L}{z} \tag{2.55}$$

where L is a constant. The metric can be now written as:

$$ds^2 = \frac{L^2}{z^2} \left(-dt^2 + d\vec{x}^2 + dz^2\right) \tag{2.56}$$

which is precisely the line element of $(d+1)$-dimensional anti-de Sitter spacetime, AdS_{d+1}, with AdS length L.
Anti de Sitter spacetime (AdS in short) is a maximal symmetric solution of Einstein equations

provided by a negative cosmological constant Λ. Maximally symmetric solutions enjoy the maximal number of indipendent *killing vectors*, *i.e.* isometries generators, which in d dimensions is fixed to be $\frac{d(d+1)}{2}$.
Maximally symmetric solutions are defined by a curvature tensor of the form:

$$R_{abcd} = \frac{\mathcal{R}}{d(d+1)} \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right) \tag{2.57}$$

and they are isotropic and homogeneous. Because of these properties the Ricci scalar $\mathcal{R}$ turns out to be constant:

$$\mathcal{R} = d\,(d+1)\,\mathcal{K} \tag{2.58}$$

where $\mathcal{K}$ is the *curvature constant.*
It follow directly that the Ricci and Riemann tensors become:

$$\begin{aligned} R_{ab} &= d\,\mathcal{K}\, g_{ab}\,, \\ R_{abcd} &= \mathcal{K}\, \left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right)\,. \end{aligned} \tag{2.59}$$

properties which define such a metric space a space of constant curvature.
Given the curvature constant, the spacetime dimensionality and signature, there is an unique maximally symmetric space. Taken an action of the form:

$$\mathcal{S} = \frac{1}{16\,\pi\, G_N} \int d^{d+1}x\,\sqrt{-g}\,\left(R - \Lambda\right) \tag{2.60}$$

one can prove that:

$$\mathcal{R} = \frac{d+1}{d-1}\,\Lambda = -d\,(d+1)\,\mathcal{K} \tag{2.61}$$

The $\Lambda = 0$ solution is flat space (Minkowski), for $\Lambda > 0$ we get de Sitter space dS_{d+1} and finally for $\Lambda < 0$ we have Anti de Sitter spacetime AdS_{d+1}[9].
All of these spaces can be realized as the set of solutions of a quadratic equation in a six-dimensional ($d + 2$-dimensional generically) space with suitable signature. Let us for simplicity focus on the five dimensional Anti de Sitter spacetime AdS_5 (the generical $d + 1$ case can be extrapolated without much difficult). AdS_5 can be isometrically embedded into an hyperboloid (see fig.2.13) defined by:

$$x_0^2 + x_5^2 - x_1^2 - x_2^2\, x_3^2 - x_4^2 = \mathcal{R}\,, \qquad \mathcal{R}^2 = -\frac{12}{\Lambda} \tag{2.62}$$

in a 6-dimensional space of signature $\mathbb{R}^{(2,4)}$ with line element:

$$ds^2 = -\,dx_0^2 - dx_5^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{2.63}$$

From this definition it is obvious that AdS_5 has isometry group $O(2, 4)$ (generically for AdS_{d+1} it would have $O(2, d)$ isometry group) which is exactly the conformal group in four dimensions!
A set of coordinates is given by:

$$\begin{aligned} x_0 &= \mathcal{R}\, \cosh\rho\, \cos\tau\,, \\ x_5 &= \mathcal{R}\, \cosh\rho\, \sin\tau\,, \\ x_i &= \mathcal{R}\, \sinh\rho\, \hat{x}_i\,, \qquad \sum \hat{x}_i^2 = 1 \end{aligned} \tag{2.64}$$

[9] This is true in Lorentzian signature. On the contrary with Euclidean signature $\Lambda > 0$ corresponds to the sphere S^{d+1} with isometries $SO(d + 2)$ and $\Lambda > 0$ to the Hyperboloid $\mathcal{H}^{d+1}$ with isometries $SO(1, d + 1)$.

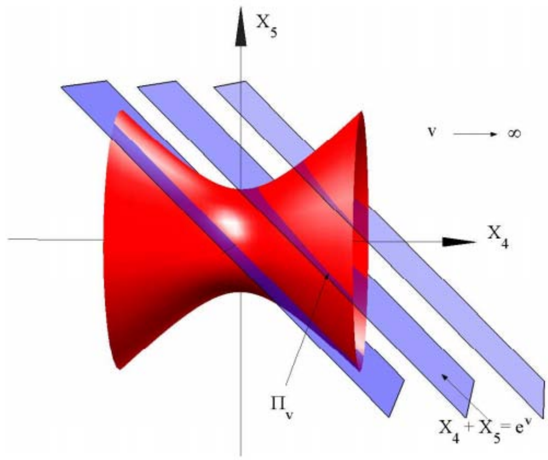

Figure 2.13: AdS_{d+1} spacetime defined through the $(d+2)$ hyperboloid and definition of Poincaré coordinates.

and the metric reads:

$$ds^2 = \mathcal{R}^2 \left(- \cosh^2 \rho \, d\tau^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3 \right) \tag{2.65}$$

where Ω_3 is the line element of a three-sphere which for a generic AdS_{d+1} would be a $(d-1)$-sphere.
It is easy to verify that $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$ cover the Minkowskian hyperboloid exactly once, and for this reason these coordinates are called *global*. Note that time is periodic and therefore we have close time-like curves. To avoid this we can take the universal cover where $\tau \in \mathbb{R}$: we shall always refer to AdS_5 as this universal cover.
We can find a second set of coordinates given by a four dimensional Lorentz vector y_μ and a fifth coordinate $u > 0$ by a redefinition:

$$\begin{aligned} x_0 &= \frac{1}{2\,u} \left(1 + u^2 \, (\mathcal{R}^2 + \vec{y}^2 - t^2) \right) , \\ x_5 &= \mathcal{R} \, u \, t \, , \\ x_{1,2,3} &= \mathcal{R} \, u \, y_{1,2,3} \, , \\ x_4 &= \frac{1}{2\,u} \left(1 - u^2 \, (\mathcal{R}^2 - \vec{y}^2 + t^2) \right) . \end{aligned} \tag{2.66}$$

which brings the metric to the form:

$$ds^2 = \mathcal{R}^2 \left(\frac{du^2}{u^2} + u^2 \, dy^\mu \, dy_\mu \right) \tag{2.67}$$

This matric has slices (at constant u) isomorphic to 4-dim Minkowski space and for this reason these coordinates are called *Poincaré* coordinates. The foliation is done along the u coordinate which runs from zero to infinity and the Minkowski slices are warped by a factor u^2 which means that an observer living on a Minkowski slice sees all lengths rescaled by a factor of u according to its position in the fifth dimension.
The $u = \infty$ position is called the "boundary" of AdS. To be mathematically precise, it represents

a *conformal boundary* meaning that is its conformally equivalent metric $d\tilde{s}^2 = ds^2/u^2$ to have a boundary $\mathbb{R}^{1,3}$ at $u = \infty$. On th contrary the antithetic position $u = 0$ defines the AdS *horizon* where the killing vector $\frac{\partial}{\partial t}$ has zero norm. These coordinates are very convenient since they contain a Minkowski slice, and we shall use them in most of our applications of the AdS/CFT correspondence. However, they cover only half of the hyperboloid; $u = 0$ does not correspond to a singularity and the metric can be extended beyond the horizon (using for example global coordinates).
We skip here various details about the conformal-causal structures of AdS spacetime and its Penrose diagram.
There are other forms of the metric in Poincaré coordinates that are commonly used. They all differ by a redefnition of the fifth coordinate u. For example with $u = 1/z = e^r$ we have:

$$ds^2 = \mathcal{R}^2 \left(\frac{dz^2 + dy_\mu \, dy^\mu}{z^2} \right) = \mathcal{R}^2 \left(dr^2 + e^{2\,r} \, dy_\mu \, dy^\mu \right) \tag{2.68}$$

The boundary is now at $z = 0$ and $r = 1$ and the horizon at $z = 1$ and $r = -\infty$.
The most important point about AdS spacetime is the fact that its isometry group $S0(2, d-1)$ corresponds exactly with the conformal group in d dimensions. This is a strong argument based on symmetries which somehow AdS_{d+1} could be dual to a conformal field theory living in d dimensions which is the actual statement of the AdS/CFT correspondence.

Counting of d.o.f. in AdS

As an additional proof of the possibile duality between AdS spacetime in $d + 1$ dimensions and CFT in d dimensions we also show that the number of degrees of freedom in the two sides matches. We start by considering a QFT in d dimensions and we do it introducing an IR and UV cutoffs namely: a lattice spacing ϵ and a finite size R of the spatial box. In this way the number of "cells" is given by $\left(\frac{R}{\epsilon}\right)^{d-1}$ and defining c_{QFT} the number of degrees of freedom for lattice space, *i.e. central charge*, the total number of d.o.f.s of the QFT can be written as:

$$N_{dof}^{QFT} = \left(\frac{R}{\epsilon} \right)^{d-1} c_{QFT} \tag{2.69}$$

For a SU(N) gauge theory, where the fields are $N \times N$ matrices, the counting for the large N limit gives $C_{SU(N)} \sim N^2$.
On the contrary regarding the gravitational bulk theory the number of degrees of freedom is given by:

$$N_{dof}^{AdS} = \frac{A_\partial}{4\,G_N} \tag{2.70}$$

where A_∂ is the area of the region at the boundary.
Given the AdS metric the latter can be computed as:

$$A_\partial = \int_{\mathbb{R}^{d-1}, z=\epsilon} d^{d+1}x \sqrt{-g} = \left(\frac{L}{\epsilon} \right)^{d-1} \int_{\mathbb{R}^{d-1}} d^{d-1}x \tag{2.71}$$

The last term $\int_{\mathbb{R}^{d-1}} d^{d-1}x$ represents the volume $V_{\mathbb{R}^{d-1}}$ which is infinite. If we request, as before, a cutoff regulator R such a volume becomes:

$$V_{\mathbb{R}^{d-1}} = R^{d-1} \tag{2.72}$$

such that the boundary area reads:

$$A_\partial = \left(\frac{R\,L}{\epsilon}\right)^{d-1} \tag{2.73}$$

If we restore the units, namely the Planck length $G_N = (l_p)^{d-1} = \frac{1}{M_p^{d-1}}$, the total number of degrees of freedom in AdS spacetime can be written as:

$$N_{dof}^{AdS} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_p}\right)^{d-1} \tag{2.74}$$

We can finally compare the number of degrees of freedom in the two pictures $N_{dof}^{QFT}, N_{dof}^{AdS}$ and check that they match.
A classical limit for gravity implies that:

$$\left(\frac{L}{l_p}\right)^{d-1} \gg 1 \tag{2.75}$$

which means that the curvature scale is small in Planck units.
Comparing the expressions we can therefore conclude that a QFT has a classical gravity description whenever c_{QFT} is large, *i.e.* in the large N limit.
In the next section we will enter deeper in this connection and we will describe in details the map known as the *dictionary*.

2.4 General aspects of the duality and its dictionary

In the previous sections we have already outlined some important features of the $\text{AdS}_{d+1}/\text{CFT}_d$ which for simplicity we are going to summarized here:

- The extra radial direction z plays the role of the energy scale. We thus see that physical processes in the bulk with identical proper energies but occurring at different radial positions correspond to different gauge theory processes with energies that scale as $E = 1/z$. In other words, a gauge theory process with a characteristic energy E is associated with a bulk process localized at $z = 1/E$. In a conformal theory, there exist excitations of arbitrarily low energies. This is reflected in the bulk in the fact that the geometry extends all the way to $z = \infty$. For a confining theory with a mass gap m, the geometry would end smoothly at a finite value $z_0 \sim 1/m$. Similarly, at a finite temperature T, which provides an effective IR cutoff, the spacetime will be cut off by an event horizon at a fnite $z_0 \sim 1/T$.

- In order to ignore the stringy and quantum corrections on the gravitational side (*Bottom-Up Holography*) the corresponding CFT has to be taken in the limit of large N and large coupling.

- Isometries of the gravitational solution correspond to the symmetry group of the dual CFT. We often say that gauged or local symmetries in the bulk are mapped into global symmetries in the boundary. To be more precise on the gravity side the global symmetries arise as large gauge transformations and they correspond to the global symmetries of the CFT. There is, sometimes, the possibility of having local symmetries in the dual CFT using the so-called *Alternative quantization* we will present in the following.

Field/operator correspondence

We can finally get into the map between the operators $\mathcal{O}_i$ of the conformal field theory and the bulk fields ϕ_i of the gravitational side and show how to extract informations from one side to the other in order to make statements about physical quantities like correlation functions.
The gravitational side, defined in $d+1$, is described by a bulk action:

$$\mathcal{S}_{bulk}\,(g_{\mu\nu}\,,\,A_\mu\,,\,\phi\,,\dots) \tag{2.76}$$

including fields of different spins. In the example we consider a spin-2 field $g_{\mu\nu}$, a vector field A_μ and a scalar ϕ. One can of course include fermionic fields with non-integer spins or even higher spin fields; we will avoid such cases for simplicity and because not necessary for the rest of the work.
On the other side, the conformal field theory is defined by a set of *primary operators* organized into a lagrangian:

$$\mathcal{L}_{CFT}\,(\mathcal{O}_i) \tag{2.77}$$

The main idea is that a field ϕ defined in the bulk is associated to an operator of the CFT with the same quantum numbers and their coupling show up via a boundary term.
From the CFT perspective one can write an operator deformation, due to a source ϕ_0, as:

$$\mathcal{L}_{CFT} + \int d^d x\,\phi_0\,\mathcal{O} \tag{2.78}$$

Just from standard QFT arguments it turns out that:

$$e^{W(\phi_0)} \,=\, \langle\, e^{\int\,\phi_0\,\mathcal{O}}\,\rangle_{QFT} \tag{2.79}$$

where $W(\phi_0)$ is the functional generator of the correlation functions for the operator $\mathcal{O}$.
The latter can be indeed obtained with the usual functional derivative prescription:

$$\langle\underbrace{\mathcal{O}\dots\mathcal{O}}_{1\,,\dots\,,\,n}\rangle_C \,=\, \frac{\delta^n\,W}{\delta\,\phi_0^n}|_{\phi_0=0} \tag{2.80}$$

where the index C stays for connected (see a QFT textbook for more details).
From the gravitational point of view the source $\phi_0(x)$ will be the boundary value of the bulk field $\phi_(x,r)$ living in a $d+1$ dimensional spacetime. That said we can finally write down the fundamental equation of the AdS/CFT correspendce known as the **GPKW** (Gubser, Polyakov, Klebanov, Witten) master rule [64, 65]

$$\boxed{e^{W(\phi_0(x)} \,=\, \langle\, e^{\int\,\phi_0(x)\,\mathcal{O}}\,\rangle_{QFT} \,=\, e^{\mathcal{S}_{AdS}[\phi(x,r)]} \,=\, \mathcal{Z}_{gravity}\,[\phi(x,r)_{boundary} \,=\, \phi_0(x)\,]} \tag{2.81}$$

Some immediate comments are in order:

i. The duality relates an off-shell theory in d dimensions to an on-shell theory in $d+1$.

ii. The equations of motions in the bulk are generically of the 2nd order type and they therefore need two boundary conditions to define a unique solution. One of them has to be fixed at the boundary where one would naively impose:

$$\phi(x, boundary) \,=\, \phi_0(x) \tag{2.82}$$

It is easy to see that this cannot be a consistent boundary condition in AdS spacetime, one will need indeed to impose a more complicated condition which will take the general form:

$$\phi(x, boundary) \;=\; h(r)\,\phi_0(x) \tag{2.83}$$

where $h(r)$ is a function of the only radial coordinate whose structure will depend on the conformal dimension of the field ϕ.
On the there side, the second BC is fixed at the horizon where one usually imposes regularity or the so-called *ingoing BC*.

So far we have not specified how the actual map selects the dual couple $\{\phi_0, \mathcal{O}\}$! The most robust argument to find such couples is given by symmetries and by the requirement that both the bulk field and the CFT operator have to bring the same quantum numbers accordingly to the $O(2, d-1)$ group.
As a direct example, we can write down:

$$\mathcal{L}_{CFT} \,+\, \int d^dx\,\sqrt{g}\;(g_{\mu\nu}\,T^{\mu\nu} \,+\, A_\mu\,J^\mu \,+\, \phi\,\mathcal{O}\,) \tag{2.84}$$

which already shows us part of the map:

$$\begin{array}{llcll} graviton & g_{\mu\nu} & \longrightarrow & stress\,tensor & T^{\mu\nu} \\ gauge\,field & A_\mu & \longrightarrow & current & J^\mu \\ scalar\,field & \phi & \longrightarrow & scalar\,operator & \mathcal{O} \end{array} \tag{2.85}$$

where for example in a gauge theory $\mathcal{O} = F_{\mu\nu}F^{\mu\nu}$.
Generically we can have different set of gauge symmetries in the bulk associated to the various fields, for example:

$$\begin{array}{llcl} graviton & g_{\mu\nu} & \longrightarrow & diffeomorphisms \\ gauge\,field & A_\mu & \longrightarrow & U(1) \end{array} \tag{2.86}$$

Gauge invariance relates to the conservation of the corresponding currents in the CFT and it fixes their conformal dimensions to the one of conserved quantities. A mass term in the bulk would generically break such gauge invariance and would modify, as we will explain in details, the conformal dimension of the correspondent operator $\mathcal{O}$ which would aquire an anomalous part signaling its non conservation.
All the details about the physics in the bulk, the specific features of the dictionary and the computation of the CFT correlations functions will be introduced in the next two sections using three benchmarck models of the AdS/CFT correspondence: the bulk scalar field, the Reissner Nordstrom Black Hole solution and the original Holographic Superconductor model. We spoil some details of the dictionary, which are going to be explained through the examples, in table 2.4.

2.5 More on linear response and correlation functions

Let's underline a bit more in details the analysis of the correlation functions in the context of the AdS/CFT correspondence inspired by the *linear response theory* of common QFTs.

The Dictionary	
AdS$_{d+1}$	**CFT**$_d$
$d+1$ dimensions	d dimensions
radial dimension r	energy scale μ
fields $\phi_I(r,x)$	operators $\mathcal{O}_I(x)$
spin $\mathcal{J}$	spin $\mathcal{J}$
mass m^2	conformal dimensions Δ
gauged symmetries	global symmetries
gauge invariance	currents conservation
confining geometry $r_0 \sim 1/m_{gap}$	mass gap m_{gap}
Hawking temperature T	QFT temperature T
metric $g_{\mu\nu}$	stress tensor $T^{\mu\nu}$
gauge field A_μ	current J^μ - charge density ρ
diffeomorphism invariance	stress tensor conservation
black hole instabilites	QFT phase transitions

Table 2.1: Sketch of the AdS/CFT dictionary.

What we are after are the n-points correlation functions of the form:

$$\langle\, \mathcal{O}(x_1)\, \dots\, \mathcal{O}(x_n)\, \rangle \tag{2.87}$$

In order to do that we have to deform the QFT lagrangian with an external source:

$$\mathcal{L} \,\to\, \mathcal{L} \,+\, \mathcal{J}(x)\, \mathcal{O}(x) \,=\, \mathcal{L} \,+\, \mathcal{L}_{\mathcal{J}} \tag{2.88}$$

and define the so-called generating functional:

$$\mathcal{Z}_{QFT} \,=\, \langle\, e^{\int \mathcal{L}_{\mathcal{J}}}\, \rangle_{QFT} \tag{2.89}$$

It follows that the n-points functions can be computed as:

$$\langle \prod_i \mathcal{O}(x^i)\, \rangle \,=\, \prod_i \frac{\delta}{\delta\, \mathcal{J}(x_i)}\, \log \mathcal{Z}_{QFT}|_{\mathcal{J}=0} \tag{2.90}$$

Let now consider a bulk field $\phi(z,x)$ living in AdS with the boundary located at $z=0$ in a suitable choice of coordinates.
The field ϕ_0 defined as (we will give more details about it in the following section):

$$\phi_0(x) \,=\, \lim_{z\to 0}\, z^{\Delta - d}\, \phi(z,x) \tag{2.91}$$

is related to the source of an operator $\mathcal{O}$ of the CFT where Δ is the dimension of the latter.
Then, the AdS/CFT prescription for the generating functional is:

$$\mathcal{Z}_{QFT} \,=\, \langle\, exp\left[\int\, \phi_0\, \mathcal{O}\right] \rangle_{QFT} \,=\, \mathcal{Z}_{gravity}[\phi \to \phi_0] \tag{2.92}$$

where $\mathcal{Z}_{gravity}[\phi \to \phi_0]$ is the partition function (i.e. the path integral) in the gravity theory evaluated over all functions which have the value ϕ_0 at the boundary of AdS:

$$\mathcal{Z}_{gravity}[\phi \to \phi_0] \,=\, \sum_{\{\phi \to \phi_0\}} e^{\mathcal{S}_{gravity}} \tag{2.93}$$

In the limit in which classical gravity dominates, we can proceed with a saddle-point approximation of such an exponential such that the previous sum reduces just to the classical solution:

$$\mathcal{Z}_{QFT} \approx e^{\mathcal{S}^{on-shell}_{gravity}[\phi \to \phi_0]} \tag{2.94}$$

One should be careful when evaluating the on-shell gravity action because it typically diverges and has to be renormalized following the procedure of holographic renormalization [66].
Thus, the classical action must be substituted by a renormalized version, which will be denoted by $\mathcal{S}^{ren}_{gravity}$ and the generating functional becomes:

$$log\, \mathcal{Z}_{QFT} = \mathcal{S}^{ren}_{gravity}[\phi \to \phi_0] \tag{2.95}$$

At the end of the story the n-point functions can be computed by computing derivatives with respect to the source ϕ_0 as:

$$\langle\, \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \,\rangle = \frac{\delta^{(n)}\, \mathcal{S}^{ren}_{gravity}[\phi]}{\delta\phi_0(x_1) \ldots \delta\phi_0(x_n)}\Big|_{\phi_0=0} \tag{2.96}$$

One point function

We start considering the 1-point function, *i.e.* expectation value, of a CFT operator $\mathcal{O}$ in the presence of an external source ϕ_0:

$$\langle\, \mathcal{O}(x) \,\rangle = \frac{\delta \mathcal{S}^{ren}_{grav}[\phi]}{\delta\phi_0} = \lim_{z\to 0} z^{d-\Delta} \frac{\delta \mathcal{S}^{ren}_{grav}[\phi]}{\delta\phi(z,x)} \tag{2.97}$$

where Δ is again the conformal dimension of the CFT operator $\mathcal{O}$.
Let the gravitational action having a generic form of the type:

$$\mathcal{S}_{grav} = \int dz\, d^d x\, \mathcal{L}\, [\phi\, ,\, \partial\phi] \tag{2.98}$$

Under a general change $\phi \to \phi + \delta\phi$ the classical action varies (after some integrations by parts) as:

$$\delta\, \mathcal{S}_{grav} = \int dx\, d^d x \left[\left(\underbrace{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\,(\partial_\mu \phi)} \right)}_{EOMs} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\,(\partial_\mu \phi)}\, \delta\phi \right) \right] \tag{2.99}$$

The first term vanishes just because of the Euler-Lagrange equations of motion. As the boundary is set at a finite cutoff $z = \epsilon$ we can therefore write:

$$\delta\, \mathcal{S}^{on-shell}_{grav} = \int_\epsilon^\infty dz \int d^d x\, \partial_z \left(\frac{\partial \mathcal{L}}{\partial\,(\partial_z \phi)}\, \delta\phi \right) = - \int d^d x\, \frac{\partial \mathcal{L}}{\partial\,(\partial_z \phi)}\, \delta\phi|_{z=\epsilon} \tag{2.100}$$

We can now define:

$$\Pi = - \frac{\partial \mathcal{L}}{\partial\,(\partial_z \phi)} \tag{2.101}$$

which is the conjugate momentum of the field ϕ if we take z as the "time" direction.
This implies we can rewrite:

$$\delta\, \mathcal{S}^{on-shell}_{grav} = \int d^d x\, \Pi(\epsilon, x)\, \delta\phi(\epsilon, x) \tag{2.102}$$

After undergoing the usual renormalization procedures (see [66]) we can define a renormalized action:

$$\mathcal{S}^{ren} \,=\, \mathcal{S}^{on-shell}_{grav} \,+\, \mathcal{S}^{ct} \tag{2.103}$$

where the second term is the action for the counterterms, local boundary terms which take care of making the total action finite. We can also defined a renormalized momentum:

$$\Pi^{ren} \,=\, -\,\frac{\delta \mathcal{S}^{ren}}{\delta\phi(z,x)} \tag{2.104}$$

Finally we can combine our results and realize that the 1-point function of the operator $\mathcal{O}$ in presence of a source ϕ_0 can be derived as:

$$\langle\, \mathcal{O}(x)\,\rangle_{\phi_0} \,=\, \lim_{z\to 0}\, z^{d\,-\,\Delta}\, \Pi^{ren}(z,x)\,. \tag{2.105}$$

Linear response and 2-point functions

Before discussing the computation of the 2-points functions which will be the main characters of our work we have to spend some words about *linear response theory* in common QFT language (see for example [67] for more details).
The field theory path integral representation of the one-point function with a source can be written as:

$$\langle\, \mathcal{O}(x)\,\rangle_{\phi_0} \,=\, \int [D\Psi]\, \mathcal{O}(x)\, e^{\mathcal{S}_E[\Psi]} \,+\, \int d^d y\, \phi_0(y)\, \mathcal{O}(y) \tag{2.106}$$

where Ψ denotes the fields content of the QFT and $\mathcal{S}_E$ the Euclidean version of the QFT action. We can now expand the previous expression at linear order in the source ϕ_0 to extract the leading contribution to the response:

$$\langle\, \mathcal{O}(x)\,\rangle_{\phi_0} \,=\, \langle\, \mathcal{O}(x)\,\rangle_{\phi_0=0} \,+\, \int d^d y\, \langle\, \mathcal{O}(x)\, \mathcal{O}(y)\,\rangle\, \phi_0(y) \,+\, \dots \tag{2.107}$$

where $\dots$ stands for higher order corrections at least quadratic in the source ϕ_0.
We can then identify the euclidean 2-points function, *i.e. Green Function*, for the operator $\mathcal{O}$ as:

$$\mathcal{G}_E(x\,-\,y) \,=\, \langle\, \mathcal{O}(x)\, \mathcal{O}(y)\,\rangle \tag{2.108}$$

We can consider normal ordered operators such that $\langle\, \mathcal{O}(x)\,\rangle_{\phi_0=0} \,=\, 0$ or just subtract to $\mathcal{O}$ its vacuum expectation value (VEV) at zero source. In this way $\langle\, \mathcal{O}(x)\,\rangle_{\phi_0}$ measures the fluctuations of the observable away from the expectation value, i.e. the linear response of the system to the external perturbation ϕ_0.
We can thereofore write:

$$\langle\, \mathcal{O}(x)\,\rangle_{\phi_0} \,=\, \int d^d y\, \mathcal{G}_E(x\,-\,y)\, \phi_0(y) \tag{2.109}$$

that in momentum space gets the form of a normal product:

$$\langle\, \mathcal{O}(k)\,\rangle_{\phi_0} \,=\, \mathcal{G}_E(k)\, \phi_0(k) \tag{2.110}$$

Finally, in momentum space, the 2-points function assumes the easy form:

$$\mathcal{G}_E(k) \,=\, \frac{\langle\, \mathcal{O}(k)\,\rangle_{\phi_0}}{\phi_0(k)} \tag{2.111}$$

In the framework of AdS/CFT computing a 2-points function reduces to:

$$\langle\, \mathcal{O}(k)\, \mathcal{O}(0)\,\rangle \,=\, \mathcal{G}_E(k) \,=\, \lim_{z\to 0}\, z^{d\,-\,\Delta}\, \frac{\Pi^{ren}(z,k)}{\phi_0(z,k)} \tag{2.112}$$

We will give explicit examples of this sort of computations in the following sections.

2.6 Example 1: The scalar field

We here consider the simplest case possible to show explicitely how the dictionary works.
We consider AdS spacetime in $d+1$ dimensions with the following metric definition:

$$ds^2 \,=\, \frac{L^2}{z^2}\,\Big(dz^2 \,+\, dx_\mu\, dx^\mu\Big) \tag{2.113}$$

with the correspondent boundary located at $z = 0$.
We then consider a massive scalar on top of this geometry whose action reads:

$$\mathcal{S} \,=\, -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,\Big[g^{MN}\,\partial_M\,\phi\,\partial_N\,\phi \,+\, m^2\,\phi^2\Big] \tag{2.114}$$

The equations of motion coming from the latter can be written down as:

$$\frac{1}{\sqrt{-g}}\,\partial_M\,\Big(\sqrt{-g}\,g^{MN}\,\partial_M\,\phi\Big) \,-\, m^2\,\phi \,=\, 0\,. \tag{2.115}$$

and once we take into account the geometry definition:

$$z^{d+1}\,\partial_z\,\Big(z^{1-d}\,\partial_z\,\phi\Big) \,+\, z^2\,\delta^{\mu\nu}\,\partial_\mu\,\partial_\nu\,\phi \,-\, m^2\,L^2\,\phi \,=\, 0\,. \tag{2.116}$$

We proceed with Fourier transforming the boundary coordinates x^μ:

$$\phi(z, x^\mu) \,=\, \int \frac{d^dk}{(2\,\pi)^d}\,e^{i\,k\,\cdot\,x}\,f_k(z) \tag{2.117}$$

This equation, approaching the UV boundary $z = 0$, admits power law solutions of the form $f_k(z) \sim z^\beta$ where β satisfies the following indicial equation:

$$\beta\,(\beta \,-\, d) \,-\, m^2\,L^2 \,=\, 0\,. \tag{2.118}$$

leading to (see fig.2.14):

$$\beta_\pm \,=\, \frac{d}{2} \,\pm\, \sqrt{\frac{d^2}{4} \,+\, m^2\,L^2} \tag{2.119}$$

After going back to the x^μ coordinates, the generic solution for the scalar field close to the boundary takes the form:

$$\phi(x, z) \,\sim\, A(x)\,z^{d\,-\,\Delta} \,+\, B(x)\,z^{\Delta} \tag{2.120}$$

where we have identified:

$$\Delta \,=\, \beta_+ \,=\, \frac{d}{2} \,+\, \nu\,, \qquad \nu \,=\, \sqrt{\frac{d^2}{4} \,+\, m^2\,L^2} \tag{2.121}$$

Let's try to analyze better the situation keeping first the mass m^2 in the regime where $d-\Delta > 0$ (*Standard quantization*) (see fig.2.14).
In this case it easy to see that $d - \Delta = \Delta_-$ is the dominant contribution at the boundary and that we are not allowed to naively take the field-operator identification defining the source for the operator $\mathcal{O}$ as the boundary value of the field ϕ because that would be always null. The right map (in the standard quantization) reads:

$$\phi_0(x) \,=\, A(x) \,=\, \lim_{z\to 0}\, z^{\Delta\,-\,d}\,\phi(z, x) \tag{2.122}$$

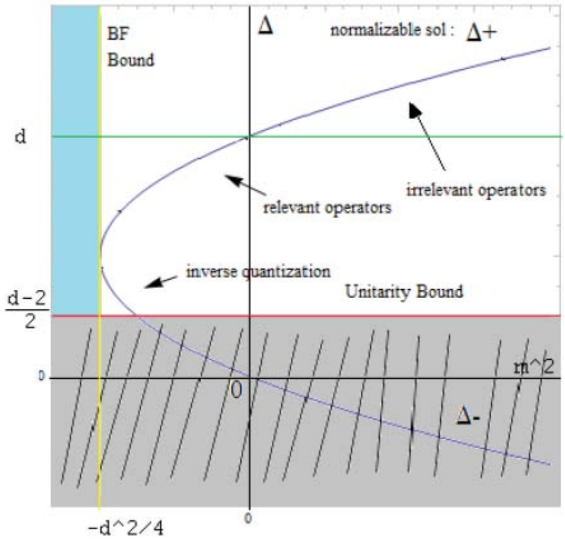

Figure 2.14: Scalar field in AdS_{d+1}. Conformal dimension and bounds.

In this way we make sure that such a value is always finite and well defined and it corresponds to the source ϕ_0.
That assumed, it easy to see (after some renormalization procedure we will not discuss, see [66]) that the on-shell action reduces to a boundary term of the form:

$$\mathcal{S}_{bdy} \sim \int d^d x \sqrt{-\gamma}\, A(x)\, B(x) \;=\; \int d^d x \sqrt{-\gamma}\, \phi_0(x)\, \mathcal{O}(x) \tag{2.123}$$

It is now clear that the leading $A(x)$ and subleading coefficient $B(x)$ of the field expansion coincide with the source ϕ_0 and the expectation value $\langle\mathcal{O}\rangle$ of the CFT operator $\mathcal{O}$.
Introducing a UV cutoff $z = \epsilon$ we can proove that the boundary action becomes:

$$\mathcal{S}_{bdy} \sim \int d^d x \left(\frac{L}{\epsilon}\right)^{2\,d} \phi(\epsilon, x)\, \mathcal{O}(\epsilon, x) \tag{2.124}$$

where $\phi(\epsilon, x) = \epsilon^{d-\Delta}\, \phi_0(x)$.
By plugging the latter in the boundary action we get:

$$\mathcal{S}_{bdy} \sim L^d \int d^d x\, \phi_0(x)\, \epsilon^{-\Delta}\, \mathcal{O}(\epsilon, x) \tag{2.125}$$

In order to understand ”who” is Δ we have to act with a scaling transformation as:

$$t' \,=\, \lambda\, t\,, \qquad x^{i\,\prime} \,=\, \lambda\, x^i\,, \qquad z' \,=\, \frac{z}{\lambda}\,. \tag{2.126}$$

In order to preserve the boundary expansion, the source ϕ_0 has to transform accordingly to:

$$\phi_0'(x') \,=\, \lambda^{d-\Delta}\, \phi_0(x) \tag{2.127}$$

Under such a scaling transformation the action becomes:

$$\int d^d x' \sqrt{-\gamma}\, \phi_0'(x')\, \mathcal{O}'(x') \;=\; \int d^d x \sqrt{-\gamma}\, \phi_0(x)\, \lambda^{\Delta}\, \mathcal{O}'(x') \tag{2.128}$$

which means that the operator $\mathcal{O}$, in order to keep the action scale invariant, has to transform as:

$$\lambda^{\Delta}\,\mathcal{O}'(x') \,=\, \mathcal{O}(x) \tag{2.129}$$

which defines its conformal dimension to be exactly $[\mathcal{O}] \,=\, \Delta$!
As a consequence $\mathcal{O}(\epsilon, x)$ can be indentified as the wave function renormalization of the operator $\mathcal{O}$ as we go deep into the bulk, namely as we run the RG flow towards the IR regime.
Note that depending on the value of Δ we can have 3 distinct situations:

- $\Delta > d$: the operator $\mathcal{O}$ is *irrelevant.* This means that its effects are going to be milder and milder towards the IR fixed point. This gets translated into the bulk as the fact that its size diminuishes going towards the interior of AdS spacetime $z \to \infty$.
- $\Delta = d$: the operator $\mathcal{O}$ is *marginal.* Its "size" and importante do not change under the RG flow.
- $\Delta < d$: the operator $\mathcal{O}$ is *relevant.* Its effects, and its size in the bulk, grow into the IR having a strong effect on the IR fixed point which will be indeed governed by this class of relevant operators.

This constitues a very elegant geometrization of the concept of RG flow and relevance/irrelevance of operators as a dynamical evolution in an extra dimension scale.
Note that so far we have not given any fundamental reason why $A(x)$ should be the source whereas $B(x)$ the expectation value of the operator $\mathcal{O}$. What is clear is they pair up into the product $\phi_0\mathcal{O}$ in a way that the sum of their dimensions is always ugual to the number of CFT spacetime dimensions:

$$\beta_{+} \,+\, \beta_{-} \,=\, d \tag{2.130}$$

Several comments are in order; a detailed description can be found in [68, 69].
The first comment is that in order to have a real power β, corresponding to a real conformal dimension Δ, we need to impose the so-called BF bound [70] which for a scalar states:

$$m^2 \,\geq\, -\left(\frac{d}{2\,L}\right)^2 \tag{2.131}$$

The mass squared of the scalar field can be negative $m^2 \,<\, 0$, namely the scalar field in AdS spacetime can be tachyonic but has to satisfy this bound. If not, various instabilities in the bulk side can arise, some of which have a clear and interesting interpretation in the CFT side we will explore later.
Finally, a good criterium to identify who is the expectation value of the operator is its finiteness. $\langle\mathcal{O}\rangle$ has to be the so called *normalizable* modes , a finite value on which we can integrate to perform the path integral of our CFT. The requirement of having the scalar field mode normalizable (see [71] for details about how to define such a norm in curved spacetime) fixes its power $\sim z^{\beta}$ to satisfy:

$$\beta \,>\, \frac{d\,-\,2}{2} \tag{2.132}$$

This lower bound coincides with the so called *unitarity bound* which the conformal dimension of the scalar has to satisfy in order for the CFT to retain unitarity (see [63]).
It is easy to see that there could be two choices:

- *Standard quantization*: $A(x) = \phi_0(x)$ and $B(x) = \langle \mathcal{O}(x) \rangle$. It is easy to see that this choice, that we discussed already, is always doable because the correspondent modes is always normalizable and satisfying the unitarity bound.
- *Alternative/inverse quantization*: $B(x) = \phi_0(x)$ and $A(x) = \langle \mathcal{O}(x) \rangle$. One can see (fig.2.14) that there is a small window $-d^2/4 < m^2 < -d^2/4 + 1$ where also the other mode is normalizable and can be therefore identified as the expectation value of the operator $\mathcal{O}$.

These two choices are actually not indipendent and they correspond to an SL(2,Z) transformation at the level of the boundary CFT [72]. There are also interesting features emerging from the two different quantization schemes dealing with the nature of the symmetries in the dual CFT side. With the alternative quantization it is indeed possible to make the global symmetry of the CFT side emergent and local; this fact has also some important phenomenological implications in the study for example of the so-called holographic superconductors we will analyze in the following [73, 74] and in the study of the holographic fractional quantum hall effect [75].
It is not always possible to perform such a different quantization but several examples are discussed also in the context of a U(1) vector bulk field [76] and also of the bulk metric in some particular circumstances [77].
Whenever not explicitely said we will stick to the standard quantization for the rest of the paper.

Higher Spin fields

For the sake of completeness we will just make a short excursion considering also higher spin fields in AdS.
For higher order p-forms $A_{\mu_1,\ldots,\mu_p}$ the indicial equations in AdS reads:

$$(\Delta - p)(\Delta + p - d) = m^2 L^2 \tag{2.133}$$

fixing the solutions to be:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d - 2p}{1}\right)^2 + m^2 L^2} \tag{2.134}$$

A known example is:

i. the gauge field A_μ ($p = 1$) for which:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d - 2}{2}\right)^2 + m^2 L^2} \tag{2.135}$$

Note that in the case of $m = 0$, when the U(1) gauge symmetry is unbroken, the conformal dimension of the current operator J^μ is equal to $\Delta = d - 1$ which is indeed the conformal dimension for a conserved current.

One can also consider symmetric fields, like the spin-2 metric field $g_{\mu\nu}$ for which:

$$\Delta(\Delta - d) = m^2 L^2 \tag{2.136}$$

such that in the massless case, enjoying diffeomorphism invariance, the dimension of the dual stress tensor operator is $\Delta = d$ which implies indeed a conserved stress tensor $T^{\mu\nu}$.

