THE CAMASSA-HOLM EQUATION AS THE LONG-WAVE LIMIT OF THE IMPROVED BOUSSINESQ EQUATION AND OF A CLASS OF NONLOCAL WAVE EQUATIONS

H.A. Erbay* and S. Erbay

Department of Natural and Mathematical Sciences, Faculty of Engineering, Ozyegin University, Cekmekoy 34794, Istanbul, Turkey

A. Erkip

Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla 34956, Istanbul, Turkey

Abstract

In the present study we prove rigorously that in the long-wave limit, the unidirectional solutions of a class of nonlocal wave equations to which the improved Boussinesq equation belongs are well approximated by the solutions of the Camassa-Holm equation over a long time scale. This general class of nonlocal wave equations model bidirectional wave propagation in a nonlocally and nonlinearly elastic medium whose constitutive equation is given by a convolution integral. To justify the Camassa-Holm approximation we show that approximation errors remain small over a long time interval. To be more precise, we obtain error estimates in terms of two independent, small, positive parameters ϵ and δ measuring the effect of nonlinearity and dispersion, respectively. We further show that similar conclusions are also valid for the lower order approximations: the Benjamin-Bona-Mahony approximation and the Korteweg-de Vries approximation.

1 Introduction

In the present paper we rigorously prove that, in the long-wave limit and on a relevant time interval, the right-going solutions of both the improved Boussinesq (IB) equation

$$u_{tt} - u_{xx} - \delta^2 u_{xxtt} - \epsilon(u^2)_{xx} = 0, \tag{1}$$

and, more generally, the nonlocal wave equation

$$u_{tt} = \beta_\delta * (u + \epsilon u^2)_{xx} \tag{2}$$

are well approximated by the solutions of the Camassa-Holm (CH) equation

$$w_t + w_x + \epsilon w w_x - \frac{3}{4}\delta^2 w_{xxx} - \frac{5}{4}\delta^2 w_{xxt} - \frac{3}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}) = 0. \tag{3}$$

In the above equations, $u = u(x,t)$ and $w = w(x,t)$ are real-valued functions, ϵ and δ are two small positive parameters measuring the effect of nonlinearity and dispersion, respectively, the symbol $*$ denotes convolution in the x-variable, $\beta_\delta(x) = \frac{1}{\delta}\beta(\frac{x}{\delta})$ is the kernel function. It should be noted that (3) can be written in a more standard form by means of a coordinate transformation. That is, in a moving frame defined by $\bar{x} = \frac{2}{\sqrt{5}}(x - \frac{3}{5}t)$ and $\bar{t} = \frac{2}{3\sqrt{5}}t$, (3) becomes

$$v_{\bar{t}} + \frac{6}{5}v_{\bar{x}} + 3\epsilon v v_{\bar{x}} - \delta^2 v_{\bar{t}\bar{x}\bar{x}} - \frac{9}{5}\epsilon\delta^2(2v_{\bar{x}}v_{\bar{x}\bar{x}} + v v_{\bar{x}\bar{x}\bar{x}}) = 0, \tag{4}$$

with $v(\bar{x},\bar{t}) = w(x,t)$. Also, by the use of the scaling transformation $U(X,\tau) = \epsilon u(x,t),\ \ x = \delta X,\ \ t = \delta\tau$, (1) and (3) can be written in a more standard form with no parameters, but the above forms of (1) and (3) are more suitable to deal with small-but-finite amplitude long wave solutions.

In the literature, there have been a number of works concerning rigorous justification of the model equations derived for the unidirectional propagation of long waves from nonlinear wave equations modeling various physical systems. One of these model equations is the CH equation [4, 14, 15] derived for the unidirectional propagation of long water waves in the context of a shallow water approximation to the Euler equations of inviscid incompressible fluid flow. The CH equation has attracted much attention from researchers over the years. The two main properties of the CH equation are: it is an infinite-dimensional completely integrable Hamiltonian system and it captures wave-breaking of water waves (see [5, 6, 7, 17] for details). A rigorous justification of the CH equation for shallow water waves was given in [7].

In a recent study [11], the CH equation has been also derived as an appropriate model for the unidirectional propagation of long elastic waves in an infinite, nonlocally and nonlinearly elastic medium (see also [12]). The constitutive behavior of the nonlocally and nonlinearly elastic medium is described by a convolution integral (we refer the reader to [9, 10] for a detailed description of the nonlocally and nonlinearly elastic medium) and in the case of quadratic nonlinearity the one-dimensional equation of motion reduces to the nonlocal equation given in (2). Moreover, the nonlocal equation (that is, the equation of motion for the medium) reduces to the IB equation (1) for a particular choice of the kernel function appearing in the integral-type constitutive relation (see Section 5 for details). In order to derive formally the CH equation from the IB equation, an asymptotic expansion valid as nonlinearity and dispersion parameters, that is ϵ and δ, tend to zero independently is used in [11]. It has been also pointed out that a similar formal derivation of the CH equation is possible by starting from the nonlocal equation (2).

The question that naturally arises is under which conditions the unidirectional solutions of the nonlocal equation are well approximated by the solutions of the CH equation and this is the subject of the present study. Given a solution of the CH equation we find the corresponding solution of the nonlocal equation and show that the approximation error, i.e. the difference between the two solutions, remains small in suitable norms on a relevant time interval. We conclude that the CH equation is an appropriate model equation for the unidirectional propagation of nonlinear dispersive elastic waves. The methodology used in this study adapts the techniques in [3, 7, 13].

We note that, in the terminology of some authors, our results are in fact consistency-existence-convergence results for the CH approximation of the IB equation and, more generally, of the nonlocal equation. We refer to [3] and the references therein for a detailed discussion of these concepts.

As it is pointed above, the general class of nonlocal wave equations contains the IB equation as a member. Therefore, to simplify our presentation, we start with the CH approximation of the IB equation and then extend the analysis to the case of the general class of nonlocal wave equations. Though our analysis is mainly concerned with the CH approximations of the IB equation and the nonlocal equation, our results apply as well to the Benjamin-Bona-Mahony (BBM) approximation. We also show how to use our results to justify the Korteweg-de Vries (KdV) approximation.

The structure of the paper is as follows. In Section 2 we observe that the solutions of the CH equation are uniformly bounded in suitable norms for all values of ϵ and δ. In Section 3 we estimate the residual term that arises when we plug the solution of the CH equation into the IB equation. In

Section 4, using the energy estimate based on certain commutator estimates, we complete the proof of the main theorem. In Section 5 we extend our consideration from the IB equation to the nonlocal equation and we prove a similar theorem for the nonlocal equation. Finally, in Section 6 we give error estimates for the long-wave approximations based on the BBM equation [2] and the KdV equation [16].

Throughout this paper, we use the standard notation for function spaces. The Fourier transform of u, defined by $\widehat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-i\xi x}dx$, is denoted by the symbol $\widehat{u}$. The symbol $\|u\|_{L^p}$ represents the L^p $(1 \le p < \infty)$ norm of u on $\mathbb{R}$. The symbol $\langle u, v\rangle$ represents the inner product of u and v in L^2. The notation $H^s = H^s(\mathbb{R})$ denotes the L^2-based Sobolev space of order s on $\mathbb{R}$, with the norm $\|u\|_{H^s} = \left(\int_{\mathbb{R}}(1+\xi^2)^s|\widehat{u}(\xi)|^2 d\xi\right)^{1/2}$. The symbol $\mathbb{R}$ in $\int_{\mathbb{R}}$ will be suppressed. C is a generic positive constant. Partial differentiations are denoted by D_t, D_x etc.

