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2014-11

Noncontractible loops in the dense O(n) loop model on the cylinder

Noncontractible loops in the dense O(*n*) loop model on the cylinder

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A lattice model of critical dense polymers O(*n*) is considered for finite cylinder geometry. Due to the presence of noncontractible loops with a fixed fugacity ξ, the model at $n = 0$ is a generalization of the critical dense polymers solved by Pearce, Rasmussen, and Villani. We found the free energy for any height N and circumference L of the cylinder. The density ρ of noncontractible loops is obtained for $N \to \infty$ and large L. The results are compared with those found for the anisotropic quantum chain with twisted boundary conditions. Using the latter method, we derived ρ for any O(*n*) model and an arbitrary fugacity.

I. INTRODUCTION

The dense O(*n*) loop model [1] is defined by drawing two arcs in each elementary cell of the square lattice: the two possible states of the cell are shown in Fig. 1. The lines on the whole lattice with appropriate boundary conditions form a system of closed loops with the Boltzmann weight *n* ascribed to every loop. Two particular cases, $n = 1$ and 0, are of particular interest. At $n = 1$, the model can be mapped to the bond percolation problem, the six-vertex model, and the XXZ quantum chain [2,3]. At $n = 0$, the bulk loops disappear and the set of lines is converted into the model of critical dense polymers [4]. This model is a recent representative of the more general two-dimensional polymer theory initiated by Saleur and Duplantier in the context of a conformal field theory [5–7]. The dense polymer model is the first member $\mathcal{LM}(1,2)$ of the Yang-Baxter integrable series of logarithmic minimal models. In the scaling limit, the central charge is $c = -2$ and the conformal weights yield the Kac formula for the infinitely extended Kac table [4].

Pearce, Rasmussen, and Villani [8] have solved the dense polymer model on a cylinder using the single-row transfer matrix and the inverse identity for the transfer matrix. They solved the inverse identity for different boundary conditions, including the case in which noncontractible loops are allowed. The structure of the inversion identity dictates its solvability at fugacity 2 of the noncontractible loops. The fixed fugacity prevents the evaluation of the free energy as a function of fugacity, so the average number of loops on the cylinder remains unknown. At the same time, the statistics of noncontractible loops is important as it is related to the conformal properties of the model in the continuous limit. The standard tool for determination of the central charge c of a conformal field theory, corresponding to a given lattice model at the critical point, is the $1/L$ expansion of the free energy of an infinitely long cylinder of finite perimeter L. The two leading terms of

the expansion have the form

$$F = f_{\text{bulk}}L - \frac{\pi c_{\text{eff}}}{6L}, \tag{1}$$

where f_{bulk} is the bulk density of the free energy per unit length, and c_{eff} is an effective central charge, which is a combination of the true central charge c and a correction term depending on the boundary conditions [9]. It will be shown below that the contribution of the noncontractible loops in F is of the order of $1/L$. Therefore, c_{eff} depends on the presence of such loops and their fugacity in the case of two-dimensional lattice models, and on a twist parameter in the corresponding quantum chains.

In this paper, we solve the model of dense polymers on the cylinder by calculating the partition function of a spanning web model on a finite cylinder in the presence of cycles winding around the cylinder. Our aim is to evaluate the grand partition function of the dense polymers model at arbitrary fugacity of the noncontractible loops and to find their density per unit height of the cylinder.

For an infinite cylinder of perimeter L, there is an alternative, albeit simpler, method to compute the free energy and the density of the noncontractible loops for any O(*n*) model at arbitrary fugacity. The states of the O(*n*) model can be defined in terms of a connectivity condition for the points of intersection between the loops and a horizontal line cutting the loops at these points. Two points are connected by a link if there exists a line between them via the half-space above the cut. For instance, imposing periodic boundary conditions in the horizontal direction on the lattice in Fig. 2 and specifying the boundary conditions at the upper edge, we obtain three minimal links between points 1 and 6, 2 and 3, and 4 and 5, which are points of intersection between loops and the bottom line of the lattice belonging to the upper half-plane. A typical configuration of links for a larger lattice is given in Fig. 3. The link representation allows definitions of the transfer matrix and the Hamiltonian as elements of the periodic Temperley-Lieb algebra [10,11]. The same algebra has a matrix representation that takes us to the XXZ spin-1/2 quantum chain with a twist depending on the fugacity. The ground-state energy is known analytically [13] for large values of L for any

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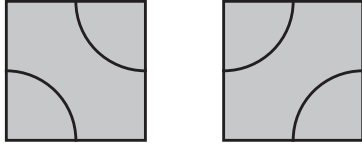*Corresponding author: brankov@theor.jinr.ru

FIG. 1. Elementary cells.

FIG. 3. A link configuration.

II. THE DENSITY OF NONCONTRACTIBLE LOOPS OBTAINED FROM THE *XXZ* QUANTUM CHAIN

anisotropy related to the parameter n of the O(n) model. The Temperley-Lieb algebra has a quotient with a free parameter that can be identified with the fugacity of noncontractible loops, so we can compute their density from the ground state of the Hamiltonian. In the special case of $n = 0$, one can use a Jordan-Wigner transformation and obtain the ground-state energy for any finite L explicitly.

The paper is organized as follows. In Sec. II, we present the calculation of the density of noncontractible loops using the *XXZ* quantum chain. We start with this presentation since, unlike the lattice case, the calculation is almost trivial. We also show that the probability distribution of noncontractible loops is not Gaussian. In Sec. III, we consider the special case $n = 0$. Due to the absence of contractible loops, the O(n) loop model can be mapped on the spanning webs model. In the latter model, one can generalize the Kirchhoff theorem and bring the calculation to that of determinants. The details of the proof are given in the Appendix. The calculation of the partition function on an $L \times N$ torus is presented in Sec. IV, and the result is given by Eq. (31). The case of finite cylinders of height N and perimeter L, with different boundary conditions on the top and bottom of the cylinder, is briefly mentioned in Sec. V. Finally, in Sec. VI, we consider the case of an infinite cylinder by taking N to infinity, and we compare the thus obtained results for the density of the noncontractible loops to those of Sec. II.
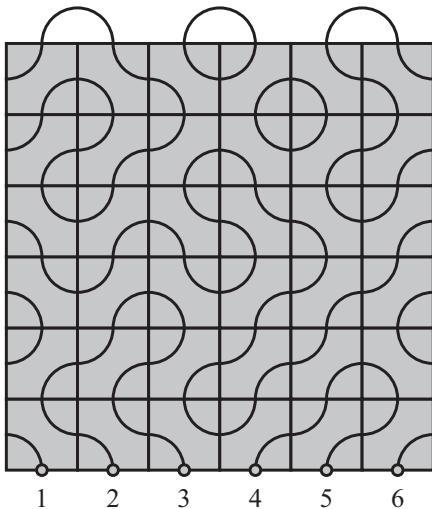

FIG. 2. Loops on the horizontally periodic lattice. One noncontractible loop joins the left and the right sides of the figure.
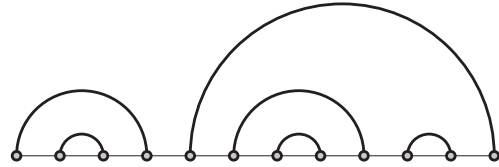
We remind the reader of a few facts about the *periodic* Temperley-Lieb algebra (PTL) and some of its representations [11]. This algebra provides the key to the calculation of the density of noncontractible loops. The PTL has L generators e_i ($i = 1,2,\ldots,L$) satisfying the relations

$$e_i^2 = x e_i, \quad e_i e_{i\pm 1} e_i = e_i,$$
$$e_i e_j = e_j e_i \, (|i - j| > 1), \quad e_{i+L} = e_i, \tag{2}$$

where x is a parameter.