One can eventually generalize the same construction to fermionic fields in the bulk and getting for example that for a spin 1/2 field:

$$\Delta = \frac{d}{2} + |m\,L| \tag{2.137}$$

Scalar field correlation functions

We now apply what we learned in the previous generic sections to compute the 1-point and 2-points functions for a massive scalar field in the bulk governed by the following action:

$$\mathcal{S} = -\frac{\eta}{2}\int dz\, d^d x\,\sqrt{-g}\,\left[g^{MN}\,\partial_M\,\phi\,\partial_N\,\phi + m^2\,\phi^2\right] \tag{2.138}$$

where η is just a normalization constant.
Using the equations of motions we can easily identify the on-shell action as:

$$\mathcal{S}^{on-shell} = -\frac{\eta}{2}\int x\, d^d x\,\partial_M\,\left[\sqrt{-g}\,\phi\, g^{MN}\,\partial_N\,\phi\right] \tag{2.139}$$

Taking the usual UV cutoff $z = \epsilon$ such an on-shell action can be rewritten in the form:

$$\mathcal{S}^{on-shell} = \frac{\eta}{2}\int d^d x\,\left(\sqrt{-g}\,\phi\, g^{zz}\,\partial_z\,\phi\right)_{z=\epsilon} \tag{2.140}$$

and the conjugate momentum Π becomes:

$$\Pi = -\frac{\partial\mathcal{L}}{\partial\left(\partial_z\phi\right)} = \eta\,\sqrt{-g}\, g^{zz}\,\partial_z\,\phi \tag{2.141}$$

All in all the on-shell action takes the simple form:

$$\mathcal{S}^{on-shell} = \frac{1}{2}\int_{z=\epsilon} d^d x\,\Pi(z,x)\,\phi(z,x) \tag{2.142}$$

Fourier transforming in the x coordinates:

$$\Pi(z,x^\mu) = \int \frac{d^d k}{(2\,\pi)^d}\, e^{i\,k\,\cdot\,x}\,\Pi_k(z)\,, \qquad \phi(z,x^\mu) = \int \frac{d^d k}{(2\,\pi)^d}\, e^{i\,k\,\cdot\,x}\, f_k(z) \tag{2.143}$$

such an expression becomes:

$$\mathcal{S}^{on-shell} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\,\Pi_{-k}(z=\epsilon)\, f_k(z=\epsilon) \tag{2.144}$$

Taking into account the UV expansion of the scalar field (2.120) close to the AdS boundary, the conjugate in momentum space takes the form of:

$$\Pi_{-k}\,(z) \approx_{z\to 0} \eta\, L^{d-1}\left[(d-\Delta)\,A(-k)\,z^{-\Delta} + \Delta\, B(-k)\,z^{\Delta - d}\right] \tag{2.145}$$

Using these results we can now compute the on-shell action, keeping all the non vanishing terms at the boundary, and we get:

$$\mathcal{S}^{on-shell} = \frac{eta}{2}\,L^{d-1}\int \frac{d^d k}{(2\pi)^d}\left[\epsilon^{-2\nu}\,(d-\Delta)\,A(-k)\,A(k) + d\,A(-k)\,B(k)\right] \tag{2.146}$$

Notice that the first term is clearly divergent $\sim \epsilon^{-2\nu}$.
It is easy to prove that to cancel such a divergence we have to choose a local counterterm of the form:

$$\mathcal{S}_{ct} \,=\, -\frac{\eta}{2}\,\frac{d\,-\,\Delta}{L}\int_{z=\epsilon} d^dx\,\sqrt{-\gamma}\,\phi^2 \tag{2.147}$$

or, equivalently, in momentum space:

$$\mathcal{S}_{ct} \,=\, -\frac{\eta}{2}\,(d\,-\,\Delta)\,L^{d-1}\int\frac{d^dk}{(2\pi)^d}\,\Big[\epsilon^{-2\nu}\,A(-k)\,A(k)\,+\,2\,A(-k)\,B(k)\Big] \tag{2.148}$$

where γ is the induced metric at the boundary of AdS spacetime.
The full renormalized action, obtained by summing up the on-shell one with the proper counterterms, is given by:

$$\mathcal{S}^{ren} \,=\, \frac{\eta}{2}\,L^{d-1}\,(2\,\Delta\,-\,d)\int\frac{d^dk}{(2\pi)^d}\,A(-k)\,B(k) \tag{2.149}$$

We can now extract the one-point function by functional deriving the renormalized action with respect to the source ϕ_0. Note that in the standard quantization scheme the source is identified with the leading mode $\phi_0(x) = A(x)$. We must be careful because such a coefficient $B(x)$ depends functionally on $A(x)$ itself.
To illustrate it better, let consider $f_k(z)$ for a generic z as:

$$f_k(z) \,=\, A(k)\,\phi_1(z,k)\,+\,B(k)\,\phi_2(z,k) \tag{2.150}$$

where $\phi_{1,2}$ are independent solutions (explicitely shown in the following) of the f_k equation normalized in a way that their behaviour at the boundary reads:

$$\phi_1(z,k)\,\sim\,z^{d\,-\,\Delta}\,,\qquad \phi_2(z,k)\,\sim\,z^{\Delta}\,. \tag{2.151}$$

To determine completely the f_k solution we need to impose an additional boundary condition at the AdS horizon located at $z = \infty$. As it will be clearer soon, this fixes uniquely the ratio B/A to a value which is independent of the value of the field ϕ at the boundary.
Let us call that ratio:

$$\chi \,=\, \frac{B}{A} \tag{2.152}$$

Clearly we can write down the renormalized action:

$$\mathcal{S}^{ren} \,=\, \frac{\eta}{2}\,L^{d-1}\,(2\,\Delta\,-\,d)\int\frac{d^dk}{(2\pi)^d}\,\phi_0(k)\,\chi(k)\,\phi_0(-k) \tag{2.153}$$

where we have used the fact that the leading term $A(k)$ can be identified with the source $\phi_0(k)$.
We can immediately compute the 1-point function as:

$$\langle\,\mathcal{O}(k)\,\rangle_{\phi_0} \,=\, (2\pi)^d\,\frac{\delta\mathcal{S}^{ren}}{\delta\phi_0(-k)} \,=\, 2\,\nu\,\eta\,L^{d-1}\,B(k) \tag{2.154}$$

where we have used that $2\Delta\,-\,d\,=2\,\nu$.
We finally find out that the 1-point function of the operator $\mathcal{O}$, namely its VEV, is encoded in the subleading contribution near the boundary of the bulk dual field ϕ, *i.e.* $B(k)$. It follows immediately that the 2-point function, or **Green Function**, for the same operator can be identified with:

$$\mathcal{G}_E(k) \,=\, 2\,\nu\,\eta\,L^{d-1}\,\frac{B(k)}{A(k)} \tag{2.155}$$

In other words, the ratio between the subleading and the leading contributions of the bulk field provides the Green Function for the correspondent CFT operator dual to such a field!
Let us be more explicit about it. We first consider a redefinition of the function f_k:

$$f_k(z) \,=\, z^{d/2}\, g_k(z) \tag{2.156}$$

It is easy to check that such new function has to satisfy the following equation:

$$z^2\,\partial_z^2\, g_k \,+\, z\,\partial_z\, g_k \,-\, \Big(\nu^2 \,+\, k^2\, z^2\Big)\, g_k \,=\, 0\,. \tag{2.157}$$

which is nothing else than the modified Bessel equation, whose two independent solutions can be taken to be $g_k = I_{\pm\nu}(kz)$, where $I_{\pm\nu}$ are the modified Bessel functions. Note that such modified Bessel functions behave for small argument (*i.e.* close to the AdS boundary) as:

$$I_{\pm\nu}(z) \,\sim\, \frac{1}{\Gamma(1\,\pm\,\nu)}\,\left(\frac{z}{2}\right)^{\pm\nu} \tag{2.158}$$

Going back to the original f_k function we can therefore define the two independent solutions $\phi_{1,2}$ as:

$$\phi_1(z) \,=\, \Gamma(1\,-\,\nu)\,\left(\frac{k}{\nu}\right)^{\nu}\, z^{d/2}\, I_{-\nu}(kz)\,, \qquad \phi_2(z) \,=\, \Gamma(1\,+\,\nu)\,\left(\frac{k}{\nu}\right)^{-\nu}\, z^{d/2}\, I_{\nu}(kz)\,. \tag{2.159}$$

One can check that close to the boundary these choices has the correct asymptotic power law behaviours assumed previuosly.
All in all the f_k solution can be written down as:

$$f_k(z) \,=\, z^{d/2}\,\left[\Gamma(1\,-\,\nu)\,\left(\frac{k}{\nu}\right)^{\nu}\, z^{d/2}\, I_{-\nu}(kz) \,+\, \Gamma(1\,+\,\nu)\,\left(\frac{k}{\nu}\right)^{-\nu}\, I_{\nu}(kz)\right] \tag{2.160}$$

Let us now impose the other regularity boundary condition at the horizon $z = \infty$, namely that such a solution turns out to be finite at that location. Noticing that the modified Bessel functions behave as:

$$I_{\pm\nu} \,\approx\, \frac{e^z}{\sqrt{2\,\pi\,z}} \tag{2.161}$$

for large argument $z \to \infty$, such a regularity condition fixes uniquely the ratio B/A to be:

$$\frac{B(k)}{A(k)} \,=\, -\,\frac{\Gamma(1\,-\,\nu)}{\Gamma(1\,+\,\nu)}\,\left(\frac{k}{2}\right)^{2\,\nu} \,=\, \frac{\Gamma(-\,\nu)}{\Gamma(\nu)}\,\left(\frac{k}{2}\right)^{2\,\nu} \tag{2.162}$$

as we stated before.
Using this result we can write down the Euclidean Green Function for the scalar operator $\mathcal{O}$ dual to the ϕ bulk field as:

$$\mathcal{G}_E(k) \,=\, 2\,\nu\,\eta\,L^{d-1}\,\frac{\Gamma(-\,\nu)}{\Gamma(\nu)}\,\left(\frac{k}{2}\right)^{2\,\nu} \tag{2.163}$$

We can now rewrite such a 2-point function in position space[10] as:

$$\langle\,\mathcal{O}(x)\,\mathcal{O}(0)\,\rangle \,=\, \frac{2\,\nu\,\eta\,L^{d-1}}{\pi^{d/2}}\,\frac{\Gamma\left(\frac{d}{2}\,+\,\nu\right)}{\Gamma(-\nu)}\,\frac{1}{|x|^{2\,\Delta}} \tag{2.165}$$

[10] We make use of the formula:

$$\int\,\frac{d^dk}{(2\pi)^d}\,e^{i\,k\,x}\,k^n \,=\, \frac{2^n}{\pi^{d/2}}\,\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(-\,\frac{n}{2}\right)}\,\frac{1}{|x|^{d+n}} \tag{2.164}$$

which is indeed what we expect in a conformal field theory for a primary operator of dimension Δ!

2.7 Example 2: Reissner Nordstrom Black Hole

So far we focused our attention to the original formulation of the AdS/CFT correspondence which maps a conformal field theory to a pure AdS bulk geometry. As we already explained, conformal field theories are very particular "beasts" dealing with quantum critical points or very fine tuned QFTs. This is not for example the case for a generic condensed matter system which usually lives at finite temperature T and finite charge density ρ. In such a way we of course introduce a scale into the problem breaking the original conformal invariance of the full theory. Such a deformations (if relevant) make the theory to depart from the original UV conformal fixed point and to undergo an RG flow towards another infrared fixed point. The AdS/CFT correspondence can be generalized easily to describe also these situations such that its name can be mutated into the more generic one of **Gauge-gravity duality**.
From the bulk point of view the departure from conformal invariance renders the spacetime geometry different from the pure AdS case, which is recovered just asymptotically in the UV. The bulk spacetime encodes directly the RG flow due to such a deformation and it is able to encode directly, also via additional fields, the features of the non-conformal QFT.
The easiest and most important example we are going to analyze in this section is the so-called **Reissner Nordstrom black hole** which is the dual gravitational picture of a QFT at finite temperature and finite charge density. This example is the first application we consider of the AdS/CMT correspondence and it has been subject of a huge amount of research under lots of directions. For generic discussions about its role among the applications to condensed matter we refer to [78, 79].
Temperature T and chemical potential μ clearly break scale invariance such that the original AdS spacetime has to be modified to the most generic form:

$$ds^2 \,=\, \frac{L^2}{z^2}\,\Big(-\,f(z)\,dt^2\,+\,g(z)\,dz^2\,+\,h(z)\,dx_\mu\,dx^\mu\Big) \tag{2.166}$$

which still preserves rotational symmetry and spacetime translations.
Not all these three functions we introduced are meaningful; it is indeed straightfroward to realize that the form of $g(z)$ it is just a gauge choice which can be set via a coordinate transformation $z \to \hat{z}(z)$. We want to retain scale invariance at high energy; this fixes the functions f, h to asymptote 1 close to the UV boundary of the geometry, which will as a consequence described by the AdS geometry. Moreover let us underline that a possible choice $f(z) \neq h(z)$ would break Lorentz invariance ab initio and it is possible (in the context of General relativity) just adding some extra matter content[11].

Warming up with the Schwarzschild-AdS solution

In order to make a precise definition of the function f, h which determine the bulk geometry one has to rely on the Einstein's equations of motion:

$$R_{\mu\nu}\,=\,-\,\frac{d}{L}\,g_{\mu\nu} \tag{2.167}$$

[11]In other setup, like Horava-Lifshitz gravity, matter would not be required to support non-relativistic solution.

Plugging the previous ansatz into the Einstein's equations one discover the famous Schwarzschild-AdS solution:

$$ds^2 \,=\, \frac{L^2}{z^2}\left(-\,f(z)\,dt^2 \,+\, \frac{dz^2}{f(z)} \,+\, dx_\mu\,dx^\mu\right) \tag{2.168}$$

where $f(z)$ takes the name of *emblackening factor* and it reads:

$$f(z) \,=\, 1 \,-\, \left(\frac{z}{z_h}\right)^d \tag{2.169}$$

A new dimensionless parameter z_h/L is introduced in the theory and has to be interpreted from the dual side. In more details, because $f(z \to 0) = 1$, AdS is recovered asymptotically in the UV while the geometry gets deformed towards the IR. At $z = z_h$ the geometry enjoys the presence of an *event horizon*[12] at which the g_{tt} term vanishes and makes the surface $z = z_h$ infinitely redshifted with respect to an asymptotic observer. An event horizon is associated to a **Black Hole** object which immediately suggests that the IR physics we have just found corresponds to placing the scale invariant theory at a finite temperature.

To convince ourselves we give a sketchy argument due to Gibbons and Hawking in [80]. Within a semiclassical regime we can think of the partition function of the bulk theory as a path integral over metrics. In a saddle point approximation (motivated by large N limit) we can write down:

$$\mathcal{Z} \,=\, e^{-\mathcal{S}_E(g*)} \tag{2.170}$$

where $\mathcal{S}_E(g*)$ is the Euclidean action evaluated at the saddle.

In order to have a well defined variational problem we have to include the Gibbons-Hawking boundary term (see for example [81]) and a counterterm part to render such an action finite. All in all we get:

$$\mathcal{S}_E \,=\, -\,\frac{1}{2\,\kappa^2}\int d^{d+1}x\,\sqrt{g}\,\left(R \,+\, \frac{d(d-1)}{L^2}\right) \,+\, \frac{1}{2\,\kappa^2}\int_{boundary} d^dx\,\sqrt{\gamma}\,\left(-\,2\,\mathcal{K} \,+\, \frac{2\,(d-1)}{L}\right) \tag{2.171}$$

where γ is the induced metric at the boundary and $\mathcal{K}$ the trace of the extrinsic curvature $\mathcal{K} = \gamma^{\mu\nu}\nabla_\mu n_\nu$ with n_μ an outward pointing unit normal vector to the boundary.

After Wick rotating the time coordinate $\tau = it$ in order for the bulk solution to be regular at $z = z_h$ (without any conical singularity) we must periodically identified τ with periodicity:

$$\tau \,\sim\, \tau \,+\, \frac{4\,\pi}{|f'(z_h)|} \tag{2.172}$$

It is a well known fact that studying field theory with a periodically identified Euclidean time corresponds to considering the theory in equilibrium at a finite temperature. The temperature is the inverse of the periodicity. We can thereofore define the BH temperature as:

$$T \,=\, \frac{|f'(z_h)|}{4\,\pi} \tag{2.173}$$

that for the Schwarzschild-AdS black hole simply reads:

$$T_{scBH} \,=\, \frac{d}{4\,\pi\,z_h} \tag{2.174}$$

[12]It is a boundary in spacetime beyond which events cannot affect an outside observer. In layman's terms, it is defined as "the point of no return", *i.e.*, the point at which the gravitational pull becomes so great as to make escape impossible, even for light.

and it corresponds to the temperature T of the dual theory.
In a generical black hole geometry the temperature can be defined through the *surface gravity* κ (in units of $\hbar$) as:

$$T = \frac{2\,\pi}{\kappa} \tag{2.175}$$

nevertheless for all the cases considered in this thesis the formula (2.173) will be enough.
In a scale invariant theory at finite temperature and in equilibrium there is no other scale to be compared with the temperature. Therefore, all nonzero temperatures should be equivalent[13] There are only two inequivalent temperatures: zero and nonzero.
Given the temperature definition we can now define the partition function explicitely as:

$$\mathcal{S}_E = -\frac{(4\pi)^d\,L^{d-1}}{2\,\kappa^2\,d^d}\,V_{d-1}\,T^{d-1} \tag{2.177}$$

In order for the semiclassical approximation to be reliable we need the coefficient of such a action to be parametrically large, which means $L^{d-1}/\kappa^2 \gg 1$, *i.e.* a weakly curved configuration (in Plank units). Note this matches with the previous identifications and the large N limit of the dual field theory.
We can now define the Free energy for the theory:

$$\mathcal{F} = -T\,\log\mathcal{Z} = T\,\mathcal{S}_E[g*] = -\frac{(4\pi)^d\,L^{d-1}}{2\,\kappa^2\,d^d}\,V_{d-1}\,T^d \tag{2.178}$$

and from it deriving all the thermodynamical quantities of interest.
The entropy s can be for example derived as:

$$s = -\frac{\partial\mathcal{F}}{\partial T} = \frac{(4\pi)^d\,L^{d-1}}{2\,\kappa^2\,d^{d-1}}\,V_{d-1}\,T^{d-1} \tag{2.179}$$

As a check of our computation we can note that this expression for the entropy is equal to the area of event horizon divided by $4G_N$, where in our conventions Newton's constant is $G_N = \kappa^2/8\pi$. This area-entropy relation is universally expected to be true for event horizons and takes the name of *Area law*.

The Reissner-Nordstrom solution

Condensed matter systems are usually characterized by another important parameter which controls the amount of charge carriers present in the material and which is encoded by the *charge density* ρ. The presence of a finite charge density is associated to the existence of a U(1) symmetry in the theory which provides such a conserved quantity.
Despite the U(1) symmetry being generically local, there are good reasons to work in an approximation where we make it global (we forget about photons); see [78] for details about this approximation and its possible failures. As already discussed before, in order to consider a QFT with a global U(1) symmetry at the boundary we have to introduce a correspondent gauged U(1) symmetry in the bulk of the gravitational dual[14].

[13] In the Schwarzschild-AdS geometry it is easy to see indeed that the scaling:

$$(z, t, x^i) \;\to\; z_h\,(z, t, x^i) \tag{2.176}$$

eliminates completely the z_h parameter from the metric.

[14] As already said, it turns out that there exists also the possibility of working with a local U(1) at the boundary where the gauge field is a composite emergent object [73] through the so called *alternative quantization*.

The latter can be done introducing an additional Maxwell field A_μ into the bulk spacetime and defining the so-called *Einstein-Maxwell* action:

$$\mathcal{S} \,=\, \int d^{d+1}x\,\sqrt{-g}\,\left[\frac{1}{2\,\kappa^2}\,\left(R+\frac{d\,(d-1)}{L^2}\right)-\frac{1}{4\,g^2}\,F^2\right] \tag{2.180}$$

where $F = dA$ is the gauge field strength.
In thermal equilibrium there are two new background scales we can now introduce in the field theory in a way that preserves rotational symmetry. One is a chemical potential $\mu = A_t^{(0)}$ and the other, which only preserves rotational symmetry in $2+1$ dimensions, is a background magnetic field $B = F_{xy}^{(0)}$. As we saw previously with the temperature, T, these new scales must cause deformations away from a pure AdS spacetime as we move away from the boundary and into the IR region. For simplicity we will avoid considering an external magnetic field for the rest of this thesis despite being an appealing and interesting direction.
What we are after is a solution of the equations of motion following from the Einstein-Maxwell action (2.180) with a non trivial radial profile for the gauge field A_μ:

$$A \,=\, A_t(z)\,dt \tag{2.181}$$

Looking for solutions, we can identify the **Reissner-Nordstrom-AdS black hole** described by:

$$\begin{aligned} ds^2 \,&=\, \frac{L^2}{z^2}\,\left(-\,f(z)\,dt^2+\frac{dz^2}{f(z)}+dx_\mu\,dx^\mu\right)\,,\\ f(z) \,&=\, 1-\left(1+\frac{z_h^2\,\mu^2}{\gamma^2}\right)\,\left(\frac{z}{z_h}\right)^d+\frac{z_h^2\,\mu^2}{\gamma^2}\,\left(\frac{z}{z_h}\right)^{2\,(d-1)}\,,\\ A_t(z) \,&=\, \mu\,\left[1-\left(\frac{z}{z_h}\right)^{d-2}\right]\,. \end{aligned} \tag{2.182}$$

where we defined:

$$\gamma \,=\, \frac{(d-1)\,g^2\,L^2}{(d-2)\,\kappa^2} \tag{2.183}$$

which is a dimensionless measure of the relative strengths of the gravitational and Maxwell forces.
Let's analyze in details the gauge field solution. Its asymptotic behaviour close to the UV boundary is of the form:

$$A_t(z) = \mu\,-\,\rho\,z^{d-2} \tag{2.184}$$

where μ is the leading contribution and ρ is the subdominant term close to the boundary. The subleading contribution, within the standard quantization choice, has to be identified as the response of the system to a chemical potential source and it represents indeed the charge density of the system:

$$\rho \,\equiv\, \text{charge density} \tag{2.185}$$

As discussed in generality before the regularity condition at the horizon fixes a relation between the leading and subleading term which is just a IR data. In this case the one form A will not be well defined unless A_t vanishes at the horizon[15] [82]. This imply the relation:

$$\rho \,=\, \frac{\mu}{z_h^{d-2}} \tag{2.186}$$

[15] In more technical words, if $A_t(z_h)$ were finite one could obtain a finite Wilson loop $\oint A$ around the vanishing Euclidean time circle, indicating that the gauge connection is singular.

which determines the static charge susceptibility of the dual system $\chi_E = \frac{\partial \rho}{\partial \mu}$.
The temperature of the BH solution is given by:

$$T \,=\, \frac{1}{4\,\pi\,z_h}\left(d \,-\, \frac{(d-2)\,z_h^2\,\mu^2}{\gamma^2}\right) \tag{2.187}$$

An important feature now is that the temperature can become zero continuously. Recall that with no chemical potential we could scale out z_h and hence all nonzero temperatures were equivalent. Here we can again do it, but we are left with the scale set by μ and therefore with the dimensionless ratio T/μ, which can be continuously taken to zero. In a scale invariant theory all dimensionless equilibrium quantities can only depend on temperature and chemical potential through this ratio - there are no other scales.
We can again compute the Euclidean action[16] and working in the gran canonical ensemble, with μ fixed, define the *Grand Potential* $\Omega = -T\log\mathcal{Z}$ as:

$$\Omega \,=\, \mathcal{F}\left(\frac{T}{\mu}\right)\,V_{d-1}\,T^d \tag{2.188}$$

where the function $\mathcal{F}$ can be easily extracted and it is a nontrivial output from AdS/CFT.
One can again derive all the various thermodynamical quantities and conclude that the macroscopic thermodynamic potential is given by:

$$\Omega \,=\, E \,-\, T\,s - \,\mu\,\rho \tag{2.189}$$

where E is the energy of the system encoded in the BH emblackening factor (for more details see [78]).
As last remarks, let's note the following:

- The temperature and chemical potential deformations are IR effects which are invisible at high energies $\omega \gg T, \mu$.
- The charge density ρ can be indentified as the conjugate momentum to A_t, *i.e.* Π_{A_t}.
- The RN black hole at temperature $T \ll \mu$ represents a nice example of RG flow. Its geometry interpolates between a UV fixed point dual to the AdS_{d+1} asymptotics and an infrared near horizon geometry given by $\mathrm{AdS}_2 \times \mathbb{R}^2$. The latter enjoys an emergent scale invariance and it is dual to a non relativistic scale invariant fixed point which features the so-called *local quantum criticality* [83].
- The RN black hole shows an entropy s which is finite at $T = 0$ and which seems to violate the common thermodynamic rules. It is believed that such a finiteness is connected to a large degeneracy of the CFT ground state probably linked with the large N limit (see [79]).

Holographic conductivity

Once defined the suitable background solutions featuring finite temperature T and finite charge density ρ we are ready to compute, through linear response technique, the electric conductivity holographically. A good reference is given by [84].

[16] No additional counterterms are necessary because the Maxwell field falls off sufficiently quickly near the boundary in the dimensions of interest ($d \geq 3$).

The conductivity is defined as the response of the system to an external oscillating electric field of the form:

$$E_x = \int \frac{d\omega}{2\pi} \, \mathrm{e}^{-i\,\omega,t} \, E_x(\omega) \tag{2.190}$$

which for simplicity is taken to point in just one direction. Such an external field will source a current:

$$J_x = \int \frac{d\omega}{2\pi} \, \mathrm{e}^{-i\,\omega,t} \, J_x(\omega) \tag{2.191}$$

such that the frequency dependent electric condutivity, *i.e.* the optical conductivity, can be defined as the ratio:

$$\sigma(\omega) = \frac{J_x(\omega)}{E_x(\omega)} \tag{2.192}$$

Because we are working in Fourier space, σ is generically complex. The real part captures what you would intuitively call the conductivity (or inverse resistivity) of the system: it describes the dissipation of the current. The imaginary part is the reactive part.
In the language of Linear response theory, the conductivity can be compute through *Kubo formulas* as the 2-points function of the J_x operator:

$$\sigma(\omega) = -\frac{i}{\omega} \, G^R_{J_x\,J_x}(\omega) \tag{2.193}$$

where R indicates the retarded Green function.
Now we want to take the RN solutions and perturb it with an external oscillating electric field. This is a source for the current J_x of which we would like to compute the response $\langle J_x \rangle$.
For simplicity, and because most of the strongly coupled materials of interest are layered, we focus on a $3+1$ bulk.
We introduce an electric field in the x direction turning on a source $A_x = \frac{E}{i\,\omega}\, e^{i\,\omega\,t}$ on the boundary. This indeed corresponds to an electric field $F_{tx} = \dot{A}_x = E\, e^{i\,\omega\,t}$ as we wanted. In the bulk, all of this gets translated into the radial profile of an A_x perturbation of the form:

$$A_x(z) = \frac{E}{i\,\omega}\, e^{i\,\omega\,t} + \langle J_x \rangle \, z + \dots \tag{2.194}$$

where the subleading term $\langle J_x \rangle$ can be derived solving the equations of motion for the system.
These calculations were first performed in [85] for the Schwarzchild black hole and in [78] for the Reissner-Nordstrom black hole.
One can show that sourcing A_x in this way will also turn on the metric component g_{tx}, but no further fields. The Maxwell equation is:

$$\left(f(z)\, A_x'(z) \right)' + \frac{\omega^2}{f(z)}\, A_x = -\frac{A_t'(z)\, z^2}{L^2} \left(g_{tx}' + \frac{2}{z}\, g_{tx} \right) \tag{2.195}$$

along with the constraint coming from Einstein's equations:

$$g_{tx}' + \frac{2}{z}\, g_{tx} + \frac{4\,L^2}{\gamma^2}\, A_t'(z)\, A_x(z) = 0 \tag{2.196}$$

We can use this latter constraint to eliminate the metric, leaving us with a single second order equation of motion for A_x:

$$\left(f(z)\, A_x'(z) \right)' + \frac{\omega^2}{f(z)}\, A_x = \frac{4\,\mu^2\, z^2}{\gamma^2\, z_h^2}\, A_x(z) \tag{2.197}$$

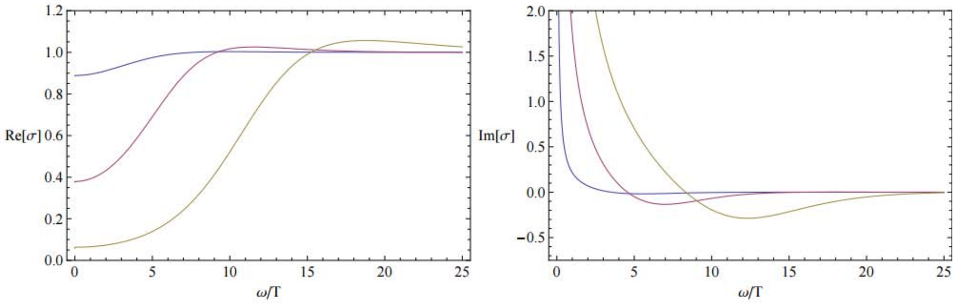

Figure 2.15: The real (left) and imaginary (right) parts of the electrical conductivity computed via AdS/CFT as described in the text. The conductivity is shown as a function of frequency. Different curves correspond to different values of the chemical potential at fixed temperature. The gap becomes deeper at larger chemical potential. Figure is taken from [78]

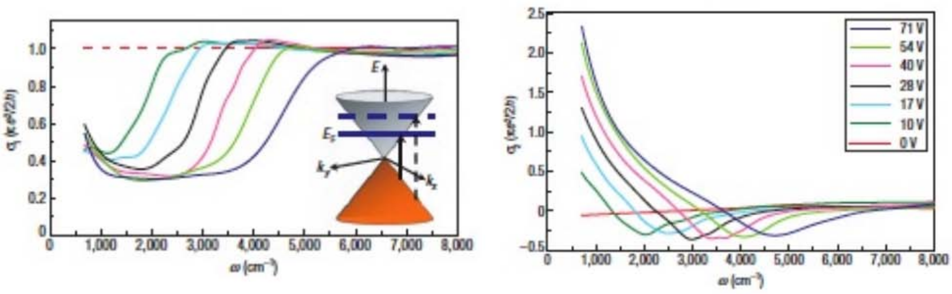

Figure 2.16: Experimental plots of the real (top) and imaginary (bottom) parts of the electrical conductivity in graphene as a function of frequency. The different curves correspond to different values of the gate voltage. The inset in the upper plot shows an interband transition that is accessible at energies above $2\,E_f$. Plots taken from [87].

What is left is solving this differential equation with ingoing boundary conditions at the horizon and determine the response $\langle J_x \rangle$ in terms of the source E_x. The choice of ingoing boundary conditions is fixed by the AdS/CFT dictionary and it corresponds to compute the retarded correlator; for more details see [86].
The ratio is the wanted optical conductivity, which we can write (after fixing the couplings to 1 for simplicity) as:

$$\sigma(\omega) \;=\; \frac{A_x'}{i\,\omega\,A_x}|_{boundary} \tag{2.198}$$

Although the equation of motion (2.197) cannot be solved analytically, it is a simple matter to solve it numerically. The results are shown in fig.2.15. Let's compare the results coming from AdS/CFT 2.15 with the experimental results in 2.16. The similarity is striking!
Let us focus on the real part of the conductivity, the imaginary part can be determined from the real part through the Kramers-Kronig relations. There are three features in the data:

i. At large frequencies the conductivity tends to a constant.

ii. At low frequencies the (real part of the) conductivity is depleted below a scale set by the chemical potential.

iii. At very small frequencies the conductivity starts to rise again.

The fact that the conductivity tends to a constant at large frequencies in both figures is consistent with the fact that the conductivity is dimensionless in $2+1$ dimensions and does not rely on relativistic invariance.
The second fact relies on the fact that the real part of the conductivity is the dissipative part of the conductivity and measures the presence of charged states as a function of energy. The drop in the real part of the conductivity therefore corresponds to a drop in the density of excitations at energies below the chemical potential. In graphene there is a simple explanation for this fact. The chemical potential sets the size of the Fermi surface. At zero momentum (we are computing the conductivity at zero momentum) the only available single particle excitations are when an electron jumps between different bands. This is illustrated in the inset of figure 2.16. In graphene, such an excitation has energy $2E_F$, where E_F is the Fermi energy and is proportional to the chemical potential μ. Therefore, the dissipative conductivity will be Boltzman suppressed up to an energy scale set by μ. Given that the same structure is observed in figure 2.15, in AdS/CFT, one is lead to wonder if there may also be an effective Fermi surface in the strongly coupled theories studied via AdS/CFT.
The last feature is the only important difference between the AdS/CFT results and the experimental plots. We know it is there because the imaginary part of the conductivity has a pole as $\omega \to 0$. The Kramers-Kronig relations imply that the real part must therefore have a delta function. The divergence of the conductivity at low frequencies is directly related to conservation of momentum and it will be one of the most important point of the present thesis. In other words, we have a system with a background charge density and with translational invariance. If you subject the system to a constant, $\omega = 0$, electric field then the charge density will necessarily accelerate. But, because there is translational invariance, there is momentum conservation. This means that there is no way for the charges to dissipate their momentum and the current will persist. This is the origin of the delta-function!
In graphene, momentum conservation is broken by the presence of impurities and the ionic lattice. These effects introduce a momentum relaxation timescale τ so that at low frequencies one has a Drude peak described by:

$$\sigma(\omega) = \sigma_0 + \frac{\rho^2}{\epsilon + P} \frac{1}{1/\tau - i\omega} \tag{2.199}$$

which clearly reproduces the AdS/CFT results in the translational invariant limit $\tau \to \infty$.
Introducing the effects of impurities and momentum dissipation into the framework of holography will represent the main topic of this thesis and will permit to reconcile with the experimental expectations.

2.8 Example 3: Holographic Superconductors

As introduced in the first chapter in recent years several materials exploiting high temperature superconductivity have been experimentally found and studied. In these materials the mechanism behind the SC transition is thought to be different from the BCS electron pairing driven via phonons interactions. Moreover it seems that because of the high temperature at which the SC instability appears this new pairing mechanism has to be related to a very strongly coupled phenomenon, strong enough to survive at such high temperatures. All in all, it is extremely

interesting to find out a possible strongly coupled model for superconductivity explaining the behaviour of these new materials. In this direction AdS/CFT turned out to be a possible tool to adress that kind of questions.
The first holographic models for superconductivity have been introduced in [88, 89] and during the past years a huge amount of work has been done and several well done reviews are available [90–93].
The physics of Superconductivity is mainly defined by the two following features:

- An infinite DC ($\omega = 0$) conductivity.
- The repulsion of the magnetic field (*Meissner effect*).

From a more theoretical perspective superconductivity can be associated to the spontaneous symmetry breaking (SSB) of the U(1) symmetry associated to charge conservation. We thus need an operator charged under the U(1) symmetry which aquires a non null vacuum expectation value (VEV). The minimal setup, known as *S-wave* superconductivity, refers to the identification of such a field with a scalar operator of spin 0 and it will constitute our benchmark playground. There is a priori no constraint in limiting the charge operatore which breaks the U(1) symmetry to be a scalar; on the contrary indeed we can consider a charged spin 1 operator (*P-Wave* superconductivity) or eventually a spin 2 charged operator as well (*P-Wave* superconductivity). In what follows we concentrate on the simplest S-wave example; see [94, 95] for the P-wave case and [96] for the D-wave one.

Superconducting instability and condensate

We consider Einstein-Maxwell theory together with a charged (complex) scalar field. A minimal Lagrangian (in $d+1$ dimensions) for such a system is:

$$\mathcal{L} = \frac{1}{2\,\kappa^2}\left(R + \frac{d\,(d-1)}{L^2}\right) - \frac{1}{4\,g^2}\,F^2 - |\,\nabla\phi - i\,q\,A\,\phi\,|^2, - m^2\,|\phi|^2 - V(|\phi|)\,. \qquad (2.200)$$

We will immediately specialise to the case $V(|\phi|) = 0$, again for simplicity. We also specialise for concreteness to the case of $d = 3$ dimensions for the boundary field theory.
The scalar ϕ constitutes the dual bulk field of the charged operator which will condense, *i.e.* getting a finite expectation value (VEV) $\langle\mathcal{O}\rangle$, and break the global $U(1)$ symmetry of the CFT producing the SC state. Note that because the $U(1)$ symmetry will results global on the boundary it would be more appropriate to call that phase a *superfluid* and not a SC; a real SC state can be achieved using the alternative quantization techniques [73].
The normal, *i.e.* non-superconducting, state of the theory will correspond to the solution where such a scalar field is trivially null. In this case we recover the usual Einstein-Maxwell theory and the simplest background we might consider is simply the Schwarzschild-AdS metric, corresponding to a scale invariant theory at finite temperature with zero charge density. This choice is too poor. We noted previously that in a scale-invariant theory all nonzero temperatures are equivalent. In particular, there cannot be a preferred critical temperature, T_c, at which something special happens. In order to have a critical temperature, another scale must be introduced. If we wish to avoid adding any new ingredients into our theory, the simplest way to introduce a scale is to work at a finite chemical potential μ. By dimensional analysis this allows $T_c \sim \mu$. Once introduced the chemical potential we end up with the Reissner-Nordstrom-AdS black hole solution which will be the desidered gravitational dual for our normal phase.

Given the background the question is whether this phase is unstable under the formation of a charge condensate. The scalar field would behave at the boundary as:

$$\phi \approx \phi_0 \left(\frac{z}{L}\right)^{\Delta_-} + \phi_1 \left(\frac{z}{L}\right)^{\Delta_+} \tag{2.201}$$

Using the standard quantization, the coefficient of Δ_- would result the dominant contribution at the boundary and therefore identified as the source for the operator $\mathcal{O}$ while the second term, the subleading one, would define the desired VEV of such operator:

$$\phi_1 \equiv \langle \mathcal{O} \rangle \tag{2.202}$$

The spontaneous breaking of the U(1) symmetry would correspond to a normalisable solution ($\phi_0 = 0$, *i.e.* no source for $\mathcal{O}$) which develops dynamically a non trivial VEV:

$$\langle \mathcal{O} \rangle \neq 0 \tag{2.203}$$

This would correspond to an instability of the RN black hole driven by the bulk scalar ϕ and would represent our SC phase.
Note that such a solution has to be searched imposing infalling boundary condition at the horizon such that the instability would be represented in the quasinormal mode spectrum of the BH as a pole of the Green function appearing in the upper complex plane [97], *i.e.* a mode growing exponentially in time.
To search for instability at a critical temperature one perturbs the Reissner-Nordstrom background by a scalar field $\phi = \phi(r)e^{-i\omega t}$ whose equation of motion reads:

$$-z^4 \left(\frac{f}{z^2}\phi'\right)' - \frac{z^2}{f}\left(\omega + q\,\mu\left(1 - \frac{z}{z_h}\right)\right)\phi + m^2 L^2 \phi = 0\,. \tag{2.204}$$

At this point we are only interested in determining whether the instability arises and the critical temperature at which at it first appears. At the critical temperature the unstable mode would be exactly located at the origin of the complex plane $\omega = 0$ and we can therefore search for a static solution of such a equation. At the critical temperature $T = T_c$ we expect to find a normalisable and static solution of (2.204).
With some rescaling of coordinates one can check that the full system is characterized just by three dimensionless parameters:

$$\gamma\, q\,, \qquad \Delta\,, \qquad \frac{\gamma\, T}{\mu}\,. \tag{2.205}$$

One therefore scans through values of Δ and γq, and for each value determines numerically whether equation (2.204) admits a normalisable solution with $\omega = 0$ for some critical value of $\gamma T/\mu$. The result of this scan, from reference [98], are shown in fig. 2.17. Despite the full phase diagram has to be computed numerically the criterium for instability can be extracted analytically exploiting some features of the extremal $T = 0$ solution.
In particular at $T = 0$ the normal phase geometry interpolates between the AdS_4 in the ultraviolet and the $AdS_2 \times \mathbb{R}^2$ in the infra-red. The radius of the AdS_2 region is $L_2^2 = L^2/6$. The scalar ϕ aquires also an effective mass:

$$M_e f f^2 L^2 = m^2 L^2 + q^2 g^{tt} A_t^2 L^2 \tag{2.206}$$

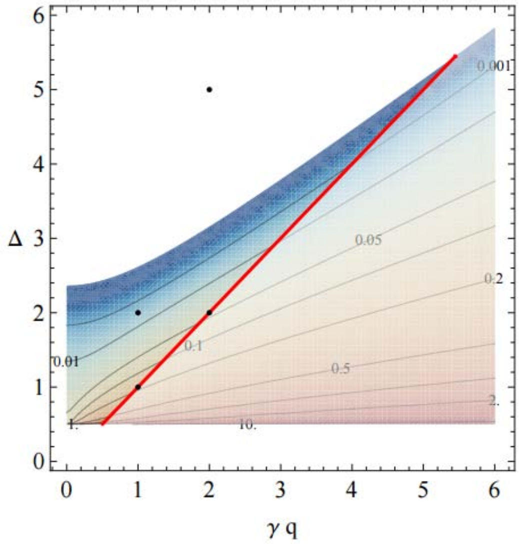

Figure 2.17: The critical temperature T_c as a function of charge γq and dimension Δ. Contours are labeled by values of $\gamma T_c/\mu$. The top boundary is a line of quantum critical points separating normal and superconducting phases at $T = 0$. The bottom boundary of the plot is the unitarity bound $\Delta = 1/2$ at which T_c diverges. Figure taken from [98].

which contains an additional contribution[17] from the gauge field configuration. Because of the presence of g^{tt} this new term appears to be negative such that at the extremal horizon the scalar can violate the BF bound:

$$M_eff^2\, L^2 \, < \, -\frac{1}{4} \tag{2.207}$$

and produce an instability.
Working out the instability criterium explicitely we end up with the following:

$$q^2\,\gamma^2 \, \geq \, 3 \, + \, 2\,\Delta\,(\Delta \, - \, 3) \tag{2.208}$$

which is in perfect agreement with the results of fig. 2.17.
If we continue to cool the theory down below the critical temperature T_c at which the bulk scalar field becomes unstable, we must switch to a different spacetime background. As the low temperature phase has a condensate for the operator $\mathcal{O}$, the bulk scalar field ϕ will be nonvanishing and will start to grow towards the $T = 0$ limit producing a strong backreaction effects on the geometry.
This leads to the following ansatz, describing a charged "hairy" black hole[18] :

$$ds^2 \, = \, \frac{L^2}{z^2}\left(-\,f(z)\,e^{-\,\chi(z)}\,dt^2 \, + \, \frac{dz^2}{f(z)} \, + \, dx_\mu\,dx^\mu\right) \tag{2.209}$$

together with:

$$A \, = \, A_t(z)\,dt\,, \qquad \phi = \phi(z)\,. \tag{2.210}$$

[17]Note that this contribution is not null just at the horizon and it vanishes at the boundary such that in the UV $M_eff = m$.

[18]This seems naively to violate the famous "no hair theorems". Nevertheless those are not valid anymore once we work in AdS spacetime so there is no contrast at all.

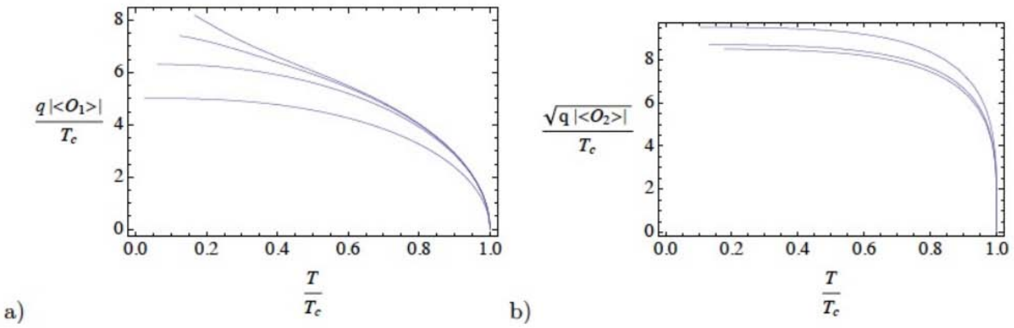

Figure 2.18: The condensate as a function of the temperature for the case $\Delta = 1$ (left) and $\Delta = 2$ (right). In curve (a), from bottom to top, $\gamma q = 1, 3, 6, 12$. In curve (b), from top to bottom, $\gamma q = 3, 6, 12$. Figure taken from [89].

