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# ABSTRACTS

# Stability and Caputo Fractional Dini Derivative of Lyapunov Functions for Caputo Fractional Differential Equations

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## 1 Introduction

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}_{t_0}^{c}D^q x = f(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $x, x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$, $t_0 \geq 0$.

The goal of the paper is study the stability properties of zero solution of the system FrDEs (1).

The stability of fractional order systems is quite recent. There are several approaches in the literature to study stability, one of which is the Lyapunov approach. We introduce the class $\Lambda$ of Lyapunov-like functions which will be used to investigate the stability of (1).

**Definition 1.** Let $t_0, T \in \mathbb{R}_+ : \ T > t_0$, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. We will say that the function $V(t,x) : [t_0, T) \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda([t_0,T), \Delta)$ if $V(t,x) \in C([t_0,T) \times \Delta, \mathbb{R}_+)$ is locally Lipschitzian with respect to its second argument and $V(t,0) \equiv 0$.

Results on stability in the literature via Lyapunov functions could be divided into two main groups:

- continuously differentiable Lyapunov functions (see, for example, the papers [4], [7]). Different types of stability are discussed using the Caputo derivative of Lyapunov functions which depends significantly of the unknown solution of the fractional equation. This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive;
- continuous Lyapunov functions (see, for example, the papers [5], [6]) in which the authors use the *Dini fractional derivative* along the FrDE by

$$ {}^{c}D_+^q V(t,x) = \limsup_{h \to 0} \frac{1}{h^q} \big[ V(t,x) - V(t-h, x - h^q f(t,x)) \big]. \tag{2}$$

  The "fractional Dini derivative" (2) is a strange operator since it is local and in some cases it is totally different than the used derivatives in ordinary case ($q = 1$).

**Example 1.** Let $V(t,x)=\frac{x^2}{(t+1)^2}$ , $x\in\mathbb{R}$. Then using (2) we get $D^qV(t,x)=\frac{2xf(t,x)}{(t+1)^2}$ which is different than the used derivative in the ordinary case

$$DV(t,x)=\frac{2x}{(t+1)^2}\,f(t,x)+x^2\Big(\frac{1}{(t+1)^2}\Big)'.$$

We introduce the derivative of the Lyapunov function based on the Caputo fractional Dini derivative of a function. We define the generalized *Caputo fractional Dini derivative* of Lyapunov like function $V(t,x)$ along the system FrDE (1) by (see [1]):

$$\begin{aligned}{}^{c}_{(1)}D^q_+V(t,x)=\limsup_{h\to 0^+}\frac{1}{h^q}\bigg\{&V(t,x)-V(t_0,x_0)-\\ &-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}qCr\big[V(t-rh,x-h^qf(t,x))-V(t_0,x_0)\big]\bigg\}\ \text{ for }\ t\geq t_0, \quad (3)\end{aligned}$$

where $t\in(t_0,T)$, $x,x_0\in\Delta$, and there exists $h_1>0$ such that $t-h\in[t_0,T)$, $x-h^qf(t,x)\in\Delta$ for $0<h\leq h_1$, $\Delta\subset\mathbb{R}^n$.

**Example 2.** Let $V(t,x)=\frac{x^2}{(t+1)^2}$ , $x\in\mathbb{R}$ and $t_0=0$, $x_0=0$. Then using (3) we get the Caputo fractional Dini derivative ${}^{c}_{(1)}D^q_+V(t,x)=\frac{2xf(t,x)}{(t+1)^2}+x^2D^q_0\,\frac{1}{(t+1)^2}$ . which is slightly different than the ordinary case $q=1$.

## 2 Comparison Results for Scalar FrDE

The base of the main results in study stability properties of FrDE (1) is the application of Caputo fractional Dini derivative (3) and some comparison results.

**Lemma 1.** *Assume the following conditions are satisfied:*

1. *the function $x^*(t)=x(t;t_0,x_0)$, $x^*\in C^q([t_0,T],\Delta)$ is a solution of the FrDE (1), where $\Delta\subset\mathbb{R}^n$, $0\in\Delta$;*
2. *the function $V\in\Lambda([t_0,T],\Delta)$ and for any points $t\in[t_0,T]$, $x\in\Delta$ the inequality ${}^{c}_{(1)}D^q_+V(t,x)\leq -c(\|x\|)$ holds, where $c\in\mathcal{K}$.*

*Then for $t\in[t_0,T]$ the inequality $V(t,x^*(t))\leq V(t_0,x_0)-\frac{1}{\Gamma(q)}\int\limits_{t_0}^{t}(t-s)^{q-1}c(\|x^*(s)\|)\,ds$ holds.*

## 3 Stability Results

Several sufficient conditions for stability, uniform stability, asymptotic stability of zero solution of the system FrDE (1) are obtained.

**Theorem 1** ([1])**.** *Assume:*
*There exists a function $V\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ such that*

(i) *for any points $t\geq 0$ and $x\in\mathbb{R}^n$ the inequality ${}^{c}_{(1)}D^q_+V(t,x)\leq -c(\|x\|)$ holds, where $c\in\mathcal{K}$;*

(ii) *$b(\|x\|)\leq V(t,x)\leq a(\|x\|)$ for $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$, where $a,b\in\mathcal{K}$.*

*Then the zero solution of the FrDE (1) is uniformly asymptotically stable.*

The introduced Caputo fractional Dini derivative (3) is appropriately transformed to the generalized *Caputo fractional Dini derivative w.r.t. to ITD* of the function $V(t,x)$:

$$
\begin{aligned}
{}_{t_0}^{c}D_{(1)}^{q}V(t,x,y,\eta,x_0,y_0) = \lim_{h\to 0^+}\sup\frac{1}{h^q}\bigg[V(t,y-x)-V(t_0,y_0-x_0)-\\
-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}qCr\Big(V\big(t-rh,y-x-h^q(f(t+\eta,y)-f(t,x))\big)-V(t_0,y_0-x_0)\Big)\bigg], \quad (4)
\end{aligned}
$$

where $t,t_0\in I$, $y-x,y_0-x_0\in\Delta$, and there exists $h_1>0$ such that $t-h\in I$, $y-x-h^q(f(t+\eta,y)-f(t,x))\in\Delta$ for $0<h\le h_1$ and $\eta\in B_H$.

The Caputo fractional Dini derivative w.r.t. to ITD is applied to study practical stability with initial time difference for FrDE (1) (see [3]).

The base of the main results is the following result.

**Lemma 2** (Shift solutions in the nonautonomous FrDE [3])**.** *Let the function $x\in C^q(\mathbb{R}_+,\mathbb{R}^n)$, $a\ge 0$, be a solution of the initial value problem for FrDE*

$$ {}_{a}^{c}D^{q}x(t)=f(t,x(t)) \ \text{ for } \ t>a, \quad x(a)=x_0. \tag{5}$$

*Then the function $\tilde{x}(t)=x(t+\eta)$ satisfies the initial value problem for the FrDE*

$$ {}_{b}^{c}D^{q}x=f(t+\eta,x) \ \text{ for } \ t>b, \quad x(b)=x_0, \tag{6}$$

*where $b\ge 0$, $\eta=a-b$.*

One of the obtained sufficient conditions are formulated below:

**Theorem 2** (Uniform practical stability [3])**.** *Let the following conditions be satisfied:*

1. *The function $g\in C[[t_0,\infty)\times\mathbb{R}\times B_H,\mathbb{R}]$, $g(t,0,0)\equiv 0$ and for any parameter $\eta\in B_H$ there exists a positive number $M_\eta$ such that for any $\varepsilon\in[0,M_\eta]$ and $v_0\in\mathbb{R}$ the IVP for the scalar FrDE ${}_{\tau_0}^{c}D^{q}x(t)=f(t,x(t))$, $t>\tau_0$, $x(\tau_0)=y_0$ has a solution $u(t;t_0,v_0,\eta,\varepsilon)\in C^q([t_0,\infty),\mathbb{R})$, where $H>0$ is a given number.*

2. *There exists a function $V\in\Lambda(\mathbb{R}_+,S(A))$ such that*

   (i) *$b(\|x\|)\le V(t,x)\le a(\|x\|)$ for $(t,x)\in\mathbb{R}_+\times S(A)$, where $a,b\in\mathcal{K}$;*

   (ii) *for any $t_0\in\mathbb{R}_+$, $x,y,x_0,y_0\in\mathbb{R}^n$ : $y-x\in S(A)$, $y_0-x_0\in S(A)$ and $\eta\in B_H$ the inequality ${}_{t_0}^{c}D_{(1)}^{q}V(t,x,y,\eta,x_0,y_0)\le g(t,V(t,y-x),\eta)$ for $t\ge t_0$ holds, where $A>0$ is a given number.*

3. *The scalar FrDE ${}_{\tau_0}^{c}D^{q}x(t)=f(t,x(t))$ for $t>\tau_0$, $x(\tau_0)=y_0$ with $\varepsilon=0$ is uniformly parametrically practically stable with respect to $(a(\lambda),b(A))$, where the constant $\lambda\in(0,A)$ is given so that $a(\lambda)<b(A)$.*

*Then the system of FrDE* (1) *is uniformly practically stable with ITD with respect to* $(\lambda,A)$ .

Also, the system of fractional differential equations with noninstantaneous impulses is defined, the Caputo fractional Dini derivative (3) is appropriately transformed and stability of the zero solution is studied in [2].

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# Travelling Wave Solutions of Integro-Differential Equation Arising in Nano-Structures

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## 1 Introduction

The demand for smaller and faster devices has encouraged technological advances resulting in the ability to manipulate matter at nanoscales that have enabled the fabrication of nanoscale electromechanical systems. With the advances in materials synthesis and device processing capabilities, the importance of developing and understanding nanoscale engineering devices has dramatically increased over the past decade. Computational Nanotechnology has become an indispensable tool not only in predicting, but also in engineering the properties of multi-functional nano-structured materials. The presence of nano-inclusions in these materials affects or disturbs their elastic field at the local and the global scale and thus greatly influences their mechanical properties.

Let $G \in R^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano–holes, nano–inclusions) subjected to time–harmonic load on the boundary $\partial G$. Note that heterogeneities are of macro size if their diameter is greater than $10^{-6}m$, while heterogeneities are of nano–size if their diameter is less than $10^{-7}m$.

The aim is to find the field in every point of $M = G\backslash I$, $I$ and to evaluate stress concentration around the inhomogeneities.

Using the methods of continuum mechanics the problem can be formulated in terms of boundary value problem for a system of 2-nd order differential equations (see [1, Chapter 2])

$$\begin{gathered} c_{44}^N \Delta u_3^N + e_{15}^N \Delta u_4^N - \rho^N u_{3,tt} = 0, \\ e_{15}^N \Delta u_3^N - \varepsilon_{15}^N \Delta u_4^N = 0, \end{gathered} \tag{1}$$

where $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is Laplace operator with respect to $t$, $N = M$ for $x \in M$ and $N = I$ for $x \in I$; $u_3^N$ is mechanical displacement, $u_4^N$ is electric potential, $\rho^N$ is the mass density, $c_{44}^N > 0$ is the shear stiffness, $e_{15}^N \neq 0$ is the piezoelectric constant and $\varepsilon_{11}^N > 0$ is the dielectric permittivity.

Assume that the interface between the nano-inclusion $I$ and its surrounding matrix $M$ is regarded as thin material surface $S$ that possesses its own mechanical parameters $c_{44}^I$, $e_{15}^I$, $\varepsilon_{11}^I$.

We shall consider the case when $I$ is a nano-hole and boundary conditions on $S$ are

$$t_j^M = \frac{\partial \sigma_{lj}^S}{\partial l} \ \text{ on } S, \tag{2}$$

where $\sigma^S_{lj}$ is generalized stress [1], $j = 3, 4$, $l$ is the tangential vector. Then we shall study boundary value problem (BVP) (1) with boundary conditions (2).

There are no numerical results for dynamic behavior of bounded piezoelectric domain with heterogeneities under anti-plane load. Validation is done in [1] for infinite piezoelectric plane with a hole, in [2] for isotropic bounded domain with holes and inclusions and in [3] for piezoelectric plane with nano-hole or nano-inclusion. In Section 2 we shall construct CNN model for the BVP (1), (2). In section 3 we shall find travelling wave solutions of this model and we shall provide validation.

## 2 Cellular Nonlinear Network (CNN) Model of the BVP

In [1] fundamental solutions of the BVP (1), (2) are found using the Fourier transform. Then using the Gauss theorem and proceeding as in [1] from the BVP a system of integro-differential equations (IDE) is obtained for the unknowns $u_{3,4}$ on $S$. This system has the following general form

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} - C_1 \int\limits_S f(u(t,x))\,dt, \quad t \in [0,1], \tag{3}$$

where $C_1$ is a constant depending on the $\rho^M$, $c^M_{44} > 0$, $e^M_{15} \neq 0$ and $\varepsilon^M_{11} > 0$, $D$ is diffusion coefficient.

Then the CNN model [4] for the IDE (3) can be written as

$$\frac{du_{ij}}{dt} = DA_1 * u_{ij} - C_1 \int\limits_S f(u_{ij}(t))\,dt, \quad 1 \le i \le n, \quad j = 3, 4, \tag{4}$$

where $A_1$ is 1-dimensional discretized Laplacian template, $*$ is convolution operator.

We shall take the output of the IDE CNN model (4) as a piecewise linear function [4]:

$$y(u_{ij}) = au_{ij} + b\big(|u_{ij} - V_p| - |u_{ij} - V_v|\big) - b\big(|u_{ij} + V_p| - |u_{ij} + V_v|\big) = N(u_{ij}), \quad j = 3, 4, \tag{5}$$

where $a > 0$, $b < 0$ are constants, $V_p$, $V_v (0 < V_p < V_v)$ are the peak and valley voltages of the CNN, and as one can notice the output function is symmetric with respect to the origin. The graph of the output function is given on Figure 1 below.

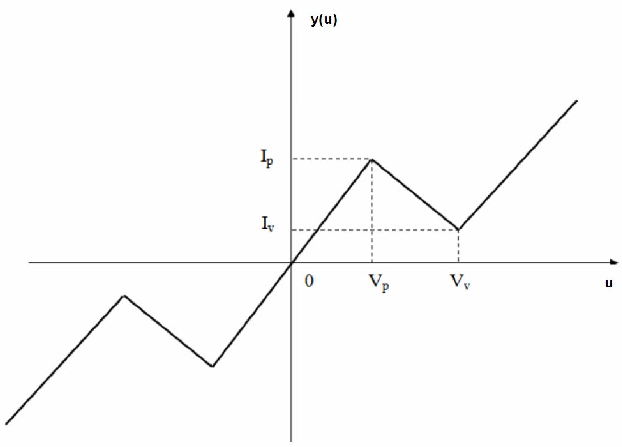


**Figure 1.** Graph of the output function (5) for the CNN model.

# 3 Travelling Wave Solutions of IDE CNN Model

We shall study traveling wave solutions of IDE CNN model (4) of the form

$$u_i = \Phi(i - ct), \tag{6}$$

for some continuous function $\Phi : \mathbf{R}^1 \to \mathbf{R}^1$ and some unknown real number $c$. Let us denote $s = i - ct$. Let us substitute (6) in the IDE CNN model (4). Therefore $\Phi(s, c)$ and $c$ satisfy the equation of the form

$$-c\Phi'(s, c) = \Phi(s - 1, c) - 2\Phi(s, c) + \Phi(s + 1, c) - C_1 \int\limits_S f(\Phi(s, c))\, dt. \tag{7}$$

Our aim in this note is to study traveling wave solution of the IDE CNN model (4). We consider solution of equation (7). The following theorem about travelling wave solution of our IDE CNN model holds.

