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## Fast Track Communications

# Cyclic representations of the periodic Temperley–Lieb algebra, complex Virasoro representations and stochastic processes

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## Abstract

An  $N\binom{L}{L/2}$ -dimensional representation of the periodic Temperley–Lieb algebra  $TL_L(x)$  is presented. It is also a representation of the cyclic group  $Z_N$ . We choose  $x = 1$  and define a Hamiltonian as a sum of the generators of the algebra acting in this representation. This Hamiltonian gives the time evolution operator of a stochastic process. In the finite-size scaling limit, the spectrum of the Hamiltonian contains representations of the Virasoro algebra with complex highest weights. The  $N = 3$  case is discussed in detail. We discuss briefly the consequences of the existence of complex Virasoro representations for the physical properties of the systems.

Keywords: stochastic process, Virasoro algebra, Temperley–Lieb algebras

PACS numbers: 03.65.Bz, 03.67.–a, 05.20.–y, 05.30.–d

(Some figures may appear in colour only in the online journal)

The periodic Temperley–Lieb algebra  $PTL_L(x)$  was introduced by Levy [1] in 1991 in order to explain some regularities observed in the spin 1/2  $XXZ$  quantum chain with periodic boundary conditions [2]. The algebra has  $L$  generators and depends on a parameter  $x$ . Various quotients of this algebra were studied by Martin and Saleur [3]. Renewed interest in  $PTL_L(x)$  arose in the last few years in the context of logarithmic conformal field theory [4] and [5]. Lately, stochastic processes describing nonlocal asymmetric exclusion processes have been studied using representations of the same algebra [6].

In the present work, we consider a new quotient and cyclic representations of the algebra. As usual, one can define a Hamiltonian expressed in terms of generators of  $PTL_L(x)$ . If we take  $|x| < 2$ , use cyclic representations, and consider the finite-size limit of the spectra of the Hamiltonian, we can show that they can be expressed in terms of complex representations of the Virasoro algebra. To our knowledge, this is the first time that such representations have been seen in physical problems.

The  $PTL_L(x)$  algebra has  $L$  generators  $e_k$  ( $k = 1, 2, \dots, L$ ) satisfying the relations [1]

$$e_k^2 = xe_k, \quad e_k e_{k\pm 1} e_k = e_k, \quad [e_k, e_l] = 0 \quad (|k - l| > 1), \quad (1)$$

and  $e_{k+L} = e_k$ .

For simplicity we take  $L$  even. We consider the quotient

$$(AB)^N A = A, \quad (2)$$

where

$$A = \prod_{j=1}^{L/2} e_{2j}, \quad B = \prod_{j=0}^{L/2-1} e_{1+2j}. \quad (3)$$

In the definition (2),  $A$  and  $B$  can be interchanged. The case  $N = 1$  is one of the quotients of [3]:

$$ABA = \alpha A \quad (4)$$

with  $\alpha = 1$ . Representations of the quotient (4) in terms of quantum chains were discussed in [1] and in [10]. Notice that on choosing  $\alpha = \exp(i2\pi r/N)$  with  $r = 0, 1, 2, \dots, N - 1$  in (4), one obtains  $N$  independent representations of the quotient (4). In what follows, we present different representations of the same quotient.

We now show that  $PTL_L(x)$  has  $Z_N$  cyclic link representations ( $Z_N$  is the cyclic group of order  $N$ ). Consider  $N$  copies ( $n = 0, 1, 2, \dots, N - 1$ ) of periodic link patterns. Each link pattern is one of the  $\binom{L}{L/2}$  configurations of nonintersecting arches joining  $L$  sites on a circle. One can think of having the circle on a cylinder. Each copy  $n$  is labeled with  $n$  circles on the same cylinder with no sites on them (noncontractible loops). In figure 1 we show the six configurations for  $L = 4$  and  $n = 2$ . The open arches and the circles join on the unseen side of the cylinder.

With a few exceptions, the generators  $e_k$  act on the configurations of a given copy in the standard way [7].

