Abstract

Consider a graph $G$ on $n$ vertices satisfying the following Ore-type condition: for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) > 3n/2$. We conjecture that if we color the edges of $G$ with 2 colors then the vertex set of $G$ can be partitioned to two vertex-disjoint monochromatic cycles of distinct colors. In this paper we prove an asymptotic version of this conjecture.

1 Background, summary of results.

In this paper, we consider the problem of partitioning the vertices of edge-colored graphs into monochromatic cycles. For simplicity, a colored graph means an edge-
colored graph in this paper. In this context it is conventional to accept *empty graphs* and *one-vertex graphs* as a cycle (of any color) and also *any edge* as a cycle (in its color). With this convention one can define the *cycle partition number* of any colored graph $G$ as the minimum number of vertex disjoint monochromatic cycles needed to cover the vertex set of $G$. For complete graphs, [6] posed the following conjecture.

**Conjecture 1.** *The cycle partition number of any $t$-colored complete graph $K_n$ is $t$.***

The $t = 2$ case of this conjecture was stated earlier by Lehel in a stronger form, requiring that the colors of the two cycles must be different. After some initial results [2, 8], Luczak, Rödl and Szemerédi [19] proved Lehel’s conjecture for large enough $n$, which can be considered as a birth of certain advanced applications of the Regularity Lemma. A more elementary proof, still for large enough $n$, was obtained by Allen [1]. Finally, Bessy and Thomassé [5] found a completely elementary inductive proof for every $n$.

The $t = 3$ case of Conjecture 1 was solved asymptotically in [13]. Pokrovskiy [21] showed recently (with a nice elementary proof) that the path partition number of any 3-colored $K_n$ is at most three (for any $n \geq 1$). Later Pokrovskiy [22] surprisingly found a counterexample to Conjecture 1 for all $t \geq 3$. However, in the counterexample all but one vertex can be covered by $t$ vertex disjoint monochromatic cycles, so perhaps the following weaker statement holds.

**Conjecture 2.** *For every integer $t \geq 2$ there exists a constant $c = c(t)$ such that for any $t$-colored graph $G$ there are $t$ vertex disjoint monochromatic cycles of $G$ that cover at least $n - c$ vertices.*

For general $t$, the best bound for the cycle partition number is $O(t \log t)$, see [9]. Note that it is far from obvious that the cycle partition number of $K_n$ can be bounded by any function of $t$.

In [3] we addressed the extension of the cycle and path partition numbers from complete graphs to arbitrary graphs $G$.

Recently, Schelp [23] suggested in a posthumous paper to strengthen certain Ramsey problems from complete graphs to graphs of given minimum degree. In particular, he conjectured that with $m = R(P_n, P_n)$, minimum degree $3m/4$ is sufficient to find a monochromatic path $P_n$ in any 2-colored graph of order $m$.

1 Influenced by this, in [3] we posed the following

**Conjecture 3.** *If $G$ is an $n$-vertex graph with $\delta(G) > 3n/4$ then in any 2-edge-coloring of $G$, there are two vertex disjoint monochromatic cycles of different colors, which together cover $V(G)$.***

1 Some progress towards this conjecture have been done in [14] and [4].
That is, the above mentioned Bessy-Thomassé result [5] would hold for graphs with minimum degree larger than $3n/4$. Note that the condition $\delta(G) \geq 3n/4$ is sharp (see [3]). Indeed, consider the following $n$-vertex graph, where $n = 4m$. We partition the vertex set into four parts $A_1, A_2, A_3, A_4$ with $|A_i| = m$. There are no edges from $A_1$ to $A_2$ and from $A_3$ to $A_4$. Edges in $[A_1, A_3], [A_2, A_4]$ are red and edges in $[A_1, A_4], [A_2, A_3]$ are blue, inside the classes any coloring is allowed. In such an edge-colored graph, there are no two vertex disjoint monochromatic cycles of different colors covering $G$, while the minimum degree is $3m - 1 = 3n/4 - 1$.

In [3] we proved Conjecture 3 in the following asymptotic sense.

**Theorem 1.** For every $\eta > 0$, there is an $n_0(\eta)$ such that the following holds. If $G$ is an $n$-vertex graph with $n \geq n_0$ and $\delta(G) \geq (\frac{3}{4} + \eta)n$, then every 2-edge-coloring of $G$ admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of $G$.

The proof of Theorem 1 followed a method of Luczak [18]. The crucial idea of this method is that “cycles” or “paths” in a statement to be proved are replaced by “connected matchings”. In a connected matching, the edges of the matching are in the same component of the graph. We prove first this weaker result, then we apply this to the cluster graph of a regular partition of the target graph. Through several technical details, the regularity of the partition is used to “lift back” the connected matching of the cluster graph to a path or cycle in the original graph.

In this paper we go one step further and consider graphs satisfying an Ore-type degree condition instead of a minimum degree condition. Here we call a degree condition Ore-type if it gives a lower bound on the degree sum for any two non-adjacent vertices. There has been a lot of efforts in trying to extend results from minimum degree conditions to Ore-type conditions. The first result of this type was proved by Ore [20]: If for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) \geq n$, then $G$ is Hamiltonian. Some other results of this type include for example [7] (Ore-type conditions for $k$-ordered Hamiltonian graphs), [15] (Ore-type results on equitable colorings) or [16] (Ore-type versions of Brooks’ theorem).