**Theorem 1.** *Let $\Phi(s, c)$ be a solution of (7) satisfying the following conditions*

$$\lim_{s\to-\infty} \Phi(s, c) = 0, \quad \lim_{s\to\infty} \Phi(s, c) = 1.$$

*Then*

(i) *If $c = c^* < 0$, $\Phi(s, c)$ is a stable travelling wave solution of IDE CNN model.*

(ii) *If $c = c_* > 2$, $\Phi(s, c)$ is unstable travelling wave solution.*

We shall skip the proof due to the lack of space.

Traveling wave solution for our IDE CNN model (4) is given on Figure 2. We use the following parameter set for the numerical simulation.Material parameters of the matrix are for transversely isotropic piezoelectric material PZT4 are: elastic stiffness: $c_{44}^M = 2.56 \times 10^{10}\, N/m^2$; piezoelectric constant: $e_{15}^M = 12.7\, C/m^2$; dielectric constant: $\varepsilon_{11}^M = 64.6 \times 10^{-10}\, C/Vm$; density: $\rho^M = 7.5 \times 10^3\, kg/m^3$.

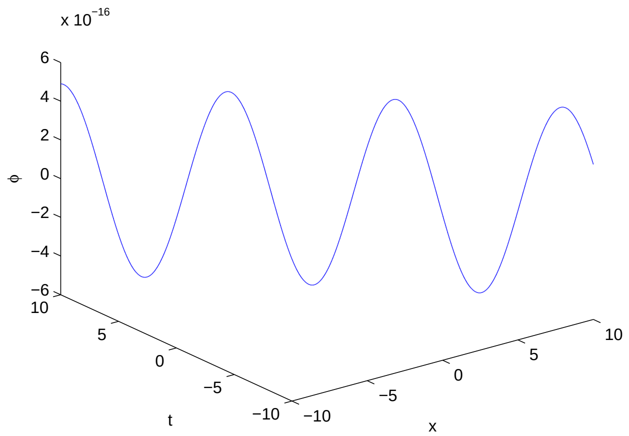


**Figure 2.** Traveling wave solution of IDE CNN model (4).

The characteristic that is of interest in nano-structures is normalized Stress Concentration Field (SCF) $(\sigma/\sigma_0)$ and it is calculated by the following formula

$$\sigma = -\sigma_{13} \sin(\varphi) + \sigma_{23} \cos(\varphi), \tag{8}$$

where $\varphi$ is the polar angle of the observed point, $\sigma_{ji}$ is the stress (2) near $S$. The applied load is time harmonic uni-axial along vertical direction uniform mechanical traction with frequency $\omega$ and amplitude $\sigma_0 = 400 \times 10^6\, N/m^2$ and electrical displacement with amplitude $D_0 = k\, \frac{\varepsilon_{11}^M}{e_{15}^M}\, \sigma_0$.

The validation of our model is provided below on Figure 3 for the parameter sets given above.

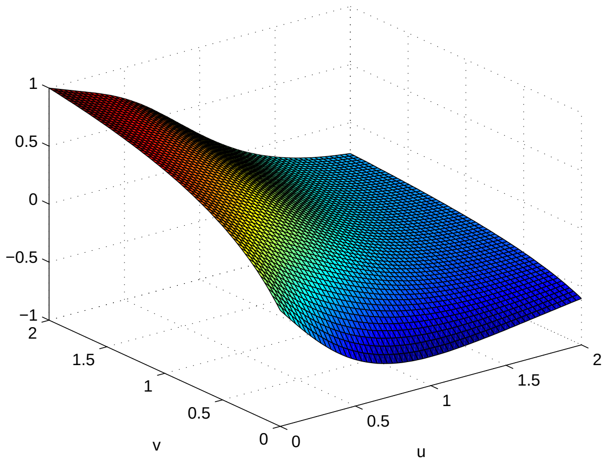


**Figure 3.** Validation – dynamic SCF at observed point.

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# On the Well-Possedness of the Cauchy Problem and the Lyapunov Stability for Systems of Generalized Ordinary Differential Equations

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Let $A_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval non-degenerated in the point. Consider the Cauchy problem

$$dx(t) = dA_0(t)\cdot x(t) + df_0(t), \quad x(t_0) = c_0. \tag{1}$$

Let $x_0$ be the unique solution of problem (1).

Along with the Cauchy problem (1) consider the sequence of the Cauchy problems

$$dx(t) = dA_k(t)\cdot x(t) + df_k(t), \quad x(t_k) = c_k \ \ (k = 1,2,\dots), \tag{$1_k$}$$

where $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $f_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $t_k \in I$ $(k = 1,2,\dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated also by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from the unified viewpoint.

In [2–4] the sufficient conditions are given for problem $(1_k)$ to have a unique solution $x_k$ for sufficiently large $k$ and

$$\lim_{k\to+\infty} \|x_k - x_0\|_s = 0. \tag{2}$$

In the present paper, the necessary and sufficient conditions are established for the sequence of the Cauchy problems $(1_k)$ $(k = 1,2,\dots)$ to have the above-mentioned property. Obtained here results are based on the concept given in [8] and they differ from the analogous ones given in [3].

Moreover, we consider the question of relationship between the Lyapunov stability of system given in (1) and the well-possedness of the Cauchy problem (1). Presented below results are more general than analogous ones obtained in [4].

The following notations and definitions will be used.

$\mathbb{R} = ]-\infty,+\infty[\,$, $\mathbb{R}_+ = [0,+\infty[\,$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$O_{n\times m}$ is the zero $n\times m$ matrix.

$I_n$ is an identity $n\times n$ matrix.

$\bigvee\limits_a^b(X)$ is the sum of total variations of the components $x_{ij}$ $(i = 1,\dots,m;\ j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$; $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of $X$ at the point $t \in I$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}(I;\mathbb{R}^{n\times m})$ is the space of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_s = \sup\{\|X(t)\| :\ t \in I\}$.

$\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ for which the restriction on $[a,b]$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b] \subset I$.

$\widetilde{C}_{loc}(I;\mathbb{R}^n)$ is the set of all vector-functions $x : I \to \mathbb{R}^n$ which are absolutely continuous on every closed interval $[a,b]$ from $I$.

$L(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are Lebesgue-integrable;

$L_{loc}(I;\mathbb{R}^{n\times m})$ is the set of matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are Lebesgue integrable on every closed interval from $I$.

We introduce the operators. If $X \in \mathrm{BV}_{loc}(I,\mathbb{R}^{l\times n})$ and $Y : I \to \mathbb{R}^{n\times m}$, then we put

$$\mathcal{B}(X,Y)(t) \equiv X(t)Y(t) - X(t_0)Y(t_0) - \int\limits_{t_0}^{t} dX(\tau)\cdot Y(\tau),$$

$$\mathcal{I}(X,Y)(t) \equiv \int\limits_{t_0}^{t} d(X(\tau) + \mathcal{B}(X,Y)(\tau))\cdot X^{-1}(\tau).$$

If $X \in \mathrm{BV}(I\,;\mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in I$ $(j=1,2)$, and $Y \in \mathrm{BV}([a,b]\,;\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(t_0) \equiv O_{n\times m}$,

$$\mathcal{A}(X,Y)(t) \equiv Y(t) - Y(t_0) + \\ + \sum_{t_0<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\, d_1Y(\tau) - \sum_{t_0\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\, d_2Y(\tau).$$

A vector-function $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ is said to be a solution of the generalized differential system given in (1) if

$$x(t) - x(s) = \int\limits_{s}^{t} dA_0(\tau)\cdot x(\tau) + f_0(t) - f_0(s) \ \text{ for } \ s<t; \ \ s,t \in I,$$

where integral is understand in the Kurzweil sense [9].

Without loss of generality, we assume that either $t_k < t_0$ $(k=1,2,\dots)$ or $t_k = t_0$ $(k=1,2,\dots)$ or $t_k > t_0$ $(k=1,2,\dots)$.

**Definition 1.** We say that the sequence $(A_k, f_k; t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k=1,2,\dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{3}$$

problem $(1_k)$ has a unique solution $x_k$ for any sufficient large $k$ and condition (2) holds.

**Theorem 1.** *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=0,1,\dots)$, $t_0 \in I$ and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that*

$$\begin{gathered}\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \ \textit{ for } \ t \in I, \quad (-1)^j(t-t_0) < 0 \ \textit{ and for} \\ t = t_0 \ \textit{ if } \ j \in \{1,2\} \ \textit{ is such that } \ (-1)^j(t_k - t_0) > 0 \ \ (k=1,2,\dots),\end{gathered} \tag{4}$$

$$\lim_{k\to+\infty} t_k = t_0. \tag{5}$$

*Then*

$$\big((A_k, f_k; t_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{6}$$

*if and only if there exists a sequence of matrix-functions* $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that*

$$\inf\big\{|\det(H_0(t))|:\ t\in I\big\}>0, \tag{7}$$

$$\lim_{k\to+\infty} H_k(t_k)=H_0(t_0), \tag{8}$$

$$\lim_{k\to+\infty} \|H_k-H_0\|_s=0, \tag{9}$$

$$\lim_{k\to+\infty}\sup_{t\in I}\Big\{\big\|(\mathcal{I}(H_k,A_k)(t)-\mathcal{I}(H_0,A_0)(t))\big\|\big(1+\big|\mathop{\vee}_{t_0}^{t}(\mathcal{I}(H_k,A_k))\big|\big)\Big\}=0 \tag{10}$$

*and*

$$\lim_{k\to+\infty}\sup_{t\in I}\Big\{\big\|(\mathcal{I}(H_k,f_k)(t)-\mathcal{I}(H_0,f_0)(t)\big\|\big(1+\big|\mathop{\vee}_{t_0}^{t}(\mathcal{I}(H_k,A_k))\big|\big)\Big\}=0. \tag{11}$$

**Definition 2.** The Cauchy problem (1) is called well-possed if condition (6) holds for every sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ and $t_k$ $(k=1,2,\dots)$ for which there exists a sequence $H_k$ $(k=0,1,\dots)$ such that conditions (4), (5) and (7)–(11) hold.

The statements of Theorem 1 mean that the Cauchy problem (1) is well-possed.

**Definition 3.** The Cauchy problem (1) is called weakly well-possed if condition (6) holds for every sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ and $t_k$ $(k=1,2,\dots)$ for which there exists a sequence $H_k$ $(k=0,1,\dots)$ such that conditions (4), (5), (7)–(9) and

$$\lim_{k\to+\infty}\Big(\big\|\mathcal{I}(H_k,A_k)-\mathcal{I}(H_0,A_0)\big\|_s+\big\|\mathcal{I}(H_k,f_k)-\mathcal{I}(H_0,f_0)\big\|_s\Big)=0$$

hold.

Consider now the Lyapunov stability question on the set $I=[0,+\infty[$.

**Definition 4.** A solution $x_0$ of the system given in (1) is called uniformly stable if for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution $x$ of system (1), satisfying the inequality

$$\|x(t_0)-x_0(t_0)\|<\delta \tag{12}$$

for some $t_0\in\mathbb{R}_+$, admits the estimate $\|x(t)-x_0(t)\|<\delta$ for $t\geq t_0$.

**Definition 5.** Let $\xi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function such that $\lim\limits_{t\to+\infty}\xi(t)=+\infty$. A solution $x_0$ of the system given in (1) is called $\xi$-exponentially asymptotically stable if there exists a positive number $\eta$ such that for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution $x$ of system (1), satisfying inequality (12) for some $t_0\in\mathbb{R}_+$, admits the estimate

$$\|x(t)-x_0(t)\|<\varepsilon\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\ \text{ for }\ t\geq t_0.$$

Note that the exponentially asymptotic stability (see [3]) is a particular case of the $\xi$-exponentially asymptotic stability if we assume $\xi(t)\equiv t$.

**Definition 6.** The system given in (1) is called stable in one or another sense if every its solution is stable in the same sense.

**Definition 7.** The matrix-function $A_0$ is called stable in one or another sense if the system $dx(t)=dA_0(t)\cdot x(t)$ is stable in the same sense.

**Theorem 2.** *Let* $A_0\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $f_0\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ *be such that*

$$\lim_{t\to+\infty}\sup\mathop{\vee}_{t}^{\nu(\xi)(t)}\mathcal{A}(A_0,A_0)<+\infty\ \ and\ \ \lim_{t\to+\infty}\mathop{\vee}_{t}^{\nu(\xi)(t)}\mathcal{A}(A_0,f_0)=0,$$

*where* $\nu(\xi)(t)=\sup\{\tau\geq t:\ \xi(\tau)\leq\xi(t)+1\}$. *Then* $\xi$*-exponentially asymptotically stability of* $A_0$ *guarantees the well-possedness of problem* (1) *on* $\mathbb{R}_+$.

**Theorem 3.** *Let $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $f_0 \in \mathrm{BV}(\mathbb{R}_+;\mathbb{R}^n)$. Then uniformly stability of $A_0$ guarantees the weakly well-possedness of problem* (1) *on $\mathbb{R}_+$.*

We realize the above-given results for the Cauchy problem for ordinary differential systems. Given here results are more general than obtained in [1, 5–8].

Let $\mathcal{P}_0 \in L_{loc}(I,\mathbb{R}^{n\times n})$ and $q_0 \in L_{loc}(I,\mathbb{R}^n)$. Let $x_0 \in \widetilde{C}_{loc}(I;\mathbb{R}^n)$ be the unique solution of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)x + q_0(t), \quad x(t_0) = c_0. \tag{13}$$

Consider the sequence of the Cauchy problems

$$\frac{dx}{dt} = \mathcal{P}_k(t)x + q_k(t), \quad x(t_k) = c_k \ \ (k = 1,2,\dots). \tag{$13_k$}$$

The system $(13_k)$ is the particular case of system $(1_k)$ if we assume that $A_k(t) \equiv \int\limits_{t_0}^{t} \mathcal{P}_k(\tau)\,d\tau$ and $f_k(t) \equiv \int\limits_{t_0}^{t} q_k(\tau)\,d\tau$ for every $k \in \{0,1,\dots\}$. Therefore, the results given below immediately follow from the analogous ones presented above.

**Definition 8.** We say that the sequence $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ belongs to the set $\mathcal{S}(\mathcal{P}_0, q_0, t_0)$ if condition (2) holds for every $c_0 \in \mathbb{R}^n$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ satisfying the condition (3), where $x_k$ is the unique solution problem $(13_k)$.