The hairy black holes are then found by plugging this ansatz into the Einstein-Maxwell-scalar equations, following from the action (2.200), and solving numerically. We will not describe the details of the numerical procedure.
Given the solution $\phi(z)$ one can read off the expectation value $\mathcal{O}$ from the subleading coefficient of its UV boundary expansion. The result for several values of q and for a particular value of the mass squared is shown in figure 2.18, taken from [89]. In figure 2.18 we see how the condensate appears at $T = T_c$ and tend to a finite value towards $T = 0$. The numerics become unreliable at very low temperatures and unfortunately do not let us determine, for instance, the fate of the increase of the condensate at low temperature in the left hand plot at large q.
The transition appears to be of second order type and the condensate indeed close to the phase transition grows like:

$$\sim (T - T_c)^{1/2} \tag{2.211}$$

which is a know result from *mean field theory.*
As a remark, this model seems to be in contradiction with the famous Mermin-Wagner theorem which states that continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leq 2$. This result is evaded because of the large N limit we took from the principle; the fluctuations are suppressed in the large N limit and at finite N, the phase transition we have just described will become a crossover.

Conductivity

We can now use the same procedure explained for the RN case to compute the electric conductivity across the SC phase transition. The result for the real (dissipative) part of the conductivity at low temperatures for several values of q and for a particular value of m^2 is shown in fig.2.19. The main features of the conductivity are:

- There is a gap ω_G at low frequencies.
- The conductivity tends to the normal phase value at large frequencies.
- There is a δ function at the origin $\omega = 0$.

The absence of electric current dissipation for frequencies $\omega < \omega_G$ is indicative of a gap in the spectrum of charged excitations. The superconducting gap appears to be tied to the presence of

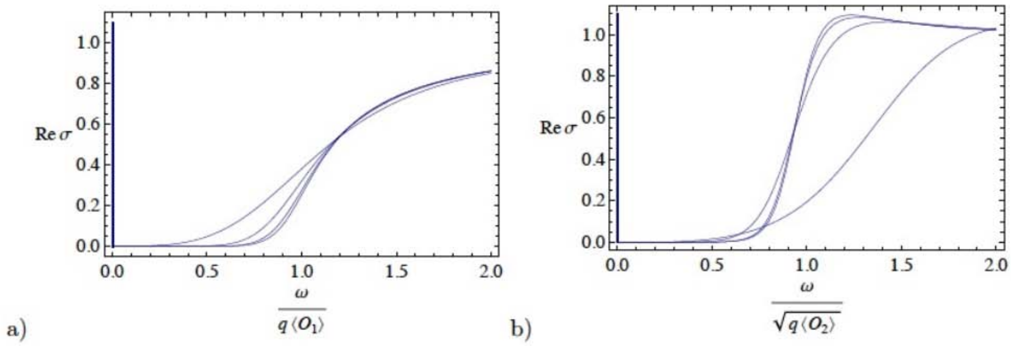

Figure 2.19: The real (dissipative) part of the electrical conductivity at low temperature in the presence of a $\Delta = 1$ (left) and $\Delta = 2$ (right) condensate. The temperature taken was $T = 0.03\gamma q\mathcal{O}$ and $T = 0.03\sqrt{\gamma q\mathcal{O}}$, respectively, and the charges were $\gamma q = 1, 3, 6, 12$. The curves with steeper slope correspond to larger γq. Figure taken from [89].

a condensate and it is typical of superconducting systems. For instance, an important prediction of weakly coupled BCS theory is that $\omega_g/T_c \approx 3.5$ which is indeed roughly observed in many conventional superconductors. We can see that a range of values is possible, although in the probe limit the value $\omega_G/T_c \approx 8$ appears to be fairly robust and it is amusing that this value is close to that reported in some measurements of the high-Tc cuprates. This higher value is one of the signal that we are after a strongly coupled SC far from the usual BCS description. In weakly coupled theory the gap would satisfy $\omega_G = 2E_G$ where E_G is the energy gap in the charged spectrum. It turns out that in the holographic model the story is different and this relation does not hold. Looking at the conductivity we can indeed extract that:

$$\sigma(\omega \to 0) \,=\, e^{-\,E_G/T} \tag{2.212}$$

where E_G does not correspond to $\omega_G/2$ as predicted by the weakly coupled logic.
The δ function in the broken phase is a typical feature of a SC state connected to the presence of a charged condensate. The same feature in the normal phase is linked to the translational invariance of the background and it constitutes a problem for a realistic description. It is indeed hard to disentangle the two infinities and read of the superfluid density from the coefficient of the δ function.
The resolution of this issue will be one of the main concerns of this thesis. Introducing momentum dissipation in the original holographic superconducting models will provide a more realistic description and will open the room for extracting more physical quantities to be compared with experiments. We will come back on this issue in the section devoted to the original results of this thesis.
We avoid any, although interesting, discussion about the role of the magnetic field in the framework of holographic superconductors which would lead to the identification of the Meissner effect and the classification of the SC phases into type I and type II. The interested reader can find details about this topic in one of the reviews proposed.

Chapter 3

Massive Gravity

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Contents

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For everybody in their busy lives, you need to invest in sharpening your tools, and you need to invest in longevity.

Ryan Holmes

General Relativity (GR) is one of the most successful, elegant and shining scientific result of the last century [99]. Its agreement with experiments is incredibly high; from the confirmation of the deflection of light led by Eddington in 1919 to the most recent gravitational waves detection announced by LIGO collaboration just this year [100]. Nevertheless, some of the large distance properties of our universe, such as the origin of its late time acceleration, still remain not explained and consistently incorporated in the framework of General Relativity. As a consequence, soon after its discovery, an increasing industry of scientific efforts arose with the target of studying its possible modifications and the corresponding effects on physical observables. One of such attempts, *i.e.* **massive gravity**, refers to the proposal of modifying the large distance GR dynamics giving the graviton a mass and relaxing the full diffeomorphism invariance. Such a modification, mainly motivated by cosmology, provides for new degrees of freedom but also possible pathologies one should take care of.

In this chapter we will go through the history, the consequences and the formal definition of a theory of a propagating massive spin 2 field. We will conclude underlying and describing the brandnew and unexpected role that the Gauge Gravity duality can give to massive gravity theories. We will be mainly following the three excellent reviews [101–103].

3.1 Modifying (linearized) GR: Fierz Pauli

We know there is gravity because apples fall from trees. We can observe gravity in daily life. But what if we could throw an apple to the edge of the universe at incredibly high speed? Or what if the apple were incredibly heavy? What would we observe?
God does not play dice with the universe and Einstein knew it. Just from the *Equivalence Principle* and the idea of *General Covariance* he has been able to describe the full non-linear dynamics of spacetime in a remarkably elegant theory of gravity known as **General Relativity**. Long range forces are mediated by massless particle of definite helicity and the requirement of having a generic coordinate transformations invariant theory for a helicity $h = 2$ massless particle leads directly to a unique answer, GR indeed. Although coordinates invariance and/or the equivalence principle are historically the motivations and pillars of GR they are not the real underlying principles of the theory. In modern language we now know that GR can be uniquely defined by the following statement: *GR is the only self-consistent theory of a non trivially interacting massless particle with helicity 2!* [104][1]
Everything follows from this statement and not vice versa.
Einsten connected in an elegant way the effects of gravity with the geometrical structure of spacetime, making it a dynamical object in rather elegant mathematical formalism. The framework allows to reconcile classical Newton's law with the requirements of Special Relativity and represents the current description of classical gravitation in modern physics. GR is of course not a UV complete (and renormalizable) theory; it represents an effective field theory with a UV cutoff set at the so called Planck scale M_p. Beyond that scale quantum effects become large and have to be considered; despite promising candidates, a consistent theory of quantum gravity is still lacking. To be more precise, within the theory of GR we can distinguish three different regimes (see 3.1):

- $r < \frac{1}{M_p}$: this is the Quantum regime where GR is not reliable anymore and a UV completion is needed.
- $\frac{1}{M_p} < r < r_s \sim \frac{M}{M_p^2}$: this is the classical non-linear regime where gravity can be treated as a classical theory but in its full non-linear fashion. This is the regime where we will be working.
- $r > r_s$: classical linear regime; at these distances GR can be linearized and the theory simplified.

Despite the universal consensus on GR accuracy and efficiency, in recent years increasing interest to modify it and test its modifications appeared. The biggest motivation comes from Supernova data [106, 107]: the Universe is accelerating and the origin of this acceleration is still unknown.

[1] Note this can be also proved by means of more modern methods such as the so-called *Soft theorem* [105]

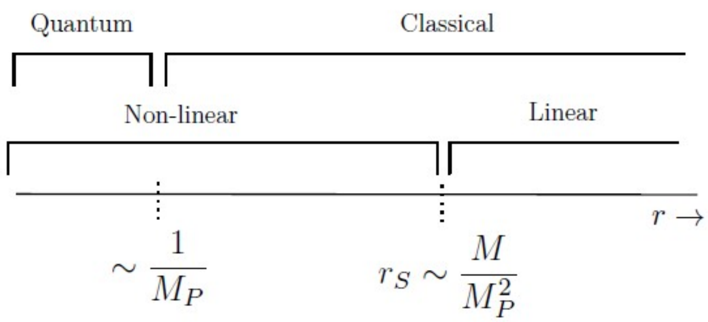


Figure 3.1: Regimes of GR. From [102].

If ones trust GR, a density of $\rho \sim 10^{-29}\, g/cm^3$ *dark matter/energy* has to be present in our Universe. The nature of such a density and its fundamental origin remains unexplained. Nevertheless, the easiest way to accomodate such a presence is through the introduction of a cosmological constant term Λ in the Einstein's equations which relates to that density as:

$$\rho \sim M_p^2\,\Lambda \tag{3.1}$$

It follows an incredibly small value of such a constant $\Lambda/M_p^2 \sim 10^{-65}$, which leads to the known *cosmological constant problem.* This is very analogous of the hierarchy problem present in the Standard Model in relation to the Higgs mass. Such a small value is not protected by any simmetry and it is therefore technically not natural unless one invokes some sort of Anthropic principles [108].

Since GR is the unique theory of an interacting massless spin 2 particle, in order to modify it we need to break some of the assumptions. One possibility is to make the force mediator to be massive in analogy to what happens to the gauge bosons in the Electroweak theory. Giving a mass to the graviton accounts for additional degrees of freedom besides the 2 usual polarizations of the massless graviton; now the gravitational excitations contain more (and eventually dangerous) d.o.f[2].

Once the graviton aquires a mass, the gravitational force takes the Yukawa form $\sim \frac{1}{r}e^{-mr}$ and at distances $r \geq \frac{1}{m}$ drops off compared to the GR expectation. Therefore one could explain the accelaration of our universe fixing the graviton mass to be order of the Hubble constant $r \sim H$. Now the small value of the cosmological constant translates into the ratio m/M_p being very small and here it comes the novelty. Since the $m = 0$ case provides for an enhancement of the symmetries of the system, namely diffeomorphism invariance, such as small value is consequently protected by such a symmetry and no longer unnatural. Building at linear level a theory of non-interacting massive graviton is a pretty simple task and was already achieved in 1939 by Fierz and Pauli [109]. Promoting such a construction to a full non linear and interacting theory is a way more challenging task which has been pursued for decades and just in recent years has encountered some positive answers.

Fierz Pauli action

At linearized level there are a priori two possible mass terms one can think of, $h_{\mu\nu}^2 = h_{\mu\nu}h^{\mu\nu}$

[2]Note this is not the less minimal modification one can do to GR; one can simply add a scalar degree of freedom like in the so called *Scalar-Tensor theories* (such as $f(R)$ gravity).

and $h = Tr(h_{\mu\nu})$, such that a generic mass deformation of the linearized Einstein-Hilbert action takes the form[3]:

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{\mathcal{MASS}}, \qquad \text{with} \qquad \mathcal{L}_{\mathcal{MASS}} \sim m^2 \left(h_{\mu\nu}^2 - A\,h^2\right) \tag{3.3}$$

The introduction of such a mass term clearly breaks the original diffeomorphism invariance of the theory:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\zeta_{\nu)} \tag{3.4}$$

and introduces new degrees of freedom.
Performing the specific analysis of the action for the decomposed graviton field[4] we discover that:

- There is a tensor helicity 2 mode with healthy kinetik term which corresponds to the usual massless graviton with 2 polarizations.
- There is an additional helicity 1 mode, accounting for two additional degrees of freedom, which anyway is not dynamical in the sense that enters in the action without time derivatives[5].
- Two scalar helicity 0 propagating degrees of freedom appear in the spectrum of theory as well.

This analysis accounts for 6 degrees of freedom. In particular the 2 scalars d.o.f are one more compared to the only expected helicity 0 state of a massive spin 2 field, which is expected to have (in $d = 4$) 5 degrees of freedom. The extra scalar mode is the dangerous character and it represents a ghosty excitation unless the A parameter is fixed to be $A = 1$ (Fierz Pauli Tuning)[6]. This is confirmed by the fact that with the choice $A = 1 - a$ the extra scalar would have a mass $m_g^2 = \frac{3-4\,a}{2\,a} m^2$ which indeed goes to infinity whenever $a \to 0$ decoupling the relative mode (for details see [110]). The final result is the so called **Fierz-Pauli** action, which takes the form:

$$\mathcal{L}_{\mathcal{FP}} \sim m^2 \left(h_{\mu\nu}^2 - h^2\right) \tag{3.5}$$

and represents the only healthy massive modification of the Einstein-Hilbert action.
We underline that what discussed here so far is valid on Minkowski background, for Lorentz invariant systems and at linear level. This is anyway a good exercise which already suggests that the most severe problems of MG are connected with the scalar sector. Indeed the helicity 0 mode is the responsable of most of the consistency issues and the phenomenology features of MG theories as we will see.

The Stückelberg trick and the vDVZ discontinuity

In order to analyze the features and the possible issues of MG theories is very convenient to use

[3]The linearized Einstein-Hilbert action reads:

$$\mathcal{L}_{EH} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \tag{3.2}$$

[4]It is convenient to do that with the Hamiltonian formalism and/or using the Stückelberg trick.

[5]To be precise it enters quadratically and not as a Langrange multiplier, so it represents a real additional, even if not dynamical, degree of freedom.

[6]This is true only at the linearized level. We will see soon that once we promote the theory to be fully non-linear the extra d.o.f. will re-appear.

the so called *Stückelberg trick*. Because it will be of fundamental importance for all the rest of the thesis we make a parenthesis and we introduce the idea considering a simpler case, the one of a massive spin 1 gauge field A_μ.
Let's start from an action for a massive U(1) vector field of the form:

$$\mathcal{S} \,=\, \int d^dx \left(-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,m^2\,A_\mu A^\mu + A_\mu J^\mu\right) \tag{3.6}$$

whose mass term explicitely breaks the gauge symmetry:

$$\delta A_\mu \,=\, \partial_\mu\,\Lambda \tag{3.7}$$

and propagates (in $d = 4$) 3 degrees of freedom.
It is straightforward to notice that the limit $m \to$ of such an action is not smooth, in the sense that it does not conserve the number of degrees of freedom (a massless gauge field in $d = 4$ has 2).
The Stückelberg trick consists in introducing a new scalar degree of freedom ϕ such that the new action restores the original gauge symmetry, it is dinamically equivalent to the original one and no d.o.f gets lost while driving the mass to zero. This can be achieved by the following:

$$A_\mu \,\to\, A_\mu \,+\, \partial_\mu\,\phi \tag{3.8}$$

Note that: i)this is not a change of field variables; ii) this is not a decomposition of the vector field A_μ in its longitudinal and transverse parts; iii) this is not a gauge transformation.
Rescaling $\phi \to \frac{1}{m}\phi$ the new action takes the form:

$$\mathcal{S} \,=\, \int d^dx \left(-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,m^2\,A_\mu A^\mu + A_\mu J^\mu \,-\, m\,A_\mu\,\partial^\mu\phi - \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{m}\,\phi\,\partial_\mu J^\mu\right) \tag{3.9}$$

and it enjoys the gauge symmetry:

$$\delta A_\mu \,=\, \partial_\mu\Lambda\,, \qquad \delta\phi \,=\, -\Lambda\,. \tag{3.10}$$

If we now go back to $\phi = 0$, using the so called *unitary gauge*, we exactly recover the previous theory for a massive U(1) field.
The Stückelberg trick is a terrific illustration of the fact that gauge symmetry is a complete sham! This just constitutes a redundancy in the description and the catch is that removing such a redundancy is not always a smart thing to do.
Now the limit $m \to 0$ is smooth and no degrees of freedom are lost. Note that if the current J is not conserved, *i.e.* $\partial_\mu J^\mu \neq 0$, the scalar in that limit becomes strongly coupled to the divergence of the source and such a limit does not exist. Otherwise, if the current is conserved, what we get in the $m \to 0$ is the original U(1) Maxwell theory plus a massless decoupled scalar field.
We can now repeat the same exercise for the massive graviton starting from the FP action:

$$\mathcal{L}_{m=0} \,-\, \frac{1}{2}\,m^2\,\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \,+\, \kappa\,h_{\mu\nu}T^{\mu\nu} \tag{3.11}$$

and introducing a new Stückelberg vector field as:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu \tag{3.12}$$

We can go on with the same procedure but we would immediately realize that this is not enough. Indeed in the $m \to 0$ limit we would be left with a massless graviton and a massless vector accounting for 4 d.o.f. , one less compared to the 5 of a massive spin 2 field. We have indeed to introduce an additional scalar field through:

$$A_\mu \to A_\mu + \partial_\mu \phi \tag{3.13}$$

such that the full theory now enjoy the gauge symmetry:

$$\begin{aligned} \delta h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu, \qquad \delta A_\mu \ = \ -\zeta_\mu \,, \\ \delta A_\mu \ = \ \partial_\mu \Lambda \,, \qquad \delta\phi \ = \ -\Lambda \,. \end{aligned} \tag{3.14}$$

If we renormalize the Stückelberg fields as $A_\mu \to \frac{1}{m} A_\mu$ and $\phi \to \frac{1}{m^2}\phi$ we end up with the following action:

$$\begin{aligned} \mathcal{S} \ - \ \mathcal{S}_{m=0} \ = \int d^d x \Big[-\frac{1}{2}\, m^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2\right) - \frac{1}{2}\, F_{\mu\nu} F^{\mu\nu} - 2\, m \left(h_{\mu\nu}\, \partial^\mu A^\nu - h\, \partial_\mu A^\mu\right) \\ - 2\big(h_{\mu\nu}\, \partial^\mu \partial^\nu \phi - h\, \partial^2 \phi\big) + \kappa\, h_{\mu\nu}\, T^{\mu\nu} - \frac{2}{m}\, \kappa\, A_\mu\, \partial_\nu\, T^{\mu\nu} + \frac{2}{m^2}\, \kappa\, \phi\, \partial_\mu \partial_\nu\, T^{\mu\nu} \Big] \end{aligned} \tag{3.15}$$

If now we take the $m \to 0$ limit (assuming the source to be conserved) we get:

$$\mathcal{L} \ - \ \mathcal{L}_{m=0} \ = \ -\frac{1}{2}\, F_{\mu\nu} F^{\mu\nu} - 2 \left(h_{\mu\nu}\, \partial^\mu \partial^\nu \phi - h\, \partial^2 \phi \right) + \kappa\, h_{\mu\nu}\, T^{\mu\nu} \tag{3.16}$$

This represents a theory with 5 propagating d.o.f. indeed where the vector one is completely decoupled from the others but there is a direct kinetik mixing between the tensor and the scalar modes.
We can unmix the modes using the following fields redefinition (which actually is the linearized form of a *conformal transformation*):

$$h_{\mu\nu} \ = \ h'_{\mu\nu} \ + \ \eta_{\mu\nu}\, \frac{2}{d-2}\, \phi \tag{3.17}$$

In this way the previous action transforms into:

$$\mathcal{L} \ - \ \mathcal{L}_{m=0} \ = \ -\frac{1}{2}\, F_{\mu\nu} F^{\mu\nu} \ - \ 2\, \frac{d-1}{d-2}\, \partial_\mu \phi \partial^\mu \phi \ + \ \kappa\, h'_{\mu\nu}\, T^{\mu\nu} \ + \ \frac{2}{d-2}\, \kappa\, \phi\, T \tag{3.18}$$

Now we have 5 proper propagatin d.o.f. but the scalar modes is still coupled (not kinetikally now) to the tensor one through the trace of the stress tensor $T = Tr[T_{\mu\nu}]$ and this coupling survives in the $m \to 0$ limit !
This is the origin of the so called **zDVZ discontinuity** [111, 112] which refers to the fact that generically:

$$MG(m = 0) \neq \text{ massless gravity} \tag{3.19}$$

and that the observable predictions of massive gravity in the $m \to 0$ limit are different to the ones expected by General Relativity. Because of this coupling the scalar field ϕ does not affect the light bending but it does affect for example the Newtonian potential (the emission of gravitational radiation is also affected for example [113]).
As a side note, with the Stückelberg trick one can also proove that violating the FP tuning of the mass term leads to a ghost. This is indeed reflected in the fact that the scalar Stückelberg would aquire a four derivatives term $\sim (\Box\phi)^2$, indicating the presence of 2 propagating d.o.f., one of which ghostlike [110, 114]. To this extent, Fierz Pauli is the exact combination that cancels such a term, up to total derivatives.

Massive gravitons on curved space

We focused our analysis on a Minkowski background, let us now describe the same phenomenon on a curved spacetime of the Einstein type $R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}$. The linearized action, together with the Fierz Pauli mass term, reads:

$$\begin{aligned} \mathcal{S} = \int d^d x \Big[&-\frac{1}{2} \nabla_\alpha h_{\mu\nu} \nabla^\alpha h^{\mu\nu} + \nabla_\alpha h_{\mu\nu} \nabla^\nu h^{\mu\alpha} - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla_\mu h \nabla^\mu h \\ &+ \frac{R}{d} \left(h_{\mu\nu} - \frac{1}{2} h \right) - \frac{1}{2} \left(h_{\mu\nu} - h \right) + \kappa h_{\mu\nu} T^{\mu\nu} \Big] \end{aligned} \qquad (3.20)$$

where ∇_α is now the proper covariant derivative.
Note that the curvature R seems to give a mass to the graviton but the term does not appear in the FP tuning form[7]. For most of the choices of m^2 this action presents 5 propagating d.o.f, namely a propagating massive graviton. Anyway for some particular choices this is not the case:

i. $m = 0$: the diffeomorphism invariance $\delta h_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$ is restored and the propagating degrees of freedom are just 2, the ones of a massless graviton;

ii. for the particular choice $R = \frac{d(d-1)}{d-2} m^2$ a scalar gauge symmetry $\delta h_{\mu\nu} = \nabla_\mu \nabla_\nu \lambda + \frac{1}{d-2} m^2 g_{\mu\nu}$ (where $g_{\mu\nu}$ comes from the usual conformal transformation $h_{\mu\nu} = h'_{\mu\nu} + \frac{2}{d-1} m^2 \phi g_{\mu\nu}$) appears and the d.o.f. reduce to 4: these theories take the name of *partially massless theories*.

To understand well what is going on we perform the Stückelberg trick along with the usual conformal transformation and we get the following action on curved spacetime:

$$\begin{aligned} \mathcal{S} = \int d^d x \, \mathcal{L}_{m=0}(h') + \sqrt{-g} \Big[&-\frac{1}{2} m^2 \left(h'_{\mu\nu} h'^{\mu\nu} - h'^2 \right) - \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} + \frac{2}{d} m^2 \, R \, A_\mu A^\mu \\ &- 2 \, m^2 \left(h'_{\mu\nu} \nabla^\mu A^\nu - h' \nabla_\mu A^\mu \right) + 2 m^2 \left(\frac{d-1}{d-2} m^2 - \frac{R}{d} \right) \left(2 \phi \nabla_\mu A^\mu + h' \phi \right) \\ &- 2 m^2 \left(\frac{d-1}{d-2} m^2 - \frac{R}{d} \right) \left((\partial \phi)^2 - m^2 \frac{2d}{d-2} \phi^2 \right) + \kappa h'_{\mu\nu} T^{\mu\nu} + \frac{2}{d-2} m^2 \kappa \phi T \Big] \, . \end{aligned} \qquad (3.21)$$

It is immediate to realize that for the specific combination $R = \frac{d(d-1)}{d-2} m^2$ (along with the requirement of $T = Tr[T_{\mu\nu}] = 0$) the scalar field ϕ completely dissapear from the game leaving just 4 d.o.f. in the theory.
One very important consequence of being in curved spacetime is the absence of the vDVZ discontinuity [116–118], namely the fact that in curved spacetime the $m \to 0$ limit is smooth and no degree of freedom is lost. In order to see that we need to rescale the vector field as $A_\mu \to \frac{1}{m} A_\mu$ and then take the limit $m \to 0$ keeping R fixed and finite. In this case is clear that there is no need of introducing an additional scalar Stückelberg ϕ because a mass term for the vector is already present:

$$\mathcal{S} = \int d^d_x \, \mathcal{L}_{m=0} + \sqrt{-g} \, (-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{2 \, R}{d} \, A_\mu A^\mu + \kappa \, h_{\mu\nu} T^{\mu\nu}) \qquad (3.22)$$

such that no degrees of freedom are lost (massive gravity=GR+massive vector).
The massive vector completely decouples from the metric such that no vDVZ discontinuity appears. As a side, note that the mass term $m_v^2 \sim \frac{2 \, R}{d}$ would be tachyonic in dS space while healthy in AdS space.

[7]It is actually a tricky business to define what ”massless’ means in a curved spacetime, see for example [115].

3.2 Non-linear massive gravity

We concentrated so far our attention on the linearized theory of gravity without considering any interactions or non-linearities. We introduce them in this section and we look how to extend the previous considerations at full non-linear level.

General Relativity

General Relativity at full non linear level is described by the famous **Einstein-Hilbert (EH) action**:

$$\mathcal{S} \;=\; \frac{1}{2\,\kappa^2}\int d^dx\,\sqrt{-g}\,R \tag{3.23}$$

which enjoys full diffeomorphism gauge invariance[8]

$$g_{\mu\nu}(x)\;\rightarrow\;\frac{\partial f^a}{\partial x^\mu}\,\frac{\partial f^b}{\partial x^\nu}\,g_{\alpha\beta}(f(x)) \tag{3.25}$$

The following field equation reads:

$$R_{\mu\nu}-\frac{1}{2}\,R\,g_{\mu\nu}\;=\;0 \tag{3.26}$$

where R is defined as the curvature, namely the trace of the Ricci tensor $R_{\mu\nu}$.
The easiest solution for such a system is provided by flat space $g_{\mu\nu}=\eta_{\mu\nu}$ and one can expand the previous action around the flat space solution $g_{\mu\nu}=\eta_{\mu\nu}+\,2\,\kappa\,h_{\mu\nu}$ getting something like:

$$\mathcal{S}\;\sim\;\underbrace{\partial^2\,h^2}_{\mathcal{S}_2}+\underbrace{\kappa\,\partial^2\,h^3\;+\cdots+\;k^n\,\partial^2\,h^{2+n}}_{interactions}+\dots \tag{3.27}$$

where $\mathcal{S}_2$ constitutes the action for a massless spin 2 field in Minkwoski space and the higher non linear terms take into account the interactions. This is an expansion in powers of (κh) which states that the linearized approximation $\mathcal{S}_2$ is only true whenever $\kappa h\ll 1$.
If one now starts with the linearized theory of gravity adding interactions through higher order terms, the requirement of preserving gauge invariance strongly constraints the theory such that, after being fully resummed, it takes the form of the Einstein-Hilbert action [104]. This is somehow the magic and the unicity theorem for General Relativity.
We started from an action defined on a non dynamical and flat background $\eta_{\mu\nu}$ but if we now perform the redefinition $h_{\mu\nu}\rightarrow g_{\mu\nu}-\eta_{\mu\nu}$ we discover that the background metric completely disappears from the action. The fully interacting EH action turns out to be background indipendent! (this will not be true in the case of massive gravity)
Eventually one can add an additional term to the Einsten-Hilbert action:

$$\mathcal{S}\;=\;\frac{1}{2\,\kappa^2}\int d^dx\,\sqrt{-g}\,\left(R-2\,\Lambda\right) \tag{3.28}$$

where Λ is the so called *Cosmological constant*[9].
One can conclude that the only possible interactions for a massless graviton propagating on an

[8] Infinitesimally we can write down $f^\mu(x)=x^\mu+\zeta^\mu$ such that the linearized gauge transformation takes the form:

$$\delta g_{\mu\nu}=\mathcal{L}_\zeta g_{\mu\nu}=\nabla_\mu\zeta_\nu+\nabla_\nu\zeta_\mu \tag{3.24}$$

where ζ^μ is the gauge parameter and $\mathcal{L}_\zeta$ its Lie derivative.

[9] In this case the fields equations are modified into $R_{\mu\nu}=\frac{R}{d}g_{\mu\nu}$ with $\Lambda=\frac{d-2}{d}\,R$.

Einstein space (the only space on which a free graviton can consistently propagate [119]) should be of the form "Einstein-Hilbert+cosmological constant".

Non Linear Massive Gravity

With non linear massive gravity we mean a non linear theory whose expansion around a fixed background results to be of the massive Fierz Pauli form. Unlike in GR where diffemorphisms invariance forces the structure of the theory here the choice is not unique; there is no symmetry protecting so any interaction is a priori allowed.
We start by considering non linear GR plus a non linear FP term of the form (we again define $h_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$):

$$\mathcal{S} \,=\, \frac{1}{2\,\kappa^2} \int d^d x \left[\sqrt{-g}\, R \,-\, \sqrt{-f}\, \frac{m^2}{4}\, f^{\mu\alpha} f^{\nu\beta} \left(h_{\mu\nu} h_{\alpha\beta} - h_{\mu\alpha} h_{\nu\beta}\right)\right] \tag{3.29}$$

where $f_{\mu\nu}$ is a fixed background metric on which the linear massive graviton propagates and whose presence breaks diffeomorphism invariance. Note as it would be impossible to build up such a term just with the metric $g_{\mu\nu}$ since its trace would be constant; two distinct structures are therefore needed. After writing down the corresponding equations of motion one realizes that if $f_{\mu\nu}$ satisfies Einstein equations then $g_{\mu\nu} = f_{\mu\nu}$ is always a solution of the full system.
More generically, reorganizing the action in term of the full metric g, we can write down an action:

$$\mathcal{S} \,=\, \frac{1}{2\,\kappa^2} \int d^d x \left[\sqrt{-g}\, R \,-\, \sqrt{-g}\, \frac{m^2}{4}\, V\left(g, h\right)\right] \tag{3.30}$$

with a potential $V = V_2 + V_3 + \cdots + V_n$:

$$\begin{aligned} V_2 \,&=\, FP \,=\, \left\langle h^2 \right\rangle \,-\, \langle h \rangle^2 \\ V_3 \,&=\, c_1 \left\langle h^3 \right\rangle \,+\, c_2 \left\langle h^2 \right\rangle \langle h \rangle \,+\, c_3 \left\langle h^3 \right\rangle \\ V_n \,&=\, \ldots \end{aligned} \tag{3.31}$$

where $\langle\rangle$ stands for the trace with respect to the full metric g.
The first term V_2 reduces to the Fierz Pauli structure while all the terms V_n with $n > d$ turn out to be redundant and can be fixed to 0. Note that we avoid the introduction of derivative terms for the same reasons we do in GR, namely because they are not relevant in the low energy effective description we are interested in.
If we now consider a spherical solution of such a theory we realize we can perform a non-linearities expansion where the expansion parameter reads r_V/r (such that for $r \gg r_V$ the linear approximation is reliable) with:

$$r_V \,=\, \left(\frac{G\,M}{m^4}\right)^{1/5} \tag{3.32}$$

known as the **Vainshtein radius** [120]. Note that in the zero mass term $m = 0$ the Vainshtein radius diverges and the solution is fine up to arbitrarily large distances.
Vainshtein argued in [120] that the perturbative expansion breaks down at a certain radius and that we can say nothing about the non linear behaviour of massive gravity in the massless limit; therefore maybe the vDVZ discontinuity is just an artifact of the linear perturbation theory and not a property of the full non linear structure!

Let's go back now to the full non linear Fierz Pauli action considered in flat space background $f_{\mu\nu} = \eta_{\mu\nu}$:

$$\mathcal{S} \,=\, \frac{1}{2\,\kappa^2}\int d^dx\left[\sqrt{-g}\,R\,-\,\frac{m^2}{4}\,\nu^{\mu\alpha}\nu^{\nu\beta}\,(h_{\mu\nu}h_{\alpha\beta}-h_{\mu\alpha}h_{\nu\beta})\right] \tag{3.33}$$

We found in the previous sections that the free theory carries the correct 5 d.o.f. in $d = 4$, because the time component h_{00} appeared in the action as a Lagrange multiplier. This is no longer true once non linearities are taken into account!

In order to prove that it is convenient to use the so called ADM (Arnowitt,Deser and Misner) formalism [121,122] choosing a spacelike slicing of spacetime by hypersurfaces Σ_t and deconstructing the metric into:

$$\begin{aligned} g_{00} \,&=\, -N^2\,+\,g^{ij}\,N_i\,N_j\,, \\ g_{0i} \,&=\, N_i\,, \qquad g_{ij} \,=\, g_{ij}\,. \end{aligned} \tag{3.34}$$

Under this foliation the Einstein-Hilbert action gets the form:

$$\frac{1}{2\,\kappa^2}\int d^dx\,\sqrt{-g}\,N\,\Big[\mathcal{R}\,-\,K^2\,+\,K_{ij}K^{ij}\Big] \tag{3.35}$$

where $\mathcal{R}$ is the curvature of the subspace defined by the metric g_{ij} and $K_{ij} = \frac{1}{2N}(\dot{g_{ij}} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature of the spatial hypersurfaces.

We can then define the conjugate momenta p_{ij}:

$$p_{ij}\,=\,\frac{\delta\mathcal{L}}{\delta g_{ij}}\,=\,\frac{1}{2\kappa^2}\,\sqrt{-g}\,\left(K^{ij}\,-\,K\,g^{ij}\right) \tag{3.36}$$

and through Legendre transform the corresponding Hamiltonian of the system:

$$\mathcal{H}\,=\,\int_{\Sigma_t} d^{d-1}x\,N\,C1\,+\,N_i\,C^i \tag{3.37}$$

with:

$$\begin{aligned} C\,&=\,\sqrt{-g}\,\Big[\mathcal{R}\,+\,K^2\,-\,K^{ij}K_{ij}\Big]\,, \\ C^i\,&=\,2\,\sqrt{-g}\,\nabla_j\,\left(K^{ij}\,-\,K\,h^{ij}\right)\,, \\ K_{ij}\,&=\,\frac{2\,\kappa^2}{\sqrt{-g}}\,\left(p_{ij}\,-\,\frac{1}{d-2}\,h_{ij}\right)\,. \end{aligned} \tag{3.38}$$

At zero graviton mass $m = 0$ the action is a pure constraint and the Hamiltonian vanishes, as a sign of diffeomorphism invariance. The shift and the lapse N and N_i are indeed Lagrange multipliers and force $C = C_i = 0$. All in all the Hamiltonian analysis confirms that also at non linear level GR propagates 2 real degrees of freedom, the ones of a massless graviton.

Let's study now the FP mass term in this language[10]. The full action now gets modified and

[10] The FP mass term $\nu^{\mu\alpha}\nu^{\nu\beta}\,(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})$, using $h_{ij} = g_{ij} - \delta_{ij}$ reads:

$$\delta^{ik}\delta^{jl}\,(h_{ij}h_{kl} - h_{ik}h_{jl}) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_j \tag{3.39}$$

becomes:

$$\mathcal{S} = \frac{1}{2\,\kappa^2}\int d^dx\,\Big(p^{ab}\,\dot{g_{ab}} - N\,C - N_i\,C^i - \frac{m^4}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_j\right]\Big) \tag{3.40}$$

In the $m \neq 0$ case N and N_i enter quadratically into the action (but with no time derivatives); this means they are no longer Lagrange multipliers but auxiliary fields. We can solve for those fields:

$$N = \frac{C}{m^2\,\delta^{ij}h_{ij}}\,, \qquad N_i = \frac{1}{m^2}\left(g^{ij} - \delta^{ij}\right)^{-1} C^j \tag{3.41}$$

and plug them back into the action. The resulting action does not contain any constraints nor gauge symmetry and as a consequence all the degrees of freedom result active. The corresponding Hamiltonian is not vanishing:

$$\mathcal{H} = \frac{1}{2\,\kappa^2}\int d^dx\,\frac{1}{2\,m^2}\,\frac{C^3}{\delta^{ij}h_{ij}} + \frac{1}{2\,m^2}\,C^i\left(g^{ij} - \delta^{ij}\right)^{-1} C^j + \frac{m^4}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij}\right] \tag{3.42}$$

In $d = 4$ there are therefore 6 propagating real degrees of freedom despite the 5 of the linearized theory!
Additionally the previous Hamiltonian is not bounded [123] and as a consequence an instability linked to the extra d.o.f. appears [124]. The instability would be represented by a ghosty excitation with a precise mass and takes the name of **Boulware-Deser Ghost** (BD) [123]. Its mass will be infinite, and therefore its presence not dangerous, in flat space but on the contrary it would result finite, and as a consequence problematic, on a non trivial background.
In [125] the unavoidable presence of the BD ghost was claimed, but that was too quick! Non trivial and additional interactions could possibly eliminate the presence of the BD ghost [126].

Non linear Stückelberg formalism

We consider now the non linear theory with full diffeomorphism invariance and we extend the Stückelberg trick of the previous section to the non linear scenario.
The full finite gauge transformation for gravity reads:

$$g_{\mu\nu}(x) \to \frac{\partial f^a}{\partial x^\mu}\,\frac{\partial f^b}{\partial x^\nu}\,g_{ab}\left(f(x)\right) \tag{3.43}$$

where $f(x)$ is an arbitrary gauge function. The mass term of MG breaks this invariance. In order to restore it we need to use the Stückelberg trick in full glory and we can do it in several ways. One way refers to the trick of replacing the metric $g_{\mu\nu}$ with an invariant object $G_{\mu\nu}$:

$$g_{\mu\nu}(x) \to G_{\mu\nu} = \frac{\partial Y^a}{\partial x^\mu}\,\frac{\partial Y^b}{\partial x^\nu}\,g_{ab}\left(Y(x)\right) \tag{3.44}$$

constructed through the Stückelberg fields $Y^\mu(x)$ which transform as the inverse of f under a gauge transformation $Y^\mu(x) \to f^{-1}\left(Y(x)\right)^\mu$[11]. The drawback of this method is that the Stückelberg

[11] For more details about this possibility the reader can look for example at [102].

expansion involves an infinite number of higher order terms in $h_{\mu\nu}$. Another possibility is to introduce the Stückelberg fields through the background metric $f_{\mu\nu}$ and then allow the full metric $g_{\mu\nu}$ to transform covariantly. This method is preferred for massive gravity theories written in terms of a potential $V(g,h)$ as introduced before and does not bring in an infinite expansion in higher powers of $h_{\mu\nu}$.
We then perform the replacement:

$$f_{\mu\nu}(x) \,\rightarrow\, \tilde{f}_{\mu\nu}(x) \,=\, f_{ab}\,(Y(x))\,\partial_\mu Y^a\,\partial_\nu Y^b \tag{3.45}$$

The $Y^a(x)$ fields introduced, despite the misleading α index, are to transform as scalars under diffeomorphisms:

$$Y^a(x) \,\rightarrow\, Y^a\,(f(x)) \tag{3.46}$$

or infinitesimally:

$$\delta Y^a \,=\, \zeta^\nu\,\partial_\nu\,Y^a \tag{3.47}$$

Given that the fields Y transform as scalars the background metric $f_{\mu\nu}$ transformas now like a proper metric tensor. As an immediate consequence the full metric $g_{\mu\nu} = f_{\mu\nu} + h_{\mu\nu}$ transforms covariantly under diffeomorphisms and any action which is a scalar function of $f_{\mu\nu}$ and $g_{\mu\nu}$ can again enjoy gauge invariance in this way.
Starting from a generic massive gravity action like:

$$\mathcal{S} \,=\, \frac{1}{2\,\kappa^2}\int d^dx\left[\sqrt{-g}\,R \,-\, \sqrt{-g}\,\frac{m^2}{4}\,V\,(g,h)\right] \tag{3.48}$$

we first lower all the indices on the $h_{\mu\nu}$ in the potential and we then replace $h_{\mu\nu}$ with:

$$h_{\mu\nu}(x) \,\rightarrow\, H_{\mu\nu}(x) \,=\, g_{\mu\nu}(x) \,-\, f_{ab}\,(Y(x))\,\partial_\mu Y^a\,\partial_\nu Y^b \tag{3.49}$$

Expanding $Y^a = x^a - Z^a$ we have:

$$H_{\mu\nu} \,=\, h_{\mu\nu} \,+\, f_{\nu a}\,\partial_\mu\,Z^a \,+\, f_{\mu a}\,\partial_\nu\,Z^a \,-\, f_{ab}\partial_\mu Z^a\,\partial_\nu Z^b \,+\,\ldots \tag{3.50}$$

where the $\ldots$ stand for terms containing derivatives of $f_{\mu\nu}$ and therefore vanishing upon the choice $f_{\mu\nu} = \eta_{\mu\nu}$.
Under infinitesimal transformations we have[12]

$$\begin{aligned}\delta Z^a \,&=\, -\zeta^a \,+\, \zeta^\nu\partial_\nu Z^a,\\ \delta h_{\mu\nu} \,&=\, \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu \,+\, \mathcal{L}_\zeta h_{\mu\nu}\,.\end{aligned} \tag{3.52}$$

where the covariant derivatives are with respect to the background metric $f_{\mu\nu}$ and the indices are lowered using the same metric.
It is somehow convenient to redefine $Z_\mu \rightarrow Z_\mu + \partial_\mu\phi$ to extract the helicity 0 mode. In this way around the flat space background $f_{\mu\nu} = \eta_{\mu\nu}$ the Stückelberg trick looks like[13] :

$$H_{\mu\nu} \,=\, h_{\mu\nu} + \partial_\nu Z_\mu + \partial_\mu Z_\nu + 2\,\partial_\mu\partial_\nu\phi - \partial_\mu\,Z^a\,\partial_\nu Z_a - \partial_\mu Z^a\,\partial_\nu\partial_a\,\phi - \partial_\mu\,\partial^a\phi\,\partial_\mu Z_a - \partial_\mu\partial^a\phi\,\partial_\nu\partial_a\phi \tag{3.55}$$

[12]Note that at linear level:

$$\begin{aligned}\delta Z^a \,&=\, -\zeta^a\,,\\ \delta h_{\mu\nu} \,&=\, \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu\,.\end{aligned} \tag{3.51}$$

which reproduces what we already know in the linear case.