We are going to consider the L-even case only. The PTL is infinite-dimensional, therefore we take a quotient that makes it finite-dimensional:

$$ABA = \alpha^2 A, \tag{3}$$

where

$$A = \prod_{i=1}^{L/2} e_{2i}, \quad B = \prod_{i=0}^{L/2-1} e_{1+2i}. \tag{4}$$

We are interested in two representations of the PTL with the quotient (3). The first one is the spin representation, in which

$$e_i = \sigma_i^+ \sigma_{i+1}^- e^{i\phi/L} + \sigma_i^- \sigma_{i+1}^+ e^{-i\phi/L} - \frac{\cos(\gamma)}{2}\sigma_i^z \sigma_{i+1}^z$$
$$+ \frac{i}{2}\sin(\gamma)\big(\sigma_{i+1}^z - \sigma_i^z\big) + \frac{\cos(\gamma)}{2}, \tag{5}$$

where

$$x = 2\cos(\gamma), \quad \alpha = 2\cos(\phi/2), \tag{6}$$

and $\sigma^{\pm,z}$ are the Pauli matrices [10] and ϕ is the boundary twist parameter.

In the second representation, the generators act in the vector space of periodic link patterns. Each link pattern is one of the $\binom{L}{L/2}$ configurations of nonintersecting arches joining L sites on a circle. One can visualize the circle on a cylinder. Besides the link patterns on the same cylinder, one takes m circles with no sites on them to represent m noncontractible loops. In Fig. 4, we show the six configurations for $L = 4$ and $m = 2$. With few exceptions, the generators e_i act on the configurations in the standard way (as in the nonperiodic Temperley-Lieb algebra) [12]. In Fig. 5, one sees the action of the generator e_2 on one of the configurations of Fig. 4. The factor x appears due to a contractible loop. The exceptions occur if one consider configurations having an arch of the size L of the system and if the generator acts on the bond between the ends of the arch [see Fig. 6(a)]. The action of e_2 on the third configuration of Fig. 4 produces a new circle and one
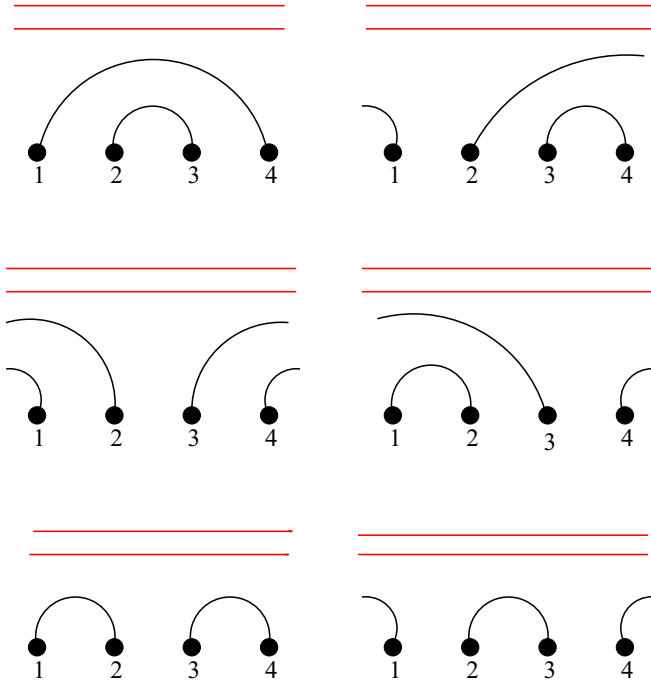
FIG. 4. (Color online) The six link pattern configurations for $L = 4$ sites on a cylinder and two circles without sites (two noncontractible loops shown in red). The open arcs and circles meet behind the cylinder.

gets a configuration with $m = 3$. Instead of considering configurations with various numbers of noncontractible loops, we are going to consider configurations with no noncontractible loops, but as a result of the action of e_2, we multiply by fugacity α instead of adding a noncontractible loop; see Fig. 6(b). With this rule, one obtains a representation of the PTL with the quotient (3), and it relates α to the fugacity of noncontractible loops.

We consider the Hamiltonian

$$H = -\sum_{i=1}^{L} e_i. \tag{7}$$

Using the representation in terms of link paths of the PTL, this Hamiltonian is equal up to a factor to the Hamiltonian H' obtained from the transfer matrix of the O(n) models [8]. This factor is equal to the sound velocity $v_s = \frac{\pi}{\gamma} \sin(\gamma)$:
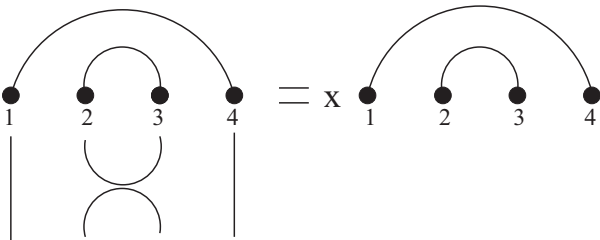
$$H' = H/v_s. \tag{8}$$

FIG. 5. The action of the e_2 generator acting on the bond between the sites 2 and 3 in one of the configurations appearing in Fig. 4. The factor x is due to the contractible loop.
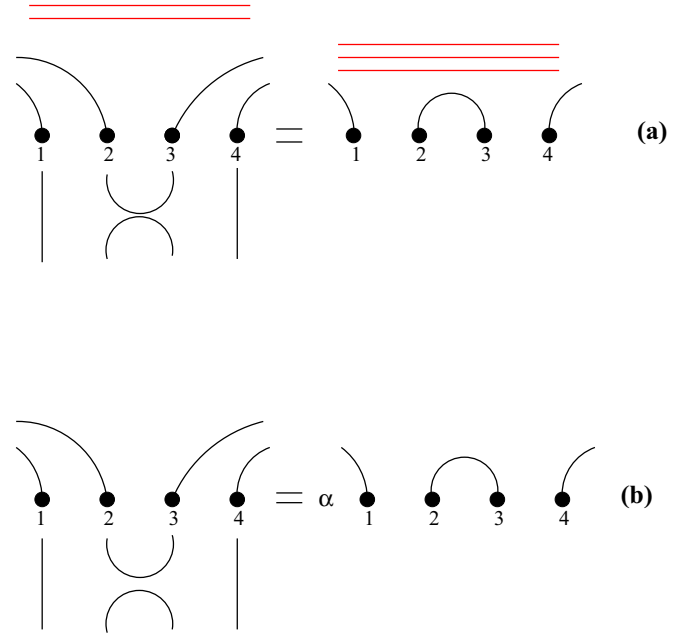

FIG. 6. (Color online) (a) The action of the e_2 generator acting on the bond between the sites 2 and 3, which are the end points of an arc of the size of the system. (b) The action of e_2 gives a factor α to the new configuration.

