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Théorie de la mesure dans la dynamique des sous-groupes de $\mathrm{Diff}^{\omega}(S^1)$

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Résumé

Dans cette thèse, nous établissons un théorème de rigidité topologique pour une large classe de sous-groupes du groupe de difféomorphismes analytiques réels préservant l'orientation du cercle $\mathrm{Diff}^{\omega}(S^1)$. En effet, les objets principaux étudiés dans cette thèse sont les sous-groupes localement C^2-non-discrets de type fini de $\mathrm{Diff}^{\omega}(S^1)$.

Dans le premier Chapitre, on donne des rappels sur la relation entre la théorie de la mesure et les systèmes dynamiques et on donne aussi des rappels sur les définitions et les propriétés des espaces hyperboliques, des groupes hyperboliques et des leurs bords.

Le deuxième Chapitre contient des définitions précises pour la plupart des notions pertinentes pour cette thèse, revisite les résultats concernant la théorie de Shcherbakov-Nakai sous une forme adaptée à nos besoins et fournit une description des dynamiques topologiques associées au sous-groupe localement C^2-non-discret de $\mathrm{Diff}^{\omega}(S^1)$.

Le troisième Chapitre est consacré à la preuve du Théorème A "le théorème de rigidité topologique". Dans la première section de ce chapitre, on démontre le Théorème A dans divers cas particuliers, dont le cas où le groupe a une orbite finie et le cas où le groupe est résoluble mais non-abélien. Il restera alors démontrer le Théorème A dans le cas dit "générique" et cela sera l'objet du restant de ce chapitre. Dans la deuxième section de ce chapitre, nous construisons une suite de difféomorphismes de G_1 convergeant vers l'identité dans C^2-topologie sur l'intervalle $I \subset S^1$. Dans la dernière section de ce chapitre, nous allons démontrer le Théorème A modulo la Proposition 3.3.3. En effet, le Théorème 3.3.1 sera prouvé et ce théorème constitue un énoncé plus forte que celui du Théorème A.

L'énoncé principal du quatrième Chapitre est le Théorème 4.2.1. La démonstration du Théorème 4.2.1 est une combinaison des faits standards sur les groupes hyperboliques avec l'existence d'une mesure μ sur G_1 donnant lieu à une mesure stationnaire absolument continue. Ce théorème entraînera la démonstration du Théorème B.

Finalement, l'Annexe contient une réponse partielle dans la catégorie analytique à une question posée dans [De]. L'annexe se termine ensuite par un résumé du rôle joué par l'hypothèse de régularité (C^{ω}) dans cette thèse.

Mots-clefs

Théorie de la mesure, mesures stationnaires, sous-groupes de $\mathrm{Diff}^{\omega}(S^1)$, groupe localement non-discret, rigidité topologique, théorie ergodique

Abstract

In this thesis we establish a topological rigidity theorem for a large class of subgroups of the group $\mathrm{Diff}^{\omega}(S^1)$ consisting of (orientation-preserving) real analytic diffeomorphisms of the circle S^1. Indeed, the primary object studied in this thesis are finitely generated, locally C^2-non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$.

In the first Chapter, we briefly recall several basic facts in the relation between measure theory and dynamical systems and recall the definitions and basic properties of hyperbolic spaces, hyperbolic groups and their boundaries.

The second Chapter contains accurate definitions for most of the notions relevant for this thesis, revisits results related to Shcherbakov-Nakai theory in a form adapted to our needs and provides a description of the topological dynamics associated with a locally C^2-non-discrete subgroup of $\mathrm{Diff}^{\omega}(S^1)$.

The third Chapter is devoted to proving Theorem A ”topological rigidity theorem”. In the first section of this chapter, we prove Theorem A in various special cases, including the case where the group has a finite orbit as well as the case in which the group is solvable but non-abelian. It will then prove Theorem A in the case called ”generic” and this will be the subject of the remainder of this chapter. In the second section of this chapter, we construct an explicit sequence of diffeomorphisms in G_1 converging to the identity in the C^2-topology on the interval $I \subset S^1$. In the last section of this chapter, we shall prove Theorem A modulo Proposition 3.3.3. In fact, Theorem 3.3.1 will be proved and this theorem provides a statement fairly stronger than what is strictly needed to derive Theorem A.

The main statement in the fourth Chapter is Theorem 4.2.1. The proof of Theorem 4.2.1 is combined standard facts about hyperbolic groups with the existence of a measure μ on G_1 giving rise to an absolutely continuous stationary measure. This theorem will lead to the proof of Theorem B.

In the end, the Appendix contains a partial answer in the analytic category to a question raised in [De]. The appendix then ends with a summary of the role played by the regularity assumption (C^{ω}) in this thesis.

Keywords

Measure theory, stationary measures, subgroups of $\mathrm{Diff}^{\omega}(S^1)$, locally non-discrete group, topological rigidity, ergodic theory

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Contents

Chapter 0

Introduction

In this thesis we establish a topological rigidity theorem for a large class of subgroups of the group $\mathrm{Diff}^{\omega}(S^1)$ consisting of (orientation-preserving) real analytic diffeomorphisms of the circle S^1. Indeed, the primary object studied in this thesis are finitely generated, locally C^2-non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$. As is often the case, our choice of restricting attention to finitely generated groups of orientation-preserving diffeomorphisms is made only to help us to focus on the main difficulties of the problem. On the other hand, the regularity assumption (C^{ω}) required from our diffeomorphisms however is a far more important point although it can substantially be weakened in several specific contexts. In this direction some possible extensions of our results to, say, smooth diffeomorphisms, are briefly discussed in the Appendix.

A group $G \subset \mathrm{Diff}^{\omega}(S^1)$ is said to be *locally C^2-non-discrete* if there is an open, non-empty interval $I \subset S^1$ and a sequence g_j of elements in G satisfying the following conditions:

- We have $g_j \neq \mathrm{id}$ for every $j \in \mathbb{N}$.
- The sequence formed by the restrictions $g_{j|I}$ of the diffeomorphisms g_j to the interval I converges in the C^2-topology to the identity on I; see Chapter 2 for further detail.

For the time being it suffices to know that locally C^2-non-discrete groups form a large class of finitely generated subgroups of $\mathrm{Diff}^{\omega}(S^1)$. After stating the main results of this thesis, we will provide some non-trivial information on the nature of these groups.

Recall that two subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$ are said to be *topologically conjugate* if there is a homeomorphism $h : S^1 \to S^1$ such that $G_2 = h^{-1} \circ G_1 \circ h$, i.e. to every element $g_{(1)} \in G_1$ there corresponds a unique element $g_{(2)} \in G_2$ such that $g_{(2)} = h^{-1} \circ g_{(1)} \circ h$ and conversely. Now we have:

Theorem A. *Consider two finitely generated, non-abelian subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$. Suppose that these groups are locally C^2-non-discrete. Then every homeomorphism $h : S^1 \to S^1$ satisfying $G_2 = h^{-1} \circ G_1 \circ h$ coincides with an element of $\mathrm{Diff}^{\omega}(S^1)$.*

Theorem A answers one of the questions raised in [R4]. When this theorem is combined with Theorem 4.2.1, we also obtain:

Theorem B. *Suppose that Γ is a finitely generated hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$ and consider two topologically conjugate faithful representations $\rho_1 : \Gamma \to \text{Diff}^{\omega}(S^1)$ and $\rho_2 : \Gamma \to \text{Diff}^{\omega}(S^1)$ of Γ in $\text{Diff}^{\omega}(S^1)$. Assume that $G_1 = \rho_1(\Gamma) \subset \text{Diff}^{\omega}(S^1)$ is locally C^2-non-discrete. Assume also the existence of a non-degenerate measure μ on G_1 having finite entropy and giving rise to an absolutely continuous stationary measure ν_1 for G_1. Then every (orientation-preserving) homeomorphism $h : S^1 \to S^1$ conjugating the representations ρ_1 and ρ_2 coincides with an element of $\text{Diff}^{\omega}(S^1)$.*

The main assumptions of Theorems A and B, namely the fact that our groups are locally C^2-non-discrete, cannot be dropped. Indeed, counterexamples for the previous statements in the context of discrete groups can be obtained in a variety of ways. For example, two cocompact representations in $\text{PSL}\,(2, \mathbb{R})$ of the fundamental group of the genus g compact surface ($g \geq 2$) are always topologically conjugate. However these representations are not C^1-conjugate unless they define the same point in the Teichmuller space. A wider family of counterexamples can be obtained by means of Schottky (free) groups. In fact, a Schottky group on two generators acting on S^1 gives rise to an action that is *structurally stable* in $\text{Diff}^{\omega}(S^1)$. Thus, by perturbing the generators inside $\text{Diff}^{\omega}(S^1)$, we obtain numerous actions that are topologically but not C^1 conjugate to the initial Schottky group (cf. [Su] and references therein).

In the case of Theorem B, there is however an additional assumption regarding the existence of an absolutely continuous stationary measure μ and this deserves a few comments (the reader is referred to Chaptre 4 for accurate definitions). Consider then a locally C^2-non-discrete group $G \subset \text{Diff}^{\omega}(S^1)$. To abridge the discussion assume that G leaves no probability measure on S^1 invariant. Alternatively the reader may simply assume that G is isomorphic to a hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$. For this type of groups, the existence of absolutely continuous stationary measures is widely believed to hold in great - if not in full - generality. This belief is based on the existence of a few promising strategies to construct absolutely continuous stationary measures even though carry any of them out to full extent involves some subtle analysis. For example, it is generally believed that a Sullivan's type construction of a discrete analogue for the Brownian motion should lead to the desired absolutely continuous stationary measure. This line of attack can further be detailed by relying on the more recent and general construction carried out by Connel and Muchnik in [C-M] which essentially reduces the problem to showing the existence of suitable *spike-like* diffeomorphism in the group G; see [C-M] for detail. In turn a natural strategy to show that G contains sufficiently many spike-like diffeomorphism consists of exploiting the denseness properties of locally C^2-non-discrete groups as stated in [R5]. The central difficulty arising in this context stems from the fact that the mentioned "approximation" properties of G are somehow *local* whereas the use of spikes as formulated in [C-M] requires a global control on the effect on certain density functions. To overcome this difficulty we need to show that "approximating sequences" as in [R5] can be constructed while keeping global control on the behavior of the diffeomorphism. Since any attempt at conducting this type of analysis here would clearly take us too far from the central ideas in this work, it seems better to defer this discussion to elsewhere and simply add the corresponding assumption to the statement of Theorem B.

To complement the preceding discussion about Theorem B, we also note that topologically rigidity does not hold in general when the group Γ is $\mathbb{Z}$. A counterexample is provided by Arnold's well-known construction of analytic diffeomorphisms of S^1 topologically conjugate to irrational rotations by singular homeomorphisms. Indeed, the group generated by an irrational rotation is clearly non-discrete. Concerning the possibility of generalizing Theorems A and B to higher rank abelian groups, the reader is referred to the discussions in [Mo] and [Y]. On the other hand, by virtue of the work of Kaimanovich and his collaborators, Theorem B still holds true for other type of groups including relatively hyperbolic ones; cf. [C-M] and its references.

The above theorems also have consequences of considerable interest in the theory of secondary characteristic classes of (real analytic) foliated S^1-bundles. For example, Theorem A yields the following result.

Corollary C. *Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two analytic foliated S^1-bundles. Assume that these foliated S^1-bundles are topologically conjugate and that the holonomy groups of $(M_1, \mathcal{F}_1)$ and of $(M_2, \mathcal{F}_2)$ are locally C^2-non-discrete. Then the Godbillon-Vey classes of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ coincide.*

Concerning Corollary C, it is well known that Godbillon-Vey classes are invariant by homeomorphisms that are transversely of class C^2 (see [C-C]). By virtue of Theorem A, every topological conjugacy between $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ will necessarily be regular in the transverse direction.

The remainder of this introduction contains an overview of our approach to the proofs of Theorems A and B including the main connections with previous works as well as some interesting examples.

Very roughly speaking, the results in this thesis are obtained by blending the technique of "vector fields in the closure of groups", developed in [Sh] and [N1] for subgroups of $\text{Diff}\,(\mathbb{C}, 0)$ and in [R1] for subgroups of $\text{Diff}^{\omega}(S^1)$, with results related to stationary measures on S^1, see [DKN-1], [An], [K-N] and with measure-theoretic boundary theory for groups [De], [Ka], and [C-M]. We will follow a chronological order to explain the various connections between these works.

First, Shcherbakov and Nakai [Sh], [N1] have independently studied the dynamics of non-solvable subgroups of $\text{Diff}\,(\mathbb{C}, 0)$ and they observed the existence of certain vector fields whose local flows were "limits" of actual elements in the pseudogroup (see Chapter 2 for detail). Then Ghys [G1] noted that non-solvable subgroups of $\text{Diff}\,(\mathbb{C}, 0)$ always contain (non-trivial) sequences of elements converging to the identity. In analogy with the case of finite dimensional Lie groups, he suggested that the existence of vector fields with similar properties should be a far more general phenomenon and he went on to discuss the topological dynamics of the analogous groups of circle diffeomorphisms.

In the case of the circle, the program proposed by Ghys was fairly accomplished in [R1]. In this thesis, vector fields whose local flows are limits of actual elements in the initial group are said to belong to the *closure of the group* (see Chapter 2 for proper definitions). The role of "locally non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$" was emphasized and it was shown that these locally non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ admit non-zero vector fields in their closure. As an application of these vector fields, the following theorem was also proved in [R1]:

Theorem (**[R1]**). *There exists a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^{\omega}(S^1)$ with the following property. Assume that G_1 (resp. G_2) is a non-solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ generated by diffeomorphisms $g_{1,1}, \ldots, g_{1,N}$ (resp. $g_{2,1}, \ldots, g_{2,N}$) lying in $\mathcal{U}$. If $h : S^1 \to S^1$ is a homeomorphism satisfying $g_{2,i} = h^{-1} \circ g_{1,i} \circ h$ for every $i = 1, \ldots, N$, then h coincides with an element of $\mathrm{Diff}^{\omega}(S^1)$.*

This theorem can be thought of as a local version of Theorem A. In fact, the assumption that h takes a generating set formed by elements "close to the identity" to elements that are still close to the identity gives the statement in question an intrinsic *local character*. For example, the above theorem from [R1] is satisfactory for deformations/pertubations problems but falls short of answering the same question for general groups admitting generating sets in the fixed neighborhood $\mathcal{U}$ unless the mentioned sets are, in addition, conjugated by h. This type of difficulty was pointed out and discussed in [R4] and the method of [R1] suggests that these rigidity phenomena should hold for general *locally non-discrete subgroups of* $\mathrm{Diff}^{\omega}(S^1)$ (again see Chapter 2 for accurate definitions). The original motivation of the present work was then to shed some light on these issues.

It is mentioned in [R1] that the main example of locally non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ is provided by non-solvable groups admitting a finite generating set contained in $\mathcal{U} \subset \mathrm{Diff}^{\omega}(S^1)$, as follows from Ghys's results in [G1]. Conversely the main examples of groups that are *locally discrete* are provided by Fuchsian groups. The problem about understanding how the subgroups of $\mathrm{Diff}^{\omega}(S^1)$ are split in locally discrete and locally non-discrete ones is then unavoidably raised.

Soon it became clear that locally non-discrete groups were, indeed, very common (see for example [R3]). The problem of finding *locally discrete* subgroups of $\mathrm{Diff}^{\omega}(S^1)$ beyond the context of Fuchsian groups, however, proved to be much harder. Recently, however, much progress has been made towards the understanding of the structure of locally discrete groups thanks to the works of Deroin, Kleptsyn, Navas, and their collaborators, see [DKN-2] and the survey [DFKN] for some up-to-date information. Meanwhile it was also observed in [R5] that the Thompson-Ghys-Sergiescu subgroup of $\mathrm{Diff}^{\infty}(S^1)$ is locally discrete. Whereas this example is only smooth, as opposed to real analytic, the observation in question connects with the fundamental notion of *expandable point* and this requires a more detailed explanation.

Fix a group G of diffeomorphisms of S^1. A point $p \in S^1$ is said to be *expandable* (for the group G) if there is an element $g \in G$ such that $|g'(p)| > 1$. Among "large" (e.g. non-solvable) subgroups of $\mathrm{Diff}^{\omega}(S^1)$ all of whose orbits are dense, $\mathrm{PSL}\,(2, \mathbb{Z})$ constitutes the simplest example of group exhibiting one *non-expandable* point. In turn, when it comes to *locally non-discrete* groups having all orbits dense, it is observed in [R5] that all points are expandable. In particular, Thompson-Ghys-Sergiescu group must be locally discrete since it exhibits non-expandable points while having all orbits dense. Hence, a method to produce locally discrete groups consists of finding groups with non-expandable points. In a recent and interesting paper [AFKMMNT] V. Kleptsyn and his collaborators have made significant progress in these questions, finding in particular *free subgroups* of $\mathrm{Diff}^{\omega}(S^1)$ which are not conjugate to Fuchsian groups and still possess non-expandable points.

Nonetheless, the full understanding of locally discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ was not yet reached (see [DKN-2] and [DFKN] for further information). To continue our discussion, we shall then restrict ourselves to the related problem of understanding "rigidity" of topological conjugations between subgroups of $\mathrm{Diff}^{\omega}(S^1)$ which, ultimately, constitutes the actual purpose of this thesis. In the sequel, we then consider two topologically conjugate subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$. Since topological rigidity is targeted, the examples provided in the beginning of the introduction indicate that one of the groups, say G_1, should be assumed to be C^2-locally non-discrete. At this level, Theorem A fully answers the question provided that G_2 is locally C^2-non-discrete as well. Thus, to make further progress, we need to investigate whether a locally C^2-non-discrete group G_1 can be topologically conjugate to a locally C^2-discrete subgroup G_2. Following our above stated results, the state-of-art of this problem can be summarized as follows.

First we assume once and for all that G_1 (and hence G_2) is *minimal* i.e. all of its orbits are dense in S^1. Moreover these groups are also assumed to be non-abelian. The material presented in Chapter 2 and Section 3.1 of this thesis shows that this assumption can be made without loss of generality. Theorem B also settles the question when the groups are of hyperbolic type, up to the technical condition on the existence of absolutely continuous stationary measures. Also, if G_2 is conjugate to a Fuchsian group, then a conjugating homeomorphism h between G_1 and G_2 cannot exist as pointed out in [R4]. These general statements apart, the existence of non-expandable points plays again a role in the problem. Thus we may consider the obvious alternative

- All points in S^1 are expandable for G_2.
- G_2 has at least one non-expandable point.

In the first case, an unpublished result of Deroin asserts that the (locally C^2-discrete group) G_2 is essentially a Fuchsian group. Therefore the preceding implies that a conjugating homeomorphism between G_1 and G_2 cannot exist (cf. [R4]). Alternately, the non-existence of topological conjugation between G_1 and G_2 can directly be derived from Theorem 3.3.1 in Section 3.3. In fact, the argument in Section 3.3 relies only on the following assumptions:

1. G_1 is locally C^2-non-discrete.
2. G_1 is minimal and non-abelian.
3. Every point in S^1 is expandable for G_2.

The fact that the argument of Section 3.3 depends only on the conditions above will also be useful in Chaptre 4 for the proof of Theorem B.

In closing, recall that a classical problem that lends further interest to regularity properties of homeomorphisms conjugating groups actions is the possibility of having different Godbillon-Vey characteristic classes. In the case of (global) groups acting on S^1, our results are satisfactory for locally C^2-non-discrete. On the other hand, in the locally discrete case, this problem is difficult even if the groups in question arise from Fuchsian groups and we refer the reader to [G2] and its references for further information.

To finish the introduction, let us provide an overview of the structure of this thesis. In the first Chapter, we briefly recall several basic facts in the relation between measure theory and dynamical systems and recall the definitions and basic properties of hyperbolic spaces, hyperbolic groups and their boundaries.

The second Chapter begins accurate definitions for most of the notions relevant for this thesis. and it then goes on by reviewing some results related to Shcherbakov-Nakai theory in a form adapted to our needs. The last Section of this chapter, namely Section 2.4, provides a description of the topological dynamics associated with a locally C^2-non-discrete subgroup of $\mathrm{Diff}^{\omega}(S^1)$. This description faithfully parallels the corresponding results established in [G1] for the case of groups admitting a generating set "close to the identity".

The third Chapter is devoted to proving Theorem A. The first Section of this chapter, namely Section 3.1, we prove Theorem A in different types of special situations. These include the case where the groups G_1 and G_2 have finite orbits as well as the case in which these groups are solvable but non-abelian. The results of Section 3.1 are implicitly used throughout the thesis since they allow us to restrict our discussion to a sort of "generic case" for the group G_1; see Proposition 3.1.6. Roughly speaking, this generic situation is such that we can fix and interval $I \subset S^1$ and, for every $\varepsilon > 0$, we can find a finite collection of elements in G_1 satisfying the following conditions:

- Diffeomorphisms in this collection are ε-close to the identity in the C^2-topology on I.
- The collection of these diffeomorphisms generated a non-solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$.

The study of this last generic case will be the object of Section 3.2, Section 3.3 and Chaptre 4.

The second Section of this chapter, namely Section 3.2, we construct an explicit sequence of diffeomorphisms in G_1 converging to the identity in the C^2-topology on the above mentioned interval I. As explained in the beginning of Section 3.2, this construction is necessary to yield a sequence converging to the identity for which we can control the mentioned convergence rate while also estimating the growing rate of the sequence formed by the corresponding higher order derivatives. In fact, the reader will note that the very definition of a locally C^2-non-discrete group provides us with a sequence converging to the identity in the C^2-topology on some non-empty interval. This definition however does not give us any estimate on, for example, the C^3-norm of the diffeomorphisms in this sequence (see Section 3.2 for a detailed discussion). In the construction of a specific sequence converging to the identity for which estimates on the growing rate of higher derivatives are also available, we will take advantage of the fact that we can select finitely many elements of G_1 generating a non-solvable group and being arbitrarily close to the identity on a fixed interval I; cf. Proposition 3.1.6.

And, in Section 3.3, we shall prove Theorem A modulo Proposition 3.3.3 whose proof is deferred to Chaptre 4. In fact, in this section Theorem 3.3.1 will be proved and this theorem provides a statement stronger than what is strictly needed to derive Theorem A in the "generic case" (which will be the only case under discussion after Section 3.1). As to Proposition 3.3.3, the reader will note that its use can be avoided modulo working with bounded distortion estimates for iterates of diffeomorphisms possessing parabolic fixed points, as done in [R1]. The interest of Proposition 3.3.3 lies primarily in the fact that it makes the discussion significantly shorter by allowing us to focus exclusively on hyperbolic fixed points which, in turn, are linearizable [St].

In the fourth Chapter we collect essentially all the results in this thesis for which Ergodic theory appears to be an indispensable tool. First we shall use this material to prove Proposition 3.3.3 so as to fully round off the discussion in Section 3.3. Then we shall state and prove Theorem 4.2.1 which reduces Theorem B to Theorem A. We also note that the proof of Proposition 3.3.3 relies heavily on [DKN-1] and, in fact, this proposition is a straightforward consequence of the proof of "Théorème F" in [DKN-1]. The proof of Theorem 4.2.1 is more involved as it combines standard facts about hyperbolic groups with results from [De] and from [Ka] and still depends in a crucial way on the existence of absolutely continuous stationary measures as assumed in Theorem B.

Finally the Appendix contains a partial answer in the analytic category to a question raised in [De]. The argument exploits the construction carried out in Section 3.2. The appendix then ends with a summary of the role played by the regularity assumption (C^ω) in this thesis. In particular, we highlight some specific problems whose solutions would lead to non-trivial generalizations of our statements to less regular groups of diffeomorphisms.

Chapter 1

Preliminaries

1.1 The relation between measure theory and dynamical systems

The theory of dynamical systems is a mathematical discipline closely intertwined with most of the main areas of mathematics. It is the study of the orbit structure of self-maps and flows with emphasis on properties invariant under coordinate changes. Its concepts, methods, and paradigms greatly stimulate research in many sciences. The origins of the filed of dynamical systems lie in the study of movement through time of some physical system. Hence, we come to the formal definition of a *dynamical system*.

Definition 1.1.1 *A dynamical system, denoted by* (X, f)*, consists of a non-empty set* X *called phase space, whose elements represent possible state of the system, and a collection of self-mapping* $\{f^t | f^t : X \to X\}$.

Note that the collection of maps cannot be arbitrary. In fact, the collection of maps must have a group or a semigroup structure.

The filed of dynamical systems comprises various disciplines according to the category of the phase space and self-maps considered. This theory is inseparably connected with several other areas : ergodic theory, smooth dynamics and topological dynamics. But we are interested mainly in measurable dynamics, or more classically, ergodic theory.

In virtually all situations of interest the phase space of a dynamical system possesses a certain structure which the evolution law respects. Different structure give rise to theories dealing with dynamical systems that preserve those structures. Let us mention the most important of those theories.

If the phase space possesses the structure of a smooth manifold, for example a domain or a closed surface in a Euclidean space, and the self-maps are diffeomorphisms of such manifolds and iterates of differentiable maps then the discipline is *smooth dynamical systems* or differentiable dynamics.

If the phase space is a topological space, usually a metrizable compact or locally compact space, and the self-maps are continuous transformations of such spaces then the field is *topological dynamics*.

Finally, In the case before us, if the phase space is a measure space, that is, a space with a finite or σ-finite measure μ and the self-maps are measure-preserving then the field is *measurable dynamics.*

Example 1.1.2 Rotations of the circle: *The space S^1 will be the circle of circumference 1 and the self-maps R_θ are rotations by an angle θ. In this case, the space and the collection of maps can be identified as the same object, the group of rotations of the circle. Lebesgue measure m on the circle is invariant under rotations, see example 1.1.12. This is a particular case of a compact group acting on itself by left multiplication and its unique invariant Haar probability measure.*

The purpose of the following section is to present the basic definitions and easier results of *measure theory.* For a more detailed of measure theory, see for example [Bt], and more thorough of ergodic theory, see [Pe].

1.1.1 Basics measure theory

Given the set X, we single out a family $\mathcal{P}(X)$ of subsets of X which are "well-behaved" in a certain technical sense. To be precise, we shall assume that this family contains the empty set $\varnothing$ and the entire set X, and that $\mathcal{P}(X)$ is closed under complementation and countable unions.

Definition 1.1.3 *A σ-algebra (or a σ-field) of subsets of X is a set $\mathcal{B}$ of subsets of X i.e. $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying the following conditions:*

1. *X belong to $\mathcal{B}$.*
2. *If B belong to $\mathcal{B}$, then so is the complement $X \setminus B$.*
3. *If $\{B_n\}$ is a sequence of sets in $\mathcal{B}$, then so is $\bigcup\limits_{n\in\mathbb{N}} B_n$.*

These properties imply that $\varnothing \in \mathcal{B}$ and if $B_1, ..., B_n \in \mathcal{B}$ then so is $\bigcap\limits_{i=1}^{n} B_i$. This is also true for infinite collection.

An ordered pair $(X, \mathcal{B})$ consisting of a set X and a σ-algebra $\mathcal{B}$ of subsets of X is called a *measurable space.* The elements of $\mathcal{B}$ are called *measurable set.*

If X is a topological space, we define the *Borel σ-algebra* as the smallest σ-algebra containing all open subsets of X.

Definition 1.1.4 *A measure μ is a function defined on a σ-algebra $\mathcal{B}$ such that assigns to each elements in $\mathcal{B}$ a non-negative number, $\mu : \mathcal{B} \to \mathbb{R}_{\geq 0}$, satisfying the following conditions:*

1. *$\mu(\varnothing) = 0$.*

2. *μ is countably additive in the sense that if $\{B_n\}$ is any disjoint sequence* [1] *of $\mathcal{B}$, then*

$$\mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n).$$

A set of measure zero is called a *null set.* A set $B \in \mathcal{B}$ is said to have *total measure* if its complement $X \setminus B$ has measure zero.

Definition 1.1.5 *Two measures μ and ν on $(X, \mathcal{B})$ belong to the same measure class, if they have the same sets of measure zero.*

If μ is a measure, we say a property is true for μ-almost everywhere (or μ-a.e.) if the set where this property fails has measure zero.

A measurable set B with positive measure is called an *atom* if contains no set of smaller but positive measure, i.e. if $\mu(B) > 0$ and for any measurable subset $A \subset B$ with $\mu(A) < \mu(B)$ then the set A has measure zero. A measure space without atoms is called *non-atomic.*

A measure μ is *finite* if there exists a sequence $\{B_n\}$ of $\mathcal{B}$ with $X = \bigcup_{n\in\mathbb{N}} B_n$ and such that $\mu(B_n) < +\infty$ for all n (more generally, $\mu(X) < +\infty$). Then a triple $(X, \mathcal{B}, \mu)$ is called *finite measure space.* In practice, we will usually normalize a finite measure by assuming that $\mu(X) = 1$. With this normalization, μ is called a probability measure on $(X, \mathcal{B})$ and a triple $(X, \mathcal{B}, \mu)$ is called probability space. For a probability measure, note that $0 \leq \mu(B) \leq 1$ for all $B \in \mathcal{B}$.

The Lebesgue measure: We denoted by $\mathcal{M}$ the σ-algebra of subset of $\mathbb{R}^n$ generated by open sets and null sets. We can define a measure $m : \mathcal{M} \to \mathbb{R}_{\geq 0}$, called *Lebesgue measure*, as the following where sets in $\mathcal{M}$ will be called Lebesgue measurable.

Definition 1.1.6 *For any subset A of $\mathbb{R}^n$, we can define its outer measure $m^*(A)$*

$$m^*(A) = \inf\{\sum_{B\in\mathcal{C}} Vol(B) \;\; : \; \mathcal{C}\, is\, a\, countable\, collection\, of\, boxes\, whose\, union\, covers\, A\}$$

where B is a set, called box, of the form

$$B = \prod_{i=1}^{n} [a_i, b_i].$$

The volume $Vol(B)$ of this box is defined to be

$$\prod_{i=1}^{n} (b_i - a_i).$$

[1] This means that $B_i \cap B_j = \varnothing$ for all $i \neq j$.

We then define the set A to be *Lebesgue measurable* if for every subset B of $\mathbb{R}^n$,

$$m^*(B) = m^*(B \cap A) + m^*(B \setminus A)\,.$$

These Lebesgue measurable sets form a σ-algebra, and the *Lebesgue measure* is defined by $m(A) = m^*(A)$ for any Lebesgue measurable set A. This measure also has the following properties:

- Suppose that $A \in \mathcal{M}$ and $x \in \mathbb{R}^n$ then $m(A + x) = m(A)$.
- $m(\bigcup\limits_{n\in\mathbb{N}} A_n) \leq \sum\limits_{n\in\mathbb{N}} m(A_n)$.
- If $A, B \in \mathcal{M}$ and $A \subset B$ then $m(A) \leq m(B)$.
- A is called a null set if and only if $m(A) = 0$.
- If $A \in \mathcal{M}$, then for any $\varepsilon > 0$, then there exist an open set U containing A such that $m(U \setminus A) < \varepsilon$.