Generalizing Conjecture 3 for graphs satisfying an Ore-type condition here we pose

**Conjecture 4.** If $G$ is an $n$-vertex graph such that for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) > 3n/2$, then in any 2-edge-coloring of $G$, there are two vertex disjoint monochromatic cycles of different colors, which together cover $V(G)$.

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2When the edges are colored, a connected red matching is a matching in a red component.
Here we prove Conjecture 4 in the following asymptotic sense.

**Theorem 2.** For every $\eta > 0$, there is an $n_0(\eta) = n_0$ such that the following holds. If $G$ is an $n$-vertex graph with $n \geq n_0$ such that for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) \geq (\frac{3}{2} + \eta)n$, then every 2-edge-coloring of $G$ admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of $G$.

The proof follows the same method as outlined above. The relaxed version of Theorem 2 for connected matchings is stated and proved in Section 2 (Theorem 3).

## 2 Partitioning into connected matchings

In this section we prove the relaxed version of our theorem for connected matchings instead of cycles.

**Theorem 3.** Let $G$ be an $n$-vertex graph, where $n$ is even and $G$ satisfies the following Ore-type condition: for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) \geq 3n/2$. If the edges of $G$ are 2-colored with red and blue, then there exist a red connected matching and a vertex-disjoint blue connected matching, which together form a perfect matching of $G$.

**Proof:** Let $C_1$ be a largest monochromatic component, say red. Let $D = V \setminus V(C_1)$.

**Case 1:** Assume $|V(C_1)| < n$, i.e. $D \neq \emptyset$. Let $A$ be those vertices in $C_1$ that are adjacent to $D$ by a blue edge. We claim that $A \cup D$ is a connected blue component. Assume to the contrary that there is a cut in blue $(A_1 \cup D_1, A_2 \cup D_2)$, where $|A_1 \cup D_2| \geq |A_2 \cup D_1|$. Now there is no edge between $V(C_1) \setminus A_2$ and $D_2$. There is no red edge by the definition of $D$ and no blue edge by the assumption on the cut. Therefore if $u$ is a vertex in $V(C_1) \setminus A_2$ and $v$ is a vertex in $D_2$ (clearly both sets are non-empty), then $\deg(u) + \deg(v) \geq 3n/2$. On the other hand, $v$ is non-adjacent to all vertices of $V(C_1) \setminus A_2$ and $u$ is non-adjacent to $D_2$. Therefore $\deg(u) + \deg(v) \leq n - 1 - |D_2| + n - 1 - |V(C_1) \setminus A_2| \leq 2n - 2 - |D_2| - |V(C_1) \setminus A| - |A_1| < 2n - n/2 = 3n/2$, a contradiction (here we used the assumption on the size of $A_1 \cup D_2$).

Let $C_2$ be this blue component covering $D = V \setminus V(C_1)$. Let $u$ be a vertex of $C_1 \setminus C_2$ and $v$ be a vertex of $C_2 \setminus C_1$. Let $|V(C_1) \setminus V(C_2)| = p$ and $|V(C_2) \setminus V(C_1)| = q$, where $p \geq q > 0$ by the choice of $C_1$. There is no edge between $u$ and $v$, in fact between $C_1 \setminus C_2$ and $C_2 \setminus C_1$. Therefore $\deg(u) + \deg(v) \geq 3n/2$. On the other hand $n - 1 - q \geq \deg(u)$ and $n - 1 - p \geq \deg(v)$. It yields $2n - 2 - (p + q) \geq \deg(u) + \deg(v) \geq 3n/2$. Therefore $p + q < n/2$ and $|V(C_1) \cap V(C_2)| > n/2$. 


If $|V(C_1)| = n$, then define $C_2$ as a largest blue component in $G$. Now $p = |V(C_1) \setminus V(C_2)|$, $q = |V(C_2) \setminus V(C_1)| = 0$.

Case 2: $|V(C_1)| = n$ and $p \leq n/2$. Now $|V(C_1) \cap V(C_2)| \geq n/2$, just as above.

Therefore in what follows, we unify the proof for the two cases we described so far. Let $G_1$ be the graph, which we obtain from $G$ by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$ and the red edges induced by $V(C_2) \setminus V(C_1)$ (if these exist).

We claim there is a perfect matching in $G_1$. Assume the contrary. By Tutte’s theorem there exists a set $X$ of vertices in $G_1$ such that the number of odd components in $G_1 \setminus X$ is larger than $|X|$, which implies that $|X| < n/2$. Let all the components of $G_1 \setminus X$ (not just the odd ones) be $D_1, D_2, \ldots, D_\ell$ in increasing order of their size, $\ell \geq |X| + 1$. Note that $\ell \geq 2$ always holds, even for $X = \emptyset$, as $n$ is even. Let $d_i = |V(D_i)|$ for $i = 1, \ldots, \ell$ and $x = |X|$.

We claim that $(V(C_1) \cap V(C_2)) \cap (\cup_{i=1}^{\ell} D_i) = \emptyset$. Assume to the contrary that $u \in C_1 \cap C_2$ and $u \in D_i$. Let $v$ be a vertex in a different $D_j$ (using $\ell \geq 2$). Clearly $u$ and $v$ are non-adjacent in $G_1$, but also in $G$ since we have not deleted any edge adjacent to $u$. Therefore $deG_1(u) + deG(v) \geq 3n/2$. Notice $deG_1(u) = deG(v)$. Now subtract the number of deleted edges adjacent to $v$, which is at most $p$ or $q$ depending on the position of $v$. We get $deG_1(u) + deG(v) \geq n$, since both $p$ and $q$ are at most $n/2$.

