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Minimal immersions of Riemannian manifolds in products of space forms

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\textbf{ABSTRACT}

In this paper, we give natural extensions to cylinders and tori of a classical result due to T. Takahashi \cite{8} about minimal immersions into spheres. More precisely, we deal with Euclidean isometric immersions whose projections in $\mathbb{R}^N$ satisfy a spectral condition of their Laplacian.

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1. Introduction

An isometric immersion $f: M^m \to N^n$ of a Riemannian manifold $M$ in another Riemannian manifold $N$ is said to be \textit{minimal} if its mean curvature vector field $H$ vanishes. The study of minimal surfaces is one of the oldest subjects in differential geometry, having its origin with the work of Euler and Lagrange. In the last century, a series of works have been developed in the study of properties of minimal immersions, whose ambient space has constant sectional curvature. In particular, minimal immersions in the sphere $S^n$ play an important role in the theory, as for example the famous paper of J. Simons \cite{7}.

Let $f: M^m \to \mathbb{R}^n$ be an isometric immersion of an $m$-dimensional manifold $M$ into the Euclidean space $\mathbb{R}^n$. Associated with the induced metric on $M$, it is defined the Laplace operator $\Delta$ acting on $C^\infty(M)$. This Laplacian can be extended in a natural way to the immersion $f$. A well-known result by J. Eells and J.H. Sampson \cite{3} asserts that the immersion $f$ is minimal if and only if $\Delta f = 0$. The following result, due to T. Takahashi \cite{8}, states that the immersion $f$ realizes a minimal immersion in a sphere if and only if its coordinate functions are eigenfunctions of the Laplace operator with the same nonzero eigenvalue.

\textbf{Theorem 1}. Let $F: M^m \to \mathbb{R}^{n+1}$ be an isometric immersion such that

$$\Delta F = -mcF$$

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for some constant $c \neq 0$. Then $c > 0$ and there exists a minimal isometric immersion $f : M^m \to S^n_c$ such that $F = i \circ f$.

O. Garay generalized Theorem 1 for the hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ satisfying $\Delta f = Af$, where $A$ is a constant $(n + 1) \times (n + 1)$ diagonal matrix. He proved in [4] that such a hypersurface is either minimal or an open subset of a sphere or of a cylinder. In this direction, J. Park [6] classified the hypersurfaces in a space form or in Lorentzian space whose immersion $f$ satisfies $\Delta f = Af + B$, where $A$ is a constant square matrix and $B$ is a constant vector. Similar results were obtained in [1], where the authors study and classify pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which satisfy the condition $\Delta f = Af + B$, where $A$ is an endomorphism and $B$ is a constant vector. We would point out that this problem is strongly connected to other related topics as immersions whose mean curvature satisfies a polynomial equation in the Laplacian, biharmonic submanifolds, finite type submanifolds and others. For the last, we refer to [2], where the author discusses the problem of determining the geometrical structure of a submanifold knowing some simple analytic information.

In this work we shall deal with an isometric immersion $f : M^m \to \mathbb{R}^N$ of a Riemannian manifold $M^m$ into the Euclidean space $\mathbb{R}^N$. If the submanifold $f(M)$ is contained in a cylinder $S^n_c \times \mathbb{R}^k \subset \mathbb{R}^N$ or in a torus $S^n_c \times S^k_d \subset \mathbb{R}^N$, we shall call that the immersion $f$ realizes an immersion in a cylinder or in a torus, respectively. Motivated by recent works on the submanifold theory in the product of space forms [5], we obtain theorems that give us necessary and sufficient conditions for an isometric immersion $f : M^m \to \mathbb{R}^N$ to realize a minimal immersion in a cylinder or in a torus (cf. Theorems 4 and 8).