The Dirac measure: Let $\mathcal{B}$ be any σ-algebra of subsets of X. As an example, we can take the biggest σ-algebra $\mathcal{P}(X)$ consisting of all subsets of X. The *Dirac measure* δ_x is defined for a given $x \in X$ and any measurable set $B \in \mathcal{B}$ by

$$\delta_x(B) = \begin{cases} 1 & if \;\; x \in B \\ 0 & if \;\; x \notin B \end{cases}$$

More generally, given a finite or countably infinite sequence of point $x_i \in X$ and given weights $\omega_i > 0$ with sum $\sum \omega_i = 1$, we can form the *probability measure* $\mu = \sum \omega_i \delta_{x_i}$, defined by the formula

$$\mu(B) = \sum \omega_i \delta_{x_i}(B) = \sum \{\omega_i; x_i \in B\}.$$

We shall introduce the *Lebesgue integral* for non-negative measurable functions. In fact, not every function is integrable. There exist a collection of functions called *measurable functions*.

Definition 1.1.7 *Let $(X, \mathcal{B}, \mu)$ be a measure space. A function f on X to $\mathbb{R}_{\geq 0}$ is said to be $\mathcal{B}$-measurable (or simply measurable) if for every real number a the set*

$$f^{-1}(-\infty, a) = \{x \in X | f(x) < a\} \in \mathcal{B}.$$

More generally, let X and X' be two sets, and $\mathcal{B}$ and $\mathcal{B}'$ σ-algebras of subsets of X and X' respectively. A map $f : X \to X'$ is called *measurable* with respect to $\mathcal{B}$ and $\mathcal{B}'$ if for every $B' \in \mathcal{B}'$ the pre-image

$$f^{-1}(B') = \{x \in X : f(x) \in B'\} \in \mathcal{B}.$$

We will write $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ in this case. If X and X' are topological spaces, we call $f : X \to X'$ measurable if it is measurable with respect to the Borel σ-algebras of X and X'. Every continuous function is measurable.

Given a measure μ on a measurable space $(X, \mathcal{B})$ and given a measurable map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$, the *puch-forward* $f_*(\mu)$ is a measure on $(X', \mathcal{B}')$ defined by the formula

$$f_*(\mu)(B') = \mu(f^{-1}(B'))$$

for every $B' \in \mathcal{B}'$. As an example, note that $f_*(\delta_x) = \delta_{f(x)}$.

A measurable map f is *non-singular* if the pre-image of every set of measure zero has measure zero.

Definition 1.1.8 *Let X be a set and f be a transformation defined on X. Then a subset B of X is said to be f-invariant if $f^{-1}(B) = B$.*

Definition 1.1.9 *Suppose that $\mathcal{B}$ is a σ-algebra of X and μ is a finite measure defined on $\mathcal{B}$. Consider a measurable transformation f from $(X, \mathcal{B})$ to itself. The measure μ on $(X, \mathcal{B})$ is called f-invariant, or f is called a measure preserving if $f_*(\mu) = \mu$, or in the other word for each $B \in \mathcal{B}$, we have the set $f^{-1}(B) \in \mathcal{B}$ and*

$$\mu(f^{-1}(B)) = \mu(B). \tag{1.1}$$

If f is an invertible measurable transformation and its inverse is measurable non-singular, then the iterates f^n, $n \in \mathbb{Z}$, form a group of measurable transformations. Moreover, condition 1.1 is equivalent to $\mu(f(B)) = \mu(B)$ for $B \in \mathcal{B}$.

Measure spaces $(X, \mathcal{B}, \mu)$ and $(X', \mathcal{B}', \mu')$ are *isomorphic* if there is a subset X_1 of total measure in X, a subset X_1' of total measure in X' and an invertible bijection $f : X_1 \to X_1'$ such that f and f^{-1} are measurable and measure preserving with respect to $\mathcal{B}$ and $\mathcal{B}'$. In addition, an isomorphism from a measure space into itself is an *automorphism*.

Integrable Functions:

Let $(X, \mathcal{B}, \mu)$ be a measure space. For $A \in \mathcal{B}$, we denote by χ_A the *characteristic function* of set A, defined as

$$\chi_A(x) = \begin{cases} 1 & when\ x \in A \\ 0 & when\ x \in X \setminus A \end{cases}$$

We define the integral of characteristic function of set $A \in \mathcal{B}$ as

$$\int_X \chi_A \, \mathrm{d}\mu = \mu(A).$$

We say that a function $f : X \to \mathbb{R} \cup \{\infty\}$ is *simple* if it can be written in the form $f = \sum\limits_{j=1}^{n} c_j \chi_{A_j}$ where $c_j \in \mathbb{R}$ for some finite collection $A_j \in \mathcal{B}$. We say that a simple function f is integrable, is usually written $\int_X f \, \mathrm{d}\mu$, if

$$\int_X f \, \mathrm{d}\mu < +\infty$$

and we define the integral of such a function as

$$\int_X f\,\mathrm{d}\mu = \sum_{j=1}^{n} c_j \int_X \chi_{A_j}\,\mathrm{d}\mu = \sum_{j=1}^{n} c_j \mu(A_j).$$

The integral of a non-negative measurable function $f : X \to \mathbb{R} \cup \{\infty\}$, possibly attaining the value ∞ at some points, is defined as the following

$$\int_X f\,\mathrm{d}\mu = \sup\{\int_X g\,\mathrm{d}\mu : g \textit{ is simple and } g(x) \leq f(x) \textit{ for all } x \in X\}.$$

Definition 1.1.10 *The integral of any measurable function (not necessarily positive) is defined by the formula*

$$\int_X f\,\mathrm{d}\mu = \int_X f^+\,\mathrm{d}\mu - \int_X f^-\,\mathrm{d}\mu$$

provided that $\int_X f^+\,\mathrm{d}\mu < +\infty$ *and* $\int_X f^-\,\mathrm{d}\mu < +\infty$. *Where* $f = f^+ - f^-$ *is the unique decomposition of* f *into the difference of two non-negative functions, given explicitly by*

$$f^+(x) = \max\{f(x), 0\} = \begin{cases} f(x) & if\ \ f(x) > 0 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$f^-(x) = \max\{-f(x), 0\} = \begin{cases} -f(x) & if\ \ f(x) < 0 \\ 0 & \textit{otherwise} \end{cases}$$

So that $|f| = f^+ + f^-$.

Example 1.1.11 *Given* $u \in \mathbb{R}^n$, *let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be the translation*

$$f(x) = x + u.$$

Clearly, f *is invertible. We also consider the Lebesgue measure* m *on* $\mathbb{R}^n$. *Then we have for each* $B \in \mathcal{B}$

$$m(f(B)) = \int_{f(B)} 1\,\mathrm{d}m = \int_B |\det\,\mathrm{d}_x f|\,\mathrm{d}m(x) = \int_B 1\,\mathrm{d}m = m(B).$$

The measure m *is* f*-invariant. In the other words, the translations of* $\mathbb{R}^n$ *preserve Lebesgue measure.*

Example 1.1.12 *For any* $\theta \in \mathbb{R}$, *define the rotation of the circle* $R_\theta : S^1 \to S^1$ *by* θ *to be the map*

$$R_\theta(x) = x + \theta \bmod 1.$$

Without loss of generality, we assume that $\theta \in [0, 1]$. *We define a measure* μ *on* S^1 *by the formula*

$$\mu(B) = m(B)$$

with $\mu(S^1) = 1$, for each $B \subset [0,1]$ in the borel σ-algebra in $\mathbb{R}$. We have also $R_\theta^{-1}(B) = B - \theta$, where

$$B - \theta = \{x - \theta : x \in B\}.$$

Therefore

$$\mu(R_\theta^{-1}(B)) = m(B - \theta) = m(B) = \mu(B).$$

Since the translations of $\mathbb{R}^n$ preserve Lebesgue measure, see example 1.1.11. This show the rotations of the circle preserve the measure μ.

Example 1.1.13 *The Gauss map $f : [0,1] \to [0,1]$ is defined by*

$$f(x) = \begin{cases} 1/x \bmod 1 & if\ \ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$$

preserves the measure μ in $[0,1]$ defined by

$$\mu(B) = \int_B \frac{1}{1+x}\,\mathrm{d}x.$$

Absolutely continuous measure: Let μ and ν be two measures defined on a fixed σ-algebra $\mathcal{B}$ of subsets of space X. Then we say that ν is *absolutely continuous* with respect to μ if $\nu(A) = 0$ for any $A \in \mathcal{B}$ whenever $\mu(A) = 0$.

Example 1.1.14 *Let $f \geq 0$ be a measurable function on $\mathbb{R}$ with finite total integral. Then the measure $\mu(A) = \int_A f(x)\mathrm{d}x$ is absolutely continuous with respect to the Lebesgue measure.*

It is clear that example 1.1.14 is an absolutely continuous measure. It is less clear, that essentially every absolutely continuous measure is the result an integration. This is the content of the important *Radon-Nikodym theorem*:

Theorem 1.1.15 *(Radon-Nikodym) Let μ and ν be two measures on a common σ-algebra $\mathcal{B}$ of subsets of space X and μ be σ-finite (i.e. X is the countable union of measurable sets with finite measure). Then ν is absolutely continuous with respect to μ if and only if there exists a measurable function $f : X \to \mathbb{R}_{\geq 0}$ such that*

$$\nu(A) = \int_A f(x)\mathrm{d}\mu(x)$$

for any A in the σ-algebra $\mathcal{B}$. This function is μ-a.e. unique. □

1.1.2 Ergodic theory

There are various definitions for ergodic theory because it uses techniques from many fields such as statistical mechanics, measure theory, number theory, vector fields on manifolds and many more.

Ergodic [2] theory is the study of statistical properties of dynamical systems relative to an invariant measure on the underlying space of the dynamical system. The word ergodic was introduced by *Ludwing Boltzman* in the context of the classical statistical mechanics. His ergodic hypothesis: the time average is equal to the space average.

Unlike topological dynamics, which studies the behaviour of individual orbits (for example periodic orbits), ergodic theory is concerned with the behaviour of the system on a set of total measure and with induced action in spaces of measurable functions. At its simplest form, see definition 1.1.1, a dynamical system is a function f defined on a set X. The iterates of the map are defined by induction $f^0 := \mathrm{id}$, $f^n := f \circ f^{n-1}$, and the aim of the theory is to describe the behaviour of $f^n(x)$ as $n \to \infty$.

The following classical result of *Poincaré* implies that recurrence is generic property of orbits of measure preserving dynamical systems helps to demonstrate the importance of invariant measures in dynamics.

Definition 1.1.16 *Suppose that $f : X \to X$ is a μ-measure preserving transformation. A point $p \in B \subset X$ is called recurrent for f with respect to μ-measurable set B if the set of return times $R(x) = \{n | f^n(x) \in B, n \in \mathbb{N}\}$ is infinite.*

Theorem 1.1.17 *(Poincaré Recurrence Theorem) Let μ is a finite measure and $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ is a μ-measure preserving transformation. Suppose that $B \subset X$, is μ-measurable, have $\mu(B) > 0$. Then μ-almost every point x of B is recurrent for f with respect to μ-measurable set B.*

Proof. Let

$$A = \{x \in B | f^n(x) \in B \ for\ infinitely\ many\ n\}$$

then we have to show that $\mu(B \setminus A) = 0$.

If we write

$$C = \{x \in B | f^n(x) \notin B \ \forall n \geq 1\}$$

then we have

$$B \setminus A = \bigcup_{k=0}^{\infty} (f^{-k}(C) \cap B).$$

Thus we have the estimate

[2] From two Greek words "ergon"(work) and "odos"(path).

$$\begin{aligned}\mu(B\setminus A) &= \mu\Big(\bigcup_{k=0}^{\infty}(f^{-k}(C)\cap B)\Big)\\ &\le \mu\Big(\bigcup_{k=0}^{\infty} f^{-k}(C)\Big)\\ &\le \sum_{k=0}^{\infty}\mu(f^{-k}(C)).\end{aligned}$$

Since μ is an invariant measure, then the sequence of measurable sets

$$C, f^{-1}(C), f^{-2}(C), \dots$$

have measure equal to $\mu(C)$, i.e. $\mu\big(f^{-k}(C)\big) = \mu(C)\ \forall k \ge 0$, it suffices to show that $\mu(C) = 0$.

Since the previous sets have the same measure, hence these sets cannot be disjoint. For if they were disjoint, their union would have infinite measure, which is impassible. Therefore, we can find integers $n > m \ge 0$ so that

$$f^{-m}(C) \cap f^{-n}(C) \ne \varnothing.$$

Choosing z lies in this intersection then $f^m(z) \in C$ and $f^{n-m}\big(f^m(z)\big) = f^n(z) \in C \subset B$, which contradicts the definition of C. Thus $f^{-m}(C)$ and $f^{-n}(C)$ are disjoint. Since $\{f^{-k}(C)\}_{k=0}^{\infty}$ is a disjoint family, thus we must have $\mu(C) = 0$. $\square$

Given any function $f : X \to X$, any orbit of point $x \in X$ by f and any real valued function $\varphi : X \to \mathbb{R}$. We can try to form the limit

$$A(x) = \lim_{n\to\infty}\frac{1}{n}\Big(\varphi(x) + \varphi(f(x)) + \ldots + \varphi(f^{n-1}(x))\Big)$$

If this limit exists, it is called the *time average* of φ over the forward orbit of x.

Now suppose that $(X, \mathcal{B}, \mu)$ is a finite measure space, that $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ is a measurable transformation and that φ is an integrable function, i.e. $\varphi \in L^1(X, \mathcal{B}, \mu)$. Then the *space average* of φ is defined simply to be the ratio $\Big(\int_X \varphi\,\mathrm{d}\mu\Big)/\mu(X)$ of the integral $\int_X \varphi(x)\,\mathrm{d}\mu(x)$ to the total measure $\mu(X)$.

A basic question in dynamics is the problem of understanding when space averages are equal to time averages.

Theorem 1.1.18 *(Birkhoff Ergodic Theorem) Let f be a measure-preserving transformation in a finite measure space $(X, \mathcal{B}, \mu)$. For any integrable function φ, the time average*

$$A(x) = \lim_{n\to\infty}\frac{1}{n}\Big(\varphi(x) + \varphi(f(x)) + \ldots + \varphi(f^{n-1}(x))\Big) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))$$

is defined for almost every x. furthermore A is integrable with respect to μ, $A \circ f(x) = A(x)$ for almost every x, and satisfies

$$\int_X A \, \mathrm{d}\mu = \int_X \varphi \, \mathrm{d}\mu.$$

□

Poincaré Recurrence theorem gives us the conditions under which the elements in a measurable set $B \in \mathcal{B}$ return again and again to a measurable set B. That mean it asserts that for μ almost all point $x \in B$ the forward orbit of x for a measure preserving transformation f returns to B infinitely often. However, *Birkhoff's Ergodic Theorem* deals with the behaviour of $\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x))$ for μ-*a.e.* $x \in X$, and for $\varphi \in L^1(X, \mathcal{B}, \mu)$. The most noteworthy consequence of *Birkhoff's Ergodic Theorem* is the special case where f is an *ergodic transformation.*

Definition 1.1.19 *Let μ be a finite measure on $(X, \mathcal{B})$. A measurable preserving transformation $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ is said to be ergodic, with respect to the measure class of μ, if every f-invariant set $B \in \mathcal{B}$ is either $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.*

Proposition 1.1.20 *(Ergodic Theorem) Let f be an ergodic transformation in a finite measure space $(X, \mathcal{B}, \mu)$. For any integrable function φ, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \Big(\int_X \varphi \, \mathrm{d}\mu \Big) / \mu(X).$$

Proof. Using Birkhoff Ergodic Theorem, we get the following

$$\int_X A \, \mathrm{d}\mu = \int_X \varphi \, \mathrm{d}\mu.$$

Since f is an ergodic transformation, we have the following lemma:

Lemma 1.1.21 *A measure preserving transformation $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ for a finite measure μ is ergodic if and only if, every measurable function φ which is f-invariant is μ-a.e. constant.*

Proof. Suppose that only f-invariant functions are μ-*a.e.* constant. A set B is f-invariant only if the characteristic function χ_B of set B is f-invariant. Since $\chi_B(x)$ takes only 1 or 0, then χ_B must equal to 0, except on a set whose measure is 0, or to 1, except on a set whose measure is 1. Hence, $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

It only remains to consider the case where φ is a f-invariant measurable function that is not μ-*a.e.* constant, then there is a constant $c \in \mathbb{R}$ such that $B = \varphi([0, c[)$, then $\mu(B) > 0$ and $\mu(X \setminus B) > 0$. Hence the set B is f-invariant. □

Going back to the proof of our proposition, we have every measurable function φ is μ-*a.e.* constant, then so is the time average A.

$$\begin{aligned}
\frac{1}{\mu(X)}\int_X \varphi(x)\,\mathrm{d}\mu(x) &= \frac{1}{\mu(X)}\int_X A(x)\,\mathrm{d}\mu(x)\\
&= \frac{1}{\mu(X)}\int_X \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))\,\mathrm{d}\mu(x)\\
&= \frac{1}{\mu(X)}[\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))]\int_X \mathrm{d}\mu(x)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))
\end{aligned}$$

we got the desired result. □

Corollary 1.1.22 *Suppose that f is an ergodic transformation in a finite measure space and $B \in \mathcal{B}$ is a measurable set. We consider $\{f^i(x)|0 < i < n-1\}$ the first n points in forward orbit of $x \in B$. Let $N_n(x)$ be the number of those points which lie in B*

$$N_n(x) = \#\bigg(B \cap \{f^i(x)|0 < i < n-1\}\bigg).$$

Then μ-a.e. $x \in X$

$$\lim_{n\to\infty}\frac{N_n(x)}{n} = \frac{\mu(B)}{\mu(X)}.$$

Proof. Let $\varphi(x) = \chi_B(x)$. We get the following

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_B(f^i(x)) = \lim_{n\to\infty}\frac{N_n(x)}{n}$$

then

$$\lim_{n\to\infty}\frac{N_n(x)}{n} = \frac{\int_X \chi_B(x)\mathrm{d}\mu(x)}{\mu(X)} = \frac{\mu(B)}{\mu(X)}.$$

□

Example 1.1.23 *The rotation of the circle $R_\theta : S^1 \to S^1$ by θ is ergodic with respect to the Lebesgue measure m if and only if θ is irrational.*

Proof. Suppose that $\theta \in \mathbb{Q}$ and write $\theta = p/q$ for $p, q \in \mathbb{Z}$ with $q \neq 0$. Define

$$\varphi(x) = e^{2\pi i q x} \in L^1(X, \mathcal{B}, m)$$

Then φ is not constant but it is R_θ-invariant

$$\varphi\Big(R_\theta(x)\Big) = e^{2\pi i q(x+\frac{p}{q})} = e^{2\pi i(qx+p)} = \varphi(x)$$

Showing that R_θ is not ergodic.

If $\theta \in \mathbb{Q}$ then for any $\varepsilon > 0$ there exist $n, l, k \in \mathbb{Z}$ with $n \neq l$ and $|n\theta - l\theta - k| < \varepsilon$. It follows that $\alpha = (n-l)\theta - k$ lies within $(0, \varepsilon)$ but is not zero, and so the set $\{0, \alpha, 2\alpha, \ldots\}$ considered in S^1 is ε-dense (that is, every point of S^1 lies within ε of a point in this set). Thus $(\mathbb{Z}\theta + \mathbb{Z})/\mathbb{Z} \subseteq S^1$ is dense.

Suppose that $B \in \mathcal{B}$ is R_θ-invariant Then for any $\varepsilon > 0$ choose a function $\varphi \in C(S^1)$ with

$$\|\varphi - \chi_B\|_{L^1} = \int_{S^1} \Big(\varphi(x) - \chi_B(x)\Big) \mathrm{d}m(x) < \varepsilon.$$

By invariance of B we have

$$\|\varphi \circ R_\theta^n - \varphi\|_{L^1} = \|\varphi \circ R_\theta^n - \chi_B + \chi_B - \varphi\|_{L^1} < 2\varepsilon$$

for all $n \in \mathbb{N}$. Since φ is continuous, it follows that

$$\|\varphi \circ R_t - \varphi\|_{L^1} \leq 2\varepsilon$$

for all $t \in \mathbb{R}$. Since the rotations of the circle preserve the Lebesgue measure m, then by Fubini's theorem we have

$$\|\varphi - \int_{S^1} \varphi(x) \mathrm{d}m(x)\|_{L^1} = \int | \int \varphi(x) - \varphi(x+t) \mathrm{d}t | \mathrm{d}x \leq \|\varphi \circ R_t - \varphi\|_{L^1} \leq 2\varepsilon.$$

Therefore

$$\|\chi_B - m(B)\|_{L^1} \leq \|\chi_B - \varphi\|_{L^1} + \|\varphi - \int_{S^1} \varphi(x) \mathrm{d}m(x)\|_{L^1} + \| \int_{S^1} \varphi(x) \mathrm{d}m(x) - m(B)\|_{L^1} < 4\varepsilon.$$

Since this holds for every $\varepsilon > 0$ we deduce that χ_B is constant and therefore $m(B) \in \{0, 1\}$. Thus the rotation of the circle R_θ is ergodic with respect to Lebesgue measure for θ irrational. □

1.2 Hyperbolic groups

Introduced by *Gromov* in the 1980's, hyperbolic groups are a fundamental topic in geometric group theory. The fundamental idea in geometric group theory is to study groups as automorphisms of geometric spaces (metric spaces), and as a special case, to study the group itself (with its canonical self-action) as a geometric space. This is accomplished most directly by means of the *Cayley graph* construction. The idea of a hyperbolic group generalises on the much earlier work of *Dehn* on surface groups and also parts of the small cancellation theory of *Tartaskii, Greendlinger* and *Lyndon-Schup*. Hyperbolicity is a relaxed notion of negative curvature. The general theory of hyperbolic groups is based on geometric arguments which eventually lead to remarkable algebraic and analytical phenomena.

In this section we will give the definition and basic properties of hyperbolic spaces, hyperbolic groups and their boundaries. For careful proofs and a more detailed discussion the reader is referred to [G-H].

1.2.1 Hyperbolic metric spaces

Let S be a finite set of generators for the group Γ. Then every element γ of Γ can be expressed as a *word* in the generators, that is $\gamma = s_1^{r_1} s_2^{r_2} \dots s_n^{r_n}$ where $s_1, s_2, \dots, s_n \in S$ and $r_1, r_2, \dots, r_n = \pm 1$. The natural number n is called the length of this word. The *length* of an element γ of Γ with respect to the generators S, written $|\gamma|_S$, is the length of the shortest word in elements of S and their inverses representing the element γ. We define the distance $d_S(\gamma_1, \gamma_2)$ between two elements γ_1 and γ_2 of Γ to be the length of the shortest word representing $\gamma_1^{-1}\gamma_2$, that is $d_S(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|_S$.

(Γ, d_S) is a metric space, *i.e.* a group Γ together with a symmetric non-negative real-valued function d_S on $\Gamma \times \Gamma$ which vanishes precisely on the diagonal, and which satisfies the triangle inequality. The fact that the left action is by isometries can now be seen by noticing that

$$d_S(\gamma_3\gamma_1, \gamma_3\gamma_2) = |(\gamma_3\gamma_1)^{-1}\gamma_3\gamma_2|_S = |\gamma_1^{-1}\gamma_2|_S = d_S(\gamma_1, \gamma_2)$$

The metric d_S is related to the *Cayley graph* $\mathcal{G}(\Gamma, S)$ [3] in a natural way. We can identify Γ with the set of vertices of $\mathcal{G}(\Gamma, S)$ and two vertices γ_1, γ_2 ($\gamma_1 \neq \gamma_2$) are adjacent in Γ if and only if $d_S(\gamma_1, \gamma_2) = 1$, in other words $\gamma_1^{-1}\gamma_2 \in S$ or $\gamma_2^{-1}\gamma_1 \in S$. More generally, if γ_1, γ_2 are joined by a path of length n in $\mathcal{G}(\Gamma, S)$, then we can express $\gamma_1^{-1}\gamma_2$ as a word of length n in S, so $d_S(\gamma_1, \gamma_2) \leq n$. The converse is also true, if $\gamma_1^{-1}\gamma_2$ can be expressed as a word of length n in S, then γ_1, γ_2 can be joined by a path of length n in $\mathcal{G}(\Gamma, S)$. Hence $d_S(\gamma_1, \gamma_2)$ is precisely the length of a shortest path in $\mathcal{G}(\Gamma, S)$ from γ_1 to γ_2.

Definition 1.2.1 *We say that a metric space* (Γ, d_S) *is a geodesic metric space if for all* γ_1, γ_2 *in* Γ *there is an isometric map (geodesic segment)* σ *from the interval* $[0, d_S(\gamma_1, \gamma_2)]$ *into* Γ *such that* $\sigma(0) = \gamma_1$ *and* $\sigma\big(d_S(\gamma_1, \gamma_2)\big) = \gamma_2$.

We also extend the definition of a geodesic to isometric map $\sigma : [0, \infty) \to \Gamma$ is called *geodesic ray* and to isometric map $\sigma : \mathbb{R} \to \Gamma$ is called *bi-infinite geodesic.*

The most serious shortcoming of this construction of metric d_S is its dependence on the choice of a generating set S. Different choices of generating set S give rise to different spaces $\mathcal{G}(\Gamma, S)$ which are typically not even homeomorphic. The standard resolution of this issue is to coarsen the geometric category in which one works.

Definition 1.2.2 *Let* (X, d_1) *and* (Y, d_2) *be metric spaces. A map* $f : X \to Y$ *(not assumed to be continuous) is a quasi-isometric map if there are constants* $C \geq 0$ *and* $\lambda > 0$ *so that*

$$\frac{1}{\lambda} d_1(x_1, x_2) - C \leq d_2\big(f(x_1), f(x_2)\big) \leq \lambda d_1(x_1, x_2) + C$$

for all $x_1, x_2 \in X$.

[3] Dehn also called this the 'Gruppenbild'.

Example 1.2.3 *:*

1. The metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are always quasi-isometric if S and S' are finite generating sets for a group Γ. Indeed, let λ be the maximum length of any element of S expressed as a word in S'or vice versa. Then the identity map $\Gamma \to \Gamma$ is a $(\lambda, 0)$-quasi isometry form (Γ, d_S) to $(\Gamma, d_{S'})$ and vice versa.

2. $(\mathbb{Z}, d)$ and $(\mathbb{R}, d)$ are quasi-isometric, where d is the usual metric. The natural map $f : \mathbb{Z} \to \mathbb{R}$ is an isometry, so a $(1, 0)$-quasi isometry. We can define a $(1, \frac{1}{2})$-quasi isometry $g : \mathbb{R} \to \mathbb{Z}$ by $g(x) = [x]$ [4].

3. More generally, let Γ be a group with a finite generating set S, and let $\mathcal{G}(\Gamma, S)$ be the corresponding Cayley graph. We can regard $\mathcal{G}(\Gamma, S)$ as a topological space in the usual way, and indeed we can make it into a metric space by identifying each edge with a unit interval $[0, 1] \subset \mathbb{R}$ and defining $d(x, y)$ to be the length of the shortest path joining x to y. This coincides with the metric d_S when x and y are vertices. Since every point of $\mathcal{G}(\Gamma, S)$ is in the $\frac{1}{2}$-neighbourhood of some vertex, (Γ, d_S) and $(\mathcal{G}(\Gamma, S), d)$ we see that are quasi-isometric for this choice of d.

A (λ, C)-quasi-geodesic in X is the image of a (λ, C)-quasi-isometric map of bounded interval $I \subset \mathbb{R}$ into X. We also extend the definition of a quasi-geodesic to quasi-isometric map $\sigma : [0, \infty) \to \Gamma$ is called *quasi-geodesic ray* and to quasi-isometric map $\sigma : \mathbb{R} \to \Gamma$ is called *bi-infinite quasi-geodesic.*

Definition 1.2.4 *A metric space (X, d) is proper if closed metric balls of bounded radius are compact, equivalently, for each point x the function $d(x, .) : X \to \mathbb{R}$ is proper.*

we will give version of Gromov's hyperbolicity criterion.

Definition 1.2.5 *Suppose that (X, d) is a metric space. Given a base-point $\omega \in X$. The Gromov [5] product on X based at ω is the non-negative real number defined by the formula*

$$(x \cdot y)_\omega = \frac{1}{2}\big(d(x, \omega) + d(y, \omega) - d(x, y)\big).$$

Definition 1.2.6 *Let δ be a non-negative real number. The metric space (X, d) is said to be δ-hyperbolic if*

$$(x \cdot y)_\omega \geq \min\big((x \cdot z)_\omega, (y \cdot z)_\omega\big) - \delta$$

for every $x, y, z \in X$ and for every choice of a base-point ω.

Definition 1.2.7 *The metric space (X, d) is said to be hyperbolic if there exists a real number δ such that (X, d) is δ-hyperbolic.*

[4] The floor function.

[5] Inner.

Theorem 1.2.8 *Let (X, d_1) and (y, d_2) be geodesic metric spaces that are quasi-isometric to one another. If (X, d_1) is hyperbolic, then so is (y, d_2) (and conversely).*

The proof of this theorem can be found, for example, in [G-H]. The hyperbolicity of metric spaces is a quasi-isometry invariant, which is important because the metric space defined by a finitely generated group is only well-defined up to quasi-isometry.

The case in which we are interested here is when the geodesic metric space under consideration is the Cayley graph $\mathcal{G}(\Gamma, S)$ of a group Γ with respect to a finite generating set S. In general, (Γ, d_S) is not a geodesic metric space, since d_S takes values in $\mathbb{N}$. However, the geometric realization of the Cayley graph $(\mathcal{G}(\Gamma, S), d)$ is geodesic, with respect to the natural metric which is quasi-isometric to (Γ, d_S).