2 Uniform Estimates for the Solutions of the Camassa-Holm Equation

In this section, we observe that the solutions $w^{\epsilon,\delta}$ of the CH equation are uniformly bounded in suitable norms for all values of ϵ and δ. This is a direct consequence of the estimates proved by Constantin and Lannes in [7] for a more general class of equations, containing the CH equation as a special case.

For convenience of the reader, we rephrase below Proposition 4 of [7]. To that end, we first recall some definitions from [7]: *(i)* For every $s \ge 0$, the symbol $X^{s+1}(\mathbb{R})$ represents the space $H^{s+1}(\mathbb{R})$ endowed with the norm $|f|^2_{X^{s+1}} = \|f\|^2_{H^s} + \delta^2\|f_x\|^2_{H^s}$, and *(ii)* the symbol $\mathcal{P}$ denotes the index set

$$\mathcal{P} = \{(\epsilon,\delta) : 0 < \delta < \delta_0,\ \epsilon \le M\delta\}$$

for some $\delta_0 > 0$ and $M > 0$. Then, Proposition 4 of [7] is as follows:

Proposition 1. *Assume that* $\kappa_5 < 0$ *and let* $\delta_0 > 0$, $M > 0$, $s > \frac{3}{2}$, *and* $w_0 \in H^{s+1}(\mathbb{R})$. *Then there exist* $T > 0$ *and a unique family of solutions*

$\left\{w^{\epsilon,\delta}\right\}_{(\epsilon,\delta)\in\mathcal{P}}$ *to the Cauchy problem*

$$\begin{aligned} & w_t + w_x + \kappa_1 \epsilon w w_x + \kappa_2 \epsilon^2 w^2 w_x + \kappa_3 \epsilon^3 w^3 w_x + \delta^2 \left(\kappa_4 w_{xxx} + \kappa_5 w_{xxt}\right) \\ & \quad - \epsilon\delta^2(\kappa_6 w w_{xxx} + \kappa_7 w_x w_{xx}) = 0, \qquad (5) \\ & w(x,0) = w_0(x) \qquad (6) \end{aligned}$$

(with constants κ_i $(i = 1, 2, ..., 7)$*) bounded in* $C\left([0, \frac{T}{\epsilon}], X^{s+1}(\mathbb{R})\right) \cap C^1\left([0, \frac{T}{\epsilon}], X^s(\mathbb{R})\right)$.

We refer the reader to [7] for the proof of this proposition. Furthermore, T of the existence time T/ϵ is expressed in [7] as

$$T = T\left(\delta_0, M, |w_0|_{X^{s+1}_{\delta_0}}, \frac{1}{\kappa_5}, \kappa_2, \kappa_3, \kappa_6, \kappa_7\right) > 0.$$

Obviously, the CH equation (3) is a special case of (5) where $\kappa_1 = 1$, $\kappa_2 = \kappa_3 = 0$, $\kappa_4 = -\frac{3}{4}$, $\kappa_5 = -\frac{5}{4}$ and $2\kappa_6 = \kappa_7 = -\frac{3}{2}$. In subsequent sections we will need to use uniform estimates for the terms $\left\|w^{\epsilon,\delta}(t)\right\|_{H^{s+k}}$ and $\left\|w_t^{\epsilon,\delta}(t)\right\|_{H^{s+k-1}}$ with some $k \geq 1$. Proposition 1 provides us with such estimates, nevertheless to avoid the extra δ^2 term in the X^{s+1}-norm, we will use a weaker version based on the inclusion $X^{s+k+1} \subset H^{s+k}$. Furthermore, for simplicity, we take $\delta_0 = M = 1$. We thus reach the following corollary:

Corollary 1. *Let* $w_0 \in H^{s+k+1}(\mathbb{R})$, $s > 1/2$, $k \geq 1$. *Then, there exist* $T > 0$, $C > 0$ *and a unique family of solutions*

$$w^{\epsilon,\delta} \in C\left([0, \frac{T}{\epsilon}], H^{s+k}(\mathbb{R})\right) \cap C^1\left([0, \frac{T}{\epsilon}], H^{s+k-1}(\mathbb{R})\right)$$

to the CH equation (3) with initial value $w(x,0) = w_0(x)$, *satisfying*

$$\left\|w^{\epsilon,\delta}(t)\right\|_{H^{s+k}} + \left\|w_t^{\epsilon,\delta}(t)\right\|_{H^{s+k-1}} \leq C,$$

for all $0 < \delta \leq 1$, $\epsilon \leq \delta$ *and* $t \in [0, \frac{T}{\epsilon}]$.

3 Estimates for the Residual Term Corresponding to the Camassa-Holm Approximation

Let $w^{\epsilon,\delta}$ be the family of solutions mentioned in Corollary 1 for the Cauchy problem of the CH equation with initial value $w_0 \in H^{s+k+1}(\mathbb{R})$. In this section we estimate the residual term that arises when we plug $w^{\epsilon,\delta}$ into the IB equation. Obviously, the residual term f for the IB equation is

$$f = w_{tt} - w_{xx} - \delta^2 w_{xxtt} - \epsilon(w^2)_{xx}, \tag{7}$$

where and hereafter we drop the indices ϵ, δ in u and w for simplicity.

Using the CH equation we now show that the residual term f has a potential function. We start by rewriting the CH equation in the form

$$w_t + w_x = -\epsilon w w_x + \frac{3}{4}\delta^2 w_{xxx} + \frac{5}{4}\delta^2 w_{xxt} + \frac{3}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx}). \tag{8}$$

Using repeatedly (8) in (7) we get

$$\begin{aligned} f =&(D_t - D_x)\left[-\epsilon w w_x + \frac{3}{4}\delta^2 w_{xxx} + \frac{5}{4}\delta^2 w_{xxt} + \frac{3}{4}\epsilon\delta^2 D_x(\frac{1}{2}w_x^2 + w w_{xx})\right] \\ &- \delta^2 w_{xxtt} - \epsilon(w^2)_{xx} \\ =&\epsilon^2 D_x^2(\frac{w^3}{3}) - \frac{3}{8}\epsilon^2\delta^2\left[D_x^2(w_x^2 + 2w w_{xx})\right] \\ &+ \frac{1}{16}\delta^4\left[(D_x^2 D_t - 3D_x^3)(3w_{xxx} + 5w_{xxt})\right] \\ &+ \frac{3}{32}\epsilon\delta^4\left[(D_x^3 D_t - 3D_x^4)(w_x^2 + 2w w_{xx})\right] \\ &+ \frac{1}{4}\epsilon\delta^2 D_x\left[(-3wD_x^2 + 2w_{xx} + w_x D_x)(w_t + w_x)\right]. \end{aligned} \tag{9}$$

After some straightforward calculations we write $f = F_x$ with

$$\begin{aligned} F =&\epsilon^2(\frac{w^3}{3})_x - \frac{1}{8}\epsilon^2\delta^2\left[3(w_x^2 + 2w w_{xx})_x - 3w(w^2)_{xxx} + 2w_{xx}(w^2)_x + w_x(w^2)_{xx}\right] \\ &+ \frac{1}{16}\delta^4\left[(D_x D_t - 3D_x^2)(3w_{xxx} + 5w_{xxt})\right] \\ &+ \frac{1}{32}\epsilon\delta^4\left[3(D_x^2 D_t - 3D_x^3)(w_x^2 + 2w w_{xx})\right. \\ &\left.+2(-3wD_x^2 + 2w_{xx} + w_x D_x)(3w_{xxx} + 5w_{xxt})\right] \\ &+ \frac{1}{32}\epsilon^2\delta^4\left[(-9wD_x^3 + 6w_{xx}D_x + 3w_x D_x^2)(w_x^2 + 2w w_{xx})\right]. \end{aligned}$$