In the spin representation of the PTL, using a similarity transformation, the Hamiltonian H' can be written as

$$H' = -\frac{1}{v_s} \left[\sum_{i=1}^{L-1} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ - \frac{\cos(\gamma)}{2} \sigma_i^z \sigma_{i+1}^z \right) \right.$$
$$+ \left(\sigma_L^+ \sigma_1^- e^{i\phi} + \sigma_L^- \sigma_1^+ e^{-i\phi} \right) - \frac{\cos(\gamma)}{2} \sigma_L^z \sigma_1^z$$
$$\left. + \frac{L \cos(\gamma)}{2} \right]. \tag{9}$$

This Hamiltonian, which is the XXZ quantum chain with a twist ϕ, is integrable and its ground-state and energy spectrum is known [13]. In particular, the ground-state energy is

$$E'(\phi, \gamma, L) = e'_\infty L + \left(\frac{\phi^2}{4(\pi - \gamma)} - \frac{\pi}{6} \right) \frac{1}{L} + o(1/L), \tag{10}$$

where e'_∞ is the bulk energy density. Notice that the choice $\alpha = 2$ ($\phi = 0$) and $\gamma = \pi/2$ used in Ref. [8] corresponds to the XX model. From (10) one can get two quantities of interest. First, taking into account that the density of the spin current is

$$J_i = i(\sigma_i^+ \sigma_{i+1}^- - \sigma_i^- \sigma_{i+1}^+), \tag{11}$$

its average value at large values of L is [see Eqs. (5) and (7)]

$$J^z = -\frac{\partial E'}{\partial \phi} = -\frac{\phi}{2(\pi - \gamma)} \frac{1}{L}. \tag{12}$$

Since the coefficient in Eq. (12) is dimensionless, we expect it to be universal. A second quantity of interest is the density of

noncontractible loops:

$$\rho_L(\gamma, \alpha) = -\alpha \frac{\partial E'}{\partial \alpha} = -2 \cot(\phi/2) J^z = \frac{\phi \cot(\phi/2)}{\pi - \gamma} \frac{1}{L}. \tag{13}$$

Notice that the density of noncontractible loops is proportional to the current density (a physical explanation of this observation is still missing). We observe also that the dependence on n in $O(n)$ has a very simple form.

There is a simple way to check if the probability distribution of noncontractible loops is Gaussian or not. Using Eqs. (6) and (10), one can compute all the moments M_n of the probability distribution. They are all of order L^{-1}. One can check that, for example, the identity

$$M_3 = 2M_1 M_2 - M_1^3, \tag{14}$$

which is valid for a Gauss distribution, is not satisfied.

Using the relation (6), one obtains for $x = 0$ ($\gamma = \pi/2$, no contractible loops)

$$\rho_L(\alpha) = \frac{4\alpha \arccos(\alpha/2)}{\pi L \sqrt{4 - \alpha^2}} \quad \text{for} \quad \alpha \leqslant 2 \tag{15}$$

and

$$\rho_L(\alpha) = \frac{4\alpha \operatorname{Arch}(\alpha/2)}{\pi L \sqrt{\alpha^2 - 4}} \quad \text{for} \quad \alpha > 2. \tag{16}$$

Expressions (15) and (16) are going to be compared to those obtained from the spanning webs model presented in the next sections.

For the same case ($x = 0$) only, one has a simple expression for the ground-state energy valid for any value of L:

$$E'(\phi, \pi/2, L) = -\cos(\phi/L)/\sin(\pi/L). \tag{17}$$

Using this relation, one can compute the density of noncontractible loops for any fugacity and any size of the system L.

III. THE SPANNING WEBS MODEL ON THE ROTATED SQUARE LATTICE

Another representation of the $O(n)$ model relates loop configurations to clusters of bonds on sublattices of the original lattice. The square lattice of sites with integer coordinates can be divided into two sublattices: black and white. For sites of the black sublattice, the sum of coordinates is even, while for sites of the white sublattice it is odd. The bijection between loop and bond configurations is shown in Fig. 7. The neighboring sites of each sublattice are connected by a bond if it does not intersect the borderlines of the elementary cell. Each connected cluster of bonds in the bulk of the lattice is situated inside a loop. Each bulk cluster on the black sublattice is surrounded
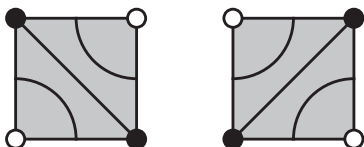

FIG. 7. The bijection between bonds and elementary cells.
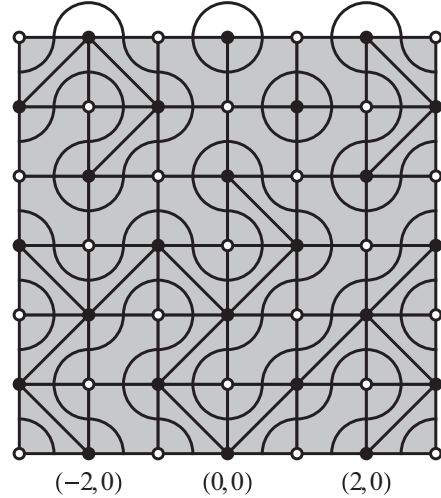

FIG. 8. The bond configuration corresponding to the loop configuration from Fig. 2.

by a connected cluster of bonds on the white sublattice and vice versa. The clusters of bonds corresponding to the loop configuration in Fig. 2 are shown in Fig. 8.

At $n = 0$, the absence of bulk loops contractible to a point implies the absence of isolated clusters of bonds on the black or white sublattice. The allowed bond configurations are the so called spanning webs, i.e., the graphs containing all vertices of a sublattice, with noncontractible loops arising from the periodic boundary conditions and spanning trees connected either to open boundaries or to the noncontractible loops. Clearly, due to the bijection between bonds and elementary cells, the presence of every noncontractible loop in the bulk of the cylinder is equivalent to the presence of a pair of noncontractible polymers in the dense polymer representation. If there are no noncontractible loops in the system, polymers can propagate from the top to the bottom of the cylinder. These polymers are treated as defect lines that separate clusters of bonds from one another.