[13]Note that ϕ always enter with two derivatives meaning there is an additional global galilean symmetry:

$$\phi(x) \rightarrow c \,+\, b_\mu\,x^\mu \tag{3.53}$$

along with the infinitesimal gauge transformation:

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + \mathcal{L}_\zeta h_{\mu\nu} \,, \\ \delta Z_\mu &= \partial_\mu \Lambda - \zeta_\mu + \zeta^\nu \partial_\nu Z_\mu \,, \\ \delta\phi &= -\Lambda \,. \end{aligned} \tag{3.56}$$

Despite the several ways of introducing the Stückelberg fields at non linear level and the various forms under which a non linear theory of massive gravity can show up at the end of the day:

$$\mathcal{S} = \frac{1}{2\,\kappa^2} \int d^d x \left[\sqrt{-g}\, R - \sqrt{-g}\, \frac{m^2}{4}\, V\,(g,h)\right] \tag{3.57}$$

is the most general Lorentz invariant massive gravity action and any Lorentz invariant massive gravity theory has a unitary gauge in which the action looks like exactly like that with an opportune potential V.

Non linear MG analysis

Let us now apply the non linear Stückelberg trick we learnt to the massive gravity case. For simplicity we will just consider $d = 4$ and the background metric to be flat $f_{\mu\nu} = \eta_{\mu\nu}$. In this case the mass term appears as:

$$\mathcal{S}_M = -\frac{M_P^2}{2}\frac{m^2}{4} \int d^4x\, \eta^{\mu\nu}\, \eta^{ab}\, (h_{\mu a}\, h_{\nu b} - h_{\mu\nu}\, h_{ab}) \tag{3.58}$$

We then introduce the Stückelberg trick defined previously in (3.55) and we canonically normalized the fields as:

$$\hat{h} = \frac{1}{2}\, M_P\, h\,, \qquad \hat{Z} = \frac{1}{2}\, m\, M_P\, Z\,, \qquad \hat{\phi} = \frac{1}{2}\, m^2\, M_P\, \phi \tag{3.59}$$

Upon these substitutions and redefinitions we got a bunch of higher order terms in these fields suppressed by various scales. We of course assume that $m \ll M_p$ such that quantum effects result negligible. The scalar field ϕ appears always with two derivatives while the Z field and the h one respectively with one and zero derivatives. All in all a generic term with n_h powers of h, n_Z powers of Z and n_ϕ of ϕ takes the following structure:

$$\sim m^2\, M_P^2\, h^{n_h}\, (\partial Z)^{n_Z} \left(\partial^2 \phi\right)^{n_\phi} \sim \Lambda_\lambda^{4-n_h-2n_Z-3n_\phi}\, h^{n_h}\, (\partial Z)^{n_Z} \left(\partial^2 \phi\right)^{n_\phi} \tag{3.60}$$

where the suppression scale reads:

$$\Lambda_\lambda = \left(M_P\, m^{\lambda-1}\right)^{1/\lambda}\,, \qquad \lambda = \frac{3\, n_\phi + 2\, n_Z + n_h - 4}{n_\phi + n_Z + n_h - 2}\,. \tag{3.61}$$

The larger the λ parameter, the smaller this scale (since $m/M_P < 1$). We are considering only interaction terms which means $n_h + n_Z + n_\phi \geq 3$. Therefore the term suppressed by the smallest scale is the cubic scalar one $n_\phi = 3, n_Z = n_h = 0$:

$$\sim \frac{\left(\partial\hat{\phi}\right)^3}{\Lambda_5^5} \tag{3.62}$$

In addition there is also a global shift symmetry:

$$Z_\mu(x) \to Z_\mu + c_\mu \tag{3.54}$$

which is suppressed by the scale $\Lambda_5 = (M_p\, m^4)^{1/5}$.
This lower scale represents the cutoff of the effective field theory. To focus on it we perform the following *decoupling limit*:

$$\begin{aligned} m &\to 0\,, \qquad T \to \infty\,, \\ M_P &\to \infty\,, \qquad \Lambda\,,\, T/M_p \text{ fixed}\,. \end{aligned} \tag{3.63}$$

Upon performing this limit all the interaction terms go to zero except the scalar cubic term which is the responsible of the stroung coupling! For what follows we do not need to consider any vector or tensor mode and we can just rely on the identification:

$$H_{\mu\nu} \,=\, 2\,\partial_\mu\,\partial_\nu\,\phi \,+\, \partial_\mu\,\partial^a\,\phi\,\partial_\nu\,\partial_a\,\phi \tag{3.64}$$

along with the usual conformal transformation $h_{\mu\nu} = h'_{\mu\nu} + m^2\,\phi\,\eta_{\mu\nu}$. After that, the scalar action, up to total derivatives, reads:

$$\mathcal{S}_\phi \,=\, \int d^4x \left(-3\,\left(\partial\hat{\phi}\right)^2 + \frac{2}{\Lambda_5^5}\left[\left(\Box\hat{\phi}\right)^3 - \left(\Box\hat{\phi}\right)\left(\partial_\mu\partial_\nu\hat{\phi}\right)^2\right] + \frac{1}{M_P}\,\hat{\phi}\,T\right) \tag{3.65}$$

From this action we can understand the origin of the breakdown of the linear expansion at the Vainshtein radius we described before. The scalar couples to the gravitational sector through the trace of the stress tensor T. Imaging an heavy point source of mass M we have:

$$\hat{\phi} \sim \frac{M}{M_P}\,\frac{1}{r} \tag{3.66}$$

That assumed, the non linear term is suppressed in comparison to the linear one by a factor:

$$\xi_{NL} \,=\, \frac{\partial^4\,\hat{\phi}}{\Lambda_5^5} \,\sim\, \frac{M}{M_P}\,\frac{1}{\Lambda_5^5\,r^5} \tag{3.67}$$

Whenever this factor becomes of order one $\xi_{NL} \approx 1$ non linearities become important and the linear approximation breaks down. This happens at a distance:

$$r_V \,\sim\, \left(\frac{M}{M_P}\right)^{1/5}\,\frac{1}{\Lambda_5} \,\sim\, \left(\frac{G\,M}{m^4}\right)^{1/5}\,. \tag{3.68}$$

such that whenever $r \leq r_V$ the linear approximation is reliable. Note that this is exactly the result we cited in previous sections which can be derived also from the perturbative spherical solution due to a localized heavy source of mass M.
The non linear action (3.65) clearly contains higher derivative terms and the corresponding equations of motion result to be fourth order. This means that there are more d.o.f. present [110] and that by Ostrogradski's theorem[14] [127, 128] one of those is a ghost!
Note that at linear order the higher derivative scalar terms are not visible indeed the linear theory propagates only 5 d.o.f. and it is completely healthy. Following [125] we consider a classical background solution $\Phi(r)$ which satisfies the equations of motion of $\hat{\phi}$ and we expand the action at quadratic order in the perturbation $\psi = \hat{\phi} - \Phi$. The action schematically takes the form:

$$\mathcal{L}_\psi \,\sim\, -\left(\partial\psi\right)^2 \,+\, \frac{(\partial^2\Phi)}{\Lambda_5^5}\,\left(\partial^2\Psi\right)^2 \tag{3.69}$$

[14] A non-degenerate Lagrangian containing time derivatives of higher than the first corresponds to a linearly unstable Hamiltonian associated with the Lagrangian via the usual Legendre transform.

This is a 4th order action and therefore propagates two linear degrees of freedom: one is stable and massless while the other one is a ghost whose mass is proportional to the scale in front of the higher derivative term. More in details, we have that:

$$m^2_{ghost} \sim \frac{\Lambda_5^5}{\partial^2 \Phi(r)} \tag{3.70}$$

On a flat background or at very large distances the ghost mass becomes infinite, the ghost freezes and this explains why in the linear theory we do not see it. We are working with an effective field theory with cutoff Λ_5 meaning that every ghost whose mass is bigger than such a cutoff is not problematic. This condition fixes a distance r_{ghost} at which indeed $\partial^2\Phi \sim \Lambda_5^5$ happens.
If we consider a source mass M at distance $r \gg r_V$ the solution takes the form $\Phi(r) \sim \frac{M}{M_P}\frac{1}{r}$ and the corresponding ghost radius reads:

$$r_{ghost} \sim \left(\frac{M}{M_P}\right)^{1/3} \frac{1}{\Lambda_5} \gg r_V \sim \left(\frac{M}{M_P}\right)^{1/5} \frac{1}{\Lambda_5} \tag{3.71}$$

meaning that r_{ghost} is generically bigger than the Vainshtein radius r_V. We will see that it will be at a distance comparable to the one at which also the quantum effects cannot be neglected anymore.
Non linearities, as we already stated, can resolve the vDVZ discontinuity through the so called **Vainshtein mechanism**. Far outside the Vainshtein radius, where the linear approximation works well, the field $\hat{\phi}$ has the usual Coulomb profile $\sim 1/r$; on the contrary whereas the non linearities become relevant and the cubic term dominates the behaviour gets modified in such a way that:

$$\begin{cases} \hat{\phi} \sim \frac{M}{M_P}\frac{1}{r} & \text{if } r \gg r_V \\ \hat{\phi} \sim \left(\frac{M}{M_P}\right)^{1/2} \Lambda_5^{5/2}\, r^{3/2} & \text{if } r \ll r_V \end{cases} \tag{3.72}$$

At distances much below the Vainshtein radius, the ghost mass m_{ghost} becomes very small, and the ghost starts to mediate a long range force. Usually a scalar field mediates an attractive force, but due to the ghost's wrong sign kinetic term, the force mediated by it is repulsive. In fact, it cancels the attractive force due to the longitudinal mode, the force responsible for the vDVZ discontinuity, and so general relativity is restored inside the Vainshtein radius.
Following [129], upon some field redefinitions the action results of the form:

$$\mathcal{L} = -\left(\partial\hat{\phi}\right)^2 + (\partial\Psi)^2 + \Lambda_5^{5/2}\,\Psi^{3/2} + \frac{1}{M_P}\,\hat{\phi}\,T + \frac{1}{M_P}\,\Psi\,T \tag{3.73}$$

where $\hat{\phi}$ is the healthy longitudinal mode and Ψ is the ghost. Both the field are coupled to the stress tensor T; the $\hat{\phi}$ field is free and has the usual profile $\sim 1/r$ while the ghost has a particular self interaction term $\sim \Psi^{3/2}$. Its profile contains two competing terms which become comparable at $r = r_V$. At radii smaller than that the linear term dominates such that the Ψ profile is also $\sim 1/r$. This profile generates a repulsive Coulomb force which exactly cancels the attractive one mediated by $\hat{\phi}$ such that in sum there are no extra forces (due to extra scalar degrees of freedom) beyond gravity in this region ($r \ll r_V$).
On the other way the funny non linear term dominates in the other limit $r \gg r_V$ such that $\Psi \sim \left(\frac{M}{M_P}\right)^2 \frac{1}{\Lambda_5^5\, r^6}$ and therefore the ghost profile is negligible in this region compared to the $\hat{\phi}$ profile. Thus the ghost ceases to be active beyond the Vainshtein radius and the longitudinal mode $\hat{\phi}$ generates a fifth force in addition to gravity. This is known as a *screening mechanism*, a

mechanism by which a light scalar degrees of freedom is made inactive at short distances through non-linearities.
Massive gravity is an effective field theory with non-renormalizable operators suppressed by the scale Λ_5. The amplitude corresponding to the scattering of longitudinal gravitons with energy E goes like $\mathcal{A} \sim \left(\frac{E}{\Lambda_5}\right)^{10}$. This amplitude becomes order 1, and thus the theory strongly coupled, when $E \sim \Lambda_5$ which represents indeed the maximal cutoff of the theory. We do not enter the details about the quantum corrections to the massive gravity theory but we can say that at a distance:

$$r_Q \sim \left(\frac{M}{M_P}\right)^{1/3} \frac{1}{\Lambda_5} \tag{3.74}$$

we cannot trust anymore the classical solution because quantum effects become important. This distance is parametrically larger than the Vainshtein radius where non linearities get relevant. Unlike the case in GR, there is no intermediate regime where the linear approximation breaks down but quantum effects are still small, so there is no sense in which a non-linear solution to massive gravity can be trusted for making real predictions in light of quantum mechanics. Thus there is no regime for which GR is a good approximation; the theory transitions directly from the linear classical regime with a long range fifth force scalar, to the full quantum regime. Finally, the radius r_Q is the same as the radius r_{ghost} where the ghost mass drops below the cutoff, so it is consistent to ignore the ghost since it lies beyond the reach of the quantum effective theory. The various regions are shown in Figure 3.2.

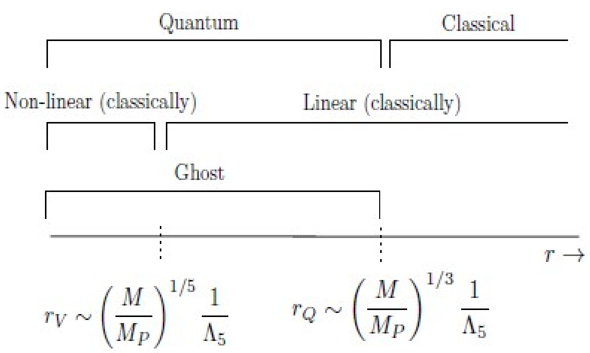


Figure 3.2: Regimes of massive gravity. Note the differences with the GR case presented in fig.3.1.

3.3 dRGT and LV massive gravity

dRGT massive gravity

As we saw in the previous section, Vainshtein [120] proved that the extra degree of freedom responsible for the vDVZ discontinuity gets screened by its own non linear interactions which dominate over the linear terms in the massless limit. That was the resolution for the vDVZ puzzle but the presence of a ghost in massive gravity theories was thought to be unavoidable [125], with the effects of rendering MG theories just a sick exercise of style. That argument was too quick and the past decade has seen a revival of interest in massive gravity with the realization that this BD ghost could be avoided either in a model of soft massive gravity (not a single massive

pole for the graviton but rather a resonance) as in the DGP (Dvali-Gabadadze-Porrati) model or its extensions [130–132], or in a three-dimensional model of massive gravity as in "new massive gravity" (NMG) [133] or more recently in a specific ghost-free realization of massive gravity (also known as **dRGT** in the literature) [134].
In this section we will focus on this last attempt achieved in 2010 when de Rham, Gabadadze, and Tolley constructed, order by order, a theory of massive gravity with coefficients tuned to avoid the Boulware-Deser ghost by packaging all ghostly (i.e., higher-derivative) operators into total derivatives which do not contribute to the equations of motion [134]. That definitely was a breakthrough and the complete absence of the Boulware-Deser ghost, to all orders and beyond the decoupling limit, was subsequently proven by Fawad Hassan and Rachel Rosen [135, 136].
The action for the dRGT theory, in the metric language[15], reads:

$$\mathcal{S}_{dRGT} \,=\, \frac{M_P^2}{2} \int d^4x \, \sqrt{-g} \, \left(R \,+\, \frac{m^2}{2} \sum_{n=0}^{4} \alpha_n \, \mathcal{L}_n \, \left[\, \mathcal{K} \, [g, f] \,\right] \right) . \tag{3.75}$$

where M_P is the Planck mass, f the background metric and m the graviton mass. In the following we will make use of the notation for the overall potential of massive gravity:

$$\mathcal{U} \,=\, -\, \frac{M_p^2}{4} \, \sqrt{-g} \, \sum_{n=0}^{4} \alpha_n \, \mathcal{L}_n \, \left[\, \mathcal{K} \, [g, f] \,\right] \tag{3.76}$$

such that the full Lagrangian for the theory reads:

$$\mathcal{L}_{dRGT} \,=\, M_P^2 \, \mathcal{L}_{GR}[g] \,-\, m^2 \, \mathcal{U}[g, f] \, . \tag{3.77}$$

where $\mathcal{L}_{GR}[g]$ is the usual Einstein-Hilbert action for the dynamical metric $g_{\mu\nu}$.
dRGT theory breaks diffeomorphism invariance which can be restored through the Stückelberg mechanism $f \to \tilde{f}$ as explained in 3.45.
In this formulation $\mathcal{L}_0$ corresponds to the cosmological constant, $\mathcal{L}_1$ to a tadpole, $\mathcal{L}_2$ to a mass term and $\mathcal{L}_{3,4}$ to allowed non linear self interactions. The previously introduced $\mathcal{K}$ matrix is defined in terms of the dynamical and background metrics g and f as:

$$\mathcal{K}^\mu_{\ \nu}[g, f] \,=\, \delta^\mu_{\ \nu} \,-\, \left(\sqrt{g^{-1} \, f} \right)^\mu_{\ \nu} \tag{3.78}$$

and the correspondent Langragians $\mathcal{L}_n$ by:

$$\begin{aligned}
\mathcal{L}_0 \, [\mathcal{K}] \,&=\, 4! \, , \\
\mathcal{L}_1 \, [\mathcal{K}] \,&=\, 3! \, [\mathcal{K}] \, , \\
\mathcal{L}_2 \, [\mathcal{K}] \,&=\, 2! \, \Big([\mathcal{K}]^2 \,-\, [\mathcal{K}^2] \Big) \, , \\
\mathcal{L}_3 \, [\mathcal{K}] \,&=\, \Big([\mathcal{K}]^3 \,-\, 3 \, [\mathcal{K}] \, [\mathcal{K}^2] \,+\, 2 \, [\mathcal{K}^3] \Big) \, , \\
\mathcal{L}_4 \, [\mathcal{K}] \,&=\, \Big([\mathcal{K}]^4 \,-\, 6 \, [\mathcal{K}^2] \, [\mathcal{K}]^2 \,+\, 3 \, [\mathcal{K}^2]^2 \,+\, 8 \, [\mathcal{K}] \, [\mathcal{K}^3] \,-\, 6 \, [\mathcal{K}^4] \Big) \, .
\end{aligned} \tag{3.79}$$

This construction can be easily generalized to the d-dimensional case where we would have the massive gravity potential up to order $\mathcal{L}_d$.
This fined tuned massive gravity theory results to be **ghost-free** and therefore a fully non linear

[15]We will not making use of the vielbein language at all through this work.

healthy theory of massive gravity. Despite the long controversy there are now various methods to check the absence of the BD ghost in dRGT massive gravity; we will follow the Hamiltonian analysis in the ADM languange.
Let's perform the usual ADM decomposition:

$$ds^2 \,=\, -\,N^2\,dt^2 \,+\, \gamma_{ij}\,\left(dx^i \,+\, N^i\,dt\right)\,\left(dx^j \,+\, N^j\,dt\right) \tag{3.80}$$

where N is the usual lapse, N^I the shift and γ_{ij} the metric of the 3-dimensional subspace.
We then define as usual the conjugate momenta:

$$p_{ij} \,=\, \frac{\partial\,\sqrt{-g}\,\mathcal{R}}{\partial\,\dot{\gamma_{ij}}} \tag{3.81}$$

In the general relativity case the Hamiltonian reduces in terms of the 12 phase space variables to:

$$\mathcal{H}_{GR} \,=\, N\,R_0(p,\gamma) \,+\, N^i\,R_i(p,\gamma) \tag{3.82}$$

such that both the shift and the lapse play the role of Lagrange multipliers; they therefore provide first class constraints which remove 2 d.o.f. each one.
This amounts to:

$$(2\,\times\,6)\,-\,2\,\text{lapse constraints}\,-\,2\,\times\,3\,\text{shift constraints}\,=\,4\,=\,2\,\times\,2 \tag{3.83}$$

corresponding to 2 left indipendent degrees of freedom, *i.e.* the two polarizations of the massless graviton.
Let's consider a massive gravity theory with generic potential $\mathcal{U}$ which is a function of the metric. Since the potential does not contain any derivatives of the metric the definition of the conjugate momenta stay unchanged. The massive gravity modification translates directly into a potential at the level of the Hamiltonian density:

$$\mathcal{H} \,=\, N\,R_0(p,\gamma) \,+\, N^i\,R_i(p,\gamma) \,+\, m^2\,\mathcal{U}(\gamma_{ij}, N^i, N) \tag{3.84}$$

If the potential $\mathcal{U}$ depends non linearly on the shift and the lapse these would not be Lagrange multipliers anymore[16]. If that is the case no first constraints are present and one is left with 6 indipendent degrees of freedom: the two tensor polarizations of GR plus two "vector" and two "scalar" polarizations. The sixth additional d.o.f. is the unwanted BD ghost. The only way this counting can be wrong is if the constraints for the shift and the lapse cannot be inverted for the shift and the lapse themselves, and thus at least one of the equations of motion from the shift or the lapse imposes a constraint on the three-dimensional metric γ_{ij}. This was performed in a fully non linear fashion in [126].
To this respect Fierz Pauli theory is particular because *at linear level* the lapse N remains linear still acting as a Lagrange multiplier. The shift on the other hand appears non linearly and it does not impose any constraint such that all in all we are left with $2 + 3 = 5$ d.o.f. at linear level, which are the usual for a massive graviton. We already know that non-linearly this is no longer true and the BD ghost appears. At non linear level it was proved in [125] that there is no potential $\mathcal{U}$ which would prevent the lapse from entering non-linearly. While this result is definitely correct, it does not however imply the absence of a constraint generated by the set

[16] If they are non-linear, they still appear at the level of the equations of motion, and so they do not propagate a constraint for the metric but rather for themselves.

of shift and lapse $N_\mu = (N, N_i)$. Indeed there is no reason to believe that the lapse should necessarily be the quantity to generate the constraint necessary to remove the BD ghost. Rather it can be any combination of the lapse and the shift.
In order to understand how this can be possible we prefer to give an illustrative example which was first presented in [137]. Consider the following Hamiltonian:

$$\mathcal{H}_{ex} = N\, C_0(\gamma, p) + N^i\, C_i(\gamma, p) + m^2\, \mathcal{U} \tag{3.85}$$

with the potential defined by:

$$\mathcal{U} = V(\gamma, p)\, \frac{\gamma_{ij}\, N^i\, N^j}{2\, N} \tag{3.86}$$

In this case neither the lapse nor the shift enter linearly, so one would naively think that the presence of the BD ghost in unavoidable. However solving for the shift and pulling back into the Hamiltonian we get:

$$\mathcal{H} = N \left(C_0(\gamma, p) - \frac{\gamma_{ij}\, C^i\, C^j}{2\, m^2\, V(\gamma, p)} \right) \tag{3.87}$$

where the lapse N magically appears linearly generating a constraint. In reality there is no need of integrating out the shift to realize it; one can just look at the Hessian:

$$L_{\mu\nu} = \frac{\partial^2 \mathcal{H}}{\partial N^\mu\, \partial N^\nu} = m^2\, \frac{\partial^2 \mathcal{U}}{\partial N^\mu\, \partial N^\nu} \tag{3.88}$$

In the example we consider one has:

$$det(L_{\mu\nu}) = 0 \tag{3.89}$$

meaning that the Hessian can not be inverted and the equations of motions cannot be solved for all the shift and the lapse. One of these equation of motions would therefore represent a constraint for the 3-d phase space variables. Note that in this case the constraint is not associated to any kind of symmetry.
Finally one could have reached the same conclusion performing the following change of variables $N_i \to n_i = \frac{N_i}{N}$ upon which the Hamiltonian would read:

$$\mathcal{H} = N \left(C_0(\gamma, p) + n^i\, C_i(\gamma, p) + m^2\, V(\gamma, p)\, \frac{\gamma_{ij}\, n^i\, n^j}{2} \right) \tag{3.90}$$

which is again linear in the lapse N!
To summarize the necessary condition to eliminate the presence of the BD ghost is that the determinant of the Hessian $L_{\mu\nu}$ vanishes as explained in [134].
This is indeed the case for the ghost-free dRGT theory as shown in [135]!
Since the derivation is pretty long and technical we prefer to leave it to the references.

Lorentz-Violating Massive Gravity

So far we have focused our attention on Lorentz preserving massive gravity theories, showing explicitely all their possible problems connected with the existence of an extra ghosty scalar mode, *i.e.* the BD ghost. If one stick to Lorentz invariant situations the only viable theory is the famous dRGT massive gravity, but as soon as this assumption gets relaxed a plethora of new possibilities appear. They go under the name of *Lorentz violating massive gravity theories* and they are nicely reviewed and described in [138, 139].
Naively one would expect these models to be less pathologic than their Lorentz invariant version

because they admit the possibility of preserving certain subgroups of the diffeomorphism one and indeed that is the case. We do not enter the discussion about the phenomenological viability of breaking Lorentz simmetry for which we refer to [140] but we will just accept it as a theoretical possibility.
The most generic Lorentz violating massive gravity theory in $d = 4$ which preserves the Euclidean group of the 3-dimensional subspace has the following lagrangian [141]:

$$\mathcal{L}_m \;=\; \frac{M_P^2}{4}\,\Big(m_0^2\, h_{00}\, h_{00} \,+\, 2\, m_1^2\, h_{0i}\, h_{0i} \,-\, m_2^2\, h_{ij}\, h_{ij} \,+\, m_3^2\, h_{ii}\, h_{jj} \,-\, 2\, m_4^2\, h_{00}\, h_{ii}\Big) \tag{3.91}$$

where, as before, $h_{\mu\nu}$ are the perturbations around the Minkowki background.
With these notations the Fierz Pauli theory is defined by:

$$\text{FP:} \qquad m_0 \,=\, 0\,, \;\; m_1 \,=\, m_2 \,=\, m_3 \,=\, m_4 \,=\, m_G\,. \tag{3.92}$$

The lagrangian in the tensor sector now contains a term:

$$\mathcal{L}_m^{TT} \;=\; -\,\frac{m_2^2}{4}\, h_{ij}^{TT}\, h_{ij}^{TT} \tag{3.93}$$

where m_2 qualifies indeed as the mass for the tensor gravitons and in order to avoid tachyonic excitations its square would better be positive:

$$m_2^2 \geq 0 \tag{3.94}$$

In the vector sector both m_1 and m_2 appears whereas all the other masses are present in the scalar action. There are rigid constraints on the masses to avoid ghosts, tachyons and other instability issues. One can consider all the possible combinations and classify the healthy, and numerous, phases of lorentz violating massive gravity which are theoretically consistent [139]. The resulting zoology can be efficiently classified by looking at the residual gauge symmetries [139]. Despite the recent efforts in discussing the possible UV completions of such theories (see for example [142]) we just rely on them as low energy effective field theory with a certain cutoff and we completely avoid such a topic.
The convenient way to describe these theories is in a language closer to the simmetry breaking mechanism by introducing Stückelberg fields. We focus just on theories where Lorentz invariance is broken down to the subgroup of spatial rotations. We introduce a set of four scalar fields (ϕ^0, ϕ^a), *i.e.* the Goldstones, with $a = 1, 2, 3$. These scalars enjoy an internal SO(3) symmetry and they couple to the metric in a covariant way. Additional symmetries have to be imposed on this sector to avoid patologies. The *spontaneous* breaking of Lorentz invariance occurs when these fields aquire background values which depend on space-time coordinates. As an example, working on Minkowski space, the background fields are:

$$\begin{aligned} \bar{\phi}^0 \;&=\; a\,\Lambda^2\, t\,, \\ \bar{\phi}^a \;&=\; b\,\Lambda^2\, x^a\,. \end{aligned} \tag{3.95}$$

where Λ is a parameter with mass dimension and a,b are numerical coefficients of order one[17]. The previous is a solution of the system if the those fields enter in the action just with derivative terms. The latter property automatically implies that the Lagrangian is invariant under a shift symmetry $\phi^a \to \phi^a + \lambda^a$ with constant λ^a. Likewise to preserve the SO(3) symmetry of the 3-dimensional

[17] In these conventions the ϕ fields have mass dimension 1.

subspace we have to ask the Lagrangian to be invariant under the rotations $\phi^i \to \Lambda^i{}_j \phi^j$.
All in all the most generic action for the gravity+scalars system is constrained to be of the form:

$$\mathcal{S} \,=\, \mathcal{S}_{EH} \,+\, \int d^4x \,\sqrt{-g}\,\Lambda^2\, F(Y, V^i, X^{ij})\,. \tag{3.96}$$

with:

$$\begin{aligned} Y \,&=\, \frac{1}{\Lambda^4}\, g^{\mu\nu}\, \partial_\mu \phi^0\, \partial_\nu \phi^0\,, \\ V^i \,&=\, \frac{1}{\Lambda^4}\, g^{\mu\nu}\, \partial_\mu \phi^0\, \partial_\nu \phi^i, \\ X^{ij} \,&=\, \frac{1}{\Lambda^4}\, g^{\mu\nu}\, \partial_\mu \phi^i\, \partial_\nu \phi^j\,. \end{aligned} \tag{3.97}$$

Internal indices i, j, k have to be contracted in the action with $\delta_{ij}, \epsilon_{ijk}$.
For reasons which will be clearer in the following in this work we will only consider theories with non trivial scalar profiles just in two spatial directions x, y. In this language this corresponds to fix $a = 0$ and therefore making the field ϕ^0 disappear from the game. In this dimensionality the symmetries fix the action to be just a function of two scalar object constructing from the X^{ij} matrix, its trace and its determinant:

$$X \,=\, Tr[X^{ij}]\,, \qquad Z \,=\, det[X^{ij}]\,. \tag{3.98}$$

All in all the most generic (within the assumptions we made) Lorentz violating theory of massive gravitons takes the form:

$$\mathcal{S} \,=\, \mathcal{S}_{EH} \,+\, \int d^4x \,\sqrt{-g}\,\Lambda^2\, V(X, Z)\,. \tag{3.99}$$

where V is a generic potential function.
There is a very strong analogy between the LV theories of massive gravity and the EFT for spontaneous Lorentz symmetry breaking (*i.e.* the EFT for fluids and solids) [143–146]. The latter are defined in flat Minkowski space but their construction and action is exactly the same. Recasting LV MG theories in the language of General Relavity + a scalar sector is not only very convenient from the point of view of the consistency and healthiness checks but from the phenomenological perspective too. One can indeed classify again the various phases using the internal symmetries of the scalar sector. As a simple example, we can distinguish *solid* from *fluids* in this way; whereas solids enjoy just internal translational symmetry, fluids are invariant also under internal volume-preserving diffeomorphism:

$$\begin{aligned} &\text{SOLIDS:} \qquad \{\,\phi^i \to \phi^i \,+\, c^i\,\}\,, \\ &\text{FLUIDS:} \qquad \{\,\phi^i \to \phi^i \,+\, c^i\,\} \,+\, \{\,\phi^i \to \xi^i(\phi^j)\,\} \qquad \text{with} \qquad det(\partial \xi^i / \partial \phi^j) \,=\, 1\,. \end{aligned}$$

This fact has a strong influence on the allowed action. The larger symmetry , which fluids enjoy, forces the action to be a function of the only determinant Z:

$$\mathcal{L}_{solid} \,\sim\, V(X, Z]\,, \qquad \mathcal{L}_{fluids} \,\sim\, V(Z]\,. \tag{3.100}$$

and in the language of (3.91) constraints the mass of the traceless transverse part of the graviton to vanish, *i.e.* $m_2 = 0$. This account to say that, despite the solid case, there are no propagating

transvers phonons in a fluid.
Note that this formulation in terms of scalars with non vanishing V.E.V.s allows to construct the most generic massive gravity theories and is able to reproduce also the Lorentz Invariant case[18] and for example the dRGT scenario. An important point to make is that a theory defined as in (3.99) is way more general than the dRGT case and that this is consistent and allowed thanks to the breaking of Lorentz simmetry.
We will make use of these theories defined on an Anti de Sitter background in the context of the Gauge-Gravity duality in order to mimick particular Condensed Matter situations.

3.4 A jump into AdS-CMT

dRGT-CMT

Systems with perfect translational simmetry cannot dissipate momentum. As a consequence whenever a finite density of charge carrier is present in such a system the correspondent electric DC conductivity $\sigma_{DC} = \sigma(\omega = 0)$ is infinite. In a weakly coupled fashion this can be easily seen from the DC formula given by the Drude Model:

$$\sigma_{DC} \,=\, \frac{n\, e^2\, \tau}{m} \tag{3.101}$$

where τ is the already mentioned collision/relaxation time coming from the equation which controls the dynamic of the momentum $\vec{p}$:

$$\frac{d\vec{p}}{dt} \,=\, e\, \vec{E} \,-\, \frac{\vec{p}}{\tau} \tag{3.102}$$

Whenever translational symmetry is preserved, momentum cannot be relaxed meaning that the relaxation time $\tau = \infty$. It follows directly from (3.101) that the electric DC conductivity is infinite[19]. Hydrodynamics arguments give that in the presence of a conserved momentum operator the low frequency conductivity reads:

$$\sigma(\omega) \,=\, s\, T \left(\delta(\omega) \,+\, \frac{i}{\omega}\right) \tag{3.104}$$

and it is characterized indeed by a δ function at zero frequency which in the presence of momentum dissipation gets smoothed out into the so-called *Drude Peak*:

$$\sigma(\omega) \,=\, \frac{\sigma_{DC}}{1 \,-\, i\, \omega\, \tau} \tag{3.105}$$

revealing a pole shifted in the lower half of the imaginary axes.
The same phenomenon can be re-expressed in modern language [147] stating that whenever the current operator $\vec{J}$ has a finite overlap with the momentum operator $\vec{P}$ (meaning the susceptibility

[18]One has just to fix the vevs of the scalars to be all the same such that Lorentz invariance is restored.
[19]Note that this infinite is significantly different from the one encountered in a Superconducting medium:

$$\sigma \,\sim\, \frac{\rho_S\, i}{\omega} \tag{3.103}$$

where the delta function is due to a Bose-Einstein condensation mechanism.

$\chi_{\vec{J}\vec{P}} \neq 0$) and momentum is a conserved quantity than the conductivity coming from the $\vec{J}\vec{J}$ correlator through the Kubo formula:

$$\sigma = \lim_{\omega \to 0} \frac{Im\, G^R_{\vec{J}\vec{J}}(\omega)}{\omega} \tag{3.106}$$

has an infinite DC value at zero frequency.
Holography does not evade such a generic prescription and holographic systems with translational symmetry, like the Reissner-Nordstrom benchmark model described in the previous section, indeed show an infinite DC conductivity[20].
In the recent years various ways of avoiding the infinite DC conductivity have been introduced by treating the charge carriers in the probe limit [148,149] (*i.e.* as a small part in a larger system of neutral fields where they can dump momentum), or by introducing spatial inhomogeneities thereby breaking translational invariance explicitly [150,151]. The first scenario consists in freezing the fluctuations of the metric. The dual field theory has then strictly speaking no energy momentum tensor and there is no overlap between the current and the momentum operators. In the second case there is an explicitly inhomogeneous background which involves hard numerical efforts in order to solve complicated systems of PDEs.
Motivated by the main goal of building a framework for translational symmetry breaking and momentum dissipation in holography without the need for complex numerical computations Massive gravity was introduced in the context of holography in [152].
The AdS-CFT dictionary tell us that the metric field $g_{\mu\nu}$ in the bulk is dual to the Stress Tensor operator $T_{\mu\nu}$ of the correspondent dual boundary CFT. Momentum (density) operator is defined as T^{0i} and it is part of such a object whose conservation reads:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.107}$$

In more details, translational symmetry (in the spatial coordinates) implies that momentum is a conserved quantity. From the point of view of the AdS-CFT correspondence the conservation of the stress tensor is encoded in the gauge symmetry of the metric field, *i.e.* diffeomorphism invariance. It is therefore clear that one method to make momentum be not conserved consists in breaking (at least the spatial part) diffeomorphims invariance in the bulk.
On the other hand the temporal part of the Stress Tensor T^{00} encodes the energy density of the system and it is kept to be conserved. This forces univoquely the symmetry breaking pattern we are interested in. Working in $d = 3 + 1$ dimensions $\{t, r, x, y\}$, we break translational symmetry in the spatial directions $\{x, y\}$ expressed in the linear diffeomorphism transformation:

$$x^i \to x^i + \zeta^i\,, \qquad i = x\,, y\,. \tag{3.108}$$

but we preserve the temporal part of such a transformation.
In the charged black brane background described by the RN solution, Ward identities for translational invariance in the x direction imply a shift symmetry in the g_{tx} field. The simplest option to break such a symmetry is to add a mass term for the graviton:

$$\mathcal{L}_m \sim \sqrt{-g}\, m^2\, g^{tx}\, g_{tx} \tag{3.109}$$

[20]Strictly speaking the numerical procedure does not show the δ function in the real part of the conductivity but just a pole $1/\omega$ in the imaginary part. Through Kramers-Kronig relations one can then argue the presence of the δ function in the real part.

The idea is to replace the usual Einstein-Maxwell theory in the bulk with dRGT-Maxwell generalization, namely:

$$\mathcal{S} = \frac{1}{2\,\kappa^2}\int d^4x\,\sqrt{-g}\,\left[\underbrace{R + \Lambda - \frac{L^2}{4}\,F^2}_{Einstein-Maxwell} + m^2\sum_{i=1}^{4} c_i\,\mathcal{U}_i(g,f)\right] \tag{3.110}$$

where f is the usual fixed reference metric, c_i are constants and $\mathcal{U}_i$ are polynomials of the eigenvalues of the matrix $\mathcal{K} = \sqrt{g^{\mu\alpha}\,f_{\nu\alpha}}$ defined as[21]

$$\begin{aligned}
\mathcal{U}_1 &= [\mathcal{K}]\,,\\
\mathcal{U}_2 &= [\mathcal{K}^2] - [\mathcal{K}]^2\,,\\
\mathcal{U}_3 &= [\mathcal{K}^3] - 3\,[\mathcal{K}^2]\,[\mathcal{K}] + 2\,[\mathcal{K}^3]\,,\\
\mathcal{U}_4 &= [\mathcal{K}^4] - 6\,[\mathcal{K}^2]\,[\mathcal{K}]^2 + 8\,[\mathcal{K}^3]\,[\mathcal{K}] + 3\,[\mathcal{K}^2]\,[\mathcal{K}]^2 - 6\,[\mathcal{K}^4]\,.
\end{aligned} \tag{3.111}$$

This massive gravity construction is built to avoid the already discussed BD ghost and in the limit $m \to 0$ it boils down to the usual translational invariant Einstein-Maxwell setup. To implement the wanted symmetry breaking pattern we choose the refence metric to be (in the basis (t, r, x, y)) :

$$f^{SP}_{\mu\nu} = diag(0, 0, 1, 1) \tag{3.112}$$

where SP stands for spatial[22]. In this way the mass term $\sim m^2\mathcal{U}(g, f^{SP})$ preserves general covariance in the t,u, coordinates but breaks it in the two spatial dimensions x,y. This is exactly what we need and it corresponds to allow momentum (but not energy) density to dissipate in the dual picture.

Because of this choice and the number of dimensions we are working on only $\mathcal{U}_{1,2}$ are indipendent objects and the generic mass term considered takes the form of:

$$\sim m^2\left[\alpha\,[\mathcal{K}] + \beta\,\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right)\right] \tag{3.113}$$

With these assumptions the charged BH solution reads:

$$\begin{aligned}
ds^2 &= L^2\,\frac{1}{r^2}\left(\frac{dr^2}{f(r)} - f(r)\,dt^2 + dx^3 + dy^2\right)\,,\\
A(r) &= A_t(r)\,dt = \mu\left(1 - \frac{r}{r_h}\right)dt\,.
\end{aligned} \tag{3.114}$$

where the emblackening factor is:

$$f(r) = 1 + \alpha\,L\,\frac{m^2}{2}\,r + \beta\,m^2\,r^2 - M\,r^3 + \frac{\mu^2}{4\,r_h^2}\,r^4 \tag{3.115}$$

The mass of the BH object M is fixed in such a way that $f(r_h) = 0$ and r_h is the proper event horizon of such a BH.

[21] The square root in $\mathcal{K}$ is understood to denote the matrix square root and the rectangular brakets the matrix trace.

[22] In the Stückelberg language this corresponds to switch on just the two spatial Stückelberg fields ϕ^x, ϕ^y. We will come back on this point later.