On the other hand $deG_1(u) \leq d_i - 1 + x$ and $deG_1(v) \leq d_j - 1 + x$. Therefore $deG_1(u) + deG_1(v) \leq d_i + d_j + 2x - 2 \leq n - 1$, since $d_i + d_j + 2x - 1$ is at most the number of vertices. This contradiction implies that $(V(C_1) \cap V(C_2)) \subseteq X$. However, this is impossible since $|V(C_1) \cap V(C_2)| \geq n/2$ and $x < n/2$. Therefore $G_1$ contains a perfect matching.

Case 3: $|V(C_1)| = n$ and $p > n/2$, so the largest blue component has size at most $n/2$. Again we get $G_1$ from $G$ by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$. We claim again that there is a perfect matching in $G_1$ and use the same set-up as above. First we show that $V(C_2) \subseteq X$. As before, we select a hypothetical vertex $u$ in $C_2 \cap D_i$ and a vertex $v$ in a different component $D_j$. Clearly $u$ and $v$ are non-adjacent in $G_1$, but also in $G$ since we have not deleted any edge adjacent to $u$. If there were at least $n/2$ blue edges adjacent to $v$, then we would find a blue component larger than $C_2$. Therefore $deG(v) - deG_1(v) < n/2$ and $deG(u) + deG(v) \geq 3n/2$ implies $deG_1(u) + deG_1(v) \geq n$. On the other hand, this is impossible since $deG_1(u) + deG_1(v) \leq d_i + d_j + 2x - 2 \leq n - 1$ as in the argument above. Therefore $V(C_2) \subseteq X$.

This implies that there is no blue component larger than $x$.

Notice that any potential edge in $G$ between two components of $G_1 \setminus X$ is a blue edge inside $C_1 \setminus C_2$ that was deleted. Let $H$ be the graph formed by the vertices in $V \setminus X$, and these crossing blue edges in $C_1 \setminus C_2$. Since $x < n/2$, we have $|V(H)| > n/2$.

Case 3.a: Assume $x \leq n/4$. 

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We claim that \( H \) is connected in the blue graph. Otherwise there exists a blue cut \((A, B)\) of \( H \), where \( A \cap D_1 \) is non-empty as well as \( B \cap (\cup_{i=2}^d D_i) \). Indeed, let us take a blue cut \((A, B)\) of \( H \), where \( A \cap D_1 \) is non-empty. If \( B \cap (\cup_{i=2}^d D_i) = \emptyset \), then \( B \subseteq D_1 \) and we reverse the roles of \( A \) and \( B \). Let \( u \) be a vertex in \( A \cap D_1 \) and \( v \) be a vertex in \( B \cap D_i \) for some \( i > 1 \). Now \( u \) and \( v \) are non-adjacent vertices in \( G \). Therefore \( \deg_G(u) + \deg_G(v) \geq 3n/2 \). Since the largest blue component has size at most \( x \), there are at most \( x - 1 \) deleted blue edges at \( u \) or \( v \). Therefore \( 3n/2 - 2x + 2 \leq \deg_G(u) + \deg_G(v) \leq d_1 - 1 + d_i - 1 + 2x \leq n - 1 \). This contradicts \( x \leq n/4 \), so \( H \) is indeed connected in blue. But then this is a larger blue component than \( x \), a contradiction, \( G_1 \) does have a perfect matching.

**Case 3.b:** Assume \( n/4 < x \leq n/2 \).

Here \( d_1 \leq 2 \), otherwise there would be too many vertices, since \( n \geq x + d_1 \ell \geq n/4 + 3(n/4 + 1) > n \), a contradiction.

We claim that \( H \) is connected in the blue graph. This again leads to a contradiction, since we have a larger blue component than \( x \). Assume the contrary and let \( A \) be a blue component in \( H \) that intersects \( D_1 \) and \( H|_{V(H) \setminus V(A)} \cap (\cup_{i=2}^d D_i) \) is non-empty.

Again, if \( (V(H) \setminus V(A)) \subseteq D_1 \), then we take a blue component in \( V(H) \setminus V(A) \) and that will play the role of \( A \). Let \( u \in A \cap D_1 \). Let \( B = H|_{V(H) \setminus V(A)} \). Now \((A, B)\) is a cut of \( H \). Let \( v \in B \cap D_i \), where \( i > 1 \) and \( i \) is as small as possible. Now \( u \) and \( v \) are non-adjacent in \( G \) and therefore \( \deg_G(u) + \deg_G(v) \geq 3n/2 \). On the other hand using the cut \((A, B)\), we get: \( \deg_G(u) \leq n - |B| + d_1 - 1 \) and \( \deg_G(v) \leq n - |A| + d_i - 1 \). It implies \( \deg_G(u) + \deg_G(v) \leq 2n - (|A| + |B|) + d_1 - 1 + d_i - 1 = n + x + d_1 - 1 + d_i - 1 \). However this leads to a contradiction if \( x + d_1 - 1 + d_i - 1 < n/2 \). In what follows we prove this last inequality.