Note that, except for the term $D_x^3 D_t^2 w$, F is a combination of terms of the form $D_x^j w$ with $j \leq 5$ or $D_x^l D_t w$ with $l \leq 4$. By taking $k = 5$ it immediately follows from Corollary 1 that all of the terms in F, except $D_x^3 D_t^2 w$, are uniformly bounded in the H^s norm. To deal with the term $D_x^3 D_t^2 w$, we first rewrite the CH equation in the form

$$w_t = \mathcal{Q}\left[-w_x - \epsilon w w_x + \frac{3}{4}\delta^2 w_{xxx} + \frac{3}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right], \tag{10}$$

where the operator $\mathcal{Q}$ is

$$\mathcal{Q} = \left(1 - \frac{5}{4}\delta^2 D_x^2\right)^{-1}. \tag{11}$$

Then, applying the operator $D_x^3 D_t$ to (10) and using (8) we get

$$\begin{aligned} D_x^3 D_t w_t =& D_x^3 D_t \mathcal{Q}\left[-w_x - \epsilon w w_x + \frac{3}{4}\delta^2 w_{xxx} + \frac{3}{4}\epsilon\delta^2(2w_x w_{xx} + w w_{xxx})\right] \\ =& D_t\left[-\mathcal{Q}\left(w_{xxxx} + \epsilon\left(w w_x\right)_{xxx}\right)\right. \\ & \left. + \frac{3}{4}\delta^2 \mathcal{Q} D_x^2 w_{xxxx} + \frac{3}{4}\epsilon\delta^2 \mathcal{Q} D_x^2 (2w_x w_{xx} + w w_{xxx})_x\right]. \end{aligned}$$

We note that the operator norms of $\mathcal{Q}$ and $\mathcal{Q}\delta^2 D_x^2$ are bounded:

$$\|\mathcal{Q}\|_{H^s} \leq 1 \quad \text{and} \quad \left\|\delta^2 \mathcal{Q} D_x^2\right\|_{H^s} \leq \frac{4}{5}.$$

The use of these bounds and uniform estimate for $D_x^3 D_t^2 w$ yield

$$\left\|D_x^3 D_t^2 w\right\|_{H^s} \leq C\left\|D_x^4 w_t\right\|_{H^s} \leq C\left\|w_t\right\|_{H^{s+4}}. \tag{12}$$

As all the terms in F have coefficients ϵ^2, $\epsilon^2\delta^2$, δ^4, $\epsilon\delta^4$ or $\epsilon^2\delta^4$ (with $0 < \epsilon \leq \delta \leq 1$) we obtain the following estimate for the potential function

$$\|F(t)\|_{H^s} \leq C\left(\epsilon^2 + \delta^4\right)\left(\|w\|_{H^{s+5}} + \|w_t\|_{H^{s+4}}\right). \tag{13}$$

Using Corollary 1 with $k = 5$, we obtain:

Lemma 3.1. *Let $w_0 \in H^{s+6}(\mathbb{R})$, $s > 1/2$. Then, there is some $C > 0$ so that the family of solutions $w^{\epsilon,\delta}$ to the CH equation (3) with initial value $w(x,0) = w_0(x)$, satisfy*

$$w_{tt} - w_{xx} - \delta^2 w_{xxtt} - \epsilon(w^2)_{xx} = F_x$$

with

$$\|F(t)\|_{H^s} \leq C\left(\epsilon^2 + \delta^4\right),$$

for all $0 < \epsilon \leq \delta \leq 1$ and $t \in [0, \frac{T}{\epsilon}]$.

4 Justification of the Camassa-Holm Approximation

In this section we prove Theorem 4.2 given below. We have the well-posedness result for the IB equation (1) in a general setting [8, 10]:

Theorem 4.1. *Let $u_0, u_1 \in H^s(\mathbb{R}),\ \ s > 1/2$. Then for any pair of parameters ϵ and δ, there is some $T^{\epsilon,\delta} > 0$ so that the Cauchy problem for the IB equation (1) with initial values $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ has a unique solution $u \in\ C^2\left([0, T^{\epsilon,\delta}], H^s(\mathbb{R})\right)$.*

The existence time $T^{\epsilon,\delta}$ above may depend on ϵ and δ and it may be chosen arbitrarily large as long as $T^{\epsilon,\delta} < T^{\epsilon,\delta}_{\max}$ where $T^{\epsilon,\delta}_{\max}$ is the maximal time. Furthermore, it was shown in [10] that the existence time, if it is finite, is determined by the L^∞ blow-up condition

$$\lim_{t\to T^{\epsilon,\delta}_{\max}} \sup \|u\,(t)\|_{L^\infty} = \infty.$$

We now consider the solutions w of the CH equation with initial data $w(x,0) = w_0$. Then we take $w_0(x)$ and $w_t(x,0)$ as the initial conditions for the IB equation (1), that is,

$$u(x,0) = w_0(x), \quad u_t(x,0) = w_t(x,0).$$

Let u be the corresponding solutions of the Cauchy problem defined for the IB equation (1) with these initial conditions. Since $w_0 \in H^{s+6}(\mathbb{R})$, clearly $u(x,0), u_t(x,0) \in H^s(\mathbb{R})$. Recalling from Corollary 1 that the guaranteed existence time for w is T/ϵ, without loss of generality we will take $T^{\epsilon,\delta} \le T/\epsilon$.

In the course of our proof of Theorem 4.2, we will use certain commutator estimates. We recall that the commutator is defined as $[K, L] = KL - LK$. We refer the reader to [17] (see Proposition B.8) for the following result.

Proposition 2. *Let $q_0 > 1/2,\ \ s \ge 0$ and let σ be a Fourier multiplier of order s.*

1. *If $\ 0 \le s \le q_0 + 1$ and $w \in H^{q_0+1}$ then, for all $g \in H^{s-1}$, one has*

$$\|[\sigma(D_x), w]g\|_{L^2} \le C\|w_x\|_{H^{q_0}}\|g\|_{H^{s-1}},$$

2. *If* $-q_0 < r \le q_0 + 1 - s$ *and* $w \in H^{q_0+1}$ *then, for all* $g \in H^{r+s-1}$*, one has*

$$\|[\sigma(D_x), w]g\|_{H^r} \le C\|w_x\|_{H^{q_0}}\|g\|_{H^{r+s-1}}.$$

For the reader's convenience we now restate the two estimates of the above proposition as follows. Let $\Lambda^s = (1 - D_x^2)^{s/2}$ and take $w \in H^{s+1}$, $g \in H^{s-1}$ and $h \in H^s$. Then, for $q_0 = s$, the first estimate above yields

$$\langle[\Lambda^s, w]g, \Lambda^s h\rangle \le C\|w\|_{H^{s+1}}\|g\|_{H^{s-1}}\|h\|_{H^s}. \tag{14}$$

Similarly, for $q_0 = s$ and $-s < r \le 1$, we obtain from the second estimate that

$$\begin{aligned}\langle\Lambda[\Lambda^s, w]h, \Lambda^{s-1}g\rangle \le& C\|\Lambda[\Lambda^s, w]h\|_{L^2}\|\Lambda^{s-1}g\|_{L^2}\\ \le& C\|[\Lambda^s, w]h\|_{H^1}\|g\|_{H^{s-1}}\\ \le& C\|w\|_{H^{s+1}}\|h\|_{H^s}\|g\|_{H^{s-1}}.\end{aligned} \tag{15}$$