In the limit $m = 0$ we recover the usual RN solution; otherwise we have two new parameters in the system α, β.
The temperature of such a background is given by:

$$T \,=\, \frac{1}{4\,\pi\,r_h}\,\left(3\,-\,\left(\frac{\mu\,r_h}{2}\right)\,+\,m^2\,r_h\,(\alpha\,L\,+\,\beta\,r_h)\right) \tag{3.116}$$

and the geometry represents a finite density state with definite entropy s, energy density ϵ and charge density ρ which satisfy the usual first law of thermodynamics:

$$d\epsilon \,=\, T\,ds\,+\,\mu\,d\rho \tag{3.117}$$

As we will see later the linearized perturbations of the metric around the background will gain a position dependent mass of the form:

$$m^2(r)\,=\,-\,2\,\beta\,-\,\frac{\alpha\,L}{r} \tag{3.118}$$

where r is the radial coordinate and the UV is fixed at $r = 0$.
It is immediately clear that one requisite of stability is dictated by imposing that such a mass is positive and real. That would correspond to require that the momentum relaxation time τ is positive.
The solution asymptotes from the AdS_4 boundary to an infrared $AdS_2\times\,R_2$ geometry; this means that the correlation functions are conformal in the UV while exhibiting local criticality in the IR.
On top of this background we can run the machinery to compute holographically the conductivity, namely perturbing the solution as the following:

$$\begin{aligned} ds^2\,&\rightarrow\,ds^2\,+\,g_{tx}(r)\,e^{i\,\omega\,t}\,+\,g_{rx}(r)\,e^{i\,\omega\,t}\,, \\ A(r)\,&\rightarrow\,A(r)\,+\,a_x(r)\,e^{i\,\omega\,t}\,dx \end{aligned} \tag{3.119}$$

The Maxwell equation becomes:

$$(f\,a_x')'\,+\,\frac{\omega^2}{f}\,a_x\,=\,-\,\frac{A_t'\,r^2}{L^2}\,\left(g_{tx}'\,+\,\frac{2}{r}\,g_{tx}\,-\,i\,\omega\,g_{rx}\right) \tag{3.120}$$

Meanwhile the t-x and r-x components of the Einstein equations read:

$$\begin{aligned} \left(g_{tx}'\,+\,\frac{2}{r}\,g_{tx}\,-\,i\,\omega\,g_{rx}\,+\,A_t'\,L^2\,a_x\right)'\,&=\,\frac{m^2(r)}{f}\,g_{tx}\,, \\ \left(g_{tx}'\,+\,\frac{2}{r}\,g_{tx}\,-\,i\,\omega\,g_{rx}\,+\,A_t'\,L^2\,a_x\right)\,&=\,-\,\frac{i\,f\,m^2(r)}{\omega}\,g_{rx}\,, \end{aligned} \tag{3.121}$$

These two equations are no longer equivalent, like it happens in the usual translational symmetric case, and therefore we are obliged to turn on also the g_{rx} component which is usually set consistently to 0.
Nevertheless the two equations imply the constraint:

$$\frac{i\,\omega\,m^2(r)}{f}\,g_{tx}\,=\,\left(m^2(r)\,f\,g_{rx}\right)' \tag{3.122}$$

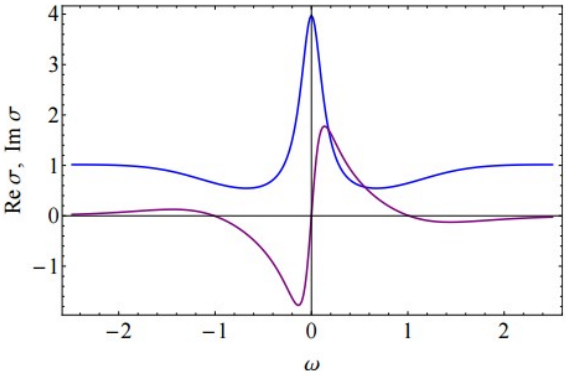


Figure 3.3: AC electric conductivity $\sigma(\omega)$ for the dRGT holographic model. The plot is taken from [152].

which can used to eliminate g_{tx} as in the usual RN case.
All in all, after redefining $\tilde{g}_{rx} = f g_{rx}$ we are left with the two differential equations:

$$\begin{aligned} (f\, a_x')' \,+\, \frac{\omega^2}{f}\, a_x \,&=\, \rho^2\, r^2\, a_x \,+\, \frac{\rho\, m^2\, r^2}{i\,\omega\, L^2}\, \tilde{g}_{rx}\,, \\ \frac{1}{r^2}\left(\frac{r^2\, f}{m^2}\,(m^2\, \tilde{g}_{rx})'\right)' \,+\, \frac{\omega^2}{f}\, \tilde{g}_{rx} \,&=\, \rho\, L^2\, i\,\omega\, a_x \,+\, m^2\, \tilde{g}_{rx}\,. \end{aligned} \tag{3.123}$$

where $\rho = \mu/r_h$ is the charge density of the system.
These equations can be used to extract numerically the electric conductivity through the usual prescription:

$$\sigma(\omega) \,=\, \frac{a_x'}{i\,\omega\, a_x}|_{UV} \tag{3.124}$$

There has been extensive effort in studying numerically the electric (and not only) conductivity in the context of massive gravity theories [152–156].
The, not surprising result, is that indeed the DC conductivity gets finite and the δ function coming from momentum conservation gets broaden up. Benchmark results are shown in fig.3.3 from ref. [152]. It is also possible, through methods which we will explain in details in the next sections, to extract the DC value of the electric conductivity for this setup [157]:

$$\sigma_{DC} \,=\, \frac{1}{e^2}\left(1 \,+\, \frac{\rho^2\, r_h^2}{m^2(r_h)}\right) \tag{3.125}$$

with $m^2(r_h) \,=\, -\,2\,\beta \,-\, \frac{\alpha\, L}{r_h}$.
The DC value contains two different terms with their own physical meaning; for the moment we just notice how the second term looks like very similar to the Drude formula we have already encountered many times during our journey. We will give more details about this formula in the following.

In order to understand better the dynamics of the momentum dissipation introduced by the presence of a graviton mass a convenient way is to rely on an hydrodynamical low energy description [153]. One can indeed prove that the low energy dynamics of such a theory is governed by a modified conservation law for energy-momentum $T^{\mu\nu}$ such that for small perturbations around the equilibrium state the fluid gets described by:

$$\partial_a T^{at} = 0 , \qquad \partial_a T^{ai} = -(\epsilon + P)\tau^{-1} u^i = -\frac{T^{ti}}{\tau} . \tag{3.126}$$

where where ϵ, P and u^i are the energy density, pressure and velocity of the near-equilibrium field theory state, and the constant τ is the characteristic timescale of momentum relaxation in the theory.
Such a relaxation time scales turns out to be inversely proportional (up to thermodynamical quantities) to the graviton mass:

$$\tau \sim \frac{1}{m^2} \tag{3.127}$$

This identification provides therefore a physical meaning to this instability related to the $m^2 \geq 0$ constraint: the state is unstable when $\tau < 0$ because it absorbs momentum at a constant rate, rather than dissipating it, and thus small perturbations of the state will grow exponentially in time.
In the limit of small graviton mass $m^2/\mu^2 \ll \omega/\mu \ll 1$, where momentum conservation is violated in a minor way, the conductivity is equivalent to the one predicted by the simple Drude model:

$$\sigma(\omega) = \frac{\sigma_{DC}}{1 - i\,\omega\,\tau} \tag{3.128}$$

However the full expression, which can be derived perturbatively in m^2, deviates from the Drude formula and it contains corrections to the latter (see fig.3.4 taken from [153]). The inclusion of these corrections results in a transfer of spectral weight from the Drude peak to higher frequencies, and a reduction in the phase of σ from the Drude value. It is anyway pretty amazing that massive gravity, at small graviton mass, namely at weak momentum dissipation, provides an strongly coupled analogue of the Drude model.
Note how the identification of the graviton mass with the inverse of the relaxation time τ is in perfect agreement with the formula for the DC conductivity, whose second term $\sim \frac{\rho^2}{m^2} \sim \rho^2\tau$ takes indeed the common Drude form.

It has been later analyzed through a quasinormal modes analysis [155] that whenever momentum dissipates slowly there is a well defined, *coherent* collective excitation in the conductivities, and a crossover between sound-like and diffusive transport at small and large scales. On the contrary, when momentum dissipates quickly, there is no such a excitation and diffusion dominates at all scales leading to an *incoherent* behaviour. This confirms the previous expectations that for small graviton mass the conductivity takes a Drude-like form with a purely imaginary pole $\omega = -i\Gamma = -i\tau^{-1}$ dominating the conductivity. The current is carried by a long-lived collective excitation whose decay rate Γ is parametrically larger the the others. Such an excitation is of course produced by the existence of an almost-conserved operator (indeed momentum) which couples to the current J. In the incoherent case the conductivity can't be approximated by a single dominating pole near the origin which is well separated from the others and will appear approximately constant with no localized features. In fig.3.5 from [155] we provide a sketch of the two different situations.

Finite DC conductivity was already found in the context of holography working with the so-called holographic lattices [150, 151, 158–162]. These models consist of Einstein-Maxwell action plus a

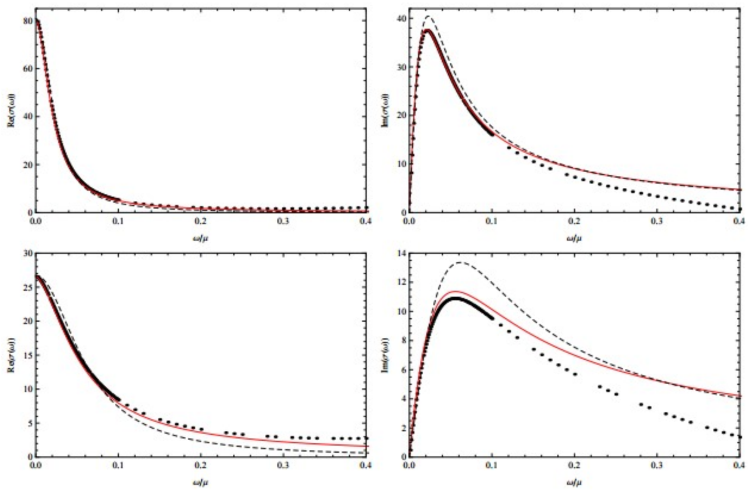


Figure 3.4: AC electric conductivity $\sigma(\omega)$ in presence of a massive gravity term taken from [153]. The black dots are the exact numerical values, the dashed black line is the Drude formula and the solid red line is the holographic formula which contains corrections to the Drude conductivity. **Top:** Small graviton mass regime $m \ll \mu$; **Bottom:** Strong momentum dissipation regime $m \gg \mu$.

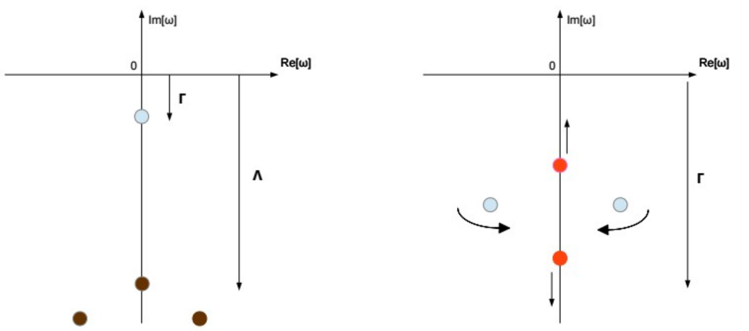


Figure 3.5: Schematic representation of the quasinormal modes locations in the massive gravity system. **Left:** For small graviton mass, *i.e.* slow momentum dissipation, there is a dominant pole close to the horigin whose decay rate is parametrically larger than all the other ones $\Gamma < \Lambda$. **Right:** There is no dominant pole and all the poles lie at a distance from the real axes $\sim \Lambda$; diffusion is governing the transport. For details see [155].

neutral scalar field:

$$\mathcal{S} \,=\, \int d^4x\,\sqrt{-g}\,\left[\frac{1}{2\,\kappa^2}\,\left(R\,+\,\frac{6}{L^2}\right)\,-\,\frac{1}{4\,e^2}\,F_{\mu\nu}F^{\mu\nu}\,-\,\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,-\,\frac{m^2}{2}\,\phi^2\right]\,. \tag{3.129}$$

The idea is to break translational invariance introducing a spatially modulated source for the operator $\mathcal{O}$ dual to the scalar ϕ.
For static solutions the near-boundary expansion of the scalar ϕ reads:

$$\phi_0(r,x,y)\,\sim\,\phi_-(x,y)\,\left(\frac{r}{L}\right)^{\Delta_-}\,+\,\phi_+(x,y)\,\left(\frac{r}{L}\right)^{\Delta_+} \tag{3.130}$$

where $\Delta_\pm \,=\, \frac{3}{2}\,\pm\,\sqrt{\frac{9}{4}\,+\,m^2L^2}$. Assuming standard quantization, meaning fixing the value of ϕ_- and identifying it as the source for the $\mathcal{O}$ operator, we choose to work with the striped source:

$$\phi_-\,=\,\epsilon\,\cos(k_L\,x) \tag{3.131}$$

where ϵ is a small parameter which will allow us to treat the lattice perturbatively. Turning on this source is equivalent to turning on a spatially modulated potential in the boundary theory, somewhat analogous to the optical lattices in cold atom experiments. The radial profile of the lattice is dynamical and determined by the equations of motion themselves. The bulk solution takes the form $\phi(r,x,y) = \epsilon\,\phi_0(r)\cos(k_L\,x)$ where the radial profile satisfies the equation:

$$\frac{d}{dr}\,\left(\frac{f}{r^2}\,\frac{d\phi_0}{dr}\right)\,-\,\frac{k_L^2}{r^2}\,\phi_0\,-\,\frac{m^2\,L^2}{r^4}\,\phi_0\,=\,0\,. \tag{3.132}$$

with $f(r)$ the emlackening factor of the charged black brane.
We will choose $m^2 < 0$ such that the operator $\mathcal{O}$ will be relevant with the profile $\phi_0(r)$ growing in the infra-red. The method consists in treating such a periodic scalar deformations as a small perturbations and solve the system at leading order in ϵ.
The main result is that at leading order the graviton aquires a mass:

$$\mathcal{S}_{eff}\,=\,\frac{1}{2}\,\int\,d^4x\,\sqrt{-g}\,M^2(r)\,g_{tx}\,g^{tx} \tag{3.133}$$

where the effective mass is radially dependent and it reads:

$$M^2(r)\,=\,\frac{1}{2}\,\epsilon^2\,k_L^2\,\,\phi_0(r)^2 \tag{3.134}$$

It has the same form as the mass terms arising in massive gravity model, albeit with a different radial profile.
This suggests a deep connection between holographic models for lattices and/or explicit disorder (in this case the scalar profile is not periodic but a more complicated and random function) and massive gravity theories. At least at leading order those models are "equivalent" (we will come back to this topic with more details) to massive gravity, which seems to realize an effective and efficient description of momentum dissipation in the context of holography. Despite the first attempts were focused on the dRGT choice it should be now clear that such a theory is not the most general massive gravity theory we can construct and that in absence of Lorentz invariance there are way more possibilities available. We will return to this aspect.

Dissipating momentum via Stückelberg fields

The holographic stress tensor obeys ([163]) a conservation equation which reads:

$$\nabla_i \langle T^{ij} \rangle \;=\; \nabla^j \,\psi^{(0)} \,\langle O \rangle \;+\; F^{(0)ij} \,\langle J_i \rangle \,. \tag{3.135}$$

where i,j label the boundary spacetime directions. This Ward identity suggests a route to holographic momentum relaxation ($\nabla_i \langle T^{ij} \rangle \neq 0$) by turning on spatially dependent source terms for the scalar ψ, meaning $\nabla^j \,\psi^{(0)} \neq 0$.
This is not surprising since spatially dependent sources have been utilised to construct holographic lattices which exhibit finite DC conductivity [150, 151, 158–162]. In all those scenarios the stress tensor becomes dependent on the spatial coordinates x^i and the correspondent Einstein equations turn out to be PDEs whose numerical integration is not trivial. Anyway the scalar(s) ψ enter into the stress tensor just via first derivatives:

$$T^{ij} \;\sim\; \nabla^i \,\psi \,\nabla^j \,\psi \tag{3.136}$$

showing the presence of a scalar field shift symmetry. Therefore the idea is to exploit such a symmetry noticing that if we turn on sources for the scalars which are linear in the boundary coordinates:

$$\psi^{(0)i} \;\sim\; \beta_i \, x^i \tag{3.137}$$

the stress tensor is blind to the boundary coordinates and one can find homogenous bulk solutions for the system. In general though, such a configuration will not be isotropic. To render it isotropic we need to introduce a totals of $\tilde{d}$ scalar fields ψ^I where $\tilde{d}$ is the number of spatial dimensions of the boundary. We can then arrange their sources such that the bulk solution is also isotropic.
In particular let's consider the following action:

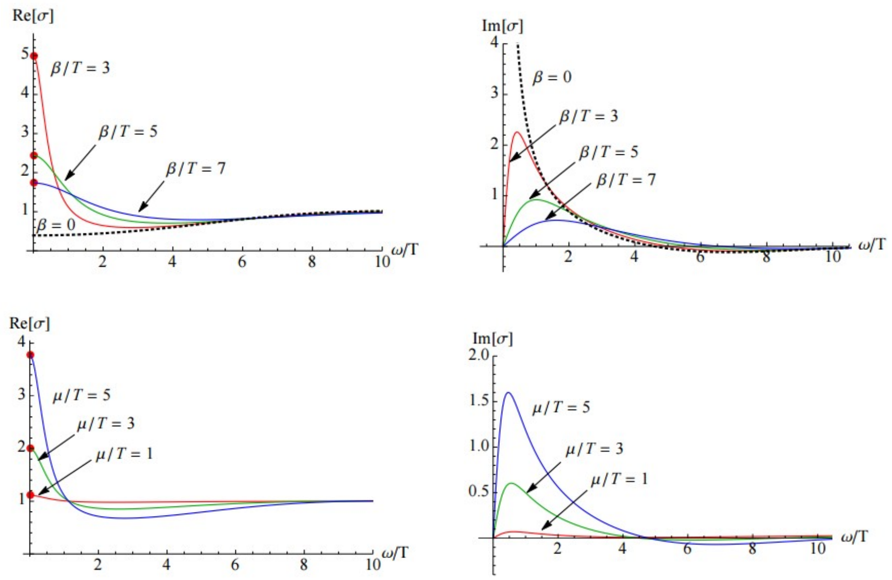
$$\mathcal{S} \;=\; \int d^{d+1}x \,\sqrt{-g}\,\left[R \,-\, 2\,\Lambda \,-\, \frac{1}{2}\sum_{I=1}^{\tilde{d}} (\partial\psi^I)^2 \,-\, \frac{1}{4}\,F^2 \right] \tag{3.138}$$

where $\Lambda = -d(d-1)/(2L^2)$ is the d-dimensional cosmological constant.
The model admit a bulk solution which reads:

$$\begin{aligned}
ds^2 \;&=\; -\,f(r)\,dt^2 \,+\, \frac{dr^2}{f(r)} \,+\, r^2\,\delta^a_b\,dx^a\,dx^b\,, \qquad A \;=\; A_t(r)\,dt\,, \qquad \psi_I \;=\; \beta_I\,x^I\,,\\
f(r) \;&=\; r^2 \,-\, \frac{\beta^2}{2\,(d-2)} \,-\, \frac{m_0}{r^{d-2}} \,+\, \frac{(d-2)\,\mu^2}{2\,(d-1)}\,\frac{r_h^{2(d-2)}}{r^{2(d-2)}}\,,\\
A_t(r) \;&=\; \mu\left(1 \,-\, \frac{r_h^{d-2}}{r^{d-2}}\right)\,.
\end{aligned} \tag{3.139}$$

where for simplicity we have fixed $\beta_I = \beta$ in order to retain isotropy and the BH mass can be identified with the usual condition $f(r_h) = 0$. These solutions have been first investigated in [164] and in the anisotropic case in [165].
We can again perturb this background in order to compute the conductivity and, after some easy manipulations (see [163]), we are left with the following two equations:

$$\begin{aligned}
r^{3-d}\,(r^{d-3}\,f\,a_x')' \,+\, \frac{\omega^2}{f}\,a_x \;&=\; (d-2)^2\,\mu^2\,\frac{r_h^{2(d-2)}}{r^{2(d-1)}}\,a_x \,+\, i\,(d-2)\,\mu\,\frac{r_h^{d-2}}{r^{2(d-1)}}\,\phi\,,\\
r^{d-1}\,\left(r^{1-d}\,f\,\phi'\right)' \,+\, \frac{\omega^2}{f}\,\phi \;&=\; -\,i\,(d-2)\,\beta^2\,\mu\,\frac{r_h^{d-2}}{r^2}\,a_x \,+\, \frac{\beta^2}{r^2}\,\phi\,.
\end{aligned} \tag{3.140}$$



(a) Re σ (b) Im σ

Figure 3.6: AC electric conductivity $\sigma(\omega)$ in presence of a massive gravity term β coming from the scalar model with profile $\psi^I = \beta x^I$. Plots are taken from [154]; the β parameter is the graviton mass but it does not correspond to the β parameter presented in the main text. **Top:** $\mu/T = 6$; **Bottom:** $\beta/T = 3$.

where a_x is the perturbation of the vector field A_μ and ϕ is the metric perturbation. The results are shown for various value of the mass β in figures 3.6 taken from [154].
Also in this case, with appropriate methods, we can derive the zero frequency value of the conductivity, which reads:

$$\sigma_{DC} \,=\, r_h^{d-3}\,\left(1\,+\,(d-2)^2\,\frac{\mu^2}{\beta^2}\right)\,. \tag{3.141}$$

where d is the number of boundary spacetime dimensions ($d = 3$ in the previous examples). From this formula, comparing with the massive gravity scenario, it is clear that such scalars provide a mass for the graviton:

$$m^2 \sim \beta^2 \tag{3.142}$$

which is proportional to their vev.
Despite at the beginning this model was supposed to be indipendent of the massive gravity results, and even incompatible [163] , it has been realized later on that there is a clear link between these two models (see next chapter) which are indeed ”equivalent”. These scalars represent nothing else than the additional degrees of freedom which breaking diffeomorphism invariance produces; in other words they are the Stückelberg fields themselves.
Being the simplest model available to introduce momentum dissipation into the holographic scenario, it surely contains some shortcomings like the apparent insensitivity of the DC conductivity (at fixed chemical potential μ) to the temperature T. This represents a big obstacle to try to classify the dual CFTs in terms of metals and insulators and to face the phenomenology of strongly correlated materials.
One way to overcome this issue is to complicate the scalar sector introducing new terms allowed

by shift symmetry. Inspired by the DBI action which arises in the tensionless limit of extended objects such as thin branes, in [166] new terms proportional to the square root of the scalar kinetik term have been introduced:

$$\mathcal{L} \sim -c_{1/2} \sum_I \sqrt{(\partial\psi_I)^2} \tag{3.143}$$

The bulk solution can still be retained to be homogeneous and isotropic but the transport and thermodynamical properties of the system get modified. In particular the DC conductivity now becomes:

$$\sigma_{DC} = r_h^{d-3} \left(1 + \frac{(d-2)^2 \mu^2}{\tilde{\beta} + \tilde{\alpha}/r_h}\right) \tag{3.144}$$

where the $\tilde{\beta}$ parameter is due to the linear term for the scalar $(\partial\psi^I)^2$ as in [163] while the new α term is entirely due to the introduction of the new square root terms $\sqrt{(\partial\psi_I)^2}$ and it is proportional indeed to its coefficient $\tilde{\alpha} \sim c_{1/2}$.
The first thing to notice is that now in $d = 3$ the DC conductivity, thanks to the $\tilde{\alpha}$ term, becomes temperature dependent and can be suitable to phenomenological discussions. It was indeed claimed in [166] that the square root term is exactly the one associated to a linear growth of the resistivity $\rho_{DC} = 1/\sigma_{DC}$ at small temperatures:

$$\rho_{DC} \sim T \tag{3.145}$$

which hints to what observed experimentally in the so-called *strange metals* where the resistivity does not scale quadratically with the temperature $\rho \sim T^2$, as imposed by the *Fermi Liquid theory*, but it does linearly.
Moreover, adding this new term, the model defined through the scalars seems to give a DC formula which, upon the correct identifications, corresponds exactly to the one extracted by dRGT theory 3.125. It has been showed in [166] that the two theories are indeed equivalent at linear level, giving the same DC transport coefficients, but they still differ when finite frequency and momentum are considered, namely at non linear level. We will get back to the interpetation and the explanation of this issue.

DC quantities

The particulary simple setup provided by the massive gravity effective description allows a strong analytical control as well. The DC, *i.e.* zero frequency, part of the conductivity can be indeed derived analytically using various techniques inspired by the *membrane paradigm* [167]. It has been first derived in [157] and then refined with more efficient methods in [168,169]. Since the DC conductivity will represent one of the main character of the next chapter, containing the original results of this thesis, we will enter in the details of both the available methods following their historical order. We will focus only on the electric conductivity although the same methods are suitable to extract the full thermo-electric conductivities and their scalings also in the presence of a magnetic background field and more complicated situations [168–179].
Let's go back to the fluctuations equations for the dRGT model 3.123; we can rewrite them as:

$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} a_x \\ \tilde{g}_{rx} \end{pmatrix} + \frac{\omega^2}{f} \begin{pmatrix} a_x \\ \tilde{g}_{rx} \end{pmatrix} = \mathcal{M} \begin{pmatrix} a_x \\ \tilde{g}_{rx} \end{pmatrix} \tag{3.146}$$

where $L_{1,2}$ are linear differential operators and $\mathcal{M}$ is the "mass matrix":

$$\mathcal{M} = \begin{pmatrix} \rho^2 r^2 & m^2 r^2/i\omega \\ \rho\, i\omega & m^2 \end{pmatrix} \tag{3.147}$$

where m^2 is importantly a function of the radial coordinate r.
The key point is that $det(\mathcal{M}) = 0$! This means that even at finite charge density $\rho \neq 0$ there is a particular combinations of the fields which does not evolve radially in the limit of $\omega \to 0$. Notice that the fields which diagonalize the mass matrix do not diagonalize the differential operators matrix; nevertheless the existence of such a massless mode will be enough to compute the DC conductivity analytically.
The eigenmodes read:

$$\lambda_1 \,=\, \left(1\,+\,\frac{\rho^2\,r^2}{m^2}\right)^{-1}\,\left[a_x\,-\,\frac{\rho\,r^2}{i\,\omega}\,\tilde{g}_{rx}\right]\,,$$
$$\lambda_2 \,=\, \left(1\,+\,\frac{\rho^2\,r^2}{m^2}\right)^{-1}\,\left[\frac{\rho}{m^2}\,a_x\,+\,\frac{\tilde{g}_{rx}}{i\,\omega}\right]\,. \tag{3.148}$$

where λ_1 corresponds to the vanishing eigenvalue of the matrix $\mathcal{M}$.
Since λ_1 corresponds in the UV, at $r = 0$, to the vector field perturbation a_x, we can therefore compute the conductivity as:

$$\sigma(\omega) \,=\, \frac{\lambda_1'}{i\,\omega\,\lambda_1}|_{UV} \tag{3.149}$$

The equation for such a mode reads[23]:

$$\left[f\,\left(1\,+\,\frac{\rho^2\,r^2}{m^2}\right)\,\lambda_1'\,-\,\frac{\rho\,f\,r^4}{m^2}\,\left(\frac{m^2}{r^2}\right)'\,\lambda_2\right]'\,+\,\frac{\omega^2}{f}\,\left(1\,+\,\frac{\rho^2\,r^2}{m^2}\right)\,\lambda_1\,=\,0 \tag{3.150}$$

which makes evident that in the DC limit $\omega \to 0$ there is a conserved quantity Π:

$$\Pi\,=\,f\,\left(1\,+\,\frac{\rho^2\,r^2}{m^2}\right)\,\lambda_1'\,-\,\frac{\rho\,f\,r^4}{m^2}\,\left(\frac{m^2}{r^2}\right)'\,\lambda_2 \tag{3.151}$$

We can therefore define a DC membrane conductivity associated to each radial slice r:

$$\sigma(r)\,=\,\lim_{\omega\to 0}\,\frac{\Pi}{i\,\omega\,\lambda_1}|_r \tag{3.152}$$

At $r = 0$ this expression coincides with the electric conductivity and because this quantity does not evolve radially[24] we can compute it at whichever radial position and in particular at the horizon $r = r_h$. This can be easily achieved remembering that both fields obey ingoing boundary conditions $a_x \sim \tilde{g}_{rx} \sim f(r)^{-i\omega/4\pi T}$ at the horizon. With this in mind, we find out that, at the horizon position, the λ_2 term in 3.151 vanishes while the λ_1' term survives. All in all we end up with an analytic expression for the DC conductivity:

$$\sigma_{DC}\,=\,1\,+\,\frac{\rho^2\,r_h^2}{m^2(r_h)} \tag{3.153}$$

which has already introduced before in 3.125.
Whenever the second term dominates it is expected to give rise to a standard Drude form; in contrast, when the second term fails to be parametrically larger than the first, we have an

[23] Since the λ fields do not diagonalize the derivative terms, the correspondent equations are not decoupled.
[24] There are some caveats to actually prove it in details that such statement is true; see [157].

incoherent metal.
We can now generalize such a model and consider a way more general setup where we can still compute with analytical methods the DC conductivity following the work of [168]. We now consider the more generic action given by:

$$\mathcal{S} \,=\, \int d^4x\,\sqrt{-g}\,\left\{R\,-\,\frac{1}{2}\,\left[(\partial\phi)^2\,+\,\Phi_1(\phi)\,(\partial\chi_1)^2\,+\,\Phi_2(\phi)\,(\partial\chi_2)^2\right]\,-\,V(\phi)\,-\,\frac{Z(\phi)}{4}\,F^2\right\} \tag{3.154}$$

where χ_I are the scalars field we were considering before (we allow them to be different) and ϕ is an additional scalar, *i.e.* the dilaton. We assume that the model admits a unit radius AdS_4 solution with $\phi = 0$ (and $V(0) = -6$ along with $Z(0) = 1$).
The solutions that we shall consider all lie within the ansatz:

$$\begin{aligned} ds^2 \,&=\, -\,U\,dt^2\,+\,U^{-1}\,dr^2\,+\,e^{2\,V_1}\,dx^2\,+\,e^{2\,V_2}\,dy^2\,,\\ A\,&=\,a\,dt\,,\qquad \chi_1\,=\,k_1\,x\,,\qquad \chi_2\,=\,k_2\,y\,. \end{aligned} \tag{3.155}$$

where U, V_I, a and ϕ are only functions of r. In general the solutions are anisotropic, $V_1 \neq V_2$ but we can enforce isotropy choosing $k_1 = k_2$ and $\Phi_1 = \Phi_2$.
We will furthermore assume that there exists an event horizon at $r = r_h$ where the functions have the following behaviour:

$$\begin{aligned} U\,&\sim\,4\,\pi\,T\,(r\,-\,r_h)\,+\,\dots \qquad V_I\,=\,V_h\,+\,\dots\\ a\,&\sim\,a_h\,(r\,-\,r_h)\,+\dots \qquad \phi\,\sim\,\phi_h\,+\,\dots \end{aligned} \tag{3.156}$$

where T is the temperature of the BH background.
On the other side at the UV AdS_4 position $r \to \infty$ we assume that:

$$\begin{aligned} U\,&\sim\,r^2\,+\,\dots \qquad e^{V_I}\,=\,r^2\,+\,\dots\\ a\,&\sim\,\mu\,-\,\frac{\rho}{r}\,+\dots \qquad \phi\,\sim\,\lambda\,r^{\Delta-3}\,+\,\dots \end{aligned} \tag{3.157}$$

with $\Delta < 3$.
The current density $J^a = (J^t, J^x, J^y)$ in the dual field theory has the form:

$$J^a\,=\,\sqrt{-g}\,Z(\phi)\,F^{ar} \tag{3.158}$$

where the right side is evaluated at the boundary $r \to \infty$.
The only non trivial component of the maxwell equation within our ansatz reads:

$$\text{maxwell equation:}\;\; \sqrt{-g}\,\nabla_r\left(\sqrt{-g}\,Z(\phi)\,F^{rt}\right)\,=\,0\,. \tag{3.159}$$

which looks like indeed as the conservation of the charge density:

$$\rho\,=\,e^{V_1\,+\,V_2}\,Z(\phi)\,a' \tag{3.160}$$

In order to compute the electric conductivity we have to introduce an external electric field through the small perturbations:

$$\begin{aligned} a_x\,&=\,-E\,t\,+\,\delta a_x(r)\,,\\ g_{tx}\,&=\,\delta g_{tx}(r)\,,\\ g_{rx}\,&=\,e^{2\,V_1}\,\delta g_{rx}(r)\,,\\ \chi_1\,&=\,k_1\,x\,+\,\delta\chi_1(r)\,. \end{aligned} \tag{3.161}$$

Again the only non trivial component of the Maxwell equation at linear order in these fluctuations is $\nabla_r\,(\sqrt{-g}\,Z(\phi)\,F^{rx})$; therefore we can deduce that the following quantity:

$$J \,=\, -\,e^{V_2\,-\,V_1}\,Z(\phi)\,U\,\delta a_x' \,-\, \rho\,e^{-\,2\,V_1}\,\delta g_{tx} \tag{3.162}$$

is radially conserved and it corresponds at the boundary with the electric current J_x in response to the electric field E.
We then consider the linearized Einstein equations. There is just one of them relevant for the present discussion[25] and it can be algebrically solved giving the constraint:

$$\delta g_{rx} \,=\, \frac{E\,\rho\,e^{-\,V_1,-\,V_2}}{k_1^2\,\Phi_1(\phi)\,U} \,+\, \frac{\delta\chi_1'}{k_1} \tag{3.163}$$

Under some assumptions (see [168]) we can derive the regular behaviour of the various fields at the horizon getting:

$$\begin{aligned}\delta a_x \,&\sim\, -\,\frac{E}{4\,\pi\,T}\,\ln(r\,-\,r_h)\,+\,\ldots\\ \delta g_{tx} \,&\sim\, -\,\frac{E\,\rho\,e^{V_1\,-\,V_2}}{k_1^2\,\Phi_1(\phi)}\,+\,\ldots\end{aligned} \tag{3.164}$$

At this stage we can already define the electric conductivity as $\sigma \,=\, J/E$ and computing the following quantity at the horizon we are left with the generic DC formula:

$$\sigma_{DC} \,=\, \left[\frac{Z(\phi)\,s}{4\,\pi\,e^{2\,V_1}} \,+\, \frac{4\,\pi\,\rho^2}{k_1^2\,\Phi_1(\phi)\,s}\right]_{r=r_h}. \tag{3.165}$$

where $s = 4\pi e^{V_1+V_2}$ is the entropy density of the unperturbed black hole.
Note that the scalar model of [163] we consider before can be re-obtained fixing:

$$\phi \,=\, 0\,, \qquad \Phi_i \,=\, 1\,, \qquad Z \,=\, 1\,, \qquad V \,=\, -6\,. \tag{3.166}$$

along with $k_1 = k_2 = k$ obtaing the well known:

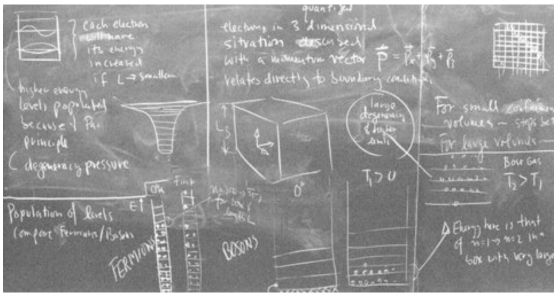
$$\sigma \,=\, 1 \,+\, \frac{\mu^2}{k^2} \tag{3.167}$$

Nowadays, since the first paper [152] in 2013, there has been a huge amount of activity, efforts and results in the present field which of course we are not able to cover in this small room. We refer to the bilbiography for more details.
We will enter more advanced and hot topics in the next chapter where we will summarize the original results presented in the published papers by the author and collaborators.

[25]Nevertheless of course the full system of equations is consistent.

Part II

Results



However beautiful the strategy, you
should occasionally look at the results.

Winston Churchill

In this section we summarize the major original results generated in the papers constituiting the main body of this Ph.D thesis. We will be rather minimal and schematic. For details on the computations and the procedures we refer directly to the papers themselves.

After the realization that Massive Gravity could represent an effective tool to introduce momentum dissipation in the framework of holography [152] lots of efforts and works have been done in the direction of understanding and improving such a tool under various aspects:

- Understanding the tool [153, 155, 173, 174, 180–183];
- Enlarging the tool with additional ingredients (anisotropies, dilaton) or more generic models [184–187];
- Exploiting the Stuckelberg mechanism to simplify the model to GR + scalar sector [163, 166, 188];
- Identifying efficient methods to extract the thermoelectric transport coefficients both numerically and analytically [154, 156, 157, 168, 176, 179, 189–194];
- Studying the stability and the consistency requirements for such a MG theories living in AdS spacetime [195–197];
- Applying MG to various phenomenological applications [198–207];
- Establishing and studying the possible existence of universal bounds on physical quantities in these frameworks [208–217].

In the following we will review the major contributions of our work to the subject.

3.5 Momentum dissipation & Massive Gravity: towards a more generic framework

The idea of exploring more generic Massive Gravity models in the context of the AdS/CMT applications has been initiated in [218] and done later in a more sistematic way in [219]. We are going to present the papers with a historical perspective of how ideas came out at that time.

Towards more generic MG models and their dual

Massive gravity model can be thought as a family of infra-red deformations of General Relativity described generically by the introduction of a new "mass term" of the form:

$$\sim\, m^2\, V\left(P_{\mu\nu}\, g^{\mu\nu}\right) \tag{3.168}$$

where $P_{\mu\nu}$ is an external fixed metric of convenience.
In order to break just the translational invariance of the dual CFT the projector P has to be choosen non null only in the spatial coordinates $P_{ij} = \delta_{ij}$ providing a Lorentz violating version of massive gravity (LVMG).
The new term (3.168) introduces extra degrees of freedom into the spectrum of the theory which it turns out convenient to explicitate via covariantizing the theory and restoring the original gauge symmetry. The minimal amount of fields which has to be added is a collection of scalar fields ϕ^I transforming under an internal Euclidean group of translations and rotations in field-space.
All in all, the most generic action[26] for the MG theory (in 3+1 dimensions) can be written down as:

$$\mathcal{S}_{MG} \,=\, M_P^2 \int d^4x,\, \sqrt{-g}\,\left[\frac{R}{2} \,+\, \frac{3}{L^2} \,-\, m^2\, V\left(X\right)\right] \tag{3.169}$$

where $X = \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I$.
This action admits a simple solution where the scalar field get a VEV linear in the spatial coordinates:

$$\phi^I \,=\, \alpha\, \delta^I_i\, x^i \tag{3.170}$$

With this choice the original $P_{\mu\nu}$ projects just on the spatial coordinates and assumes the form $\sim \partial_\mu \phi^I \partial_\nu \phi^I$.
The crucial point is that most of the literature to date unnecessarily restrict to a very narrow families of potential V. In particular most of the work has been focused on dRGT MG [152] and on the choices $V(X) \sim X, \sqrt{X}$ [163, 166]. Nevertheless, in the Lorentz violating case the set of consistent choices is definitely bigger and unexplored.
We therefore consider action (3.169) and we study the consistency of the model and the phenomenological implications which a completely generic potential $V(X)$ can produce in the transport properties of the dual field theory.
In order to avoid ghosty excitations and other pathological instabilities the potential V has to satisfy some requirements which can be derived performing an analysis of the fluctuations in the decoupling limit. Perturbing the Goldstone fields $\phi^I = \bar{\phi}^I + \delta\phi^I$ and expanding the corresponding action up to 2nd order we get:

$$V(\bar{X})\, \partial_\mu \delta\phi^i \partial^\mu \delta\phi^i \,+\, \bar{X}\, V''(\bar{X})\, \left(\partial_i \delta\phi^i\right)^2 \tag{3.171}$$

Absence of ghosts, then, leads to monotonic potentials:

$$V'(\bar{X}) \,>\, 0 \tag{3.172}$$

The local (sound) speed of longitudinal phonons is:

$$c_S^2 \,=\, 1 \,+\, \frac{\bar{X}\, V''(\bar{X})}{V'(\bar{X})} \tag{3.173}$$

[26] This statement is not totally true. We will see soon that an additional term can be introduced [219] and can assume an important role for various aspects.

which is safest to require it to be everywhere positive to guarantee the absence of gradient instabilities[27].
No further constraints arise from the vector and tensor sectors nor at nonlinear level in the scalars, which is not surprising since a substantial advantage of the Lorentz non-invariant mass terms is that they can be free from the Boulware-Deser ghost. The requirement of having an asymptotical AdS spacetime implies a further constraint on the potential which has to vanish at the boundary $u = 0$. This gets translated into the condition:

$$\lim_{X \to 0} V(X) \,=\, 0 \tag{3.174}$$

All in all, an economic and safe way to satisfy all constraints is to assume that $V(X)$ is a positive definite polynomial function of X. We will therefore consider the benchmark example:

$$V(X) \,=\, X \,+\, \beta\, X^N\,, \qquad \beta > 0\,, \quad N \,>\, 1 \tag{3.175}$$

which is rich enough to give rise to new and interesting phenomenological results. It is somehow very generic because it can be thought as the Taylor expansion of any function $V(X)$ satisfying at non linear level the requirements above.
In any asymptotically AdS solution $\bar{X} \,=\, u^2\, \alpha^2$ and close to the AdS boundary ($u = 0$) the Goldstone gradient $\bar{X}$ vanishes. This implies that the mass term for metric modes is also vanishing. So this is a weaker form of massive gravity than is usually discussed in cosmology - the Compton wavelength is at most of order the curvature radius of the spacetime. In the CFT interpretation, the stress tensor $T_{\mu\nu}$ does not develop an anomalous dimension. Still this is enough to lead to momentum relaxation in the CFT.
Additionally, noticing that X is getting smaller at the boundary, it is clear that at large temperature T (u small) the physics will be dominated by the smallest power of X appearing in the potential V. On the contrary at small temperature T (u large) the higher powers, *i.e.* non linear new corrections, get important and can abruptly affect the physics of the system.
In CFT language, the scalars ϕ^I correspond multiplet of operators $\mathcal{O}^I$ with internal shift symmetries and which are somehow related with phonons and impurities. A consistent interpretation seems to be that the strength α of the linear vevs $\langle \mathcal{O}^I \rangle = \alpha \delta^I_i x^i$ is the density of homogeneously-distributed impurities. The fluctuations $\delta\mathcal{O}^I$ around this distribution are CFT operators that create "phonon" excitations[28].
In order to study the transport properties of the dual CFT, we need to add one ingredient, the charge carriers. So we assume that the CFT also contains a conserved current operator J_μ. This is implemented in the gravity dual by adding to the model a Maxwell field,

$$\mathcal{S} \,=\, \mathcal{S}_{MG} \,-\, \frac{M_p^2}{4} \int d^4x\, \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu} \tag{3.176}$$

The dual of the CFT ground state at finite charge density ρ, temperature T and impurity strength α is a planar black brane (BB) with:

$$ds^2 \,=\, \frac{L^2}{u^2} \left(\frac{du^2}{f(u)^2} \,-\, f(u)\, dt^2 \,+\, dx^2 \,+\, dy^2 \right)$$

[27] This represents a local speed of sound in the bulk which does not coincide, at least a priori, with the speed of sound of any CFT excitations.

[28] This point is quite subtle because from the point of view of the CFT side the breaking of translational symmetry is definitely explicit and not spontaneous. There is anyway a limit in which these excitations can be thought as "pseudo-goldstone" á la pions. We avoid a deep discussion on this point at this stage.