If \( d_1 = 2 \), then the inequality simplifies to \( x + d_i < n/2 \). Notice that \( d_i \leq d_\ell \) and \( d_\ell \leq n - 3x \), since \( \ell \geq x + 1 \) and \( 2 = d_1 \leq d_j \) for any \( j \). Using this and \( n/4 < x \) we get \( x + d_i \leq x + d_\ell \leq n - 2x < n/2 \).

The other possibility is \( d_1 = 1 \). We have to show \( x + d_i - 1 < n/2 \) or \( x + d_i \leq n/2 \). Actually note that in the above the inequality \( x + |D_i \cap A| < n/2 \) already leads to a contradiction, so it is sufficient to prove this. Firstly if \( i = \ell \), then \( A \supseteq \cup_{j=1}^{i-1} D_j \) by the choice of \( i \). This is a contradiction if \( |A| > x \), since now \( A \) is larger than a largest blue component \( C_2 \). The only exception is \( d_j = 1 \) for \( 1 \leq j \leq \ell - 1 \) and \( A \cap D_\ell = \emptyset \). However in that case \( x + |D_i \cap A| = x + 0 < n/2 \) holds. Secondly \( i < \ell \).

Now \( \sum_{j \neq i, j \neq \ell} d_j \geq x - 1 \) and \( d_i + d_\ell \geq 2d_i \). If strict inequality holds in one of these, then we get the following: \( n \geq x + \sum_{j \neq i, j \neq \ell} d_j + d_i + d_\ell > 2x - 1 + 2d_i \), which implies \( n/2 \geq x + d_i \) as claimed. The only case left is \( i = \ell - 1, 1 = d_1, \ldots, d_{\ell-1} \) and \( d_\ell = d_i \). However this is impossible since now the number of vertices is \( 2x - 1 + 2d_i \), but we started with an even \( n \).

All these contradictions prove the existence of a perfect matching in \( G_1 \). Since the red and blue halves are both connected, we proved our theorem. \( \square \)
3 Applying the Regularity lemma.

As in many applications of the Regularity Lemma, one has to handle irregular pairs, that translates to exceptional edges in the reduced graph. A graph $G$ on $n$ vertices is $\varepsilon$-perturbed if at most $\varepsilon \binom{n}{2}$ of its edges are marked as exceptional (or perturbed). For a perturbed graph $G$, let $G^-$ denote the graph obtained by removing all perturbed edges. First we need a perturbed version of Theorem 3. These perturbation arguments are fairly standard modifications of the original argument (see e.g. [14]). We give all details to be self-contained.

**Theorem 4.** For every $\eta > 0$, there exist $n_0 = n_0(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)(\ll \eta)$ such that the following holds. Suppose that $\varepsilon \leq \varepsilon_0$ and $G$ is a 2-edge-colored $\varepsilon$-perturbed graph on $n \geq n_0$ vertices and $G$ satisfies the following Ore-type condition: for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\deg(x) + \deg(y) \geq (3/2 + \eta)n$. All but at most $6\sqrt{\varepsilon}n$ vertices of $G$ can be covered by the vertices of a red connected matching and a vertex-disjoint blue connected matching in $G^-$.

**Proof:** We may assume that $n$ is sufficiently large and $\varepsilon \ll \eta$. Let us start by "trimming" the graph, i.e. by deleting those vertices of $G$ that are adjacent to at least $\sqrt{\varepsilon}n$ exceptional edges. There are less than $\sqrt{\varepsilon}n$ such vertices. We may remove one more arbitrary vertex to guarantee that the number of remaining vertices is even. This way we get a slightly smaller graph $G_\varepsilon$ on $n'$ vertices, where $n'$ is even. Secondly we delete the remaining exceptional edges to form the graph $G^-_\varepsilon$. In what follows we mimic the proof of Theorem 3 replacing $G$ by $G^-_\varepsilon$.

Let $C_1$ be a largest monochromatic component in $G^-_\varepsilon$, say red. We have $n' > (1 - \sqrt{\varepsilon})n$. Let $D = V(G^-_\varepsilon) \setminus V(C_1)$.