Definition 1.2.9 *(Hyperbolic group) A finitely generated group Γ is said to be hyperbolic if there is a finite generating set S of Γ such that the Cayley graph $\mathcal{G}(\Gamma, S)$ is hyperbolic with respect to the metric d_S.*

Example 1.2.10 *Any finite group G is hyperbolic, because its Cayley graphs are all bounded. For any integer $n \geq 1$, the free group F_n of rank n is hyperbolic, because it has Cayley graphs that are trees. Moreover, if a group has a free subgroup of finite index, then it is quasi-isometric to a free group, and hence hyperbolic.*

Geometric considerations give access to various properties of hyperbolic groups. Here are some of these properties:

1. A hyperbolic group Γ is finitely presented.
2. A hyperbolic group Γ contains only finitely many conjugacy classes of torsion elements.
3. The growth function of a hyperbolic group Γ, relatively to an arbitrary finite generating set S, is rational. Let us recall that the growth function of a finitely generated group Γ, and that S is a finite generating set for Γ, is the formal power series $g(t) = \sum c_n t^n$, where c_n is defined by
$$c_n = \#\{\gamma \in \Gamma | d_S(\gamma, \mathrm{id}) \leq n\}$$

1.2.2 The boundary of a hyperbolic group

The aim of this section is to define a boundary ∂X for a hyperbolic metric space X, which gives a compactification $\overline{X} = X \cup \partial X$ when X is complete and locally compact. Let X be a hyperbolic space, geodesic and proper, with a base-point ω. We say that a sequence $\{x_n\}$ of elements in X *converges at infinity* if the Gromov product $(x_n \cdot x_p)_\omega$ tends to ∞ when (n, p) converges to ∞. It is clear that this definition does not depend on the choice of ω since $|(x \cdot y)_\omega - (x \cdot y)_{\omega'}| \leq d(\omega, \omega')$. We define the relation

$$\{x_n\}\mathcal{R}\{y_n\} \Leftrightarrow \lim_{n\to\infty}(x_n \cdot y_n)_\omega = \infty.$$

The restriction of $\mathcal{R}$ to the set of sequences that converge at infinity $S_\infty(X)$ is an equivalence relation.

Definition 1.2.11 *The boundary of X is $\partial X = S_\infty(X)/\mathcal{R}$. If a is a point in ∂X, we say that a sequence $\{x_n\}$ of points in X converges to a if $a = [\{x_n\}]_\mathcal{R}$.*

It is easy to ensure that this definition does not depend on the choice of a base-point ω.

If X is proper and δ-hyperbolic, and ω is any base-point, then every equivalence class contains a geodesic ray starting at ω. For, if $\sigma : [0, +\infty[\to X$ is a geodesic ray such that $d(\omega, \sigma(t)) = t$, then there exists a point a in ∂X such that the sequence $x_n = \sigma(t_n)$ converges to a for every sequence $\{t_n\}$ of non-negative real numbers such that $t_n \to \infty$. We shall write $a = \sigma(\infty)$. In the same manner, every bi-infinite geodesic $\sigma : \mathbb{R} \to X$ defines two distinct points $\sigma(+\infty)$ and $\sigma(-\infty)$ of ∂X.

Example 1.2.12 *The real line $\mathbb{R}$, with the usual metric and 0 as the base-point, is compactified in this way by adding two points, $+\infty$ and $-\infty$. If $\{\lambda_n\} \in S_\infty(\mathbb{R})$ then either $\lambda_n > 0$ for almost all $n \in \mathbb{N}$ or $\lambda_n < 0$ for almost all n. This defines two distinct equivalence classes of sequences which we call $+\infty$ and $-\infty$ respectively and $\partial X = \{+\infty, -\infty\}$ as expected.*

We may extend the *Gromov product* to a continuous function $(a \cdot b)_\omega : \partial X \times \partial X \to [0, \infty]$ by

$$(a \cdot b)_\omega = \inf \limsup_{n,p\to\infty} (x_n \cdot y_p)_\omega$$

for all $a, b \in \partial X$, where the infimum is taken over all sequences $\{x_n\}$ and $\{y_p\}$ in X such that $a = \lim\limits_{n\to\infty} x_n$ and $b = \lim\limits_{p\to\infty} y_p$. This allows us to define the topology on the boundary of X. For any $a \in \partial X$ and $k \geq 0$ we define the set

$$\mathcal{V}(a, k) = \{b \in \partial X | (a \cdot b)_\omega \geq k\}.$$

We now endow ∂X with a topology by setting the basis of neighbourhoods for any $a \in \partial X$ to be the collection

$$\{\mathcal{V}(a, k) | k \geq 0\}.$$

It is not hard to show that the resulting topology does not depend on the choice of a base-point ω. Moreover, we have

Proposition 1.2.13 *For any $a \in \partial X$, the collection $\{\mathcal{V}(a, k) | k \geq 0\}$ is a basis topology. Moreover, ∂X is compact.* □

One of the main properties of ∂X that depends only on the class of quasi-isometry of ∂X. We have the following important:

Proposition 1.2.14 *The quasi-isometry f between two proper hyperbolic geodesic spaces (X, d_1) and (Y, d_2) extends to a canonical homeomorphism of their boundaries $\partial f : \partial X \to \partial Y$.* □

In particular, let Γ be a hyperbolic group. Then for some (and therefore for any) finite generating set S of Γ the Cayley graph $\mathcal{G}(\Gamma, S)$ is hyperbolic with respect to the metric d_S. Clearly, $\mathcal{G}(\Gamma, S)$ is a proper geodesic metric space. We define the boundary $\partial\Gamma$ of Γ as the boundary of the space metric $\mathcal{G}(\Gamma, S)$. Since the change of a generating set induces a quasi-isometry of the cayley graphs, the topological type of $\partial\Gamma$ does not depend on the choice of S.

Let us also note that the action of Γ on $\mathcal{G}(\Gamma, S)$ by left translations is an isometric action which induces a (continuous) action of Γ on $\partial\Gamma$. This action on $\partial\Gamma$ does not depend on the choice of the finite generating set S of Γ. The dynamical system $(\partial\Gamma, \Gamma)$ is therefore canonically associated to the hyperbolic group Γ.

Example 1.2.15 *:*

1. A hyperbolic group has empty boundary if and only if it is a finite group.
2. If Γ is an infinite cyclic group, then $\partial\Gamma = \{+\infty, -\infty\}$.
3. If F_n is a free group, $n > 2$, then ∂F_n is a Cantor set, that is, ∂F_n is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.
4. The ordinary hyperbolic plane $\mathbb{H}^2$, when considered in the Poincaré disc model, has a natural boundary "the circle at infinity" $\partial\mathbb{H}^2 = S^1 = \{x \in \mathbb{R}^2, \|x\| < 1\}$ which is a compact space. Similarly, hyperbolic space $\mathbb{H}^n$ of dimension n has a natural boundary which is S^{n-1}.

Also implicit in the above examples is the existence of a natural topology on $\partial\Gamma$. In fact, we can define a metric on $\partial\Gamma$ which will depend on our choice of base-point ω, although the resulting topology does not.

Definition 1.2.16 *Choose $\varepsilon > 0$, we define the measure of separation of the points in $\partial\Gamma$ by*

$$\rho_\varepsilon(a, b) = e^{-\varepsilon(a \cdot b)_\omega}$$

for all $a, b \in \partial\Gamma$.

In general ρ_ε is not a metric on $\partial\Gamma$, because it does not satisfy the triangle inequality

$$\rho_\varepsilon(a, b) \leq (1 + \varepsilon') \max\left(\rho_\varepsilon(a, c), \rho_\varepsilon(c, b)\right)$$

with $\varepsilon' = e^{\varepsilon\delta} - 1$.

The geodesic boundary of Γ admits a family of *visual metrics* d_ε defined as follows. Pick a positive parameter ε. Given points $a, b \in \partial\Gamma$ consider various chains $C = (a_0, \ldots, a_n)$ (where n varies) so that $a_0 = a$ and $a_n = b$. Given such a chain, define

$$d_\varepsilon(a, b) = \inf \sum_{i=1}^{n} \rho_\varepsilon(a_{i-1}, a_i)$$

where the infimum is taken over all chains connecting a and b.

Then d_ε is a metric on $\partial\Gamma$ for sufficiently small ϵ, that is $\varepsilon' < \sqrt{2} - 1$. It turns out that $\partial\Gamma$ is a compact space.

Definition 1.2.17 *Let f be an homeomorphism between two metric spaces (X, d_1) and (Y, d_2). Then f is K-quasi-conformal if*

$$\limsup_{r\to 0} \frac{\sup\{d_2(f(x), f(x'))|d_1(x,x') = r\}}{\inf\{d_2(f(x), f(x'))|d_1(x,x') = r\}} \leq K$$

for all $x \in X$.

We know that an isometry of a hyperbolic space X induces a homeomorphism of ∂X. It turns out that this homeomorphism is $K(\varepsilon, \delta)$-quasi-conformal for the metric d_ϵ where $K(\varepsilon, \delta)$ is a constant converges to 1 when $\varepsilon \to 0$. In some ways, we can say that the action of an isometry at infinity is conformal. Similarly, it is ensured that the homeomorphism of ∂X induced an quasi-isometry of X is a homeomorphism K-quasi-conformal for some constant K.

As meaning that, we say that the boundary of a hyperbolic group has a quasi-conformal natural structure.

Moreover, a subgroup of Γ is non-elementary if its action on $\partial\Gamma$ does not fix a finite set. In other word, a hyperbolic group Γ is said to be elementary if it is finite or finite extension of $\mathbb{Z}$. We say that a probability measure μ on Γ is non-elementary if the subgroup generated by its support is itself non-elementary.

Let μ be a probability measure on Γ. Since Γ acts by homeomorphisms on the compact space $\partial\Gamma$, then it admits a *stationary measure* (see [Ka]). In other word, there exists a probability measure ν_Γ on $\partial\Gamma$ associated to the action of Γ on $\partial\Gamma$ such that $\mu * \nu_\Gamma = \nu_\Gamma$, i.e. for every Borel set $B \subset \partial\Gamma$, we have

$$\nu_\Gamma(B) = \sum_{\gamma\in\Gamma} \mu(\gamma)\nu_\Gamma(\gamma^{-1}(B)).$$

If μ is non-elementary, then this measure ν_Γ is unique and has no atom.

We can define the *Busemann function* as follows:

Definition 1.2.18 *The Busemann function $\beta_a(\cdot,\cdot)$ relative to the point $a \in \partial\Gamma$ is defined by following: for any $\gamma_1, \gamma_2 \in \Gamma$ we have*

$$\beta_a(\gamma_1, \gamma_2) = \sup \Big\{ \limsup_{n\to\infty} \left[d_S(\gamma_1, a_n) - d_S(\gamma_2, a_n)\right] \Big\}$$

where the supremum is taken over all sequences $\{a_n\}$ in Γ which tends to $a \in \partial\Gamma$.

Proposition 1.2.19 *Let β_a the Busemann function relative to $a \in \partial\Gamma$ and let $\gamma_1, \gamma_2 \in \Gamma$. Then $\lim_{\gamma_1\to a} \beta_a(\gamma_1, \gamma_2) = -\infty$ and $\lim_{\gamma_1\to b} \beta_a(\gamma_1, \gamma_2) = +\infty$ for all $b \in \partial\Gamma$.* □

Let $a \in \partial\Gamma$ and $\omega \in \Gamma$, the Gromov product relative to the point a and based at ω is defined by the formula

$$(\gamma_1 \cdot \gamma_2)_{a,\omega} = \frac{1}{2}\big(\beta_a(\gamma_1, \omega) + \beta_a(\gamma_2, \omega) - d_S(\gamma_1, \gamma_2)\big)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. We may extend the Gromov product relative to the point a and based at ω to points of $\partial\Gamma$ by

$$(b \cdot c)_{a,\omega} = \inf \limsup_{n,p\to\infty} (x_n \cdot y_p)_{a,\omega}$$

for all $b, c \in \partial\Gamma$, where the infimum is taken over all sequences $\{x_n\}$ and $\{y_p\}$ in Γ such that $b = \lim_{n\to\infty} x_n$ and $c = \lim_{p\to\infty} y_p$.

In fact, we can define the family metrics $d_{\varepsilon,a,\omega}$ on $\partial\Gamma \setminus \{a\}$ for a fixed (small) $\varepsilon > 0$ and $\omega \in \Gamma$ as follows. For all $b, c \in \partial\Gamma \setminus \{a\}$, we put

$$\rho_{\varepsilon,a,\omega}(b, c) = e^{-\varepsilon(b\cdot c)_{a,\omega}}$$

and consider various chains $C = (b_0, \ldots, b_n)$ (where n varies) so that $b_0 = b$ and $b_n = c$. Given such a chain, define

$$d_{\varepsilon,a,\omega}(b, c) = \inf \sum_{i=1}^{n} \rho_{\varepsilon,a,\omega}(b_{i-1}, b_i)$$

where the infimum is taken over all chains connecting b and c.

If $\varepsilon' = e^{-1200\varepsilon\delta} - 1$, then we have the following proposition which its proof can be found, for example, in [G-H].

Proposition 1.2.20 *If $\varepsilon > 0$ is small enough, then $d_{\varepsilon,a,\omega}$ is a metric on $\partial\Gamma \setminus \{a\}$ with*

$$(1 - 2\varepsilon')\rho_{\varepsilon,a,\omega}(b, c) \leq d_{\varepsilon,a,\omega}(b, c) \leq \rho_{\varepsilon,a,\omega}(b, c).$$

Moreover, in particular, there is some constant $K(\varepsilon, \delta) \geq 1$ converges to 1 when $\varepsilon \to 0$ such that

$$\frac{1}{K(\varepsilon,\delta)} e^{-\varepsilon\beta_a(\omega,\omega')} \leq \frac{d_{\varepsilon,a,\omega'}(b, c)}{d_{\varepsilon,a,\omega}(b, c)} \leq K(\varepsilon, \delta) e^{-\varepsilon\beta_a(\omega,\omega')}$$

for all $\omega' \in \Gamma$ and $b, c \in \partial\Gamma \setminus \{a\}$. □

Chapter 2

Locally non-discrete groups of $\mathrm{Diff}^{\omega}(S^1)$

2.1 Basic definitions

The definition of locally non-discrete groups is implicit in [R1] and formulated in [R3] and in [De]. In the analytic case, it reads as follows:

Definition 2.1.1 *A subgroup G of $\mathrm{Diff}^{\omega}(S^1)$ is said to be locally C^m-non-discrete if there is a non-empty open interval $I \subseteq S^1$ and a sequence of elements $\{g_i\} \subset G$, $g_i \neq \mathrm{id}$ for every $i \in \mathbb{N}$, whose restrictions $\{g_{i|I}\}$ to I converge to the identity in the C^m-topology (as maps from I to S^1).*

Naturally, a group $G \subset \mathrm{Diff}^{\omega}(S^1)$ is called *locally C^m-discrete* if it fails to satisfy the conditions of Definition 2.1.1. Unless otherwise stated, the terminology used in this paper is such that every *interval* is open, connected and non-empty. In what follows we shall mainly work with locally C^2-non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$.

Concerning Definition 2.1.1 and the corresponding sequence $\{g_i\}$ of diffeomorphisms in G, the reader will note that the condition $g_i \neq \mathrm{id}$ ensures that the restriction of g_i to the interval I does not coincide with the identity either since our diffeomorphisms are real analytic. The analogous definition becomes therefore slightly more technical for groups of, say, smooth diffeomorphism; cf. [De].

It also useful to adapt Definition 2.1.1 to the context of pseudogroups. However, even in the analytic category, the case of pseudogroups exhibits a difficulty analogous to the one pointed out above for groups of smooth diffeomorphisms since the domain of definition of an element in a pseudogroup may be disconnected.

Definition 2.1.2 *Consider an open set $U \subset \mathbb{R}$ along with a pseudogroup Γ of analytic diffeomorphisms from open subsets of U to $\mathbb{R}$. The pseudogroup Γ is said to be locally C^m-non-discrete if there is an interval (open, connected and non-empty) $I \subset U$ and a sequence of maps $\{g_i\} \subset \Gamma$ satisfying the following conditions:*

1. *For every $i \in \mathbb{N}$, the interval I is contained in the domain of definition of g_i viewed as element of the pseudogroup Γ.*

2. *The restriction $g_{i|I}$ of g_i to I does not coincide with the identity map.*

3. *The sequence $\{g_{i|I}\}$ formed by the restrictions of the g_i to I converge to the identity in the C^m-topology (as maps from I to $\mathbb{R}$).*

Remark 2.1.3 In the case of pseudogroups of maps defined on the real line, the above mentioned issue involving possibly disconnected domains of definitions can be avoided in many cases including, for example, when the pseudogroup has a finite generating set all of whose elements are defined on all of U. In higher dimensions however pseudogroups having elements with disconnected domains of definitions are very common and cannot easily be avoided.

For the discussion in this paper we have opted for including Condition 2 in our assumptions so as to have a definition that can immediately be generalized. Yet the reader will note that for our purposes this condition is of little importance since our attention can be restricted to pseudogroups induced by suitable restrictions of actual groups of diffeomorphisms of the circle.

2.2 Some examples of locally discrete/non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$

Example 1: Suppose that G is a non-abelian group generated by diffeomorphisms $f, g \in \mathrm{Diff}^{\omega}(S^1)$ sharing a common fixed point. One such group G is necessarily locally C^2-non-discrete (in fact locally C^{∞}-non-discrete), as follows from Shcherbakov-Nakai theory as expounded in Section 2.3.

The reader will however note that a group generated by a *random choice* of $n \geq 2$ diffeomorphisms $f_1, \ldots, f_n$ in $\mathrm{Diff}^{\omega}(S^1)$ is such that the stabilizer of every point in S^1 is cyclic (possibly trivial). From this point of view, groups $G \subset \mathrm{Diff}^{\omega}(S^1)$ as in *Example 1* are somehow rather special. Nonetheless, when it comes to providing non-trivial examples of locally non-discrete groups, the following result due to Ghys [G1] is far more satisfying.

Example 2: Let $\mathrm{Diff}^{\omega}(S^1)$ be equipped with the analytic topology (see [G1]). Then there is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{\omega}(S^1)$ of the identity with the following property: every non-solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ generated by a finite set $g_1, \ldots, g_n$ contained in $\mathcal{U}$ is C^{∞}-non-discrete.

Ghys result is a non-linear generalization to $\mathrm{Diff}^{\omega}(S^1)$ of the classical Zassenhaus lemma valid for Lie groups of finite dimension. In fact, according to Zassenhaus, in every finite dimensional Lie group, there is a neighborhood U of the identity such that every discrete subgroup Γ generated by a finite set contained in U must be nilpotent.

Example 3: Every subgroup $G \subset \mathrm{Diff}^{\omega}(S^1)$ having a *Cantor set* as minimal set is necessarily locally C^2-discrete, as follows from Proposition 2.4.1.

On the other hand, recall that the group $\mathrm{PSL}\,(2,\mathbb{R}) = \mathrm{SL}\,(2,\mathbb{R})/\{\mathrm{id}, -\mathrm{id}\}$ (where id stands for the identity matrix) has a natural action on $S^1 \simeq \mathbb{R} \cup \{\infty\}$ given by

$$x \mapsto \frac{ax+b}{cx+d} \;\; ; \;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}\,(2,\mathbb{R}).$$

This action being analytic, the group $\mathrm{PSL}\,(2,\mathbb{R})$ becomes identified with a subgroup of $\mathrm{Diff}^{\omega}(S^1)$ (up to the choice of the identification $S^1 \simeq \mathbb{R} \cup \{\infty\}$). By thinking of $\mathrm{PSL}\,(2,\mathbb{R})$ as a subgroup of $\mathrm{Diff}^{\omega}(S^1)$, we can easily produce interesting examples of locally discrete and locally non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$, cf. below.

Example 4: Recall that a subgroup $\Gamma \subset \mathrm{PSL}\,(2,\mathbb{R})$ is said to be a *Fuchsian group* if it is discrete as subset of $\mathrm{PSL}\,(2,\mathbb{R})$ when the latter is equipped with its standard topology of *Lie group*.

Since $\mathrm{PSL}\,(2,\mathbb{R})$ is identified with a subgroup of $\mathrm{Diff}^{\omega}(S^1)$, every subgroup $\Gamma \subset \mathrm{PSL}\,(2,\mathbb{R})$ acts on S^1 as well. The following proposition is elementary and well-known.

Proposition 2.2.1 *Consider a subgroup* $\Gamma \subset \mathrm{PSL}\,(2,\mathbb{R})$ *be identified with a subgroup of* $\mathrm{Diff}^{\omega}(S^1)$ *as in the above discussion. Then* Γ *is locally* C^2*-discrete if and only if* Γ *is a Fuchsian group.*

Example 5: In connection with *Examples 1* and *3*, it is natural to wonder about locally discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ whose action on S^1 is *minimal*, i.e. has all orbits dense (cf. Proposition 2.4.2).

It is easy to construct Fuchsian groups acting minimally on S^1. For example, consider a hyperbolic structure on the genus g surface Σ_g ($g \geq 2$). The surface Σ_g is then covered by the hyperbolic disc $\mathbb{D}$ and the fundamental group $\Pi_1(\Sigma_g)$ acts on $\mathbb{D}$ so as to induce an analytic action on $S^1 \simeq \partial\mathbb{D}$. Since Σ_g is compact, these follows that every such action on S^1 is minimal.

A much harder question is to find locally C^2-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ which acts minimally on S^1 and *are not made out of* Fuchsian groups. This leads us to the far more elaborate classes of examples below.

Example 6: Ghys and Sergiescu have realized Thompson group T as a subgroup of $\mathrm{Diff}^{\infty}(S^1)$ acting minimally on S^1. The reader will note that Thompson-Ghys-Sergiescu subgroup of $\mathrm{Diff}^{\infty}(S^1)$ is not conjugate to any finite covering of a Fuchsian group. This group is however locally discrete, as observed in [R5]. In fact, Thompson-Ghys-Sergiescu group was probably the first known example of a locally discrete group which is genuinely different from Fuchsian groups.

Naturally the inconvenient in the example provided by Thompson-Ghys-Sergiescu group lies in the fact that it is a subgroup of smooth diffeomorphisms and therefore is not a subgroup of $\mathrm{Diff}^{\omega}(S^1)$. As a matter of fact, to the best of my knowledge, it is an open question whether or not Thompson group T can be realized as a subgroup of $\mathrm{Diff}^{\omega}(S^1)$. In any event, the existence of locally discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ genuinely different from Fuchsian groups was recently established V. Kleptsyn and his collaborators in [AFKMMNT].

2.3 Vector fields in the closure of pseudogroups

Vector fields whose local flow can be approximated by elements in the initial group (pseudogroup) constitute a very important tool to investigate the dynamics associated with *locally non-discrete* groups (pseudogroups). The idea of approximating a flow by elements in a group/pseudogroup is made accurate by the following definition.

Definition 2.3.1 *Consider an open set $U \subset \mathbb{R}$ along with a pseudogroup Γ of maps from open subsets of U to $\mathbb{R}$. Consider also a vector field X defined on an interval $I \subset U$ and let Ψ_X denote its local flow. The vector field X is said to be (contained) in the C^m-closure of Γ if, for every interval $I_0 \subset I$ and for every $t_0 \in \mathbb{R}_+$ such that Ψ_X^t is defined on I_0 for $0 \leq t \leq t_0$, there exists a sequence of maps $\{g_i\} \subset \Gamma$ satisfying the conditions below:*

- *For every $i \in \mathbb{N}$, the interval I_0 is contained in the domain of definition of g_i viewed as element of the pseudogroup Γ.*
- *The sequence $\{g_{i|I_0}\}$ formed by the restrictions of the g_i to I_0 converge to $\Psi_X^{t_0} : I_0 \to \mathbb{R}$ in the C^m-topology (where $m \in \mathbb{N} \cup \{\infty\}$).*

Unless otherwise mentioned, whenever we mention a vector field X belonging to the closure of a pseudogroup Γ it is implicitly assumed that this vector field does not vanish identically. It is clear from the definitions that a pseudogroup containing some (non-identically zero) vector field in its C^m-closure cannot be *locally C^m-discrete*.

Before going further into the structure of the topological dynamics of locally C^2-non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$, let us quickly revisit some results established by Shcherbakov and Nakai for pseudogroups of holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$; see [N1], [Sh]. The discussion below is slightly simplified by the fact that only local diffeomorphisms having real coefficients will be considered. Let $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ denote the group of germs of orientation-preserving analytic diffeomorphisms fixing $0 \in \mathbb{R}$. Here by *orientation-preserving* it is meant that every $g \in \mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ satisfies $g'(0) > 0$.

First, we have:

Lemma 2.3.2 *Let Γ be a pseudogroup generated by finitely many elements of* $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ *and denote by Γ_0 the group of germs at $0 \in \mathbb{R}$ corresponding to Γ. Assume that Γ_0 is not abelian. Then Γ contains analytic vector fields in its closure. In particular Γ is locally C^{∞}-non-discrete.*

Proof. The proof is split in two cases according to whether or not Γ_0 is fully constituted by germs of diffeomorphisms tangent to the identity.

Assume first the existence of an element in Γ_0 such that $g'(0) \neq 1$. Since g preserves the orientation of S^1, we have $g'(0) = \lambda > 0$. Thus up to replacing g by its inverse g^{-1}, we can assume that $g'(0) = \lambda \in (0,1)$. In this case, there are local (analytic) coordinates where $g(x) = \lambda\, x$; see [St]. Since Γ_0 is not abelian, there also exists another element $g_1 \neq \mathrm{id}$ belonging to $D^1\Gamma_0$. Though $g_1 \neq \mathrm{id}$, the derivative of g_1 at $0 \in \mathbb{R}$ equals 1 since $g_1 \in D^1\Gamma_0$ is a product of commutators. Now, by repeating the standard argument of Shcherbakov-Nakai with elements of Γ_0 having the form $\lambda^{-N(k)}.g_1(\lambda^{N(k)}x)$, it is well known that a suitable choice of the integers $N(k)$ leads to an analytic vector field X in the C^∞-closure Γ, see for example [N1]. The reader will note that the mentioned vector field X is defined around $0 \in \mathbb{R}$ which in general does not happen for Shcherbakov-Nakai vector fields. The proof of the lemma is therefore completed provided that Γ_0 contains an element which is not tangent to the identity.

We now consider the case where every element g in Γ_0 satisfies $g'(0) = 1$. Since Γ_0 is not abelian, there must exist elements $g_1,\, g_2 \in \Gamma_0$, $g_1,\, g_2 \neq \mathrm{id}$, having different contact orders with the identity. These two elements can then be used to produce a vector field of Shcherbakov-Nakai in the C^∞-closure of Γ, see for example [N1]. The lemma is proved. □

Lemma 2.3.3 *Consider a pseudogroup Γ generated by finitely many elements of* $\mathrm{Diff}^\omega(\mathbb{R},0)$ *and denote by Γ_0 the group of germs at $0 \in \mathbb{R}$ corresponding to Γ. Then the following are equivalent:*

1. *Γ_0 is an infinite cyclic group unless it is reduced to the identity.*
2. *Γ is locally C^m-discrete, for every $m \in \mathbb{N} \cup \{\infty\}$.*
3. *Γ does not contain vector fields in its C^m-closure, for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. Owing to Lemma 2.3.2 we can assume that Γ_0 is abelian otherwise none of the above statements holds. Since the elements in $\Gamma \subset \mathrm{Diff}^\omega(\mathbb{R},0)$ are assumed to preserve the orientation of $\mathbb{R}$, it follows at once that every element different from the identity in Γ_0 has infinite order. Assuming once and for all that Γ_0 is not reduced to the identity, consider an element $g \neq \mathrm{id}$ in Γ_0. Modulo replacing g by its inverse g^{-1}, we can assume that $g'(0) \leq 1$. Let us then split the discussion into two cases.

Case 1. Suppose there is $g \in \Gamma_0$ such that $g'(0) = \lambda < 1$. Again Sternberg's result [St] implies the existence of local analytic coordinates where $g(x) = \lambda\, x$. Since Γ_0 is abelian, there follows that every element of Γ_0 coincides with a linear map of type $x \mapsto c\, x$ in the above coordinates, where $c \in \mathbb{R}_+^*$ is a constant. In other words, Γ_0 is naturally identified with a multiplicative subgroup of $\mathbb{R}_+^*$. The mutual equivalence of the above statements follows at once.

Case 2. Suppose now that every element in Γ_0 is tangent to the identity. Let then $g \neq \mathrm{id}$ be an element of Γ_0 and denote by Y the formal vector field whose time-one map coincides with g.

Let $\mathcal{T}$ be the sets of those values of $t \in \mathbb{R}$ for which the formal flow Ψ_Y^t of Y actually defines an element of Γ_0. Clearly $\mathcal{T}$ is an additive subgroup of $\mathbb{R}$. Moreover, it is well known that the formal power series defining Y will be convergent provided that the set $\mathcal{T}$ is not discrete in $\mathbb{R}$, see [Ba], [Ec].

Since Γ_0 is abelian, it embeds in the 1-parameter group generated by the formal flow of Y. In fact, the formal flow of Y is known to contain the germs of all elements in $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ commuting with g. There follows that Γ_0 is infinite cyclic if and only if $\mathcal{T}$ is a discrete subgroup of $\mathbb{R}$. In this case, there also follows that Γ is locally C^m-discrete and that Γ contains only trivial vector fields in its C^m-closure (for every $r \in \mathbb{N} \cup \{\infty\}$). Conversely, if $\mathcal{T}$ is not discrete in $\mathbb{R}$, then it must be dense. Furthermore the formal vector field Y turns out to be analytic ([Ba], [Ec]). It is now immediate to check that Y itself is contained in the C^{∞}-closure of Γ. The lemma is proved. □

Shcherbakov-Nakai vector fields for non-solvable subgroups of $\mathrm{Diff}\,(\mathbb{C}, 0)$ were the first genuinely non-linear situation where vector fields in the closure of (countable) groups were proven to exist. Subgroups of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ (or even of $\mathrm{Diff}\,(\mathbb{C}, 0)$) are obviously special, as opposed to groups of $\mathrm{Diff}^{\omega}(S^1)$, in the sense that their elements share a same fixed point, namely the origin. In addition to the existence of free discrete subgroups in $\mathrm{Diff}^{\omega}(S^1)$, the absence of a common fixed point for elements in free subgroups of $\mathrm{Diff}^{\omega}(S^1)$ is the main obstacle to extend to this context the results obtained in [N1], [Sh]. This difficulty was overcome for the first time in [R1]. The following lemma singles out the key point that is common to all constructions of vector fields having similar properties (for detailed explanations see [R4]).

Lemma 2.3.4 *Suppose that the pseudogroup Γ consisting of local diffeomorphisms from open sets of an open (non-empty) interval $J \subset \mathbb{R}$ to $\mathbb{R}$ contains a sequence of elements $\{\tilde{g}_i\}$ satisfying the following conditions:*

1. *For every i, $\tilde{g}_i$ is defined on a fixed non-empty open interval I. Moreover the restriction $\tilde{g}_{i|_I}$ of $\tilde{g}_i$ to I is different from the identity for every i.*

2. *The sequence of local diffeomorphisms $\tilde{g}_{i|_I}$ converges to the identity in the C^m-topology.*

3. *There is a uniform constant C such that*

$$\|\tilde{g}_i - \mathrm{id}\|_{m,I} \leq C\, \|\tilde{g}_i - \mathrm{id}\|_{m-1,I}$$

where $\|\,.\,\|_{m,I}$ (resp. $\|\,.\,\|_{m-1,I}$) stands for the C^m-norm (resp. C^{m-1}-norm) of $\tilde{g}_i - \mathrm{id}$ on I.

Then there is a (non-identically zero) vector field X contained in the C^{m-1}-closure of Γ.