The reformulation of the dense polymer model in terms of bond configurations leads us to the standard problem of spanning graphs on the square lattice. The cycle-free spanning graphs are called spanning trees; the graphs containing a number of cycles are called spanning webs. The enumeration of spanning trees is traced back to the classical Kirchhoff theorem [14,19]. The spanning web model appears in statistical mechanics as the Temperley representation [15] of the dimer model solved by Kasteleyn [16] and by Temperley and Fisher [17]. A particular case we consider here is the spanning web model on a cylinder with noncontractible cycles supplied by fugacity ξ. This decoration needs a generalization of the Kirchhoff theorem. A similar model, considered as the (1,2) logarithmic minimal model, has been solved in Ref. [18], where the fugacity of noncontractible loops was first introduced. A basic accent of the present work is the calculation of the density of noncontractible cycles in a finite geometry.

We consider an oriented labeled graph $\mathcal{G} = (V, E)$ with a vertex set V and a set of bonds E. Vertices are the sites of a finite square lattice rotated by $\pi/4$ and wrapped on a cylinder.
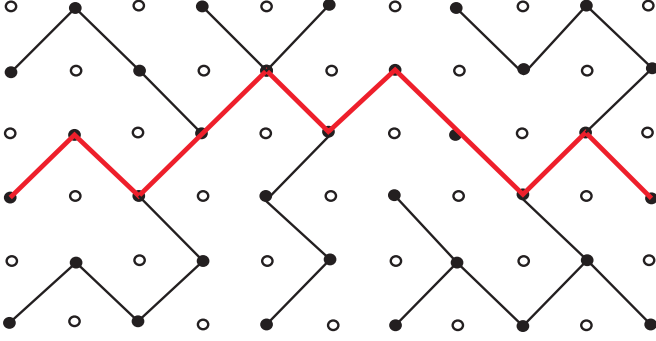
FIG. 9. (Color online) Spanning web with a single noncontractible loop (shown in red).

The graph $\mathcal{G} = (V, E)$ can be considered as a sublattice of square superlattice $\mathcal{L}$ with standard orientation, containing N rows and L columns of cells. Vertices of the superlattice are shown in Fig. 9 as open and filled circles. The vertex set V is the sublattice of filled circles. For convenience, in the remainder we call N the "height" and L the "perimeter" of the cylinder. Let the cells $\mathbf{r}_{ij} = (i, j)$ of $\mathcal{L}$ be labeled by the integer coordinates $i = 1, \ldots, L$ and $j = 1, \ldots, N$, so that the row $\{\mathbf{r}_{i1} = (i, 1) : i = 1, \ldots, L\}$ is the bottom boundary of the cylinder and the row $\{\mathbf{r}_{iN} = (i, N) : i = 1, \ldots, L\}$ is its top boundary. For the sake of convenience, both L and N are chosen to be even. The vertex set V of the rotated square lattice $\mathcal{G}$ then consists of the vertices of the sublattice of $\mathcal{L}$ with, say, an even sum of the horizontal and vertical coordinates, i.e., $V = \{\mathbf{r}_{ij} = (i, j) : i + j = \text{even}\}$. Explicitly, we have

$$V = \bigcup_{i=1}^{L/2} \bigcup_{j=1}^{N/2} \{(2i - 1, 2j - 1), (2i, 2j)\}. \tag{18}$$

The edges in E we take oriented from a site $(i, j) \in V$ to its nearest neighbors on the right-hand side, $(i + 1, j + 1)$ and $(i + 1, j - 1)$. We call this direction "positive," and the opposite one, from a site $(i, j) \in V$ to its nearest neighbors on the left-hand side, $(i - 1, j + 1)$ and $(i - 1, j - 1)$, we call "negative."

We find it convenient to analyze the construction of spanning web configurations on the above-oriented graph by using the arrow representation; see, e.g., [19]. Accordingly, to each vertex $\mathbf{r} \in V$ we attach an arrow directed along one of the bonds $(\mathbf{r}, \mathbf{r}')$ incident to it. Each arrow defines a directed bond $(\mathbf{r} \rightarrow \mathbf{r}')$ and each configuration of arrows $\mathcal{A}$ on $\mathcal{G}$ defines a spanning directed graph (digraph) $\mathcal{G}_{\text{sd}}(\mathcal{A})$ with a set of bonds $E_{\text{sd}}(\mathcal{A}) = \{(\mathbf{r} \rightarrow \mathbf{r}') : \mathbf{r}, \mathbf{r}' \in V\}$ depending on $\mathcal{A}$.

A cycle of length k is a sequence of directed bonds $(\mathbf{r}_1, \mathbf{r}_2), (\mathbf{r}_2, \mathbf{r}_3), (\mathbf{r}_3, \mathbf{r}_4), \ldots, (\mathbf{r}_k, \mathbf{r}_1)$, where all $\mathbf{r}_j$, $1 \leqslant j \leqslant k$, are distinct. If both $(\mathbf{r} \rightarrow \mathbf{r}')$ and $(\mathbf{r}' \rightarrow \mathbf{r})$ belong to the same spanning web, we say that it contains a cycle of length 2. Our aim is to study sets of spanning digraphs with no other cycles than those that wrap the cylinder. The relevant configurations will be enumerated with the aid of a generating function defined as the determinant of an appropriately constructed weight matrix:

$$Z_{L,N}(\omega) = \det D_{L,N}(\omega). \tag{19}$$

In this respect, the derivation of the above relationship, see the Appendix, is merely a generalization of the matrix Kirchhoff theorem [14].

The elements $[D_{L,N}(\omega)]_{\alpha,\beta}$, $\alpha, \beta = 1, \ldots, n = LN/2$, of the matrix $D_{L,N}$, associated with the graph $\mathcal{G}$, are explicitly given as

$$[D_{L,N}(\omega)]_{\alpha,\beta} = \begin{cases} z_\alpha & \text{if} & \alpha = \beta, \\ -b & \text{if} & \mathbf{r}_\beta \text{ is a right neighbor of } \mathbf{r}_\alpha, \\ -b^{-1} & \text{if} & \mathbf{r}_\beta \text{ is a left neighbor of } \mathbf{r}_\alpha, \\ 0 & & \text{otherwise.} \end{cases}$$
$$\tag{20}$$

Here z_α is the order of vertex $\mathbf{r}_\alpha$ in $\mathcal{G}$, $b = \omega^{1/L} e^{-i\pi/L}$, and the condition "$\mathbf{r}_\beta$ is a right (left) neighbor of $\mathbf{r}_\alpha$" means that if $\mathbf{r}_\alpha = (i, j)$, then $\mathbf{r}_\beta = (i + 1, j \pm 1)$ [$\mathbf{r}_\beta = (i - 1, j \pm 1)$]. Note that all closed paths that do not wrap the cylinder contain an equal number of edges with either orientations, hence their weight in $\det D$ remains the same as in $\det \Delta$. Therefore, all the configurations that contain such closed paths (contractible cycles) cancel out in the expansion of $\det D$. On the other hand, cycles generated by off-diagonal elements that wrap the cylinder change their sign, because they contain edges oriented in one direction exceeding by L the number of edges in the opposite direction. This amounts to the total factor of $b^L = -\omega$ or $b^{-L} = -\omega^{-1}$ depending on the orientation. Therefore, each noncontractible cycle with a given orientation is counted twice, however with a different weight—once it enters into the determinant expansion with the unit weight, being generated by diagonal elements of the matrix D, and the second time it enters with a factor ω or ω^{-1} (depending on the orientation) as generated by off-diagonal elements of that matrix. Thus, the total number of noncontractible cycles, irrespective of their origin and orientation, is given by the coefficient in front of the corresponding power of $\omega + \omega^{-1} + 2 \equiv \xi$ in the series expansion of the partition function. In general, besides the noncontractible cycles, the average number of which is controlled by fugacity ξ, the spanning digraph contains tree subgraphs connected to the cycles. All branches of the trees can be generated only by the diagonal elements of D and, hence, carry unit weight.