**Case 1:** Assume $|V(C_1)| < n'$, i.e. $D \neq \emptyset$. Let $A$ be those vertices in $C_1$ that are adjacent to $D$ by a blue edge. We claim that $A \cup D$ is a connected blue component. Assume to the contrary that there is a cut $(A_1 \cup D_1, A_2 \cup D_2)$, where $|A_1 \cup D_2| \geq |A_2 \cup D_1|$. Now again there is no edge in $G^-_\varepsilon$ between $V(C_1) \setminus A_2$ and $D_2$ as before, but now there might be some exceptional edges in $G$. However if either $|A_1| \geq \sqrt{\varepsilon}n$ or $|D_2| \geq \sqrt{\varepsilon}n$, then we certainly find a pair $u, v$ that are non-adjacent in $G$ as well (so $(u, v)$ cannot be an exceptional edge) and $u \in A_1$ and $v \in D_2$. In the remaining case we have $|A_2 \cup D_1| \leq |A_1 \cup D_2| < 2\sqrt{\varepsilon}$. But then clearly $V(C_1) \setminus A_2 > \sqrt{\varepsilon}n$ and therefore we find a non-adjacent pair $u, v$ in $G$ such that $u$ is a vertex in $V(C_1) \setminus A_2$ and $v$ is a vertex in $D_2$. Now for this appropriate pair of vertices $\deg_G(u) + \deg_G(v) \geq (3/2 + \eta)n$. On the other hand, $v$ is non-adjacent to all vertices of $V(C_1) \setminus A_2$ and $u$ is non-adjacent to $D_2$ in the graph $G^-_\varepsilon$. Therefore $\deg_G(u) + \deg_G(v) < n - 1 - |D_2| + n - 1 - |V(C_1) \setminus A_2| + 2\sqrt{\varepsilon}n \leq 2n - 2 - |D_2| - |V(C_1) \setminus A_2| - |A_1| + 2\sqrt{\varepsilon}n < 2n - n/2 + 2\sqrt{\varepsilon}n = 3n/2 + 2\sqrt{\varepsilon}n$, a contradiction using $2\sqrt{\varepsilon} \ll \eta$ (here we used the assumption on the size of $A_1 \cup D_2$).
Let $C_2$ be this blue component of $G^-_e$ covering $V(G^-_e) \setminus V(C_1)$. Let $|V(C_1) \setminus V(C_2)| = p$ and $|V(C_2) \setminus V(C_1)| = q$, where $p \geq q > 0$ by the choice of $C_1$. We claim that $p + q < (1/2 - \eta/2)n'$. This clearly holds if $p, q < \sqrt{\varepsilon}n$. Otherwise $p \geq \sqrt{\varepsilon}n$ or $q \geq \sqrt{\varepsilon}n$. Therefore, we find a pair of vertices $u$ and $v$ such that $u \in V(C_1) \setminus V(C_2)$, $v \in V(C_2) \setminus V(C_1)$ and $u$ and $v$ are non-adjacent in $G$. Thus by the Ore-type condition we have $\deg_G(u) + \deg_G(v) \geq (3/2 + \eta)n$. On the other hand $n' - 1 - q \geq \deg_{G^-_e}(u)$ and $n' - 1 - p \geq \deg_{G^-_e}(v)$. We can also use that $\deg_{G^-_e}(u) + \sqrt{\varepsilon}n \geq \deg_G(u)$ and $\deg_{G^-_e}(v) + \sqrt{\varepsilon}n \geq \deg_G(v)$. Now $(3/2 + \eta)n \leq \deg_G(u) + \deg_G(v) \leq \deg_{G^-_e}(u) + \deg_{G^-_e}(v) + 2\sqrt{\varepsilon}n \leq 2n' - 2 - (p + q) + 2\sqrt{\varepsilon}n \leq 3n/2 + n'/2 - 2 - (p + q) + 2\sqrt{\varepsilon}n$. It yields $n'/2 + 2\sqrt{\varepsilon}n - \eta n > p + q$. Therefore $p + q < (1/2 - \eta/2)n'$ since $\varepsilon \ll \eta$. This implies $|V(C_1) \cap V(C_2)| > (1/2 + \eta/2)n'$.

If $|V(C_1)| = n'$, then define $C_2$ as a largest blue component in $G^-_e$. Now $p = |V(C_1) \setminus V(C_2)|$, $q = |V(C_2) \setminus V(C_1)| = 0$.

**Case 2:** $|V(C_1)| = n'$ and $p \leq (1/2 - \eta/2)n'$. Then again $|V(C_1) \cap V(C_2)| \geq (1/2 + \eta/2)n'$, just as above.

Therefore in what follows, we unify the proof for the two cases we described so far. Let $G_1$ be the graph, which we obtain from $G^-_e$ by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$ and the red edges induced by $V(C_2) \setminus V(C_1)$ (if these exist).

We claim there is a perfect matching in $G_1$. Assume the contrary. By Tutte’s theorem there exists a set $X$ of vertices in $G_1$ such that the number of odd components in $G_1 \setminus X$ is larger than $|X|$, which implies that $|X| < n/2$. Let all the components of $G_1 \setminus X$ (not just the odd ones) be $D_1, D_2, \ldots, D_\ell$ in increasing order of their size, $\ell \geq |X| + 1$. Note that $\ell \geq 2$ always holds, even for $X = \emptyset$, as $n'$ is even. Let $d_i = |V(D_i)|$ for $i = 1, \ldots, \ell$ and $x = |X|$.

We claim $|(V(C_1) \cap V(C_2)) \cup \bigcup_{i=1}^{\ell} D_i| \leq 2\sqrt{\varepsilon}n$. Assume the contrary. Now we want to copy the corresponding part of the proof of Theorem 3. Although some non-adjacent $u, v$ pairs in $G_1$ might be connected by an exceptional edge in $G$, the size of $|(V(C_1) \cap V(C_2)) \cup \bigcup_{i=1}^{\ell} D_i|$ now assures that we find a non-adjacent pair as follows. We can find an index $j$ such that $D_j \cap (V(C_1) \cap V(C_2)) = \emptyset$. We can think of $U$ as a collection of potential $u$’s. Let $D_j = \bigcup_{i=1, i \neq j}^{\ell} D_i$. If $|D_j| \geq \sqrt{\varepsilon}n$, then pick any vertex $u \in U$. There are less than $\sqrt{\varepsilon}n$ exceptional edges adjacent to $u$. Therefore, we find a vertex $v \in D_j$ that is non-adjacent to $u$ in $G$. If $|D_j| < \sqrt{\varepsilon}n$, then $|U| \geq \sqrt{\varepsilon}n$. Now pick a vertex $v \in D_j$ (using $\ell \geq 2$). There are less than $\sqrt{\varepsilon}n$ exceptional edges adjacent to $v$. Therefore, we find a vertex $u \in U$ that is non-adjacent to $v$ in $G$.