2. Preliminaries

Let $M^m$ be a Riemannian manifold and $h \in C^\infty(M)$. The hessian of $h$ is the symmetric section of Lin($TM \times TM$) defined by

$$\text{Hess } h(X, Y) = XY(h) - \nabla_X Y(h)$$

for all $X, Y \in TM$. Equivalently,

$$\text{Hess } h(X, Y) = \langle \nabla_X \text{grad } h, Y \rangle$$

where $X, Y \in TM$ and grad $h$ is the gradient of $h$. The Laplacian $\Delta h$ of a function $h \in C^\infty(M)$ at the point $p \in M$ is defined as

$$\Delta h(p) = \text{trace Hess } h(p) = \text{div} \text{ grad } h(p).$$

Consider now an isometric immersion $f : M^m \to \mathbb{R}^n$. For a fixed $v \in \mathbb{R}^n$, let $h \in C^\infty(M)$ be the height function with respect to the hyperplane normal to $v$, given by $h(p) = \langle f(p), v \rangle$. Then

$$\text{Hess } h(X, Y) = \langle \alpha_f(X, Y), v \rangle$$

for any $X, Y \in TM$. For an isometric immersion $f : M^n \to \mathbb{R}^n$, by $\Delta f(p)$ at the point $p \in M$ we mean the vector

$$\Delta f(p) = (\Delta f_1(p), \ldots, \Delta f_n(p)),$$

where $f = (f_1, \ldots, f_n)$. Taking traces in (1) we obtain
\[ \Delta f(p) = mH(p), \]  

where \( H(p) \) is the mean curvature vector of \( f \) at \( p \in M \).

3. Minimal submanifolds in \( S^n_c \times \mathbb{R}^k \)

Let \( S^n_c \) denote the sphere with constant sectional curvature \( c > 0 \) and dimension \( n \). We use the fact that \( S^n_c \) admits a canonical isometric embedding in \( \mathbb{R}^{n+1} \) as

\[ S^n_c = \{ X \in \mathbb{R}^{n+1} : \langle X, X \rangle = 1/c \}. \]

Thus, \( S^n_c \times \mathbb{R}^k \) admits a canonical isometric embedding

\[ i : S^n_c \times \mathbb{R}^k \to \mathbb{R}^{n+k+1}. \]

Denote by \( \pi : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+1} \) the canonical projection. Then, the normal space of \( i \) at each point \( z \in S^n_c \times \mathbb{R}^k \) is spanned by \( N(z) = c(\pi \circ i)(z) \), and the second fundamental form of \( i \) is given by

\[ \alpha_i(X, Y) = -c(\pi X, Y)\pi \circ i. \]

If we consider a parallel orthonormal frame \( E_1, \ldots, E_{n+k+1} \) of \( \mathbb{R}^{n+k+1} \) such that

\[ \mathbb{R}^k = \text{span}\{E_{n+2}, \ldots, E_{n+k+1}\}, \]  

we can express the second fundamental form \( \alpha_i \) as

\[ \alpha_i(X, Y) = -c \left( \langle X, Y \rangle - \sum_{i=n+2}^{n+k+1} \langle X, E_i \rangle \langle Y, E_i \rangle \right) \pi \circ i. \]  

(4)

The following result shows that minimal immersions of an \( m \)-dimensional Riemannian manifold into the cylinder \( S^n_c \times \mathbb{R}^k \) are precisely those immersions whose \( n+1 \) first coordinate functions in \( \mathbb{R}^{n+k+1} \) are eigenfunctions of the Laplace operator in the induced metric.

**Proposition 2.** Let \( f : M^m \to S^n_c \times \mathbb{R}^k \) be an isometric immersion and set \( F = i \circ f \), where \( i : S^n_c \times \mathbb{R}^k \to \mathbb{R}^{n+k+1} \) is the canonical inclusion. Let \( E_1, \ldots, E_{n+k+1} \) be a parallel orthonormal frame of \( \mathbb{R}^{n+k+1} \) as in (3). Then \( f \) is a minimal immersion if and only if

\[ \Delta F = -c \left( m - \sum_{j=n+2}^{n+k+1} \|T_j\|^2 \right) \pi \circ F, \]  

(5)

where \( T_j \) denotes the orthogonal projection of \( E_j \) onto \( TM \).