We are now ready to prove the main result for the CH approximation of the IB equation (an extension of the following theorem to the nonlocal equation will be given in Section 5 (see Theorem 5.2)):

Theorem 4.2. *Let* $w_0 \in H^{s+6}(\mathbb{R})$, $s > 1/2$ *and suppose that* $w^{\epsilon,\delta}$ *is the solution of the CH equation (3) with initial value* $w(x, 0) = w_0(x)$*. Then, there exist* $T > 0$ *and* $\delta_1 \le 1$ *such that the solution* $u^{\epsilon,\delta}$ *of the Cauchy problem for the IB equation*

$$\begin{aligned}&u_{tt} - u_{xx} - \delta^2 u_{xxtt} - \epsilon(u^2)_{xx} = 0\\ &u(x, 0) = w_0(x), \qquad u_t(x, 0) = w_t^{\epsilon,\delta}(x, 0),\end{aligned}$$

satisfies

$$\|u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)\|_{H^s} \le \; C\left(\epsilon^2 + \delta^4\right)t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ *and all* $0 < \epsilon \le \delta \le \delta_1$.

Proof. We fix the parameters ϵ and δ such that $0 < \epsilon \le \delta \le 1$. Let $r = u - w$. We define

$$T_0^{\epsilon,\delta} = \sup\left\{t \le T^{\epsilon,\delta} : \|r(\tau)\|_{H^s} \le 1 \;\text{ for all }\; \tau \in [0, t]\right\}. \tag{16}$$

We note that either $\left\|r\left(T_0^{\epsilon,\delta}\right)\right\|_{H^s} = 1$ or $T_0^{\epsilon,\delta} = T^{\epsilon,\delta}$. Moreover, in the latter case we must have $T_0^{\epsilon,\delta} = T^{\epsilon,\delta} = T/\epsilon$ by the discussion above about for the maximal time $T_{\max}^{\epsilon,\delta}$. For the rest of the proof we will drop the superscripts ϵ, δ to simplify the notation. Henceforth, we will take $t \in [0, T_0^{\epsilon,\delta}]$. Obviously, the function $r = u - w$ satisfies the initial conditions $r(x,0) = r_t(x,0) = 0$. Furthermore, it satisfies the evolution equation

$$\left(1 - \delta^2 D_x^2\right) r_{tt} - r_{xx} - \epsilon \left(r^2 + 2wr\right)_{xx} = -F_x,$$

with the residual term $F_x = w_{tt} - w_{xx} - \delta^2 w_{xxtt} - \epsilon(w^2)_{xx}$ that was already estimated in (13)). We define a function ρ so that $r = \rho_x$ with $\rho(x,0) = \rho_t(x,0) = 0$. This is possible since r satisfies the initial conditions $r(x,0) = r_t(x,0) = 0$ (see [10] for details). In what follows we will use both ρ and r to further simplify the calculation. The above equation then becomes

$$\left(1 - \delta^2 D_x^2\right) \rho_{tt} - r_x - \epsilon \left(r^2 + 2wr\right)_x = -F. \tag{17}$$

Motivated by the approach in [13], we define the "energy" as

$$\begin{aligned} E_s^2(t) &= \frac{1}{2}\left(\|\rho_t(t)\|_{H^s}^2 + \delta^2 \|r_t(t)\|_{H^s}^2 + \|r(t)\|_{H^s}^2\right) + \epsilon \left\langle \Lambda^s(w(t)r(t)), \Lambda^s r(t)\right\rangle \\ &\quad + \frac{\epsilon}{2}\left\langle \Lambda^s r^2(t), \Lambda^s r(t)\right\rangle. \end{aligned} \tag{18}$$

Note that

$$|\langle \Lambda^s(wr), \Lambda^s r\rangle| \le C \|r(t)\|_{H^s}^2, \quad \text{and} \quad \left|\left\langle \Lambda^s r^2, \Lambda^s r\right\rangle\right| \le \|r(t)\|_{H^s}^3 \le \|r(t)\|_{H^s}^2,$$

where we have used (16). Thus, for sufficiently small values of ϵ, we have

$$E_s^2(t) \ge \frac{1}{4}\left(\|\rho_t\|_{H^s}^2 + \delta^2 \|r_t\|_{H^s}^2 + \|r\|_{H^s}^2\right),$$

which shows that $E_s^2(t)$ is positive definite. The above result also shows that an estimate obtained for E_s^2 gives an estimate for $\|r(t)\|_{H^s}^2$. Differentiating $E_s^2(t)$ with respect to t and using (17) to eliminate the term ρ_{tt} from the resulting equation we get

$$\begin{aligned} \frac{d}{dt}E_s^2 =& \frac{d}{dt}\left(\epsilon \langle \Lambda^s(wr), \Lambda^s r\rangle + \frac{\epsilon}{2}\left\langle \Lambda^s r^2, \Lambda^s r\right\rangle\right) - \epsilon \left\langle \Lambda^s(r^2 + 2wr), \Lambda^s r_t\right\rangle \\ & - \langle \Lambda^s F, \Lambda^s \rho_t\rangle \\ =& \epsilon\left[\langle \Lambda^s(w_t r), \Lambda^s r\rangle - \langle \Lambda^s(wr), \Lambda^s r_t\rangle + \langle \Lambda^s r, \Lambda^s(w r_t)\rangle + \langle \Lambda^s(r r_t), \Lambda^s r\rangle \right.\\ & \left. - \frac{1}{2}\left\langle \Lambda^s r^2, \Lambda^s r_t\right\rangle\right] - \langle \Lambda^s F, \Lambda^s \rho_t\rangle. \end{aligned} \tag{19}$$

The first term in the parentheses and the last term are estimated as

$$\begin{aligned} \langle \Lambda^s(w_t r), \Lambda^s r\rangle \leq & C\, \|r\|^2_{H^s} \leq CE_s^2 \\ \langle \Lambda^s F, \Lambda^s \rho_t\rangle \leq & \|F\|_{H^s}\|\rho_t\|_{H^s} \leq C\left(\epsilon^2+\delta^4\right) E_s, \end{aligned}$$

respectively, where we have used Lemma 3.1. We rewrite the second and the third terms in the parentheses in (19) as

$$\begin{aligned} -\langle \Lambda^s(wr), \Lambda^s r_t\rangle + \langle \Lambda^s r, \Lambda^s(wr_t)\rangle = & \int \left[-\Lambda^s(wr)\Lambda^s r_t + \Lambda^s r \Lambda^s(wr_t)\right] dx \\ = & -\langle [\Lambda^s, w]r, \Lambda^s r_t\rangle + \langle [\Lambda^s, w]r_t, \Lambda^s r\rangle. \end{aligned} \tag{20}$$

Furthermore, using the commutator estimates (14)-(15) we get the following estimates for the two terms in (20):

$$\langle [\Lambda^s, w]r, \Lambda^s r_t\rangle = \langle \Lambda[\Lambda^s, w]r, \Lambda^{s-1} r_t\rangle \leq C\|w\|_{H^{s+1}}\|r\|_{H^s}\|r_t\|_{H^{s-1}}, \tag{21}$$

$$\langle [\Lambda^s, w]r_t, \Lambda^s r\rangle \leq C\|w\|_{H^{s+1}}\|r\|_{H^s}\|r_t\|_{H^{s-1}}. \tag{22}$$