**Theorem 4.** *Let $P_k \in L(I,\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$, $q_k \in L(I;\mathbb{R}^n)$ $(k = 0,1,\dots)$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ satisfy condition* (5)*. Then*

$$\big((\mathcal{P}_k, q_k, t_k)\big)_{k=1}^{+\infty} \in S(\mathcal{P}_0, q_0, t_0) \tag{14}$$

*if and only if there exists a sequence of matrix-functions $H_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$ such that conditions* (7)–(9),

$$\lim_{k\to+\infty} \sup_{t\in I} \bigg\{ \bigg\| \int\limits_{t_0}^{t} (\mathcal{P}_k^*(\tau) - \mathcal{P}_0^*(\tau))\,d\tau \bigg\| \bigg(1 + \bigg| \int\limits_{t_0}^{t} \|\mathcal{P}_k^*(\tau)\|\,d\tau \bigg|\bigg) \bigg\} = 0 \tag{15}$$

*and*

$$\lim_{k\to+\infty} \sup_{t\in I} \bigg\{ \bigg\| \int\limits_{t_0}^{t} (q_k^*(\tau) - q_0^*(\tau))\,d\tau \bigg\| \bigg(1 + \bigg| \int\limits_{t_0}^{t} \|\mathcal{P}_k^*(\tau)\|\,d\tau \bigg|\bigg) \bigg\} = 0 \tag{16}$$

*hold, where*

$$\mathrm{P}_k^*(t) \equiv \big(H_k'(t) + H_k(\tau)P_k(t)\big)H_k^{-1}(t), \quad q_k^*(t) \equiv \big(H_k'(t) + H_k(\tau)q_k(t)\big)H_k^{-1}(t).$$

**Definition 9.** The Cauchy problem (13) is called well-possed if condition (14) holds for every sequence $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ and $t_k$ $(k = 1,2,\dots)$ for which there exists a sequence $H_k$ $(k = 0,1,\dots)$ such that conditions (7)–(9), (15) and (16) hold, where $\mathcal{P}_k^*$ and $q_k^*$ are matrix- and vector-functions defined in Theorem 4.

**Definition 10.** The Cauchy problem (1) is called weakly well-possed if condition (14) holds for every sequences $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ and $t_k$ $(k = 1,2,\dots)$ for which there exists a sequence $H_k$ $(k = 0,1,\dots)$ such that conditions (7)–(9) and

$$\lim_{k\to+\infty} \sup_{t\in I} \bigg\{ \bigg\| \int\limits_{t_0}^{t} \big(\mathcal{P}_k^*(\tau) - \mathcal{P}_0^*(\tau)\big)\,d\tau \bigg\| + \bigg\| \int\limits_{t_0}^{t} \big(q_k^*(\tau) - q_0^*(\tau)\big)\,d\tau \bigg\| \bigg\} = 0$$

hold, where $\mathcal{P}_k^*$ and $q_k^*$ are the matrix- and vector-functions defined in Theorem 4.

**Theorem 5.** *Let $\mathcal{P}_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $q_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ be such that*

$$\lim_{t\to+\infty} \sup \int\limits_t^{\nu(\xi)(t)} \|\mathcal{P}_0(\tau)\|\,d\tau < +\infty \ \text{ and } \ \lim_{t\to+\infty} \int\limits_t^{\nu(\xi)(t)} \|q_0(\tau)\|\,d\tau = 0,$$

*where $\nu(\xi)(t) = \sup\{\tau \geq t : \ \xi(\tau) \leq \xi(t) + 1\}$. Then $\xi$-exponentially asymptotically stability of $\mathcal{P}_0$ guarantees the well-possedness of problem* (13) *on $\mathbb{R}_+$.*

**Theorem 6.** *Let $\mathcal{P}_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $q_0 \in L(\mathbb{R}_+, \mathbb{R}^n)$. Then uniformly stability of $\mathcal{P}_0$ guarantees the weakly well-possedness of problem* (13) *on $\mathbb{R}_+$.*

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# On Asymptotic Behavior of Solutions to Nonlinear Differential Equations with a Small Right-Hand Side

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## 1 Introduction

The problem of asymptotic behavior of solutions to nonlinear differential equations with an exponentially small or power-law small right-hand sides is investigated.

Consider the equation

$$y^{(n)} + p(x)|y|^k \operatorname{sgn} y = F(x), \quad n \geq 2, \quad k > 1, \tag{1}$$

with continuous functions $p(x)$ and $F(x)$.

Equation (1) with $F(x) = 0$ was investigated from different points of view (see, for example, [8], [4] and the bibliography therein). In particular, the asymptotic behavior of its solutions vanishing at infinity is described. If the function $F(x)$ is sufficiently small, it is possible to describe the asymptotic behavior of vanishing at infinity solutions to equation (1), too. Previous results are published in [1]– [6]. Results of this type for ordinary differential equations and their systems can be useful also to investigate some problems for partial differential equations (see, for example, [7]).

Note that there exist notions of asymptotic equivalence different from the one used here (cf. [10]–[17]).

## 2 Main results

In this section results on asymptotic equivalence of solutions to differential equations with different right-hand sides are formulated.

### 1 Exponentially equivalent right-hand sides

**Theorem 2.1** (see [6])**.** *Let $f(x)$, $g(x)$, and $p(x)$ be bounded continuous functions defined in a neighborhood of $+\infty$. Suppose $y(x)$ is a solution to the equation*

$$y^{(n)} + p(x)|y|^k \operatorname{sgn} y = f(x)\, e^{-\beta x} \tag{2}$$

*with $n \geqslant 2$, $k > 1$, $\beta > 0$ and $y(x) \to 0$ as $x \to +\infty$. Then there exists a unique solution $z(x)$ to the equation*

$$z^{(n)} + p(x)|z|^k \operatorname{sgn} z = g(x)\, e^{-\beta x} \tag{3}$$

*such that $|z(x) - y(x)| = O(e^{-\beta x})$ as $x \to +\infty$.*

To prove this result we use the following lemmas.

**Lemma 2.1.** *If a function $y(x)$ and its $n$-th derivative $y^{(n)}(x)$ both tend to zero as $x \to +\infty$, then the same is true for all of its lower-order derivatives $y^{(j)}(x)$, $0 < j < n$.*

**Lemma 2.2.** *Suppose a function $y(x)$ satisfies the inequality $|y^{(j)}(x)| \geq W > 0$ on a segment $I$ of length $\Delta$. Then there exists a segment $I' \subset I$ of length $4^{-j}\Delta$ with $|y(x)| \geq W(2^{-1-j}\Delta)^j$ satisfied for all $x \in I'$.*

**Lemma 2.3.** *Let $y(x)$ be a solution to equation* (2) *tending to zero as $x \to +\infty$. Then*

$$y(x) = \mathbf{J}^n \big[e^{-\beta x} f(x) - p(x)|y(x)|^k \operatorname{sgn} y(x)\big],$$

*where the operator $\mathbf{J}$ takes each sufficiently rapidly decreasing function $\varphi(x)$ to its primitive function vanishing at infinity:*

$$\mathbf{J}[\varphi](x) = -\int\limits_x^\infty \varphi(\xi)\, d\xi.$$

**Corollary 2.1.** *Suppose the function $F(x)$ in equation* (1) *satisfies the condition*

$$|F(x)| \le Ce^{-\beta x}, \quad C > 0, \quad \beta > 0, \tag{4}$$

*and $p(x)$ is a bounded continuous function. Then for any solution $y(x)$ to equation* (1) *tending to zero as $x \to \infty$ there exists a solution $z(x)$ to equation* (1) *with F(x)=0 such that*

$$|y(x) - z(x)| = O(e^{-\beta x}), \quad x \to \infty.$$

**Remark 2.1.** Note that if $p(x) \to p_0 \neq 0$ as $x \to \infty$, for $n = 2$ [8] and $n \in \{3, 4\}$ ( [3] and [4], Ch.I, Section 5.4) asymptotic behavior of all solutions to equation (1) with $F(x) = 0$ is described. In particular, if $(-1)^n p_0 < 0$, then all nontrivial vanishing at infinity solutions $z(x)$ to equation (1) with $F(x) = 0$ satisfy

$$z(x) = C\, x^{-\alpha}(1 + o(1)), \quad x \to \infty, \quad \text{with} \alpha = \frac{n}{k-1}, \quad C = \Big(\frac{1}{p_0} \prod_{j=0}^{n-1} (\alpha + j)\Big)^{\frac{1}{k-1}}.$$

As for $n \ge 5$, solutions with the above asymptotic behavior also exist if $p(x)$ tends to $p_0$ quickly enough. This was proved in [4] (Ch.I, Theorem 5.3) for the function $p$ depending on $x, y, y', \dots, y^{(n-1)}$ and satisfying rather cumbersome conditions, which are reduced, in the case $p(x)$, to the condition $p(x) = p_0 + O(x^{-\gamma})$ with some $\gamma > 0$.

So, we can obtain asymptotic behavior of solutions to equation (1) vanishing at $+\infty$.

**Theorem 2.2.** *Suppose $2 \le n \le 4$, $p(x) \to p_0 \neq 0$ as $x \to \infty$, $(-1)^n p_0 < 0$, and $f(x)$ satisfies condition* (4)*. Then any solution $y(x)$ to equation* (1) *tending to zero as $x \to \infty$ behaves as*

$$y(x) = C\, x^{-\alpha}(1 + o(1)), \quad x \to \infty. \tag{5}$$

*If $n \ge 5$ and $p(x) = p_0 + O(x^{-\gamma})$ as $x \to \infty$ with $\gamma > 0$, then there exists a solution to equation* (1) *satisfying* (5)*.*

The following theorems, which were formulated in [1]– [6], can proved similarly.

**Theorem 2.3** (see [2, Ch. 2, pp. 15–16])**.** *Consider the equations*

$$y^{(2n)} + (-1)^n x^\sigma |y|^k \operatorname{sgn} y = F(x), \tag{6}$$

$$z^{(2n)} + (-1)^n x^\sigma |z|^k \operatorname{sgn} z = 0 \tag{7}$$

*with $\sigma > 0$, $n \ge 1$, $k > 1$.*

*Suppose $|F(x)| = O(e^{-\beta x})$, $\beta > 0$, $x \to \infty$, and $y(x)$ is a solution to equation* (6) *with $\lim\limits_{x\to\infty} y(x) = 0$. Then there exists a unique solution $z(x)$ to equation* (7) *such that*

$$|y(x) - z(x)| = O(e^{-\beta x}), \quad x \to \infty.$$

Straightforward calculations show that the function $y(x) = C(x-x_0)^{-\alpha}$ with $\alpha = \frac{n}{k-1}$, $C = (\prod\limits_{j=0}^{n-1}(\alpha+j))^{\frac{1}{k-1}}$, and arbitrary $x_0$ is a solution to the equation

$$y^{(n)} + (-1)^{n-1}|y(x)|^k \operatorname{sgn} y = 0, \ \ n \geq 2, \ \ k > 1. \tag{8}$$

It was proved for this equation with $n = 2$ [8] and $3 \leq n \leq 4$ [3] that all its Kneser solutions, i.e. those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j \, y^{(j)}(x) > 0$ for $0 \leq j < n$, have the above power form. However, it was also proved [9] that for any $N$ and $K > 1$ there exist an integer $n > N$ and $k \in (1; K)$ such that equation (1) has a solution $y(x) = (x-x_0)^{-\alpha} \ h(\log{(x-x_0)})$, where $h$ is a positive periodic non-constant function on $\mathbf{R}$.

In [5] existence of that type of solutions was investigated for some fixed $n$.

**Theorem 2.4.** *Suppose* $12 \leq n \leq 14$*. Then there exists* $k > 1$ *such that equation* (8) *has a solution* $y(x)$ *satisfying*

$$y^{(j)}(x) = (x-x_0)^{-\alpha-j} h_j\big(\log(x-x_0)\big), \ \ j = 0, 1, \dots, n-1,$$

*with periodic positive non-constant functions* $h_j$ *on* $\mathbf{R}$ *and arbitrary* $x_0 \in \mathbf{R}$.

So, the following Theorem is proved.

**Theorem 2.5.** *If* $12 \leq n \leq 14$*,* $f(x)$ *satisfies* (4)*, then there exist* $k > 1$ *and a solution to the equation*

$$y^{(n)} + (-1)^{n-1}|y(x)|^k \operatorname{sgn} y = F(x),$$

*satisfying the condition*

$$\big|y(x) - (x-x_0)^{-\alpha} h\big(\log(x-x_0)\big)\big| = O(e^{-\beta x}), \ \ x \to \infty,$$

*with some periodic positive non-constant function* $h$ *on* $\mathbf{R}$.

## 2 Power-law small potential

**Theorem 2.6.** *Suppose the function* $F(x)$ *in equation* (1) *satisfies the condition*

$$|F(x)| \leq Cx^{-\sigma}, \ \ C > 0, \ \ \sigma > n, \tag{9}$$

*and* $p(x)$ *is a bounded continuous function.*

*Then for any solution* $y(x)$ *to equation* (1) *tending to zero as* $x \to \infty$ *there exists a solution* $z(x)$ *to equation* (1) *with* $F(x) = 0$ *such that*

$$|y(x) - z(x)| = O(x^{n-\sigma}), \ \ x \to \infty.$$

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# Kiguradze-type and Belohorec-type Oscillation Theorems for Second Order Nonlinear Dynamic Equations on Time Scales

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Consider the second order nonlinear dynamic equations

$$x^{\Delta\Delta}+p(t)x^{\alpha}(\sigma(t))=0, \tag{1}$$

where $p\in C(\mathbb{T},R)$, $t\in\mathbb{T}$ is a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T}=\infty$, $\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}$ and $\alpha\neq 1$, $\alpha>0$ is the quotient of odd positive integers. Equation (1) is called superlinear if $\alpha>1$ and sublinear if $0<\alpha<1$. We call an equation oscillatory if all its continuable solutios are oscillatory.

When $\mathbb{T}=\mathbb{R}$, the dynamic equation (1) is the second order nonlinear differential equation

$$x''(t)+p(t)x^{\alpha}(t)=0. \tag{2}$$

When $\mathbb{T}=\mathbb{N}_0$, the dynamic equation (1) is the second order nonlinear difference equation

$$\Delta^2x(n)+p(t)x^{\alpha}(n+1)=0. \tag{3}$$

When $p(t)$ is nonnegative, stronger oscillation results exist for the nonlinear equation (2) when $\alpha\neq 1$, notably the following:

**Theorem 1** (Atkinson [2])**.** *Let $\alpha>1$. Then* (2) *is oscillatory if and only if*

$$\int^{\infty} tp(t)\,dt=\infty. \tag{4}$$

**Theorem 2** (Belohorec [10])**.** *Let $0<\alpha<1$. Then* (2) *is oscillatory if and only if*

$$\int^{\infty} t^{\alpha}p(t)\,dt=\infty. \tag{5}$$

When $p(t)$ is allowed to take on negative values, for $\alpha>1$, Kiguradze [1] proved that (4) is sufficient for the differential equation (2) to be oscillatory and for $0<\alpha<1$ Belohorec [11] proved that (5) is a sufficient for the differential equation (2) to be oscillatory. These results have been further extended by Kwong and Wong [12].

When $p(n)$ is nonnegative, J. W. Hooker and W. T. Patula [5, Theorem 4.1], A. Mingarelli [6], respectively proved that

**Theorem 3.** *Let $\alpha>1$. Then* (3) *is oscillatory if and only if*

$$\sum_{1}^{\infty}(n+1)p(n)=\infty. \tag{6}$$

**Theorem 4.** *Let $0 < \alpha < 1$. Then* (3) *is oscillatory if and only if*

$$\sum_{1}^{\infty} (n+1)^{\alpha} p(n) = \infty.$$

In this paper, when $p(t)$ is allowed to take on negative values, we obtain the following results.