**Proof.** The second fundamental forms of \( f \) and \( F \) are related by

\[ \alpha_F(X, Y) = i_\ast \alpha_f(X, Y) + \alpha_i(f_\ast X, f_\ast Y) \]

for all \( X, Y \in TM \). From (4) we get that

\[ \alpha_F(X, Y) = i_\ast \alpha_f(X, Y) - c \left( \langle X, Y \rangle - \sum_{j=n+2}^{n+k+1} \langle X, T_j \rangle \langle Y, T_j \rangle \right) \pi \circ F, \]
where $T_j$ denotes the orthogonal projection of $E_i$ onto $TM$. Taking traces and using (2) yields

$$\Delta F = m_i H^f - c\left( m - \sum_{j=n+2}^{n+k+1} \|T_j\|^2 \right) \pi \circ F,$$

and the conclusion follows. $\square$

**Remark 3.** In case $f : M^m \to S^n_c \times \mathbb{R}$, a tangent vector field $T$ on $M$ and a normal vector field $\eta$ along $f$ are defined by

$$\frac{\partial}{\partial t} = f_* T + \eta,$$

where $\frac{\partial}{\partial t}$ is a unit vector field tangent to $\mathbb{R}$. In this case, $f$ is a minimal immersion if and only if

$$\Delta F = -c(m - \|T\|^2) \pi \circ F.$$

The next result states that any isometric immersion of a Riemannian manifold $M^m$ into Euclidean space $\mathbb{R}^{n+k+1}$, whose Laplacian satisfies a condition as in (5), arises for a minimal isometric immersion of $M$ into some cylinder $S^n_c \times \mathbb{R}^k$.

**Theorem 4.** Let $F : M^m \to \mathbb{R}^{n+k+1}$ be an isometric immersion and let $E_1, \ldots, E_{n+k+1}$ be a parallel orthonormal frame in $\mathbb{R}^{n+k+1}$ such that

$$\Delta F = -c\left( m - \sum_{j=n+2}^{n+k+1} \|T_j\|^2 \right) \left( F - \sum_{j=n+2}^{n+k+1} \langle F, E_j \rangle E_j \right),$$

for some constant $c \neq 0$, where $T_j$ denotes the orthogonal projection of $E_j$ onto the tangent bundle $TM$. Then $c > 0$ and there exists a minimal isometric immersion $f : M^m \to S^n_c \times \mathbb{R}^k$ such that $F = i \circ f$.

**Proof.** Since $\Delta F = mH$ by (2), the assumption implies that the vector field

$$N = F - \sum_{j=n+2}^{n+k+1} \langle F, E_j \rangle E_j$$

is normal to $F$. On the other hand,

$$\langle N, E_j \rangle = \left( F - \sum_{l=n+2}^{n+k+1} \langle F, E_l \rangle E_l, E_j \right) = \langle F, E_j \rangle - \langle F, E_j \rangle = 0$$

for all $n + 2 \leq j \leq n + k + 1$. Hence, for any $X \in TM$ we have

$$X \langle N, N \rangle = 2 \left( F, X - \sum_{j=n+2}^{n+k+1} \langle F, X, E_j \rangle E_j, N \right) = 0,$$
and it follows that \( \langle N, N \rangle = r^2 \) for some constant \( r \). Now we claim that
\[
\Delta \|F\|^2 = 2 \sum_{j=n+2}^{n+k+1} \|T_j\|^2.
\] (6)