We rewrite the fourth and fifth terms in the parentheses in (19) as

$$\begin{aligned} \langle \Lambda^s(rr_t), \Lambda^s r\rangle & -\frac{1}{2}\left\langle \Lambda^s r^2, \Lambda^s r_t\right\rangle \\ & = \left\langle \Lambda^{s-1}\left(1-D_x^2\right) r, \Lambda^{s-1}(rr_t)\right\rangle - \frac{1}{2}\left\langle \Lambda^{s-1}\left(1-D_x^2\right) r^2, \Lambda^{s-1} r_t\right\rangle \\ & = \left\langle \Lambda^{s-1} r, \Lambda^{s-1}(rr_t)\right\rangle - \frac{1}{2}\left\langle \Lambda^{s-1}(r^2 - 2r_x^2), \Lambda^{s-1} r_t\right\rangle \\ & \quad - \left(\left\langle \Lambda^{s-1}(rr_t), \Lambda^{s-1} r_{xx}\right\rangle - \left\langle \Lambda^{s-1} r_t, \Lambda^{s-1}(rr_{xx})\right\rangle\right). \end{aligned}$$

Then, if we group the first two terms together and the last two terms together in the above equation, we obtain the following estimates

$$\begin{aligned} \left|\left\langle \Lambda^{s-1} r, \Lambda^{s-1}(rr_t)\right\rangle - \frac{1}{2}\left\langle \Lambda^{s-1}(r^2-2r_x^2), \Lambda^{s-1} r_t\right\rangle\right| \leq & C\|r\|^2_{H^{s-1}}\|r_t\|_{H^{s-1}}, \\ \leq & C\|r\|^2_{H^s}\|r_t\|_{H^{s-1}}, \\ \left|\left\langle \Lambda^{s-1}(rr_t), \Lambda^{s-1} r_{xx}\right\rangle - \left\langle \Lambda^{s-1} r_t, \Lambda^{s-1}(rr_{xx})\right\rangle\right| \leq & C\|r\|_{H^s}\|r_t\|_{H^{s-1}}\|r_{xx}\|_{H^{s-2}} \\ \leq & C\|r\|^2_{H^s}\|r_t\|_{H^{s-1}}. \end{aligned}$$

Note that the second line follows from (20) and (21) where w, r, r_t are replaced, respectively, by r, r_t, r_{xx} and s by $s-1$. Also, we remind that

$$\|r_t\|_{H^{s-1}} = \|\rho_{xt}\|_{H^{s-1}} \leq \|\rho_t\|_{H^s} \leq CE_s(t)$$

and $\|r\|_{H^s} \leq 1$. Combining all the above results we get from (19) that

$$\frac{d}{dt}E_s^2(t) \leq C\left(\epsilon E_s^2(t) + \left(\epsilon^2+\delta^4\right)E_s(t)\right).$$

As $E_s(0)=0$, Gronwall's inequality yields

$$\begin{aligned} E_s(t) \leq & \frac{\epsilon^2+\delta^4}{\epsilon}\left(e^{C\epsilon t}-1\right) \leq Ce^{CT}\left(\epsilon^2+\delta^4\right)t \\ \leq & C'\left(\epsilon^2+\delta^4\right)t \quad \text{for} \quad t \leq T_0^{\epsilon,\delta} \leq \frac{T}{\epsilon}. \end{aligned}$$

Finally recall that $T_0^{\epsilon,\delta}$ was determined by the condition (16). The above estimate shows that $\left\|r\left(T_0^{\epsilon,\delta}\right)\right\|_s \leq C'\left(\epsilon^2+\delta^4\right)T_0^{\epsilon,\delta} < 1$ for $\epsilon \leq \delta$ small enough. Then $T_0^{\epsilon,\delta} = T^{\epsilon,\delta}$ and furthermore $T^{\epsilon,\delta} = \frac{T}{\epsilon}$, and this concludes the proof. □

We want to conclude with some remarks about the above proof.

Remark 1. *Theorem 4.2 shows that the approximation error is $\mathcal{O}\left((\epsilon^2+\delta^4)t\right)$ for times of order $\mathcal{O}(\frac{1}{\epsilon})$. Consequently, the CH approximation provides a good approximation to the solution of the IB equation for large times.*

Remark 2. *The key step is to use the extra ϵ terms in the energy E_s^2, where we have adopted the approach in [13]. This allows us to replace $\|r_t\|_{H^s}$ by $\|r_t\|_{H^{s-1}}$ hence avoiding the loss of δ in our estimates. The proofs in [13] work for integer values of s, whereas via commutator estimates our result holds for general s. The standard approach of taking the energy as*

$$\tilde{E}_s^2(t) = \frac{1}{2}\left(\|\rho_t(t)\|_{H^s}^2 + \delta^2\|r_t(t)\|_{H^s}^2 + \|r(t)\|_{H^s}^2\right)$$

would give the estimate

$$\tilde{E}_s(t) \leq \left(\epsilon^2+\delta^4\right)\frac{\epsilon}{\delta}\left(e^{C\frac{\epsilon}{\delta}t}-1\right),$$

in turn implying $\tilde{E}_s(t) \leq C'\left(\epsilon^2+\delta^4\right)t$ for times $t \leq \frac{\delta}{\epsilon}T$, that is, only for relatively shorter times.

5 The Nonlocal Wave Equation

In this section we return to the nonlocal equation (2) and extend the analysis of the previous sections concerning the IB equation (1) to (2). We will very briefly sketch the main features of the nonlocal equation, referring the reader to [10] for more details. In [10], for the propagation of strain waves in a one-dimensional, homogeneous, nonlinearly and nonlocally elastic infinite medium the following wave equation was proposed (here we restrict our attention to the quadratically nonlinear equation):

$$U_{\tau\tau} = \beta * (U + U^2)_{XX} \tag{23}$$

where $U = U(X, \tau)$ is a real-valued function. Following the assumptions in [10], the kernel function $\beta(X)$ is even and its Fourier transform satisfies the ellipticity condition

$$c_1 \left(1 + \eta^2\right)^{-r/2} \leq \widehat{\beta}(\eta) \leq c_2 \left(1 + \eta^2\right)^{-r/2} \tag{24}$$

for some $c_1, c_2 > 0$ and $r \geq 2$, where η is the Fourier variable corresponding to X. Then the convolution can be considered as an invertible pseudodifferential operator of order r. The following result on the local well-posedness of the Cauchy problem was originally given in [10]:

Theorem 5.1. *Let $r \geq 2$ and $s > 1/2$. For $U_0, U_1 \in H^s(\mathbb{R})$, there is some $\tau^* > 0$ such that the Cauchy problem for (23) with initial values $U(X, 0) = U_0(X)$, $U_\tau(X, 0) = U_1(X)$ has a unique solution $U \in C^2([0, \tau^*], H^s(\mathbb{R}))$.*

Moreover, as in the case of the IB equation, the L^∞ blow-up condition

$$\lim_{\tau \to \tau_{\max}^-} \sup \|U(\tau)\|_{L^\infty} = \infty$$

determines the maximal existence time if it is finite.