IV. THE PARTITION FUNCTION ON A TORUS

To calculate the generating function $Z_{L,N}(\omega)$ in expression (19), we make some initial transformations that allow us to easily diagonalize the matrix $D_{L,N}(\omega)$. First of all, the decomposition (18) of the vertex set V of the rotated square lattice $\mathcal{G}$ suggests its rearrangement by combining all pairs of nearest neighbors $(2i - 1, 2j - 1)$ and $(2i, 2j)$, $i = 1, 2, \ldots, L' = L/2$, $j = 1, 2, \ldots, N' = N/2$ into two-site unit cells. Thus we obtain a square $L' \times N'$ array of $LN/4$ unit cells with the connectivity of a triangular lattice. Under neglect of the boundary effects at the top and bottom of the cylinder, we describe the weighted connectivity of the sites in a unit cell with their neighbors in $\mathcal{G}$, taking into account the bond orientation, by introducing the following

2×2 matrices:

$$a(0,0) = \begin{pmatrix} 0 & b^{-1} \\ b & 0 \end{pmatrix}, \quad a(1,0) = \begin{pmatrix} 0 & 0 \\ b^{-1} & 0 \end{pmatrix},$$

$$a(0,1) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad a(1,1) = \begin{pmatrix} 0 & 0 \\ b^{-1} & 0 \end{pmatrix},$$

$$a(-1,0) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad a(0,-1) = \begin{pmatrix} 0 & b^{-1} \\ 0 & 0 \end{pmatrix},$$

$$a(-1,-1) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Now the matrix $D_{L,N}(\omega)$, see Eq. (20), can be written as

$$[4I_2 - a(0,0)] \otimes I_{L'} \otimes I_{N'} - a(1,0) \otimes R_{L'} \otimes I_{N'}$$
$$- a(-1,0) \otimes R_{L'}^T \otimes I_{N'} - a(0,1) \otimes I_{L'} \otimes R_{N'}$$
$$- a(0,-1) \otimes I_{L'} \otimes R_{N'}^T - a(1,1) \otimes R_{L'} \otimes R_{N'}$$
$$- a(-1,-1) \otimes R_{L'}^T \otimes R_{N'}^T. \quad (21)$$

Here R_M is the $M \times M$ matrix,

$$R_M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (22)$$

and R_M^T is the matrix transposed of R_M. Now we note that both R_M and R_M^T are diagonalized by the similarity transformation

$$S_M^{-1} R_M S_M = \text{diag}\{e^{i2\pi m/M}, m = 1,2,\dots,M\},$$

where the S_M is the matrix with elements

$$(S_M)_{n,m} = M^{-1/2} e^{i2\pi mn/M}, \quad m,n = 1,2,\dots,M. \quad (23)$$

Since $R_M R_M^T = I_M$, we have

$$S_M^{-1} R_M^T S_M = \text{diag}\{e^{-i2\pi m/M}, m = 1,2,\dots,M\}.$$

Therefore, with the similarity transformation generated by the matrix $I_2 \otimes S_{L'} \otimes S_{N'}$, we can diagonalize the matrix $D_{L,N}(\omega)$ in the $L' = L/2$ and $N' = N/2$ -dimensional subspaces. Then for the determinant we readily obtain

$$\det D_{L,N}(\omega) = \prod_{m=1}^{L'} \prod_{n=1}^{N'} \det Q(2\pi m/L', 2\pi n/N'), \quad (24)$$

where $Q(\theta_1, \theta_2)$ is the 2×2 matrix,

$$Q(\theta_1, \theta_2) = 4I_2 - a(0,0) - a(1,0)e^{i\theta_1} - a(-1,0)e^{-i\theta_1}$$
$$- a(0,1)e^{i\theta_2} - a(0,-1)e^{-i\theta_2}$$
$$- a(1,1)e^{i\theta_1 + i\theta_2} - a(-1,-1)e^{-i\theta_1 - i\theta_2}. \quad (25)$$

It is convenient to cast its determinant in the form

$$\det Q(\theta_1, \theta_2)$$
$$= 4\cos^2(\theta_2/2)\left[\frac{4}{\cos^2(\theta_2/2)} - 2 - 2\cos(\theta_1 + \delta)\right], \quad (26)$$

where $\delta = 2i \ln b = (i \ln \omega + \pi)/L'$ Thus, from Eq. (24) we obtain the ω-dependent part of the partition function

$$Z_{L',N'}(\omega)$$
$$= \prod_{m=1}^{L'} \prod_{n=1}^{N'} \left[\frac{4}{\cos^2(\pi n/N')} - 2 - 2\cos(2\pi m/L' + \delta)\right]. \quad (27)$$

Next we set

$$\frac{4}{\cos^2(\pi n/N')} - 2 = A^2(\pi n/N') + A^{-2}(\pi n/N'), \quad (28)$$

where

$$A^2(\pi n/N') = \frac{[1 + |\sin(\pi n/N')|]^2}{\cos^2(\pi n/N')}. \quad (29)$$

Now, making use of the identity

$$\prod_{m=1}^{L'} [A^2 + A^{-2} - 2\cos(2\pi m/L' + \delta)]$$
$$= A^{2L'} + A^{-2L'} - 2\cos(L'\delta), \quad (30)$$

with $\delta = (i \ln \omega + \pi)/L'$, we perform exactly the product over m:

$$Z_{L',N'}(\omega) = \prod_{n=1}^{N'} \left[\frac{[1 + \sin(\pi n/N')]^L}{\cos^L(\pi n/N')}\right.$$
$$\left. + \frac{[1 - \sin(\pi n/N')]^L}{\cos^L(\pi n/N')} + \omega + \omega^{-1}\right]. \quad (31)$$