Proof. For every i, we consider the vector field X_i defined on I by the formula

$$X_i = \frac{1}{\|\tilde{g}_i - \mathrm{id}\|_{m,I}}\, (\tilde{g}_i(x) - x)\partial/\partial x\,.$$

It follows at once that the C^m-norm of X_i on I is bounded by 1 and, in addition, that this bound is attained in the closure of I. In turn, condition 3 above shows that the C^{m-1}-norm of X_i is bounded from below by a positive constant. In fact, we have

$$0 < \frac{1}{C} \leq \|X_i\|_{m-1,I}$$

for every $i \in \mathbb{N}$. Owing to Ascoli-Arzela theorem, and modulo passing to a subsequence, the sequence of vector fields $\{X_i\}$ converges in the C^{m-1}-topology towards a C^{m-1}-vector field X. Furthermore, X is not identically zero since it must verify $\|X\|_{m-1,I} \geq 1/C > 0$. Now a standard application of Euler polygonal method shows that the vector field X is contained in the C^{m-1}-closure of Γ in the sense of Definition 2.3.1. The lemma is proved. □

Lemma 2.3.4 will often be used in the context where $m = 2$. The method originally put forward in [R1] is summarized by Proposition 2.3.5 below; see also [R4] and [DKN-2].

Proposition 2.3.5 *Consider a pseudogroup Γ consisting of maps from an interval $J \subset \mathbb{R}$ to $\mathbb{R}$. Assume that Γ satisfies the two conditions below.*

- *There is a sequence of elements $\{g_i\} \subset \Gamma$ such that all the maps g_i are defined on J and none of them coincides with the identity on J. Moreover, this sequence converges to the identity on the C^m-topology on J.*
- *There is an element $f \in \Gamma$ possessing a hyperbolic fixed point $p \in J$.*

Then there is an open interval $I \subset J$ containing p and a sequence of elements $\{\tilde{g}_i\}$ in Γ satisfying the conditions of Lemma 2.3.4 (in particular, all the diffeomorphisms $\tilde{g}_i$ are defined on I and none of them coincides with the identity on I).

Proof. We shall sketch the argument since extensions of this basic idea will play an important role in Sections 3.2 and 3.3. It suffices to consider the case $m = 2$. By assumption, we have $f(p) = p$ and $f'(p) = \lambda \in (0, 1)$. Since f is analytic, there is a local coordinate x around p where $f(x) = \lambda\, x$ [St]. Let then $I \subset J$ be an interval containing p whose closure is contained in the domain of definition of the coordinate x. First, we have the following:

Claim 1. Without loss of generality, we can assume that $g_i(p) \neq p$ for every i.

Proof of Claim 1. Suppose that $g_i(p) = p$ for all but finitely many i. If, for some large enough i, we have $g_i'(p) = 1$ then by considering elements of the form $\{\lambda^{-N}\, g_i(\lambda^N x)\}$ (with i fixed), we can obtain a Shcherbakov-Nakai vector field defined *on a neighborhood of p* and contained in the C^∞-closure of Γ; see for example [N1]. The existence of this vector field actually suffices for our purposes, yet we point out that the sequence of elements $\{\lambda^{-N}\, g_i(\lambda^N x)\}$ satisfies the conditions of the lemma.

There follows from the preceding that the proposition holds provided that there is some g_i not commuting with f and satisfying $g_i(p) = p$. Hence it only remains to consider the possibility of having all the diffeomorphisms g_i commuting with f and satisfying $g_i(p) = p$ (modulo dropping finitely many terms of the initial sequence). Since g_i commutes with f, it must be given on I and in the coordinate x by $g_i(x) = \lambda_i \, x$. However the sequence $\{\lambda_i\}$ converges to 1 since $\{g_i\}$ converges C^2 (in fact C^{∞}) to the identity. In other words, the sequence $\{g_i\}$ satisfies all the conditions in the statement. □

Considering the last possibility discussed in the proof of the above claim, the reader will note that the C^1-closure of Γ contains a flow consisting of linear maps $x \mapsto \Lambda \, x$ for every $\Lambda \in \mathbb{R}^*$. Indeed, for every i, $\lambda_i \neq 1$ since $g_i \neq \mathrm{id}$. There follows that the multiplicative group of $\mathbb{R}^*$ generated by the collection of all λ_i is dense in $\mathbb{R}^*$ what, in turn, ensures that the mentioned vector field lies in the C^1-closure (indeed in the C^{∞}-closure) of Γ.

Going back to the proof of our proposition, in what follows we assume that $g_i(p) \neq p$ for every $i \in \mathbb{N}$. Next, let κ_i be a sequence of positive integers going to infinity to be determined later. Set

$$\tilde{g}_i = f^{-\kappa_i} \circ g_i \circ f^{\kappa_i} = \lambda^{-\kappa_i} \, g_i(\lambda^{\kappa_i} x) \, .$$

Note that the second derivative $\tilde{g}_i''$ of $\tilde{g}_i$ at a point x is simply $\tilde{g}_i''(x) = \lambda^{\kappa_i} \, g_i''(\lambda^{\kappa_i} x)$ provided that both sides are defined. This simple formula shows that $\sup_{x \in I} \|\tilde{g}_i''(x)\|$ decreases as κ_i increases. On the other hand the absolute value of $\lambda^{-n} \, g_i(0)$ increases monotonically with n and becomes unbounded as $n \to \infty$ since $g_i(0) \neq 0$. Therefore the C^1-norm of $\tilde{g}_i - \mathrm{id}$ on I also increases with n. Thus, for every i fixed, we can find $\kappa_i \in \mathbb{N}^*$ so that the following estimate holds:

$$\sup_{x \in I} \|\tilde{g}_i''(x)\| \leq \sup_{x \in I} \{\|\tilde{g}_i - \mathrm{id}\| + \|\tilde{g}_i' - 1\|\} \, .$$

For these choices of κ_i we immediately obtain

$$\|\tilde{g}_i - \mathrm{id}\|_{2,I} < 2\|\tilde{g}_i - \mathrm{id}\|_{1,I}$$

proving the proposition. □

2.4 Topological dynamics of locally non-discrete subgroups

The material presented in this section is very closely related to the description in [G1] of the topological dynamics associated with groups generated by diffeomorphisms close to the identity. In fact, our purpose is to prove the following:

Proposition 2.4.1 *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a locally C^2-non-discrete group. Then either G has a finite orbit or every orbit of G is dense in S^1. Moreover, the set of points in S^1 having finite orbit under G is itself finite. Finally, if I is a connected interval in the complement of this set and G_I denote the subgroup of G consisting of those diffeomorphisms fixing I, then the action of G_I on I has all orbits dense in I.*

Since our assumptions are slightly more general than those used in [G1], we shall provide below a detailed proof for Proposition 2.4.1. We begin by recalling a well-known proposition; see for example [C-C], [Nv].

Proposition 2.4.2 *Denote by* $\mathrm{Homeo}\,(S^1)$ *the group of homeomorphisms of the circle and consider a subgroup* $G \subset \mathrm{Homeo}\,(S^1)$*. Then one of the following holds:*

1. *The group G possesses a finite orbit in S^1.*
2. *The G-orbit of every point $p \in S^1$ is dense in S^1.*
3. *There is a Cantor set $K \subset S^1$ invariant by G and such that the G-orbit of every point $p \in K$ is dense in K. This set is unique and contained in the closure of the G-orbit of every point $p \in S^1$.* □

Consider now a subgroup G of $\mathrm{Diff}^{\omega}(S^1)$. When G possesses a finite orbit the statement of Proposition 2.4.2 can be strengthened as follows. Since G has a finite orbit, rotation numbers of the elements in G take values in some finite set. There follows that the subgroup G_0 of G consisting of those diffeomorphisms fixing every point in the mentioned finite orbit has finite index in G. In particular G_0 is not reduced to the identity unless G is a finite group. Assuming that G is not finite and choosing $g \in G_0$, $g \neq \mathrm{id}$, there follows that the set *of all points in S^1 possessing finite orbit under G* must be finite since it is contained in the set of *fixed points of g*. Hence, we have proved:

Lemma 2.4.3 *Assume that the group G is infinite but has a finite orbit $\mathcal{O}_p$. Denote by* $\mathrm{Per}\,(G) \subset S^1$ *the set of periodic points "the set consisting of those points $q \in S^1$ whose orbit under G is finite". Then* $\mathrm{Per}\,(G)$ *is a finite set. In particular, G possesses a finite index subgroup G_0 whose elements fix every single point in* $\mathrm{Per}\,(G)$. □

Dealing with subgroups of $\mathrm{Diff}^{\omega}(S^1)$ having finite orbits will naturally involve groups of analytic diffeomorphism of the interval $[0,1]$ (i.e. the group of diffeomorphisms from $[0,1]$ to $[0,1]$ fixing the endpoints 0 and 1). In this direction, the following statement is attributed to G. Hector (see [G1] for a proof).

Proposition 2.4.4 (**G. Hector**) *Let G_I denote a group consisting of orientation-preserving real analytic diffeomorphisms of* $[0,1]$*. Suppose that the only points in* $[0,1]$ *that are fixed for every element in G_I are precisely the endpoints* 0 *and* 1*. Suppose also that G is neither trivial nor an infinite cyclic group. Then the orbit of every point* $p \in (0,1)$ *is dense in* $(0,1)$. □

We are now able to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. The core of the proof consists of showing that a subgroup $G \subset \mathrm{Diff}^{\omega}(S^1)$ leaving invariant a Cantor set $K \subset S^1$ must be locally C^2-discrete. Equivalently a locally C^2-non-discrete group cannot leave a Cantor set invariant. We begin by proving this assertion. Let G be a locally C^2-non-discrete subgroup of $\mathrm{Diff}^{\omega}(S^1)$ and assume for a contradiction that G leaves invariant some Cantor set $K \subset S^1$.

Recall that by hypothesis the group G is locally C^2-non-discrete. In other words, assume the existence of an interval $I \subset S^1$ along with a sequence of elements in $\{g_i\} \subset G$ satisfying the following:

1. $g_i \neq \mathrm{id}$ for every $i \in \mathbb{N}$ (since G is constituted by analytic diffeomorphisms this condition implies that the restriction $g_{i|_I}$ of g_i to I does not coincide with the identity on I).

2. The sequence of restricted maps $g_{i|_I} : I \to S^1$ converges to the identity on the C^2-topology over I.

Assume by contradiction that there is a minimal Cantor set $K \subset S^1$ invariant by G. Proposition 2.4.2 ensures that K is the unique minimal set of G in S^1. Furthermore K and the whole of S^1 are the only non-empty closed subsets of S^1 that are invariant by G.

Now we have:

Claim. The intersection $I \cap K$ is not empty.

Proof of the Claim. Suppose that $I \cap K = \emptyset$ and denote by $\widetilde{I}$ the connected component of $S^1 \setminus K$ containing I. The endpoints of $\widetilde{I}$ belong to K and are automatically fixed by every element of the subgroup G_I of G defined by

$$G_I = \{g \in G \,;\; g(\widetilde{I}) \cap \widetilde{I} \neq \emptyset\,\}\,.$$

Thus, modulo dropping finitely many terms of the sequence $\{g_i\}$, we can assume that every g_i fixes a chosen endpoint p of $\widetilde{I}$. Consider a neighborhood U of p and the pseudogroup Γ_U induced on U by restrictions of elements in G_I. Since G_I fixes p, we can also consider the group Γ_p of germs at p of elements in Γ_U. In turn, since $p \in K$ and K is invariant by G, there follows that the C^m-closure of Γ_U contains neither (standard) Shcherbakov-Nakai vector fields (asymptotically defined on an one-sided interval starting at p) nor vector fields defined on neighborhood of p. Clearly Γ_p is not trivial since it contains the germs at p of the diffeomorphisms g_i. Lemma 2.3.3 then ensures that Γ_p must be infinite cyclic. Next, on a neighborhood of p all diffeomorphisms g_i are locally given as maps induced by a unique (possibly formal) local flow Ψ at specific times t_i. The additive subgroup of $\mathbb{R}$ generated by the times t_i must be discrete, otherwise the local flow Ψ would actually be defined for all $t \in \mathbb{R}$ and the associated analytic vector field would be in the closure of Γ_p which is known to be impossible. Being discrete, the subgroup of $(\mathbb{R}, +)$ generated by the times t_i has a generator $t_0 > 0$. Thus, the dynamics of the group G_I on $\widetilde{I}$ consists of the iterations of a single diffeomorphism having the endpoints of $\widetilde{I}$ fixed. In particular, the orbit of every point in $\widetilde{I}$ by the diffeomorphism in question converges to a fixed point of this diffeomorphism. This contradicts the assumption that the sequence $\{g_i\}$ converges to the identity on $I \subset \widetilde{I}$. This ends the proof of Claim. $\square$

To complete the proof of the proposition, we proceed as follows. According to a classical theorem due to Sacksteder [C-C], [Nv], there is a point $p \in K$ and a diffeomorphism $f \in G$ such that $f(p) = p$ and $0 < |f'(p)| < 1$. Since $I \cap K$ is not empty and the dynamics of G on K is minimal, there is no loss of generality in supposing that $p \in I \cap K$. Now, by considering the pseudogroup Γ generated on I by f and by the sequence of maps $g_{i|_I}$, Proposition 2.3.5 ensures the existence of a nowhere zero vector field X defined about p and contained in the C^1-closure of Γ. This yields a contradiction since K is a Cantor set supposed to be invariant by G and, hence, by Γ. The resulting contradiction then proves our claim that a locally C^2-non-discrete group $G \subset \mathrm{Diff}^{\omega}(S^1)$ cannot leave a Cantor set $K \subset S^1$ invariant.

Now there only remains to discuss further the case in which G has a finite orbit. The very assumption that G is locally C^2-non-discrete implies that G cannot be finite. Thus the set $\mathrm{Per}\,(G)$ of Lemma 2.4.3 is finite. Let I be a connected component of $S^1 \setminus \mathrm{Per}\,(G)$ and consider the subgroup G_I of G consisting of diffeomorphisms fixing I. To finish the proof of Proposition 2.4.1 is suffices to check that the action of G_I on I has all orbits dense. Owing to Proposition 2.4.4, if this does not happen then G_I must be infinite cyclic. Assuming that G_I is infinite cyclic, this group is also locally non-discrete. Lemma 2.3.3 then shows that the orbits of G_I on I are still dense. Proposition 2.4.1 is proved. □

Chapter 3

Topological Rigidity

In this chapter we shall apply the vector fields whose local flow can be approximated by elements in the locally non-discrete groups to show that one of the main results of this thesis ”Theorem A”.

Theorem A. *Consider two finitely generated non-abelian topologically conjugate subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$. Suppose that these groups are locally C^2-non-discrete. Then every homeomorphism $h : S^1 \to S^1$ satisfying $G_2 = h^{-1} \circ G_1 \circ h$ coincides with an element of $\mathrm{Diff}^{\omega}(S^1)$.*

Recall that two subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$ are said to be *topologically conjugate* if there is a homeomorphism $h : S^1 \to S^1$ such that $G_2 = h^{-1} \circ G_1 \circ h$, i.e. to every element $g_{(1)} \in G_1$ there corresponds a unique element $g_{(2)} \in G_2$ such that $g_{(2)} = h^{-1} \circ g_{(1)} \circ h$ and conversely.

Theorem A answers one of the questions raised in [R4].

3.1 Rigidity in the presence of points with large stabilizers and related cases

The purpose of this section is to prove Theorem A in some specific cases related, for example, to the existence of finite orbits for a non-solvable group (say G_1). We shall also settle the case in which G_1 is an actual solvable group. This material will reduce the proof of Theorem A to a *generic situation* where, roughly speaking, the group G_1 is not solvable and every point in S^1 has cyclic (possibly trivial) stabilizer; cf. Proposition 3.1.6. The generic situation described by Proposition 3.1.6 is, however, substantially harder and will be detailed in the subsequent sections of this thesis.

In the sequel, consider a locally C^2-non-discrete subgroup G_1 of $\mathrm{Diff}^{\omega}(S^1)$. Then fix an interval $I \subseteq S^1$ and a sequence $\{g_{1,i}\}$ of elements in G_1 whose restrictions $\{g_{1,i|I}\}$ to I converge to the identity in the C^2-topology (with ($g_{1,i} \neq \mathrm{id}$) for every i). Next, let G_2 be another subgroup of $\mathrm{Diff}^{\omega}(S^1)$ that happens to be *topologically conjugate* to G_1. The reader is reminded that the conjugating homeomorphism h is assumed to preserve the orientation of the circle.

Having fixed the sequence $\{g_{1,i|I}\}$, we consider subgroup $G_{1,r} \subset G_1$ generated by the elements $g_{1,1}, \ldots, g_{1,r}$ for every fixed value of $r \in \mathbb{N}$ (notation: $G_{1,r} = \langle g_{1,1}, ..., g_{1,r} \rangle$). In the subsequent discussion, we shall be allowed to "redefine" the sequence $\{g_{1,i|I}\}$ by dropping finitely many terms of it and then setting $g_{1,i} = g_{1,i+i_0}$ for every $i \in \mathbb{N}$ and for a certain $i_0 \in \mathbb{N}$.

Throughout this section the group G_1 is assumed to be non-abelian. Furthermore, we shall often assume that the following condition holds:

(FOG_r) *For every $r \in \mathbb{N}$, the group $G_{1,r}$ possesses a finite orbit while these groups are not finite themselves.*

Lemma 3.1.1 *Under Condition (FOG_r) and up to redefining the sequence $\{g_{1,i|I}\}$ by dropping finitely many terms of it, there is a finite set $P = \{p_1, ..., p_l\} \subset S^1$ whose points p_j, $i = 1, \ldots, l$, are fixed for all the groups $G_{1,r}$.*

Proof. Let $P_1 \subset S^1$ be the set of points having finite orbit for $G_{1,1}$. Owing to Lemma 2.4.3, the set P_1 consists of finitely many points. Naturally, for every $r \geq 1$, the set of points with finite orbit under the group $G_{1,r}$ is contained in P_1 since $G_{1,1} \subset G_{1,r}$. Denoting by $P_r \subset S^1$ the set of points having finite orbit under $G_{1,r}$, we have $P_1 \supset P_2 \supset \cdots$ so that the intersection

$$P = \bigcap_{r=1}^{\infty} P_r$$

is contained in P_1. Furthermore this intersection is not empty since our assumption ensures that none of the sets P_r is empty. Thus, to prove the lemma, it suffices to show that the diffeomorphisms $g_{1,i}$ fix all points in P provided that i is sufficiently large. For this let I_1 denote a connected component of $S^1 \setminus P$ having non-empty intersection with the open interval I. Since $\{g_{1,i}\}$ converges to the identity on I, for i large enough we must have $g_{1,i}(I_1) \cap I_1 \neq \emptyset$. Since, on the other hand, the set P is invariant under $g_{1,i}$, it follows at once that $g_{1,i}$ fixes every point in P. The lemma is proved. □

Let us assume that Condition (FOG_r) holds. Again up to dropping finitely many terms of the sequence $\{g_{1,i|I}\}$, Lemma 3.1.1 yields a finite set P all of whose points are fixed by each diffeomorphism $\{g_{1,i|I}\}$.

Next, let us also consider the group G_2 along with the homeomorphism h. We begin by letting $g_{2,i} = h^{-1} \circ g_{1,i} \circ h$ for every $i \in \mathbb{N}$. We also pose $G_{2,r} = \langle g_{2,1}, \ldots, g_{2,r} \rangle$. Next recall that $P = \{p_1, ..., p_l\}$ and let $q_j = h^{-1}(p_j)$, for $j = 1, \ldots, l$. It is cleat that the set $Q = \{q_1, \ldots, q_l\}$ is constituted by fixed points of $G_{2,r}$ for every $r \in \mathbb{N}$.

Now let $p_1 \in P$ and $q_1 = h^{-1}(p_1) \in Q$ be fixed. From what precedes, the stabilizer of p_1 (resp. q_1) contains all of $G_{1,r}$ (resp. $G_{2,r}$) for every $r \in \mathbb{N}$. Now we shall consider a few different possibilities involving the algebraic structure of the groups $G_{1,r}$.

Proposition 3.1.2 *Under the preceding conditions. If, for some $r \in \mathbb{N}$, the group $G_{1,r}$ is not solvable, then the conjugating homeomorphism h coincides with a real analytic diffeomorphism of S^1.*

Proof. Let Γ_1 (resp. Γ_2) denote the germ of $G_{1,r}$ (resp. $G_{2,r}$) about p_1 (resp. q_1). Naturally both groups Γ_1, Γ_2 can be identified with non-solvable subgroups of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ which are (locally) topologically conjugate by a homeomorphism induced by the restriction of h. A result due to Nakai [N2] ensures then that h is real analytic on a neighborhood of $0 \simeq p_1$. Since p_1 is an arbitrary point in P, we conclude that h is analytic on a neighborhood of every point in P. Finally, up to choosing n even larger if needed, we can assume that $G_{1,n}$ has dense orbits on the connected components of $S^1 \setminus P$, cf. Proposition 2.4.4. From this it promptly follows that the local analytic character of h about points in P extends to all of S^1. The proof of our proposition is over. □

In view of Proposition 3.1.2 whenever Condition (FOG_r) is satisfied, we can assume without loss of generality that all the groups $G_{1,r}$ are solvable. Note that these groups may as well be abelian since the assumption that the group G_1 is not abelian does not immediately imply the same holds for the groups $G_{1,r}$.

Next we shall drop Condition (FOG_r) and work instead with the assumption that all the groups $G_{1,r}$ are solvable, possibly after dropping finitely many terms from the sequence $\{g_{1,i}\}$. In the sequel the following well-known lemma on solvable subgroups of $\mathrm{Diff}^{\omega}(S^1)$ will be needed.

Lemma 3.1.3 *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$. Then either G has a finite orbit or it is topologically conjugate to a group of rotations.*

Proof. Since G is solvable, its action on S^1 preserves a probability measure μ. Hence the support $\mathrm{Supp}\,(\mu)$ of μ is a closed subset of S^1 invariant by G. Consider a minimal set $\mathcal{M}$ for G contained in $\mathrm{Supp}\,(\mu)$. In view of Proposition 2.4.2, $\mathcal{M}$ must be of one of the following types: the entire circle, a finite set or a Cantor set. Suppose first that $\mathcal{M}$ coincides with all of S^1. Then by parameterizing the circle by the integral of the Radon-Nikodym derivative, a topological conjugation between G and a rotation group of S^1 can be constructed (in particular G is abelian). In turn, if the support of μ is a finite set, then this set is invariant by G so that this group has finite orbits. Hence the proof of our lemma is reduced to checking that $\mathcal{M}$ cannot be a Cantor set. This last assertion follows from Sacksteder's theorem; see [C-C], [Nv]. In other words, if $\mathcal{M}$ is a Cantor set, then there is an element $g \in G$ and a point $p \in \mathcal{M}$ such that p is a hyperbolic fixed point for g. Now, since g preserves μ and $p \in \mathrm{Supp}\,(\mu) = \mathcal{M}$, there follows that the point p must have strictly positive μ-mass. However the measure μ is invariant by G and finite which, in turn, forces the orbit of p to be finite itself thus completing the proof of the lemma. □

Remark 3.1.4 *The reader will note that the Lemma 3.1.3 also holds for solvable subgroups of $\mathrm{Diff}^{1+\mathrm{lip}}(S^1)$.*

Proposition 3.1.5 *Up to dropping finitely many terms of the sequence $\{g_{1,i}\}$, suppose that $G_{1,r}$ is an infinite solvable group for every $r \in \mathbb{N}$. Suppose, in addition, the existence of $r_0 \in \mathbb{N}$ for which G_{1,r_0} has a finite orbit. Then the homeomorphism h conjugating G_1 to G_2 coincides with an analytic diffeomorphism of S^1.*

Proof. Note that G_{1,r_0} cannot be conjugate to a group of rotations. Thus for every $r \geq r_0$, Lemma 3.1.3 ensures that $G_{1,r}$ is a solvable group having a finite orbit. By intersecting these finite orbits over $r \geq r_0$, we derive the existence of a non-empty finite set all of whose points have finite orbit under $G_{1,r}$, for every $r \in \mathbb{N}$.

Let $p \in S^1$ be a point having finite orbit under $G_{1,r}$ for every $r \in \mathbb{N}$. Denote by $G_{1,r}^{(p)}$ the stabilizer of p in $G_{1,r}$. Clearly $G_{1,r}^{(p)}$ induces a solvable subgroup $\Gamma_{p,r}$ of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$. Besides, with suitable identifications, the restriction of h to a neighborhood of p topologically conjugates $\Gamma_{p,r}$ to another subgroup $\Gamma_{q,r}$ of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$. Again the proof of the proposition becomes reduced to checking that the homeomorphism (still denoted by h) conjugating $\Gamma_{p,r} \subset \mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ to $\Gamma_{q,r} \subset \mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ must be analytic on a neighborhood of $0 \in \mathbb{R}$. For this, let us consider the following possibilities:

Case 1. Suppose that $\Gamma_{p,r}$ (an thus $\Gamma_{q,r}$) is not abelian. From the description of solvable subgroups of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$, there follows that solvable non-abelian subgroups of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ have elements f_1, g_1 satisfying the following conditions (see for example [EISV]):

- f_1 has a hyperbolic fixed point at $0 \in \mathbb{R}$.
- g_1 is tangent to the identity at $0 \in \mathbb{R}$ (though $g_1 \neq \mathrm{id}$).

As shown in Lemma 2.3.2, a suitable sequence of elements having the general form $f_1^{-N} \circ g_1 \circ f_1^N$ leads to analytic vector field in the C^{∞}-closure of $\Gamma_{p,r}$ (see [N2]), then the local diffeomorphisms f_1, g_1 can be combined to construct a (non-identically zero) analytic vector field X_1 defined *on a neighborhood of* $0 \in \mathbb{R}$ and contained in the closure of $\Gamma_{p,r}$. A similar vector field X_2 can be defined by means of the elements $f_2 = h^{-1} \circ f_1 \circ h$ and $g_2 = h^{-1} \circ g_1 \circ h$ of $\Gamma_{q,r}$. Indeed the above mentioned structure of solvable subgroups of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$ also ensures that $f_2 = h^{-1} \circ f_1 \circ h$ is hyperbolic whereas $g_2 = h^{-1} \circ g_1 \circ h$ is tangent to the identity. By using the fact that h conjugates the actions of $\Gamma_{p,r}$, $\Gamma_{q,r}$, there follows from the indicated constructions that h conjugates X_1 to X_2 in a time-preserving manner. Thus h must be analytic about $0 \in \mathbb{R}$ and this establishes the proposition in the first case.

Case 2. Suppose now that $\Gamma_{p,r}$ (and thus $\Gamma_{q,r}$) is an infinite abelian subgroup for every $r \in \mathbb{N}$. Then all the groups $G_{1,r}^{(p)}$ are abelian as well ($r \in \mathbb{N}$). Now define the abelian group $G_{1,\infty}$ by letting

$$G_{1,\infty} = \bigcup_{r=1}^{\infty} G_{1,r}^{(p)} \, .$$

Next fix an element $f \in G_{1,\infty}$ different from the identity. Clearly $f(p) = p$ so that the set of fixed points of f is non-empty and finite since f is analytic. Because $G_{1,\infty}$ is an abelian group, the set of fixed points of f is preserved by $G_{1,\infty}$. Therefore every diffeomorphism in the sequence $\{g_{1,i}\}$ must fix every point that is fixed by f provided that i is very large. Therefore the stabilizer $G_{1,\infty}^{(p)}$ of p in $G_{1,\infty}$ is an abelian group containing all the diffeomorphisms $g_{1,i}$ for large enough i. There follows that $G_{1,\infty}^{(p)}$ is non-discrete since $\{g_{1,i}\}$ converges to the identity on I. Since $G_{1,\infty}^{(p)}$ naturally induces a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{R}, 0)$, we can resort to Lemma 2.3.2 in Section 2.3 to produce analytic vector fields X_1, X_2 defined around p and q respectively that are conjugated by h in a time-preserving manner. Hence h is again real analytic around $0 \in \mathbb{R}$. The proposition is proved. □

To finish this section we shall establish a last reduction to the proof of Theorem A in the form of Proposition 3.1.6. This proposition will also summarizes the preceding lemmata. To state it, recall that $I \subset S^1$ is a fixed interval for which G_1 contains a sequence of elements $\{g_{1,i}\}$, ($g_{1,i} \neq \mathrm{id}$), whose restrictions $\{g_{1,i|I}\}$ to I converge to the identity in the C^2-topology.

Proposition 3.1.6 *To prove Theorem A, we can assume without loss of generality that all of the following hold:*

- *There is $N \in \mathbb{N}$ for which the group generated by $\{g_{1,1}, \ldots, g_{1,N}\}$ is not solvable.*
- *For every given $\varepsilon > 0$ (and up to dropping finitely many terms from the sequence $\{g_{1,i}\}$), all the diffeomorphisms $g_{1,1}, \ldots, g_{1,N}$ are ε-close to the identity in the C^2-topology on the interval I.*
- *No point $p \in S^1$ is simultaneously fixed by all the diffeomorphisms $g_{1,1}, \ldots, g_{1,N}$.*
- *In general, every finite subset generating a non-solvable subgroup of G_1 cannot have a common fixed point.*

To prove Proposition 3.1.6, let us assume once and for all that $\varepsilon > 0$ is given. We also assume without loss of generality that the sequence of diffeomorphisms $\{g_{1,i}\}$ is constituted by diffeomorphisms ε-close to the identity in the C^2-topology on the interval I.

Assume there is $r_0 \in \mathbb{N}$ such that the group G_{1,r_0} is not solvable. Owing to Proposition 3.1.2, Theorem A holds provided that the non-solvable group G_{1,n_0} possesses a finite orbit. More generally, Proposition 3.1.2 also justifies the last assertion in the statement of Proposition 3.1.6. In other words, to establish Proposition 3.1.6 it suffices to show that Theorem A holds provided that all the groups $G_{1,r}$ are solvable ($r \in \mathbb{N}$). This will be our aim in the remainder of this section.

To begin with, recall the general fact that every finite subgroup of $\mathrm{Diff}^{\omega}(S^1)$ is analytically conjugate to a rotation group. Also Lemma 3.1.3 informs us that every *infinite solvable group* having no finite orbit is topologically conjugate to a subgroup of the rotation group. By virtue of Proposition 3.1.5, we can therefore assume that each individual group $G_{1,r}$ is abelian and topologically conjugate to a group of rotations.