**Theorem A.** *Let $\alpha > 1$ and there exist a real number $\beta$, $0 < \beta \le 1$ such that $\int\limits_{t_0}^{\infty} (\sigma(t))^{\beta} p(t)\, \Delta t = \infty$. Then* (1) *is oscillatory.*

**Theorem B.** *Let $0<\alpha<1$ and there exist a real number $\beta$, $0<\beta\le 1$ such that $\int\limits_{t_0}^{\infty} (\sigma(t))^{\alpha\beta} p(t)\, \Delta t = \infty$. Then* (1) *is oscillatory.*

From Theorem A and Theorem B, we can get the following corollaries.

**Corollary 5.** *Let $\alpha > 1$ and $p(t)$ be allowed to take on negative values. Then* (3) *is oscillatory if*

$$\sum_{1}^{\infty} (n+1) p(n) = \infty.$$

**Corollary 6.** *Let $0 < \alpha < 1$ and $p(t)$ be allowed to take on negative values. Then* (3) *is oscillatory if*

$$\sum_{1}^{\infty} (n+1)^{\alpha} p(n) = \infty.$$

**Example 7.** The superlinear difference equation

$$\Delta^2 x(n) + \Big[\frac{a}{(n+1)^{b+1}} + \frac{c(-1)^n}{(n+1)^b}\Big] x^{\alpha}(n+1) = 0, \ \ \alpha > 1,$$

for $a > 0$, $0 < b \le 1$, is oscillatory. In [3], this result is shown to be true only for $0 < b < 1$ and $0 < bc < a < c(1-b)$.

**Example 8.** The sublinear difference equation

$$\Delta^2 x(n) + \Big[\frac{1}{(n+1)^{c+1}} + \frac{b(-1)^n}{(n+1)^c}\Big] x^{\alpha}(n+1) = 0, \ \ 0 < \alpha < 1,$$

is oscillatory if $0 \le c \le \alpha$, and is nonoscillatory if $c > \alpha$ (using Theorem 2.1 in [7]).

To prove Theorem A and Theorem B, we need the following Lemmas.

**Lemma 9.** *Suppose that $\alpha > 1$ and $x(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Then we have*

$$\int\limits_{T}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\, \Delta s \le \frac{x^{-\alpha+1}(T)}{\alpha - 1}.$$

*Proof.* Using the Pötzsche chain rule [4, Theorem 1.90], we get that

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} = -\int\limits_{0}^{1} \frac{dh}{(x(s) + h\mu(s) x^{\Delta}(s))^{\alpha}}\, x^{\Delta}(s) = -\int\limits_{0}^{1} \frac{dh}{(hx(\sigma(s)) + (1-h)x(s))^{\alpha}}\, x^{\Delta}(s). \tag{7}$$

When $x^{\Delta}(s) \geq 0$, that is $x(\sigma(s)) \geq x(s)$, from (7) we have

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\int\limits_0^1 \frac{dh}{(hx(\sigma(s)) + (1-h)x(\sigma(s)))^{\alpha}}\, x^{\Delta}(s) = -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{8}$$

When $x^{\Delta}(s) \leq 0$, that is $x(\sigma(s)) \leq x(s)$, from (7) we also have

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\int\limits_0^1 \frac{dh}{(hx(\sigma(s)) + (1-h)x(\sigma(s)))^{\alpha}}\, x^{\Delta}(s) = -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{9}$$

So from (8) and (9), we get that for $s \in [T,\infty)_{\mathbb{T}}$

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{10}$$

Integrating (10) from $T$ to $t$, we get

$$\int\limits_T^t \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,\Delta s \leq -\int\limits_T^t \Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta}\Delta s = \frac{x^{-\alpha+1}(T)}{\alpha-1} - \frac{x^{-\alpha+1}(t)}{\alpha-1} \leq \frac{x^{-\alpha+1}(T)}{\alpha-1}\,. \qquad \square$$

Similarly, we have

**Lemma 10.** *Suppose that* $0<\alpha<1$ *and* $x(t)>0$ *for* $t \in [T,\infty)_{\mathbb{T}}$. *Then we have*

$$\int\limits_T^t \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\,\Delta s \geq -\frac{x^{1-\alpha}(T)}{1-\alpha}\,,$$

*and*

$$\int\limits_T^t \frac{(x^{\alpha}(s))^{\Delta}x(\sigma(s))}{x^{\alpha}(s)x^{\alpha}(\sigma(s))}\,\Delta s \geq -\frac{\alpha x^{1-\alpha}(T)}{1-\alpha}\,.$$

The complete proofs of Theorem A and Theorem B are in [8] and [9].

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# On the Structure of Upper Frequency Spectra of Linear Differential Equations

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Let us consider the linear $n$th-order homogeneous differential equation ($n \in \mathbb{N}$)

$$y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_{n-1}(t)\dot{y} + a_n(t)y = 0, \quad t \in \mathbb{R}_+ \stackrel{\text{def}}{=} [0, +\infty), \tag{1}$$

with continuous coefficients $a_i(\,\cdot\,)\colon \mathbb{R}_+ \to \mathbb{R}$, $i = \overline{1,n}$. Identifying equation (1) and its row of coefficients $a = a(\,\cdot\,) = (a_1(\,\cdot\,), \dots, a_n(\,\cdot\,))$, we denote equation (1) also by $a$. For the set of all nonzero solutions of equation (1) we use the notation $S_*(a)$.

The following definitions were given by I. N. Sergeev [1], [2].

**Definition 1.** For an arbitrary solution $y(\,\cdot\,) \in S_*(a)$ and a time $t > 0$ the expression $\nu(y,t)$ with either $\nu = \nu^0$ or $\nu = \nu^-$ or $\nu = \nu^+$ is understood as follows.

(a) The number $\nu^0(y,t)$ of zeros of the function $y(\,\cdot\,)$ on the interval $(0,t)$.

(b) The number $\nu^-(y,t)$ of sign alternations of the functions $y(\,\cdot\,)$ on the interval $(0,t)$. (A point $\tau > 0$ is called a sign alternation point of the function $y(\,\cdot\,)$ if in every sufficiently small neighborhood of $\tau$ the function takes values of different signs).

(c) The total number $\nu^+(y,t)$ of roots of the function $y(\,\cdot\,)$ on the interval $(0,t)$; here each root of the function $y(\,\cdot\,)$ is counted with regard of their multiplicity.

It is easy to see that $\nu^0(y,t)$, $\nu^-(y,t)$, and $\nu^+(y,t)$ are finite integer numbers for every nonzero solution $y(\,\cdot\,)$ and $t > 0$.

**Definition 2.** The upper frequencies of zeros, signs, and roots of a solution $y(\,\cdot\,) \in S_*(a)$ are defined as

$$\widehat{\nu}^{\,0}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,0}(y(\,\cdot\,);t), \quad \widehat{\nu}^{\,-}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,-}(y(\,\cdot\,);t), \ \text{ and } \ \widehat{\nu}^{\,+}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,+}(y(\,\cdot\,);t),$$

respectively.

**Definition 3.** The upper frequency spectra $\widehat{\nu}^{\,0}(S_*(a)$, $\widehat{\nu}^{\,-}(S_*(a)$, and $\widehat{\nu}^{\,+}(S_*(a)$ of zeros, signs, and roots of equation (1) are defined as the sets of upper frequencies of zeros, signs, and roots of all solutions belonging to $S_*(a)$, respectively.

Generally speaking, upper frequencies (2) can be equal to $+\infty$ for some solutions of equation (1) with unbounded coefficients.

For symbols $\nu = \nu^0$, $\nu^-$, and $\nu^+$, respectively, functions $\widehat{\nu}(\,\cdot\,)\colon \mathbb{R}^n \setminus \{\mathbf{0}\} \to \overline{\mathbb{R}}_+$ are defined as $\widehat{\nu}(\alpha) \stackrel{\text{def}}{=} \widehat{\nu}[y_\alpha]$, where $y_\alpha(\,\cdot\,)$ is a solution of equation (1) such that $(y_\alpha(0), \dot{y}_\alpha(0), \dots, y_\alpha^{(n-1)}(0))^{\mathrm{T}} = \alpha$, and $\overline{\mathbb{R}}_+ \stackrel{\text{def}}{=} [0, +\infty]$ is a nonnegative semi-axis of the extended real number line $\overline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \sqcup \{-\infty, +\infty\}$. The functions $\widehat{\nu}^{\,0}(\,\cdot\,)$, $\widehat{\nu}^{\,-}(\,\cdot\,)$, and $\widehat{\nu}^{\,+}(\,\cdot\,)$ are called functions of zeros, signs, and roots of equation (1), respectively.

As it follows from Sturm's theorem and was noted in [1], [2], the upper frequency spectra consist of zero for an arbitrary first-order equation (1) and of the same nonnegative number for an arbitrary second-order equation (1). Let us present some results dedicated to the structure of the upper frequency spectra of higher order equations. For arbitrary positive incommensurable numbers $\omega_2 > \omega_1$ there exists [3] a fourth-order autonomous equation, whose upper frequency spectra coincide with segment $[\omega_1, \omega_2]$. There exists [4] a third-order periodic equation whose upper frequency spectra contain the same segment. In [5] a third-order equation was constructed whose upper frequency spectra are equal to $[0,1] \cap \mathbb{Q}$, where the symbol $\mathbb{Q}$ stands for the set of rational numbers. Moreover in the paper [5] another one third-order equation was obtained whose upper frequency spectra consist of $([0,1] \cap \mathbb{I}) \cup \{0\}$, where by $\mathbb{I}$ we denote the set of irrational numbers of the real number line $\mathbb{R}$.

We naturally encounter the problem as to what the upper frequency spectra and the functions of zeros, signs, and root are. In the report under the assumption that zero belongs to the upper frequency spectra of equation (1) the complete description of the spectra are obtained. Here we also give an improvable description of the functions $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^-(\,\cdot\,)$, and $\widehat{\nu}^+(\,\cdot\,)$ in terms of Baire classes.

To formulate the theorem of our report let us briefly give some necessary notations and definitions. Let $\mathcal{M}$ be an arbitrary set and $N$ be some class of its subsets. It is said that a function $f(\,\cdot\,)\colon \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the class $(^*, N)$ if for every $r \in \overline{\mathbb{R}}$ Lebesgue set $[f(\,\cdot\,) \geqslant r]$ (i.e. a preimage $f^{-1}([r,+\infty])$ of the segment $[r,+\infty]$) belongs to the class $N$. In the report we consider mainly Borel subsets of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ of orders zero, one, and two [6]. Closed and open sets are said to be Borel sets of zero order. Borel sets of order one are sets of type $F_\sigma$ or $G_\delta$ which are, respectively, countable unions of closed sets and countable intersections of open sets. Borel sets of the second order are set of type $F_{\sigma\delta}$ (the countable intersections of $F_\sigma$-sets) or sets of type $G_{\delta\sigma}$ (the countable unions of $G_\delta$-sets). Borel sets of an arbitrary finite order are defined in a similar manner by induction. A set is said to be a Borel set of the exact order $k$ if it is a Borel set of the $k$th order but it isn't Borel set of order $k-1$.

A set $\mathcal{A} \subset \mathbb{R}$ is called a Suslin set [7, p. 213], [8, p. 489] of the number line $\mathbb{R}$ if it is a continuous image of irrational numbers $\mathbb{I}$ with the subspace topology. The class of Suslin sets contains the class of Borel sets as a proper subclass, and at the same time it is a proper subclass of the class of Lebesgue measurable sets. A set $\mathcal{A} \subset \overline{\mathbb{R}}$ is called a Suslin set of the extended real number line if it can be represented as an union of a Suslin set of $\mathbb{R}$ and some subset (including the empty subset) of two-element set $\{-\infty, +\infty\}$.

**Theorem 1.** *The following inclusions $\widehat{\nu}^-(\,\cdot\,) \in (^*, G_\delta)$ and $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^+(\,\cdot\,) \in (^*, F_{\sigma\delta})$ hold.*

From Theorem 1 it follows that the function $\widehat{\nu}^-(\,\cdot\,)$ belongs to the second Baire class and the functions $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^+(\,\cdot\,)$ belong to the third Baire class. The following theorem is a simple consequence of Theorem 1 and the definition of Suslin sets.

**Theorem 2.** *The upper frequency spectra $\widehat{\nu}^0(S_*(a))$, $\widehat{\nu}^-(S_*(a))$, and $\widehat{\nu}^+(S_*(a))$ of zeros, signs, and roots of equation* (1) *are Suslin sets of the nonnegative semi-axis $\overline{\mathbb{R}}_+$.*

Under the assumption that zero belongs to the upper frequency spectra the converse of Theorem 2 was obtained.

**Theorem 3.** *For an arbitrary Suslin set $\mathcal{A} \subset \overline{\mathbb{R}}_+$ containing zero there exists a third-order differential equation* (1) *whose upper frequency spectra of zeros, signs, and roots are equal to $\mathcal{A}$.*

The following theorem shows that the assertion of Theorem 1 is improvable.

**Theorem 4.** *There exist a number $r > 0$ and a third-order differential equation* (1) *such that the Lebesgue set $[\widehat{\nu}^-(\,\cdot\,) \geqslant r]$ of its function of signs is a Baire set of the exact first order, also there exists another third-order differential equation* (1) *such that the Lebesgue sets $[\widehat{\nu}^0(\,\cdot\,) \geqslant r]$ and $[\widehat{\nu}^+(\,\cdot\,) \geqslant r]$ of its functions of zeros and roots, respectively, are Baire sets of the exact second order.*

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# Convergence Analysis of Difference Schemes for Generalized Benjamin–Bona–Mahony–Burgers Equation

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We consider the initial boundary-value problem for the 1D nonlinear Generalized Benjamin–Bona–Mahony–Burgers (GBBM-Burgers) equation

$$\frac{\partial u}{\partial t}-\alpha\,\frac{\partial^2 u}{\partial x^2}+\beta\,\frac{\partial u}{\partial x}+\frac{\partial (u)^m}{\partial x}-\frac{\partial^3 u}{\partial x^2\partial t}=0,\ \ (x,t)\in Q, \tag{1}$$

$$u(0,t)=u(1,t)=0,\ t\in[0,T),\quad u(x,0)=\varphi(x),\ x\in[0,1], \tag{2}$$

where $u(x,t)$ represents the velocity of fluid in the horizontal direction $x$, $Q=(0,1)\times(0,T]$, $\alpha>0$, $\beta$ are constants and $m\ge 2$ is an integer.

Assume that the solution of this problem belongs to the fractional-order Sobolev space $W_2^k(Q)$, $k>1$, whose norms we denote by $\|\cdot\|_{W_2^k(Q)}$.

In [1], Che et al. have investigated a three-level unconditionally stable difference scheme for the problem (1), (2) and ascertained second-order convergence under assumption that the exact solution belongs to $\mathcal{C}^{4,3}(\overline{Q})$.

In this article, two-level scheme is constructed to find the values of the unknown function on the first level, besides the term $\partial(u)^2/\partial x$ is approximated by the offered in [2, 3] way. For the upper layers, as in [1], the known approximations are used for derivatives. The error estimate is derived using certain well-known techniques (see, e.g. [4, 5]).