Now we may use the Ore-type condition $\deg_G(u) + \deg_G(v) \geq (3/2 + \eta)n$. Notice $\deg_G(u) \geq \deg_{G^-_e}(u) - \sqrt{\varepsilon}n$. Let us subtract the number of deleted non-exceptional edges adjacent to $v$, which is at most $p$ or $q$ depending on the position of $v$. We get $\deg_G(u) + \deg_G(v) \geq \deg_{G^-_e}(u) - \sqrt{\varepsilon}n + \deg_{G^-_e}(v) - p - \sqrt{\varepsilon}n \geq (3/2 + \eta)n - 2\sqrt{\varepsilon}n - p \geq n$, since both $p$ and $q$ are less than $n/2$ and $\varepsilon \ll \eta$. On the other hand
\[ \deg_{G_1}(u) \leq d_i - 1 + x \quad \text{and} \quad \deg_{G_1}(v) \leq d_j - 1 + x. \]

Therefore \(\deg_{G_1}(u) + \deg_{G_1}(v) \leq d_i + d_j + 2x - 1 - 1 \leq n' - 1 \leq n - 1\), since \(d_i + d_j + 2x - 1\) is at most the number of vertices in \(G_1\). This contradiction implies \(|(V(C_1) \cap V(C_2)) \cap \bigcup_{i=1}^t D_i| \leq 2\sqrt{\varepsilon}n\).

Using this we get \(n/2 + 2\sqrt{\varepsilon}n > x + 2\sqrt{\varepsilon}n \geq |V(C_1) \cap V(C_2)| \geq (1/2 + \frac{\eta}{2})n' > (1/2 + \eta/2)(1 - \sqrt{\varepsilon})n.\) However, this is a contradiction since \(\varepsilon \ll \eta\). Therefore \(G_1\) contains a perfect matching.

**Case 3:** \(|V(C_1)| = n'\) and \(p > (1/2 - \eta/2)n'\), so the largest blue component of \(G_x\) has size at most \((1/2 + \eta/2)n'\). We claim in this case that there is a matching in \(G_1\) covering all but at most \(5\sqrt{\varepsilon}n\) vertices of \(G_1\). We use the same set-up and notation as previously. Thus by Tutte's theorem now we have slightly more components than before: \(\ell \geq |X| + 5\sqrt{\varepsilon}n.\) This implies \(x = |X| < n'/2 - 2\sqrt{\varepsilon}n\).

We can show \(|V(C_2) \cap \bigcup_{i=1}^t D_i| \leq 2\sqrt{\varepsilon}n\) as before. Assume the contrary and as above select a vertex \(u \in C_2 \cap D_i\) and a vertex \(v\) in a different component \(D_j\). Clearly \(u\) and \(v\) are non-adjacent in \(G_1\), but also in \(G\) since we have not deleted any edge adjacent to \(u\). If there were at least \((1/2 + \eta/2)n\) \((\geq (1/2 + \eta/2)n')\) deleted non-exceptional blue edges adjacent to \(v\), then we would find a blue component larger than \(C_2\). Therefore \(\deg_{G_1}(v) - \deg_{G_1}(u) < (1/2 + \eta/2)n + \sqrt{\varepsilon}n, \deg_{G_1}(u) \geq \deg_{G_1}(u) - \sqrt{\varepsilon}n\) and \(\deg_{G_1}(u) + \deg_{G_1}(v) \geq (3/2 + \eta)n\) imply \(\deg_{G_1}(u) + \deg_{G_1}(v) \geq (1 + \eta/2)n - 2\sqrt{\varepsilon}n > n.\) On the other hand, this is impossible since \(\deg_{G_1}(u) + \deg_{G_1}(v) \leq d_i + d_j + 2x - 1 - 1 \leq n' - 1 < n\) as in the previous argument. Therefore \(|V(C_2) \cap \bigcup_{i=1}^t D_i| \leq 2\sqrt{\varepsilon}n\). This implies that there is no blue component larger than \(x + 2\sqrt{\varepsilon}n\).

Notice that any non-exceptional edge in \(G\) between two components of \(G_x \setminus X\) is a blue edge inside \(C_1 \setminus C_2\) that was deleted. Let \(H\) be the graph formed by the vertices in \(V(G_1) \setminus X\), and these crossing non-exceptional blue edges in \(C_1 \setminus C_2.\) Now we have \(x = |X| < n'/2 - 2\sqrt{\varepsilon}n\) and therefore \(|V(H)| > n'/2 + 2\sqrt{\varepsilon}n.\)

**Case 3.a:** Assume \(x \leq (1 + \eta)n/4.\)

We claim that \(H\) is connected in the blue graph except for possibly \(2\sqrt{\varepsilon}n\) vertices. This leads to the final contradiction, since \(|V(H)| - 2\sqrt{\varepsilon}n > n'/2 > x + 2\sqrt{\varepsilon}n.\) (We found a blue connected component larger than the size of a largest.)