Consider again the group $G_{1,\infty} = \bigcup_{r=1}^{\infty} G_{1,r}$. In the present setting, $G_{1,\infty}$ is clearly an infinite locally non-discrete abelian group all of whose orbits are infinite. Although it is infinitely generated, the action of $G_{1,\infty}$ still preserves a probability measure μ_{∞}. To check this claim consider a probability measure μ_r invariant by $G_{1,r}$. Next take μ_{∞} as an accumulation point of the sequence $\{\mu_r\}$. The fact that $G_{1,r} \subset G_{1,r+1}$ promptly implies that μ_{∞} must be invariant by $G_{1,r}$ for every $r \in \mathbb{N}$. Since $G_{1,\infty}$ has no finite orbit, it follows that μ_∞ has no atomic component so that its support must coincide with all of S^1 (recall that the support cannot be a Cantor set thanks to Sacksteder theorem [C-C], [Nv]). Hence, the Radon-Nikodym derivative of this measure allows us to construct a topological coordinate H on S^1 in which $G_{1,\infty}$ is a group of rotations. Next we have:

Lemma 3.1.7 *In the topological coordinate H, the group $G_{1,\infty}$ is a dense subgroup of the group of all rotations of S^1.*

Proof. Consider the map $\rho : G_{1,\infty} \to \mathbb{R}/\mathbb{Z}$ assigning to an element $g \in G_{1,\infty}$ its rotation number. Because $G_{1,\infty}$ is an abelian group, the map ρ is a homomorphism so that its image $\rho(G_{1,\infty}) \subset S^1$ is a dense set of S^1 viewed as a multiplicative group.

Moreover, the homomorphism ρ is injective since, in the coordinate H, the rotation corresponding to an element $g \in G_{1,\infty}$ is nothing but the rotation of angle equal to the rotation number of G. The lemma then follows from the fact that the subgroup $\rho(G_{1,\infty})$ is clearly infinite. □

The next lemma is also elementary.

Lemma 3.1.8 *Suppose that $g : S^1 \to S^1$ is a homeomorphism of the circle that commutes with a dense set E of rotations. Then g is itself a rotation.*

Proof. Consider the circle equipped with the standard euclidean metric d induced from $\mathbb{R}$ by means of the identification $S^1 = \mathbb{R}/\mathbb{Z}$. To show that g is a rotation amounts to check that g is an isometry of d. Hence, chosen an interval J with endpoints x, y, we need to show that the length of $g(J)$ equals to the length of J. If this were not true, then there would exist $J \subset S^1$ such that the length $L(J)$ of J would be strictly smaller than the length $L(g(J))$ of $g(J)$. Now, since E is a dense set of rotations, we can find an element $\sigma \in E$ such that $\sigma(g(J)) \subset J$. Thus the map $\sigma \circ g$ maps J strictly inside itself and must therefore have a fixed point $p \in J \subset S^1$. Furthermore $\sigma \circ g$ commutes with all rotations in E so that the orbit of p by elements in E must consist of fixed points for $\sigma \circ g$. However, since the orbit of p by all rotations in E is clearly dense in S^1, there follows that $\sigma \circ g$ coincides with the identity. The resulting contradiction proves the lemma. □

Let us close this section with the proof of Proposition 3.1.6

Proof of Proposition 3.1.6. The proof amounts to showing that the initial sequence of diffeomorphisms $\{g_{1,i}\} \subset G_1$ can be chosen so as to ensure that for large enough $r \in \mathbb{N}$ the group $G_{1,r}$ cannot be topologically conjugate to a group of rotations. For this, consider a finite generating set $f_{1,1}, \ldots, f_{1,s}$ for G_1. Given the initial sequence $\{g_{1,i}\} \subset G_1$, we consider all diffeomorphisms of the form $g_{1,j,i} = f_{1,j}^{-1} \circ g_{1,i} \circ f_{1,j}$ where $j \in \{1, \ldots, s\}$ and $i \in \mathbb{N}$. Next, the indices j, i can be reorganized to ensure that all the diffeomorphisms $g_{1,j,i}$ are actually contained in the initial sequence $\{g_{1,i}\}$. With this new definition of the sequence $\{g_{1,i}\}$, the following holds:

Claim. The group $G_{1,\infty}$ is no longer topologically conjugate to a group of rotations.
Proof. By construction the group $G_{1,\infty}$ consists of elements having the form $f_{1,j}^{-1} \circ \tilde{g}_k \circ f_{1,j}$, where $\tilde{g}_k \in \mathrm{Diff}^\omega(S^1)$ is a certain sequence of diffeomorphisms converging to the identity on I (in the C^2-topology). Suppose for a contradiction that $G_{1,\infty}$ is abelian without finite orbits. Now fixed k, the elements g_k and $f_{1,j}^{-1} \circ g_k \circ f_{1,j}$, $j = 1, \ldots, s$ have all the same rotation number. What precedes then ensures that all these elements are the same. Indeed, it was seen that the "rotation number homomorphism" from $G_{1,\infty}$ to S^1 is one-to-one. In other words, for every $k \in \mathbb{N}$ and every $j = 1, \ldots, s$ the diffeomorphisms g_k and $f_{1,j}$ do commute.

Now recall the existence of a topological coordinate H where $G_{1,\infty}$ is identified to a group of rotations that happens to be dense in the group of all rotations of S^1. Let Γ be the subgroup of $G_{1,\infty}$ generated by all the elements g_k, $k \in \mathbb{N}$ and note that Γ is dense the group of all rotations of S^1 as well. Finally, always working in the coordinate H, the generators $f_{1,1}, \ldots, f_{1,s}$ of G_1 commute with all elements in Γ. Lemma 3.1.8 then ensures that every $f_{1,j}$ is itself another rotation in the coordinate H. Hence the group G_1 must be abelian and this yields the desired contradiction. $\square$

Now the proposition results from the repeating word-by-word the preceding discussion. $\square$

3.2 Convergence estimates for sequences of commutators

This section is devoted to providing an algorithmic way to construct diffeomorphisms converging to the identity on a suitably fixed interval. This algorithmic construction will allow for a more effective use of the assumption that our groups are locally non-discrete and it is convenient to add some explanation in this direction. Consider a locally C^2-non-discrete group $G \subset \mathrm{Diff}^\omega(S^1)$. This means there is an interval $I \subset S^1$ and a sequence of elements $\{g_i\} \subset G$ satisfying the conditions of Definition 2.1.1. Definition 2.1.1 however has the inconvenient that the sequence $\{g_i\}$ is *a priori given* and this prevents us from having any additional control on the behavior of the diffeomorphisms g_i. For example, we have no information whatsoever on the higher order derivatives of g_i and, in particular, no information on the growing rate of the sequence $\|g_i\|_3$, where $\| \, . \, \|_3$ stands for the C^3-norm. In the context of Theorem A, if $\{g_{1,i}\}$ is a sequence as above for the group G_1, then the corresponding sequence $g_{2,i} = h^{-1} \circ g_{1,i} \circ h$ of elements in G_2 is known to converge to the identity only in the C^0-topology. Nonetheless to derive non-trivial implications on the regularity of h, it is natural to look for sequences as above such that $\{g_{2,i}\}$ forms a convergent sequences in stronger topologies as well. The main immediate virtue of the construction carried out below is to yield some estimates on the growing rate of the sequence formed by higher order derivatives of g_i. These estimates will prove to be crucial for the proof of Theorem A. Finally we also point out that the mentioned construction will enable us to give a partial answer to some questions raised in [De], cf.Appendix.

To make our discussion accurate, we place ourselves in the context of a locally C^2-

non-discrete group $G \subset \mathrm{Diff}^{\omega}(S^1)$ satisfying the conditions in Proposition 3.1.6. Hence we fix some interval $I \subset S^1$ and a collection $S \subset G$ of elements $\overline{g}_1, \ldots, \overline{g}_N$ generating a non-solvable subgroup. The diffeomorphisms $\overline{g}_i$, $i = 1, \ldots, N$ are also assumed to be ε-close to the identity in the C^2-topology on the interval I, where the value of $\varepsilon > 0$ will be fixed only later. Our purpose is to produce an explicit sequence of diffeomorphisms converging to the identity out of the finite set $S = \{\overline{g}_1, \ldots, \overline{g}_N\}$. In turn, the idea to obtain the desired sequence consists of iterating commutators. This will be a slight refinement of the method employed by Ghys in [G1] which relies on a fast iteration technique. Indeed, the difficulty in proving convergence to the identity of sequences of iterated commutators lies in the fact that an estimate of the C^m-norm of a commutator $[f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1}$ requires estimates on the C^{m+1}-norm of f_1, f_2. To establish the convergence of a sequence of "iterated commutators" becomes therefore tricky as at each step there is an intrinsic loss of one derivative. It is thus natural to try to overcome this difficulty by means of some suitable fast iteration scheme. The method of Ghys [G1] consists then of using holomorphic extensions and the topology of uniform convergence for these extensions in order to take advantage of Cauchy formula. Owing to Cauchy formula, we can substitute the loss of one derivative by the loss of a portion of the domain of definition: hence we only need to check that the size of the region lost in the domain of definition at each step of the iteration scheme decreases fast enough to ensure that some non-empty domain is kept at the end.

Since we will work only with C^2-convergence the same fast iteration scheme is not available, albeit some adaptations are still possible. We prefer however to introduce a slightly more elaborated iterative procedure which avoids fast convergence estimates. The idea is to add a step of *renormalization* at each stage of the commutator iteration. This renormalization step has a regularizing effect on derivatives of order two or greater. A simplified version of the same idea was already used in the proof of Proposition 2.3.5. One advantage of our procedure is to avoid the loss of derivatives; other advantages will become clear in the course of the discussion and these include the convergence rate to the identity of the resulting sequence; see Remark 3.2.7.

After this brief overview of the upcoming discussion, we begin to provide accurate definitions. We shall work with the pseudogroup generated by $S = \{\overline{g}_1, \ldots, \overline{g}_N\}$ on the interval $I \subset \mathbb{R}$ where $\overline{g}_1, \ldots, \overline{g}_N$ generate a non-solvable group. Also, and whereas we shall primarily think of $\overline{g}_1, \ldots, \overline{g}_N$ as maps defined on I, it is sometimes useful to keep in mind that the maps in questions are nothing but restrictions to I of global analytic diffeomorphisms of S^1 (still denoted by $\overline{g}_1, \ldots, \overline{g}_N$, respectively).

According to Ghys [G1], with the set $S = \{\overline{g}_1, \ldots, \overline{g}_N\}$ is associated a *sequence of sets* $S(k)$, $k = 1, 2, \ldots$, inductively defined as follows:

- $S(0) = S$
- $S(k)$ is the set whose elements are commutators of the form $[\tilde{f}_i^{\pm 1}, \tilde{f}_j^{\pm 1}]$ where $\tilde{f}_i \in S(k-1)$ and $\tilde{f}_j \in S(k-1) \cup S(k-2)$ ($\tilde{f}_j \in S(0)$ if $k = 1$).

There follows from [G1] that the resulting sequence of sets $S(k)$ is never reduced to the identity since $S = \{\overline{g}_1, \ldots, \overline{g}_N\}$ generates a non-solvable group. This also yields the following:

Lemma 3.2.1 *For every $k \in \mathbb{N}$, the subgroup generated by $S(k) \cup S(k-1)$ is non-solvable.*

Proof. Assume there were $k \in \mathbb{N}$ such that $\Gamma = \langle S(k) \cup S(k-1) \rangle$ is solvable, where $\langle S(k) \cup S(k-1) \rangle$ stands for the group generated by $S(k) \cup S(k-1)$. Since $\Gamma \subset \mathrm{Diff}^{\omega}(S^1)$, there follows that Γ is, indeed, metabelian, i.e. its derived group $D^1\Gamma$ is abelian. Recalling that $D^1\Gamma$ is the group generated by all commutators of the form $[\gamma_1, \gamma_2]$ where $\gamma_1, \gamma_2 \in \Gamma$, there follows that the sets $S(k+1)$ and $S(k+2)$ are contained in $D^1\Gamma$. Since $D^1\Gamma$ is abelian, the definition of the sequence of sets $\{S(k)\}$ promptly implies that the set $S(k+3)$ must coincide with $\{\mathrm{id}\}$. Hence the initial group generated by $\overline{g}_1, \ldots, \overline{g}_N$ must be solvable. The resulting contradiction proves the lemma. $\square$

By virtue of Proposition 3.1.6, we obtain the following corollary:

Corollary 3.2.2 *In order to prove Theorem A, we can assume that the elements in $S(k) \cup S(k+1)$ do not share a common fixed point (and this for every $k \in \mathbb{N}$).* $\square$

From now on, we set $I = [-a, a] \subset \mathbb{R}$, $a > 0$, with the obvious identifications. Given $\varepsilon > 0$, we permanently fix a set of diffeomorphisms $\overline{g}_1, \ldots, \overline{g}_N$ generating a non-solvable group and ε-close to the identity in the C^2-topology on I. The value of $\varepsilon > 0$ convenient for our purposes will only be fixed later. In the remainder of the section these conditions are assumed to hold without further comments.

Unless otherwise mentioned, in what follows we shall say that $f : I' \subseteq I \subset \mathbb{R} \to \mathbb{R}$ is a diffeomorphism meaning that f is a diffeomorphism from $I' \subset \mathbb{R}$ to $f(I') \subset \mathbb{R}$. Let us begin our discussion by stating a simple general lemma.

Lemma 3.2.3 *Given $\epsilon_0 > 0$ small and $m \geq 1$, there is a neighborhood $\mathcal{U}_0^m$ of the identity in the C^m-topology on I such that the commutator $[f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1}$ between diffeomorphisms $f_1, f_2 \in \mathcal{U}_0^m$ satisfies the two conditions below:*

- *Viewed as an element of the pseudogroup generated by f_1, f_2 on I, the map $[f_1, f_2]$ is well defined on $[-a + 5\epsilon_0, a - 5\epsilon_0]$.*
- *There is a constant $C > 0$ such that the C^{m-1}-distance $\|[f_1, f_2] - \mathrm{id}\|_{m-1,[-a+5\epsilon_0,a-5\epsilon_0]}$ from $[f_1, f_2]$ to the identity on the interval $[-a + 5\epsilon_0, a - 5\epsilon_0]$ satisfies the estimate*

  $$\|[f_1, f_2] - \mathrm{id}\|_{m-1,[-a+5\epsilon_0,a-5\epsilon_0]} < C \, \|f_1 - \mathrm{id}\|_{m,[-a,a]} \, \|f_2 - \mathrm{id}\|_{m,[-a,a]}$$

  where $\|f_1 - \mathrm{id}\|_{m,[-a,a]}$ (resp. $\|f_2 - \mathrm{id}\|_{m,[-a,a]}$) stands for the C^m-distance from f_1 (resp. f_2) to the identity on the interval $I = [-a, a]$.

The reader will note that the constant C in the above lemma depends only on the neighborhood $\mathcal{U}_0^m$. In particular C does not increase *when the neighborhood is reduced.*

We now focus on the case $m = 2$ (see Appendix for a more general discussion). Since we can always reduce $\varepsilon > 0$, the neighborhood $\mathcal{U}_0^2$ can be chosen as

$$\mathcal{U}_0^2 = \{f \in C^2([-a,a]) \; ; \; \|f - \mathrm{id}\|_{2,[-a,a]} < \varepsilon\} \tag{3.1}$$

where $C^2([-a,a])$ stands for the space of C^2-functions defined on $[-a,a]$ and taking values in $\mathbb{R}$. For this neighborhood $\mathcal{U}_0^2$, the constant provided by Lemma 3.2.3 will be denoted by C and the value of C does not increase when ε decreases.

Proof of Lemma 3.2.3. We will prove the Lemma on the case $m = 2$. Clearly $[f_1, f_2] - \mathrm{id} = (f_1 \circ f_2 - f_2 \circ f_1) \circ (f_2 \circ f_1)^{-1}$ so that we have

$$\|[f_1, f_2] - \mathrm{id}\|_1 \leq \|f_1 \circ f_2 - f_2 \circ f_1\|_1 . [1 + \|f_1^{-1}\|_1 (1 + \|f_2^{-1}\|_1)] \, . \tag{3.2}$$

Set $\Delta f_1 = f_1 - \mathrm{id}$, $\Delta f_2 = f_2 - \mathrm{id}$ and note also that $f_1 \circ f_2$ is well defined on $[-a+\epsilon_0, a-\epsilon_0]$. We will note that the constant $C = [1 + \|f_1^{-1}\|_1 (1 + \|f_2^{-1}\|_1)]$ depends only on the neighborhood $\mathcal{U}_0^2$. Furthermore $f_1 \circ f_2$ satisfies

$$f_1 \circ f_2 = \mathrm{id} + \Delta f_1 + \Delta f_2 + (\Delta f_1 \circ f_2 - \Delta f_1) \, .$$

Therefore we conclude that

$$\begin{aligned} \|f_1 \circ f_2 - f_2 \circ f_1\|_1 &= \|(\Delta f_1 \circ f_2 - \Delta f_1) - (\Delta f_2 \circ f_1 - \Delta f_2)\|_1 \\ &\leq \|(\Delta f_1 \circ f_2 - \Delta f_1)\|_1 + \|(\Delta f_2 \circ f_1 - \Delta f_2)\|_1 \, . \end{aligned} \tag{3.3}$$

Now, for $i, j = 1, 2$, the Mean Value Theorem yields

$$\|(\Delta f_i \circ f_j - \Delta f_i)\|_1 \leq \sup_{x \in [-a,a]} \|(D^1 \Delta f_i)\|_1 . \|\Delta f_j\|_1 \, .$$

In turn, $\sup_{x \in [-a,a]} \|(D^1 \Delta f_1)\|_1 \leq \|\Delta f_1\|_2$ (resp. $\sup_{x \in [-a,a]} \|(D^1 \Delta f_2)\|_1 \leq \|\Delta f_2\|_2$). There follows that

$$\|(\Delta f_i \circ f_j - \Delta f_i)\|_1 \leq \|f_i - \mathrm{id}\|_1 . \|f_j - \mathrm{id}\|_2$$

for $i, j = 1, 2$. Combining these estimates to Estimates (3.2) and (3.3), we conclude that

$$\|[f_1, f_2] - \mathrm{id}\|_1 \leq \; 2\, C \, \|f_1 - \mathrm{id}\|_2 . \|f_2 - \mathrm{id}\|_2$$

where $\|[f_1, f_2] - \mathrm{id}\|_1$ is taken over the interval $[-a + 5\epsilon_0, a - 5\epsilon_0]$ while $\|f_i - \mathrm{id}\|_2$ is taken over the interval $[-a, a]$, for $i = 1, 2$. The lemma is proved. □

Now we state a simple complement to Lemma 3.2.3:

Lemma 3.2.4 *Up to reducing $\varepsilon > 0$, for every pair $f_1, f_2 \in \mathcal{U}_0^2$ the second derivative $D^2[f_1, f_2]$ of the commutator $[f_1, f_2]$ on the interval $[-a + 5\epsilon_0, a - 5\epsilon_0]$ satisfies the estimate*

$$\sup_{x \in [-a+5\epsilon_0, a-5\epsilon_0]} \left| D^2[f_1, f_2] \right| \leq \; 5 \max_{x \in (-a,a)} \{ \left| D^2 f_1 \right| , \left| D^2 f_2 \right| \}$$

where $D^2 f_j$ stands for the second derivative of f_j, $j = 1, 2$.

Proof. The proof is elementary and we shall summarize the argument. For $j = 1, 2$, the very definition of $\mathcal{U}_0^2$ yields (see (3.1))

$$1 - \varepsilon \leq |D_x^1 f_j| \leq 1 + \varepsilon \quad \text{and} \quad \frac{1}{1+\varepsilon} \leq \frac{1}{|D_x^1 f_j|} \leq \frac{1}{1-\varepsilon}$$

for every $x \in [-a, a]$. Concerning the inverses of f_1, f_2, we also have

$$D_x^1 f_j^{-1} = \frac{1}{D^1_{f_j^{-1}(x)} f_j} \quad \text{and} \quad D_x^2 f_j^{-1} = -\frac{D^2_{f_j^{-1}(x)} f_j}{[D^1_{f_j^{-1}(x)} f_j]^3} \, .$$

Next we compute the second derivative of $[f_1, f_2]$ at a point belonging to $[-a+5\epsilon_0, a-5\epsilon_0]$. In this calculation, the points at which the several derivatives are evaluated will be omitted: since $[f_1, f_2]$ is well defined on $[-a + 5\epsilon_0, a - 5\epsilon_0]$, it suffices to know that all these points belong to the interval $(-a, a)$. Since $D^1[f_1, f_2] = D^1 f_1 . D^1 f_2 . D^1 f_1^{-1} . D^1 f_2^{-1}$, there follows

$$\begin{aligned} D^2[f_1, f_2] = D^2 f_1 .(D^1 f_2)^2 .(D^1 f_1^{-1})^2 .(D^1 f_2^{-1})^2 + D^1 f_1 . D^2 f_2 .(D^1 f_1^{-1})^2 .(D^1 f_2^{-1})^2 + \\ D^1 f_1 . D^1 f_2 . D^2 f_1^{-1} .(D^1 f_2^{-1})^2 + D^1 f_1 . D^1 f_2 . D^1 f_1^{-1} . D^2 f_2^{-1} \, . \end{aligned}$$

Therefore on $[-a + 5\epsilon_0, a - 5\epsilon_0]$, we have

$$\begin{aligned} |D^2[f_1, f_2]| &\leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^4}|D^2 f_1| + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^4}|D^2 f_2| + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^5}|D^2 f_1| + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^4}|D^2 f_2| \\ &\leq \left[\frac{(1+\varepsilon)^2}{(1-\varepsilon)^4} + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^4} + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^5} + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^4}\right] \max\{D^2 f_1, D^2 f_2\} \, . \end{aligned}$$

Up to choosing ε sufficiently small, there follows that $|D^2[f_1, f_2]| \leq 5 \max\{|D^2 f_1|, |D^2 f_2|\}$ proving the lemma. □

Let us now begin the construction of a sequence of diffeomorphisms in G converging to the identity in the C^2-topology on $I = [-a, a]$. First recall that non-solvable subgroups of $\mathrm{Diff}^\omega(S^1)$ are known to have elements with hyperbolic fixed points (see for example [E-T]). Let then $F \in G$ be a diffeomorphism satisfying $F(0) = 0$ and $F'(0) = \lambda \in (0, 1)$. The next step is to define a new sequence $\{\widetilde{S}(k)\}$ of subsets of G. The sequence $\{\widetilde{S}(k)\}$ will depend on a fixed integer $n \in \mathbb{N}^*$ which will be omitted in the notation. To define the sequence $\{\widetilde{S}(k)\}$ we proceed as follows:

- $\widetilde{S}(1)$ is the set formed by the commutators having the form $[F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-n} \circ \tilde{f}_2 \circ F^n]$ where $\tilde{f}_1$, $\tilde{f}_2 \in S$. Thus $\widetilde{S}(1) = F^{-n} \circ S(1) \circ F^n$.
- $\widetilde{S}(k)$ is the set formed by the commutators $[F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-n} \circ \tilde{f}_2 \circ F^n]$ with $\tilde{f}_1, \tilde{f}_2 \in \widetilde{S}(k-1)$ and by the commutators $[F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-2n} \circ \tilde{f}_2 \circ F^{2n}]$ with $\tilde{f}_1 \in \widetilde{S}(k-1)$ and $\tilde{f}_2 \in \widetilde{S}(k-2)$.

In other words, the sequence $\{\widetilde{S}(k)\}$ verifies $\widetilde{S}(k) = F^{-kn} \circ S(k) \circ F^{kn}$ for every $k \in \mathbb{N}$. Taking advantage of the fact that all our local diffeomorphisms are realized as global diffeomorphisms of the circle, we obtain the following:

Lemma 3.2.5 *The sequence of sets $\widetilde{S}(k)$ never degenerates into* $\{\mathrm{id}\}$.

Proof. When all the diffeomorphisms in question are globally viewed as diffeomorphisms of the circle, the set $\widetilde{S}(k)$ is conjugate to the set $S(k)$, for every $k \in \mathbb{N}$. The statement follows then from Ghys theorem claiming that the initial sequence $S(k)$ cannot degenerate into $\{\mathrm{id}\}$ provided that G is non-solvable. □

The global realizations of our diffeomorphisms ensure that the domain of definition of elements in $\widetilde{S}(k)$ are always non-empty as every diffeomorphism is clearly defined on all of S^1. However, going back to our local setting where the initial C^2-maps $\overline{g}_1, \ldots, \overline{g}_N$ are defined on $[-a, a]$ and where the domains of definition for their iterates are understood in the sense of pseudogroup, the content of the last statement becomes unclear. In other words, in the context of pseudogroups, the statement of Lemma 3.2.5 is only meaningful for those elements in $\widetilde{S}(k)$ having non-empty domain of definition *when viewed as elements of the pseudogroup in question.* In any event, the estimates developed below will show that this is always the case provided that we start with a sufficiently small $\varepsilon > 0$.

The central result of this section is Proposition 3.2.6 below. To state it accurately and to make its content promptly available to the reader, it is convenient to explicitly summarize the conditions on which this proposition is based. In particular, since we will be dealing with a statement about convergence of sequences to the identity, there is no need to guarantee that the corresponding diffeomorphisms are different from the identity. This remark enables us to formulate the desired proposition in the context of pseudogroups. Consider a fixed interval $I = [-a, a] \subset \mathbb{R}$ $(a > 0)$ along with C^2-diffeomorphisms F and $\overline{g}_1, \ldots, \overline{g}_N$ from a neighborhood of $I = [-a, a]$ to $\mathbb{R}$. Furthermore assume that the following holds:

a - We have $F(0) = 0$ and $F'(0) = \lambda \in (0, 1)$. Moreover $\overline{g}_1, \ldots, \overline{g}_N$ are ε-close to the identity in the C^2-topology on $I = [-a, a]$ (for some $\varepsilon > 0$ to be fixed later).

b - The sequence of sets $S(j)$ is defined as before starting from $S(0) = \{\overline{g}_1, \ldots, \overline{g}_N\}$.

c - There are sequences $\{f_j\}$ and $\{g_j\}$ of elements in the pseudogroup generated on I by F and by $\overline{g}_1, \ldots, \overline{g}_N$ such that $f_j \in S(j)$ for every j. Moreover for every j we have $g_j = F^{-jn} \circ f_j \circ F^{jn}$ for some fixed $n \in \mathbb{N}$.

Proposition 3.2.6 *Under the above assumptions there are $\varepsilon > 0$ and $n \in \mathbb{N}$ such that for every sequence $\{f_j\}$ as above, the corresponding sequence $\{g_j\}$, $g_j = F^{-jn} \circ f_j \circ F^{jn}$, of elements in the pseudogroup generated on I by F and by $\overline{g}_1, \ldots, \overline{g}_N$ satisfies the following:*

- *There is $b > 0$ such that the interval $[-b, b]$ is contained in the domain of definition of every diffeomorphism g_j.*

- *The sequence of diffeomorphisms $\{g_j\}$ converges to the identity in the C^2-topology on the interval $[-b, b]$.*

Recall that $\lambda = F'(0)$. Up to changing coordinates we can thus assume that $F(x) = \lambda x$ for every $x \in [-a, a]$, [St]. In these coordinates, g_j becomes $g_j = \lambda^{-jn} f_j(\lambda^{jn} x)$. Fix $\epsilon_0 > 0$ small (for example $\epsilon_0 = a/20$). We choose $\varepsilon > 0$ and $n \in \mathbb{N}$ so that all the conditions below are fulfilled:

(A) - The value of n is chosen to be the smallest positive integer for which the following conditions are satisfied:

$$0 < \lambda^n a < a - 5\epsilon_0 \quad \text{and} \quad \lambda^n < 1/20 \, .$$

(B) - Lemma 3.2.3 holds on $\mathcal{U}_0^2$ for some $C > 0$.

(C) - $\varepsilon > 0$ is small enough to ensure that Lemma 3.2.4 holds and that

$$\varepsilon \max \left\{ (\lambda^{-n} + 1)C \, , \, (\lambda^{-n} + 1) \right\} < 1/10 \, .$$

Proof of Proposition 3.2.6. Under the above conditions we are going to show that Proposition 3.2.6 holds with $b = a$. The proof is by induction. First consider a diffeomorphism $g_1 \in \widetilde{S}(1)$. By assumption, $g_1 = \lambda^{-n} f_1(\lambda^n x)$ for some f_1 given as a commutator $[\overline{g}_i, \overline{g}_j]$ for some $i, j \in \{1, \ldots, N\}$. Owing to Lemma 3.2.3, f_1 is defined on $[-a+5\epsilon_0, a-5\epsilon_0]$ when viewed as element of the pseudogroup generated by $\overline{g}_1, \ldots, \overline{g}_N$ on $[-a, a]$. Furthermore, the C^1-norm of $f_1 - \mathrm{id}$ on $[-a + 5\epsilon_0, a - 5\epsilon_0]$ satisfies

$$\|f_1 - \mathrm{id}\|_{1,[-a+5\epsilon_0,a-5\epsilon_0]} < C \, \varepsilon^2 \, . \tag{3.4}$$

Next observe that $g_1 = \lambda^{-n} f_1(\lambda^n x)$ is defined on all of $[-a, a]$ since $\lambda^n a < a - 5\epsilon_0$. Moreover, we clearly have:

$$\sup_{x\in[-a,a]} |g_1(x) - x| = \sup_{x\in[-a,a]} |\lambda^{-n} f_1(\lambda^n x) - x| = \lambda^{-n} \sup_{y\in[-a+5\epsilon_0,a-5\epsilon_0]} |f_1(y) - y| \, .$$

Similarly

$$\sup_{x\in[-a,a]} |D_x^1 g_1 - 1| = \sup_{y\in[-a+5\epsilon_0,a-5\epsilon_0]} |D_y^1 f_1 - 1| \, .$$

In particular, we obtain

$$\sup_{x\in[-a,a]} |g_1(x) - x| + \sup_{x\in[-a,a]} |D_x^1 g_1 - 1| < (\lambda^{-n} + 1) C \varepsilon^2 \, . \tag{3.5}$$

Finally, the second derivative of g_1 at a point $x \in [-a, a]$ is such that $D_x^2 g_1 = \lambda^n \, D_{\lambda^n x}^2 f_1$ so that

$$\sup_{x\in[-a,a]} |D_x^2 (g_1 - \mathrm{id})| = \sup_{x\in[-a,a]} |D_x^2 g_1| < \lambda^n 5 \max_{x\in(-a,a)} \left\{ \left| D_x^2 \overline{g}_i \right| , \left| D_x^2 \overline{g}_j \right| \right\} \leq 5 \lambda^n \varepsilon \, , \tag{3.6}$$

where we have used Lemma 3.2.4. Comparing Estimates (3.5) and (3.6), there follows that

$$\|g_1 - \mathrm{id}\|_{2,[-a,a]} \leq (\lambda^{-n} + 1)C\varepsilon^2 + 5\lambda^n \varepsilon \leq \frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

where conditions (A), (B), and (C) concerning the choices of ε, n, and the constant C were used. In particular, we see that g_1 belongs to $\mathcal{U}_0^2$. Since g_1 is an arbitrary element of $\widetilde{S}(1)$, we conclude that $\widetilde{S}(1) \subset \mathcal{U}_0^2$ so that the procedure can be iterated. Consider then $g_2 = \lambda^{-n}[\tilde{f}_{i_1}, \tilde{f}_{i_2}](\lambda^n x)$ where $\tilde{f}_{i_1}, \tilde{f}_{i_2}$ belong to $\widetilde{S}(1) \cup \{\overline{g}_1, \ldots, \overline{g}_N\}$. Repeating word-by-word, the preceding argument we eventually obtain

$$\|g_2 - \mathrm{id}\|_{2,[-a,a]} \leq \frac{\varepsilon}{2}$$

(in particular g_2 is defined on all of $[-a, a]$). However an element $g_3 \in \widetilde{S}(3)$ can be written as $g_3 = \lambda^{-n}[\tilde{f}_{i_1}, \tilde{f}_{i_2}](\lambda^n x)$ where $\tilde{f}_{i_1}, \tilde{f}_{i_2}$ now satisfy

$$\max\{\|\tilde{f}_{i_1} - \mathrm{id}\|_{2,[-a,a]} \,;\, \|\tilde{f}_{i_2} - \mathrm{id}\|_{2,[-a,a]}\} < \varepsilon/2\,.$$

Therefore, what precedes yields

$$\|g_3 - \mathrm{id}\|_{2,[-a,a]} < \frac{\varepsilon}{2^2}\,.$$

Now a straightforward induction shows that

$$\|g_{2j} - \mathrm{id}\|_{2,[-a,a]} < \frac{\varepsilon}{2^j} \tag{3.7}$$

and completes the proof of Proposition 3.2.6. □

Remark 3.2.7 Consider a sequence $g_1, g_2, \ldots$ so that $g_j \in \widetilde{S}(j)$ as above. Consider also the sequence of real numbers given by $\{\|g_j - \mathrm{id}\|_{2,[-a,a]}\}$. Estimate (3.7) shows that the subsequence of $\{\|g_j - \mathrm{id}\|_{2,[-a,a]}\}$ formed by those g_j with even order decays at least as $1/\sqrt{2}^j$. In fact, it can be shown that the entire sequence $\{\|g_j - \mathrm{id}\|_{2,[-a,a]}\}$ decays faster than Θ^j for every a priori given $\Theta > 0$. To check this claim, we proceed as follows.