The finite domain $[0,1]\times[0,T]$ in plane is divided into rectangle grids by the points $(x_i,t_j)=(ih,j\tau)$, $i=0,1,\dots,n$, $j=0,1,2,\dots,J$, where $h=1/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function $U$ at the node $(x_i,t_j)$ is denoted by $U_i^j$, that is $U(ih,j\tau)=U_i^j$. For the sake of simplicity sometimes we use notations without subscripts: $U_i^j=U$, $U_i^{j+1}=\widehat{U}$, $U_i^{j-1}=\check{U}$. Moreover, let

$$\overline{U}^{\,0}=\frac{U^1+U^0}{2},\quad \overline{U}^{\,j}=\frac{U^{j+1}+U^{j-1}}{2},\ \ j=1,2,\dots\,.$$

We define the difference quotients in $x$ and $t$ directions as follows:

$$(U_i)_{\overline{x}}=\frac{U_i-U_{i-1}}{h}\,,\quad (U_i)_{\overset{\circ}{x}}=\frac{1}{2h}(U_{i+1}-U_{i-1}),\quad (U_i)_{\overline{x}x}=\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}\,,$$
$$(U^j)_t=\frac{U^{j+1}-U^j}{\tau}\,,\quad (U^j)_{\overset{\circ}{t}}=\frac{U^{j+1}-U^{j-1}}{2\tau}\,,\quad (U^j)_{\overline{t}t}=\frac{U^{j+1}-2U^j+U^{j-1}}{\tau^2}\,.$$

We approximate the problem (1), (2) with the help of the three-level finite-difference scheme:

$$\mathcal{L}U_i^j=0,\ \ i=1,2,\dots,n-1,\ \ j=0,1,\dots,J-1, \tag{3}$$

$$U_0^j=U_n^j=0,\ \ j=0,1,2,\dots,J,\quad U_i^0=\varphi(x_i),\ \ i=0,1,2,\dots,n, \tag{4}$$

where

$$\begin{aligned}
\mathcal{L}U^0 &:= (U^0)_t - \alpha(\overline{U}^{\,0})_{\overline{x}x} + \beta(\overline{U}^{\,0})_{\overset{\circ}{x}} + \frac{m}{m+1}\,\Lambda U^0 - (U^0)_{\overline{x}xt},\\
\mathcal{L}U^j &:= (U^j)_{\overset{\circ}{t}} - \alpha(\overline{U}^{j})_{\overline{x}x} + \beta(\overline{U}^{\,j})_{\overset{\circ}{x}} + \frac{m}{m+1}\,\Lambda U^j - (U^j)_{\overline{x}x\overset{\circ}{t}},\ \ j=1,2,\dots,\\
\Lambda U_i^0 &:= (U_i^0)^{m-1}(\overline{U}^{\,0})_{\overset{\circ}{x}} + \big((U_i^0)^{m-1}\overline{U}^{\,0}\big)_{\overset{\circ}{x}},\\
\Lambda U^j &:= (U^j)^{m-1}(\overline{U}^{\,j})_{\overset{\circ}{x}} + \big((U^j)^{m-1}\overline{U}^{\,j}\big)_{\overset{\circ}{x}},\ \ j=1,2,\dots .
\end{aligned}$$

Let $\overline{\omega} = \{x_i : i = 0,1,2,\dots,n\}$, $\omega = \{x_i : i = 1,2,\dots,n-1\}$, $\omega^+ = \{x_i : i = 1,2,\dots,n\}$. By $L_2(\omega)$ we denote the set of functions defined on the mesh $\bar{\omega}$ and equal to zero at $x = x_0$ and $x = x_n$. We define the following inner product and norms:

$$(U,V) = \sum_{x\in\omega} hU(x)V(x), \quad \|U\| = (U,U)^{1/2}.$$

Let, moreover,

$$(U,V\,] = \sum_{x\in\omega^+} hU(x)V(x), \quad \|U\,]| = (U,U\,]^{1/2}, \quad \|U\|_{W_2^1(\omega)} = \|U_{\overline{x}}]|.$$

**Theorem 1.** *Difference scheme* (3), (4) *is uniquely solvable and the following estimates hold for its solution:*

$$\|U^j\|^2 + \|U^j_{\overline{x}}\,]|^2 \le \|\varphi\|^2 + \|\varphi_{\overline{x}}]|^2, \ \ j = 1,2,\dots .$$

**Theorem 2.** *Difference scheme* (3), (4) *is absolutely stable with respect to initial data.*

**Theorem 3.** *Let the exact solution of the initial-boundary value problem* (1), (2) *belong to* $W_2^k(Q)$. *Then, the convergence rate of the finite difference scheme* (3), (4) *is determined by the estimate*

$$\|U^j - u^j\|_{W_2^1(\omega)} \le c(\tau^{k-1} + h^{k-1})\|u\|_{W_2^k(Q)}, \ \ 1 < k \le 3,$$

*where $c = c(u)$ denotes positive constant, independent of $h$ and $\tau$.*

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# Asymptotic Behavior of Solutions with Slowly Varying Derivatives of Essentially Nonlinear Second Order Differential Equations

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\exp(R(|\ln|y||))\varphi_1(y'), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[ \to ]0,+\infty[$ [1] $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to ]0,+\infty[$ $(i=0,1)$, are continuous functions, $R : ]0,+\infty[ \to ]0,+\infty[$ is a continuously differentiable function, $Y_i \in \{0,\pm\infty\}$, $\Delta_{Y_i}$ is either the interval $[y_i^0, Y_i[$ [2], or the interval $]Y_i, y_i^0]$, is considered.

We suppose also that $R$ is a regularly varying function of index $\mu$, every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index $\sigma_i$ and $0 < \mu < 1$, $\sigma_0 + \sigma_1 \ne 1$.

We call the measurable function $\varphi : \Delta_Y \to ]0,+\infty[$ a regularly varying as $z \to Y$ of index $\sigma$ if for every $\lambda > 0$ we have

$$\lim_{\substack{z\to Y\\ z\in\Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma.$$

Here $Y \in \{0,\pm\infty\}$, $\Delta_Y$ is some one-sided neighbourhood of $Y$. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [1] that regularly varying functions have the next properties.

$M_1$: The function $\varphi(z)$ is regularly varying of index $\sigma$ as $z \to Y$ if and only if it admits the representation

$$\varphi(z) = z^\sigma \theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

$M_2$: If the function $L : \Delta_{Y^0} \to ]0,+\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to ]0,+\infty[$ is slowly varying as $z \to Y$.

$M_3$: If the function $\varphi : \Delta_Y \to ]0,+\infty[$ satisfies the condition

$$\lim_{\substack{z\to Y\\ z\in\Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then $\varphi$ is regularly varying as $z \to Y$of index $\sigma$.

We call the solution $y$ of the equation (1) the $P_\omega(Y_0,Y_1,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t,\omega[ \to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \;\; (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)\,y(t)} = \lambda_0. \tag{2}$$

[1] If $\omega > 0$, we take $a > 0$.
[2] If $Y_i = +\infty$ $(Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

All $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) were investigated in [2, 3] for $\lambda_0\in\mathbb{R}\setminus\{0\}$. The necessary and sufficient conditions for the existence and asymptotic representations of such solutions as $t\uparrow\omega$ were found. The cases $\lambda_0\in\{0,\pm\infty\}$ are singular in studying of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of (1). To investigate such solutions we must put additional conditions to the right side of equation (1).

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition $S$ if for any continuous differentiable function $L:\Delta_{Y_i}\to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{zL'(z)}{L(z)}=0,$$

the next condition takes place

$$\theta(zL(z))=\theta(z)(1+o(1))\ \text{ as }\ z\to Y\ \ (z\in\Delta_Y).$$

By the statement $M_1$ and definition of $\varphi_0$ it is clear that $\varphi_0(z)|z|^{-\sigma_0}$ is slowly varying function as $z\to Y_0$ $(z\in\Delta_{Y_0})$. The sufficiently important class of $P_\omega(Y_0,Y_1,\infty)$-solutions of the equation (1) was investigated only in cases, when $R(z)\equiv1$ and the function $\varphi_0(z)|z|^{-\sigma_0}$ satisfies the condition $S$. Using (2) and statements $M_1$–$M_3$, it is easy to see that the first derivative of every $P_\omega(Y_0,Y_1,\infty)$-solution of the equation (1) is a slowly varying function as $t\uparrow\omega$. This is one of the most difficult problems in studying such solutions. For equations of the type (1) that contain, for example, functions like $\exp(\sqrt{|\ln|y||})$ or $\exp(\sqrt[m]{|\ln\|y\||})$, the asymptotic representations of $P_\omega(Y_0,Y_1,\infty)$-solutions were not established before. The aim of the work is to establish the necessary and sufficient conditions for the existence and asymptotic representations as $t\uparrow\omega$ of $P_\omega(\lambda^0_{n-1})$-solutions of the equation (1) in general case. Let us note that the function $\exp(R(|\ln|z||))$ does not satisfy the condition $S$.

We need the following subsidiary notations

$$\pi_\omega(t)=\begin{cases}t & \text{as }\ \omega=+\infty,\\ t-\omega & \text{as }\ \omega<+\infty,\end{cases}\qquad \theta_0(z)=\Psi_0(z)|z|^{-\sigma_0}.$$

We put also

$$L(t)=p(t)|\pi_\omega(t)|^{\sigma_0+1}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big),$$

$$I_0(t)=\int_{A^0_\omega}^{t}p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\theta_0\big(|\pi_\omega(\tau)|\operatorname{sign}y_0^0\big)\,d\tau,$$

$$A^0_\omega=\begin{cases}a, & \text{if }\ \displaystyle\int_a^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big)\,dt=+\infty,\\ \omega, & \text{if }\ \displaystyle\int_a^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big)\,dt<+\infty,\end{cases}$$

in case $\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|\operatorname{sign}y_0^0=Y_0$. Here we choose $b\in[a,\omega[$ so that $|\pi_\omega(t)|\operatorname{sign}y_0^0\in\Delta_{Y_0}$ as $t\in[b,\omega[$.

The following conclusions are valid for the equation (1).

**Theorem 1.** *The following conditions are necessary for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of the equation* (1)

$$Y_0=\begin{cases}\pm\infty, & \text{if }\ \omega=+\infty,\\ 0, & \text{if }\ \omega<+\infty,\end{cases}\qquad \pi_\omega(t)y_0^0y_1^0>0\ \text{ as }\ t\in[a,\omega[\,. \tag{3}$$

*If the function $\varphi_0(z)|z|^{\sigma_0}$ satisfies the condition $S$ and the statement*

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)}=0 \tag{4}$$

*is true, then the conditions* (3) *and*

$$\alpha_0 y_1^0(1-\sigma_0-\sigma_1)I_0(t)>0 \;\; as \;\; t\in[b,\omega[\,,$$
$$\lim_{t\uparrow\omega} y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}=Y_1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=0$$

*are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of the equation* (1)*. For any such solution the following asymptotic representations take place as $t\uparrow\omega$:*

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))}=\alpha_0(1-\sigma_0-\sigma_1)I_0(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

**Theorem 2.** *Let the function $\varphi_0(z)|z|^{\sigma_0}$ satisfy the condition $S$, but the statement* (4) *do not fulfilled. If*

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|\pi_\omega(t)||)L(t)}{\pi_\omega(t)L'(t)}=\infty,$$

*then the conditions* (3) *and*

$$\alpha_0 y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)|>0 \;\; for \;\; t\in[a,\omega[\,,$$
$$\lim_{t\uparrow\omega} y_1^0\exp\Big(\frac{1}{1-\sigma_0-\sigma_1}R(|\ln|\pi_\omega(t)||)\Big)=Y_1$$

*are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of the equation* (1)*. For any such solution the following asymptotic representations take place as $t\uparrow\omega$:*

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))}=\frac{|1-\sigma_0-\sigma_1|L(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

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# Multipoint Boundary Value Problem for the Linear Matrix Lyapunov Equation with Parameter

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This work is a continuation and development of [1] and the problem is investigated with the help of constructive regularization method [2, Ch. 1].

Consider the multipoint boundary value problem for the matrix equation

$$\frac{dX}{dt} = \big(A_0(t) + \lambda A_1(t)\big)X + XB(t) + F(t), \quad X \in \mathbb{R}^{n\times m} \tag{1}$$

with the condition

$$\sum_{i=1}^{k} M_i X(t_i) = 0, \quad 0 = t_1 < t_2 < \cdots < t_k = \omega, \tag{2}$$

where $A_0(t)$, $A_1(t)$, $B(t)$, $F(t)$ are matrices for class $\mathbb{C}[0,\omega]$ of corresponding dimensions, $M_i$ are given constant $(n\times n)$-matrices, $\lambda \in \mathbb{R}$.

A nonlinear problem of the type (1), (2) was studied by qualitative methods in [3].

We investigate the problem (1), (2) on the bases of the method of integral equations. We use the additive decomposition of the matrix $B(t)$ in the form $B(t) = B_1(t) + B_2(t)$, where the matrices $B_1(t)$, $B_2(t)$ are chosen in a certain way (see, for example, [2, Ch. 1]).

We introduce the following notations.

$$\gamma = \|\Phi^{-1}\|, \ \ \mu_1 = \max_t \|V(t)\|, \ \ \mu_2 = \max_t \|V^{-1}(t)\|, \ \ v_i = \|V_i\|, \ \ m_i = \|M_i\|, \ \ \varepsilon = |\lambda|,$$

$$\beta_2 = \max_t \|B_2(t)\|, \ \ \alpha_i = \max_t \|A_i(t)\| \ \ (i = 0, 1), \quad q_0 = \gamma\mu_1\mu_2(\alpha_0 + \beta_2)\omega \sum_{i=1}^{k} m_i v_i,$$

$$q_1 = \gamma\mu_1\mu_2\alpha_1\omega \sum_{i=1}^{k} m_i v_i, \quad N = \gamma\mu_1\mu_2\omega h \sum_{i=1}^{k} m_i v_i,$$

where $\Phi$ is a linear operator: $\Phi Y \equiv \sum\limits_{i=1}^{k} M_i Y V_i$; $V_i = V(t_i)$, $V(t)$ is a fundamental matrix of the equation $dV/dt = VB_1(t)$; $\|\bullet\|$ is an agreement matrix norm.

**Theorem.** *Let the operator $\Phi$ be invertible and $q_0 < 1$. Then for $|\lambda| < (1 - q_0)/q_1$ the problem (1), (2) is uniquely solvable; its solution $X(t)$ can be represented as the limit of a uniformly convergent sequence of matrix functions defined by an integral recursion relation and satisfying the condition (2); moreover, the following estimate holds*

$$\|X(t,\lambda)\| \le \frac{N}{1 - q_0 - \varepsilon q_1}\,. \tag{3}$$

*Proof.* We use a constructive method that follows from the approach in [2]. Then we have equivalent integral equation

$$X(t) = \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i \int\limits_{t_i}^{t} \Big[A(\tau)X(\tau) + X(\tau)B_2(\tau) + F(\tau)\Big]V^{-1}(\tau)\,d\tau \cdot V_i\bigg\}\bigg)V(t), \tag{4}$$

where $X(t)\equiv X(t,\lambda)$, $A(\tau)\equiv A_0(\tau)+\lambda A_1(\tau)$.

To analyze the solvability of the matrix equation (4), we use the contraction mapping principle [4, p. 605]. Next, we obtain an integral recursion relation for the approximate solution

$$X_p(t)=\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\int\limits_{t_i}^t\Big[A(\tau)X_{p-1}(\tau)+X_{p-1}(\tau)B_2(\tau)+F(\tau)\Big]V^{-1}(\tau)\,d\tau\cdot V_i\bigg\}\bigg)V(t),\quad p=1,2,\dots\,. \tag{5}$$

For the initial approximation $X_0(t)$ one can take any matrix of the class $\mathbb{C}(I,\mathbb{R}^{n\times n})$.