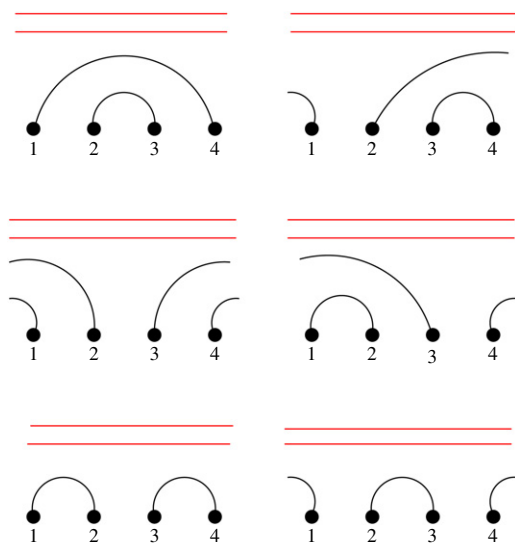
In figure 2 we show the action of  $e_2$  on one of the configurations shown in figure 1. The factor  $x$  appears due to a contractible loop. The exceptions occur if one considers, on the copy  $n$ , a configuration having an arch of the size  $L$  of the system and if the generators act on the bond between the two ends of the arch (see figure 3). The action of  $e_2$  on the third configuration in figure 1 produces a new circle and therefore gives a configuration in the copy  $n = 3$ .

What we have seen in this example is a general phenomenon. If a generator acts on a bond connecting two sites which are the end-points of an arch of length  $L$  of the copy  $n$ , one obtains a configuration belonging to the copy  $n + 1$ .

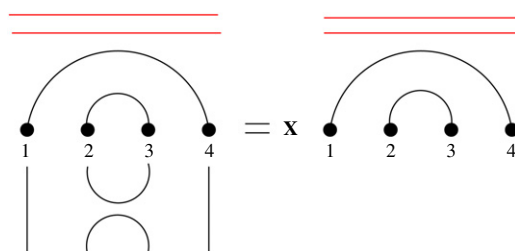
In order to get a finite-dimensional representation of the algebra, one has to take a decision. The simplest one is to identify the copy  $N$  with the copy  $N - 1$ . This possibility is illustrated in figure 4 for the case  $N = 3$  and  $L = 4$ .

It is easy to check that one obtains in this way a representation not of the quotient (2), but of a different quotient:

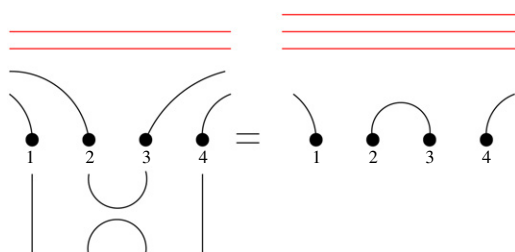
$$(AB)^N A = (AB)^{N-1} A. \quad (5)$$



**Figure 1.** The six link pattern configurations for  $L = 4$  sites on a cylinder and two circles without sites (noncontractible loops). The open arches and circles meet behind the cylinder.



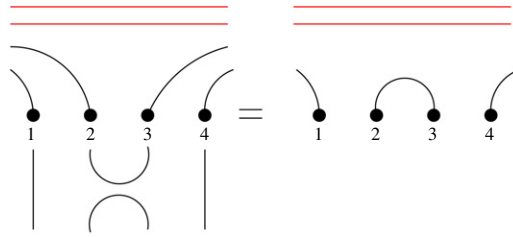
**Figure 2.** The action of the  $e_2$  generator acting on the bond between sites 2 and 3, which are not end-points of an arch of the size of the system.  $L = 4, n = 2$  in the figure.



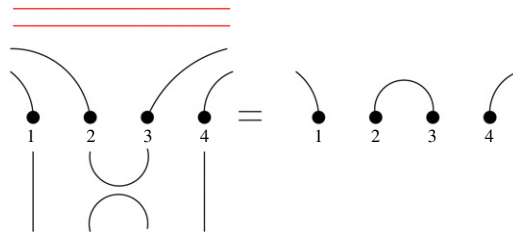
**Figure 3.** The action of the  $e_2$  generator acting on the bond between sites 2 and 3, which are the end of an arch of the size of the system  $L = 4$ . A new circle is created on the cylinder and one moves from the copy  $n = 2$  to the copy  $n = 3$ .