Strictly speaking, this expression is valid for spanning webs on a torus, since we have not considered boundary conditions at the edges of the cylinder properly. For the sake of completeness, in the next section we present the exact expressions for the partition function under closed and open boundary conditions at the top and bottom of a finite-size cylinder.
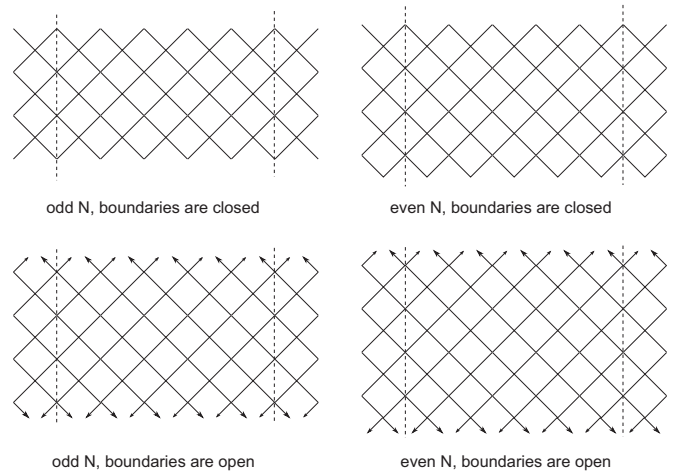

FIG. 10. Rotated square lattices with different boundary conditions on the cylinder. The two dashed vertical lines in each of the four panels represent a single line due to periodicity in the horizontal direction. For example, the top-left lattice has $L = 10$ and $N = 7$.

V. CYLINDRICAL BOUNDARY CONDITIONS

In the preceding section, we computed the partition function of the spanning webs without contractible loops on the torus. Here we consider the cases in which the boundaries at the top and bottom of cylinder are closed or open; see Fig. 10. It can be seen from the figure that there is a difference between the cases of odd and even N: in the latter case, the lower edge is shifted with respect to the top one. In our considerations $L = 2L'$ is even, whereas N is of any parity.

$$Z_{L,N}^{\text{op,op}} = \begin{cases} 2^{L'}(N+1)^{L'} \prod_{n=1}^{(N-1)/2} \prod_{m=0}^{L'-1} \left(\frac{4}{\sin^2 \frac{\pi m}{N+1}} - 2 - 2\cos(2\pi m/L') \right), & \text{odd } N, \\ \prod_{n=1}^{N/2} \prod_{m=0}^{L'-1} \left(\frac{4}{\sin^2 \frac{\pi(n-1/2)}{N+1}} - 2 - 2\cos(2\pi m/L' + \delta) \right), & \text{even } N. \end{cases} \tag{32}$$

The partition functions in the case of all the other uniform boundary conditions can be expressed in terms of $Z_{L,N}^{\text{op,op}}$ as follows:

In the case of both closed boundary conditions,

$$Z_{L,N}^{\text{cl,cl}} = \xi Z_{L,N-2}^{\text{op,op}}, \tag{33}$$

in the case mixed open-closed boundary conditions,

$$Z_{L,N}^{\text{op,cl}} = 2^{-L'} \frac{Z_{L,2N-1}^{\text{op,op}}}{Z_{L,N-1}^{\text{op,op}}}, \tag{34}$$

and for the torus,

$$Z_{L,N}^{\text{torus}} = 4^{L'} \xi \left(Z_{L,N/2-1}^{\text{op,op}} \right)^2. \tag{35}$$

From these expressions, it can be easily seen that the density of noncontractible loops (see the next section) is equal for different boundary conditions in the limit $N \to \infty$.

VI. DENSITY OF NONCONTRACTIBLE LOOPS

The average density (per unit height of the cylinder) of the noncontractible cycles as a function of the fugacity ξ is defined as

$$\rho_{L,N}(\xi) = \frac{1}{N} \xi \frac{\partial}{\partial \xi} \ln \det D_{L,N}[\omega(\xi)]. \tag{36}$$

Since we are interested in the case of the density on an infinitely long cylinder, we expect the boundary corrections to vanish in the limit $N' \to \infty$. To take this limit, we factor out another ω-independent term from $Z_{L',N'}(\omega)$,

$$Z_{L',N'}(\omega) = \left[\prod_{n=1}^{N'} \frac{[1 + \sin(\pi n/N')]^L}{\cos^L(\pi n/N')} \right]$$
$$\times \prod_{n=1}^{N'} \left[1 + (\omega + \omega^{-1}) \frac{\cos^L(\pi n/N')}{[1 + \sin(\pi n/N')]^L} \right.$$
$$\left. + \frac{\cos^{2L}(\pi n/N')}{[1 + \sin(\pi n/N')]^{2L}} \right], \tag{37}$$

Note that the diagonal elements of the matrix (20) in the case of both boundaries open are $z_\alpha = 4$, because then every vertex has four outgoing edges. In general, $z_\alpha = 2$ in the case in which vertex α belongs to a closed boundary, and $z_\alpha = 4$ when vertex α belongs to the interior of the lattice or to an open boundary.

The partition function in the case of open-open boundary conditions is [compare to (27)]

and we make use of the density definition (36):

$$\rho_{L,N}(\xi) = \frac{\xi \partial}{\partial \xi} \frac{1}{N} \sum_{n=1}^{N'} \ln \left[1 + (\xi - 2) \frac{\cos^L(\pi n/N')}{[1 + \sin(\pi n/N')]^L} \right.$$
$$\left. + \frac{\cos^{2L}(\pi n/N')}{[1 + \sin(\pi n/N')]^{2L}} \right]. \tag{38}$$

Hence, in the limit $N' \to \infty$,

$$\rho_{L,\infty}(\xi) = \xi \frac{\partial}{\partial \xi} \frac{1}{2\pi} \int_0^\pi d\phi \ln$$
$$\times \left[1 + (\xi - 2) \frac{\cos^L(\phi)}{[1 + \sin(\phi)]^L} + \frac{\cos^{2L}(\phi)}{[1 + \sin(\phi)]^{2L}} \right]$$
$$= \frac{\xi}{2\pi} \int_0^\pi \frac{g^L(\phi) \, d\phi}{1 + (\xi - 2)g^L(\phi) + g^{2L}(\phi)}, \tag{39}$$

where ($L = 2L'$ is even)

$$g(\phi \in [0,\pi]) = \frac{|\cos(\phi)|}{[1 + \sin(\phi)]} \in [0,1],$$
$$\max_{\phi \in [0,\pi]} g(\phi) = g(0) = g(\pi) = 1. \tag{40}$$

Therefore, when $L \gg 1$, the essential contribution in $\rho_{L,\infty}(\xi)$ comes from the integration over the two small intervals $0 \leqslant \phi \leqslant \epsilon$ and $\pi - \epsilon \leqslant \phi \leqslant \pi$ with $\epsilon \ll 1$. To leading order in $L \gg 1$, it suffices to take the linear term in the expansion

$$\ln g(\phi) = -\phi + O(\phi^2), \quad \phi \to 0^+, \quad \text{or} \quad \phi \to \pi, \tag{41}$$

which yields

$$\rho_{L,\infty}(\xi) \simeq \frac{\xi}{\pi} \int_0^\epsilon \frac{e^{-L\phi} d\phi}{1 + (\xi - 2)e^{-L\phi} + e^{-2L\phi}},$$
$$= \frac{\xi}{\pi L} \int_{e^{-L\epsilon}}^1 \frac{dy}{1 + (\xi - 2)y + y^2}. \tag{42}$$