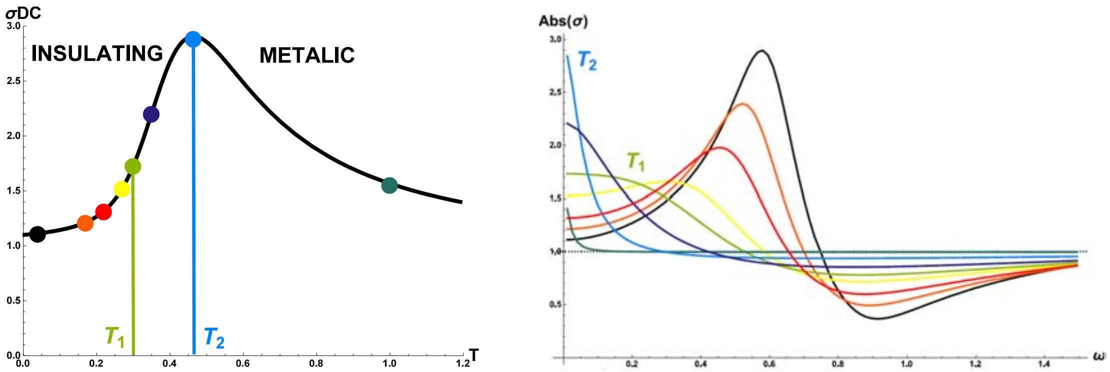


Figure 3.7: Electric conductivity. **Left:** DC conductivity in function of temperature and MIT crossover. **Right:** Optical conductivity $\sigma(\omega)$ at various temperatures $T = 1, 0.46(T_2), 0.35, 0.3(T_1), 0.27, 0.22, 0.17, 0.04$ and *Polaron* formation.

$$
\begin{aligned}
f(u) &= u^3 \int_u^{u_h} dy \left[\frac{3}{y^4} - \frac{\rho^2}{2\,L^2} - \frac{m^2\,L^2}{y^4}\, V\left(\frac{\alpha^2\,y^2}{L^2} \right) \right] \\
A_t(u) &= \mu - \rho\,u\,, \qquad \phi^I = \alpha\,\delta^I_i\,x^i\,. \qquad (3.177)
\end{aligned}
$$

where u_h stands for the BB horizon.
The regularity condition for the gauge field implies $\mu = \rho\,u_h$ and the correspondent Hawking temperature can be found to be $T = -f'(u_h)/(4\pi)$.
We want now to switch on the vector perturbations around the BB above by setting:

$$
A_\mu = \bar{A}_\mu + a_\mu\,, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad \phi^I = \bar{\phi}^I + \delta\,\phi^I \qquad (3.178)
$$

with bars denoting the background solutions.
It is convenient to work with gauge invariant variables defined by:

$$
T_i \equiv u^2\,h_{ti} - \frac{\partial_t \phi^i}{\alpha}\,, \qquad U_i = f(u) \left[h_{ui} - \frac{\partial_u \phi^i}{\alpha\,u^2} \right]\,, \qquad B_i = b_i - \frac{\phi_i}{\alpha}\,. \qquad (3.179)
$$

with $h_{ij} = \frac{1}{u^2}\,\partial_{(i} b_{j)}$.
With this choice of variables the linearized equations can be written as:

$$
\begin{aligned}
&\partial_u \left(f\,\partial_u\,a_i \right) + \left[\frac{\omega^2}{f} - k^2 - 2\,u^2\,\rho^2 \right] a_i - \frac{i\,\rho\,u^2\,(2\,\bar{m}^2 + k^2)}{\omega}\,U_i + \frac{i\,f\,\rho\,k^2}{\omega}\,\partial_u\,B_i = 0\,, \\
&\frac{1}{u^2}\,\partial_u \left[\frac{f\,u^2}{\bar{m}^2}\,\partial_u(\bar{m}^2\,U_i) \right] + \left[\frac{\omega^2}{f} - k^2 - 2\,\bar{m}^2 \right] U_i + 2\,i\,\rho\,\omega\,a_i - \frac{f'\,k^2}{u^2}\,B_i = 0 \\
&k \left\{ u^2\,\partial_u \left(\frac{f}{u^2} \partial_u B_i \right) + \left[\frac{\omega^2}{f} - k^2 - 2\,\bar{m}^2 \right] B_i + 2\,\frac{\bar{m}'}{\bar{m}}\,U_i \right\} = 0\,. \qquad (3.180)
\end{aligned}
$$

where we introduced $\bar{m}^2(u) = \alpha^2 m^2 V'(\alpha^2 u^2)$, namely the radial dependent effective grativon mass.
From the numerical integration of these equations we can extract all the transport properties of the dual CFT and in particular the DC and AC electric conductivity and the correspondent QNM spectrum using the dictionary we discussed in the previous sections.

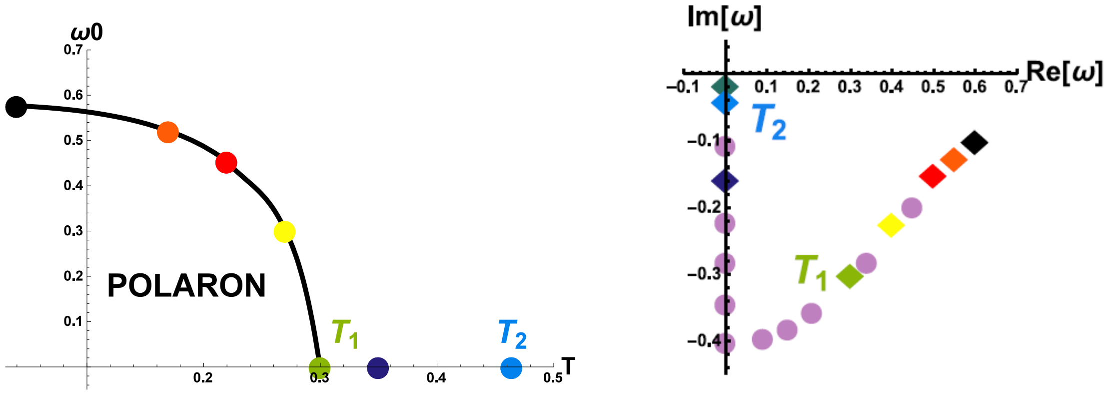


Figure 3.8: Excitations of the system. **Left:** Polaron formation. **Right:** QNM spectrum of the model. At large T, the QNM separates from the real axis with decreasing T, until it collides with the next QNM (near T_1) and forms a pair of conjugated poles with positive and negative real parts – the polaron particle/anti-particle poles. (Similar QNM collisions have been observed e.g. in [155]. In our case, it is crucial that after collision the QNM approaches the real axis.)

In the homogeneous limit $k \to 0$ we can compute the DC, $\omega = 0$, value of the electric conductivity in terms of horizon data analytically and we get:

$$\sigma_{DC} \,=\, 1 \,+\, \frac{\rho^2\, u_h^2}{\bar{m}^2(u_h)} \,=\, 1 \,+\, \frac{\rho^2\, u_h^2}{\alpha^2\, m^2\, V'(\alpha^2\, u_h^2)} \tag{3.181}$$

which generalizes the previous results in the literature.
The previously considered models [152, 163, 166] allowed just for metallic behaviour of the type $d\sigma_{DC}/dT < 0$. Within this class of more generical potential we can not only provide a dual for insulating states[29] of the type $d\sigma_{DC}/dT > 0$ but also transitions between such states and metallic phases (MIT) as in figure 3.7. From the analytical formula 3.181 we can extract the exact condition ($d\sigma/dT = 0$) for such a crossover to happen which takes the form:

$$\bar{X}\, V''(\bar{X}) \,=\, V'(\bar{X}) \tag{3.182}$$

It is easy to show that for the benchmark model (3.175) this condition translates in the definition of a critical temperature $T_2 \sim (N-2)^\gamma$ showing that one needs an high exponent $N > 2$ to have an MIT. Models with $N < 2$ tend to give metals and incoherent metals. It is anyway clear and important that generic massive gravity models seem to be able to reproduce way more than single metallic behaviour and can encorporate also strongly coupled insulating systems.
Along with the depletion of the DC conductivity at low temperatures further features appear in the optical part of the conductivity (see fig.3.7) with the appearance of a localized excitation in the mid-infrared region. This suggests the formation of a localized and propagating bound state which one is tempted to identify as a phonon-electron state known as *polaron*. Such a bound state is created at a temperature T_1 which is generically different from the MIT temperature T_2 and always smaller. It is not clear therefore if there is a close correlation between the two effects but they turn out to be certainly related. Moreover, the formation of this ”peak” is favoured

[29]Strictly speaking the proper definition for an insulator would be $\sigma(T = 0) \approx 0$. This is not the case in this kind of models because of the first unitary and temperature indipendent piece of the DC conductivity. We will therefore assume a milder definition $d\sigma_{DC}/dT > 0$. See further discussions in the following.

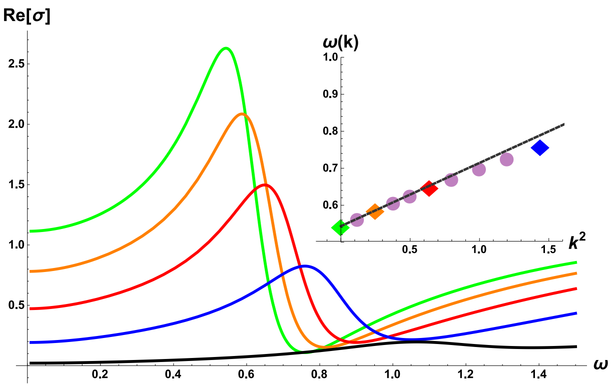


Figure 3.9: Motion of the polaron peak with wavenumber. $T = 0.04$, $k = 0,\ 0.5,\ 0.8, 1.2,\ 2$. Inset: the extracted dispersion relation.

by the amount of non linearities (which can be dialed increasing the *impurity strength* α or the non-linear power in the potential N) and by lowering the temperature T. Note that despite the formation of the *polaron* appears of the mean-field type (see fig.3.8), usually related to 2nd order continous phase transitions, here we are not in presence of any spontaneous symmetry breaking mechanism. The phenomenon is more similar to a dynamical crossover which is generated by the non linearities of the massive gravity sector.
The features of the system can be efficiently collected into its QNM spectrum (see fig.3.8). The model undergoes a transition between an overdamped regime with a clear and separated drude peak, appearing as a pole on the lower imaginary axes $\omega = -i\,\Gamma$ to an underdamped regime with a pole with both real and imaginary part $\omega = -i\,\Gamma + \nu$. Eventually such a pole, lowering the temperature, can build up a bigger and bigger real part and approach the real axes, providing the peaked feature in the optical conductivity.
The emerging bound state represents a real propagating degree of freedom, whose real part of the frequency ν stands for its effective energy/mass and the imaginary part as its decay width. One can follow such a pole and picture its dispersion relation (see fig.3.9) which turns out to be $\omega \sim k^2$ at low momentum as expected for a non lorentz invariant propagating mode.
In summary, we have seen that a simple nonlinear extension of holographic massive gravity captures interesting features of correlated materials such as polaron-localization and a (phonon-electron) interaction-driven Metal-Insulator transition (MIT). In other words, generalizing the massive gravity dual through a generic potential, introduces an effective graviton mass whose radial dependence can be arbitrary, within the consistency regime of the theory. Such a dependence gets translated in the field theory picture as the temperature dependence of the relaxation time such that the correspondent phenomenology becomes very rich and accounts also for insulating states and MIT transitions.

Phases of holographic massive gravity

In [219] we performed a systematic analysis of holographic massive gravity theories (HMG) which might store interesting condensed matter content. As already underlined, the set of consistent and healthy MG theories is way larger than the famous dRGT case firstly considered in the holographic literature. Many results are indeed not generic and this restricted view could somehow lead to misguided implications for the CM dual.
Realizing the existence of a broader class of consistent EFTs provides already an important

distinction between *solid-fluid* type massive gravity theories which can be separated by:

i. The nature of the unbroken spacetime symmetries.

ii. The elastic response of the dual system.

The first point has origins in the framework of EFT for fluids and solids which in modern language are written down through a set of phonon scalars ϕ^I (in $2+1$ dimensions, $I = 1, 2$) that enjoy internal shift and rotation symmetries for homogeneous and isotropic materials [143–146]. The internal symmetry group for solids is the two-dimensional Euclidean group of translations and rotations. For fluids, the internal group is much bigger and includes also volume preserving diffeomorphisms (VPDiffs). The scalars acquire an expectation value $\langle\phi^I\rangle = \delta^I_i\, x^i$ and break the product of the (space transformations) $\times$ (internal transformations) to the diagonal subgroup. For fluids, the preserved symmetry includes a volume preserving diagonal subgroup.
The effective Lagrangian at the lowest order in derivatives in the two cases can be written as:

$$\mathcal{L}^{solids} \,=\, V_s(X, Z)\,, \qquad \mathcal{L}^{fluids} \,=\, V_f(Z) \tag{3.183}$$

where $X = Tr[\mathcal{I}^{IJ}]$, $Z = Det[\mathcal{I}^{IJ}]$ and $\mathcal{I}^{IJ} = \partial_\mu\phi^I\partial^\mu\phi^J$.
The functions $V_{s,f}$ encode the linear and nonlinear properties of the solid and fluid, and they are free functions subject to mild consistency constraints. It is easy to realize that gauging these theories leads to graviton mass terms around the solution with non trivial vevs for the Goldstones ϕ^I. The simplest way to see this is to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and to go to the unitary gauge where the scalar fields are fixed to be equal to their background configuration. The above solid/fluid Lagrangians then become nonlinear potential terms for the metric

$$V_s\left(Tr\left[g^{ij}\right], Det\left[g^{ij}\right]\right)\,, \qquad V_f\left(Det\left[g^{ij}\right]\right) \tag{3.184}$$

where g^{ij} denotes the spatial part of the inverse metric.
The form of this action is dictated by requiring it to be invariant under certain subset of the diffeomorphisms, that do not include the spatial diffeomorphisms $x^i = \tilde{x}^i(x^j, t)$. The preserved diffeomorphisms are the ones enjoyed by the potential terms in (3.184). Both for solid and fluid MGs, these include the time-reparametrizations $t \to f(t)$ plus global translations and rotations that force the potential not to depend explicitly on x^i and to contract the spatial indices with Kronecker delta δ_{ij} .For fluid MG the potential is also invariant under the spatial VPDiffs, that forces it to be a function of $Det\left(g^{ij}\right)$ only. Importantly, as we shall see below, the VPDiff symmmetry protects the vanishing of the physical mass parameter of the metric tensor modes.
The main idea and novel contribution of [219] is to consider fully generic HMG theories and to study in details the implications of such a separation in the dual CM picture. Such a description will lead to the definition of a new physical quantity encoded in the Green function of the tensor mode, which we will identify as the *elasticity* of the system.
In the flat space language of [139] an isotropic and homgeneous LV massive gravity theory can be realized as:

$$m_0^2\, h_00^2 \,+\, 2\, m_1^2\, h_{0i}^2 \,-\, m_2^2\, h_{ij}^2 \,+\, m_3^2\, h_{ii}^2 \,-\, 2\, m_4\, h_{00}\, h_{ii} \tag{3.185}$$

and the distinction between solids and fluids, following from symmetry arguments, boils down to:

- Solids: $m_{1,2,3} \neq 0$.

- Fluids: $m_{1,3} \neq 0$ and importantly $m_2 = 0$.

Let us insist that in both cases the spatial translations are broken and both cases lead to a finite DC conductivity in the electric response. The main differences between the two types of theories are i) that the solid phases exhibit propagating transverse phonons - the Goldstone modes of the broken space translations, inhomogenenous in spatial coordinates - whereas the fluids do not; and ii) that the tensor modes are massive/massless for solid/fluid phases respectively.
Once translated such a distinction into the holographic framework we will realize that the presence or not of the tensor mode mass m_2 is indeed linked to the elastic response and in particular to the *shear modulus* of the dual CM system.
For holographic applications in condensed matter theory we are interested in massive gravity theories that allow for asymptotically AdS charged black brane solutions. The action that will be considered is the Einstein-Maxwell action with a negative cosmological constant and a graviton mass term:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(R + \frac{6}{L^2} \right) - \frac{L^2}{4} F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_\phi \right] \tag{3.186}$$

where $\mathcal{L}_\phi$ stands for the massive gravity term written through the Stückelberg fields ϕ^I.
A straightforward way of analyzing the stability and phenomenology of a general theory of massive gravity is to start with the non-covariant form of the massive gravity action with the mass term written it terms of the metric perturbation $h^{\mu\nu}$ and then restore the diffeomorphism invariance by the Stückelberg trick. The most general quadratic mass term that preserves the rotations of the transverse coordinates in the following form:

$$\begin{aligned} \mathcal{L}_\phi(h^{\mu\nu}, r) = \frac{1}{2} \Big(& m_0^2(r)(h^{tt})^2 + 2m_1^2(r)h^{ti}h^{ti} - m_2^2(r)h^{ij}h^{ij} \\ & + m_3^2(r)h^{ii}h^{jj} - 2m_4^2(r)h^{tt}h^{ii} \\ & + m_5^2(r)(h^{rr})^2 + m_6^2(r)h^{tt}h^{rr} \\ & + m_7^2(r)h^{ri}h^{ri} + m_8^2(r)h^{ti}h^{ri} + m_9^2(r)h^{rr}h^{ii} \\ & + m_{10}^2(r)h^{rt}h^{rt} + m_{11}^2(r)h^{rt}h^{ii} + m_{12}^2(r)h^{tt}h^{rt} + m_{13}^2(r)h^{rr}h^{rt} \Big) \, , \end{aligned} \tag{3.187}$$

where all the masses m_i^2 are functions of the radial coordinate r.
Note that the mass parameters defined in (3.187) can be classified with respect to the perturbations that they affect as:

i. scalar: m_0, m_{2-6}, m_{9-13}

ii. vector: m_1, m_7, m_8

iii. tensor: m_2

From the dual perspective the vector and tensor masses will be the respectively relevant for characterizing the electric and the visco-elastic responses.
The costrunction of the model follows exactly in the same way of the previous section and allow us to define the most general bulk action in AdS_4 for our HMG theories:

$$\mathcal{S}_\phi = \int d^4x \sqrt{-g} \, \mathcal{L}_\phi = - \int d^4x \sqrt{-g} \, V(X, Z) \tag{3.188}$$

which generalizes further what we analyzed in [218].
The dRGT theory considered in ref. [152] is a particular case of the latter with the Lagrangian given by:

$$V_{\text{dRGT}} = -\beta_1 \sqrt{\frac{1}{2}\left(X + \sqrt{Z}\right)} - \beta_2 \sqrt{Z}\,. \tag{3.189}$$

Let's first concentrate to the electric response of the system which is determined by the only $m_{1,7,8}$ mass terms. Consistency conditions can be derived in details (see [219]) and force:

$$m_7^2 + +2\,m_1^2 \,=\, 0\,, \qquad m_8^2 \,=\, 0\,. \tag{3.190}$$

reducing the full problem to a single mass term whose radial dependence is anyway arbitrary and linked with the structure of the potential $V(X, Z)$.
Performing the usual vector perturbations in a gauge invariant formalism we are left with the system of differential equations described by:

$$(f a')' + \omega^2 \frac{a}{f} - \frac{2\mu^2 r^2}{r_h^2} a + \frac{2 r^2 \mu}{r_h} \lambda = 0\,, \tag{3.191}$$

$$2m_1^2 \frac{1}{f M^2} \omega^2 \lambda - i\omega m_8^2 \frac{1}{M}\left(\frac{\lambda}{M}\right)' - m_7^2 \left(\frac{f}{M^2}\lambda'\right)' + \frac{r^2 \det \mathcal{P}}{2L^2}\left(\frac{\mu}{r_h} a - \lambda\right) = 0\,, \tag{3.192}$$

with $\det \mathcal{P} = m_8^4 - 8 m_1^2 m_7^2 \neq 0$ and λ being a gauge invariant field whose structure is not relevant for the following.
While the optical conductivity (finite ω) requires the numerical integration of such a system with infalling boundary conditions for the fields at the horizon, using standard techniques one can derive the DC conductivity for the generic model, which turns out to be:

$$\sigma_{DC} \,=\, 1 \,+\, \frac{\mu^2\, L^2}{m^2(r_h)} \tag{3.193}$$

where μ is the chemical potential, r_h the position of the horizon and:

$$m^2(r) \,=\, 2\, m_1^2\, r^2\, M(r)^2 \tag{3.194}$$

the only left graviton mass. Note how all the freedom of the model, namely the choice of the potential $V(X, Z)$ is just incorporated in the function $M(r)$ which can be taken generically of the form:

$$M^2(r) \,=\, L^{-2} \left(\frac{r}{L}\right)^{\nu} \tag{3.195}$$

Translating into the language of the potential $V(X, Z)$ such a function gets the form[30] :

$$M^2(r) \,=\, \frac{1}{r^2}\left(V_X \,+ 2\, r_h^2\, V_Z\right) \tag{3.197}$$

where $V_{X,Z} = \frac{\partial V}{\partial (X,Z)}$.
A phenomenologically interesting question to investigate is the different types of materials that can be described within the framework of holographic massive gravity and, in particular, their

[30]The DC conductity becomes:

$$\sigma_{DC} \,=\, 1 \,+\, \frac{\mu^2}{V_X(r_h) \,+ 2\, r^2\, V_Z(r_h)} \tag{3.196}$$

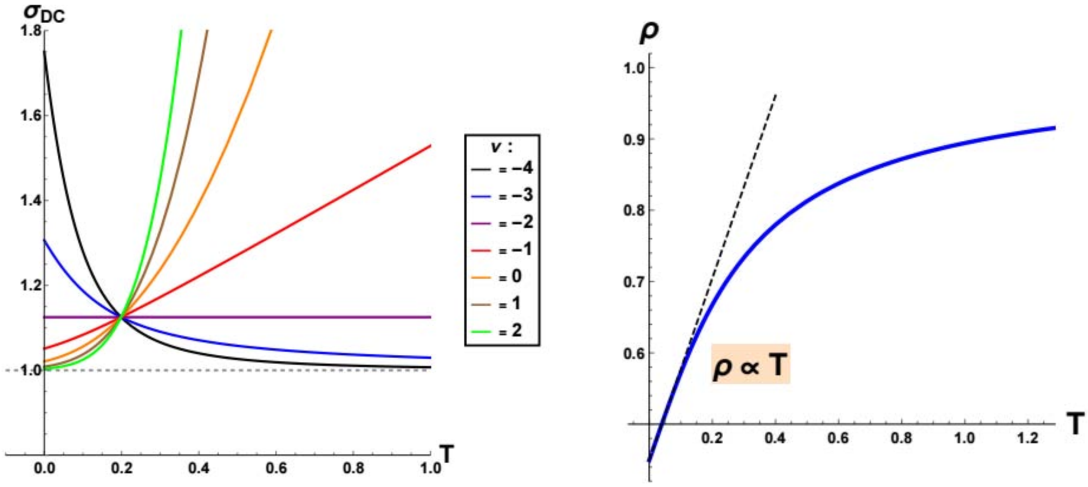


Figure 3.10: DC conductivity for a generic function $M^2(r) \sim r^\nu$. **Left:** DC conductivity for various values of ν showing the transition between a metallic to an insulating phase. **Right:** Linear in T resistivity (at low T) for the choice $\nu = -3$.

ability to conduct an electric current. The distinction between different classes of materials is well captured by the temperature dependence of their DC electric conductivity. In the models of massive holography proposed in this paper, it is controlled by the radial dependence of the mass function $M^2(r)$ and in particular by the parameter ν. The latter, which determines the nature of the dual CFT, can be related in the case of the simple potential of the form $V(X) = X^n$ to the power n as follows:

$$n = \frac{4+\nu}{2} \tag{3.198}$$

It is straightforward to realized that for the model to be consistent we need to impose $\nu > -4$ which corresponds to $n > 0$.
We can define a distinction between a metallic behaviour ($d\sigma/dT < 0$) and an insulating behaviour ($d\sigma/dT > 0$). For $\nu < -2$ the behaviour is metallic while for the opposite case, $\nu > -2$, we are in the presence of an insulator (see fig.3.10). Since the mass function (3.197) in the dRGT theory corresponds to the cases $\nu = \{-2, -3\}$, the dual materials exhibit a metallic behaviour there (this happens for the linear and square root models of [163, 166] as well). The possibility to mimic metallic and insulating behaviour (and a transition between them) in the context of holographic massive gravity has been already pointed out in [218]. We further exploit the freedom offered by the generic power ν to investigate the option of having a linear T resistivity $\rho = 1/\sigma_{DC} \sim T$. This is a special feature of strange metals which evades the usual scaling predicted by the Fermi liquid theory ($\sigma_{DC} \sim T^{-2}$). Within our class of models we observe a linear scaling in the resistivity at low temperature only for the power $\nu = -3$ as shown in fig. 3.10. As mentioned above, the $\nu = -3$ mass function coincides with the β_1 term of dRGT massive gravity. The linear scaling of the DC resistivity for this particular case in the low temperature regime has been already observed earlier in [166].
The AC conductivity can be found by numerically and details are presented in [219]. We fixed a generic mass function $M^2(r) = r^\nu$ and we performed the computation using the standard holographic methods. As already shown in fig 3.10 the power ν determines whether the model is dual to a metallic or to an insulating state. The choice of $M^2(r)$ influences also the AC

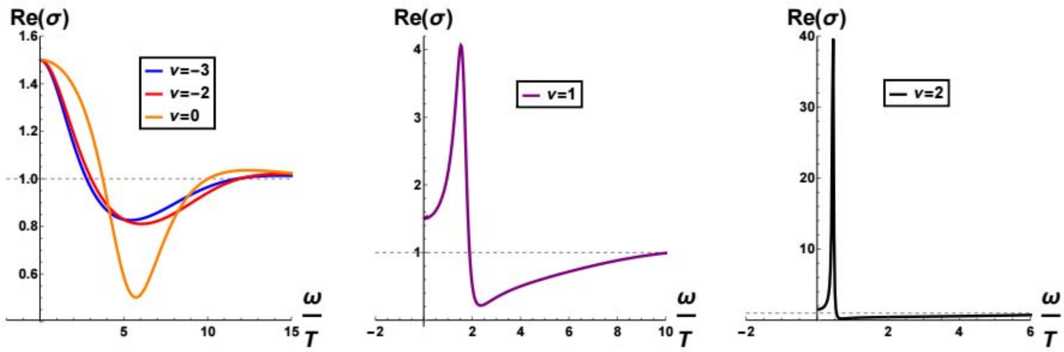


Figure 3.11: Real part of the optical conductivity for different choices of ν.

response of the dual CFT as shown in fig. 3.11. The value of the real part of the conductivity at zero frequency coincides with the analytic value for the DC conductivity given before. For $\nu \geq 0$ we observe a formation of a peak, which becomes sharply localized for higher values of ν and moves towards zero frequency. This suggests the presence of a localized excitation whose width decreases with ν very fast. This resonance shows up only in the insulating state and it is reminescent of what has been found in [218]. We associate the appearance of the peak at these values of ν with the fact that the mass functions that are more localized near the AdS boundary give rise to an explicit source of disorder that seems equivalent to the lattice disorder discussed in introduction. We find this a reasonable interpretation because such a disorder amounts to an additional source of an explicit breaking of translations that makes the otherwise exactly massless phonons become pseudo-Goldstone bosons. This is indeed what can be seen in the solid HMGs: one can identify almost massless transverse phonon poles in cases where the breaking is localized near the horizon, while as the profile of the mass function is moved towards the AdS boundary the phonons become massive. Hence, the physical phenomenon related to the appearance of peaks in the electric conductivity for $\nu \geq 0$ is expected to be the small collective field excitations - the phonons - due to the spontaneous breaking of the translational symmetry. In the presence of charge density this can be interpreted as a polaron formation as first suggested and observed in [218].

In addition to the classification of the materials into solids and fluids according to the residual symmetries that are preserved in the low energy EFT, there is another perhaps more intuitive way to distinguish them, namely, according to the type of response they exhibit under a shape deformation.
In standard mechanical response theory, the magnitude that encodes the material deformation is a displacement vector u_i, and the direct measure of the deformation is the linearized strain tensor,

$$u_{ij} = \frac{1}{2}\,\partial_{(i}u_{j)} \tag{3.199}$$

A clear distinction between fluids and solids is that they respond very differently to a constant applied shear stress (given by a traceless stress tensor T_{ij} leaving the volume of the material unchanged). For fluids, this leads to a constant shear velocity gradient (traceless part of $\dot{u}_{ij}$) and the corresponding response parameter is the shear viscosity. For solids, instead, a small constant applied shear stress leads to a constant shear strain (traceless part of u_{ij}), and the response parameter is the elasticity. Thus, a practical distinction between solids and fluids is

that the static shear elasticity (or rigidity) modulus is nonzero for solids and vanishes for fluids - fluids do not offer resistance to a constant shear deformation. This distinction between the solid and fluid phases is exactly reproduced in HMG: it is encoded exclusively in the $m_2(r)$ mass parameter, which vanishes for the fluid HMGs but not for the solid HMGs.
Indeed this mass term is just related to the derivative of the potential with respect to $X = Tr[\mathcal{I}^{ij}]$:

$$m_2(r) \sim V_X \tag{3.200}$$

and for the fluids case, where $V(X, Z) = \mathcal{H}(Z)$, it is trivially zero.
Of course, the response of different materials (and also black branes) is more complex than the simplified picture above. Materials can, for instance, exhibit both elastic and viscous (i.e., viscoelastic) response. Given that the elastic properties of HMG black branes have not yet been unveiled, we shall restrict here to the elastic response by considering only static applied stress and static deformation or strain. The full viscoelastic response can be studied by considering time dependent applied stresses.
In homogeneous and isotropic materials, the (static) elastic shear modulus or modulus of rigidity can be defined as the stress/strain ratio:

$$T^T_{ij} \,=\, G\, u^T_{ij} \tag{3.201}$$

where the superscript T stands for the traceless part. Equivalently, one can extract the modulus of rigidity, G, from a Kubo-like formula that relates it to the Fourier transform at zero frequency and wavenumber of the retarded correlator as:

$$G \,=\, \lim_{\omega\to 0}\, Re\,\langle T^T_{xy}\, T^T_{xy}\rangle^R \tag{3.202}$$

Once we have G expressed in terms of the stress tensor two-point functions, it is easy to apply the holographic prescription to extract it. The bulk field dual to T_{ij} is the traceless tensor mode of the metric perturbation $h^T_{ij}(r, t, x^k)$ whose equation of motion reads:

$$\left[f\,\partial_r^2 + \left(f' - 2\frac{f}{r}\right)\partial_r + \left(\frac{\omega^2}{f} - \,4\,r^2\,m_2^2(r)\right)\right] h^T_{ij} = 0\,. \tag{3.203}$$

For the constant and homogeneous ($\omega = 0$) response, this equation depends exclusively on the $m_2(r)$ mass parameter. The retarded Greens function is extracted as usual by solving (3.227) with ingoing boundary conditions and taking the ratio of the subleading to the leading mode. The resulting response vanishes only for $m_2 = 0$, i.e. for the fluid HMGs. In fig. 3.12 we show the dependence on temperature of the shear modulus for some representative model choice. This shows that there is a well-defined sense in which the HMG black branes enjoy a nontrivial elastic shear response and thus behave as solids. We observe that the rigidity modulus G increases with temperature, whereas most ordinary solids display the opposite dependence and G decreases with increasing T. However, in ordinary solids this behaviour occurs at roughly constant energy density while in the middle panel of fig. 3.12 the energy density is strongly increasing with T (which relates to the fact that the CFT degrees of freedom are inevitably in a relativistic regime). The ratio of G to the energy density ϵ, instead, does display the more standard decreasing in T form, so in this sense the result seems to be consistent with expectations from ordinary solids.

We have not analyzed in details all the technical part dealing with the consistency and stability of the generical HMG we considered; the interested reader can find it in [219]. All in all we made some importants points which can be summarized here:

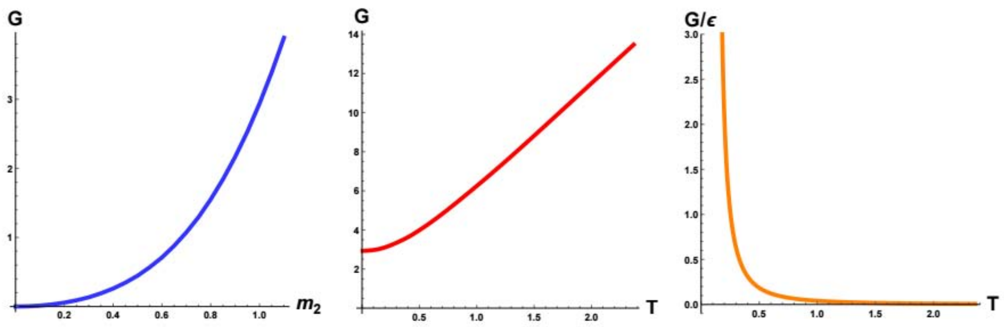


Figure 3.12: Modulus of rigidity G for the choice $\nu = -2$. **Left:** mass dependence. **Center:** temperature dependence. **Right:** temperature dependence after renormalizing the modulus of rigidity G by the T dependent energy density ϵ.

- We introduced the most generic holographic massive gravity framework, which goes beyond the dRGT case, and we showed explicitely its formulation in terms of the unitary gauge and via the Stückelberg field ϕ^I.
- We analzyed the all the possible pathological issues in details and we found out that the set of possible consistent HMG is pretty wide (and unexplored).
- We can define two types of HMG: the solid type and the fluid one. They are distinguished by the symmetry breaking pattern and by a new physical observable which refers to *elasticity*.
- We performed the first computation of the *shear modulus* showing that it is not null in solids while it turns out to be null in fluids as expected.
- We analyzed in full generality the electric response of HMG theories, showing that they are able to encode not only metallic behaviours but insulating as well.

In the next section we will give more phenomenological details about the electric response of holographic EFT for condensed matter dealing with massive gravity.

3.6 Electric response & Metal Insulator transitions

In the previous chapter we stated that HMG could mimick not only metallic phase ($d\sigma/dT < 0$) but insulating ones ($d\sigma/dT > 0$) as well. Keeping a critical attitude, this is actually not totally correct because we never had:

$$\sigma_{DC}\,(T=0)\,\approx\,0 \tag{3.204}$$

which is the proper definition for a insulator.
This issue can be derived from the generic structure of the DC conductivity coming from that class of models:

$$\sigma_{DC}\,=\,\frac{1}{e^2}\left(1\,+\,\frac{\mu^2}{\mathbb{M}^2(T)}\right) \tag{3.205}$$

where $\mathbb{M}^2(T)$ is an effective graviton mass, which depends generically on the temperature T (through the position of the horizon u_h) and on the details of the specific model considered. This

generic structure is quite robust and can be derived just through hydrodynamical arguments [175, 191].
The issue just described, has been formalized in [209] as the existence of a "universal" lower bound for the electric conductivity for HMG theories, which are thought to represent a *mean-field* description of disorder:

$$\sigma_{DC} \geq \frac{1}{e^2} \tag{3.206}$$

The presence of a lower bound of course prevents the possibility of having a proper insulating state with a very small electric conductivity at zero temperature.
From a more phenomenological point of view, this is as to say that no disorder-driven metal-insulator transitions MIT appear in such a models. On the contrary, increasing the amount of disorder, *i.e.* the graviton mass, the system passes from a good metal characterized by a clear Drude peak to an incoherent metal where no localized excitation is dominating the response. In nature increasing further disorder will turn the system into an insulating state which is the result of a *localization* mechanism. In HMG theories this is not happening (yet)!
To be more precise, it has been claimed in [209] that this is true for a large class of "simple" models, defined by the following benchmark structure:

$$\mathcal{S}_{bulk} = \int d^4x \sqrt{-g} \left(\frac{R}{2} - 2\Lambda - \frac{F_{\mu\nu}F^{\mu\nu}}{4e^2} + \mathcal{V}(\phi^I) + \dots \right) \tag{3.207}$$

where $\mathcal{V}(\phi^I)$ encodes the Lagrangian for a generic neutral translation-breaking (TB) sector.
The main point of this issue turns around the first term appearing in the generic DC formula (3.205): performing indeed a large disorder limit, $\mathbb{M} \to \infty$, the second term generically drops out but the first term remains and it is completely unaffected by the TB sector. In other words the first term is just dictated by the structure of the electromagnetic part of the action and without modifying the usual Maxwell term such a term will not vary neither with temperature nor disorder strength.
One possible way to avoid such a feature is to introduce a new degree of freedom into the model which couples directly to the Maxwell term. This is what happens for the strings inspired Einstein-Maxwell-Dilaton models whose action reads:

$$\mathcal{S}_{bulk} = \int d^4x \sqrt{-g} \left(\frac{R}{2} - 2\Lambda - \frac{Z(\Phi)}{4e^2} F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \Phi)^2 - V(\Phi) + \mathcal{V}(\phi^I) + \dots \right) \tag{3.208}$$

where the new neutral scalar field Φ is called the *dilaton*.
This new ingredient represents from the dual perspective the running of the coupling of the dual field theories and can have a strong impact on the IR features of the system. From the point of view of the DC conductivity, it modifies the generic formula (3.205) into:

$$\sigma_{DC} = \frac{1}{e^2} \left(Z(\Phi_h) + \frac{\mu^2}{\mathbb{M}^2(T)} \right) \tag{3.209}$$

where Φ_h is the value of the dilaton at the horizon.
It is clear that as a conseguence of this modification the proposed bound (3.206) is violated and depending on the dynamics of the dilaton sector, *i.e.* the form of the function $Z(\Phi)$, possible insulating states could appear. This is indeed what happens [177, 187].
In the present [220], we shall show instead that one can certainly avoid a bound like (3.206) in minimal and natural holographic models that contain the same dynamical ingredients (operators) as well as the mutual ("electron-disorder") interactions which are still allowed by the

symmetries[31]. Indeed, using EFT-like reasoning, it is clear that (3.207) is not the most general action allowed by the symmetries and the required field content. Clearly, there are additional couplings between the charge and TB sectors that can (and should) be included in the effective action. The crucial new ingredient that will be relevant for the present discussion is a direct coupling between the charge and TB sectors, which we can write schematically as:

$$Y[\phi^I]\, F_{\mu\nu}F^{\mu\nu} \tag{3.210}$$

where $Y[\phi^I]$ stands for some function of the TB sector ϕ^I (or its derivatives) alone.
Physically, even before specifying how we shall implement the TB sector and choose the Y function, it is clear that this effective interaction captures how much the TB sector affects the charge sector. This coupling, then, encodes the charged disorder - the effects from ionic impurities that directly couple to the mobile charge carriers. From the point of view of an effective description it is all the more reasonable that this kind of disorder is encoded in a direct coupling of this form.
Let us also emphasize one crucial difference between our proposal and some previous models [177, 187] that use a running dilaton Φ that couples to the charge sector through a bulk term like $Z(\Phi)F^2$. These models include a new dynamical ingredient, a scalar CFT operator $\mathcal{O}$. The BB solutions are relevant deformations of the CFT by the operator $\mathcal{O}$ that already in vacuum gives rise to confinement and therefore an insulating-like behavior. In these cases, it is hard to argue that the insulating behavior is driven by disorder. In our case, instead, there is no room for doubt. There are no more dynamical ingredients in the CFT other than the TB and the charge sector, so the BB solutions represent CFTs deformed by disorder (and finite density). At this point we can also see that the new interaction will play a role similar to the dilaton-Maxwell coupling in the sense that the physical magnitude of the charge carried by a charge carrier gets renormalized along the RG flow - in our case clearly due to disorder. Furthermore, we will show that dynamical consistency of the model requires that the renormalization is such that the conductivity is necessarily reduced at small temperatures (which is not necessarily the case for the Maxwell-dilaton coupling).
We therefore consider the minimal model in $3+1$ dimensions:

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[\frac{R}{2} - \Lambda - \frac{1}{4\,e^2}\,Y(X)\,F^2 - m^2\,V(X)\right]. \tag{3.211}$$

with $X = g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^I$ and $F^2 = F_{\mu\nu}F^{\mu\nu}$.
It is quite clear from the structure of the action (3.211), that the TB enters in two distinct ways, encoded in the functions $Y(X), V(X)$. $V(X)$ represents a neutral disorder - the disorder from neutral forms of impurities, that do not couple directly to the charge carriers. Instead, $Y(X)$ captures the effects of disorder that are felt directly by the charged sector. In other words this model provides a generalization of the previous HMG models [218, 219].
The model admits asymptotically AdS charged black brane solutions with a planar horizon topology. For arbitrary choice of V, Y they take the form:

$$ds^2 = \frac{1}{u^2}\left[-f(u)dt^2 + \frac{1}{f(u)}\,du^2 + dx^2 + dy^2\right],$$
$$f(u) = -\,u^3\int_u^{u_h}\left(\frac{\rho^2}{2\,Y\left(\alpha^2\xi^2\right)} + \frac{m^2\,V\left(\alpha^2\xi^2\right)}{\xi^4} + \frac{\Lambda}{\xi^4}\right)d\xi\,,$$

[31] There are other effective ways of reaching this goal which have been recently proposed. One can introduce non-linearities in the EM sector of the form $K(F^2)$. This model has been investigated recently in [221] and seems to give an effective holographic description of Mott Insulators. Eventually one can also couple the Ricci scalar directly to the Stückelberg sector $\sim R\mathcal{P}(\Phi^I)$ and get similar results explored in [222]

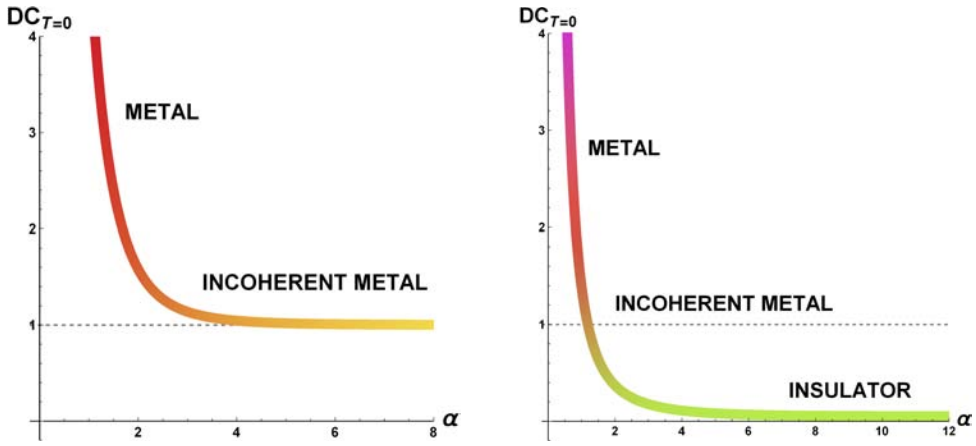


Figure 3.13: Electric DC conductivity at zero temperature for the model (3.215) dialing the disorder strength α (i.e the graviton mass); **Left:** $\kappa = 0$ (i.e. previous literature); **Right:** with the new coupling $\kappa = 0.5$ (safe region) .

$$\begin{aligned} &\phi^I = \alpha\, \delta^I_i x^i \ , \quad I = \{x, y\} \,, \\ &A_t(u) \,=\, \rho \int_u^{u_h} \frac{1}{Y(\xi^2\,\alpha^2)}\, d\xi \,, \end{aligned} \tag{3.212}$$

where u_h denotes the horizon location.
The temperature of the background geometry reads:

$$T \,=\, -\frac{\rho^2\, u_h^3}{8\,\pi\, Y\,(\alpha^2\, u_h^2)} - \frac{m^2\, V\,(\alpha^2 u_h^2)}{4\,\pi\, u_h} - \frac{\Lambda}{4\,\pi\, u_h} \tag{3.213}$$

An very important part of the present analysis concerns the conditions under which the models above are consistent – they are free from instabilities. Its main outcome is that the functions $V(X), Y(X)$ that appear in the Lagrangian are subject to the constraints:

$$V'(X) > 0 \ , \ \ Y(X) > 0 \ , \ \ Y'(X) < 0 \tag{3.214}$$

Crucially, the Maxwell-Stückelberg coupling Y is allowed (and must be positive). Not only that, it must also be a decreasing function of X. Let us emphasize that the latter condition stems solely from the requirement that the transverse vector modes have a normal (non-ghosty) kinetic term (the actual condition is slightly less restrictive, but for simplicity we shall take $Y' < 0$ which is certainly sufficient and more robust). The fact that $Y' < 0$ will have a dramatic impact on the possibility to have a MIT.