Representations of the quotient (5) might be interesting in their own right, but we did not study them here.

In order to obtain representations of the quotient (2) we have to identify the copy  $N$  not with the copy  $N - 1$  but with the copy  $n = 0$  (no noncontractible loops). See figure 5 for



**Figure 4.** One takes  $N = 3$ . The action of the generator  $e_2$  described in figure 3 is changed depending of the quotient that one chooses. In the figure, we show the choice of the quotient of equation (5). One does not change the copy, which stays as  $n = 2$ .



**Figure 5.** The same as for figure 4, but choosing the quotient given by equation (2). From the copy  $n = 2$ , one moves to the copy  $n = 0$  in order to get a representation with the symmetry  $Z_3$ .

$N = 3$  and  $L = 4$ . On adding circles without sites, this representation is also a representation of the cyclic group  $Z_N$ . One can show [8] that this representation is reducible. It splits into  $N$  representations defined by the quotients (4) with  $\alpha = \exp(i2\pi r/N)$ .

In what follows, we consider the application of cyclic representations to stochastic processes [6] taking  $x = 1$ . The Hamiltonian

$$H = \sum_{k=1}^L (1 - e_k), \tag{6}$$

gives the time evolution of the probability distribution function defined in the configuration space of the  $N$  copies of link patterns each containing  $\binom{L}{L/2}$  configurations. A detailed discussion of the spectra of  $H$  will be presented elsewhere [9]. For the remainder of this work, we consider only even values of  $L$ .

We first recall the known case  $N = 1$ . We use the spin representation of the  $PTL_L(1)$  [1, 10]:

$$e_k = \sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \frac{1}{4}(1 - \sigma_k^z \sigma_{k+1}^z) + i \frac{\sqrt{3}}{4}(\sigma_{k+1}^z - \sigma_k^z), \quad k = 1, 2, \dots, L - 1,$$

$$e_L = e^{i\frac{2\pi}{3}} \sigma_L^+ \sigma_1^- + e^{-i\frac{2\pi}{3}} \sigma_L^- \sigma_1^+ + \frac{1}{4}(1 - \sigma_L^z \sigma_1^z) + i \frac{\sqrt{3}}{4}(\sigma_1^z - \sigma_L^z). \tag{7}$$

In the scaling limit, the scaling dimensions  $\{x\}$  are obtained from the leading behavior of the energy-gap amplitudes  $E = 2\pi v_s x/L$ , where  $v_s = 3\sqrt{3}/2$  is the sound velocity. The spectrum of  $H$  in the link representation is contained in the  $S^z = \sum_{k=1}^L \sigma_k^z = 0$  sector and is known. The scaling dimensions associated with the eigenstates with momenta  $P = 2\pi p/L \pmod{\pi}$ ;  $p = 0, \pm 1, \pm 2, \dots$  are [11, 12]

$$x = (3/4)(1/3 + s)^2 - 1/12 + m + m', \quad p = m - m' \tag{8}$$

where  $s, m, m' = 0, \pm 1, \pm 2, \dots$

**Table 1.** Numerical estimates for the lowest two scaling dimensions appearing in the sector with  $r = 1$  and momentum 0 (mod  $\pi$ ). In the last line of the table, we show the results obtained from the van den Broeck and Swartz extrapolants (VBS). The complex conjugated dimensions appear in the sector with  $r = 2$ .

$L$	$x_0^1(1)$	$x_0^1(2)$
6	0.041 133 7612 + 0.089 322 2227 i	0.152 201 5017 - 0.152 201 5017 i
10	0.039 818 6156 + 0.088 342 8114 i	0.150 165 1591 - 0.117 108 7213 i
14	0.039 451 0154 + 0.088 091 2156 i	0.149 623 3621 - 0.117 517 7089 i
18	0.039 296 9971 + 0.087 992 1416 i	0.149 405 1829 - 0.117 707 0065 i
22	0.039 217 8087 + 0.087 943 6454 i	0.149 296 5628 - 0.117 811 0010 i
26	0.039 171 6570 + 0.087 916 5117 i	0.149 234 9425 - 0.117 874 6350 i
30	0.039 142 3702 + 0.087 899 8853 i	0.149 196 7343 - 0.117 916 5804 i
$\infty$	0.039 050 + 0.087 853 i	0.149 085 - 0.118 06 i

The lowest excitation is obtained if one takes  $s = -1, m = m' = 0$ :

$$x_0^0(1) = 1/4 = 0.25, \quad p = 0. \tag{9}$$

The explanation for the notation  $x_0^0(1)$  will be given in a few lines.