First observe that the choice of $\varepsilon > 0$ made in condition (C) can be modified by replacing the $1/10$ on the right side of the corresponding estimate by a sufficiently small $\delta > 0$. Note that this change does not affect either n or the constant C whereas it allows us to obtain a finer estimate than $\varepsilon/2$ for $\|g_1 - \mathrm{id}\|_{2,[-a,a]}$. Thus the same induction argument employed above now yields a new exponential decay for the sequence $\{\|g_j - \mathrm{id}\|_{2,[-a,a]}\}$ where the base depends on δ (and becomes larger when δ becomes smaller). On the other hand, we have show that every element g_j in $\widetilde{S}(j)$ satisfies $\|g_{2j} - \mathrm{id}\|_{2,[-a,a]} < \varepsilon/2^j$ so that there is $j_0 \in \mathbb{N}$ for which every element in $\widetilde{S}(j)$ satisfies the estimate in condition (C) with a fixed $\delta > 0$ in the place of $1/10$. Thus, up to dropping finitely many terms, the sequence $\{g_j\}$ converges to the identity faster than Θ^j. Since only finitely many terms have been dropped, there follows that the initial sequence $\{g_j\}$ converges to the identity faster than Θ^j. This simple observation will be useful in the next section.

Remark 3.2.8 Concerning the proof of Theorem A, we can also assume without loss of generality the existence of a sequence $\{g_j\}$ as in Proposition 3.2.6, and hence converging to the identity in the C^2-topology on some interval $I \subset S^1$, such that $g_j \neq \mathrm{id}$ for every $j \in \mathbb{N}$. Indeed, in view of Proposition 3.1.6 we can assume the existence of diffeomorphisms F and $\overline{g}_1, \ldots, \overline{g}_N$ in G satisfying the conditions of Proposition 3.1.6 on a suitable interval $I \subset S^1$ and such that $\overline{g}_1, \ldots, \overline{g}_N$ generate a non-solvable group. Since the group generated by $\overline{g}_1, \ldots, \overline{g}_N$ is not solvable, there follows that none of the sets $S(j)$ degenerate into the set containing only the identity map (see [G1]). Clearly the same applies to the sets $\widetilde{S}(j)$ (Lemma 3.2.5).

Moreover, up to passing to a subsequence, Corollary 3.2.2 allows us to assume also that $f_j(p) \neq p$ (and similarly $g_j(p) \neq p$) for every a priori given point $p \in S^1$.

3.3 Expansion, bounded distortion and rigidity

In this section we shall complete the proof of Theorem A by taking for granted Proposition 3.3.3 stated below. The proof of Proposition 3.3.3 in turn is deferred to the next chapter. We begin by recalling that the argument in [G-T] reduces the proof of Theorem A to checking that h is a diffeomorphism of class C^1. The proof of this statement under suitable conditions will be the object of the section.

To make the discussion accurate, let G_1 and G_2 be two finitely generated subgroups of $\mathrm{Diff}^{\omega}(S^1)$ that are conjugated by a homeomorphism $h : S^1 \to S^1$. By assumption, the group G_1 is locally C^2-non-discrete. In view of the material presented in the previous sections, the following conditions can be assumed to hold without loss of generality.

(1) All the orbits of G_1 are dense in S^1 (in particular G_1 has no finite orbit). The same condition is automatically verified by G_2 since the groups are topologically conjugate.

(2) There is an interval $I = [-a, a] \subset \mathbb{R} \subset S^1$ ($a \neq 0$) and an element F_1 in G_1 satisfying $F_1(0) = 0$ and $F_1'(0) = \lambda_1 \in (0, 1)$.

(3) For every $\varepsilon > 0$, we can find a finite set $\{\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}\} \subset G_1$ satisfying all the conditions below:

- $\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}$ are ε-close to the identity in the C^2-topology on I (where $I = [-a, a]$ is the above chosen interval).
- $\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}$ generate a non-solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ having no finite orbit.
- Consider the sequence $\widetilde{S}_1(k)$ defined in Section 3.2 by means of the set $\widetilde{S}_1(0) = S_1(0) = S_1 = \{\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}\}$ so that $\widetilde{S}_1(k) = F_1^{-kn} \circ S_1(k) \circ F_1^{kn}$ for every $k \in \mathbb{N}$ and a certain fixed $n \in \mathbb{N}^*$. Then every sequence of elements $\{g_{1,k}\}$ with $g_{1,k} \in \widetilde{S}(k)$ converges to the identity in the C^2-topology on the interval I.

(4) In fact, if $\{g_{1,k}\} \subset G_1$ is such that $g_{1,k} \in \widetilde{S}_1(k)$, $k \in \mathbb{N}$, then for every $\Theta \in \mathbb{R}_+^*$, we have

$$\lim_{k\to\infty} \left[\frac{\|g_{1,k} - \mathrm{id}\|_{2,[-a,a]}}{\Theta^k} \right] = 0 \,. \tag{3.8}$$

Next recall that a point $p \in S^1$ is said to be *expandable* for a given group $G \subset \mathrm{Diff}^{\omega}(S^1)$ if there is $g \in G$ such that $g'(p) > 1$. Since our diffeomorphisms preserve the orientation of S^1 the conditions $g'(p) > 1$ and $|g'(p)| > 1$ are indeed equivalent. With this terminology, we state:

Theorem 3.3.1 *Assume that G_1 satisfy all the conditions (1)–(4) above. Assume also that every point $p \in S^1$ is expandable for G_2. Then every homeomorphism $h : S^1 \to S^1$ conjugating G_1 to G_2 coincides with an element of* $\mathrm{Diff}^{\omega}(S^1)$.

The following simple lemma clarifies the connection between Theorem A and Theorem 3.3.1.

Lemma 3.3.2 *Assume that $G \subset \mathrm{Diff}^{\omega}(S^1)$ is a locally C^2-non-discrete group satisfying conditions (1)–(4) above. Then G leaves no probability measure on S^1 invariant. Moreover, every point $p \in S^1$ is expandable for G.*

Proof. Since G has all orbits dense, every probability measure invariant by G must be supported on all of S^1. Up to parameterizing S^1 by means of the corresponding Radon-Nikodym derivative, the group G becomes conjugate to a group of rotations. This is impossible since G contains elements exhibiting hyperbolic fixed points.

To establish the second part of the statement, we proceed as follows. Since G contains elements having hyperbolic fixed points, we can choose an interval $I = [-a, a]$ and an element F in G satisfying $F(0) = 0$ and $F'(0) > 1$. Furthermore, owing to Proposition 2.3.5, we can assume without loss of generality that I is equipped with a vector field X contained in the C^1-closure of G. Consider first the case of a point p lying in the interval I. Choose t_0 so that the local flow ϕ^t of X satisfies $\phi^{t_0}(p) = 0$. The diffeomorphism $\overline{f} = \phi^{-t_0} \circ F \circ \phi^{t_0}$ satisfies $\overline{f}(p) = p$ and $\overline{f}'(p) > 1$. Since X lies in the C^1-closure of G, there follows that ϕ^{t_0} is the C^1-limit of a sequence $\tilde{f}_r$ of elements in G restricted to some small neighborhood of p. Thus, for r large enough, we conclude that $(\tilde{f}_r^{-1} \circ F \circ \tilde{f}_r)'(p) > 1$ proving the statement for points in I. To finish the proof of the lemma, just note that the minimal character of G enables us to find a finite covering of S^1 by intervals satisfying the same conditions used above for the interval I. This ends the proof of lemma. □

Proof of Theorem A. It follows at once from the combination of Theorem 3.3.1 with Lemma 3.3.2. □

The proof of Theorem 3.3.1 will occupy the remainder of this section. We begin by stating Proposition 3.3.3. For this, first note that diffeomorphisms in G_1 having a hyperbolic fixed point in I are far from unique. We have fixed one of them, namely F_1. The element F_2 of G_2 verifying $F_2 = h^{-1} \circ F_1 \circ h$ has therefore a fixed point in the interval $J = h^{-1}(I)$, namely the point $q = h^{-1}(0)$. However, since h is only a homeomorphism, we cannot immediately conclude that q is hyperbolic for F_2. In fact, whereas F_2 certainly realizes a "topological contraction" on a neighborhood of q, it may happen that $F_2'(q) = 1$. The possibility of having $F_2'(q) = 1$ is a bit of an inconvenient since it would require us to work with iterations of a "parabolic map" in a context similar to the one discussed in Section 2.3. This type of difficulty, however, can be overcome with the help of Proposition 3.3.3 below. The proof of this proposition however will be deferred to the next chapter since it relies heavily on the methods of [DKN-1].

Proposition 3.3.3 *Without loss of generality, we can assume that* $F_2'(q) < 1$ *where* $q = h^{-1}(0)$.

Now consider a C^1-diffeomorphism $f : S^1 \to S^1$. Given an interval $U \subset \mathbb{R} \subset S^1$, the *distortion of f in U* is defined as

$$\varpi(f, U) = \sup_{x,y \in U} \log \frac{|f'(x)|}{|f'(y)|} = \sup_{x \in U} \log(|f'(x)|) - \inf_{y \in U} \log(|f'(y)|) \tag{3.9}$$

where $|\,.\,|$ stands for the absolute value. Furthermore, assuming that the map $x \mapsto \log(|D_x f|)$ has a Lipschitz constant $C_{\rm Lip}$, the estimate

$$\varpi(f, U) \le C_{\rm Lip} \, \mathcal{L}(U) \tag{3.10}$$

holds (where $\mathcal{L}(U)$ stands for the length of the interval U with respect to the Euclidean metric for which the length of S^1 equals 2π). Note also that the mentioned Lipschitz condition is satisfied provided that f is of class C^2 on U. On the other, given two diffeomorphisms $f_1, f_2 : S^1 \to S^1$ as above, the estimate

$$\varpi(f_1 \circ f_2, U) \le \varpi(f_1, f_2(U)) + \varpi(f_2, U) \tag{3.11}$$

also holds provided that both sides are well defined.

Let us now go back to the sequence of sets $\{\widetilde{S}_1(k)\} \subset G_1$ fixed in the beginning of the section. For every $k \in \mathbb{N}$ all the mappings in $\widetilde{S}_1(k)$ are defined on the interval $I = [-a, a]$. Next recall that this sequence was obtained as indicated in Section 3.2 by means of the finite set $\{\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}\} \subset G_1$ and of the diffeomorphism F_1. In particular $\widetilde{S}_1(k) = F_1^{-kn} \circ S_1(k) \circ F_1^{kn}$. From now on, we fix a sequence $\{g_{1,k}\} \subset G_1$ of diffeomorphisms such that $g_{1,k} \neq {\rm id}$ belongs to $\widetilde{S}_1(k)$ for every $k \in \mathbb{N}$. Consider also the corresponding sequence $\{\widetilde{S}_2(k)\} \subset G_2$ defined by means of $\{\overline{g}_{2,1}, \ldots, \overline{g}_{2,N}\} \subset G_2$ and of the diffeomorphism F_2. More precisely, we set $\overline{g}_{2,j} = h^{-1} \circ \overline{g}_{1,j} \circ h$ for every $j = 1, \ldots, N$ and $F_2 = h^{-1} \circ F_1 \circ h$ where F_2 is assumed to have a contractive hyperbolic fixed point at $j = h^{-1}(0)$ (cf. Proposition 3.3.3). Thus, for every $k \in \mathbb{N}$, we have $g_{2,k} = h^{-1} \circ g_{1,k} \circ h$. Finally we also pose $J = h^{-1}(I)$.

Next, for every $k \in \mathbb{N}$, let $\mathcal{P}_{I,k}$ denote the partition of the interval I into 5^k sub-intervals having the same size and write $\mathcal{P}_{I,k} = \{I_{1,k}, \ldots, I_{5^k,k}\}$. By means of the homeomorphism h, the partitions $\mathcal{P}_{I,k}$ induce partitions $\mathcal{P}_{J,k} = \{J_{1,k}, \ldots, J_{5^k,k}\}$ of the interval J where $J_{j,k} = h^{-1}(I_{j,k})$ for every $k \in \mathbb{N}$ and for every $j \in \{1, \ldots, 5^k\}$. Now we have:

Lemma 3.3.4 *Denote by $\varpi\left(g_{2,k}, J_{j,k}\right)$ the distortion of $g_{2,k}$ in the interval $J_{j,k}$. Then to each $k \in \mathbb{N}$ there corresponds $j \in \{1, \ldots, 5^k\}$ such that the resulting sequence $k \mapsto \varpi\left(g_{2,k}, J_{j,k}\right)$ converges to zero.*

Proof. Consider the set formed by the diffeomorphisms $\overline{g}_{2,1}, \ldots, \overline{g}_{2,N}$, F_2 along with their inverses. This set is therefore symmetric in the sense that whenever a diffeomorphism belongs to it so does the inverse of the diffeomorphism in question The semigroup generated by this set of diffeomorphisms coincides with the group generated by $\overline{g}_{2,1}, \ldots, \overline{g}_{2,N}$, and F_2. Every element in the group in question can be represented as a word in the alphabet whose letters are the diffeomorphisms in the initial symmetric set. If f represents an element in this alphabet, i.e. a letter, the map $x \mapsto \log(|D_x f|)$ is well defined on all of S^1 (since $D_x f \neq 0$ for all $x \in S^1$). These maps are also Lipschitz on all of S^1 since f is, in any event, a C^2-diffeomorphism. Fix then a positive constant C greater than the maximum among the Lipschitz constants of all the maps $x \mapsto \log(|D_x f|)$ with f belonging to the alphabet in question.

The explicit construction of the sequences $\{g_{1,k}\}$ and $\{g_{2,k}\}$ makes it clear that every diffeomorphism $g_{2,k}$ can be spelled out in the above mentioned alphabet using at most $4^k + 2nk$ letters. Next let c_1 be a constant such that $c_1 \mathcal{L}\left(J\right) > 2\pi$ (where $\mathcal{L}\left(J\right)$ stands for the length of J). Note also that every diffeomorphism f of the circle must satisfy $\mathcal{L}\left(f(J)\right) < 2\pi$.

Now fixed $k \in \mathbb{N}$, let $g_{2,k} = f_l \circ \cdots \circ f_1$ denote the above mentioned spelling of $g_{2,k}$. Thus $l \leq 4^k + 2nk$ and f_i belongs to $\{F_2^{\pm 1}, \overline{g}_{2,1}^{\pm 1}, \ldots, \overline{g}_{2,N}^{\pm 1}\}$ for every $i \in \{1, \ldots, l\}$. The subadditivity relation expressed by (3.11) combined to estimate (3.10) yields

$$\varpi\left(g_{2,k}, J\right) \leq C \sum_{i=1}^{l-1} \mathcal{L}\left(f_i \circ \cdots \circ f_1(J)\right) + C\,\mathcal{L}\left(J\right) \leq c_1 C\,\mathcal{L}\left(J\right)(4^k + 2nk)\,.$$

On the other hand, given an sub-interval $J_{j,k}$ in the partition $\mathcal{P}_{J,k}$ (so that $j \in \{1, \ldots, 5^k\}$). The preceding argument ensures that

$$\varpi\left(g_{2,k}, J_{j,k}\right) \leq C \sum_{i=1}^{l-1} \mathcal{L}\left(f_i \circ \cdots \circ f_1(J_{j,k})\right) + C\,\mathcal{L}\left(J_{j,k}\right) \leq c_1 C\,\mathcal{L}\left(J_{j,k}\right)(4^k + 2nk)\,.$$

However, we clearly have

$$\sum_{j=1}^{5^k} \left[\sum_{i=1}^{l-1} \mathcal{L}\left(f_i \circ \cdots \circ f_1(J_{j,k})\right)\right] + \sum_{j=1}^{5^k} \mathcal{L}\left(J_{j,k}\right) = \sum_{i=1}^{l-1} \mathcal{L}\left(f_i \circ \cdots \circ f_1(J)\right) + \mathcal{L}\left(J\right).$$

Hence there follows that

$$\sum_{j=1}^{5^k} \varpi\left(g_{2,k}, J_{j,k}\right) \leq c_1 C\,\mathcal{L}\left(J\right)(4^k + 2nk)\,.$$

Finally, if j realizes the minimum of $j \mapsto \varpi(g_{2,k}, J_{j,k})$ over the set $\{1, \ldots, 5^k\}$, we conclude that

$$\varpi(g_{2,k}, J_{j,k}) \le \frac{c_1 C \,\mathcal{L}(J)(4^k + 2nk)}{5^k}$$

which goes to zero as $k \to \infty$. The proof of the lemma is completed. □

As k increases, we know that $g_{2,k}(y) - y$ converges uniformly to zero on all of J. However, when we consider the sequence of sub-intervals $J_{j(k),k}$ their diameters go to zero as well. A comparison between $\sup_{y \in J_{j(k),k}} |g_{2,k}(y) - y|$ and the length $\mathcal{L}(J_{j(k),k})$ of $J_{j(k),k}$ will however be needed. In particular, we would like to claim that the sequence of quotients $\sup_{y \in J_{j(k),k}} |g_{2,k}(y) - y| / \mathcal{L}(J_{j(k),k})$ converges to zero as $k \to \infty$. At this moment, our results are not sufficient to derive this conclusion since we have no control of the ratio between the lengths of two intervals $J_{j_1(k),k}$ and $J_{j_2(k),k}$. A suitable comparison between the lengths of the mentioned intervals will however be supplied by Proposition 3.3.5 claiming that the conjugating homeomorphism h is Hölder continuous; cf. below for details.

The next basic step in the proof of Theorem 3.3.1 consists of *magnifying* the intervals $I_{j(k),k}$ and $J_{j(k),k}$ into intervals with diameters bounded from below by some strictly positive constant. To do this, we shall resort to a slightly more straightforward version of the celebrated "Sullivan's expansion strategy" as expounded in [Nv] and [S-S]. The main difficulty in applying Sullivan's type of argument to our situation lies in the fact that the procedure needs to be simultaneously applied to both groups G_1 and G_2. To overcome this problem we shall first establish that the conjugating homeomorphism $h : S^1 \to S^1$ is Hölder continuous for a suitable exponent (Proposition 3.3.5 below). Proposition 3.3.5 will subsequently be combined with the several estimates involving convergence rates for the sequences $\{g_{1,k}\}_{k \in \mathbb{N}}$ and $\{g_{2,k}\}_{k \in \mathbb{N}}$ (restricted to the intervals $I_{j(k),k}$ and $J_{j(k),k}$, respectively) to yield Theorem 3.3.1.

Recall that a map $f : U \subset S^1 \to S^1$ is said to be *α-Hölder continuous* on the interval U if the supremum

$$\sup_{x,y \in U x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \tag{3.12}$$

is finite (where the bars $|\,.\,|$ stand for the fixed Euclidean metric on S^1). The above definition is local in the sense that the length of U and of $f(U)$ are assumed to be smaller than π so that the above indicated distances are well defined. We shall say that f is α-Hölder continuous if its restriction to every interval U satisfying $\max\{\mathcal{L}(U),\ \mathcal{L}(f(U))\} < \pi$ is α-Hölder continuous on U. With this terminology, we have:

Proposition 3.3.5 *There is $\alpha > 0$ such that the homeomorphisms $h : S^1 \to S^1$ and $h^{-1} : S^1 \to S^1$ are both α-Hölder continuous.*

The idea to prove Proposition 3.3.5 is very simple and relies on the fact that none of the groups G_1 and G_2 has non-expandable points. We first provide an outline of the argument which might well be sufficient to convince readers familiar with Sullivan's expansion strategy and related results in one-dimensional dynamics. To prove that h is Hölder continuous we will rely on the fact that all points in S^1 are expandable for the group G_2. By permuting the roles of G_1 and G_2 in the subsequent discussion we would also conclude the Hölder continuity of h^{-1}.

To begin with note that the definition of α-Hölder continuity is only non-trivial when $|x-y|$ becomes arbitrarily small: indeed, if $|x-y|$ is bounded from below by some positive constant then the supremum appearing in Equation (3.12) is clearly finite. Hence to check that h is α-Hölder continuous we only need to check that

$$\limsup \frac{|h(x_i)-h(y_i)|}{|x_i-y_i|^\alpha}$$

is finite for every sequence $\{(x_i,y_i)\} \subset S^1 \times S^1$, $x_i \neq y_i$ for every $i \in \mathbb{N}$, satisfying $\lim |y_i - x_i| = 0$. Fix then some small $\delta > 0$ and consider those diffeomorphisms in G_2 expanding the interval $[h(x_i), h(y_i)]$ to a size greater than δ. The existence of these diffeomorphisms follows from the fact that G_2 does not have non-expandable points. Among the mentioned diffeomorphisms in G_2, we fix one $F_{2,i} \in G_2$ which, in addition, has a minimal spelling in the letters of the alphabet given by $\overline{g}_{2,1}, \ldots, \overline{g}_{2,N}$, F_2 and their inverses. The fact that all points in S^1 can be expanded by G_2 ensures that the number of letters involved in the spelling of $F_{2,i}$ is comparable to $-\ln(|h(y_i)-h(x_i)|)$. More precisely, there follows from Sullivan's expansion strategy that the mentioned number is bounded by some constant times $-\ln(|h(y_i)-h(x_i)|)$. On the other hand, this number is clearly bounded from below by some constant times $-\ln(|h(y_i)-h(x_i)|)$ since the derivatives of all the corresponding "letters" are uniformly bounded on S^1.

Since G_1 is conjugate to G_2, the interval $[x_i, y_i]$ can also be expanded to some uniform size (depending only on δ and on h) by using diffeomorphisms $F_{1,i}$ in G_1 whose spelling in the alphabet formed by $\overline{g}_{1,1}, \ldots, \overline{g}_{1,N}$, F_1 and their inverses uses the same number of letters as $F_{2,i}$. In particular this number must be bounded from below by some constant times $-\ln(|y_i - x_i|)$ (whether or not G_1 has non-expandable points since this is the easy direction of the above mentioned estimate). Thus the quotient

$$\frac{\ln(|h(y_i)-h(x_i)|)}{\ln(|y_i-x_i|)}$$

is bounded from below by a strictly positive constant which yields the Hölder continuity of h.

In the sequel we provide full detail on the above argument for readers less familiar with the corresponding techniques. In particular Sullivan's expansion strategy is summarized by Lemma 3.3.6 below.

We begin by recalling that to each point $p \in S^1$ there corresponds a diffeomorphism $f_{1,p} \in G_1$ with $f'_{1,p}(p) > 1$ (Lemma 3.3.2). Owing to the compactness of S^1, there is a finite covering $\mathfrak{U}_1 = \{U_{1,1}, \ldots, U_{1,s}\}$ of S^1 by open connected intervals $U_{1,i}$, $i = 1, \ldots, s$, satisfying the following conditions:

- For $2 \leq i \leq s-1$, the interval $U_{1,i}$ intersects only the intervals $U_{1,i-1}$ and $U_{1,i+1}$. The interval $U_{1,1}$ (resp. $U_{1,s}$) intersects only the intervals $U_{1,2}$ and $U_{1,s}$ (resp. $U_{1,s-1}$ and $U_{1,1}$).

- To each interval $U_{1,i}$ there corresponds a diffeomorphism $f_{1,i} \in G_1$ such that $f'_{1,i}(x) > 1$ for every $x \in U_{1,i}$ (recall that G_1 and G_2 preserve the orientation of S^1).

Let $m_1 > 1$ be given as

$$m_1 = \min_{i \in \{1,\ldots,s\}} \{ \inf_{U_{1,i}} f'_{1,i} \} .$$

Similarly let $M_1 = \max_{i \in \{1,\ldots,s\}} \{ \sup_{U_{1,i}} f'_{1,i} \}$. Next let $L > 0$ denote the minimum of the lengths of the sets $U_{1,1} \cap U_{1,s}$ and $U_{1,i} \cap U_{1,i+1}$ (for $i = 1, \ldots, s-1$) so that every interval $[a,b] \subset S^1$ of length less than L is contained in some interval U_{1,i_1} ($i_1 \in \{1, \ldots, s\}$). For $[a,b]$ as indicated, the derivative of f_{1,i_1} is not less than $m_1 > 1$ at every point in $[a,b]$ and the length $\mathcal{L}(f_{1,i_1}([a,b]))$ of $f_{1,i_1}([a,b])$ is at least $m_1 \mathcal{L}([a,b]) > \mathcal{L}([a,b])$. When $\mathcal{L}(f_{1,i_1}([a,b]))$ is still less than L, $f_{1,i_1}([a,b])$ is again contained in some interval U_{1,i_2}. Thus $f_{1,i_2}(f_{1,i_1}([a,b]))$ has length greater than or equal to $m_1^2 \mathcal{L}([a,b])$ and the procedure can be continued provided that $\mathcal{L}(f_{1,i_2} \circ f_{1,i_1}([a,b])) < L$. Thus we have proved the following:

Lemma 3.3.6 *To every interval $[a,b] \subset S^1$ of length less than L, we can assign an element $F_{1,[a,b]} \in G_1$ satisfying the following conditions:*

1. $F_{1,[a,b]} = f_{1,i_r} \circ \cdots \circ f_{1,i_1}$ *where each i_l belongs to $\{1, \ldots, s\}$.*
2. *For every $l \in \{1, \ldots, r\}$, $f_{1,i_{l-1}} \circ \cdots \circ f_{1,i_1}([a,b])$ is contained in U_{1,i_l} (where $f_{1,i_{l-1}} \circ \cdots \circ f_{1,i_1}([a,b]) = [a,b]$ if $l = 1$).*
3. *We have*

$$L \leq \mathcal{L}(F_{1,[a,b]}([a,b])) \leq L\, M_1 .$$

□

Recalling that $\mathfrak{U}_1 = \{U_{1,1}, \ldots, U_{1,s}\}$, we define a new covering $\mathfrak{U}_2$ of S^1 by letting $\mathfrak{U}_2 = \{U_{2,1}, \ldots, U_{2,s}\}$ where $U_{2,i} = h^{-1}(U_{1,i})$ for every $i = 1, \ldots, s$. To every diffeomorphism $F_{1,[a,b]} = f_{1,i_r} \circ \cdots \circ f_{1,i_1} \in G_1$ as above, we assign the corresponding diffeomorphism

$$F_{2,[a,b]} = f_{2,i_r} \circ \cdots \circ f_{2,i_1} = h^{-1} \circ F_{1,[a,b]} \circ h$$

where $f_{2,i_l} = h^{-1} \circ f_{1,i_l} \circ h$ for every $l \in \{1, \ldots, r\}$. Clearly the diffeomorphism $F_{2,[a,b]}$ takes the (small) interval $h^{-1}([a,b])$ to the interval $h^{-1}(F_{1,[a,b]}([a,b]))$ whose diameter is bounded from below by a positive constant since h is uniformly continuous (S^1 is compact). Moreover we can still define

$$M_2 = \max_{i \in \{1,\ldots,s\}} \{ \sup_{U_{2,i}} f'_{2,i} \}$$

so that $M_2 > 1$. However at this point we cannot ensure that $\inf_{U_{2,i}} f'_{2,i} > 1$ for a given $i \in \{1, \ldots, s\}$.

We are now able to complete the proof of Proposition 3.3.5.

Proof of Proposition 3.3.5. By using the above introduced coverings $\mathfrak{U}_1$ and $\mathfrak{U}_2$ of S^1, we are going to show the existence of $\alpha > 0$ so that h is α-Hölder continuous. By reversing the roles of G_1 and G_2 as indicated above, the same argument implies the α-Hölder continuity of h^{-1} as well (up to reducing $\alpha > 0$). In fact, all points in S^1 are known to be expandable for G_1 (Lemma 3.3.2) while, in the case of G_2, this condition is satisfied by assumption.

As already pointed out, the claim that h is α-Hölder continuous has a local character. More precisely, considering points $c \neq d$ in S^1, we need to find constants $C \in \mathbb{R}_+^*$ and $\alpha > 0$ such that

$$|h(d) - h(c)| \leq C\,|d - c|^{\alpha}$$

provided that $|d - c|$ is small. Here the vertical bars $|.|$ stand for the distance between the corresponding points for the fixed Euclidean metric (i.e. $|d - c| = \mathcal{L}\,([c, d])$). Owing to the previous discussion and to the fact that both h and h^{-1} are uniformly continuous since S^1 is compact, there easily follows the existence of a uniform $\tau > 0$ so that all the considerations made in the course of the proof are valid provided that $|d - c| < \tau$. We shall then proceed to prove that h is α-Hölder continuous on intervals whose length does not exceed τ which clearly implies the proposition.