We shall focus on a representative 'benchmark' model,

$$Y(X) \,=\, e^{-\kappa\, X}\,, \quad V(X) = X/(2\, m^2) \ . \tag{3.215}$$

This is by far not the most general model but it will suffice to illustrate the new features that can be modeled with this kind of coupling. Note that it suffices to take $\kappa > 0$ to satisfy all the consistency conditions. One can also anticipate that for order-one values of κ the effects from this coupling can be rather important.
Proceeding with the vector perturbations on top of the background defined in (3.212), one can

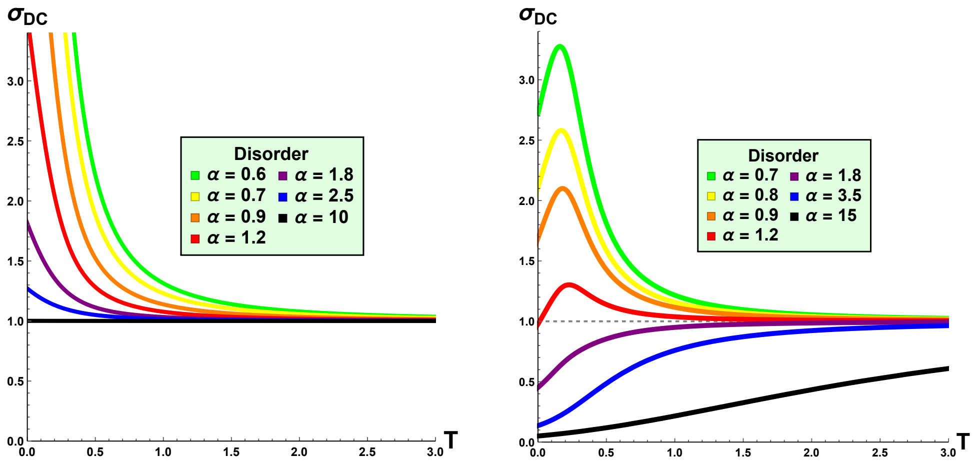


Figure 3.14: Temperature features of the DC conductivity at different disorder strengths α (i.e. graviton mass) for the model considered in (3.215) with unitary charge density; **Left:** metal-incoherent metal transition for $\kappa = 0$ (i.e. previous literature); **Right:** metal-insulator transition for $\kappa = 0.5$ (safe region).

compute numerically the optical electric conductivity and analytically its DC value. We find the following analytic result for the electric DC conductivity,

$$\sigma_{DC} \;=\; \left[Y(\bar{X}) + \frac{\rho^2\, u^2}{m_{eff}^2} \right]_{u_h} \tag{3.216}$$

with

$$m_{eff}^2 \equiv \alpha^2 \left(m^2\, V'(\bar{X}) - \rho^2\, u^4\, \frac{Y'(\bar{X})}{2\, Y^2(\bar{X})} \right)$$

where $\bar{X} = u^2\,\alpha^2$ and all quantities have to be evaluated at the horizon, $u = u_h$. The expression (3.246) encodes all the interesting phenomenology which follows.
For the benchmark model (3.215), the interesting quantities read:

$$\begin{aligned} T \;&=\; -\frac{\rho^2\, u_h^3\, e^{\kappa\,\alpha^2 u_h^2}}{8\,\pi} - \frac{\alpha^2\, u_h}{8\,\pi} + \frac{3}{4\,\pi\, u_h}, \\ \sigma_{DC} \;&=\; e^{-\kappa\,\alpha^2\, u_h^2} + \frac{2\,\rho^2\, u_h^2}{\alpha^2\, \left(\rho^2\, \kappa\, u_h^4\, e^{\kappa\,\alpha^2 u_h^2} + 1 \right)}\,. \end{aligned} \tag{3.217}$$

In this scenario we are left with only four parameters in our model: the temperature T, the charge density ρ, the neutral and charged disorder strengths α and κ.
The analysis of the electric DC conductivity in function of disorder is our primary task. The first interesting and new feature of the model deals with the DC conductivity at zero temperature which characterizes the nature of our '*material*' (i.e. metal/insulator). In the previous massive gravity models the system could be just in a metallic phase (with a sharp Drude peak) or in an incoherent metallic phase where there is no clear and dominant localized long lived excitation. The bound of [209] can be indeed rephrased with the statement that such a models fall down in an extremely incoherent metallic state for very strong disorder without undergoing a metal-insulator transition. Note that this is a quite unnatural behaviour in real-life experiments

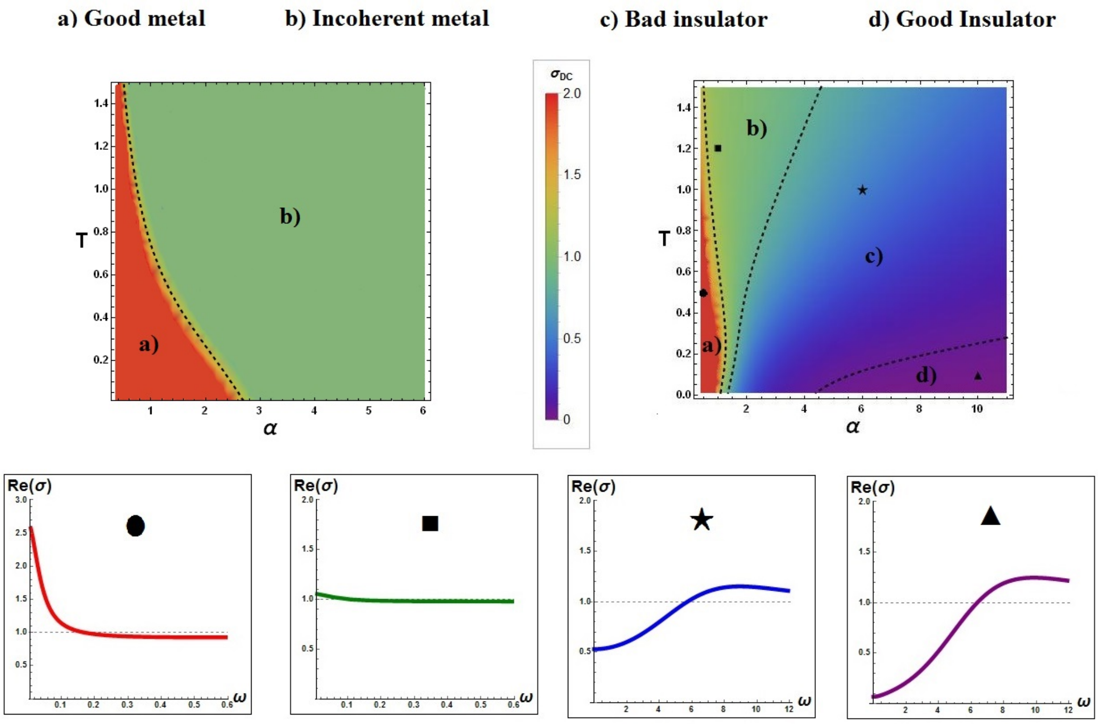


Figure 3.15: Phase diagrams for the model (**??**) with unitary charge density $\rho = 1$. Dashed lines correspond to $\sigma_{DC} = 0.1,\, 0.8\,, 1.2$ and they divide the four regions: a) good metal, b) incoherent metal, c) bad insulator and d) good insulator . **Top Left:** Temperature-disorder plane with $\kappa = 0$. **Top Right:** Temperature-disorder plane with graviton mass $\kappa = 0.5$. **Bottom:** For every region (a,b,c,d) in the phase diagrams one representative example of $\Re\,(\sigma)$ is shown. The parameters for each one of the AC plots are pinpointed in the $T-\alpha$ phase diagram (top right) and they correspond to: $[\,\bullet : (\alpha = 0.5, T = 0.5),\ \blacksquare : (\alpha = 1, T = 1.2),\ \bigstar : (\alpha = 6, T = 1),\ \blacktriangle : (\alpha = 10, T = 0.2)\,]$.

where usually strong disorder produces clear insulating behaviours. In our case the scenario is more complex and increasing the disorder one can reach an insulating state where $\sigma_{DC} \approx 0$ at zero temperature. This is basically what we mean by disorder-driven MIT and to the best of our knowledge this model is the first holographic example of such a mechanism. It is fair to say that the link between massive gravity and disorder is still very blurred and that right now massive gravity is able to capture just some few features of it. The difference between our novel results and previous literature is summarized in fig.3.13.

Most of the results of this short note are summarized in fig.3.14 where the presence of a metal-insulator transition (in contrast with previous massive gravity models) is made evident . The picture again emphasizes how we can overcome the bound $\sigma_{DC} \geq 1/e^2$ proposed in [209] (dashed line) increasing the disorder in the system and exploiting the new parameter κ.
This result also suggests that there is no universal lower-bound for the electrical conductivity at least within the holographic models. Despite the absence of any lower bound in the electric conductivity provided by our simple generalization of the implicit assumptions in [209], it should be very clear that we are not giving in a derivation of *localization*, or that localization is the phenomenon that lies behind the bad conductivity of these holographic materials. For this one would certainly need to abandon homogeneity and study more complicated models. In conclusion, the main features of the model (3.223) are summarized in Fig. 3.15 where we draw the phase diagrams in the temperature-disorder plane for $\kappa = 0$ (*i.e.* previous literature) and $\kappa = 0.5$ (which lies in the healthy region of the parameters space). For the known case $\kappa = 0$ just metallic phases are accessible and only a crossover between metals and incoherent metals can be manifested (see *e.g.* [155]). On the contrary, for the novel case, the phase diagram gets richer and incorporate several phases of matter depending on the parameters: good metal (a), bad or incoherent metal (b), bad insulator (c) and good insulator (d). Both the quantum phase transition (MIT) and the finite temperature crossover are present in the picture. For every phase of matter a representative example of optical conductivity is shown at the bottom of Fig. 3.15.

The framework of effective holographic theories can be enriched and enlarged to account for several condensed matter wisdoms and it can represent an useful tool to reproduce a large set of unexplained phenomena. The study of insulating states in this context has been initiated bringing a collection of new questions and unexplored directions.
Whether holographic models account for the presence of universal bounds/values for certain physical observables, such as the electric conductivity we considered at this stage, is a very valuable question on which we will return in the next section and expecially in the final remarks of this thesis.

3.7 The η/s bound in theories without translational symmetry

It has been long known that black brane solutions can be characterized both by thermodynamic quantities like temperature and entropy as well as hydrodynamic entities like viscosity and diffusion. In gauge/gravity duality, the hydrodynamics of the black branes is mapped to the hydrodynamic properties in the dual field theory. One of the most prominent insights that the AdS/CFT correspondence have provided for the understanding of dynamics of strongly coupled condensed matter systems is that the shear viscosity to entropy density ratio takes on a universal value[32] for all gauge theories with Einstein gravity duals [223]:

$$\frac{\eta}{s} = \frac{1}{4\,\pi} \tag{3.218}$$

This value was conjectured to set a fundamental lower bound on this ratio - the celebrated Kovtun-Son-Starinets (KSS) bound [224]. Amazingly enough, the bound seems to be satisfied for all known fluids where η/s has been measured, including examples like superfluid helium [225] and the QCD quark gluon plasma (see *e.g.* [226]).
By now it is well established that the KSS bound is violated by higher curvature corrections to the Einstein theory. In particular, the violation of the bound was observed in Einstein gravity supplemented by the quadratic Gauss-Bonnet term [227]. In terms of the Gauss-Bonnet coupling λ_{GB} the viscosity to entropy density ratio was found to be

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 4\lambda_{GB}\right]\,. \tag{3.219}$$

For a positive coupling this would imply an arbitrary violation of the bound. However, the consistency requirements on the dual field theory impose constraints on the allowed values of the Gauss–Bonnet coupling constant. In particular, it was found that the field excitations in the dual field theory allow for superluminal propagation velocities for $\lambda_{GB} > 9/100$, thus imposing a new lower bound on the viscosity to entropy ratio [228]. In the light of these results it is at present not clear whether a universal fundamental bound on the shear viscosity to entropy ratio exists. For a review on the bound violation in higher derivative theories of gravity, see [229] and references therein.
Asking ourselves about the fate of the η/s universal bound in the context of holographic theories with momentum dissipation is for sure a valuable and interesting question which we want to adress [230]. We make use of the generic HMG theories described in [219] and we compute the viscosity of the system via the Kubo formula:

$$\eta \equiv \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\;\mathcal{G}^R_{T_{ij}\,T_{ij}} \tag{3.220}$$

where $\mathcal{G}^R$ is the retarded Green's function of the stress tensor.
Note that such a generic theories possess a more complicated viscoelastic response, stress-tensor T_{ij}-displacement tensor u_{ij} relation of the type:

$$T^{(T)}_{ij} = G\,u^{(T)}_{ij} + \eta\,\dot{u}^{(T)}_{ij}\,. \tag{3.221}$$

where G is the so called modulus of rigidity, dealing with the elastic properties, which can be similarly computed through:

$$G \equiv \lim_{\omega\to 0}\mathrm{Re}\;\mathcal{G}^R_{T_{ij}\,T_{ij}}\,. \tag{3.222}$$

In terms of the two parameters defined in (3.220) and (3.222), the static mechanical response of generic isotropic materials can be depicted in the $\{G,\eta\}$ plane. The $G = 0$ axis corresponds to fluids. The $\eta = 0$ axis to non-dissipative (*e.g.* at zero temperature) solids. The rest of the two dimensional space is spanned by viscoelastic materials. As we shall see, solids dual to massive gravity black branes of [219] do lie inside this plane.

[32] We work in the units where $\hbar = k_B = 8\pi G \equiv 1$.

We consider, as in [219] holographic models defined by the generic 3 + 1 dimensional gravity theory:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}\left(R+\frac{6}{L^2}\right)-\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu}-\frac{m^2}{L^2}\,V(X,Z)\right]+\int_{r\to 0} d^3x\sqrt{-\gamma}\,K\ , \tag{3.223}$$

where L is the AdS radius, m is a dimensionless mass parameter, and

$$X \equiv \frac{1}{2}\,\mathrm{tr}[\mathcal{I}^{IJ}]\,, \qquad Z \equiv \det[\mathcal{I}^{IJ}]\,, \qquad \mathcal{I}^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J\,, \tag{3.224}$$

and the indices $I, J = \{x, y\}$ are contracted with δ_{IJ}. In (3.223), we have included the Gibbons-Hawking boundary term where γ is the induced metric on the AdS boundary, and $K = \gamma^{\mu\nu}\nabla_\mu n_\nu$ is the extrinsic curvature with n^μ - an outward pointing unit normal vector to the boundary. Around the scalar fields background $\hat{\phi}^I = \delta^I_i x^i$ the metric admits the black brane background solution

$$ds^2 = L^2\left(\frac{dr^2}{f(r)r^2}+\frac{-f(r)dt^2+dx^2+dy^2}{r^2}\right)\,, \tag{3.225}$$

with the emblackening factor given in terms of the background value of the mass potential:

$$f(r) = 1+\frac{\mu^2 r^4}{2r_h^2}+m^2\,r^3\int^r d\tilde{r}\frac{1}{\tilde{r}^4}\hat{V}(\tilde{r})\,, \tag{3.226}$$

where $\hat{V}(r) \equiv V(\hat{X},\hat{Z})$. The solution for the Maxwell field is $\hat{A}_t = \mu\,(1-r/r_h)$.
The viscoelastic response of the boundary theory in the holographic description is encoded in the transverse traceless tensor mode of the metric perturbations which obeys:

$$\left[f\partial_r^2+\left(f'-2\frac{f}{r}\right)\partial_r+\left(\frac{\omega^2}{f}-4m^2M^2(r)\frac{r^2}{L^2}\right)\right]h_\omega = 0\,. \tag{3.227}$$

where we have defined a mass function

$$M^2(r) \equiv \frac{1}{2r^2}\hat{V}_X(r)\,. \tag{3.228}$$

It is very important to emphasize here that the mass of the tensor mode is only due to the X dependence of the potential $V(X,Z)$. Hence, in the case when V is only a function of Z the graviton remains massless. In our previous work we have argued that in the case when $V = V(Z)$ the dual theory describes *fluids*, whereas the presence of an X dependence, *i.e.* when $V = V(X,Z)$, indicates that the material is a *solid* [219]. We have also shown that there is no elastic response in the case of fluids. Moreover, since for fluids the graviton mass is zero, the universality proof [167] for the viscosity to entropy ratio based on the membrane paradigm is applicable and we expect no violation of the KSS bound. Without loss of generality we therefore only consider the theories describing solids with graviton mass terms of the form

$$V(X) = X^n\,. \tag{3.229}$$

Here we are allowing for general values of n in order to see what is the impact of this parameter on the elasticity and viscosity. As already discussed this choice corresponds to a mass function of the form:

$$M^2(r) = \frac{1}{2\,L^2}\left(\frac{r}{L}\right)^\nu \tag{3.230}$$

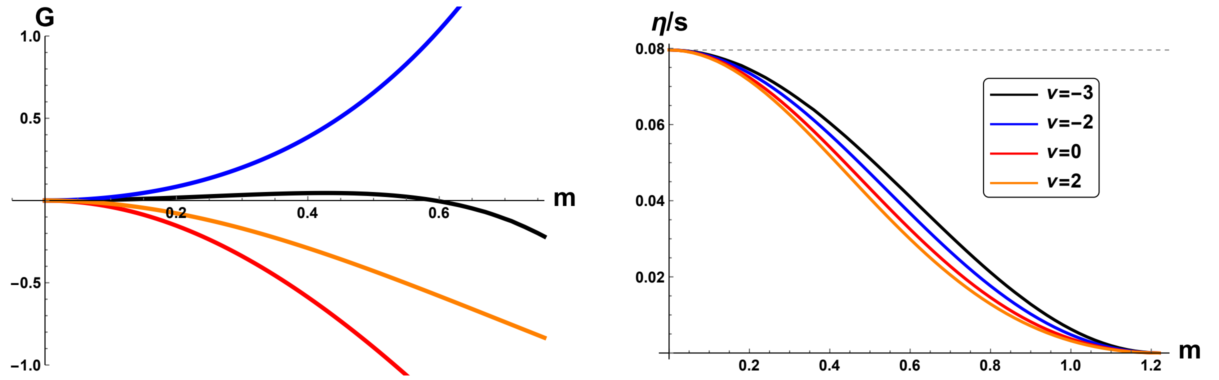


Figure 3.16: Backreacted model $V(X) = X^n$ for different $n = \frac{4+\nu}{2}$, $\nu = -3, -2, 0, 2$. **Left:** Elasticity; **Right:** η/s ratio, the horizontal dashed line shows the value $\eta/s = 1/(4\pi)$. Plots from [230].

with $n = \frac{4+\nu}{2}$.
From the equation of motion (3.227) it follows that in the near-boundary region the metric perturbations $h \equiv h_\omega$ behave as

$$\lim_{r \to 0} h = h_0 + \left(\frac{r}{L}\right)^3 h_3 + \dots \tag{3.231}$$

showing that the scaling dimension of h is $\Delta = 3$ and is independent on the radial dependence of the graviton mass. The gauge/gravity duality prescription then allows one to find the retarded Green's function as the ratio of the subleading to leading mode of the graviton:

$$\mathcal{G}^R_{T_{ij}\,T_{ij}} = \frac{2\Delta - d}{2L}\,\frac{h_3}{h_0} \tag{3.232}$$

where $d = 3$ is the number of spatial dimensions. We numerically solve the equation of motion for the graviton and extract the retarded Green's function by using the above expression. From the latter we are able to compute the complete viscoelastic response of the dual CFT. In Fig. 3.16 we show the real part of the Green's function and the viscosity to entropy density ratio as a function of the graviton mass for different values of the exponent ν. We first observe that the η/s ratio goes below the universal value $1/(4\pi) \approx 0.08$ for graviton mass parameter values $m > 0$ and thus violates the KSS bound. As expected, in the fluid regime with $m = 0$ we recover the standard universal value. The second observation that we make is that the real part of the Green's function becomes negative for all values of ν apart from $\nu = -2$. Although, negative modulus of elasticity can, in principle, be observed in nature it is always associated with instabilities. From the holographic perspective, the fact that there is an instability is not so surprising because the kinetic terms for the Stückelberg fields are non-canonical for $V(X) = X^n$ with $n > 3/2$ and $n = 1/2$ (corresponding to $\nu > -1$ and $\nu = -3$ respectively). Both the numerical and analytical results give a positive rigidity modulus for the canonical Stückelberg case, $n = 1$ ($\nu = -2$) with $V = X$, which can therefore be singled out as the most reasonable model from the phenomenological point of view.

We would like to point out that the fact that the KSS bound can be violated in theories with massive gravity duals was also noticed in [155] for the case $\nu = -2$ corresponding to $V = X/2$ and $\mu = 0$. It was then argued by the authors that this result is irrelevant for the physical viscosity

due to the fact that for graviton masses of order $m/T \gtrsim 1$ the dual field theory does not admit a coherent hydrodynamic description. Instead a crossover from the coherent hydrodynamic phase of the system to an incoherent regime occurs for graviton mass that is comparable to the black brane temperature. In the results presented in this paper we see the violation of the KSS bound also at arbitrary small values of the graviton mass where the hydrodynamic description applies [33]. We therefore believe that our findings are physically significant and suggest that the KSS bound can be violated in materials with non-zero elastic response. In general, however, we find that the question of whether or not the black branes are close to having a hydrodynamic description is not particularly relevant in the context of holographic solids. In these systems we do not expect the dynamics to be understood in terms of hydrodynamics while there does exist a well defined low energy effective field theory description of solids defined as an expansion at low frequencies and momenta.
As an additional remark, let's note (see fig.3.17) that in the extremal limit $T = 0$ the viscosity-entropy bound is null. This was also observed and discussed in details in [211] where the value of η/s at $T = 0$ has been related to the nature of the momentum dissipating sector in the deep IR. To be more precise, the graviton mass we consider does not vanish at the extremal horizon such that the momentum dissipation mechanism is still active in that limit leading to a vanishing KSS ratio. On the contrary at temperatures $T \gg m^2$ we expect the presence of a graviton mass to be completely irrelevant and indeed we recover the universal value $1/4\pi$ ((see fig.3.17).
Note also that the violation of the KSS bound in this framework is definitely stronger than the higher derivative case because no further lower bound appear at all: the η/s ratio can go down to 0! We can analyze further the situation exploiting perturbative methods which allow us to

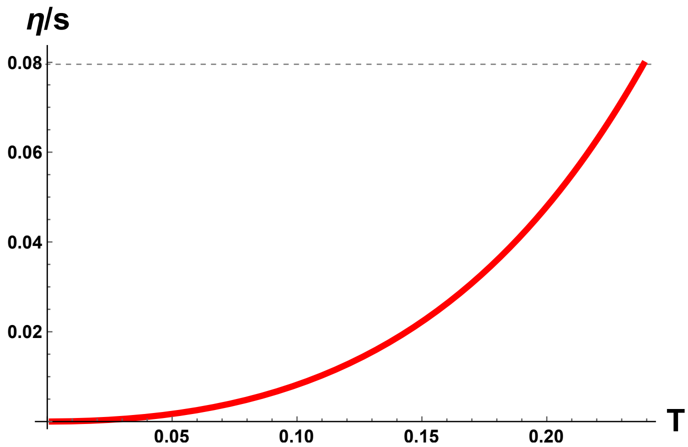


Figure 3.17: η/s in function of temperature T for the benchmark model $V(X) = X$. Note that is totally generically in our theory to have $\eta/s(T = 0) = 0$. Dashed line is the universal KKS value $1/4\pi$.

compute at small graviton mass the value of G and η analytically. What we find out (see [230] for details) is that at low momentum dissipation those two quantities take the form:

$$G = \frac{L^2}{2\, r_h^3}\, c_\nu\, m^2 + \ldots$$

[33] See [212] for a detailed hydrodynamical analysis of the η/s violation in holographic theories with momentum dissipation.

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{2}{3}\, c_\nu\, m^2\, \mathcal{H}_{\frac{1}{3}(\nu+1)} + \dots\right) \tag{3.233}$$

with $c_\nu = -\frac{2}{\nu+1}\left(\frac{r_h}{L}\right)^{4+\nu 34}$ and $\mathcal{H}_p$ the p-th harmonic number. The numerical results for the real part of the Green's function and for the η/s ratio are in good agreement with these analytic expressions for small values of the graviton mass parameter m.

In conclusion we have seen a violation of the KSS bound $\eta/s \geq 1/4\pi$ in holographic massive theory of the solid type. We suspect this violation to be correlated with the presence of a non zero shear elastic modulus G. Importantly the violation we found is not directly connected with the breaking of translational symmetry of the dual CFT because for fluid type HMG theories the bound is fully satisfied even if momentum is dissipating.

Given the putative connection between the elastic response and the violation of the KKS bound we provided it would be extremely interesting to perform experimental measures of the η/s ration in viscoelastic materials. In addition, in the spirit of the KSS conjecture, one can also wonder

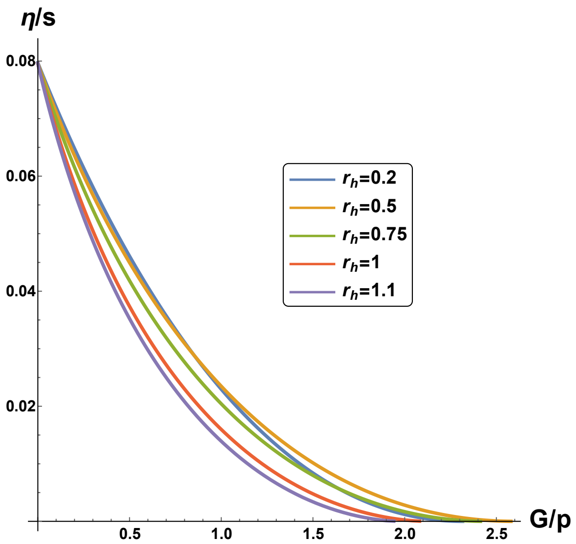


Figure 3.18: Viscosity–elasticity diagram for the $V(X) = X$ model. The value of the graviton mass and the temperature are changing along the solid lines, with $m = 0$ on the axes where $\eta/s = 1/4\pi \approx 0.08$ and $T = 0$ on the axes $\eta/s = 0$ where the rigidity G takes its maximum value.

whether or not there is any generalization of it that holds in solid systems. From dimensional analysis it is reasonable to expect that if there does exist a more general bound, it should involve the rigidity to pressure ratio, G/p, in addition to the η/s ratio. In Fig. 3.18 we plot η/s against G/p for the holographic solid with $\nu = -2$ and see a clear correlation. Keeping the KSS logic that the gravity solutions might represent the least dissipative materials, the Fig. 3.18 then suggests that there might be a more general bound in (viscoelastic) solids. At relatively large temperatures this would approximately take the form

$$4\pi\frac{\eta}{s} + \mathcal{C}\frac{G}{p} \lesssim 1$$

with $\mathcal{C}$ being an order-one constant. Another possibility for a generalized KSS bound has been proposed in [211] in connection with the rate of entropy production encoded by the viscosity η.

[34]The case $\nu = -1$ is a particular one and it has to be treated separately.

3.8 Superconductors with momentum dissipation

Metal-SC phase transitions

The holographic superconductors model [88,89] is one of the first and main result of the AdS/CMT program. Within its simplicity it describes a system which exists in two states: a superconducting state which has a nonvanishing charge condensate, and a normal state which is a perfect conductor. As a direct conseguence, already in the normal phase the static electric response, namely the DC conductivity ($\omega = 0$), is infinite. This is a straightforward consequence of the translational invariance of the boundary field theory, which leads to the fact that the charge carriers do not dissipate their momentum, and accelerate freely under an applied external electric field. This fact represents a shortcoming of the model which has to be cured in order to have a realistic metallic normal state with finite electric DC conductivity clearly distinguishable from the infinite one in the superconducting phase.
We [231] therefore introduce HMG theories into the original SC model [88,89] to take care of such a lack and to study the effects of disorder-driven momentum dissipation on the main features of the SC phase transitions, such as the critical temperature T_c and the value of the charged condensate $\langle \mathcal{O} \rangle$.
The total action of our model is :

$$I = I_1 + I_2 + I_3 \,, \tag{3.234}$$

where we have denoted the Einstein-Maxwell terms I_1, the neutral scalar terms I_2, and the charged scalar terms I_3;

$$\begin{aligned} I_1 &= \int d^{d+1}x \sqrt{-g} \left[R - 2\Lambda - \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} \right] , \\ I_2 &= -2m^2 \int d^{d+1}x \sqrt{-g}\, V\left(X\right) , \\ I_3 &= -\int d^{d+1}x \sqrt{-g} \left(|D\psi|^2 + M^2 |\psi|^2 + \kappa\, H\left(X\right) |\psi|^2 \right) . \end{aligned} \tag{3.235}$$

and we consider the following MG potentials as benchmark examples[35]

$$\textbf{model 1} : \quad V(X) = \frac{X}{2\, m^2} \,, \tag{3.236}$$

$$\textbf{model 2} : \quad V(X) = X + X^5 \tag{3.237}$$

$$\textbf{model 3} : \quad V(X) = \frac{X^N}{2\, m^2} \,, \qquad N \neq 1 \tag{3.238}$$

We have defined

$$X = \frac{1}{2}\, L^2\, g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^I \,. \tag{3.239}$$

We denote $D_\mu \psi = (\partial_\mu - i\, q\, A_\mu)\psi$ to be the standard covariant derivative of the scalar ψ with the charge q.

[35] 3.236 has been already studied in the context of holographic SC in [199,200]. We have inserted an additional coupling m^2 in front of the potential $V(X)$ which is going to be redundant for the monomial cases 3.236 and 3.238 where we decided in fact to reabsorb it into the definition of $V(X)$. In this way for those cases we are left with just one parameter α which is going to represent the disorder-strength in the system. In the case of the polinomial potential 3.237 m^2 is going to be an independent parameter in addition to α.

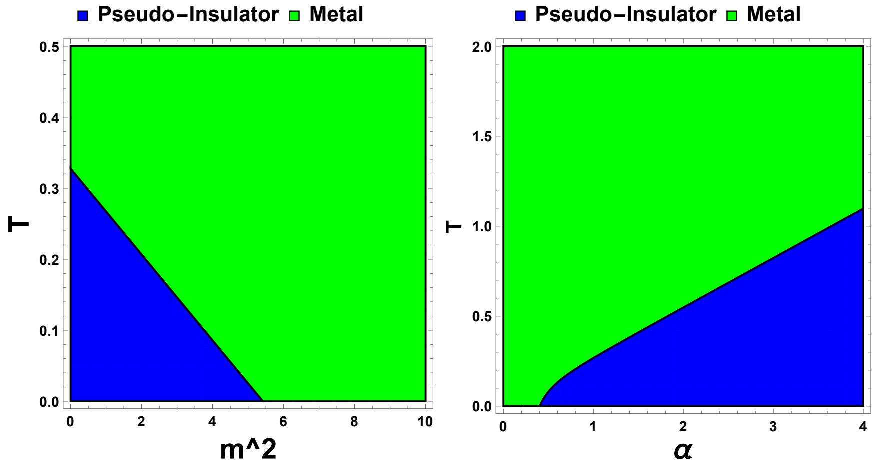


Figure 3.19: Region plots for the model (3.237) in the normal phase. We choose units where the density is $\rho = 1$. Here we have fixed $\alpha = 1$ (left plot) and $m = 1$ (right plot). The blue region is pseudo-insulating, $d\sigma_{DC}/dT > 0$, the green region is metallic, $d\sigma_{DC}/dT < 0$.

The generic ansatz we consider is given by:

$$\begin{aligned} &ds^2 = L^2 \left(-\frac{1}{u^2} f(u) e^{-\chi(u)} dt^2 + \frac{1}{u^2}(dx^2 + dy^2) + \frac{1}{u^2 f(u)} du^2 \right) \\ &\phi^I = \alpha\, \delta^I_i\, x^i \,, \qquad I, i = x, y \,. \\ &A = A_t(u) du \,, \qquad \psi = \psi(u) \,. \end{aligned} \tag{3.240}$$

where ψ can be taken to be real-valued.
Within this ansatz the EOMs coming from the generic action (3.235) read:

$$\frac{q^2\, u\, e^{\chi}\, A_t^2\, \psi^2}{f^2} - \chi' + u\, \psi'^2 = 0 \tag{3.241}$$

$$\begin{aligned} &\psi'^2 - \frac{2\, f'}{u\, f} + \frac{e^{\chi} u^2 A_t'^2}{2 f} + \frac{M^2 L^2 \psi^2}{u^2 f} + \frac{\kappa\, L^2\, H \psi^2}{u^2 f} + \frac{e^{\chi} q^2 A_t^2 \psi^2}{f^2} \\ &+ \frac{2 m^2 L^2 V}{u^2 f} + \frac{2 \Lambda L^2}{u^2 f} + \frac{6}{u^2} = 0 \end{aligned} \tag{3.242}$$

$$\frac{2 q^2 A_t \psi^2}{u^2 f} - \frac{\chi'}{2} A_t' - A_t'' = 0 \tag{3.243}$$

$$\psi'' + \left(-\frac{2}{u} + \frac{f'}{f} - \frac{\chi'}{2} \right) \psi' + \left(\frac{e^{\chi} q^2 A_t^2}{f^2} - \frac{M^2 L^2}{u^2 f} - \frac{\kappa H\, L^2}{u^2 f} \right) \psi = 0 \tag{3.244}$$

and the Hawking temperature of the black brane (3.240) is given by:

$$T = -\frac{f'(u_h)}{4\pi}\, e^{-\frac{\chi(u_h)}{2}} \,. \tag{3.245}$$

In the case of a non-trivial condensate $\psi(u)$ it is in general impossible to solve the background equations of motion analytically. However, when $\psi(u) = 0$, *i.e.* the normal phase, the solution is known and it was given in [218]. The correspondent DC electric conductivity is finite and takes

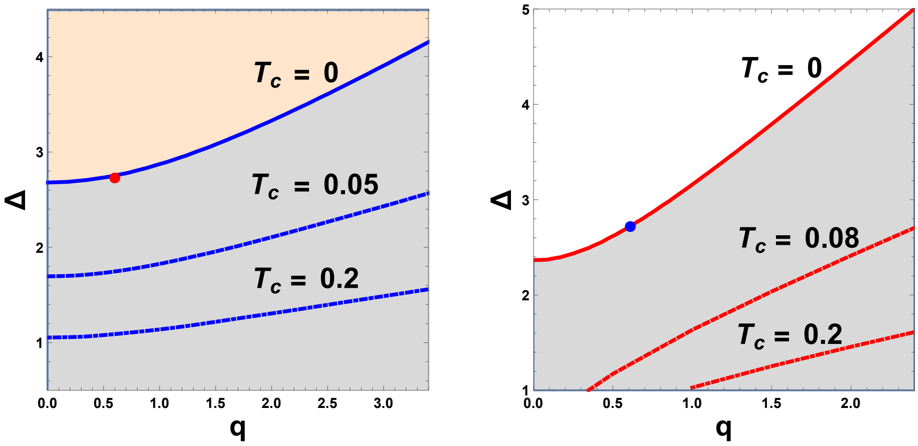


Figure 3.20: **Left:** Region and contour plots for the model (3.236) with linear potential for the neutral scalars. The region of Δ, q, satisfying the IR instability condition (3.249) is shaded in grey. The red dot is centered around $(q_d, \Delta_d) = (0.6, 2.74)$. These tuned (q, Δ) confine superconducting phase of the model (3.236) into a dome region. Notice the proximity of the red dot to the boundary of the IR instability region, resulting in $T_c(q_d, \Delta_d)$ being very small. **Right:** Region and contour plots for the model (3.237) with non-linear potential for some representative parameters.

the value:

$$\sigma_{DC} = \frac{1}{e^2}\left(1 + \frac{\rho^2\, u_h^2}{2\, m^2\, \alpha^2\, \dot{V}(u_h^2\, \alpha^2)}\right) . \tag{3.246}$$

With this formula at disposal we can already draw down the phase diagram of the normal phase accordingly to the following definitions:

- $d\sigma_{DC}/dT < 0$: metal
- $d\sigma_{DC}/dT > 0$: pseudo-insulator[36]

The phase diagram of this normal phase is already rich and can give insights towards the interpretation about the various ingredients introduced into the model. In the case of the linear Lagrangian, which goes back to the original model [199], the parameters m and α are combined into $m\,\alpha$, which can be interpreted as the strength of translational symmetry breaking. From the dual field theory point of view this is thought to be related to some sort of homogeneously distributed density of impurities, representing the disorder-strength in the material.

In the case of a more general $V(X)$, the m parameter keeps this kind of interpretation while the α one represents the strength of interactions of the neutral scalar sector. This reasoning is confirmed by the study of the phase diagrams of the system (figure 3.19) which makes evident the difference between the two parameters. Indeed, while the m parameter, which we are going to interpret as the disorder-strength of our High-Tc superconductor, enhances the metallic phase, the α one clearly reduces the mobility of the electronic sector driving the system towards the pseudo-insulating phase.

[36] We use the "pseudo" prefix to make explicit the fact that we do not have $\sigma_{DC}(T = 0) = 0$ in this phase.

The normal phase we just described is unstable towards the development of a non-trivial profile of the charged scalar field. This allows one to determine a line of the second order superconducting phase transition, $T_c(\alpha, m)$, in the boundary field theory, with broken translational symmetry. We start by considering the system at zero temperature, which we are able to study analytically. Then we proceed to studying the normal phase at a finite temperature. Upon lowering the temperature, at a certain critical value $T = T_c$, the normal phase becomes unstable. This is the point of a superconducting phase transition. We construct numerically T_c as a function of the parameters Δ, q, α (or m), for the models with various $V(X)$.

In the case of $T = 0$ the normal phase geometry interpolates between the AdS_4 in the ultraviolet and the $AdS_2 \times \mathbb{R}^2$ in the infra-red. We can apply the known analytical calculation to study the stability of the normal phase towards formation of a non-trivial profile of the scalar ψ.