If  $N \neq 1$ , the states are separated not only by the momenta but also by the  $Z_N$  representation  $\exp(i2\pi r/N)$  to which they belong ( $r = 0, 1, 2, \dots, N - 1$ ). The  $r = 0$  states, for example, are obtained by taking the sum of the same link configurations in all the  $N$  copies. We will denote by  $x_p^r(i)$  ( $i = 1, 2, \dots$ ) the scaling dimensions associated with the  $i$ th lowest energy in the sector of momentum  $P = 2\pi p/L \pmod{\pi}$  and the  $r$  representation of  $Z_N$ . In what follows, we present some results for the case  $N = 3$ .

It is known [13] that the system is integrable but the calculations are tedious; we have studied the finite-size scaling spectra numerically using up to  $L = 30$  sites. We separate the vector space into disjoint sectors labeled with the momentum  $P$  and the index  $r$  of the representation  $Z_N$ . The lowest energy in each sector is calculated by the power method. The ground state of  $H$  which corresponds to the stationary state of the stochastic process corresponds to the eigenvalue zero. The eigenfunction is in the  $p = 0, r = 0$  sector and shows no new combinatorial properties beyond those known from the  $N = 1$  case [14]. One relevant result is that the entire spectra related to the scaling dimensions  $\{x_p^0(i)\}$  coincide with the known spectra of the  $N = 1$  representation. In order to show the precision of our procedure, we have estimated the scaling dimension (9) just from the energy gap for an  $L = 30$  lattice (no extrapolations using different sizes!) and got 0.249 762 20.

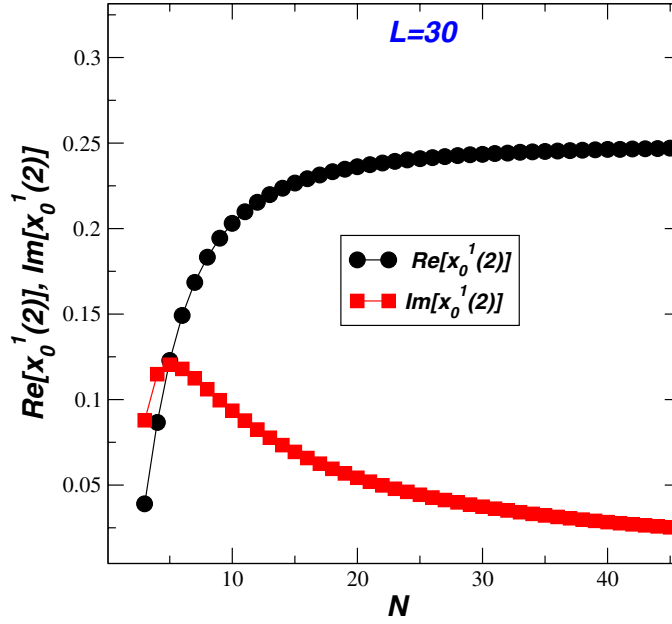
Taking  $N = 3$ , we looked at the spectra in the  $r = 1$ , and  $p = 0$  sectors and got a surprise. The extrapolants [15] for the two first excited levels gave the following complex values:

$$x_0^1(1) = 0.039 05 + 0.087 53 i, \quad x_0^1(2) = 0.149 08 - 0.118 06 i. \tag{10}$$

In order to check whether these results have anything to do with conformal invariant spectra, we looked at the  $r = 1, P = 2\pi/L \pmod{\pi}$  spectrum. If the finite-size scaling limits of the spectra are given by Virasoro representations with a complex highest weight, one should expect  $x_1^1(i) = x_0^1(i) + 1$  ( $i = 1, 2$ ). This is indeed the case, since we get

$$x_1^1(1) = 1.0391 + 0.087 55 i, \quad x_1^1(2) = 1.149 - 0.118 06 i. \tag{11}$$

In order to illustrate the precision of the estimates of the scaling dimensions, in tables 1 and 2 we give their measured values for different lattice sizes. One can see that the data converge very nicely. In the  $r = 2$  sector, one obtains the complex conjugate values of (10) and (11):  $x_p^2 = (x_p^1)^\dagger$ . The very existence of Virasoro representations is a remarkable fact, since the transition from one copy to another is a highly nonlocal operation.