Assume the contrary. Then there exists a blue cut \((A, B)\) of \(H\) where we have \(|A|, |B| > 2\sqrt{\varepsilon}n\), \(A \cap D_1\) is non-empty and \(|B \cap \bigcup_{i=1}^t D_i| \geq \sqrt{\varepsilon}n.\) Indeed, let us take a blue cut \((A, B)\) of \(H\), where \(|A|, |B| > 2\sqrt{\varepsilon}n\) and \(A \cap D_1\) is non-empty. If \(|B \cap \bigcup_{i=2}^t D_i| < \sqrt{\varepsilon}n,\) then \(|B \cap D_1| \geq \sqrt{\varepsilon}n\) and we reverse the roles of \(A\) and \(B.\) Let \(u\) be a vertex in \(A \cap D_1.\) Since \(|B \cap \bigcup_{i=2}^t D_i| \geq \sqrt{\varepsilon}n,\) we find a vertex \(v\) in \(B \cap D_1\) such that \(u\) and \(v\) are non-adjacent in \(G.\) Therefore \(\deg_{G_1}(u) + \deg_{G_1}(v) \geq (3/2 + \eta)n.\) Since the largest blue component has size at most \(x + 2\sqrt{\varepsilon}n\), there are at most \(x + 2\sqrt{\varepsilon}n - 1\) deleted non-exceptional blue edges at \(u\) or \(v.\) Therefore \((3/2 + \eta)n - 2x - 4\sqrt{\varepsilon}n + 2 - 2\sqrt{\varepsilon}n \leq \deg_{G_1}(u) + \deg_{G_1}(v) \leq d_i - 1 + d_j - 1 + 2x \leq n - 1.\) This yields \(n/2 + \eta n - 6\sqrt{\varepsilon}n \leq 2x.\) This contradicts \(x \leq (1 + \eta)n/4,\) since \(\varepsilon \ll \eta.\)

**Case 3.b:** Assume \((1 + \eta)n/4 < x < n'/2 - 2\sqrt{\varepsilon}n.\)
Here $d_1 \leq 2$, otherwise there would be too many vertices, since $n \geq x + d_1 \ell > x + 3x = 4x > (1 + \eta)n$, a contradiction.

We claim again that $H$ is connected in the blue graph except for possibly $2\sqrt{\varepsilon n}$ vertices, a contradiction again, since $|V(H)| - 2\sqrt{\varepsilon n} = n'/2 > x + 2\sqrt{\varepsilon n}$. (We found a blue connected component larger than the size of a largest.) Assume the contrary and let $A$ be a blue component in $H$ that intersects $D_1$. Let $u \in A \cap D_1$.

Let $B = H|_{V(H) \setminus V(A)}$. Now $(A, B)$ is a cut of $H$. We may assume $|B| \geq 2\sqrt{\varepsilon n}$ (since otherwise we are done) and thus $|B \cap (\cup_{i=2}^\ell D_i)| \geq \sqrt{\varepsilon n}$ (using $d_1 \leq 2$). Let $v \in B \cap D_i$ such that $u$ and $v$ are non-adjacent in $G$ and $i > 1$ is as small as possible.

Now $u$ and $v$ are non-adjacent in $G$ and therefore $\deg_G(u) + \deg_G(v) \geq (3/2 + \eta)n$.

On the other hand using the cut $(A, B)$, we get: $\deg_G(u) \leq n - |B| + d_1 - 1 + \sqrt{\varepsilon n}$ and $\deg_G(v) \leq n - |A| + d_i - 1 + \sqrt{\varepsilon n}$. It implies $(3/2 + \eta)n \leq \deg_G(u) + \deg_G(v) \leq 2n - (|A| + |B|) + d_1 - 1 + d_i - 1 + 2\sqrt{\varepsilon n} = 2n - n' + x + d_1 - 1 + d_i - 2 + 3\sqrt{\varepsilon n}$. However this is a contradiction if $x + d_1 + d_i - 2 + 3\sqrt{\varepsilon n} < n'/2 + \eta n$. Using $d_1 \leq 2$, it suffices to prove $x + d_i + 3\sqrt{\varepsilon n} < n'/2 + \eta n$, or $x + d_i < (1 + \eta)n/2$. In what follows we prove this last inequality.

Let $d_1 = 2$. Notice that $d_i \leq d_\ell$ and $d_\ell \leq n - 3x$, since $\ell > x$ and $2 = d_1 \leq d_j$ for any $j$. Using this and $(1 + \eta)n/4 < x$ we get $x + d_i \leq x + d_\ell \leq n - 2x < n - (1 + \eta)n/2 < n/2 < (1 + \eta)n/2$, as desired.

The other possibility is $d_1 = 1$. Firstly if $i = \ell$, then $A \supseteq \cup_{j=1}^{\ell - 1} D_j$. This is a contradiction since $|A| > x + 2\sqrt{\varepsilon n}$, we have a blue component that is larger than a largest blue component $C_2$. Secondly $i < \ell$. Now $\sum_{j \neq i, j \neq \ell} d_j \geq x$ and $d_i + d_\ell \geq 2d_i$.

Thus we get the following: $n \geq x + \sum_{j \neq i, j \neq \ell} d_j + d_i + d_\ell > 2x + 2d_i$, which implies $x + d_i \leq n/2 < (1 + \eta)n/2$, as claimed.