To see this, fix a point \( p \in M \) and consider a local geodesic frame \( \{X_1, \ldots, X_m\} \) in \( p \). Then
\[
\text{grad} \|F\|^2 = \sum_{\alpha=1}^{m} X_\alpha (\|F\|^2) X_\alpha = 2 \sum_{\alpha=1}^{m} \langle F_\ast X_\alpha, F \rangle X_\alpha = 2F^T.
\]
Since \( N \) is normal to \( F \), we have
\[
F^T = \sum_{j=n+2}^{n+k+1} \langle F, E_j \rangle T_j = \sum_{j=n+2}^{n+k+1} \sum_{\alpha=1}^{m} \langle F, E_j \rangle \langle E_j, X_\alpha \rangle X_\alpha,
\]
and it follows that
\[
\text{grad} \|F\|^2 = 2 \sum_{j=n+2}^{n+k+1} \sum_{\alpha=1}^{m} \langle F, E_j \rangle \langle E_j, X_\alpha \rangle X_\alpha.
\]
Therefore,
\[
\Delta \|F\|^2 = \sum_{\beta=1}^{m} \langle \nabla_{X_\beta} \text{grad} \|F\|^2, X_\beta \rangle = 2 \sum_{\alpha, \beta=1}^{m} \sum_{j=n+2}^{n+k+1} \langle \nabla_{X_\beta} \langle F, E_j \rangle \langle E_j, X_\alpha \rangle X_\alpha, X_\beta \rangle
\]
\[
= 2 \sum_{\alpha, \beta=1}^{m} \sum_{j=n+2}^{n+k+1} \langle F_\ast X_\beta, E_j \rangle \langle E_j, X_\alpha \rangle \langle X_\alpha, X_\beta \rangle
\]
\[
= 2 \sum_{\alpha=1}^{m} \sum_{j=n+2}^{n+k+1} \langle X_\alpha, T_j \rangle^2 = 2 \sum_{j=n+2}^{n+k+1} \|T_j\|^2,
\]
and this proves our claim. Finally, using the fact that
\[
\Delta \|F\|^2 = 2(\langle \Delta F, F \rangle + m),
\]
we get that
\[
\sum_{j=n+2}^{n+k+1} \|T_j\|^2 = \langle \Delta F, F \rangle + m = \left\langle -c \left( m - \sum_{j=n+2}^{n+k+1} \|T_j\|^2 \right) N, N \right\rangle + m
\]
\[
= - \left( m - \sum_{j=n+2}^{n+k+1} \|T_j\|^2 \right) cr^2 + m,
\]
and the equality above implies that \( c = 1/r^2 \). We conclude that there exists an isometric immersion \( f : M^m \to S^n_c \times \mathbb{R}^k \) such that \( F = i \circ f \), and minimality of \( f \) follows from Proposition 2. □
A simple application of Theorem 1 is to show that the Veronese surface $f : S_{1/3}^2 \rightarrow S^4$, given by

$$f(x, y, z) = \left(\frac{yz}{\sqrt{3}}, \frac{xz}{\sqrt{3}}, \frac{xy}{\sqrt{3}}, \frac{x^2 - y^2}{2\sqrt{3}}, \frac{x^2 + y^2 - 2z^2}{6}\right),$$

is a minimal surface. In this case, it is straightforward to verify that $\Delta f = -2f$. As an application of Theorem 4 we will construct an example into $S^{2n-1} \times \mathbb{R}$.

**Example 5.** Given two positive integer numbers $n$ and $k$, with $1 < k < n$, consider the immersion $f : \mathbb{R}^n \rightarrow \mathbb{R}^{2n+1}$ given by

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{k}} \left( e^{i\sqrt{k} x_1}, \ldots, e^{i\sqrt{k} x_k}, \sqrt{k} \sum_{j=k+1}^{n} x_j \right).$$

$f$ is an isometric immersion and satisfies the hypothesis of Theorem 4. In fact, in this case, we have

$$\Delta f = -k \left( f - \left< f, \frac{\partial}{\partial t} \right> \frac{\partial}{\partial t} \right),$$

where $\frac{\partial}{\partial t}$ denotes a unit vector field tangent to the factor $\mathbb{R}$. Therefore, according to Theorem 4, $f$ realizes a minimal isometric immersion into $S^{2n-1} \times \mathbb{R}$.