We therefore consider c, d as before and let $[a, b] = h([c, d])$. Without loss of generality, h preserves the orientation so that we set $a = h(c)$ and $b = h(d)$. The next step consists of expanding the interval $[a, b]$ by means of the procedure summarized by Lemma 3.3.6. With the notation used in this lemma, we find $F_{1,[a,b]} = f_{1,i_r} \circ \cdots \circ f_{1,i_1} \in G_1$ such that

$$L \leq \mathcal{L}\,(F_{1,[a,b]}([a, b])) \leq M_1 L\,. \tag{3.13}$$

Consider now the corresponding element $F_{2,[a,b]} = h^{-1} \circ F_{1,[a,b]} \circ h$ in G_2. We also set $F_{2,[a,b]} = f_{2,i_r} \circ \cdots \circ f_{2,i_1}$ as previously indicated. There exists a uniform $\delta > 0$ so that

$$\mathcal{L}\,(F_{2,[a,b]}([c, d])) \geq \delta > 0\,.$$

Indeed, just note that $F_{2,[a,b]}([c, d]) = h^{-1} \circ F_{1,[a,b]}([a, b])$ so that the claim follows from the uniform continuity of h^{-1} since $\mathcal{L}\,(F_{1,[a,b]}([a, b])) \geq L > 0$.

Consider now the number r of diffeomorphisms $f_{1,i}$ ($i \in \{1, \ldots, s\}$) appearing in the above indicated spelling of $F_{1,[a,b]}$. By construction, at each iteration of $f_{1,i}$ the corresponding interval is expanded by a factor bounded from below by $m_1 > 1$. Hence we obtain

$$\frac{|b - a|\, m_1^r}{L M_1} \leq 1\,. \tag{3.14}$$

In particular, there follows that

$$r \leq \frac{1}{\ln m_1}\left(\ln(L M_1) - \ln|b - a|\right)\,. \tag{3.15}$$

On the other hand, considering $F_{2,[a,b]} = f_{2,i_r} \circ \cdots \circ f_{2,i_1}$, there also follows that at each iteration of $f_{2,i}$ the interval in question cannot be expanded by a factor exceeding M_2. Hence, we similarly obtain $|d-c|M_2^r \geq \delta$ so that

$$1 \leq \frac{|d-c|\, M_2^r}{\delta}\,. \tag{3.16}$$

Putting together Estimates (3.14) and (3.16), we conclude that

$$|b-a| \leq \frac{LM_1}{\delta}\, |d-c| \left(\frac{M_2}{m_1}\right)^r \,. \tag{3.17}$$

Without loss of generality, we can assume $M_2 > m_1$ for otherwise the preceding estimate implies at once that h is Lipschitz. Set $\overline{c} = \ln(M_2/m_1)/\ln m_1$ so that $\overline{c} > 0$ since $M_2 > m_1$ and $m_1 > 1$. Moreover Estimate (3.17) becomes

$$\begin{aligned} |b-a| &\leq C_1|d-c| \exp(\ln |b-a|^{-\overline{c}}) \\ &\leq C_1|d-c||b-a|^{-\overline{c}} \end{aligned}$$

for some suitable constant C_1. Hence h is α-Hölder continuous for $\alpha = 1/(1+\overline{c})$. The proof of the proposition is completed. □

We are almost ready to prove Theorem 3.3.1. The last ingredient needed in our proof consists of a simple estimate for the second derivatives of $F_{1,[a,b]}$ and of $F_{2,[a,b]}$. This is as follows. Keep the preceding notation and fix again intervals $[a,b]$ and $[c,d]$ such that $h([c,d]) = [a,b]$. Then we have:

Lemma 3.3.7 *There are constants $\overline{C} > 0$ and $\beta \in (0,1)$ such that*

$$\max\left\{\sup_{x\in[a,b]} |D^2F_{1,[a,b]}(x)| \;\; ; \;\; \sup_{y\in[c,d]} |D^2F_{2,[a,b]}(y)|\right\} \leq \overline{C}|b-a|^{\ln\beta}\,.$$

Proof. Let $\overline{M}$ be a constant satisfying

$$\max_{i=1,\ldots,s}\left\{\sup_{U_{1,i}} |D^2 f_{1,i}| \;\; ; \;\; \sup_{U_{2,i}} |D^2 f_{2,i}|\right\} < \overline{M}\,.$$

First we will show the existence of constants $\overline{C}_1$ and β_1 for which $\sup_{x\in[a,b]} |D^2F_{1,[a,b]}(x)| \leq \overline{C}_1|b-a|^{\ln\beta_1}$. We begin by recalling that $F_{1,[a,b]} = f_{1,i_r} \circ \cdots \circ f_{1,i_1}$. For $x_0 \in [a,b]$ and $l \in \{1,\ldots,r-1\}$, let $x_l = f_{1,i_l} \circ \cdots \circ f_{1,i_1}(x_0)$. Thus we have $F'_{1,[a,b]}(x_0) = f'_{1,i_r}(x_{r-1})\cdots f'_{1,i_1}(x_0)$ and

$$D^2F_{1,[a,b]}(x_0) = \prod_{l=1}^{r} f'_{1,i_l}(x_{l-1}) \left[\sum_{j=1}^{r}\left(\frac{D^2 f_{1,i_j}(x_{j-1})}{f'_{1,i_j}(x_{j-1})} f'_{1,i_{j-1}}(x_{j-2})\cdots f'_{1,i_1}(x_0)\right)\right]\,.$$

Hence

$$|D^2F_{1,[a,b]}(x_0)| \leq \overline{M}M_1^{2r}\,. \tag{3.18}$$

On the other hand, recall that $r \leq (\ln LM_1 - \ln|b-a|)/\ln m_1$ (Estimate (3.15)). Setting $\overline{C_1} = \overline{M} M_1^{2\ln LM_1/\ln m_1}$, there follows that

$$|D^2 F_{1,[a,b]}(x_0)| \leq \overline{C}_1 |b-a|^{-2\ln M_1/\ln m_1} .$$

Since $M_1 \geq m_1 > 1$, the exponent $-2\ln M_1/\ln m_1$ is negative and hence has the form $\ln\beta_1$ for some $\beta_1 \in (0,1)$. This proves the first assertion. To complete the proof of the lemma it only remains to show that a similar estimate holds for $|D^2 F_{2,[a,b]}|$ on $[c,d]$. However, a repetition word-by-word of the above argument yields constants $\overline{C}_2$ and $\beta_2 \in (0,1)$ such that

$$|D^2 F_{2,[a,b]}(y_0)| \leq \overline{C}_2 |d-c|^{\ln\beta_2}$$

for every $y_0 \in [c,d]$. The desired estimate is then an immediate consequence of Proposition 3.3.5. The lemma is proved. □

Proof of Theorem 3.3.1. In what follows we keep all the notation introduced in the course of this section. Consider the interval $I \subset S^1$ (resp. $J = h^{-1}(I) \subset S^1$) and the sequence of partitions $\mathcal{P}_{I,k}$ on I (resp. $\mathcal{P}_{J,k}$ on J). More precisely, consider the sequences of intervals $k \mapsto I_{l_k,k}$ and $k \mapsto J_{l_k,k}$ where $J_{l_k,k}$ is as in Lemma 3.3.4.

Next set $I_{l_k,k} = [a_k, b_k]$ and $J_{l_k,k} = [c_k, d_k]$ so that $a_k = h(c_k)$ and $b_k = h(d_k)$. Also $\alpha > 0$ is fixed so that both homeomorphisms h and h^{-1} are α-Hölder continuous (Proposition 3.3.5). Now, for each $k \in \mathbb{N}$ fixed, let $F_{1,[a_k,b_k]}$ be the element of G_1 obtained by means of Lemma 3.3.6. Thus we have $F_{1,[a_k,b_k]} = f_{1,i_{r_k}} \circ \cdots \circ f_{1,i_1}$ where each i_l belongs to $\{1,\ldots,s\}$. Analogously we define $F_{2,[a_k,b_k]} \in G_2$ so that $F_{2,[a_k,b_k]} = h^{-1} \circ F_{1,[a_k,b_k]} \circ h$. In particular

$$F_{2,[a_k,b_k]} = f_{2,i_{r_k}} \circ \cdots \circ f_{2,i_1}$$

with $i_l \in \{1,\ldots,s\}$.

By construction, all the intervals of the form $\{F_{1,[a_k,b_k]}([a_k,b_k])\} \subset S^1$ have length comprised between L and LM_1. Hence, up to passing to a subsequence, we can assume that $F_{1,[a_k,b_k]}(a_k) \to \widetilde{a}$ and $F_{1,[a_k,b_k]}(b_k) \to \widetilde{b}$ where $\widetilde{a} \neq \widetilde{b}$. To abridge notation, we refer to this by saying that the mentioned intervals converge towards the open interval $\widetilde{I} = (\widetilde{a},\widetilde{b}) \subset S^1$. Finally set also $\widetilde{J} = h^{-1}(\widetilde{I}) = (\widetilde{c},\widetilde{d}) \subset S^1$ so that $F_{2,[a_k,b_k]}(c_k) \to \widetilde{c}$ and $F_{2,[a_k,b_k]}(d_k) \to \widetilde{d}$.

Consider now the sequences of diffeomorphisms $\{\widetilde{f}_{1,k}\} \subset G_1$ and $\{\widetilde{f}_{2,k}\} \subset G_2$ obtained by setting

$$\widetilde{f}_{1,k} = F_{1,[a_k,b_k]} \circ g_{1,k} \circ F_{1,[a_k,b_k]}^{-1} \quad \text{and} \quad \widetilde{f}_{2,k} = F_{2,[a_k,b_k]} \circ g_{2,k} \circ F_{2,[a_k,b_k]}^{-1} .$$

Claim 1. The sequence $\{\widetilde{f}_{1,k}\} \subset G_1$ (resp. $\{\widetilde{f}_{2,k}\} \subset G_2$) converges to the identity in the C^0-topology on compact parts of $\widetilde{I}$ (resp. $\widetilde{J}$).

Proof of Claim 1. Consider first the sequence $\{\widetilde{f}_{1,k}\}$ and a point $x \in \widetilde{I}$. By construction the point $y = F_{1,[a_k,b_k]}^{-1}(x)$ lies in $I_{l_k,k} = [a_k, b_k]$ provided that k is large enough. Therefore

$$\begin{aligned}
|\widetilde{f}_{1,k}(x) - x| &= |F_{1,[a_k,b_k]} \circ g_{1,k} \circ F_{1,[a_k,b_k]}^{-1}(x) - x| \\
&\leq \sup_{[a_k,b_k]} |D^1 F_{1,[a_k,b_k]}| \, |g_{1,k}(y) - y| \\
&\leq M_1^{r_k} \, |g_{1,k}(y) - y| \,.
\end{aligned}$$

However r_k is bounded by Estimate (3.15) which yields $M_1^{r_k} \leq \text{const}|b_k - a_k|^{-\ln M_1/\ln m_1}$ for some constant const. Since in addition $|b_k - a_k|$ equals 5^{-k} up to a multiplicative constant, we obtain

$$|\widetilde{f}_{1,k}(x) - x| \leq \text{Const}\, 5^{k \ln M_1/\ln m_1} |g_{1,k}(y) - y|$$

which converges to zero as $k \to \infty$ by virtue of condition (4) in the beginning of the section (here Const stands for new some suitable constant).

It remains to show the same holds for the sequence $\{\widetilde{f}_{2,k}\}$. Setting $z = h^{-1}(x)$ and $w = h^{-1}(y)$, the same argument used above yields

$$|\widetilde{f}_{2,k}(z) - z| \leq \text{Const}'\, 5^{k \ln M_2/\ln m_1} |g_{2,k}(w) - w|$$

for a new constant Const$'$. However the α-Hölder continuity of h^{-1} ensures that $|g_{2,k}(w) - w| \leq |g_{1,k}(y) - y|^{\alpha}$ so that the claim follows again from condition (4). $\square$

We now consider the problem of C^1-convergence for the sequences $\{\widetilde{f}_{1,k}\}$ and $\{\widetilde{f}_{2,k}\}$. We begin by recalling that the restriction of $\{g_{1,k}\}$ to $I_{l_k,k}$ converges C^2 (in particular C^1) to the identity. On the other hand, the restriction of $g_{2,k}$ to $J_{l_k,k}$ is known to satisfy the following conditions:

(A)
$$\frac{\sup_{w \in J_{l_k,k}} |g_{2,k}(w) - w|}{\mathcal{L}\left(J_{l_k,k}\right)} \longrightarrow 0\,.$$

(B) The sequence $\{\varpi\left(g_{2,k}, J_{l_k,k}\right)\}$ formed by the distortion of $g_{2,k}$ on $J_{l_k,k}$ converges to zero.

The reader will note that item (B) is nothing but Lemma 3.3.4 whereas item (A) follows from Proposition 3.3.5. In fact, the α-Hölder continuity of h ensures that $\mathcal{L}\left(J_{l_k,k}\right) \geq \mathcal{L}\left(I_{l_k,k}\right)^{1/\alpha}$ while the α-Hölder continuity of h^{-1} yields $\sup_{w \in J_{l_k,k}} |g_{2,k}(w) - w| \leq \sup_{y \in I_{l_k,k}} |g_{1,k}(y) - y|^{\alpha}$. Thus the mentioned limit results from condition (4).

Owing to Proposition 3.3.3 and to the fact G_1 acts minimally on S^1, we choose a point $p \in \widetilde{I}$ such that the following condition holds: there are conjugate elements $\widetilde{F}_1 \in G_1$ and $\widetilde{F}_2 \in G_2$ ($\widetilde{F}_2 = h^{-1} \circ \widetilde{F}_1 \circ h$) such that $\widetilde{F}_1$ has a hyperbolic fixed point in p whereas $\widetilde{F}_2$ has a hyperbolic fixed point in $q = h^{-1}(p)$. For the reasons already explained, we can assume without loss of generality that $\widetilde{f}_{1,k}(p) \neq p$ for every $k \in \mathbb{N}$ (which also implies that $\widetilde{f}_{2,k}(q) \neq q$).

The next step consists of estimating the derivative of $\widetilde{f}_{1,k}$ at a point $x \in \widetilde{I}$. For $y = F^{-1}_{1,[a_k,b_k]}(x)$, we clearly have $\widetilde{f}'_{1,k}(x) = D^1_{g_{1,k}(y)} F_{1,[a_k,b_k]} g'_{1,k}(y) D^1_x F^{-1}_{1,[a_k,b_k]}$. Thus,

$$\begin{aligned} |\widetilde{f}'_{1,k}(x)| &\leq |D^1_{g_{1,k}(y)} F_{1,[a_k,b_k]} - D^1_y F_{1,[a_k,b_k]}| \, |g'_{1,k}(y) D^1_x F^{-1}_{1,[a_k,b_k]}| + |g'_{1,k}(y)| \\ &\leq \sup_{[a_k,b_k]} |D^2 F_{1,[a_k,b_k]}| \, |g_{1,k}(y) - y| \, g'_{1,k}(y) + g'_{1,k}(y) \, . \end{aligned}$$

On the other hand, $|b_k - a_k|$ is bounded by a uniform constant times 5^{-k}. Thus Lemma 3.3.7 yields

$$\sup_{[a_k,b_k]} |D^2 F_{1,[a_k,b_k]}| \leq \mathrm{const}\, 5^{-k \ln \beta} \, .$$

Therefore condition (4) ensures that $\sup_{[a_k,b_k]} |D^2 F_{1,[a_k,b_k]}| \, |g_{1,k}(y) - y|$ converges to zero as k goes to infinity. Since $\{g_{1,k}\}$ converges C^1 to the identity, there follows that the restriction of $\widetilde{f}_{1,k}$ to every compact part of $\widetilde{I}$ converges C^1 to the identity as well. The claim below shows that a similar phenomenon holds for the sequence $\{\widetilde{f}_{2,k}\}$ as well.

Claim 2. The sequence $\{\widetilde{f}_{2,k}\}$ converges C^1 to the identity on $\widetilde{J}$.

Proof of Claim 2. The argument is more subtle and builds on the previous discussion. Recalling that $q = h^{-1}(p)$, we set $q_k = F^{-1}_{2,[a_k,b_k]}(q)$. Let also $\lambda_k = g'_{2,k}(q_k)$. We also immediately note that Lemma 3.3.7 still yields $\sup_{[c_k,d_k]} |D^2 F_{2,[a_k,b_k]}| \leq \mathrm{const}\, 5^{-k \ln \beta}$ for a suitable constant const. For $z \in \widetilde{J}$ and $w = F^{-1}_{2,[a_k,b_k]}(z)$, the argument used above now provides

$$|\widetilde{f}'_{2,k}(z)| \leq \sup_{[c_k,d_k]} |D^2 F_{2,[a_k,b_k]}| \, |g_{2,k}(w) - w| \, g'_{2,k}(w) + g'_{2,k}(w) \, .$$

Again $\sup_{[c_k,d_k]} |D^2 F_{2,[a_k,b_k]}| \, |g_{2,k}(w) - w| \, g'_{2,k}(w)$ converges to zero so that $|\widetilde{f}'_{2,k}(z)|$ becomes arbitrarily close to $g'_{2,k}(w)$. In turn, owing to Lemma 3.3.4, the derivative $g'_{2,k}(w)$ becomes arbitrarily close to λ_k. Finally we can assume that λ_k converges to some $\overline{\lambda} \in \mathbb{R}$ for λ_k is uniformly bounded since the lengths of the intervals $\widetilde{f}_{2,k}(\widetilde{J})$ are clearly so. Summarizing what precedes, the sequence of maps $\{\widetilde{f}'_{2,k}\}$ converges uniformly on $\widetilde{J}$ to the constant $\overline{\lambda}$. To conclude that $\overline{\lambda} = 1$, just note that the sequence of primitives $\{\widetilde{f}_{2,k}\}$ converges uniformly to the identity on $\widetilde{J}$ (Claim 1). This ends the proof of Claim 2. □

To finish the proof of Theorem 3.3.1 we proceed as follows. Consider again the sequences of maps $\{\widetilde{f}_{1,k}\} \subset G_1$ and $\{\widetilde{f}_{2,k}\} \subset G_2$. By construction, we have $\widetilde{f}_{2,k} = h^{-1} \circ \widetilde{f}_{1,k} \circ h$ for every $k \in \mathbb{N}$. Furthermore $\{\widetilde{f}_{1,k}\}$ (resp. $\{\widetilde{f}_{2,k}\}$) converges C^1 to the identity on $\widetilde{I}$ (resp. $\widetilde{J}$). From this point, the standard argument relies on *synchronized vector fields* (see [R1]). This is as follows.

Recall that $\widetilde{f}_{1,k}(p) \neq p$ (resp. $\widetilde{f}_{2,k}(q) \neq q$) for every $k \in \mathbb{N}$. Moreover there are conjugate elements $\widetilde{F}_1 \in G_1$ and $\widetilde{F}_2 \in G_2$ which have hyperbolic fixed points in p and q, respectively. In suitable local coordinates around $p \simeq 0$ (resp. $q \simeq 0$), $\widetilde{F}_1$ becomes a homothety $x \mapsto \Lambda_1 x$ (resp. $\widetilde{F}_2$, $z \mapsto \Lambda_2 z$). Here both Λ_1 and Λ_2 belong to $(0,1)$. Consider the effect of the conjugations $\widetilde{F}_1^{-j} \circ \widetilde{f}_{1,k} \circ \widetilde{F}_1^{j}$ on $\widetilde{f}_{1,k}$ for k fixed and $j \in \mathbb{N}$. As there follows from [R1] (cf. also owing to Proposition 2.3.5 in Section 2.3) if $j(k)$ is a suitably chosen sequence with $j(k) \to \infty$, the conjugate diffeomorphisms $\widetilde{F}_1^{-j(k)} \circ \widetilde{f}_{1,k} \circ \widetilde{F}_1^{j(k)}$ and $\widetilde{F}_2^{-j(k)} \circ \widetilde{f}_{2,k} \circ \widetilde{F}_2^{j(k)}$ converge in the C^1-topology, respectively on $\widetilde{I}$ and $\widetilde{J}$, to non-trivial translations. Thus, we actually obtain non-zero constant vector fields $\overline{X}_1$ and $\overline{X}_2$ contained in the C^1-closures of G_1 and G_2, respectively, and whose flows ϕ_1^t and ϕ_2^t satisfy the equation

$$h \circ \phi_2^t(z) = \phi_1^t \circ h(z)$$

whenever both sides are well defined. By fixing z and letting t takes values around $0 \in \mathbb{R}$, we conclude that h is of class C^1 on a neighborhood of $z \in \widetilde{J}$. The fact that the dynamics of G_1 and G_2 are minimal then implies that h is of class C^1 on the entire circle. The proof of Theorem 3.3.1 is completed. □

Chapter 4

Ergodic theory and conjugate groups

We shall apply some probabilistic methods to the study of topologically conjugate groups of circle diffeomorphisms. In the course of this chaptre Proposition 3.3.3 will be proved. Theorem B in the introduction will also be proved here as an immediate consequence of Theorem A combined with Theorem 4.2.1.

4.1 Hyperbolic fixed points proposition

Throughout the section, we fix two topologically conjugate subgroups G_1 and G_2 of $\mathrm{Diff}^{\omega}(S^1)$. The group G_1 is assumed to be locally C^2-non-discrete and, in fact, it is assumed to satisfy all the conditions (1)–(4) in the beginning of Section 3.3. This section is devoted to the proof of Proposition 3.3.3.

Lemma 4.1.1 *None of the groups G_1 and G_2 leaves a probability measure on S^1 invariant.*

Proof. The statement holds for G_1 thanks to Lemma 3.3.2. The conclusion concerning G_2 then arises from the fact that these two groups are topologically conjugate. □

We also know that the each of the topologically conjugate groups G_1 and G_2 acts minimally on S^1 (i.e. all their orbits are dense). Next recall that a group G acting on S^1 is said to be *proximal* if every closed interval can be mapped to intervals of arbitrarily small length by means of elements of G. Since our groups have dense orbits, the condition of being proximal is equivalent to the existence of a sequence of intervals $\{I_k\} \subset S^1$ along with a sequence of elements $\{g_k\}$ in G such that the sequences formed by the lengths of the intervals in $\{I_k\}$ and in $\{S^1 \setminus g_k(I_k)\}$ both converge to zero. On the other hand, the fact that every point in S^1 is expandable (Lemma 3.3.2), makes it clear that arbitrarily small intervals of S^1 can always be expanded by the dynamics of our group beyond some uniform positive length. We refer to this property by saying that the group is *expansive*. As pointed out by Ghys in [G3], page 362, expansive groups acting minimally on S^1 are proximal *up to a finite quotient*. More precisely, if a non-abelian group $G \subset \mathrm{Diff}^{\omega}(S^1)$ is expansive and minimal but not proximal, then there exists a homeomorphism $\varsigma : S^1 \to S^1$ having finite order $\kappa(G) \geq 2$ which satisfies the following conditions:

- ς commutes with all elements of G. In particular, G inherits a natural action on the quotient S^1/ς.
- The action induced from G on S^1/ς is proximal.

At this point we can already prove Proposition 3.3.3.

Proof of Proposition 3.3.3. The proof is actually a by-product of the proof of Theorem F in [DKN-1]. The argument will be summarized below and the reader is referred to [DKN-1] for fuller detail. As it also happens in [DKN-1], we will first treat the case where G_1 (and hence G_2) is proximal. Since G_1 acts minimally on S^1, we only need to show the existence of a diffeomorphism $F_1 \in G_1$ having a hyperbolic fixed point at some point $p \in S^1$ and such that the corresponding diffeomorphism $F_2 = h^{-1} \circ F_1 \circ h$ in G_2 has a hyperbolic fixed point at $q = h^{-1}(p)$.

Consider a finite generating set L_1 for G_1 containing elements and their inverses (i.e. L_1 generates G_1 as semigroup). Denote by $L_2 = h^{-1} \circ L_1 \circ h$ the corresponding set in G_2. The sets L_1, L_2 can be put in natural correspondence with a finite set of letters Σ and, through this identification, we equip Σ with a probability measure μ that is symmetric (i.e. gives the same mass to an element and to its inverse) and non-degenerate (i.e. every element in Σ has strictly positive μ-mass). Denote by Ω the *shift space* $\Sigma^{\mathbb{N}}$ equipped with the standard shift map $\sigma : \Omega \to \Omega$ and with the probability measure $\mathbb{P}(\Sigma) = \mu^{\mathbb{N}}$. By a small abuse of notation, we shall identify μ with measures on L_1 and on L_2. Similarly $\mathbb{P}(\Sigma)$ (resp. σ) will also be thought of as a measure (resp. shift map) in either $L_1^{\mathbb{N}}$ or $L_2^{\mathbb{N}}$. Finally, we define maps T_1 and T_2 from $\Omega \times S^1$ to $\Omega \times S^1$ by letting $T_1(\omega, x) = (\sigma(\omega), \tilde{f}_1^1(x))$ and $T_2(\omega, x) = (\sigma(\omega), \tilde{f}_1^2(x))$ where $\tilde{f}_1^1$ (resp. $\tilde{f}_1^2$) is the projection of ω in the first copy of Σ viewed with the identifications corresponding to G_1 (resp. G_2).

Next denote by ν_1 (resp. ν_2) the stationary measure of G_1 (resp. G_2) obtained from μ. In other words, ν_1 (resp. ν_2) is a probability measure on S^1 whose value on a Borel set $\mathcal{B} \subset S^1$ is given by

$$\nu_1(\mathcal{B}) = \sum_{g \in G_1} \mu(g) \nu_1(g^{-1}(\mathcal{B}))$$

(resp. $\nu_2(\mathcal{B}) = \sum_{g \in G_2} \mu(g) \nu_2(g^{-1}(\mathcal{B}))$). These stationary measures ν_1 and ν_2 are unique after [DKN-1] complemented by Lemma 4.1.1. From the uniqueness of these stationary measures, there follows that $h^* \nu_1 = \nu_2$.

On the other hand, according to Furstenberg [Fu], for all continuous function ψ on S^1, the sequence of random variables on the probability space $(\Sigma^{\mathbb{N}}, \nu_1)$

$$\xi_{1,l}(\omega) = \int_{S^1} \psi d(\tilde{f}_1^1 \cdots \tilde{f}_l^1(\nu_1))$$

(resp. $\xi_{2,l}(\omega) = \int_{S^1} \psi d(\tilde{f}_1^2 \cdots \tilde{f}_l^2(\nu_2)))$ is a martingale so that both limits

$$\omega(\nu_1) = \lim_{l\to\infty} \tilde{f}_1^1 \cdots \tilde{f}_l^1(\nu_1) \quad \text{and} \quad \omega(\nu_2) = \lim_{l\to\infty} \tilde{f}_1^2 \cdots \tilde{f}_l^2(\nu_2)$$

exist for a subset of full $\mathbb{P}(\Sigma)$-measure of Σ. Now taking into account that G_1 (and hence G_2) is proximal, there follows that the resulting limit measure $\omega(\nu_1)$ (resp. $\omega(\nu_2)$) is a Dirac mass, as originally proved in [An]; see also [K-N] and Proposition 5.2 of [DKN-1]. A topological analogue of the last assertion can be obtained as follows. Define the *contraction coefficient* $c(g)$ of a diffeomorphism (homeomorphism) g of S^1 as the infimum over $\epsilon > 0$ for which there are closed intervals U and V of sizes not greater than ϵ and such that $g(\overline{S^1 \setminus U}) = V$. With this definition, the preceding argument on Dirac masses also implies that the contraction coefficients $c_l^1(\tilde{f}_l^1 \cdots \tilde{f}_1^1)$ and $c_l^2(\tilde{f}_l^2 \cdots \tilde{f}_1^2)$ converge to zero for a set of full $\mathbb{P}(\Sigma)$-measure of Σ (Proposition 5.3 of [DKN-1]).

The rest of the proof consists of repeating word-by-word the argument detailed in Section 4.4 of [DKN-1] (aimed at the proof of Theorem F in the mentioned paper). Indeed, for a generic choice of $\omega \in \Sigma^{\mathbb{N}}$, there are a sequence of intervals U_l^1, V_l^1 (resp. U_l^2, V_l^2) whose sizes converge to zero and such that

$$\tilde{f}_l^1 \cdots \tilde{f}_1^1(S^1 \setminus U_l^1) \subset V_l^1 \quad \text{and} \quad \tilde{f}_l^2 \cdots \tilde{f}_1^2(S^1 \setminus U_l^2) \subset V_l^2 \,.$$

When U_l^1, V_l^1 are disjoint then the fixed points of $\tilde{f}_l^1 \cdots \tilde{f}_1^1$ are contained in these intervals (and the analogous conclusion holds for $\tilde{f}_l^2 \cdots \tilde{f}_1^2 \in G_2$ since G_1 and G_2 are topologically conjugate). The argument in [DKN-1] then continues by showing first that U_l^1, V_l^1 are often disjoint. In a second moment, the authors use techniques of Lyapunov exponents to control the contraction rate so as to conclude that the fixed points are of hyperbolic nature. This ends the proof of the proposition provided that G_1 is proximal.

It remains to justify the case in which G_1 is not proximal. As previously explained, in this case there exists a homeomorphism of finite order $\varsigma_1 : S^1 \to S^1$ commuting with all elements of G_1 and such that the action induced by G_1 on S^1/ς_1 is proximal. This latter action is topologically conjugate to the action induced by G_2 on S^1/ς_2 where $\varsigma_2 = h^{-1} \circ \varsigma_1 \circ h$. Whereas these proximal actions are in general only continuous, the argument of probabilistic nature used above still applies to them. In turn, those arguments based on controlling the contraction rate and on uniform hyperbolicity can still be employed in the context of the original actions of G_1 and G_2 on S^1. The pair of intervals U_l^1, V_l^1 (resp. U_l^2, V_l^2) involved in the above discussion becomes then a $\kappa(G_1)$-tuple of pair of intervals $U_{l,j}^1$, $V_{l,j}^1$ (resp. $U_{l,j}^2$, $V_{l,j}^2$) where $j = 1, \ldots, \kappa(G_1)$. The arguments of controlled contraction rate can now be repeated to ensure the hyperbolic nature of the corresponding fixed points. The proof of the proposition is completed. □

4.2 Proof of the second main result "Theorem B"

This section is devoted to the proof of Theorem 4.2.1. Throughout the section, we fix two topologically conjugate subgroups G_1 and G_2 of $\text{Diff}^\omega(S^1)$. The group G_1 is assumed to be locally C^2-non-discrete and, in fact, it is assumed to satisfy all the conditions (1)–(4)

in the beginning of Section 3.3. Assume that the group G_2 acts minimally on S^1 and leaves no probability measure invariant, cf. Lemma 4.1.1. The Theorem 4.2.1 concerns the potential existence of topologically conjugate groups G_1 and G_2 acting on S^1 with G_1 being locally C^2-non-discrete whereas G_2 is locally C^2-discrete. This discussion will lead to the proof of Theorem B in the introduction.

Theorem B. *Suppose that Γ is a finitely generated hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$ and consider two topologically conjugate faithful representations $\rho_1 : \Gamma \to \mathrm{Diff}^\omega(S^1)$ and $\rho_2 : \Gamma \to \mathrm{Diff}^\omega(S^1)$ of Γ in $\mathrm{Diff}^\omega(S^1)$. Assume that $G_1 = \rho_1(\Gamma) \subset \mathrm{Diff}^\omega(S^1)$ is locally C^2-non-discrete. Assume also the existence of a non-degenerate measure μ on G_1 having finite entropy and giving rise to an absolutely continuous stationary measure ν_1 for G_1. Then every (orientation-preserving) homeomorphism $h : S^1 \to S^1$ conjugating the representations ρ_1 and ρ_2 coincides with an element of $\mathrm{Diff}^\omega(S^1)$.*

We begin by stating Theorem 4.2.1. For the rest of this section, Γ will always denote an abstract hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$. The notion of entropy for measures as those considered in the statement of Theorem 4.2.1 is also recalled below.