Due to eq. (3.244), the effective mass M_{eff} of the scalar ψ is given by:

$$M_{eff}^2 \, L^2 = \, M^2 \, L^2 + \kappa \, H \, L^2 + q^2 \, A_t^2 \, g^{tt} \, L^2 \, . \tag{3.247}$$

Notice that at the boundary the mass of the scalar is just M^2 but at the horizon it gets an additional contribution. The normal phase is unstable towards formation of the scalar hair, if M_{eff} violates the BF stability bound in the AdS_2, namely:

$$M_{eff}^2 \, L_2^2 < -\frac{1}{4} \, . \tag{3.248}$$

In (3.248) we have denoted the AdS_2 radius as L_2.
All in all, the IR instability condition (3.248) finally reads:

$$D < 0 \, , \tag{3.249}$$

where we have defined the function D as:

$$D = \frac{1}{4} + \frac{L^2 \left(\kappa H + M^2\right) \left(L^2 m^2 \left(\alpha^2 u_h^2 \dot{V} \left(\alpha^2 u_h^2\right) - 2V \left(\alpha^2 u_h^2\right)\right) + 6\right) - q^2 \rho^2 u_h^4}{\left(L^2 m^2 \left(\alpha^2 u_h^2 \dot{V} \left(\alpha^2 u_h^2\right) - 2V \left(\alpha^2 u_h^2\right)\right) + 6\right)^2} \tag{3.250}$$

Consider the system at large temperature in a normal phase, which exists in a superconducting phase at low temperatures. Therefore as we decrease the temperature, at certain critical value T_c the superconducting phase transition occurs. If T_c is non-vanishing, then for $T < T_c$ the system is in a superconducting phase, with a non-trivial scalar condensate $\psi(u)$.
Recall that near the boundary the scalar field with mass M:

$$M = \frac{1}{L} \sqrt{\Delta(\Delta - 3)} \, , \tag{3.251}$$

behaves asymptotically as:

$$\psi(u) = \frac{\psi_1}{L^{3-\Delta}} u^{3-\Delta} + \frac{\psi_2}{L^{\Delta}} u^{\Delta} \, , \tag{3.252}$$

where ψ_1 is the leading term, identified as the source in the standard quantization.
Near the second order phase transition point $T = T_c$ the value of ψ is small, and therefore one can neglect its backreaction on the geometry. The SC instability can be detected by looking at the motion of the QNMs of ψ in the complex plane. To be more specific, it corresponds to a QNM going to the upper half of a complex plane. Exactly at the critical temperature we have a static mode at the origin of the complex plane, $\omega = 0$, and the source at the boundary vanishes,

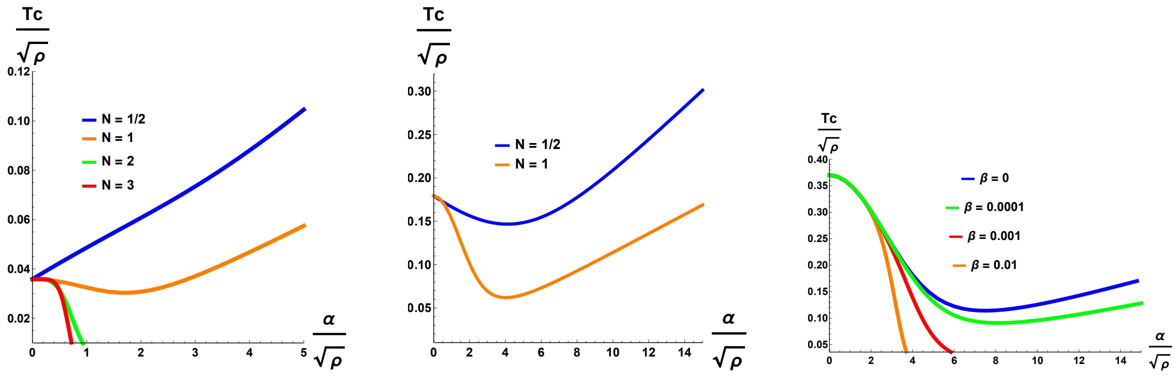


Figure 3.21: Critical temperature as a function of α for: **Left:** Model 3.238 $q = 1$, $\Delta = 2$. **Centre:** Model 3.238 $q = 3$, $\Delta = 2$. **Right:**$V(X) = X/2m^2 + \beta\, X^5/2m^2$ for different choices β. All the curves have a runaway behavior at $\alpha \to \infty$, and only the shape depends on the value of β.

$\psi_1 = 0$. We will solve numerically the equations for the whole background, and confirm this explicitly.
The scalar field is described by eq. (3.244), which in the normal phase becomes:

$$\psi'' + \left(-\frac{2}{u} + \frac{f'}{f}\right)\psi' + \left(\frac{q^2\rho^2}{f^2} - \frac{M^2L^2}{u^2 f} - \frac{\kappa H\, L^2}{u^2 f}\right)\psi = 0\,, \tag{3.253}$$

where $f(u)$ is the emblackening factor of the BH solution.
To determine the critical temperature T_c we need to find the *highest* temperature, at which there exists a solution to eq. (3.253), satisfying the $\psi_1 = 0$ condition. In this case for $T < T_c$ the system is in a superconducting state, with a non-vanishing condensate ψ_2.

With these tools, we are able to describe completely the behaviour of the critical temperature T_c in function of the various parameters of the system.
In figure 3.20 we plot the IR instability region on the (Δ, q) plane, for the model 1, (3.236) and the model 2, (3.237) for some representative parameters. The $T_c = 0$ line is in perfect agreement with the analytical BF argument while the rest of the plane is built via the numerical routine described above. Two clear statements, which are generic for all the models we considered, can be extracted from those plots:

- Increasing the charge q enhances the SC instability;
- Increasing the conformal dimension Δ of the charged scalars on the contrary disfavoures it.

In addition, the behaviour of the critical temperature in function of the disorder strength, *i.e.* graviton mass, is shown for the various models in fig.3.21. The results are pretty curious because despite momentum dissipation disfavoures the SC instability, decreasing the critical temperature T_c, at its small values the behaviour then changes drastically showing that increasing further the graviton mass one can enhance the formation of the SC phase. We do not have a clear interpretation of this fact which is anyway observed also in other models with momentum dissipation such as [199]. One clear result is that non linearities in the MG potential do decrease in a generic way the critical temperature T_c of the SC transition.

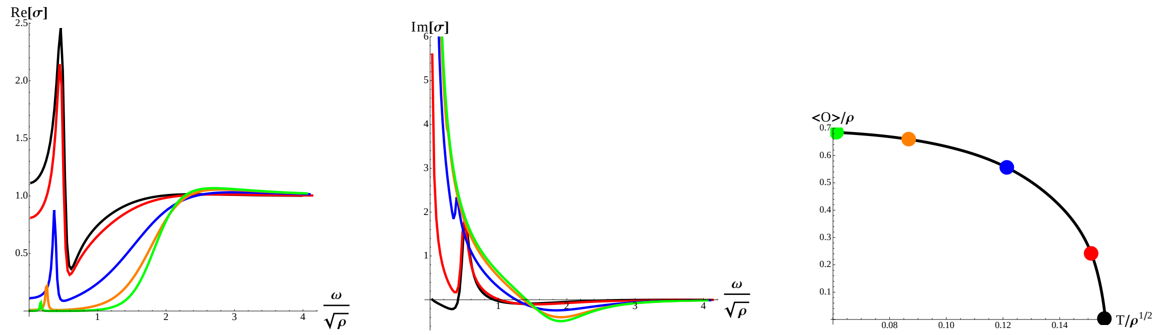


Figure 3.22: The AC conductivity for the model (3.237) for some representative parameters. Black line is at the temperature, slightly below the corresponding critical temperature $T_c/\rho^{1/2}$, and matches the result of the normal phase calculation at $T = T_c$. Red, blue, orange and green lines are for $T/\rho^{1/2} = 0.15, 0.12, 0.09, 0.06$, respectively. Notice that as we decrease the temperature, between blue and orange line, the peak in the imaginary part of the AC conductivity disappears. We call the corresponding critical temperature $T''/\rho^{1/2}$. We also provide the condensate as a function of temperature and mark the points where we calculated the AC conductivity.

We can then construct in a full fashion the complete SC solution and check via Free Energy analysis that the SC transition indeed appears and it does as a 2nd order phase transition. We skip the details of such procedure for which one can read [231].
Additionally we can compute, using the standard holographic method, the electric optical conductivity of the system across the SC transition. The results are shown in fig. 3.22 for the non linear MG model 3.237. One can notice various fact. As we expected in the normal phase the DC electric conductivity is finite, it takes the value 3.246 and it is clearly distinct from the infinite DC conductivity appearing in the SC phase. In the normal phase the AC conductivity is characterized by a mid-infrared peak which was first observed in [218] and described in this thesis. Decreasing the temperature, and letting the condensate grow, this peak gets depleted and eventually disappear at a temperature $T'' < T_c$. This suggests a possible competition between the superconducting mechanism and the momentum dissipating one. In particular it seems clear that a large superfluid density completely screens this collective excitation which in a sense gets eaten by the large condensate.

A final interesting question is studying the full phase diagram of the dual CFT which now can contain three different phases: the metallic one, the pseudo-insulating one and the SC one. The landscape of the possible outcomes is quite rich. We focus on three examples, shown in figure 3.23.

i. In the first example of fig. 3.23 a phase diagram for the non linear MG model it is shown. As expected one can provide the competition of three different phases with a quite rich phenomenology.

ii. Close to the fine tuned point we discuss one can produce a SC dome shaped region in the middle of the phase diagram which:

   - In the case of the linear potential 3.236 can be just surrounded by a metallic phase.
   - In the case of the non linear potential 3.237 can be embedded in a richer phase diagram which shows interesting features.

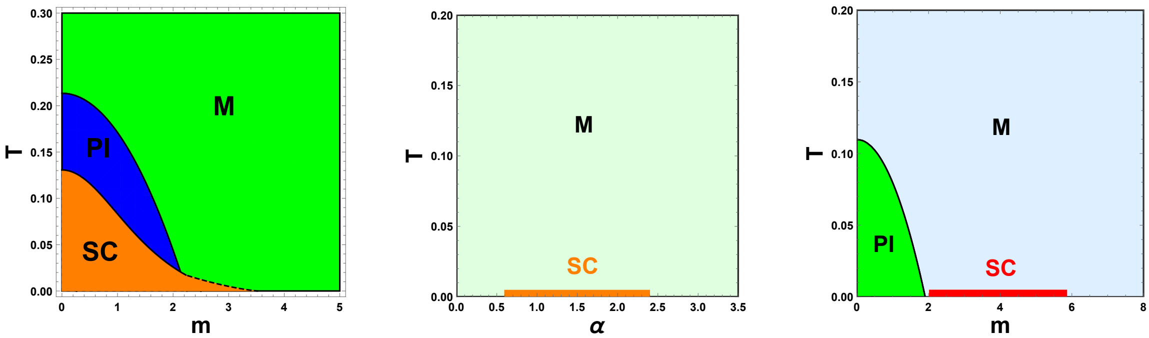


Figure 3.23: Full phase diagram for three different cases. Green region is a normal pseudo-insulating phase, grey region is a normal metallic phase, red region is a superconducting phase.**Left:** 3.237 model for some representative parameters. **Center:** 3.237 model at the fine tuned "dome-point". **Right:** 3.236 model at the fine tuned "dome-point". Plots from [231].

The first important thing to notice is that we are considering the phase diagram on the temperature-disorder strength plane, which is very different from what is usually shown for the High Tc SC experiments. There the horizontal axes is doping, which should be related to our chemical potential μ. Therefore the SC dome has no direct link with the famous CM results we are aiming to reproduce.

Additionally, because this behaviour, even if generic, is present just in a small fine tuned region in the plane $\{\Delta, q\}$ the critical temperature T_c of its edge is very small and with the actual techniques we are not able to detect it nor describe it with accuracy.

In order to get closer to the actual High Tc phase diagram we need to introduce more ingredients into the holographic model in addition of the ones we already considered.

Towards the High-Tc phase diagram

The idea is to introduce novel fields in the bulk and a more generic action governing them and check if a phase diagram, which shares similarities with the experimental one characterizing High-Tc superconductors, is actually obtainable within the holographic framework. We take inspiration from the model described in [203] adding to it an additional translational symmetry breaking sector, *i.e.* massive gravity.

We consider the following bulk degrees of freedom: the metric $g_{\mu\nu}$, two $U(1)$ gauge fields A_μ, B_μ, the complex scalar field ψ, and two neutral scalars ϕ^I, $I = x, y$. Here x, y are spatial coordinates on the boundary. We will denote the radial bulk coordinate as u. The boundary is located at $u = 0$, the horizon is located at $u = u_h$.

We want to describe a system of charge carriers, coexisting with a media of impurities. The density of the charge carriers is denoted by ρ_A and is dual to the gauge field A_μ while the density of impurity ρ_B is dual to the gauge field B_μ. The quantity

$$\mathbf{x} = \rho_B/\rho_A \tag{3.254}$$

is called the doping parameter and represents the amount of charged impurities present in the system [203].

The total action of the model is written as:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} + \mathcal{L}_c + \mathcal{L}_s\right) \tag{3.255}$$

where we fixed the cosmological constant $\Lambda = -3/L^2$, and denoted the Lagrangian densities for the charged sector [203], and the neutral scalar sector [218] as:

$$\mathcal{L}_c = -\frac{Z_A(\chi)}{4} A_{\mu\nu}A^{\mu\nu} - \frac{Z_B(\chi)}{4} B_{\mu\nu}B^{\mu\nu} - \frac{Z_{AB}(\chi)}{2} A_{\mu\nu}B^{\mu\nu} \tag{3.256}$$

$$- \frac{1}{2}(\partial_\mu\chi)^2 - H(\chi)(\partial_\mu\theta - q_A A_\mu - q_B B_\mu)^2 - V_{int}(\chi) \tag{3.257}$$

$$\mathcal{L}_s = -2m^2 V(X)\,. \tag{3.258}$$

Here the $A_{\mu\nu}$ and $B_{\mu\nu}$ stand for the field strengths of the gauge fields A_μ and B_μ respectively. Following [203] we decomposed the charge scalar as $\psi = \chi e^{i\theta}$. We also defined:

$$X = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^I\,. \tag{3.259}$$

The most general black-brane ansatz we consider is:

$$ds^2 = \frac{L^2}{u^2}\left(-f(u)e^{-\tau(u)}dt^2 + dx^2 + dy^2 + \frac{du^2}{f(u)}\right)\,, \tag{3.260}$$

$$A_t = A_t(u)\,, \qquad B_t = B_t(u)\,, \tag{3.261}$$

$$\chi = \chi(u)\,, \qquad \theta \equiv 0\,, \tag{3.262}$$

$$\phi^x = \alpha\, x\,, \qquad \phi^y = \alpha\, y\,. \tag{3.263}$$

The corresponding equations of motion are provided in the original paper [232]. The temperature of the black brane (3.260) is given by:

$$T = -\frac{e^{-\frac{\tau(u_h)}{2}} f'(u_h)}{4\pi}\,. \tag{3.264}$$

We will be considering:

$$V_{int}(\chi) = \frac{M^2\chi^2}{2}\,. \tag{3.265}$$

Solving the χ e.o.m. near the boundary $u = 0$ one obtains $\chi(u) = C_-\,(u/L)^{3-\Delta} + C_+\,(u/L)^{\Delta}$, where $(ML)^2 = \Delta(\Delta - 3)$. Here C_- is the source term, which one demands to vanish, and C_+ is the v.e.v. of the dual charge condensate operator, $C_+ = \langle\mathcal{O}\rangle$. The Δ is equal to the scaling dimension of the operator $\mathcal{O}$. Following [203]we fix the scaling dimension to be $\Delta = 5/2$.
In the normal phase the charge condensate vanishes, and the charged scalar field is trivial, $\chi \equiv 0$. Solving the background equations of motion we obtain $\tau \equiv 0$, along with:

$$f(u) = u^3 \int_{u_h}^{u} dy\, \frac{\rho_A^2(1 + \mathbf{x}^2)\, y^4 + 4\,(mL)^2\, V(\alpha^2\, y^2) - 12}{4y^4}\,, \tag{3.266}$$

$$A_t(u) = \rho_A(u_h - u)\,, \qquad B_t(u) = \rho_B(u_h - u)\,. \tag{3.267}$$

The temperature in the normal phase is given by:

$$T = \frac{12 - \rho_A^2(1 + \mathbf{x}^2)\, u_h^4 - 4\,(mL)^2\, V(\alpha^2\, u_h^2)}{16\pi u_h}\,. \tag{3.268}$$

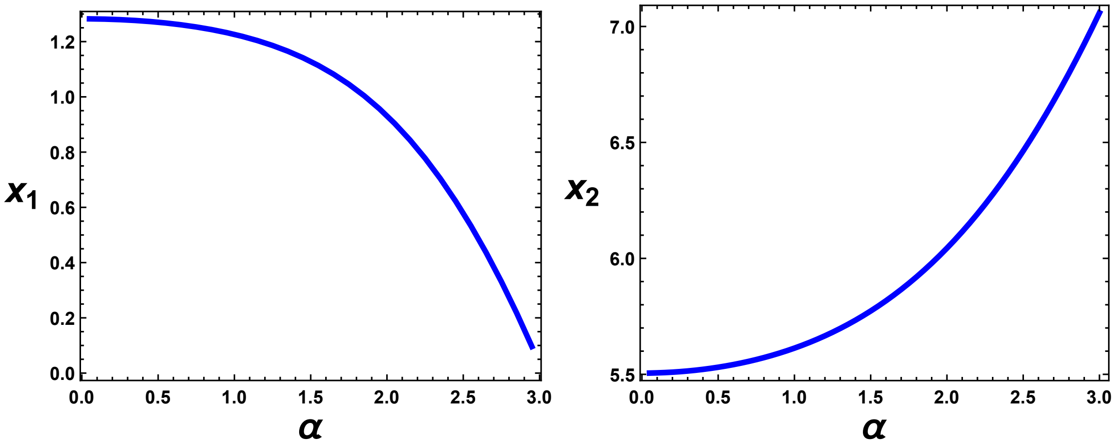


Figure 3.24: The boundaries of the zero-temperature IR instability region on the doping line for the model (3.272), with the translational symmetry broken by the neutral scalars with the linear potential $V(X) \sim X$.

Using the membrane paradigm one can calculate analytically the DC conductivity in the normal phase. Its value for a general neutral scalars Lagrangian V is given by:

$$\sigma_{DC} = 1 + \frac{\rho_A^2 u_h^2}{2\, m^2\, \alpha^2\, \dot{V}(u_h^2 \alpha^2)}\,. \tag{3.269}$$

The features of this normal phase have been already described in [218] and in the previous sections.
We can then perform the usual routine to check the instability of such a normal phase towards the spontaneous formation of a charged condensate encoded in the scalar profile $\chi(u)$. Following [203] we define the following expansion of the couplings:

$$H(\chi) = \frac{n\,\chi^2}{2}\,, \quad Z_A(\chi) = 1 + \frac{a\,\chi^2}{2}\,, \quad Z_B(\chi) = 1 + \frac{b\,\chi^2}{2}\,, \quad Z_{AB}(\chi) = \frac{c\,\chi^2}{2}\,. \tag{3.270}$$

and define the $U(1)_{A,B}$ charges to be $q_A = 1\,,\, q_B = 0$.
A natural place to start searching for superconductor is at zero temperature. The BF violation argument, within this setup, leads to the instability condition:

$$(2\,M^2 - u_0^4\,(a + 2\,c\,x + b\,x^2))\,(6 + m^2((\alpha\,u_0)^2\,\dot{V} - 2V)) - 2\,n\,u_0^4\,(q_A + q_B\,x)^2 < 0\,, \tag{3.271}$$

where dot stands for derivative of V w.r.t. its argument and u_0 for the radial position of the extremal horizon, $T(u_0)$=0.
To obtain a superconducting dome on the temperature-doping plane $(T, \mathbf{x})$, one needs to fix the parameters of the model in such a way that zero-temperature superconducting instability appears in an interval $[\mathbf{x}_1, \mathbf{x}_2]$, between two positive values $\mathbf{x}_{1,2}$ of the doping parameter. The specific model determined by the parameters:

$$a = -10\,, \quad b = -\frac{4}{3}\,, \quad c = \frac{14}{3}\,, \quad n = 1\,. \tag{3.272}$$

has been extensively studied, and it was pointed out that in the interval $\mathbf{x} \in [\mathbf{x}_1, \mathbf{x}_2]$, $\mathbf{x}_1 \simeq 1.28$, $\mathbf{x}_2 \simeq 5.51$ at zero temperature the effective mass of the scalar field χ violates the AdS_2 BF bound

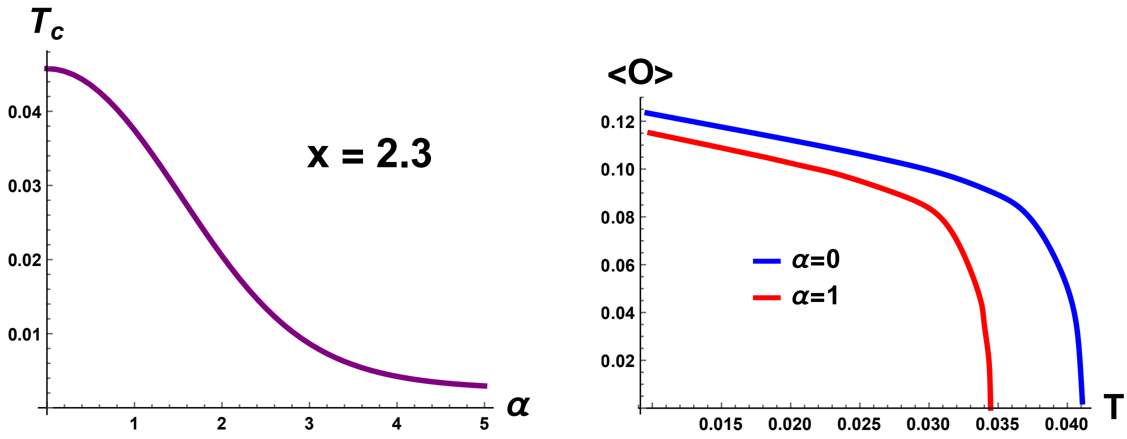


Figure 3.25: **Left:** Critical temperature $T_c(\alpha)$ for the model (3.272), with the translational symmetry broken by the neutral scalars with the linear potential $V(X) \sim X$. Here the doping is fixed to be $\mathbf{x} = 2.3$. **Right:** Condensate for the model (3.272), with the doping fixed to $\mathbf{x} = 2$, with broken translational symmetry. **Left:** The linear model linear potential $V(X) \sim X$ with $\alpha = 1$, plotted next to the translationally-symmetric system $\alpha = 0$.

in absence of momentum dissipation. Our main goal is to add the breaking of translational symmetry and study the consequences on such a SC dome found in [203].
We observe that for $\alpha \neq 0$ the instability persists, although the 'depth' of the AdS_2 BF violation becomes smaller, and therefore we expect the corresponding critical temperature of the superconducting phase transition to be lower. At zero temperature the SC range of values of the doping parameter increases when the translational symmetry breaking parameter α gets bigger. We plot the α-dependence of the boundary points of the IR instability region, $\mathbf{x}_{1,2}(\alpha)$, in figure 3.24.

The critical temperature T_c of a second-order phase transition can be determined by studying the dynamics of the scalar $\chi(u)$, considered as a probe in a finite-temperature normal phase background. In accordance with our expectations from the zero-temperature instability analyses we observe a decrease of the critical temperature with α, as shown in figure 3.25. Moreover the presence of a non null disorder strength, *i.e.* the graviton mass α, depletes the value of the condensate as well as shown in figure 3.25.

Now let us fix the value of α and plot the critical temperature as a function of the doping parameter $\mathbf{x}$, see figure 3.26. The breaking of translation symmetry preserves the superconducting dome structure exhibited by the model (3.272), and merely diminishes a little the critical temperature. Now let us consider the model with translational symmetry broken by neutral scalars governed by the non-linear non linear potential $V(X) \sim X + X^5$. We fix $\alpha = 0.5$, $m = 1$ and determine the critical temperature $T_c(\mathbf{x})$. In figure 3.26 we combine this with the temperature $T_0(\mathbf{x})$ of the metal/pseudo-insulator phase transition (MIT), and obtain the full phase diagram of the system with the superconducting phase enclosed inside a dome.

This means that even if momentum dissipation unfavores the SC phase it is still possible to achieve a SC dome-shaped region as in actual High-Tc superconductors and having a normal phase with a finite DC conductivity. The main result is to show that the SC dome-shaped region built in [203] can be completed with a simple momentum dissipation mechanism and embedded in

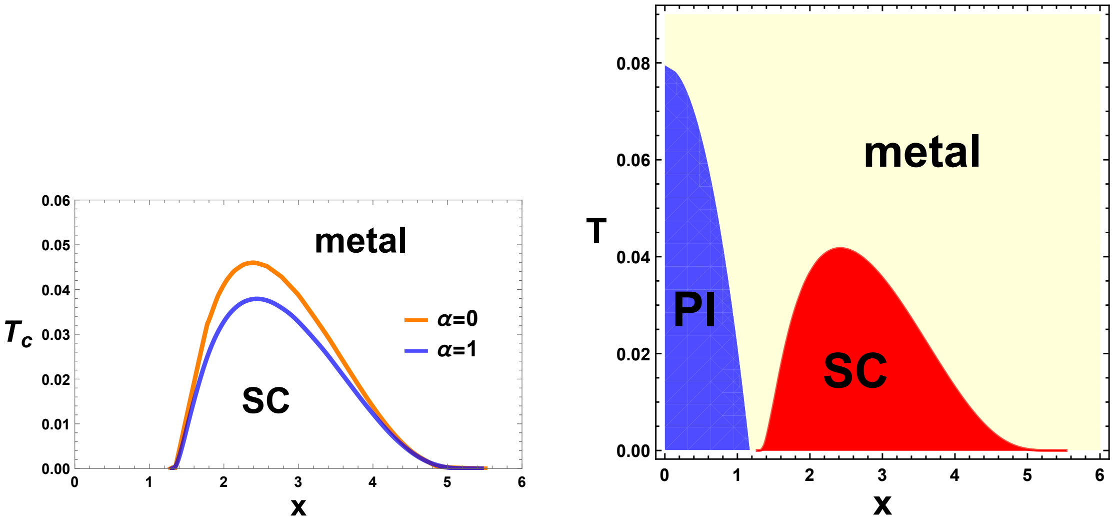


Figure 3.26: **Left:** Phase diagram in the $(T, \mathbf{x})$ plane for the model (3.272) coupled to the neutral scalars with the linear potential $V(X) \sim X$). We compare the case $\alpha = 0$ of [**?**], and the system with broken translational symmetry, at $\alpha = 1$. **Right:** Phase diagram in the $(T, \mathbf{x})$ plane for the model (3.272) coupled to the neutral scalars with the non linear potential $V(X) \sim X + X^5$. We fixed $\alpha = 0.5$ and $m = 1$.

a normal phase region featuring a finite DC conductivity. This represents a further step towards reproducing holographically the phase diagram for High-Tc superconductors [233].

Part III

Final Remarks

I think and think for months and years.
Ninety-nine times, the conclusion is false.
The hundredth time I am right.

Albert Einstein

In this last section we aim to give a critical review of what AdS/CMT did, is doing and could do in the nearby future.

What did gauge-gravity duality teach us about condensed matter physics?

What can gauge-gravity duality teach us (more) about condensed matter physics?

Quoting [234] these will be our main questions to review the actual status of the framework and its possible future achievements. A critical, and perhaps even provocative review, can be found in [235]. In the last years AdS-CMT started to become a novel effective theory approach able to capture several CM mechanisms and features opening unexpected new directions and techniques. In what is left we will wandering about various condensed matter topics with a proposing attitude discussing the impact of gauge-gravity duality on their understanding, as an **holographic effective theory for condensed matter**.

Disclaimer: from this point forward I will simply be thinking out loud. Do not take it too seriously!

3.9 Holographic effective theories for condensed matter

Strange Metals and universal scalings

The first important condensed matter issue refers to the so-called *Strange Metals* [236, 237]: a large class of materials whose transport properties obey unusual temperature scalings which are not in agreement with the Fermi Liquid Theory (FL), one of the CM last century pillars.

In short, Strange metals, which by the way realize the normal phase of most of the High-Tc superconductors materials we discussed previously (see fig.3.27), are characterized by the following (and actually also other exotic features):

$$\rho \sim T\,(\neq T^2)\,, \qquad \theta_H \sim T^2\,(= T^2)\,. \tag{3.273}$$

where ρ and θ_H are respectively the electric resistivity and the Hall angle and the scalings in curved brakets represent the FL expectations.

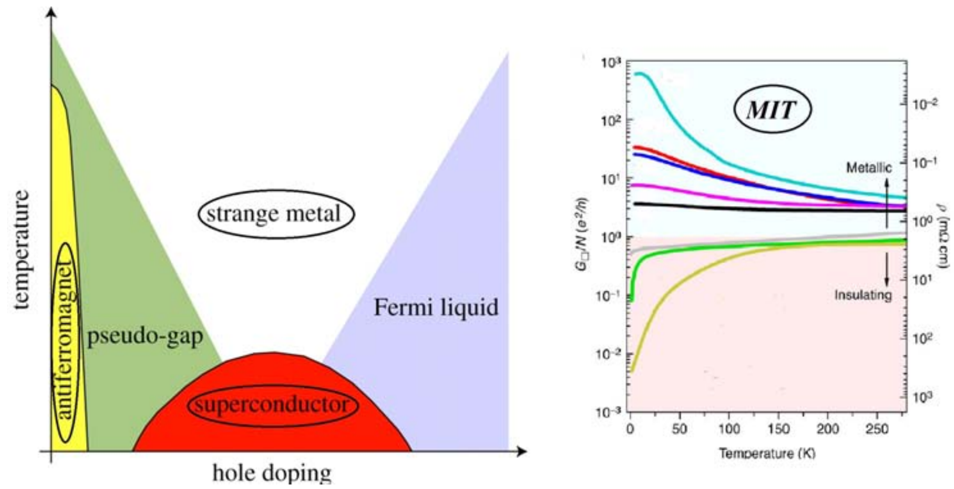


Figure 3.27: Sketch of the various open issues in CM related which the AdS-CMT started to attack. Holographic effective theories for CM could be a new and suitable tool to adress these issues: Strange Metals scaling, nature of High Tc Superconductors, mechanism underlying Strongly Correlated Insulators, Many body localization.

A possible explanation, originally proposed by Anderson in [238], refers to the possibility that the insertion of an effective spin-spin interaction leads to two different scattering rates for longitudinal and transverse modes which account for the different scalings $\sim T$, T^2 observed. Understanding and describing the Fermi Liquid scalings is one of the most pressing and early issue that the AdS-CMT program tried to adress. Importantly the achievement of such a task needs the introduction of a momentum dissipative sector into the game and it has been initiated in [177, 184, 239].
Realizing a linear in T resistivity turns out to be obtainable via the introduction of an opportune dilatonic field ϕ [184] in the context of holographic theories with broken translational symmetry. On the contrary accomodating both the linear in T resistivity and the Hall angle scaling seems to be an harder target. It [239] it was noticed that, in full generality, the conductivity and the Hall angle of an holographic system with momentum dissipation aquires the structure:

$$\sigma \,=\, \sigma_0 \,+\, \sigma_{diss}\,, \qquad \Theta_H \sim \frac{B}{\mathcal{Q}}\,\sigma_{diss}\,. \tag{3.274}$$

where σ_0 and σ_{diss} can have in general different temperature scalings.
It was therefore suggested that in order to accomodate the Stange Metals nature one needs the first term σ_0 (coinciding with the electric conductivity at zero heat current) to be the dominating one in the electric conductivity such that the scalings of dual theory would go like:

$$\rho \,\sim\, \sigma_0^{-1}\,, \qquad \Theta_H \sim \sigma_{diss}\,. \tag{3.275}$$

and the wanted phenomenology would be recovered whenever:

$$\sigma_0 \,\sim\, 1/T, \qquad \sigma_{diss} \,\sim\, 1/T^2\,. \tag{3.276}$$

Unfortunately, it has been recently pointed out [240] that, at least in the case where the charge density and the magnetic field are not relevant operators, this is not achievable.
The original dream of describing the Strange Metals phenomenology is still in the to-do list of holography [235]. Several directions are worth to consider:

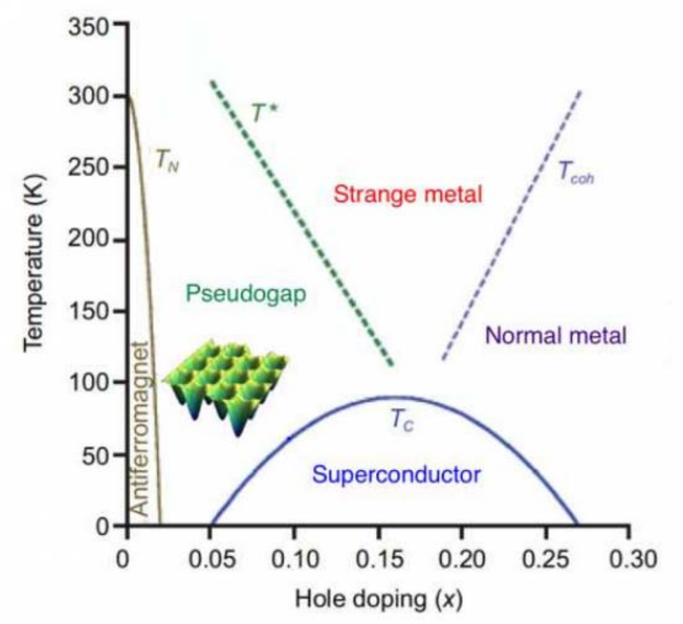


Figure 3.28: Schematic phase diagram of the high-Tc cuprates. Figure taken from [203].

- Consider the cases with the charge $\mathcal{Q}$ or the magnetic field B being relevant operators of the dual CFT and having a strong impact on the IR physics.
- Consider more generic system by adding a Chern-Simons term $\sim F \wedge *F$ and studying the effects of such a term to the transport properties of the dual material.
- Introducing additional gauge fields as in [241] or with the idea of implementing holographically some kind of spin degrees of freedom with the aim of realizing the original Anderson's proposal ([238]) of two different scattering rates.

High-Tc Superconductors

The understanding of the properties of high-temperature superconductivity, especially its microscopic origin, is significant in theory and application. In an attempt to build a consistent overall picture of those unconventional materials, mapping out and forming a rudimentary understanding of the temperature/doping phase diagram becomes a primary focus of research which holography can adress. The phase diagram is a landscape of exotic states of matter (see fig.3.28) and it contains an high degree of universality, being indeed shared by many unconventional superconductors, such as cuprates and iron-based pnictides. Although many efforts have been made, so far the phase diagram in the temperature-doping plane, putting all ingredients together, has not been assembled. The competition of various phases, namely the antiferromagnetic order, the striped order and the superconducting order, has been originally analyzed using the techniques of holography in [203] and later on in [233]. Using a rather simple model an analogue of the real phase diagram 3.28 has been built and described. As an only shortcoming, momentum dissipation have not been introduced and as a consequence the normal phase of such a diagram is not a proper metallic state with a finite DC electric conductivity. The question whether such a diagram, resembling so much the experimental data, keeps its features in the presence of momentum

dissipation is a very relevant question which can provide a step further towards the completion of a realistic phase diagram for unconventional superconductors.
As a first check, in [232] we studied whether the superconducting dome region survives in the presence of momentum dissipation and we got positive results. One can indeed build a SC dome region, as in [203], where the normal phase is realized by a proper metal with finite DC conductivity (unfortunately not a Strange Metal as in the actual critical region of the phase diagram). The question whether the other phases survive in the presence of translational symmetry breaking represents still an open problem which the holographic techniques could attach in a rather systematic way. Just time will tell us!

Mott Insulators

Strongly correlated materials are interesting because interactions play a very significant role and therefore they are not easy to describe. One can distinguish 3 different mechanisms that can be responsible for the nontrivial (electrical) response: electron-phonon (e-ph), electron-disorder (e-dis), and electron-electron (e-e) interactions. Usually, Mott insulators refer to the materials that are dominated by the latter: charge-carrier self-interactions. The heuristic picture that summarizes the Mott behaviour (sometimes referred to as Mottness) is that of an electronic traffic jam: strong enough e-e interactions should, of course, prevent the available mobile charge carriers to efficiently transport charge. Because the lack on controllable computational tools, it is worth and interesting to ask whether and how holography can consistently incorporate electron-electron interactions within its description.
The first positive results have been obtained in the context of probe fermions models [242–244], where the introduction of a dipole-interaction, was proved to lead to the dynamical formation of a gap in the Fermi surface which shares lots of features with the actual Mott insulators' nature. Under a completely different perspective, it was recently showed [221], that introducing non-linearities in the bulk charge sector can provide an interesting phenomenology. In particular, using an effective model with generic non-linear self interactions for the gauge field A_μ of the form:

$$\sim \mathcal{K}\left(F_{\mu\nu}\, F^{\mu\nu}\right) \tag{3.277}$$

it has been shown that insulating states, sharing several features with real Mott insulators, can be obtained. In addition, upon dialing the non-linearities of the system, representing the strenght of the "electrons" self-interactions, possible metal-insulator transitions could appear.
Despite the hunt for a dual for Mott insulators is still in progress, several interesting developments have been recently performed and the task is certainly at the horizon.

Localization

Strong disorder can induce insulating behaviours due to Localization mechanisms. The charged excitations in a strongly coupled medium can get localized (see fig.3.29) making the correspondent conductivities drop down.

The question whether holography can reproduce this behaviour and give some hints about its explanation has been recently considered in [209]. It was claimed, that because the existence of a lower bound in the electric conductivity for generic "simple" holographic model, gauge-gravity duality is not able to reproduce Localization. On the contrary it seems that in the limit of strong disorder, holography just turns into a very incoherent metallic state.
The existence of a generic and universal lower bound for the electric conductivity has been proven to be generically incorrect in [214, 220, 222], showing that more complicated holographic setups

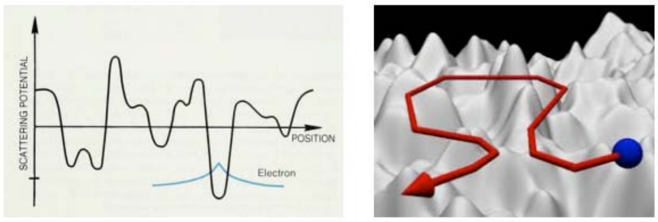


Figure 3.29: Localization of an electron in a randomly distributed scattering potential.

could reproduce insulating states driven by disorder. The nature of those insulating phases and whether they are connected to some sort of Localization mechanism is still a task in progress. It seems indeed pretty generic [245] that an analogous lower bound on the thermal conductivity remains, hinting towards the absence of localized phases in holography.
It is not clear whether this shortcoming is due to the large N limit or most likely to the strong coupling regime. The presence and the search for localization mechanisms is an open and interesting question which more generic holographic effective models can adress. Probably, the homogeneous models (massive gravity, Q-lattices, helical lattices) are not enough to capture the effects of localization, and more complicated disordered setups (for example [246, 247]) may be the answer.

Metal-Insulator transitions and Mottness

In the last years, in the holographic community, many efforts and advances have been made in the direction of finding out possible insulating states and describing quantum phase transitions between the latter and proper metallic states, *i.e.* MIT. The landscape of the different mechanisms leading to MITs is quite rich and several correlations between physical observable are present. The idea of exploring further this topic is of course intriguing with the aim of, not only identifying those mechanims, but also extracting physical results and maybe one day also possible predictions. See for example [248] for an initial holographic study about the existence of colossal magnetoresistance at the metal-insulator transition.

Universal bounds

The presence of possible universal bounds on physical observable may be a solid hint for the emergence of a universal behaviour shared by many UV fixed points, which is of course of great interest.
Without doubts the most famous universal bound in the context of strongly coupled system is the well known KKS viscosity/entropy ratio $\eta/s \geq 1/4\pi$. It has been recently claimed [211, 212, 230] that such a generic bound, which all the experimental checks respect so far, can be violated in particular theories which enjoy translational symmetry breaking. From the theoretical level, momentum dissipation does not lead directly to such a violation, but particular realizations (labelled as solid-type) do. The presence of an effective graviton mass for the transverse traceless helicity-2 mode provides indeed a violation of the bound which is arbitrary and can eventually drop down to a null value at $T = 0$. The understanding of such a violation is still in progress and the possibility of a generalization of the KSS is being already considered but without particular success. All in all, this violation, which holographic theories seem to suggest, has still to be properly understood and it can be moreover verified in possible experiments with viscoelastic

materials or solid materials in the future.
On a different, but somehow connected, line possible bounds on the diffusive constants of strongly coupled metals, known as incoherent metals, have been conjectured in [210, 249]. For simple holographic models, dealing with massive gravity theories, such a conjecture resulted to be incorrect [208]. Nevertheless, recently, an unexpected connection between chaos (and possible bounds associated with it [250]) has been analyzed in the framework of holographic theories with momentum dissipation and the results [215] are pretty promising and worth of further investigation.

What is massive gravity exactly?

Is MG effectively encoding a lattice?

Is MG an averaged version of disorder?

Despite the widespread convictions that holographic massive gravity models provide a nice effective description of translational symmetry breaking mechanisms, the fundamental meaning and intepretation of these theories are still elusive and not understood. What has been proven in [180] is that, at least at linear order, an explicit holographic lattice gives rise to a mass for the graviton. Moreover, it is also clear that HMG do not share typical features of realistic lattices such as commensurability [205] meaning that the identification of [180] is valid just at linear level and no more. There is certainly a connection between HMG and disorder. Indeed, disorder can relax momentum via scatterings that involve only low-momentum ($k \sim 0$) processes, while in a lattice you relax momentum via high momentum ($k \sim k_L$) processes. MG is exactly realizing the first scenario and this is an important fact; however, it is only one of many features of disorder. It would be extremely interesting and encouraging to test if there is any deeper connection between HMG theories and theories with explicit disorder, in particular up to the limit of strong disorder. Thinking of MG as an averaged, "mean field version", of translational symmetry breaking it would be important to check wether at that level one can distinguish between a periodic breaking (*i.e.* lattice) or a disordered one (*i.e.* impurities). This will also somehow adress the question of how many features of disordered systems MG can share or at least mimick.

Phonons physics and elasticity

Can we reproduce phonons physics and elasticity through holography?
Can massive gravity be the path?
From the CFT perspective it is pretty clear that MG realizes an explicit breaking of translational symmetry with apparent no link to phonons physics. It would be interesting to see if there is a sensible limit where the holographic description resembles closely the typical features of phonons. If that is the case one could eventually re-build the theory of elasticity in the holographic context and improve the physical understanding of HMG theories. This is certainly something that the AdS-CMT program is still missing.
Inspiring works come from the description, in flat space, of the different spontaneous breaking patterns of Poincaré symmetry analyzed in [145,146]. A first attempt of embedding such a framework into the holographic picture has been made in [219]; pursuing this path can be a promising direction to get a holographic low energy description of phonons through massive gravity theories. Similar suggestions have been made in [251] where potential connections between the theory of elasticity and linearized massive gravity have been speculated.
Certainly there is something lying over there, we just need to reveal it.

3.10 Farewell

We conclude here our long journey.

The take-home message is that the Gauge-Gravity duality provides a huge an unexplored playground where researchers from unbelievable distant fields could meet together and build up a common ground of ideas, questions and targets. It is an extraordinarily interdisciplinar setup which puts in contact the most fundamental questions and theoretical frameworks of physics with the most intriguing and exotic experimental results. As a matter of fact, such a connection may give in the future incredible outcomes both at the theoretical abstract level and at the "real-world" experimental one.

The beast got released, we just have to ride it along the uncharted path and time will tell us...

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