4. **Minimal submanifolds in the product $S^n_c \times S^k_d$**

Let us now consider two spheres $S^n_c$ and $S^k_d$ with their respective curvatures and dimensions. Using the fact that the spheres admit a canonical isometric embedding $S^n_c \subset \mathbb{R}^{n+1}$ and $S^k_d \subset \mathbb{R}^{k+1}$, the product $S^n_c \times S^k_d$ admits a canonical isometric embedding

$$i : S^n_c \times S^k_d \rightarrow \mathbb{R}^{n+k+2}.$$  \hspace{1cm} (7)

Denote by $\pi_1 : \mathbb{R}^{n+k+2} \rightarrow \mathbb{R}^{n+1}$ and $\pi_2 : \mathbb{R}^{n+k+2} \rightarrow \mathbb{R}^{k+1}$ the canonical projections. Then, the normal space of $i$ at each point $z \in S^n_c \times S^k_d$ is spanned by $N_1(z) = c(\pi_1 \circ i)(z)$ and $N_2(z) = d(\pi_2 \circ i)(z)$, and the second fundamental form of $i$ is given by

$$\alpha_i(X, Y) = -c(\pi_1 X, Y) N_1 - d(\pi_2 X, Y) N_2,$$

for all $X, Y \in T_z(S^n_c \times S^k_d)$.

Now, let $f : M^m \rightarrow S^n_c \times S^k_d$ be an isometric immersion of a Riemannian manifold $M^m$. Then, writing $F = i \circ f$, the unit vector fields $N_1 = \pi_1 \circ F$ and $N_2 = \pi_2 \circ F$ are normal to $F$. Consider a parallel orthonormal frame $E_1, \ldots, E_{n+k+2}$ of $\mathbb{R}^{n+k+2}$ such that

$$\mathbb{R}^{n+1} = \text{span}\{E_1, \ldots, E_{n+1}\} \quad \text{and} \quad \mathbb{R}^{k+1} = \text{span}\{E_{n+2}, \ldots, E_{n+k+2}\}.$$  \hspace{1cm} (8)

In terms of this frame, we can express the vector fields $N_1$ and $N_2$ as

$$N_1 = F - \sum_{j=n+2}^{n+k+2} \langle F, E_j \rangle E_j \quad \text{and} \quad N_2 = F - \sum_{i=1}^{n+1} \langle F, E_i \rangle E_i.$$  \hspace{1cm} (9)
Proposition 6. Let \( f : M^m \to S^n_c \times S^k_d \) be an isometric immersion and set \( F = i \circ f \), where \( i : S^n_c \times S^k_d \to \mathbb{R}^{n+k+2} \) is the canonical inclusion. Let \( E_1, \ldots, E_{n+k+2} \) be a parallel orthonormal frame of \( \mathbb{R}^{n+k+2} \) as in (8). Then \( f \) is a minimal isometric immersion if and only if

\[
\Delta F = -c \left( m - \sum_{j=n+1}^{n+k+2} \| T_j \|^2 \right) N_1 - d \left( m - \sum_{l=1}^{n+1} \| T_l \|^2 \right) N_2,
\]

where \( T_j \) denotes the orthogonal projection of \( E_j \) onto \( TM \).

Proof. The second fundamental forms of \( f \) and \( F \) are related by

\[
\alpha_F(X, Y) = i_* \alpha_f(X, Y) + \alpha_i(f_* X, f_* Y)
\]

for all \( X, Y \in TM \). Let \( E_1, \ldots, E_{n+k+2} \) be a parallel orthonormal frame of \( \mathbb{R}^{n+k+2} \) as in (8). Given \( X \in TM \), we can write

\[
\pi_1 X = X - \sum_{j=n+1}^{n+k+2} \langle X, T_j \rangle E_j \quad \text{and} \quad \pi_2 X = X - \sum_{l=1}^{n+1} \langle X, T_l \rangle E_l,
\]

and so, we have

\[
\langle \pi_1 X, Y \rangle = \langle X, Y \rangle - \sum_{j=n+1}^{n+k+2} \langle X, T_j \rangle \langle Y, T_j \rangle
\]

and

\[
\langle \pi_2 X, Y \rangle = \langle X, Y \rangle - \sum_{l=1}^{n+1} \langle X, T_l \rangle \langle Y, T_l \rangle.
\]