Extending the lower limit in the latter integral to $y = 0$, we finally obtain the general expression

$$\rho_{L,\infty}(\xi) = \begin{cases} \frac{\xi}{\pi L \sqrt{\xi(\xi-4)}} \ln \frac{\xi+\sqrt{\xi(\xi-4)}}{\xi-\sqrt{\xi(\xi-4)}} & \text{if} \quad (\xi-4) > 0, \\ \frac{2}{\pi L} & \text{if} \quad (\xi-4) = 0, \\ \frac{2\xi}{\pi L \sqrt{\xi(4-\xi)}} \left[\tan^{-1} \frac{\xi}{\sqrt{\xi(4-\xi)}} - \tan^{-1} \frac{\xi-2}{\sqrt{\xi(4-\xi)}} \right] & \text{if} \quad (\xi-4) < 0. \end{cases} \quad (43)$$

We are going to compare the obtained result with Eq. (15) in the interval of the loop fugacity $0 \leqslant \alpha \leqslant 2$ corresponding to real values of the twist parameter ϕ. Toward that end, we transform the bottom expression in Eq. (43),

$$\frac{2\xi}{\pi L \sqrt{\xi(4-\xi)}} \left[\tan^{-1} \frac{\xi}{\sqrt{\xi(4-\xi)}} - \tan^{-1} \frac{\xi-2}{\sqrt{\xi(4-\xi)}} \right]$$
$$= \frac{2\xi}{\pi L \sqrt{\xi(4-\xi)}} \arccos \left(\frac{\sqrt{\xi}}{2} \right) \quad (44)$$

and remember that each cycle in the spanning web model corresponds to a pair of noncontractible loops with fugacity $\xi = \alpha^2$. Then the density of noncontractible loops in the dense polymer model is

$$\rho_L(\alpha) = \frac{4\alpha \arccos(\alpha/2)}{\pi L \sqrt{(4-\alpha^2)}}, \quad (45)$$

in full agreement with (15).

For $\xi > 4$ and $\alpha > 2$, we use the formula

$$\ln[(1+x)/(1-x)] = 2\operatorname{Arth}(x) \quad (46)$$

and get

$$\rho_L(\alpha) = \frac{4\alpha \operatorname{Arch}(\alpha/2)}{\pi L \sqrt{(\alpha^2-4)}}. \quad (47)$$

We see that the formula for $\alpha > 2$ corresponds to the quantum chain result (16) with the complex twist

$$\phi = i\Omega, \quad \alpha = 2\cosh(\Omega/2). \quad (48)$$

The crucial check of Eq. (47) is the value of density $\rho_L(\alpha)$ for $\alpha = \sqrt{2}$, when the fugacity of noncontractible cycles in the spanning web model is $\xi = 2$. In this case, the noncontractible cycles enter into the partition function as two nonweighted sequences of bonds oriented clockwise and anticlockwise. Due to symmetry of the O(n) model, the spread of each sequence in the horizontal direction (that is, L by the definition of loops) coincides on average with that in the direction of the cylinder axis. Each of the two loops on the rotated lattice (Fig. 9) is separated by a loop on the dual lattice. Thus, the spanning web configuration in the vertical direction is a sequence of sandwiches of loops and dual loops of average thickness $2L$. Then, the density of web cycles is $1/(2L)$ and the density of noncontractible loops is $\rho_L(\sqrt{2}) = 1/L$.

Note added in proof. Morin-Duchesne *et al.* [20] have obtained the inversion relation for any α which should allow one to obtain our results in a different way.

APPENDIX

In this Appendix, we give a detailed derivation of the expression (19) for the generating function of all spanning digraphs on $\mathcal{G}$ which have no contractible cycles. We begin with an examination of the determinant expansion of the usual Laplace matrix Δ for the graph $\mathcal{G}$. Let the vertices $\mathbf{r} \in V$ be labeled in arbitrary order from 1 to $n = |V| = LN/2$. Then Δ has the following elements ($\alpha, \beta \in \{1, \ldots, n\}$):

$$\Delta_{\alpha,\beta} = \begin{cases} z_\alpha & \text{if} \quad \alpha = \beta, \\ -1 & \text{if} \quad \alpha \text{ and } \beta \text{ are adjacent}, \\ 0 & \text{otherwise}, \end{cases} \quad (A1)$$

where z_α is the order of vertex $\mathbf{r}_\alpha$ in the rotated square lattice $\mathcal{G}$. Since the matrix Δ has a zero eigenvalue, its determinant vanishes. On the other hand, the Leibniz formula expresses the determinant of Δ as a sum over all permutations σ of the set $\{1, 2, \ldots, n\}$:

$$\det \Delta = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \, \Delta_{1,\sigma(1)} \Delta_{2,\sigma(2)} \cdots \Delta_{n,\sigma(n)} = 0, \quad (A2)$$

where S_n is the symmetric group and $\operatorname{sgn}(\sigma) = \pm 1$ is the signature of the permutation σ. The identity permutation $\sigma = \sigma_{\text{id}}$ in Eq. (A2) yields the term $z_1 z_2 \cdots z_n$, which is equal to the number of all possible arrow configurations on $\mathcal{G}$.

In general, each permutation $\sigma \in S_n$ can be factored into a product (composition) of disjoint cyclic permutations, say $\sigma = c_1 \circ c_2 \cdots \circ c_k$. This representation partitions the set of vertices V into nonempty disjoint subsets—the orbits O_i of the corresponding cycles c_i, $i = 1, \ldots, k$. More precisely, if $O_i = \{v_{i,1}, v_{i,2}, \ldots v_{i,l_i}\} \subset V$ is the orbit of c_i, then $\cup_{i=1}^k O_i = V$ and $\sum_{i=1}^k l_i = n$, where l_i is the cardinality of the orbit O_i, or equivalently, the length of the cycle c_i. The orbits consisting of just one element, if any, constitute the set $S_{\text{fp}}(\sigma)$ of fixed points of the permutation: $S_{\text{fp}}(\sigma) = \{v = \sigma(v), v \in V\}$. In the case of the identity permutation $\sigma_{\text{id}} \in S_n$, all orbits consist of exactly one element, $O_i(\sigma_{\text{id}}) = \{v_i\} \subset V, i = 1, \ldots, n$, and $S_{\text{fp}}(\sigma_{\text{id}}) = V$. A cycle c_i of length $|c_i| = l_i \geqslant 2$ will be called a *proper cycle*. A proper cycle of length 2 corresponds to two oppositely directed edges that connect a pair of adjacent vertices: $(v_{i,1} \to v_{i,2})$, $(v_{i,2} \to v_{i,1})$. Note that the vertices of an orbit O_i of cardinality $l_i = |O_i(\sigma)| \geqslant 3$ are connected by a closed path on $\mathcal{G}$ that can be traversed in two opposite directions: if c_i is the cycle defined by $v_{i,1} \to \sigma(v_{i,1}) = v_{i,2}, \to \cdots \to \sigma(v_{i,l_i}) = v_{i,1}$, then the reverse cycle c_i' can be represented as $v_{i,l_i} \to \sigma(v_{i,l_i}) = v_{i,l_i-1}, \to \cdots \to \sigma(v_{i,1}) = v_{i,l_i}$.