All these contradictions prove the existence of a matching in $G_1$ of the desired size (covering all but at most $\sqrt{\varepsilon n} + 5\sqrt{\varepsilon n} = 6\sqrt{\varepsilon n}$ vertices of $G$). Since the red and blue halves are both connected, we proved our theorem. \qed

## 4 Building cycles from connected matchings.

Next we show how to prove Theorem 2 from Theorem 4 and the Szemerédi Regularity Lemma [24]. The material of this section is fairly standard by now (see [3, 9, 10, 11, 12, 13]) so we omit some of the details.

We need a 2-edge-colored version of the Szemerédi Regularity Lemma.\footnote{For background, this variant and other variants of the Regularity Lemma see [17].}

**Lemma 1.** For every integer $m_0$ and positive $\varepsilon$, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any $n$-vertex graph $G$, where $G = G_1 \cup G_2$
with $V(G_1) = V(G_2) = V$, there is a partition of $V$ into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that

- $m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \cdots = |V_\ell| = L$, $|V_0| < \varepsilon n$,
- apart from at most $\varepsilon \binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are $\varepsilon$-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 2$.

**Proof:** Let $\varepsilon \ll \rho \ll \eta \ll 1$, $m_0$ sufficiently large compared to $1/\varepsilon$ and $M_0$ obtained from Lemma 1. Let $G$ be a graph on $n > M_0$ vertices such that for any two non-adjacent vertices $x$ and $y$ of $G$, we have $\text{deg}(x) + \text{deg}(y) \geq (\frac{3}{2} + \eta)n$. Consider a 2-edge-coloring of $G$, that is $G = G_1 \cup G_2$. We apply Lemma 1 to $G$. We obtain a partition of $V$, that is $V = \cup_{0 \leq i \leq \ell} V_i$.

Define the following reduced graph $G^R$: The vertices $p_1, \ldots, p_\ell$ of $G^R$ correspond to the clusters, and there is an edge between vertices $p_i$ and $p_j$ if the pair $(V_i, V_j)$ is either exceptional$^4$, or if it is $\varepsilon$-regular in both $G_1$ and $G_2$ with density in $G$ exceeding $\rho$. Thus note that $G^R$ is an $\varepsilon$-perturbed graph where a non-edge is a regular pair where the density is at most $\rho$. The edge $p_ip_j$ is colored by the color, which is used on most edges from $G[V_i, V_j]$ (the bipartite subgraph of $G$ with edges between $V_i$ and $V_j$). If the pair is non-exceptional, then the density of this majority color is still at least $\rho/2$ in $G[V_i, V_j]$. This defines a 2-edge-coloring $G^R = G^R_1 \cup G^R_2$.

We claim that $G^R$ satisfies a similar Ore-type condition: for any two non-adjacent vertices $p_i$ and $p_j$ of $G^R$, we have $\text{deg}_{G^R}(p_i) + \text{deg}_{G^R}(p_j) \geq (\frac{3}{2} + \frac{\eta}{2})\ell$. Indeed, let $p_i$ and $p_j$ be non-adjacent in $G^R$ and consider the corresponding clusters $V_i$ and $V_j$. By definition the number of non-edges in $G[V_i, V_j]$ is at least $(1 - \rho)|V_i||V_j| = (1 - \rho)L^2$. For each of these non-edges we can use the Ore-condition in $G$ so we get the following estimate

$$\sum_{u \in V_i} \sum_{v \in V_j} (\text{deg}_G(u) + \text{deg}_G(v)) \geq (1 - \rho)L^2 \left(\frac{3}{2} + \frac{\eta}{2}\right)n.$$

On the other hand we can get the following upper bound for this quantity

$$\sum_{u \in V_i} \sum_{v \in V_j} (\text{deg}_G(u) + \text{deg}_G(v)) \leq L^2(\text{deg}_{G^R}(p_i) + \text{deg}_{G^R}(p_j)) + 2\varepsilon nL^2 + 2\rho nL^2,$$

where the last 2 error terms come from the edges to $V_0$, and from the regular pairs with density at most $\rho$. However, from this we get

$$\text{deg}_{G^R}(p_i) + \text{deg}_{G^R}(p_j) > \left(\frac{3}{2} + \frac{\eta}{2}\right)n \geq \left(\frac{3}{2} + \frac{\eta}{2}\right)\ell,$$

as desired.

$^4$That is, $\varepsilon$-irregular in $G_1$ or in $G_2$. Also, these edges are marked exceptional in $G^R$. 

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Applying Theorem 4 to the 2-colored and $\varepsilon$-perturbed $G^R$, we get a connected matching in $(G^R_1)^-$ and a vertex-disjoint connected matching in $(G^R_2)^-$, which together cover most of $G^R$. Finally, we lift the connected matchings back to cycles in the original graph using the following lemma in our context.

**Lemma 2.** Assume that there is a monochromatic connected matching $M$ (say in $(G^R_1)^-$) saturating at least $c|V(G^R)|$ vertices of $G^R$, for some positive constant $c$. Then in the original $G$ there is a monochromatic cycle in $G_1$ covering at least $c(1-3\varepsilon)n$ vertices.

This completes the proof. Indeed, the number of vertices left uncovered in $G$ is at most $f(\varepsilon)n \leq \eta n$, using our choice of $\varepsilon$. Here the uncovered parts come from Theorem 4, Lemma 2 and $V_0$. □

**References**


$^5$As in [10, 11, 12, 13].