Then the second fundamental form of \( F \) can be expressed by

\[
\alpha_F(X, Y) = i_* \alpha_f(X, Y) - c \left( \langle X, Y \rangle - \sum_{j=n+1}^{n+k+2} \langle X, T_j \rangle \langle Y, T_j \rangle \right) N_1 - d \left( \langle X, Y \rangle - \sum_{l=1}^{n+1} \langle X, T_l \rangle \langle Y, T_l \rangle \right) N_2.
\]

Taking traces and using (2) yields

\[
\Delta F = m H_f = m i_* H_f - c \left( m - \sum_{j=n+1}^{n+k+2} \| T_j \|^2 \right) N_1 - d \left( m - \sum_{l=1}^{n+1} \| T_l \|^2 \right) N_2,
\]

and the conclusion follows. \( \square \)

Remark 7. Observe that the torus \( S^n_c \times S^k_d \) admits a canonical isometric embedding in the sphere \( S^{n+k+1}_\kappa \), where \( \kappa = \frac{cd}{e+d} \). Therefore, any isometric immersion \( f : M^m \to S^n_c \times S^k_d \) can be seen as an isometric immersion \( \tilde{f} = i \circ f : M^m \to S^{n+k+1}_\kappa \), where \( i : S^n_c \times S^k_d \to S^{n+k+1}_\kappa \) denotes the canonical inclusion.

The next result states that any isometric immersion of a Riemannian manifold \( M^m \) into the sphere \( S^{N-1}_\kappa \subset \mathbb{R}^N \) with constant sectional curvature \( \kappa \), whose Laplacian of the coordinate functions satisfies a condition as in (10), arises as a minimal isometric immersion of \( M^m \) into a product of spheres \( S^n_c \times S^k_d \subset \mathbb{R}^N \).
Theorem 8. Let $F : M^m \rightarrow \mathbb{S}^{N-1}_c$ be an isometric immersion. Fixed a choice of two integers $n$ and $k$, with $N = n + k + 2$, let $E_1, \ldots, E_N$ be a parallel orthonormal frame in $\mathbb{R}^N$ as in (8) such that

$$
\Delta \tilde{F} = -c \left( m - \sum_{j=n+2}^{n+k+2} \|T_j\|^2 \right) N_1 - d \left( m - \sum_{i=1}^{n+1} \|T_i\|^2 \right) N_2,
$$

where $\tilde{F} = h \circ F$, $h : \mathbb{S}^{N-1}_c \rightarrow \mathbb{R}^N$ is the umbilical inclusion, $T_i$ denotes the orthogonal projection of $E_i$ onto $TM$, $N_1$ and $N_2$ are as in (9), and $c$ and $d$ are real numbers such that $\kappa = \frac{cd}{c+d}$. Then there exists a minimal isometric immersion $f : M^m \rightarrow \mathbb{S}^n_c \times \mathbb{S}^k_d$ such that $F = i \circ f$.