We proof next: the functions $X_1(t),X_2(t),\dots$ satisfy the condition (2). Consider the algorithm (5) in differential form:

$$\begin{aligned}\frac{dX_p(t)}{dt}&=X_p(t)B_1(t)+\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\Big[A(t)X_{p-1}(t)+X_{p-1}(t)B_2(t)+F(t)\Big]V^{-1}(t)V_i\bigg\}\bigg)V(t)=\\&=X_p(t)B_1(t)+\Big(\Phi^{-1}\Big\{\Phi\big[A(t)X_{p-1}(t)+X_{p-1}(t)B_2(t)+F(t)\big]V^{-1}(t)\Big\}\Big)V(t)=\\&=X_p(t)B_1(t)+\big[A(t)X_{p-1}(t)+X_{p-1}(t)B_2(t)+F(t)\big]V^{-1}(t)V(t)=\\&=X_p(t)B_1(t)+\big[A(t)X_{p-1}(t)+X_{p-1}(t)B_2(t)+F(t)\big].\end{aligned}$$

Hence we obtain the representation

$$\frac{dX_p(t)}{dt}=X_p(t)B_1(t)+\big[A(t)X_{p-1}(t)+X_{p-1}(t)B_2(t)+F(t)\big]. \tag{6}$$

From (6) we have

$$\big[A(\tau)X_{p-1}(\tau)+X_{p-1}(\tau)B_2(\tau)+F(\tau)\big]\,d\tau=dX_p(\tau)-X_p(\tau)B_1(\tau)\,d\tau. \tag{7}$$

By using (7), on the bases of (6) we obtain

$$\begin{aligned}X_p(t)&=\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\int\limits_{t_i}^t\big[dX_p(\tau)-X_p(\tau)B_1(\tau)\,d\tau\big]V^{-1}(\tau)\cdot V_i\bigg\}\bigg)V(t)=\\&=\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\int\limits_{t_i}^t(dX_p(\tau))V^{-1}(\tau)V_i-\sum_{i=1}^k M_i\int\limits_{t_i}^t X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\bigg\}\bigg)V(t)=\\&=\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\bigg(X_p(\tau)V^{-1}(\tau)\Big|_{t_i}^t+\int\limits_{t_i}^t X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\bigg)V_i-\\&\qquad-\sum_{i=1}^k M_i\int\limits_{t_i}^t X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\bigg\}\bigg)V(t)=\\&=\bigg(\Phi^{-1}\bigg\{\sum_{i=1}^k M_i\bigg(X_p(t)V^{-1}(t)-X_p(t_i)V^{-1}(t_i)+\int\limits_{t_i}^t X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\bigg)V_i-\\&\qquad-\sum_{i=1}^k M_i\int\limits_{t_i}^t X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\bigg\}\bigg)V(t)=\end{aligned}$$

$$
\begin{aligned}
&= \bigg(\Phi^{-1}\bigg\{ \sum_{i=1}^{k} \bigg( M_i X_p(t) V^{-1}(t) V_i - M_i X_p(t_i) + M_i \int_{t_i}^{t} X_p(\tau) B_1(\tau) V^{-1}(\tau)\, d\tau \cdot V_i \bigg) - \\
&\qquad - \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau) B_1(\tau) V^{-1}(\tau)\, d\tau \cdot V_i \bigg\}\bigg) V(t) = \\
&= \bigg(\Phi^{-1}\bigg\{ \sum_{i=1}^{k} M_i X_p(t) V^{-1}(t) V_i - \sum_{i=1}^{k} M_i X_p(t_i) + \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau) B_1(\tau) V^{-1}(\tau)\, d\tau \cdot V_i - \\
&\qquad - \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau) B_1(\tau) V^{-1}(\tau)\, d\tau \cdot V_i \bigg\}\bigg) V(t) = \\
&= \Big(\Phi^{-1}\Big\{ \Phi\big[X_p(t) V^{-1}(t)\big] - \sum_{i=1}^{k} M_i X_p(t_i) \Big\}\Big) V(t) = \\
&= \Big(\Phi^{-1}\Phi\big[X_p(t) V^{-1}(t)\big] - \Phi^{-1} \sum_{i=1}^{k} M_i X_p(t_i) \Big) V(t) = \\
&= \Big( X_p(t) V^{-1}(t) - \Phi^{-1} \sum_{i=1}^{k} M_i X_p(t_i) \Big) V(t) = X_p(t) - \Big(\Phi^{-1} \sum_{i=1}^{k} M_i X_p(t_i)\Big) V(t).
\end{aligned}
\tag{8}
$$

Note that the formula (8) yields

$$\sum_{i=1}^{k} M_i X_p(t_i) = 0.$$

Let us analyze the convergence of the sequence $\{X_p(t)\}_1^\infty$. By (5), we have

$$X_{p+1}(t) - X_p(t) = \mathfrak{L}(X_p) - \mathfrak{L}(X_{p-1}), \ \ p = 1, 2, \dots, \tag{9}$$

where

$$\mathfrak{L}(Y) = \bigg(\Phi^{-1}\bigg\{ \sum_{i=1}^{k} M_i \int_{t_i}^{t} \Big[A(\tau)Y(\tau) + Y(\tau)B_2(\tau) + F(\tau)\Big] V^{-1}(\tau)\, d\tau \cdot V_i \bigg\}\bigg) V(t).$$

By estimating the norm in (9), we obtain the inequality

$$\|X_p - X_{p-1}\|_C \le q^p \|X_1 - X_0\|_C, \ \ p = 1, 2, \dots, \tag{10}$$

where $q = q_0 + \varepsilon q_1$, $\|X_1 - X_0\|_C = \|\mathfrak{L}(X_0) - X_0\|_C$.

By using (10), one can show that the sequence converges uniformly with respect to $t \in [0, \omega]$ to a solution of the integral equation (4), equivalent to the problem (1), (2), and we obtain the estimates

$$
\begin{gathered}
\|X - X_r\|_C \le \frac{q^r}{1-q}\, \|X_1 - X_0\|_C, \ \ r = 0, 1, 2, \dots, \\
\|X\|_C \le \|X_0\|_C + \frac{\|X_1 - X_0\|_C}{1-q}\,.
\end{gathered}
\tag{11}
$$

From (5) we have the estimate $\|X_1\|_C \le N$ for $X_0 = 0$ , and from (11) we have the inequality (3). □

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# On the Existence of Positive Periodic Solutions to Second Order Linear Functional Differential Equations

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For linear second order functional differential equations, the periodic boundary value problem is investigated (see, for example, [1–5]). We will find unimprovable conditions for the existence of a positive solution in two cases:

1. the Green function of the periodic problem can change its sign (Theorems 2, 3, 4, Corollary 1);
2. right-hand side functions $f$ of the equations are not necessary non-negative or non-positive (Theorems 2, 5, 6, Corollary 2).

Consider the periodic boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t) \text{ for almost all } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{1}$$

where $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $f \in \mathbf{L}[0,1]$, a solution $x : [0,1] \to \mathbb{R}$ has an absolutely continuous derivative, $\mathbf{C}[0,1]$ is the space of all continuous functions $x : [0,1] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{C}} = \max\limits_{t\in[0,1]} |x(t)|$, $\mathbf{L}[0,1]$ is the space of all integrable functions $z : [0,1] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{L}} = \int\limits_0^1 |z(t)|\,dt$.

**Assumption 1.** *Let non-negative functions $q$, $r \in \mathbf{L}[0,1]$ be given,*

$$p \equiv q - r,$$

$$\mathcal{P} \equiv \int\limits_0^1 p(t)\,dt \neq 0, \quad \widetilde{p} \equiv p/\mathcal{P}.$$

*We suppose that the operator $T$ has a representation*

$$T = T^+ - T^-,$$

*where $T^+$, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ are linear bounded operators such that*

$$T^+\mathbf{1} = q, \quad T^-\mathbf{1} = r,$$

$\mathbf{1}$ *is the unit function, the operators $T^+$, $T^-$ are positive (that is, they map nonnegative functions from $\mathbf{C}[0,1]$ into almost everywhere non-negative functions from $\mathbf{L}[0,1]$).*

**Definition 1.** For every $t_1$, $t_2$ $(0 \le t_1 \le t_2 \le 1)$, define the piecewise linear function

$$g_{t_1,t_2}(s) \equiv G(t_2,s) - G(t_1,s), \quad s \in [0,1],$$

where

$$G(t,s)=\begin{cases} t(s-1) & \text{if } 0\le t\le s\le 1,\\ s(t-1) & \text{if } 0\le s<t\le 1,\end{cases}$$

is the Green function of the Dirichlet problem $\ddot{x}(t)=z(t)$, $t\in[0,1]$, $x(0)=0$, $x(1)=0$.

For every function $z\in\mathbf{L}[0,1]$, we denote

$$g_{t_1,t_2,z}(s)\equiv g_{t_1,t_2}(s)-\int\limits_0^1 z(\tau)g_{t_1,t_2}(\tau)\,d\tau,\ \ s\in[0,1],$$

$$[z]^+(s)\equiv\frac{z(s)+|z(s)|}{2},\quad [z]^-(s)\equiv\frac{|z(s)|-z(s)}{2},\ \ s\in[0,1].$$

**Theorem 1.** *Let*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{p}}]^+(t)+r(t)[g_{t_1,t_2,\widetilde{p}}]^-(t)\Big)\,dt<1. \tag{2}$$

*Then periodic problem* (1) *has a unique solution.*

**Assumption 2.** *Suppose further that* $\int\limits_0^1 f(s)\,ds\neq 0$. *Define* $\mathcal{F}\equiv\int\limits_0^1 f(s)\,ds$, $\widetilde{f}\equiv f/\mathcal{F}$.

**Theorem 2.** *Let inequality* (2) *be fulfilled.*

*If*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{f}}]^+(t)+r(t)[g_{t_1,t_2,\widetilde{f}}]^-(t)\Big)\,dt<1 \tag{3}$$

*and*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{f}}]^-(t)+r(t)[g_{t_1,t_2,\widetilde{f}}]^+(t)\Big)\,dt<1, \tag{4}$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{F}\mathcal{P})\,x(t)>0\ \ \text{for all}\ \ t\in[0,1]. \tag{5}$$

**Definition 2.** Let $\mu\ge 1$. Define the set

$$S_\mu\equiv\Big\{h\in\mathbf{L}[0,1]:\ \operatorname{vrai\,sup}_{t\in[0,1]}h(t)\le\mu\operatorname{vrai\,inf}_{t\in[0,1]}h(t)>0\Big\}.$$

**Theorem 3.** *Let inequality* (2) *be fulfilled,* $f\in S_\mu$.

*If*

$$\min\Big\{\operatorname{vrai\,sup}_{t\in[0,1]}q(t),\operatorname{vrai\,sup}_{t\in[0,1]}r(t)\Big\}+\\ +\mu\max\Big\{\operatorname{vrai\,sup}_{t\in[0,1]}q(t),\operatorname{vrai\,sup}_{t\in[0,1]}r(t)\Big\}\le 8(1+\sqrt{\mu})^2,$$

*and*

$$q+\mu r\not\equiv 8(1+\sqrt{\mu})^2,\quad r+\mu q\not\equiv 8(1+\sqrt{\mu})^2,$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t)>0\ \ \text{for all}\ \ t\in[0,1].$$

**Theorem 4.** *Let inequality* (2) *be fulfilled,* $f \in S_\mu$.
*If*

$$\min\Big\{\int_0^1 q(t)\,dt, \int_0^1 r(t)\,dt\Big\} + \sqrt{\mu}\max\Big\{\int_0^1 q(t)\,dt, \int_0^1 r(t)\,dt\Big\} \le 4(1+\sqrt{\mu}),$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \text{ for all } \ t \in [0,1].$$

**Corollary 1.** *Let* $q \equiv 0$ *or* $r \equiv 0$.
*If*

$$\operatorname{vrai\,sup}_{t\in[0,1]} |p(t)| \le 32\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big)^2, \quad |p| \not\equiv 32\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big)^2,$$

*or*

$$\int_0^1 |p(t)|\,dt \le 8\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big),$$

*then for each* $f \in S_\mu$ *a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})x(t) > 0 \ \text{ for all } \ t \in [0,1].$$

**Definition 3.** Let $\rho > 1$. Define the set

$$\Lambda_\rho \equiv \Big\{h \in \mathbf{L}[0,1]: \ h \not\equiv 0, \ \int_0^1 [h]^+(t)\,dt \ge \rho \int_0^1 [h]^-(t)\,dt\Big\}.$$

**Theorem 5.** *Let inequality* (2) *be fulfilled,* $f \in \Lambda_\rho$.
*If*

$$\max\Big\{\operatorname{vrai\,sup}_{t\in[0,1]} q(t), \operatorname{vrai\,sup}_{t\in[0,1]} r(t)\Big\} \le 8\,\frac{\rho-1}{\rho+1}\,, \ \ r \not\equiv 8\,\frac{\rho-1}{\rho+1}, \ \ q \not\equiv 8\,\frac{\rho-1}{\rho+1}\,,$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \text{ for all } \ t \in [0,1].$$

**Theorem 6.** *Let inequality* (2) *be fulfilled,* $f \in \Lambda_\rho$.
*If*

$$\rho\,\max\Big\{\int_0^1 q(t)\,dt, \int_0^1 r(t)\,dt\Big\} - \min\Big\{\int_0^1 q(t)\,dt, \int_0^1 r(t)\,dt\Big\} \le 4\,(\rho-1),$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \text{ for all } \ t \in [0,1].$$

**Corollary 2.** *Let* $q \equiv 0$ *or* $r \equiv 0$.
*If*

$$\operatorname{vrai\,sup}_{t\in[0,1]} |p(t)| \le 8\,\frac{\rho-1}{\rho+1}\,, \quad |p| \not\equiv 8\,\frac{\rho-1}{\rho+1}\,,$$

*or*

$$\int_0^1 |p(t)|\,dt \le 4\Big(1 - \frac{1}{\rho}\Big),$$

*then for each* $f \in \Lambda_\rho$ *a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})x(t) > 0 \ \text{ for all } \ t \in [0,1].$$

**Remark.** All inequalities in all these theorems and corollaries are sharp. In particular, if inequality (2) is not fulfilled, then there exists an operator $T$ such that Assumption 1 is satisfied and problem (1) does not have a unique solution. If inequality (3) or (4) is not fulfilled, then there exist an operator $T$ and a function $f$ such that Assumption 1 is satisfied and problem (1) has a solution which does not satisfy (5).

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# The Asymptotic Properties of Rapidly Varying Solutions of Second Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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The investigations of second order differential equations play an important role in the development of the qualitative theory of differential equations. Such equations have a lot of applications in different fields of science.

Many results have been obtained for equations with power nonlinearities. But in practice we often deal with differential equations not only with power nonlinearities but also with exponential nonlinearities. It happens, for example, when we study the distribution of electrostatic potential in a cylindrical volume of plasma of products of burning. The corresponding equation may be reduced to the following one
$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$
In the work of V. M. Evtukhov and N. G. Drik [3] some results on asymptotic behavior of solutions of such equations have been obtained.