We note that, under the transformation defined by

$$U(X, \tau) = \epsilon u(x, t), \qquad x = \delta X, \qquad t = \delta\tau, \tag{25}$$

(23) becomes (2) with $\beta_\delta(x) = \frac{1}{\delta}\beta(X) = \frac{1}{\delta}\beta(\frac{x}{\delta})$. Recall that the functional relationship between the Fourier transforms of $\beta(X)$ and $\beta_\delta(x)$ is as follows: $\widehat{\beta}(\eta) = \widehat{\beta}(\delta\xi) = \widehat{\beta}_\delta(\xi)$ where ξ is the Fourier variable corresponding to x. Theorem 5.1 applies for (2) with $t \in [0, T^{\epsilon,\delta}]$. Note that if we choose the

kernel function in the form $\beta_\delta(x) = \frac{1}{2\delta}e^{-|x|/\delta}$ (in which $\beta(X) = \frac{1}{2}e^{-|X|}$, $\widehat{\beta}(\eta) = (1+\eta^2)^{-1}$ and $\widehat{\beta}_\delta(\xi) = (1+\delta^2\xi^2)^{-1}$), then (2) recovers the IB equation (1).

Our aim is to prove that, in the long-wave limit, the unidirectional solutions of the nonlocal equation are well approximated by the solutions of the CH equation under certain minimal conditions on β (equivalently on β_δ). From now on, we will make the following assumptions on the moments of β:

$$\int \beta(X)dX = 1, \quad \int X^2\beta(X)dX = 2, \quad \int X^4|\beta(X)|dX < \infty. \tag{26}$$

Proposition 3. *Suppose that β satisfies the conditions in (26). Then there is a continuous function m such that*

$$\frac{1}{\widehat{\beta}(\eta)} = 1 + \eta^2 + \eta^4 m(\eta). \tag{27}$$

Proof. Since the Fourier transform of $-iX\beta(X)$ equals $\frac{d}{d\eta}\widehat{\beta}(\eta)$, (26) implies that $\widehat{\beta} \in C^4$ and

$$\widehat{\beta}(0) = \int \beta(X)dX = 1, \; (\widehat{\beta})''(0) = -\int X^2\beta(X)dX = -2. \tag{28}$$

Then $1/\widehat{\beta}(\eta) \in C^4$, $1/\widehat{\beta}(0) = 1$ and $\left(1/\widehat{\beta}\right)''(0) = 2$. As β is even, the odd moments, hence the odd derivatives of $1/\widehat{\beta}(\eta)$, vanish at $\eta = 0$. Thus the function defined as

$$m(\eta) = \frac{\frac{1}{\widehat{\beta}(\eta)} - 1 - \eta^2}{\eta^4}$$

for $\eta \neq 0$ can be extended continuously to $\eta = 0$. □

Remark 3. *The above assumption is not very restrictive in our setting. For instance, if $\int \beta(X)dX = a$ and $\int X^2\beta(X)dX = b > 0$, a suitable scaling will reduce it to the above case.*

The lower bound in (24) shows that

$$0 < \frac{1}{\widehat{\beta}(\eta)} = 1 + \eta^2 + \eta^4 m(\eta) \leq c_1^{-1}(1+\eta^2)^{r/2}.$$

Thus

$$\eta^4|m(\eta)| \leq c_1^{-1}(1+\eta^2)^{r/2} + (1+\eta^2) \leq C(1+\eta^2)^{r/2}.$$

Since $m(\eta)$ is continuous, this implies

$$|m(\eta)| \leq C(1+\eta^2)^{\frac{r-4}{2}},$$

so that m has order $r-4$. We note that under the scaling (25) we have

$$\frac{1}{\widehat{\beta}_\delta(\xi)} = 1 + \delta^2\xi^2 + \delta^4\xi^4 m(\delta\xi). \tag{29}$$

We define the pseudodifferential operators

$$MU = \mathcal{F}^{-1}\left(m(\eta)\widehat{U}(\eta)\right), \qquad M_\delta u = \mathcal{F}^{-1}\left(m\left(\delta\xi\right)\widehat{u}(\xi)\right).$$

When $r > 4$, we have

$$|m(\delta\xi)| \leq C(1+\delta^2\xi^2)^{\frac{r-4}{2}} \leq C(1+\xi^2)^{\frac{r-4}{2}},$$

so that

$$\left\|M_\delta u\right\|_{H^s} \leq C\left\|u\right\|_{H^{s+r-4}}.$$

On the other hand, when $r \leq 4$, we get

$$|m(\delta\xi)| \leq C(1+\delta^2\xi^2)^{\frac{r-4}{2}} \leq C,$$

so that

$$\left\|M_\delta u\right\|_{H^s} \leq C\left\|u\right\|_{H^s}.$$

Thus we have the uniform estimates for $M_\delta u$:

$$\left\|M_\delta u\right\|_{H^s} \leq C\left\|u\right\|_{H^{s+\sigma-4}}, \qquad \sigma = \max\{r, 4\}. \tag{30}$$

Due to (25), $MU = \epsilon M_\delta u$. Multiplying (23) by $(1 - D_X^2 + D_X^4 M)$ and (2) by $(1 - \delta^2 D_x^2 + \delta^4 D_x^4 M_\delta)$ we rewrite (23) and (2) more familiar forms

$$\left(1 - D_X^2 + D_X^4 M\right) U_{\tau\tau} - U_{XX} = \left(U^2\right)_{XX} \tag{31}$$

and

$$\left(1 - \delta^2 D_x^2 + \delta^4 D_x^4 M_\delta\right) u_{tt} - u_{xx} = \epsilon\left(u^2\right)_{xx}, \tag{32}$$

respectively.

When we apply the formal asymptotic approach given in [11] to (32) (in [11] it was used to derive the CH equation from the IB equation), we again get exactly the same result, that is, the CH equation. As remarked in [11], this follows from the observation that the extra term $\delta^4 D_x^4 M_\delta$ will only give rise to $O(\delta^4)$ terms and these terms do not affect the derivation in [11]. The following theorem gives the convergence of the formal asymptotic expansion and shows that the right-going solutions of (32) (and (2)) are well approximated by the solutions of the CH equation.

Theorem 5.2. *Let $w_0 \in H^{s+\sigma+2}(\mathbb{R})$, $s > 1/2$, $\sigma = \max\{r, 4\}$ and suppose $w^{\epsilon,\delta}$ is the solution of the CH equation (3) with initial value $w(x,0) = w_0(x)$. Then, there exist $T > 0$ and $\delta_1 \leq 1$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem for (32) (equivalently for (2))*

$$
\begin{aligned}
&\left(1-\delta^2 D_x^2+\delta^4 D_x^4 M_\delta\right) u_{tt}-u_{xx}-\epsilon(u^2)_{xx}=0,\\
&u(x,0)=w_0\left(x\right),\ u_t\left(x,0\right)=w_t^{\epsilon,\delta}\left(x,0\right),
\end{aligned}
$$

satisfies

$$\|u^{\epsilon,\delta}(t)-w^{\epsilon,\delta}(t)\|_{H^s}\leq\ C\left(\epsilon^2+\delta^4\right)t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ and all $0 < \epsilon \leq \delta \leq \delta_1$.