Proof. We first prove that $N_1$ and $N_2$ are normal to $F$. In fact, in terms of an orthonormal frame $\{X_1, \ldots, X_m\}$ of $TM$, we have

$$
\sum_{i=1}^{N} \|T_i\|^2 = \sum_{i=1}^{n+1} \|T_i\|^2 + \sum_{j=n+2}^{N} \|T_j\|^2 = m. \tag{11}
$$

Then, as $\tilde{F} = N_1 + N_2$, we can write:

$$
\Delta \tilde{F} = -c \left( \sum_{i=1}^{n+1} \|T_i\|^2 \right) N_1 - d \left( \sum_{j=n+2}^{N} \|T_j\|^2 \right) N_2 \\
= -c \left( \sum_{i=1}^{n+1} \|T_i\|^2 \right) \tilde{F} + \left( c \sum_{i=1}^{n+1} \|T_i\|^2 - d \sum_{j=n+2}^{N} \|T_j\|^2 \right) N_2. \tag{12}
$$

If

$$
c \sum_{i=1}^{n+1} \|T_i\|^2 = d \sum_{j=n+2}^{N} \|T_j\|^2,
$$

we have, by using (11), that

$$
\Delta \tilde{F} = -\frac{d}{c+d} m \tilde{F}.
$$

Thus, it follows from Theorem 1 that there exists a minimal isometric immersion $f : M^m \rightarrow \mathbb{S}^{N-1}_{c+d}$ such that $\tilde{F} = i \circ f$, where $i : \mathbb{S}^{N-1} \rightarrow \mathbb{R}^N$ is the umbilical inclusion. Suppose from now on that

$$
c \sum_{i=1}^{n+1} \|T_i\|^2 \neq d \sum_{j=n+2}^{N} \|T_j\|^2. \tag{13}
$$

As $\Delta \tilde{F} = mH$ and $\tilde{F}$ is normal to $M$, we conclude from (12) that $N_2$ is normal to $M$. Similarly we obtain that $N_1$ is normal to $M$. Now, for $n + 2 \leq j \leq n + k + 2$, we have

$$
\langle N_1, E_j \rangle = \left\langle \tilde{F} - \sum_{i=n+2}^{N} \langle \tilde{F}, E_i \rangle E_i, E_j \right\rangle \\
= \langle \tilde{F}, E_j \rangle - \langle \tilde{F}, E_j \rangle = 0.
$$
Hence, for any $X \in TM$ we have

$$X \langle N_1, N_1 \rangle = 2\left(\tilde{F}_s X - \sum_{j=n+2}^N \langle \tilde{F}_s X, E_j \rangle E_j, N_1 \right) = 0,$$

and it follows that $\langle N_1, N_1 \rangle = r^2$, for some constant $r$. The same argument gives $\langle N_2, N_2 \rangle = s^2$ for some constant $s$. Since $\tilde{F} = N_1 + N_2$ and $\Delta \|\tilde{F}\|^2 = 2(\langle \Delta \tilde{F}, \tilde{F} \rangle + m)$, we have

$$0 = \frac{1}{2} \Delta \|\tilde{F}\|^2 = \langle \Delta \tilde{F}, \tilde{F} \rangle + m$$

$$= -c \left( m - \sum_{j=n+2}^N \|T_j\|^2 \right) r^2 - d \left( m - \sum_{l=1}^{n+1} \|T_l\|^2 \right) s^2 + m$$

$$= -c \left( \sum_{l=1}^{n+1} \|T_l\|^2 \right) r^2 - d \left( \sum_{j=n+2}^N \|T_j\|^2 \right) s^2 + m.$$

Since $r^2 + s^2 = \frac{c+d}{cd}$, we can rewrite the above equation as

$$\left( c \sum_{l=1}^{n+1} \|T_l\|^2 - d \sum_{j=n+2}^N \|T_j\|^2 \right) s^2 = \frac{c+d}{cd} \sum_{l=1}^{n+1} \|T_l\|^2 - m$$

$$= d \left( c \sum_{l=1}^{n+1} \|T_l\|^2 - d \sum_{j=n+2}^N \|T_j\|^2 \right).$$

As we are assuming (13), we obtain $d = \frac{1}{2}$, and therefore $c = \frac{1}{2}$. We conclude that there exists an isometric immersion $f : M^m \to S^n_c \times S^d_B$ such that $F = i \circ f$, and minimality of $f$ follows from Proposition 6. □

References