Theorem 4.2.1 *For Γ as above, let $\rho_1 : \Gamma \to \mathrm{Diff}^\omega(S^1)$ be a faithful representation of Γ in $\mathrm{Diff}^\omega(S^1)$ and set $G_1 = \rho_1(\Gamma)$. Assume that G_1 is locally C^2-non-discrete and that there is a non-degenerate measure with finite entropy on $\Gamma \simeq G_1$ giving rise to an absolutely continuous stationary measure ν_1 on S^1. Then every subgroup $G_2 \subset \mathrm{Diff}^\omega(S^1)$ topologically conjugate to G_1 is locally C^2-non-discrete as well.*

Proof of Theorem B. Just note that the statement follows immediately from the combination of Theorem A and Theorem 4.2.1. □

In turn, the proof of Theorem 4.2.1 relies on the combination of a few deep results including Theorem 1.1 of [De] and Kaimanovich's theorem in [Ka]. For suitable background on hyperbolic groups and on measure theoretic methods in group theory, the reader is referred to [Ka], [Ve], and [G-H](see Section 1.2 for proper definitions).

Let then Γ and $G_1 = \rho_1(\Gamma)$ be as above so that G_1 is locally C^2-non-discrete. Moreover, by assumption, there is a non-degenerate measure of finite entropy on $\Gamma \simeq G_1$ leading to an absolutely continuous stationary measure for the action of G_1 on S^1. The existence of absolutely continuous stationary measures however is not needed until we effectively start the proof of Theorem 4.2.1 and for this reason we shall conduct a more general discussion for the time being.

In the sequel we assume by way of contradiction that the statement of Theorem 4.2.1 is false. Thus we can assume the existence of a locally C^2-discrete group $G_2 \subset \mathrm{Diff}^\omega(S^1)$ which is topologically conjugate to G_1. The proof of Theorem 4.2.1 relies heavily on properties of stationary measures and the structure of our argument can be described as follows (see below for proper definitions). First we can assume without loss of generality that G_2 is indeed locally C^1-discrete; cf. Lemma 4.2.2. We consider then a suitable measure μ on $\Gamma \simeq G_1 \simeq G_2$ (non-degenerate and with finite entropy) and denote by ν_1 (resp. ν_2) the corresponding stationary measure for the action of G_1 (resp. G_2) on S^1. Owing to a result due to Deroin [De], we know that the Furstenberg boundary of G_2 can essentially be identified with (S^1, ν_2). On the other hand, Kaimanovich [Ka] shows that this Furstenberg boundary can also be modeled by the geometric boundary $\partial\Gamma$ of the hyperbolic group Γ. Putting all these identifications together, we obtain a measurable isomorphism between the action of G_1 on (S^1, ν_1) and the action of Γ on $\partial\Gamma$. The action of Γ in $\partial\Gamma$ is however "locally discrete" in a C^0-sense (Lemma 4.2.4) whereas the action of G_1 on S^1 is locally non-discrete. A priori this is not a contradiction since the equivariant map between S^1 and $\partial\Gamma$ is only measurable. However, if ν_1 is absolutely continuous, the classical Lusin theorem yields topological constraints on the measurable map in question and these constraints are sufficient to derive the desired contradiction.

The preceding discussion will be made accurate in what follows. Let then Γ, G_1, and G_2 be as above and recall that ρ_1 stands for the representation $\rho_1 : \Gamma \to G_1 \subset \mathrm{Diff}^\omega(S^1)$. By post-composing ρ_1 with a conjugating homeomorphism h, we obtain another faithful representation $\rho_2 : \Gamma \to \mathrm{Diff}^\omega(S^1)$ satisfying

$$\rho_2(\gamma) = h^{-1} \circ \rho_1(\gamma) \circ h$$

for every $\gamma \in \Gamma$ and where $G_2 = \rho_2(\Gamma)$. In other words, the representations ρ_1 and ρ_2 are topologically conjugated by h. We begin with the following lemma.

Lemma 4.2.2 *Without loss of generality we can assume that the group G_2 is locally C^1-discrete.*

Proof. The proof relies on Lemma 3.3.2. In fact, according to this lemma, the only possibility for G_2 being locally C^2-discrete occurs when G_2 has a non-expanding point. Hence to prove the lemma it suffices to check that a locally C^1-non-discrete subgroup of $\mathrm{Diff}^\omega(S^1)$ expands every point $p \in S^1$.

Consider then a diffeomorphism $F_2 \in G_2$ having a hyperbolic fixed point $q \in S^1$. In local coordinates around $q \simeq 0$, we then have $F_2(x) = \lambda x$ for some $\lambda \in (0, 1)$. Next, suppose that G_2 is locally C^1-non-discrete. By using the minimal character of G_2, we then obtain a sequence $g_{2,j}$ of diffeomorphisms in G_2 ($g_{2,j} \neq \mathrm{id}$ for all $j \in \mathbb{N}$) which converges to the identity on a small interval $(-\varepsilon, \varepsilon)$ around $q \simeq 0$ (for some $\varepsilon > 0$). Again the discussion of Proposition 2.3.5 in Section 2.3 allows us to assume that $g_{2,j}(0) \neq 0$ for every $j \in \mathbb{N}$. Thus, as shown in the end of the proof of Theorem 3.3.1, there is a sequence of positive integers $m(j) \to \infty$ such that the corresponding diffeomorphisms $F_2^{-m(j)} \circ g_{2,j} \circ F_2^{m(j)}$ converge in the C^1-topology on $(-\varepsilon, \varepsilon)$ to a non-trivial translation. There also follows that the vector field $\partial/\partial x$ on $(-\varepsilon, \varepsilon)$ is contained in the C^1-closure of G_2. Since $q \simeq 0 \in (-\varepsilon, \varepsilon)$ is clearly expanding for G_2, there immediately follows that every point in $(-\varepsilon, \varepsilon)$ must be expanding for G_2. The lemma follows since G_2 acts minimally on S^1. □

Now it is convenient to revisit the notion of stationary measures in fuller detail. Consider a finite generating set $A = \{\overline{\gamma}_1, \ldots, \overline{\gamma}_r, \overline{\gamma}_1^{-1}, \ldots, \overline{\gamma}_r^{-1}\}$ for Γ containing elements and their inverses so that A generates Γ as semigroup. Given a measure μ on Γ, recall that the *entropy of* μ is defined by

$$H(\mu) = -\sum_{\gamma\in\Gamma} \mu(\gamma)\ln\mu(\gamma). \tag{4.1}$$

We then fix some non-degenerate, probability measure μ on Γ which has finite entropy and gives strictly positive mass to every element of A. Note that the measure μ *is not* required to be symmetric and, in addition, the set A can be strictly contained in the support of μ. As mentioned the possibility of choosing μ as before so as to have an absolutely continuous stationary measure for the group G_1 will only be exploited later on.

Now denote by $\partial\Gamma$ the geometric boundary of the hyperbolic group Γ, see [G-H]. The boundary $\partial\Gamma$ is a compact metric space which is effectively acted upon by the group Γ itself. Thus we often identify an element $\gamma\in\Gamma$ with the corresponding automorphism of $\partial\Gamma$ (still denoted by γ).

Since Γ is endowed with the measure μ, a unique *stationary measure* ν_Γ on $\partial\Gamma$ is associated to the action of Γ on $\partial\Gamma$ (cf. [Ka]). In other words, for every Borel set $\mathcal{B}\subset\partial\Gamma$, we have

$$\nu_\Gamma(\mathcal{B}) = \sum_{\gamma\in\Gamma} \mu(\gamma)\nu_\Gamma(\gamma^{-1}(\mathcal{B}))$$

where $\gamma(\mathcal{B})$ refers to the identification of $\gamma\in\Gamma$ with the corresponding automorphism of $\partial\Gamma$.

Next let $g_{1,i}\in G_1$ (resp. $g_{2,i}\in G_2$) be defined as $g_{1,i}=\rho_1(\overline{\gamma}_i)$ (resp. $g_{2,i} = \rho_2(\overline{\gamma}_i)$), $i=1,\ldots,r$. We also pose $A_1=\{g_{1,1},\ldots,g_{1,r},g_{1,1}^{-1},\ldots,g_{1,r}^{-1}\}$ (resp. $A_2 = \{g_{2,1},\ldots,g_{2,r},g_{2,1}^{-1},\ldots,g_{2,r}^{-1}\}$). Since both representations ρ_1 and ρ_2 from Γ to $\mathrm{Diff}^\omega(S^1)$ are one-to-one, the groups G_1 and G_2 become equipped with the probability measure μ up to the evident identifications.

Going back to the action of G_1 on S^1, Lemma 3.3.2 allows us to apply the main theorem of [DKN-1] to ensure the existence of a unique stationary measure ν_1 for G_1 (with respect to μ). The support of ν_1 is all of S^1 since G_1 is minimal. It is also well known that G_1 gives no mass to points. Analogous conclusions hold for the stationary measure ν_2 on S^1 arising from G_2 and μ. Now the combination of [De] with [Ka] yields the following.

Lemma 4.2.3 *There is a measurable isomorphism θ_2 from $(\partial\Gamma,\nu_\Gamma)$ to (S^1,ν_2).*

Proof. Whereas G_2 was initially assumed to be locally C^2-discrete, Lemma 4.2.2 shows that G_2 is, in fact, locally C^1-discrete. Recalling that the measure μ is assumed to have finite entropy, we apply Theorem 1.1 of [De] to the action of G_2 on S^1. Since G_2 is locally C^1-discrete and μ has finite entropy, all the conditions required by the theorem in question are satisfied so that the Poisson boundary of G_2 coincides with its (G_2,μ)-boundary (see [De], [C-M] for terminology).

In turn, Kaimanovich theorem in [Ka] ensures that the Poisson boundary of G_2 can be identified with $(\partial\Gamma,\nu_\Gamma)$ (recall that G_2 is isomorphic to the fixed hyperbolic group Γ).

Thus, to complete the proof of the lemma, it suffices to show that (G_2, μ)-boundary of G_2 can be identified with (S^1, ν_2). For G_2 proximal (and leaving no probability measure invariant, see Lemma 3.3.2), this is exactly the contents of [An] and [K-N]. In the general case, we have seen there is a finite topological quotient of S^1 where G_2 induces a proximal action. This quotient in endowed with a unique stationary measure ν_2'. The pair (S^1, ν_2') is the (G, μ)-boundary of the quotient owing to the result of Antonov and Kleptsyn-Nal'ski. Finally, the (G_2, μ)-boundary of G_2 can then be identified with S^1 equipped with the pull-back (still denoted by ν_2) of ν_2' by the projection map. This completes the proof of the lemma. □

It is implicitly understood in the statement of Lemma 4.2.3 that θ_2 is Γ-equivariant in the sense that $\theta_2^* \nu_2 = \nu_\Gamma$ and

$$\theta_2 \circ \gamma(x) = \rho_2(\gamma) \circ \theta_2(x) \tag{4.2}$$

for every $\gamma \in \Gamma$ and ν_Γ-almost all point $x \in \partial\Gamma$. We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $h : S^1 \to S^1$ be a homeomorphism conjugating G_1 to G_2. By way of contradiction, we have assumed that G_1 is locally C^2-non-discrete whereas G_2 is locally C^2-discrete. From now on we fix μ on $\Gamma \simeq G_1 \simeq G_2$ satisfying the previous conditions and such that, in addition, the corresponding stationary measure ν_1 for G_1 is absolutely continuous.

Recall also that ν_1 (resp. ν_2) is the unique stationary measure for G_1 (resp. G_2) with respect to μ (see [DKN-1]). From the uniqueness of the stationary measure there follows again that $h^* \nu_1 = \nu_2$.

Consider the measurable isomorphism $\theta_2 : (\partial\Gamma, \nu_\Gamma) \longrightarrow (S^1, \nu_2)$ of Lemma 4.2.3 and define a new measurable isomorphism $\theta_1 : (\partial\Gamma, \nu_\Gamma) \longrightarrow (S^1, \nu_1)$ by letting $\theta_1 = h \circ \theta_2$. The equivariant nature of θ_2 expressed by Equation (4.2) combines with the fact that $h^* \nu_1 = \nu_2$ to yield

$$\theta_1 \circ \gamma(x) = \rho_1(\gamma) \circ \theta_1(x) \tag{4.3}$$

for every $\gamma \in \Gamma$ and ν_Γ-almost all point $x \in \partial\Gamma$. Furthermore $\theta_1^* \nu_1 = \nu_\Gamma$. Up to eliminating null measure sets, we fix once and for all a Borel set $\mathcal{B} \subset \partial\Gamma$ having full ν_Γ-measure and such that Equation (4.3) holds for every $x \in \mathcal{B}$ and every $\gamma \in \Gamma$ (in particular both sides of this equation are well defined). To complete the proof of the proposition, we are going to show that the existence of θ_1 is not compatible with the fact that G_1 is locally C^2-non-discrete. To do this, we proceed as follows.

Fix an interval $I \subset S^1$ along with a sequence of elements $\{g_j\} \subset G_1$, $g_j \neq \mathrm{id}$ for every $j \in \mathbb{N}$, whose restrictions to I converge to the identity in the C^2-topology. The existence of I and of $\{g_j\}$ clearly follows from the assumption that G_1 is locally C^2-non-discrete. Now Lusin approximation theorem [Bt] ensures the existence of a Cantor set K satisfying the following conditions:

1. $K \subset I \cap \theta_1(\mathcal{B})$, i.e. K is contained in the domain of definition of θ_1^{-1}.

2. The *restriction* of θ_1^{-1} to K is continuous from K to $\partial\Gamma$ (where the reader is reminded that $\partial\Gamma$ is a compact metric space).

3. $\nu_1(K) \geq 9\nu_1(I)/10$.

Next, for each j, let $\gamma_j \in \Gamma$ be such that $\rho_1(\gamma_j) = g_j$.

Claim. There is a Cantor set $K_\Gamma \subset \partial\Gamma$ such that the restrictions of the elements γ_j to K_Γ converge uniformly to the identity.

Proof of the Claim. Since $\{g_j\}$ converges to the identity in the C^1-topology and ν_1 is absolutely continuous, there follows that $\nu_1(K \cap g_j^{-1}(K))$ converges to $\nu_1(K)$ as $j \to \infty$. Therefore, up to passing to a subsequence, we can assume that

$$K_\infty = K \cap \bigcap_{j=1}^{\infty} g_j^{-1}(K)$$

is an actual (non-empty) Cantor set. Furthermore, by construction, $K_\infty \subset K$ and $g_j(K_\infty) \subset K$ for every $j \in \mathbb{N}^*$. Finally let $K_\Gamma = \theta_1^{-1}(K_\infty)$.

To complete the proof of the claim, note that the restriction of θ_1 to K_Γ is continuous since θ_1^{-1} is continuous and one-to-one on the Cantor set K (and $K_\infty \subset K$). On the other hand, on K_Γ we have

$$\gamma_j = \theta_1^{-1} \circ g_j \circ \theta_1$$

i.e. the left hand side is well defined on K_Γ. Since θ_1 is continuous on K_Γ and θ_1^{-1} is continuous on $g_j \circ \theta_1(K_\Gamma) \subset K$, the fact that g_j converges uniformly (and actually C^1) to the identity implies the claim. □

We have just found a sequence $\{\gamma_j\}$ of elements in Γ, $\gamma_j \neq \mathrm{id}$ for every j, whose restrictions to a (non-empty) Cantor set $K_\Gamma \subset \partial\Gamma$ converge uniformly to the identity. The theorem now will follow from Lemma 4.2.4 below claiming that such a sequence cannot exist in a finitely generated hyperbolic group. □

To state Lemma 4.2.4 recall that every element $\gamma \in \Gamma$ can be identified with the corresponding automorphism of $\partial\Gamma$. Naturally γ can equally well be identified with its translation action on Γ which happens to be an isometry for the natural left-invariant metric on Γ (see [G-H]).

Lemma 4.2.4 *Let Γ be a hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$. Let K_Γ be a Cantor set contained in the boundary $\partial\Gamma$ of Γ and let $\{\gamma_j\}$ be a sequence of elements in Γ thought of as automorphisms of $\partial\Gamma$. Assume that the sequence $\{\gamma_{j,|K_\Gamma}\}$ obtained by restricting γ_j to K_Γ converges uniformly to the identity. Then we have $\gamma_j = \mathrm{id}$ for large enough $j \in \mathbb{N}$.*

Proof. The lemma is certainly well known to the specialists albeit we have not been able to find it explicitly stated in the literature. In the sequel, the reader is referred to the chapters 7 and 8 of [G-H] for background material.

Assume for a contradiction that $\gamma_j \neq \mathrm{id}$ for every $j \in \mathbb{N}$. Consider also a base point $w \in \Gamma$ along with the sequence $\gamma_j(w)$. Since γ_j acts as an isometry of Γ, there follows that the sequence $\{\gamma_j(w)\}$ leaves every compact part of Γ. Thus, up to a passing to a subsequence, we assume that $\gamma_j(w) \to b \in \partial\Gamma$.

Next fix another point $a \in \partial\Gamma \setminus K_\Gamma$, $a \neq b$, and consider the family of metrics $d_{\varepsilon,a,w'}$ on $\partial\Gamma \setminus \{a\}$ for a fixed (small) $\varepsilon > 0$ and where $w' \in \Gamma$ (see [G-H], page 141). Let β_a denote the Busemann function relative to the point $a \in \partial\Gamma$. Since $\gamma_j(w) \to b$, with $b \neq a$, there follows from the general properties of Busemann functions that

$$\beta_a(w, \gamma_j(w)) \longrightarrow -\infty$$

(cf. [G-H] page 136). In particular, there is some uniform constant C such that

$$\frac{1}{C}\exp(-\varepsilon\beta_a(w,\gamma_j(w))) \leq \frac{d_{\varepsilon,a,\gamma_j(w)}(x,y)}{d_{\varepsilon,a,w}(x,y)} \leq C\exp(-\varepsilon\beta_a(w,\gamma_j(w)))\,;$$

see [G-H], page 141. In other words, the metric $d_{\varepsilon,a,\gamma_j(w)}$ is bounded from below and by above by the metric $d_{\varepsilon,a,w}$ multiplied by suitable constants going to infinity as $j \to \infty$. However, by construction, these metrics also satisfy $d_{\varepsilon,a,\gamma_j(w)}(\gamma_j(x),\gamma_j(y)) = d_{\varepsilon,a,w}(x,y)$. Therefore

$$\frac{d_{\varepsilon,a,w}(x,y)}{d_{\varepsilon,a,w}(\gamma_j(x),\gamma_j(y))} \longrightarrow \infty$$

uniformly for every pair $x \neq y$ in $\partial\Gamma \setminus \{a\}$. The desired contradiction now arises by choosing $x \neq y \in K_\Gamma$ so that $\gamma_j(x) \to x$ and $\gamma_j(y) \to y$. The proof of the lemma is completed. $\square$

Appendix

On locally C^r-non-discrete groups

For $r \geq 2$, every subgroup G of $\mathrm{Diff}^{\omega}(S^1)$ that is locally C^r-non-discrete is clearly locally C^l-non-discrete for every $l \leq r$. A sort of converse for the above claim also holds in most cases. This is the content of the theorem below.

Theorem A.0.5 *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a non-solvable group and assume that G is locally C^2-non-discrete. Then G is locally C^{∞}-non-discrete.*

To prove Theorem A.0.5 we shall use the same technique of regularization (or renormalization) employed in Section 3.2. By assumption there is an open (non-empty) interval $I \subset S^1$ and a sequence $\{f_j\}$, $f_j \neq \mathrm{id}$ for every $j \in \mathbb{N}$, of elements in G whose restrictions to I converge to the identity in the C^2-topology. In fact, arguing as in Section 3.1, we can assume without loss of generality that the following holds: for every given $\varepsilon > 0$, there is a finite set $\overline{f}_1, \ldots, \overline{f}_N$ of elements in G satisfying the two conditions below.

- The group $G_{(\varepsilon,N)} \subset G$ generated by $\overline{f}_1, \ldots, \overline{f}_N$ is not solvable.
- For every $i \in \{1, \ldots, N\}$, the restrictions of $\overline{f}_i$ and of $\overline{f}_i^{-1}$ to the interval I are ε-close to the identity in the C^2-topology on I.

First we state:

Proposition A.0.6 *If $\varepsilon > 0$ is small enough, then the group $G_{(\varepsilon,N)}$ is locally C^r-non-discrete for every $r \in \mathbb{N}$.*

The proof of Theorem A.0.5 can be derived from Proposition A.0.6 as follows.

Proof of Theorem A.0.5. We can assume once and for all that $G_{(\varepsilon,N)}$ has no finite orbits, otherwise Theorem A.0.5 follows at once from the discussion in Section 2.3. In turn, it is clearly sufficient to prove that the subgroup $G_{(\varepsilon,N)}$ is locally C^{∞}-non-discrete provided that $\varepsilon > 0$ is small enough. This is equivalent to finding an open, non-empty interval $I_{\infty} \subset S^1$ on which "$G_{(\varepsilon,N)}$ is locally C^r-non-discrete for every $r \in \mathbb{N}$". More precisely, for every fixed $r \in \mathbb{N}$, there is a sequence $\{f_{j,C^r}\}_{j \in \mathbb{N}}$, $f_{j,C^r} \neq \mathrm{id}$ for every $j \in \mathbb{N}$, of elements in $G_{(\varepsilon,N)}$ whose restrictions to I_{∞} converge to the identity in the C^r-topology.

On the other hand, by assumption, to every $r \in \mathbb{N}$ there corresponds a non-trivial sequence $\{\tilde{f}_{j,C^r}\}_{j \in \mathbb{N}}$ of elements in $G_{(\varepsilon,N)}$ whose restriction to some open, non-empty interval I_r converges to the identity in the C^r-topology on I_r. Thus the only difficulty to derive Theorem A lies in the fact that the intervals I_r depend on r. To show that these intervals can be chosen in a uniform way, we proceed as follows.

First recall that $G_{(\varepsilon,N)}$ contains an element F exhibiting a hyperbolic fixed point. Furthermore S^1 can be covered by finitely many intervals $J_1, \ldots, J_l$ such that each interval J_i is equipped with a constant (non-zero) vector field X_i in the C^1-closure of $G_{(\varepsilon,N)}$; cf. Theorem 3.4 of [R5] (which, in particular, recovers the fact that all orbits of G are dense in S^1). By using these constant vector fields and the diffeomorphism F, we obtain a sequence F_r of elements in $G_{(\varepsilon,N)}$ satisfying the following conditions:

- The diffeomorphism F_r has an attracting hyperbolic fixed point p_r lying in I_r.
- The basin of attraction of p_r with respect to F_r has length greater than a certain $\delta > 0$ (in other words, there is $\delta > 0$ such that F_r has no other fixed point on a δ-neighborhood of p_r).

Now each interval I_r can be "re-scaled" by means of F_r so as to have length bounded from below by δ. More precisely, fixed r and $n_r \in \mathbb{N}$, the sequence of elements of $G_{(\varepsilon,N)}$ given by $j \mapsto F_r^{-n_r} \circ \tilde{f}_{j,C^r} \circ F_r^{n_r}$ clearly converges to the identity in the C^r-topology on the interval $\tilde{I}_r = F_r^{-n_r}(I_r)$. The above stated conditions on the diffeomorphisms F_r then ensure that n_r can be chosen so that $\tilde{I}_r = F_r^{-n_r}(I_r)$ has length bounded from below by $\delta > 0$. Up to passing to a subsequence, the sequence of intervals $\{\tilde{I}_r\}$ must converge to a uniform interval I_∞ satisfying the desired conditions. The proof of Theorem A.0.5 is completed. □

As in Section 3.2, we consider the sequence of sets $S(k)$ defined by means of the initial set $S = S(0) = \{\overline{f}_1, \ldots, \overline{f}_N\}$. Since the group generated by $\overline{f}_1, \ldots, \overline{f}_N$ is not solvable, none of the sets $S(k)$ is reduced to the identity diffeomorphism.

We can now prove Proposition A.0.6.

Proof of Proposition A.0.6. The proof is essentially by induction. First we are going to prove that $G_{(\varepsilon,N)}$ is locally C^3-non-discrete. To do this, we proceed as follows. Consider a fixed set $\{\overline{f}_1, \ldots, \overline{f}_N\}$ generating a non-solvable group $G_{(\varepsilon,N)}$ as before. Assume moreover that for every $i = 1, \ldots, N$, both diffeomorphisms $\overline{f}_i$ and $\overline{f}_i^{-1}$ are ε-close to the identity in the C^2-topology on I where the value of $\varepsilon > 0$ will be fixed later on.

As already seen, the group $G_{(\varepsilon,N)}$ contains an element F exhibiting a hyperbolic fixed point in I. Without loss of generality, we can assume that this fixed point coincides with $0 \in I \subset \mathbb{R}$. Furthermore in suitable coordinates, F becomes a homothety $x \mapsto \lambda x$ on all of the interval I. Still keeping the notation of Section 3.2, consider the sequence of sets $\widetilde{S}(k)$ given by $\widetilde{S}(k) = F^{-kn} \circ S(k) \circ F^{kn}$ for some $n \in \mathbb{N}^*$ fixed. We will show that the diffeomorphisms in $\widetilde{S}(k)$ converge to the identity in the C^3-topology on I provided that n is suitably chosen.

Claim. There is $n \in \mathbb{N}$ such that every non-trivial sequence $\{\tilde{f}_k\}$, with $\tilde{f}_k \in \widetilde{S}(k)$, converges to the identity in the C^3-topology on I.

Proof of the Claim. Fix a sequence $\{\tilde{f}_k\}$ as in the statement. In Section 3.2 it was seen that these elements converge to the identity in the C^2-topology. More precisely, we have

$$\|\tilde{f}_k - \mathrm{id}\|_{2,I} < \frac{\varepsilon}{\sqrt{2^k}} \tag{4}$$

for every diffeomorphism $\tilde{f}_k \in \widetilde{S}(k)$ and for a suitable fixed n. To show that convergence takes place in the C^3-topology as well, we first estimate the third derivative $D^3[f_1, f_2]$ of a commutator $[f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1}$. For this we shall use the fact that f_1, f_2 and their inverses f_1^{-1}, f_2^{-1} are C^2-close to the identity. Recall then that higher order derivatives of a composed function are given by Faà di Bruno formula which, in the present case, simply means

$$D^3(f_1 \circ f_2) = D^3_{f_2(x)} f_1 . (D_x f_2)^3 + 3 D^2_{f_2(x)} f_1 . D^2_x f_2 . D^1_x f_2 + D^1_{f_2(x)} . D^3_x f_2 \,. \tag{5}$$

Thus, if $\varepsilon > 0$ is sufficiently small, we have

$$\begin{aligned} |D^1(f_1 \circ f_2) - 1| &\leq 3 \max\{\sup_I |D^1(f_1 - \mathrm{id})|, \sup_I |D^1(f_2 - \mathrm{id})|\}; \\ D^2(f_1 \circ f_2) &\leq 3 \max\{\sup_I |D^2 f_1|, \sup_I |D^2 f_2|\}; \\ D^3(f_1 \circ f_2) &\leq 3 \max\{\sup_I |D^3 f_1|, \sup_I |D^3 f_2|\}. \end{aligned}$$

Similar estimates also hold for $D^1(f_1^{-1} \circ f_2^{-1})$, $D^2(f_1^{-1} \circ f_2^{-1})$, and $D^3(f_1^{-1} \circ f_2^{-1})$. If $\varepsilon > 0$ is small enough, then the preceding estimates can also be applied to $(f_1 \circ f_2) \circ (f_1^{-1} \circ f_2^{-1})$ so as to yield

$$D^3[f_1, f_2] \leq 10 \max\{\sup_I |D^3 f_1|, \sup_I |D^3 f_2|, \sup_I |D^3 f_1^{-1}|, \sup_I |D^3 f_2^{-1}|\} \tag{6}$$

provided that f_1, f_2, f_1^{-1}, and f_2^{-1} are ε-close to the identity in the C^2-topology. From Estimate (6), there follows that

$$\begin{aligned} D^3(F^{-n} \circ [f_1, f_2] \circ F^n) &= D^3(\lambda^{-n} . [f_1, f_2](\lambda^n x)) \\ &\leq 10 \lambda^{2n} \max\{\sup_I |D^3 f_1|, \sup_I |D^3 f_2|, \sup_I |D^3 f_1^{-1}|, \sup_I |D^3 f_2^{-1}|\} \,. \end{aligned}$$

If n is chosen so that $\lambda^{2n} < 1/10$, there follows that the third order derivatives of elements in $\widetilde{S}(1)$ are smaller than the maximum of the third order derivatives of elements in $S(0)$. This procedure can be iterated to higher order commutators by virtue of Estimate (4) so that third order derivatives of elements in $\widetilde{S}(k)$ actually decay geometrically with k. The claim results at once. □

The remainder of the proof of Proposition A.0.6 is a straightforward induction step. By repeating the previous discussion, we just need to prove that a locally C^r-non-discrete group is also locally C^{r+1}-non-discrete provided that $r \geq 2$. The argument is totally analogous to the one employed in the proof of the above claim (the general Faà di Bruno formulas can be used in the context). □

• **Final comments**. We close this Appendix by pointing out a couple of specific issues involved in our regularization scheme for iterated commutators, as explained above and in Section 3.2. First, the reader will note that the analytic assumption is not needed in order to ensure the corresponding diffeomorphisms converge to the identity. The importance of the analytic assumption lies in the fact that the sequence of sets $S(k)$ (and hence $\widetilde{S}(k)$) does not degenerate into $\{\mathrm{id}\}$. As mentioned this result is due to Ghys [G1] and has a formal algebraic nature: it depends on ensuring that a C^∞-diffeomorphism f of S^1 coincides with the identity so long there is a point in S^1 at which f is C^∞-tangent to the identity. It would be nice to know whether or not there are finitely generated pseudo-solvable, yet non-solvable, groups in $\mathrm{Diff}^\infty(S^1)$.

Finally note also that our regularization technique falls short of working in the C^1-case. Therefore, even in the analytic category, we have not proved that a locally C^1-non-discrete subgroup of $\mathrm{Diff}^\omega(S^1)$ is also locally C^∞-non-discrete. Although this statement is very likely to hold, the renormalization procedure $x \mapsto \lambda x$ used here does not decrease the first order derivative of the diffeomorphism and this accounts for the special nature of locally C^1-non-discrete groups. To overcome this difficulty, our iteration scheme must be further elaborated. This can probably be done by suitably adding further "take the commutator" steps so as to keep control on the growing rate of first order derivatives.

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