Proof. The proof follows a similar pattern to that of the proof of Theorem 4.2. The only difference is that (32) involves additional term $\delta^4 D_x^4 M_\delta u_{tt}$. Following closely the scheme in the proof of Theorem 4.2 corresponding to case of the IB equation, we now outline the proof. First we note that plugging the solution $w^{\epsilon,\delta}$ of the CH equation into (32) leads to a residual term $D_x F^M$ with $F^M = F + \delta^4 D_x^3 M_\delta w_{tt}$ where $D_x F$ is the residue term corresponding to the IB case, given in (9). Going through a cancelation process similar to the cancelations in the IB case, we get

$$\left\|D_x^3 M_\delta w_{tt}\right\|_{H^s}\leq C\left\|D_x^3 w_{tt}\right\|_{H^{s+\sigma-4}}\leq C\left\|w_t\right\|_{H^{s+\sigma-4+4}}=C\left\|w_t\right\|_{H^{s+\sigma}},$$

where we use the estimate (30) for M_δ and (12) for $D_x^3 w_{tt}$. Since $\sigma \geq 4$, we have

$$\left\|F^M(t)\right\|_{H^s}\leq C(\epsilon^2+\delta^4)\left(\|w\|_{H^{s+\sigma+1}}+\|w_t\|_{H^{s+\sigma}}\right).$$

Thus we take $k = \sigma + 1$ in Corollary 1 to get a uniform bound on F^M. The next step is to define the energy as

$$E_{s,M}^2=E_s^2+\frac{1}{2}\delta^4\left\langle\Lambda^s M_\delta D_x r_t(t),\Lambda^s D_x r_t(t)\right\rangle,$$

where E_s^2 is given by (18). We note that the extra term in $E_{s,M}^2$ is not necessarily positive. Yet recalling that $r = \rho_x$ and collecting the ρ_t and r_t terms in $E_{s,M}^2$ we have:

$$
\begin{aligned}
\|\rho_t\|_{H^s}^2 + \delta^2 \|r_t\|_{H^s}^2 - \delta^4 \left\langle \Lambda^s D_x^2 M_\delta r_t, \Lambda^s r_t \right\rangle &= \left\langle \Lambda^s \left(1 - \delta^2 D_x^2 + \delta^4 D_x^4 M_\delta\right) \rho_t, \Lambda^s \rho_t \right\rangle \\
&= \int \frac{(1+\xi^2)^s}{\widehat{\beta}(\delta\xi)} \left|\widehat{\rho_t}(\xi)\right|^2 d\xi \\
&\geq c_2^{-1} \int (1+\xi^2)^s (1+\delta^2\xi^2)^{r/2} \left|\widehat{\rho_t}(\xi)\right|^2 d\xi \\
&\geq c_2^{-1} \int (1+\xi^2)^s (1+\delta^2\xi^2) \left|\widehat{\rho_t}(\xi)\right|^2 d\xi \\
&= c_2^{-1} \left(\|\rho_t\|_{H^s}^2 + \delta^2 \|\rho_{xt}\|_{H^s}^2\right) \\
&= c_2^{-1} \left(\|\rho_t\|_{H^s}^2 + \delta^2 \|r_t\|_{H^s}^2\right).
\end{aligned}
$$

Hence $E_{s,M}^2 \geq C E_s^2$. It is straightforward to compute the time derivative of $E_{s,M}^2$ since as the extra term vanishes due to (31) and we are left with the same right-hand side as in the previous section and hence with the same conclusion. ☐

Remark 4. *We conclude from Theorem 5.2 that the comments made in Remark 1 on the precision of the CH approximation to the IB equation are also valid for the nonlocal equation.*

6 The BBM and KdV Approximations

In this section we consider the BBM equation and the KdV equation which characterize the particular cases of the CH equation and we show how the results of the previous sections can be used to obtain the results for these two equations. The analysis is similar in spirit to that of Sections 3 and 4, we therefore give only the main steps in the proofs.

6.1 The BBM Approximation

When we neglect terms of order $\epsilon\delta^2$ in the CH equation (3), we get the BBM equation

$$
w_t + w_x + \epsilon w w_x - \frac{3}{4}\delta^2 w_{xxx} - \frac{5}{4}\delta^2 w_{xxt} = 0, \tag{33}
$$

which is a well-known model for unidirectional propagation of long waves in shallow water [2]. It should be noted that, in order to write this equation in a more standard form, the term w_{xxx} can be eliminated by means of the coordinate transformation given in Section 1. Obviously, the BBM equation (33) is a special case of (5) with $\kappa_1 = 1$, $\kappa_2 = \kappa_3 = 0$, $\kappa_4 = -\frac{3}{4}$, $\kappa_5 = -\frac{5}{4}$ and $\kappa_6 = \kappa_7 = 0$. Then, for the BBM equation, Corollary 1 takes the following form:

Corollary 2. *Let $w_0 \in H^{s+k+1}(\mathbb{R})$, $s > 1/2$, $k \geq 1$. Then, there exist $T > 0$, $C > 0$ and a unique family of solutions*

$$w^{\epsilon,\delta} \in C\left([0, \frac{T}{\epsilon}], H^{s+k}(\mathbb{R})\right) \cap C^1\left([0, \frac{T}{\epsilon}], H^{s+k-1}(\mathbb{R})\right)$$

to the BBM equation (33) with initial value $w(x,0) = w_0(x)$, satisfying

$$\left\|w^{\epsilon,\delta}(t)\right\|_{H^{s+k}} + \left\|w_t^{\epsilon,\delta}(t)\right\|_{H^{s+k-1}} \leq C,$$

for all $0 < \delta \leq 1$, $\epsilon \leq \delta$ and $t \in [0, \frac{T}{\epsilon}]$.

As we did in Section 3, we plug the solution w of the Cauchy problem of the BBM equation into the IB equation. Then the residual term f is given by (7) but now w represents a solution of the BBM equation. Making use of the approach in Section 3, we obtain f corresponding to the case of the BBM approximation in the form $f = F_x$ with

$$\begin{aligned} F =& \epsilon^2 \left(\frac{w^3}{3}\right)_x - \frac{1}{4}\epsilon\delta^2 \left(6ww_{xxt} + 2w_x w_{xt} + w_t w_{xx} - 9w_x w_{xx}\right) \\ &+ \frac{1}{16}\delta^4 D_x^3 \left(5w_{tt} - 12w_{xt} - 9w_{xx}\right). \end{aligned}$$

Thus we have the BBM version of Lemma 3.1, namely the uniform estimate

$$\|F(t)\|_{H^s} \leq C\left(\epsilon^2 + \delta^4\right).$$

The rest of the proof holds and we obtain the BBM version of Theorem 4.2:

Theorem 6.1. *Let $w_0 \in H^{s+6}(\mathbb{R})$, $s > 1/2$ and suppose $w^{\epsilon,\delta}$ is the solution of the BBM equation (33) with initial value $w(x,0) = w_0(x)$. Then, there* *exist $T > 0$ and $\delta_1 \leq 1$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem for the IB equation*

$$\begin{aligned} &u_{tt} - u_{xx} - \delta^2 u_{xxtt} - \epsilon (u^2)_{xx} = 0 \\ &u(x,0) = w_0(x), \; u_t(x,0) = w_t^{\epsilon,\delta}(x,0), \end{aligned}$$

satisfies

$$\|u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)\|_{H^s} \leq \; C\left(\epsilon^2 + \delta^4\right) t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ *and all* $0 < \epsilon \leq \delta \leq \delta_1$.

Following the arguments in Section 5, we may extend Theorem 6.1 to the general class of nonlocal wave equations, namely

Theorem 6.2. *Let $w_0 \in H^{s+\sigma+2}(\mathbb{R})$, $s > 1/2$, $\sigma = \max\{r, 4\}$ and suppose $w^{\epsilon,\delta}$ is the solution of the BBM equation (33) with initial value $w(x,0) = w_0(x)$. Then, there exist $T > 0$ and $\delta_1 \leq 1$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem for the nonlocal equation*

$$\begin{aligned} &u_{tt} = \beta_\delta * (u + \epsilon u^2)_{xx} \\ &u(x,0) = w_0(x), \; u_t(x,0) = w_t^{\epsilon,\delta}(x,0), \end{aligned}$$

satisfies

$$\|u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)\|_{H^s} \leq \; C\left(\epsilon^2 + \delta^4\right) t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ *and all* $0 < \epsilon \leq \delta \leq \delta_1$.