Now we take into account that the proper cycles on $\mathcal{G}$ are of even length only, hence the signature of every permutation in the expansion of the determinant depends on the number of proper cycles in its factorization, i.e., if $\sigma = c_1 \circ c_2 \cdots \circ c_p$, where $|c_i| \geqslant 2, i = 1, \ldots, p$, then $\operatorname{sgn}(\sigma) = (-1)^p$. Thus, the terms in Eq. (A2) can be rearranged according to the number
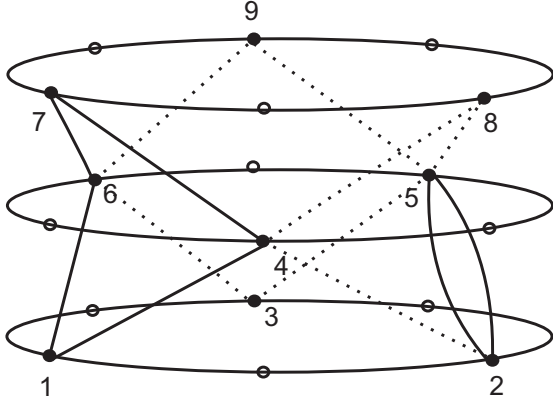
FIG. 11. Spanning digraph generated by a single term in the determinant expansion of the Laplacian matrix (see the text).

p of disjoint proper cycles as follows:

$$\prod_{i=1}^{n} z_i + \sum_{p=1}^{[n/2]} (-1)^p \sum_{\sigma=c_1\circ\cdots\circ c_p} \prod_{i=1}^{p} \Delta_{v_i,c_i(v_i)}\Delta_{c_i(v_i),c_i^2(v_i)} \cdots$$

$$\Delta_{c_i^{l_i-1}(v_i),v_i} \prod_{j\in S_{\mathrm{fp}}(\sigma)} z_j. \tag{A3}$$

Here c_i^k is the k-fold composition of the cyclic permutation c_i of even length l_i, $v_i \in O_i(\sigma)$, so that $c_i^{k-1}(v_i) \neq c_i^k(v_i)$ and $c_i^{l_i}(v_i) = v_i$. Note that all nonvanishing off-diagonal elements are equal to -1.

The above expansion reveals the following features: (i) As expected, all spanning digraphs on $\mathcal{G}$ have at least one proper cycle. (ii) Each term with $S_{\mathrm{fp}}(\sigma) \neq \emptyset$ represents a set of $\prod_{j\in S_{\mathrm{fp}}} z_j$ distinct spanning digraphs, which have in common the specified cycles $c_1, \ldots, c_p$, and they differ in the oriented edges outgoing from the vertices $j \in S_{\mathrm{fp}}(\sigma)$. These oriented edges may form cycles on their own that do not enter into the list $c_1, \ldots, c_p$. (iii) Since the sets $\cup_{i=1}^{p} O_i$ and $S_{\mathrm{fp}}(c_1, \ldots, c_p)$ are disjoint, the proper cycles formed by the oriented edges incident to the fixed points of a given permutation $\sigma = c_1 \circ c_2 \circ \cdots \circ c_p$ should enter into the enlarged list of cycles $c_1, c_2, \ldots, c_p, \ldots, c_{p'}$, $p' > p$, corresponding to the cycle decomposition of another permutation σ'.

For example, consider the determinant of the Laplacian matrix of a cylinder of height 3 and perimeter 6 shown in Fig. 11. The set of vertices V consists of filled circles marked by $1,2,\ldots,9$. The set of oriented edges is a collection of 24 inclined vectors of type $1 \to 4, 1 \to 6, 4 \to 1, 6 \to 1, \ldots$. The corresponding Leibniz expansion contains the term

$$(-1)^2(\Delta_{1,4}\Delta_{4,7}\Delta_{7,6}\Delta_{6,1})(\Delta_{2,5}\Delta_{5,2})\Delta_{3,3}\Delta_{8,8}\Delta_{9,9}, \tag{A4}$$

which represents $z_3 z_8 z_9 = 8$ spanning digraphs on $\mathcal{G}$ with two specified cycles and all possible oriented bonds outgoing from the vertices 3, 8, and 9.

As noticed first in Ref. [19], the expansion (A3) parallels in form the *inclusion-exclusion principle* in combinatorial mathematics. Indeed, let $c_1, c_2, \ldots, c_m$ be the list of all possible proper cycles on $\mathcal{G}$, labeled in an arbitrary order. Define A_i, $i = 1, 2, \ldots, m$, as the set of all spanning digraphs on $\mathcal{G}$ containing the particular cycle c_i. Then, expansion (A3) can be written in the form of the inclusion-exclusion principle:

$$\left|\cup_{i=1}^{m} A_i\right| - \sum_{i=1}^{m} |A_i| + \sum_{1\leqslant i<j\leqslant m} |A_i \cap A_j|$$

$$- \sum_{1\leqslant i<j<k\leqslant m} |A_i \cap A_j \cap A_k| + \cdots$$

$$- (-1)^{m+1}|A_1 \cap \cdots \cap A_m|, \tag{A5}$$

which holds for any finite sets $A_1, A_2, \ldots, A_m$, where $|A|$ is the cardinality of the set A. This sum equals zero because all spanning digraphs on $\mathcal{G}$ have at least one proper cycle c_i, $i = 1, 2, \ldots, m$. The first term of the expansion originates from the term $\prod_{i=1}^{n} z_i$ in Eq. (A3) and represents the set of all possible cycles formed by oriented edges incident to every vertex of $\mathcal{G}$. To obtain the number of spanning webs, one has to subtract all the digraphs having contractible cycles and leave all those with noncontractible cycles wrapping the cylinder. To keep the number of noncontractible cycles, we have to change the weights of the nondiagonal terms in such a way that every cycle-containing difference between the numbers of positive and negative steps equal to $+L$ or $-L$, depending on the orientation, enters the determinant expansion with the opposite sign and sums up with the corresponding noncontractible cycle, generated by the arrows representing the diagonal term $\prod_{i=1}^{n} z_i$. At that, all the proper contractible cycles keep their sign in the expansion (A3) in order to cancel out. This readily follows from the fact that every contractible cycle contains equal numbers of positive and negative steps. Thus, the matrix $D_{L,N}$ with elements given by (20) has all the necessary properties for the relationship (19) to define the proper generating function.

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