Exponential nonlinearities form a special class of rapidly varying nonlinearities. The consideration of the last ones is necessary for some models. Such consideration needs the establishment of the next class of functions.

The function $\varphi : [s, +\infty[ \to ]0, +\infty[$ $(s > 0)$ is called a rapidly varying [1] function of the order $+\infty$ as $z \to \infty$ if this function is measurable and the following condition is true
$$\lim_{z\to\infty} \frac{\varphi(\lambda z)}{\varphi(z)} = \begin{cases} 0, & \text{if } 0 < \lambda < 1, \\ 1, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1. \end{cases}$$
The function $\varphi$ is called a rapidly varying function of the order $-\infty$ as $z \to \infty$ if this function is measurable and
$$\lim_{z\to\infty} \frac{\varphi(\lambda z)}{\varphi(z)} = \begin{cases} -\infty, & \text{if } 0 < \lambda < 1, \\ 1, & \text{if } \lambda = 1, \\ 0, & \text{if } \lambda > 1. \end{cases}$$

The function $\varphi(z)$ is called a rapidly varying function in zero if $\varphi(\frac{1}{z})$ is a rapidly varying function of the order $+\infty$.

An exponential function is a special case of such functions. The differential equation
$$y'' = \alpha_0 p(t) \varphi(y)$$
with a rapidly varying function $\varphi$, was investigated in the work of V. M. Evtuhov and V. M. Kharkov [4]. But in the mentioned work the introduced class of solutions of the equation depends on the function $\varphi$. This is not convenient for practice.

The more general class of equations of the mentioned type is established in this work. It is a natural generalization of previous investigations.

Let us consider the differential equation
$$y'' = \alpha_0 p(t) \varphi_0(y) \varphi_1(y'), \tag{1}$$

In this equation $\alpha_0$ is $-1$ or $+1$, $p : [a,\omega[ \to ]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$ is a continuous function, $\varphi_i : \Delta_{Y_i} \to ]0,+\infty[$ $(i \in \{0,1\})$ also are continuous functions, $Y_i \in \{0,\pm\infty\}$, the intervals $\Delta_{Y_i}$, $i \in \{0,1\}$ may be of the form $[y_i^0, Y_i[$ [1], or of the form $]Y_i, y_i^0]$.

Furthermore, we assume that the function $\varphi_1$ is a regularly varying function as $z \to Y_1$ $(z \in \Delta_{Y_1})$ of the order $\sigma_1$, and the function $\varphi_0$ is twice continuously differentiable and satisfies the following limit relations

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}} \varphi_0(z) \in \{0,+\infty\}, \qquad \lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}} \frac{\varphi_0(z)\varphi_0''(z)}{(\varphi_0'(z))^2} = 1.$$

It can be proved that $\varphi_0$ is a rapidly varying function as $z \to Y_0$ $(z \in \Delta_{Y_0})$.

We introduce the following notations and definitions.

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty,\\ t-\omega, & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_1(z) = \varphi_1(z)|z|^{-\sigma_1},$$

$$I(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \int\limits_{B_\omega^0}^{t} \Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}} \operatorname{sign} y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\, d\tau,$$

$$B_\omega^0 = \begin{cases} b, & \text{if } \displaystyle\int\limits_b^\omega \Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}} \operatorname{sign} y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\, d\tau = +\infty,\\ \omega, & \text{if } \displaystyle\int\limits_b^\omega \Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}} \operatorname{sign} y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\, d\tau < +\infty, \end{cases}$$

$$I_1(t) = \int\limits_{B_\omega^1}^{t} \frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\, d\tau, \quad B_\omega^1 = \begin{cases} b, & \text{if } \displaystyle\int\limits_b^\omega \frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\, d\tau = +\infty,\\ \omega, & \text{if } \displaystyle\int\limits_b^\omega \frac{\lambda_0 |I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\, d\tau < +\infty \end{cases}$$

$$\Phi_0(z) = \int\limits_{A_\omega^0}^{z} \frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\, dy, \quad A_\omega^0 = \begin{cases} y_0^0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0} \frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\, dy = +\infty,\\ Y_0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0} \frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\, dy < +\infty, \end{cases}$$

$$\Phi_1(z) = \int\limits_{A_\omega^1}^{z} \frac{\Phi_0(\tau)}{\tau}\, d\tau, \quad A_\omega^1 = \begin{cases} y_0^0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0} \frac{\Phi_0(\tau)}{\tau}\, d\tau = +\infty,\\ Y_0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0} \frac{|\Phi_0(\tau)|}{\tau}\, d\tau < +\infty. \end{cases}$$

[1] If $Y_i = +\infty$ $(Y_i = -\infty)$, we suppose that $y_i^0 > 0$ $(y_i^0 < 0)$.

The inferior limits of the integrals are chosen in such a way that the corresponding integrals tend either to 0 or to $\infty$.

The solution $y$ of the equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution if

$$y^{(i)} : [t_0, \omega[ \to \Delta_{Y_i} \ \ (t_0 \geq a), \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)\, y(t)} = \lambda_0.$$

Let $Y \in \{0, \infty\}$, $\Delta_Y$ is some one-sided neighborhood of $Y$. The differentiable function $L : \Delta_Y \to ]0; +\infty[$ is said to be a normalized slowly varying function as $z \to Y$ $(z \in \Delta_Y)$ if

$$\lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0.$$

We say that a slowly varying function $\theta : \Delta_Y \to ]0; +\infty[$ as $z \to Y$ $(z \in \Delta_Y)$ satisfies the condition $S$ if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to ]0; +\infty[$ the following relation takes place

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

The next theorem contains necessary and sufficient conditions of existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1), and the asymptotic representations for these solutions and their derivatives of the first order as $t \uparrow \omega$.

**Theorem.** *Let $\sigma_1 \neq 1$, the function $\varphi_1$ satisfy the condition $S$, and the following limit relation be true*

$$\lim_{\substack{z \to Y_0 \\ z \in \Delta_{Y_0}}} \frac{(\frac{\Phi_1'(z)}{\Phi_1(z)})''(\frac{\Phi_1'(z)}{\Phi_1(z)})}{((\frac{\Phi_1'(z)}{\Phi_1(z)})')^2} = \gamma_0, \ \ \gamma_0 \in R \setminus \{1, 0\}.$$

*The following conditions are necessary for the existence of the $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1), where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$,*

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0; \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0, \quad y_1^0 \cdot \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1,$$

$$\lim_{t \uparrow \omega} \Phi_1^{-1}(I_1(t)) = Y_0,$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_1'(t)}{I_1(t)} = \infty, \quad \lim_{t \uparrow \omega} \frac{I_1'(t) \pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t))) \Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,.$$

*These conditions are also sufficient for the existence of the $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) if*

$$I(t) I_1(t) \lambda_0(\sigma_1 - 1) > 0 \ \text{ as } \ t \in [a; \omega[$$

*and the function* $\frac{|\pi_\omega(t)|^{1 - \frac{(2 - \gamma_0)\lambda_0}{(1 - \gamma_0)(\lambda_0 - 1)}} I_1'(t)}{I_1(t)}$ *is a normalized slowly varying function as $t \uparrow \omega$.*

*Moreover, for each such solution the following asymptotic representations take place as $t \uparrow \omega$:*

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t) \Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\, [1 + o(1)].$$

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# The Control Problem of Asynchronous Spectrum of Linear Systems with Depended Blocks of Complete Column Rank

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It is well known that, under certain conditions, a periodic differential system has periodic solutions whose period is incommensurable with the period of the system itself [1–5]. Such solutions are said to be strongly irregular [5]. The conditions of the process in which the oscillations of the system are described by strongly irregular solutions are called an asynchronous mode [6, 7], and the frequency spectrum of such solutions is referred to as the asynchronous spectrum. Asynchronous modes of oscillations are implemented in a number of various devices (see, [6] etc.). In particular, there exist systems that transform the energy of a source of high-frequency oscillations into low-frequency ones whose frequency is almost independent of the source frequency. Such systems implement a specifically defined influence on the oscillations, which leads to a periodic transport of energy from an external harmonic source designed for the generation, amplification, or transformation of oscillations. In this case, the oscillatory processes are implemented at the natural frequency of system oscillations, which is not necessarily commensurable with the frequency of the external force. Note that, even in the mid-1930s, the possibility of excitation of oscillations at frequencies with an almost arbitrary relationship with the frequency of changes of parameters was demonstrated in investigation [8] under the supervision by L. I. Mandel'shtam and N. D. Papaleksi of the parametric influence on two-circuit systems.

The problem of synthesis of such modes for linear problems was stated in [9] as a control problem for the asynchronous spectrum. This problem was solved in [10] for linearly independent column basis of some blocks of the coefficient matrix without the average value. In the present paper, we solve the control problem for the asynchronous spectrum with depended blocks of complete column rank.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $A(t)$ is a continuous $\omega$-periodic $n \times n$-matrix, $B$ is a constant $n \times n$-matrix. We assume that the control is given in the form of a linear state feedback

$$u = U(t)x \tag{2}$$

with $\omega$-periodic $n \times n$-matrix $U(t)$. The problem of finding the matrix $U(t)$ (the feedback coefficient) such that the closed system

$$\dot{x} = (A(t) + BU(t))x \tag{3}$$

has strongly irregular periodic solutions with a given frequency spectrum $L$ (the objective set) will be called the problem of control of the frequency spectrum of irregular oscillations (asynchronous spectrum)with objective set $L$.

This problem is a version of the generalization of the spectrum assignment problem in the nonstationary case. Note that, for regular oscillations, the choice of frequencies other than multiples of the frequencies of the right-hand side of system (1) is impossible.

Let $L = \{\lambda_1, \dots, \lambda'_r\}$ be an objective set of frequencies whose elements are pairwise distinct, commensurable with each other, and incommensurable with $2\pi/\omega$. Then there exists a maximum positive real number $\lambda$ such that $\lambda_1, \dots, \lambda'_r$ are multiples of $\lambda$. Set $\Omega = 2\pi/\lambda$ then the ratio $\omega/\Omega$ is irrational.

Consider the case of a singular matrix $B$, i.e. $\operatorname{rank} B = r < n \quad (n - r = d)$, and let the first $d$ rows of the matrix $B$ are zero. We denote the matrix consisting of the last $r$ rows of the matrix $B$ by $B_{r,n}$.

We represent the coefficient matrix $A(t)$ in block form corresponding to the structure of the matrix $B$. Let $A^{(11)}_{d,d}(t)$ and $A^{(21)}_{r,d}(t)$ be its left upper and lower blocks, and let $A^{(12)}_{d,r}(t)$ and $A^{(22)}_{r,r}(t)$ be its right upper and lower blocks. The subscripts show the dimension. In accordance with this representation, in turn, we split the averaged matrix $\widehat{A}$ into four blocks $\widehat{A}^{(11)}_{d,d}$, $\widehat{A}^{(21)}_{r,d}$, $\widehat{A}^{(12)}_{d,r}$, $\widehat{A}^{(22)}_{r,r}$ of the same dimensions. Let $\widetilde{A}(t) = A(t) - \widehat{A}$. Suppose that $\widehat{A}^{(12)}_{d,r} = 0$. Then $\widetilde{A}^{(12)}_{d,r}(t) = A^{(12)}_{d,r}(t)$.

Let us consider the system

$$
\begin{gathered}
\dot{x}^{[d]} = \widehat{A}^{(11)}_{d,d} x^{[d]}, \\
\dot{x}_{[r]} = \big(\widehat{A}^{(21)}_{r,d} + B_{r,n}\widehat{U}_{n,d}\big)x^{[d]} + \big(\widehat{A}^{(22)}_{r,r} + B_{r,n}\widehat{U}_{n,r}\big)x_{[r]}, \\
\widetilde{A}^{(11)}_{d,d}(t)x^{[d]} + A^{(12)}_{d,r}(t)x_{[r]} = 0, \\
(\widetilde{A}^{(21)}_{r,d}(t) + B_{r,n}\widetilde{U}_{n,d}(t))x^{[d]} + (\widetilde{A}^{(22)}_{r,r}(t) + B_{r,n}\widetilde{U}_{n,r}(t))x_{[r]} = 0,
\end{gathered} \tag{4}
$$

where

$$
\begin{gathered}
x = \operatorname{col}(x^{[d]}, x_{[r]}), \quad x^{[d]} = \operatorname{col}(x_1, \dots, x_d), \quad x_{[r]} = \operatorname{col}(x_{d+1}, \dots, x_n), \\
\widehat{U} = \big\{\widehat{U}_{n,d}, \widehat{U}_{n,r}\big\}, \quad \widetilde{U}(t) = \big\{\widetilde{U}_{n,d}(t), \widetilde{U}_{n,r}(t)\big\}.
\end{gathered}
$$

Suppose that the upper left and right blocks of the matrix $\widetilde{A}(t)$ have complete column rank

$$
\operatorname{rank}_{\mathrm{col}} \widetilde{A}^{(11)}_{d,\,d} = d, \quad \operatorname{rank}_{\mathrm{col}} A^{(12)}_{d,\,r} = r \tag{5}
$$

and are linearly depended

$$
\widetilde{A}^{(11)}_1(t)F_1 + \widetilde{A}^{(11)}_2(t)F_2 = A^{(12)}_{d,r}(t)G, \tag{6}
$$

where $\widetilde{A}^{(11)}_{d,d}(t) = \{\widetilde{A}^{(11)}_1(t), \widetilde{A}^{(11)}_2(t)\}$, $F_1$ is nonsingular $s \times s$-martix, $F_2$ is $(d-s) \times s$-martix. $G$ is a constant $r \times s$ matrix $(1 \le s \le \min\{d, r\})$).

Let $x'^{[d]} = \operatorname{col}(x_1, y_2, \dots, y_s)$, $x''^{[d]} = \operatorname{col}(x_{s+1}, \dots, x_d)$. In accordance with the representation of the vector $x[d]$ via $x'^{[d]}$ and $x''^{[d]}$ on the basis of the matrix $\widehat{A}^{(11)}_{d,d}$ we form four matrices $\widehat{A}^{(11)}_1$, $\widehat{A}^{(11)}_2$, $\widehat{A}^{(21)}_1$, and $\widehat{A}^{(22)}_1$.

By assumption (5), (6), system (4) can be represented in the form

$$
\begin{gathered}
\dot{x}'^{[d]} = (\widehat{A}^{(11)}_1 + \widehat{A}^{(11)}_2 H)x'^{[d]}, \\
H\dot{x}'^{[d]} = (\widehat{A}^{(11)}_3 + \widehat{A}^{(11)}_4 H)x'^{[d]}, \\
\dot{x}_{[r]} = (\widehat{A}^{(21)}_{r,d} + B_{r,n}\widehat{U}_{n,d})x^{[d]} + (\widehat{A}^{(22)}_{r,r} + B_{r,n}\widehat{U}_{n,r})x_{[r]}, \\
(\widetilde{A}^{(21)}_{r,d}(t) + B_{r,n}\widetilde{U}_{n,d}(t))x^{[d]} + (\widetilde{A}^{(22)}_{r,r}(t) + B_{r,n}\widetilde{U}_{n,r}(t))x_{[r]} = 0, \\
x''^{[d]} = Hx'^{[d]}, \quad x_{[r]} = Px'^{[d]} \quad (H = F_2F_1^{-1},\ P = -GF_1^{-1}),
\end{gathered} \tag{7}
$$

where

$$
x'^{[d]} = \operatorname{col}(x_1, y_2, \dots, y_s), \quad x''^{[d]} = \operatorname{col}(x_{s+1}, \dots, x_d).
$$

It follows from [4] that systems (3) and (7) are equivalent in the sense of existence of strongly irregular periodic solutions. Therefore, a strongly irregular solution of the closed system (3) is a trigonometric polynomial.