**Figure 6.** Real part (black) and imaginary part (red) of the estimated value of the scaling dimension  $x_0^1(2)$ , as a function of  $N$  for the lattice size  $L = 30$ .

**Table 2.** Numerical values of the lowest two scaling dimensions appearing in the sector with  $r = 1$  and momentum  $2\pi/L \pmod{\pi}$  (the complex conjugated dimensions appear in the sectors with  $r = 2$  and momentum  $2\pi/L \pmod{\pi}$ ). In the last line of the table, we show the results obtained from the VBS extrapolants.

$L$	$x_1^1(1)$	$x_1^1(2)$
6	$0.895\,498\,8326 + 0.0427\,352\,699\,i$	$0.948\,586\,1617 - 0.054\,330\,6613\,i$
10	$0.985\,615\,5271 + 0.069\,767\,4849\,i$	$1.072\,733\,2613 - 0.091\,752\,8656\,i$
14	$1.011\,475\,1102 + 0.078\,159\,5719\,i$	$1.109\,211\,1061 - 0.103\,830\,6091\,i$
18	$1.022\,267\,1807 + 0.081\,805\,1754\,i$	$1.124\,623\,6272 - 0.109\,146\,8356\,i$
22	$1.027\,770\,1176 + 0.083\,713\,3906\,i$	$1.132\,547\,1271 - 0.111\,946\,8246\,i$
26	$1.030\,949\,9010 + 0.084\,837\,4121\,i$	$1.137\,153\,4904 - 0.113\,601\,8590\,i$
30	$1.032\,951\,6135 + 0.085\,555\,7589\,i$	$1.140\,067\,2494 - 0.114\,661\,8390\,i$
$\infty$	$1.0391 + 0.0878\,i$	$1.149 - 0.118\,06\,i$

Notice that the scaling dimensions (10) have a smaller real part than the value (9). This observation has physical consequences. If we consider a local observable, using the mappings of the link patterns into Dyck paths, charged particles or particle–vacancy configurations [6], for large systems, the approach to the stationary state will be oscillatory. As far as we know, this is the first time that such a phenomenon has been observed, since normally the imaginary part of the energy levels decreases faster with  $L$  than the real part. There are obviously consequences for the correlation functions too. We should stress that the stochastic process with the evolution operator (6) takes place in the  $N\binom{L}{L/2}$ -dimensional vector space which is a representation of  $Z_N$  and not in the independent copies (4) with  $\alpha = \exp(i\pi r/N)$ . The spectra are related, but one has to have in mind that in a stochastic model, the wavefunctions must have real nonnegative coefficients, and that the various sectors are mixed.



We would like to mention that we have also looked at the variation of the lowest first excited state with  $N$ , keeping  $r = 1$ . The data for the second level with momentum zero (mod  $\pi$ ) are shown in figure 6. One sees that on increasing  $N$ , the real part approaches the value (9), and that the imaginary part gets smaller and smaller. This is not to say that the smallest scaling dimension cannot be found for another value of  $r$ , but the consequence of the data shown in figure 6 is that in the large  $N$  limit, the scaling dimension  $1/4$  will be found at least three times ( $r = 0, 1$ ). We have not looked at the possible existence of Jordan cells in the spectrum [16].

In [9] we will give the partition function for each sector  $r$  and for any parameter  $x$  of the definition of the algebra [1]. The case  $x = 0$  is especially interesting since in this case, the Hamiltonian is related to the transfer matrix of a classical system of  $N$  colored interacting polymers on a cylinder, generalizing the known case  $N = 1$  [4].

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