6.2 The KdV Approximation

The KdV equation [16]

$$w_t + w_x + \epsilon w w_x + \frac{\delta^2}{2} w_{xxx} = 0 \tag{34}$$

is also a well-known model for unidirectional propagation of long waves in shallow water and it has the same order of accuracy as the BBM equation. In fact, the KdV equation (34) is a special case of (5) with $\kappa_1 = 1$, $\kappa_4 = 1/2$, $\kappa_2 = \kappa_3 = \kappa_5 = \kappa_6 = \kappa_7 = 0$. However, Proposition 1 will not apply to the KdV equation because the condition $\kappa_5 < 0$ is not satisfied. Instead we refer to the following theorem proved by Alazman et al in [1]:

Theorem 6.3. *(Theorem A2 in [1]) Let $s \geq 1$ be an integer. Then for every $K > 0$, there exists $C > 0$ such that the following is true. Suppose $q_0 \in H^s$ with $\|q_0\|_{H^s} \leq K$, and let q be the solution of the KdV equation*

$$q_t + q_x + \frac{3}{2}\bar{\epsilon} q q_x + \frac{1}{6}\bar{\epsilon} q_{xxx} = 0 \tag{35}$$

with initial data $q(x,0) = q_0(x)$. Then for all $\bar{\epsilon} \in (0,1]$ and all $t \geq 0$,

$$\|q(t)\|_{H^s} \leq C.$$

Further, for every integer l such that $1 \leq 3l \leq s$, it is the case that

$$\left\|D_t^l q(t)\right\|_{H^{s-3l}} \leq C.$$

It is easy to see that the substitution

$$w = \frac{9}{2}\frac{\delta^2}{\epsilon} q, \qquad \delta^2 = \frac{\bar{\epsilon}}{3} \tag{36}$$

transforms (34) into (35). Suppose $c_1 \leq \frac{\delta^2}{\epsilon} \leq c_2$ with positive constants c_1 and c_2. Then we have

$$\|w(t)\|_{H^s} = \frac{9}{2}\frac{\delta^2}{\epsilon}\|q(t)\|_{H^s} \leq \frac{9}{2} c_2 \|q(t)\|_{H^s} \tag{37}$$

and

$$\|q_0\|_{H^s} = \frac{2}{9}\frac{\epsilon}{\delta^2}\|w_0\|_{H^s} \leq \frac{2}{9c_1}\|w_0\|_{H^s}. \tag{38}$$

We thus reach the following corollary:

Corollary 3. *Let $s + k \geq 1$ be an integer. Suppose $w_0 \in H^{s+k}$ and let $w^{\epsilon,\delta}$ be the solution of the KdV equation (34) with initial data $w(x,0) = w_0(x)$. Then there is some C such that for all $\delta^2 \in (0, \frac{1}{3}]$ and all $\epsilon \in \left[\frac{\delta^2}{c_2}, \frac{\delta^2}{c_1}\right]$ with positive constants c_1 and c_2 and all $t \geq 0$,*

$$\left\|w^{\epsilon,\delta}(t)\right\|_{H^{s+k}} + \left\|w_t^{\epsilon,\delta}(t)\right\|_{H^{s+k-3}} \leq C.$$

We next plug the solution $w^{\epsilon,\delta}$ of the KdV equation (34) into the IB equation. Again, omitting the indices ϵ, δ, the residual term f is given by

(7). Following the steps in Section 3, we obtain f corresponding to the case of the KdV approximation in the form $f = F_x$ with

$$F = D_x \left\{ \frac{1}{3}\epsilon^2 w^3 + \frac{1}{4}\epsilon\delta^2 \left[-3(w_x)^2 + 4(ww_x)_t \right] + \frac{1}{4}\delta^4(-w_{xxxx} + 2w_{xxxt}) \right\}.$$

As there are at most five derivatives of w and four derivatives of w_t in F, we will choose $k = 7$ in the corollary to get the KdV version of Lemma 3.1, namely the estimate:

$$\|F(t)\|_{H^s} \leq C\epsilon^2$$

for the residual term.

Although the above results hold for all times, to follow the approach in the previous sections we fix some $T > 0$ and restrict ourselves to the time interval $[0, \frac{T}{\epsilon}]$. As in the previous cases, the residual estimate leads to the following theorem:

Theorem 6.4. *Let $w_0 \in H^{s+7}(\mathbb{R})$, $s \geq 1$ an integer and suppose $w^{\epsilon,\delta}$ is the solution of the KdV equation (34) with initial value $w(x, 0) = w_0(x)$. Then, for any $T > 0$ and $0 < c_1 < c_2$ there exist $\delta_1^2 \leq \frac{1}{3}$ and $C > 0$ such that the solution $u^{\epsilon,\delta}$ of the Cauchy problem for the IB equation*

$$u_{tt} - u_{xx} - \delta^2 u_{xxtt} - \epsilon(u^2)_{xx} = 0$$
$$u(x, 0) = w_0(x),\ \ u_t(x, 0) = w_t^{\epsilon,\delta}(x, 0),$$

satisfies

$$\|u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)\|_{H^s} \leq\ C\epsilon^2 t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ *and all* $\delta \in (0, \delta_1]$, $\epsilon \in \left[\frac{\delta^2}{c_2}, \frac{\delta^2}{c_1}\right]$.

The result in Theorem 6.4, namely the rigorous justification of the KdV approximation of the IB equation, was already proved by Schneider [18]. The discussion in Section 5 allows us to prove a similar theorem for the general class of nonlocal wave equations. Again we have to estimate the term $D_x^3 M_\delta w_{tt}$ in the residue F^M. We get

$$\left\| D_x^3 M_\delta w_{tt} \right\|_{H^s} \leq \|w_{tt}\|_{H^{s+3+\sigma-4}} \leq C\,\|w\|_{H^{s+3+\sigma-4+6}} = C\,\|w\|_{H^{s+\sigma+5}}\,,$$

which requires taking $k = \sigma + 5$ in Corollary 3. Hence we get:

Theorem 6.5. *Let* $w_0 \in H^{s+\sigma+5}(\mathbb{R})$, $s > 1/2$, $s+\sigma$ *an integer,* $\sigma = \max\{r, 4\}$ *and suppose* $w^{\epsilon,\delta}$ *is the solution of the KdV equation (34) with initial value* $w(x,0) = w_0(x)$*. Then, for any* $T > 0$ *and* $0 < c_1 < c_2$ *there exist* $\delta_1^2 \leq \frac{1}{3}$ *and* $C > 0$ *such that the solution* $u^{\epsilon,\delta}$ *of the Cauchy problem for the nonlocal equation*

$$\begin{aligned} u_{tt} &= \beta_\delta * (u + \epsilon u^2)_{xx} \\ u(x,0) &= w_0(x),\ u_t(x,0) = w_t^{\epsilon,\delta}(x,0), \end{aligned}$$

satisfies

$$\|u^{\epsilon,\delta}(t) - w^{\epsilon,\delta}(t)\|_{H^s} \leq C\epsilon^2 t$$

for all $t \in \left[0, \frac{T}{\epsilon}\right]$ *and all* $\delta \in (0, \delta_1]$, $\epsilon \in \left[\frac{\delta^2}{c_2}, \frac{\delta^2}{c_1}\right]$.

We finally note that in the KdV case T can be chosen arbitrarily large while in the CH or the BBM cases T is determined by the equation.

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