The following assertion holds.

**Theorem.** *Let the first $d$ rows in the matrix $B$ be zero, and the remaining $r$ rows be linearly independed. Suppose that $\widehat{A}_{d,r}^{(12)} = 0$. Under assumptions* (5), (6), *the control problem for an asynchronous spectrum for system* (1) *can be reduced to finding constant matrices $\widehat{U}_{n,d}$, $\widehat{U}_{n,r}$ and $\omega$-periodic matrices $\widetilde{U}_{n,d}(t)$, $\widetilde{U}_{n,r}(t)$ such that system* (7) *has an $\Omega$-periodic solution $\operatorname{col}(x'^{[d]}(t), x''^{[d]}(t), x_{[r]}(t))$ whose frequencies form the objective set $L$. A strongly irregular solution of the closed system* (3) *is a trigonometric polynomial.*

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# Multi-Point Boundary Value Problems for Functional Differential Equations

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On the interval $[a,b]$, we consider the multi-point boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{1}$$

$$\sum_{i=1}^{n} \alpha_i u(t_i) = c, \tag{2}$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator, $q \in L([a,b];\mathbb{R})$, $\alpha_i \in \mathbb{R} \setminus \{0\}$, $a \le t_1 < t_2 < \dots < t_n \le b$ $(i = 1,\dots,n)$, and $c \in \mathbb{R}$. Here and in what follows, $C([a,b];\mathbb{R})$ and $L([a,b];\mathbb{R})$ stand for Banach spaces of continuous and Lebesgue integrable functions defined on $[a,b]$, respectively, with standard norms; $C([a,b];\mathbb{R}_+)$ and $L([a,b];\mathbb{R}_+)$ are subsets of non-negative functions of the corresponding spaces; $AC([a,b];\mathbb{R})$ is a set of absolutely continuous functions defined on $[a,b]$.

A linear bounded operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is called an $a$-Volterra operator, resp. a $b$-Volterra operator, if for arbitrary $c \in \,]a,b]$, resp. $c \in [a,b[\,$, and $v \in C([a,b];\mathbb{R})$ such that

$$v(t) = 0 \ \text{ for } \ t \in [a,c], \ \text{ resp. } \ v(t) = 0 \ \text{ for } \ t \in [c,b],$$

the equality

$$\ell(v)(t) = 0 \ \text{ for a.e. } \ t \in [a,c], \ \text{ resp. } \ \ell(v)(t) = 0 \ \text{ for a.e. } \ t \in [c,b],$$

is fulfilled.

**Notation.** Let $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ be a linear bounded operator. Then $\ell \in \mathcal{P}_{ab}$ iff it transforms a set $C([a,b];\mathbb{R}_+)$ into a set $L([a,b];\mathbb{R}_+)$; $\ell \in \mathcal{P}_{ab}^+$ iff it transforms the non-negative non-decreasing absolutely continuous functions to the non-negative functions; $\ell \in \mathcal{S}_{ab}(a)$, resp. $\ell \in \mathcal{S}_{ab}(b)$, iff every absolutely continuous function $u$ satisfying

$$u'(t) \ge \ell(u)(t) \ \text{ for a.e. } \ t \in [a,b], \ \ u(a) \ge 0,$$

resp.

$$u'(t) \le \ell(u)(t) \ \text{ for a.e. } \ t \in [a,b], \ \ (b) \ge 0,$$

admits the inequality $u(t) \ge 0$ for $t \in [a,b]$.

**Remark 1.** In the case when $\ell(u)(t) \stackrel{\text{def}}{=} p(t)u(\tau(t)) - g(t)u(\mu(t))$ with $p, g \in L([a,b];\mathbb{R}_+)$, $\tau, \mu : [a,b] \to [a,b]$ measurables functions, it can be shown that $\ell \in \mathcal{P}_{ab}^+$ iff $p(t) \geq g(t)$ and $g(t)(\tau(t) - \mu(t)) \geq 0$ for a.e. $t \in [a,b]$.

The efficient conditions guaranteeing the inclusions $\ell \in \mathcal{S}_{ab}(a)$ and $\ell \in \mathcal{S}_{ab}(b)$ can be found in [2].

The proofs of the following theorems are based on the results established in [1].

**Theorem 1.** *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that*

$$n > i_1 > i_2 > \cdots > i_k \geq 1, \tag{3}$$

*and either*

$$(-1)^r \alpha_z > 0 \ \text{ for } \ z = i_{r+1} + 1, \dots, i_r \ \ (r = 0, \dots, k) \tag{4}$$

*or*

$$(-1)^r \alpha_z < 0 \ \text{ for } \ z = i_{r+1} + 1, \dots, i_r \ \ (r = 0, \dots, k), \tag{5}$$

*where $i_0 = n$, $i_{k+1} = 0$. Let, in addition,*

$$\sum_{z=i_{2r+1}+1}^{i_{2r}} |\alpha_z| \geq \sum_{z=i_{2r+2}+1}^{i_{2r+1}} |\alpha_z|, \ \ r = 0, \dots, \Big[\frac{k-1}{2}\Big]. \tag{6}$$

*If either at least one of the inequalities in (6) is strict, or $k$ is even, or*

$$\int_I \ell(1)(t)\,dt \neq 0, \quad I = \bigcup_{r=0}^{[\frac{k-1}{2}]} [t_{i_{2r+2}+1}, t_{i_{2r}}], \tag{7}$$

*then the problem (1), (2) is uniquely solvable.*

**Theorem 2.** *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying*

$$\gamma(t) > 0 \ \text{ for } \ t \in [a,b], \tag{8}$$

$$\gamma'(t) \geq \ell(\gamma)(t) \ \text{ for a.e. } \ t \in [a,b], \tag{9}$$

*and let there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that (3) holds and either (4) or (5) is satisfied, where $i_0 = n$, $i_{k+1} = 0$. Let, in addition, (6) be fulfilled. If either at least one of the inequalities in (6) is strict, or $k$ is even, or (7) holds, then the problem (1), (2) is uniquely solvable.*

**Theorem 3.** *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that (3) holds, and either (4) or (5) be fulfilled where $i_0 = n$, $i_{k+1} = 0$. Let, in addition,*

$$\frac{\gamma(t_n)}{\gamma(a)} \sum_{z=i_1+1}^{n} |\alpha_z| \leq \sum_{z=1}^{i_k} |\alpha_z| \ \text{ if } k \text{ is odd}, \tag{10}$$

$$\frac{\gamma(t_n)}{\gamma(a)} \sum_{z=i_1+1}^{n} |\alpha_z| + \sum_{z=1}^{i_k} |\alpha_z| \leq \sum_{z=i_k+1}^{i_{k-1}} |\alpha_z| \ \text{ if } k \text{ is even}, \tag{11}$$

*and*

$$\sum_{z=i_{2r+3}+1}^{i_{2r+2}} |\alpha_z| \leq \sum_{z=i_{2r+2}+1}^{i_{2r+1}} |\alpha_z|, \ \ r = 0, \dots, \Big[\frac{k-3}{2}\Big] \ \text{ if } k \geq 3, \tag{12}$$

*where* $\gamma \in AC([a,b];\mathbb{R})$ *is a function satisfying* (8) *and* (9)[1]. *If either at least one of the inequalities in* (10)–(12) *is strict, or there exists* $I \subseteq [a,t_n]$ *with* $\operatorname{meas} I > 0$ *such that*

$$\gamma'(t) \neq \ell(\gamma)(t) \ \text{ for a.e. } t \in I, \tag{13}$$

*or*

$$\sum_{i=1}^{n} \alpha_i \gamma(t_i) \neq 0, \tag{14}$$

*or*

$$\int_I \ell(1)(t)dt \neq 0, \tag{15}$$

*where*

$$\begin{gathered} I = [t_{i_1}, t_n] \cup I_1 \cup I_2, \\ I_1 = [a, t_{i_k}] \ \text{ if } k \text{ is odd}, \quad I_1 = [a, t_{i_{k-1}}] \ \text{ if } k \text{ is even}, \\ I_2 = \bigcup_{r=0}^{[\frac{k-3}{2}]} [t_{i_{2r+3}+1}, t_{i_{2r+1}}] \ \text{ if } k \geq 3, \quad I_2 = \varnothing \ \text{ if } k < 3, \end{gathered} \tag{16}$$

*then the problem* (1), (2) *is uniquely solvable.*

**Theorem 4.** *Let* $\ell \in \mathcal{P}^+_{ab}$ *admit the representation* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *and let* $-\ell_1 \in \mathcal{S}_{ab}(b)$. *Let, moreover, there exist* $\gamma \in AC([a,b];\mathbb{R})$ *satisfying* (8) *and* (9), *and let there exist* $i_j \in \{1,\dots,n\}$ $(j = 1,\dots,k)$ *such that* (3) *holds, and either* (4) *or* (5) *be fulfilled where* $i_0 = n$, $i_{k+1} = 0$. *Let, in addition,* (10)–(12) *be satisfied. If either at least one of the inequalities in* (10)–(12) *is strict, or there exists* $I \subseteq [a,t_n]$ *with* $\operatorname{meas} I > 0$ *such that* (13) *holds, or* (14), *or* (15) *is fulfilled with* $I$ *defined by* (16), *then the problem* (1), (2) *is uniquely solvable.*

**Theorem 5.** *Let* $\ell$ *admit the representation* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell(1)(t) \geq 0$ *for a.e.* $t \in [a,b]$, *and let* $-\ell_1 \in \mathcal{S}_{ab}(b)$ *be an* $a$*-Volterra operator. Let, moreover, there exist* $\gamma \in AC([a,b];\mathbb{R})$ *satisfying* (8) *and* (9). *Let, in addition,* $t_1 = a$ *and*

$$\alpha_1\alpha_i < 0 \ \ (i = 2,\dots,n), \quad |\alpha_1| \leq \sum_{i=2}^{n} |\alpha_i|.$$

*If either*

$$|\alpha_1| < \sum_{i=2}^{n} |\alpha_i|$$

*or*

$$\int_a^{t_n} \ell(1)(t)\,dt \neq 0,$$

*then the problem* (1), (2) *is uniquely solvable.*

**Theorem 6.** *Let* $\ell$ *admit the representation* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell(1)(t) \geq 0$ *for a.e.* $t \in [a,b]$, *and let* $\ell_0 \in \mathcal{S}_{ab}(a)$ *be a* $b$*-Volterra operator. Let, moreover,* $t_n = b$ *and*

$$|\alpha_n| \geq \sum_{i=1}^{n-1} \sigma_i |\alpha_i|,$$

*where*

$$\sigma_i = \frac{1}{2}\big(1 - \operatorname{sgn}(\alpha_i\alpha_n)\big) \ \ (i = 1,\dots,n-1).$$

*Let, in addition, at least one of the following items be fulfilled:*

[1] The existence of such a function is guaranteed by [2, Theorem 1.1].

(a)
$$|\alpha_n| > \sum_{i=1}^{n-1} \sigma_i |\alpha_i|;$$

(b) *there exists* $i_0 \in \{1, \dots, n-1\}$ *such that* $\alpha_{i_0}\alpha_n > 0$*;*

(c)
$$\int_{t_1}^{b} \ell(1)(t)\,dt \neq 0.$$

*Then the problem* $(1), (2)$ *is uniquely solvable.*

**Remark 2.** Results analogous to Theorems 1–6 can be derived by a standard transformation in the case when $\ell \in \mathcal{N}_{ab}^-$, i.e. when $\ell$ transforms the non-negative non-increasing absolutely continuous functions to the non-positive functions, and when $\ell(1)(t) \leq 0$ for a.e. $t \in [a, b]$, respectively.

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# Asymptotic Representation of Solutions of $n$-th Order Ordinary Differential Equations with Regularly Varying Nonlinearities

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We consider the differential equation

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}), \tag{1}$$

where $\alpha_k \in \{-1;1\}$ $(k=\overline{1,m})$, $p_k : [a,\omega[ \to ]0,+\infty[$ $(k=\overline{1,m})$ are continuous functions, $\varphi_{kj} : \Delta Y_j \to ]0,+\infty[$ $(k=\overline{1,m};\ j=\overline{0,n-1})$ are continuous and regularly varying functions as $y^{(j)} \to Y_j$ of orders $\sigma_{kj}$, $-\infty < a < \omega \le +\infty$[1], $\Delta Y_j$ – one-sided neighborhood of $Y_j$, $Y_j$ is equal to 0, or $\pm\infty$.

**Definition.** A solution $y$ of the equation (1) is called a $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,\omega[ \subset [a,\omega[$ and satisfies the following conditions

$$y^{(j)}(t) \in \Delta_{Y_j} \ \text{ at } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y^{(j)}(t) = Y_j \ \ (j=\overline{0,n-1}), \tag{2}$$
$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0.$$

The aim of this work is to establish the conditions of the existence and asymptotic as $t \to \omega(\omega \le +\infty)$ representations of one class of $P_\omega$ solutions of $n$-th order differential equation (1) containing the right side several main terms, what means that for some $s \in \{1,\dots,m\}$ and not empty set $\Gamma \subset \{1,\dots,m\}$,

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(y^{(j)}(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(y^{(j)}(t))} = c_{ks} = const \ne 0 \ \text{ at } \ k \in \Gamma, \tag{3}$$

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(y^{(j)}(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(y^{(j)}(t))} = 0 \ \text{ at } \ k \in \{1,\dots,m\} \setminus \Gamma. \tag{4}$$

In the works by Evtukhov V. M. and Klopot A. M. [1–3] there is considered the case when in the target class of solutions the right side of equation (1) has one main term, which means that the condition (4) is satisfied for all $k \ne s$.

Let us introduce notation needed in forthcoming considerations.

From the definition of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions of the equation (1) it is clear that any such solution and all of its derivatives up to order $n$ differs from zero on an interval $[t_1,\omega[ \subset [t_0,\omega[$, and

[1]We consider that $a > 1$ when $\omega = +\infty$, and $\omega - 1 < a < \omega$ when $\omega < +\infty$.

on this interval $j+1$-th ($j \in \{0,\dots,n-1\}$) derivative of this decision is positive, if $\Delta_{Y_j}$ is left neighborhood of $Y_j$, and negative – otherwise. Given this fact enter the number

$$\nu_j = \begin{cases} 1, & \text{if } \Delta_{Y_j}\text{-left neighborhood } 0, \text{ and if } Y_j = +\infty, \text{ or } Y_j = 0, \\ -1, & \text{if } \Delta_{Y_j}\text{-right neighborhood } 0, \text{ and if } Y_j = -\infty, \text{ or } Y_j = 0, \end{cases} \quad (j = \overline{0, n-2})$$

defining accordingly signs of $j$-th and $j+1$-th derivatives of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions. At the same time, we note that for $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions of equation (1) the conditions

$$\nu_j\nu_{j+1} < 0, \ \text{ if } \ Y_j = 0, \quad \nu_j\nu_{j+1} > 0, \quad Y_j = \pm\infty \ \ (j = \overline{0, n-2}) \tag{5}$$

are satisfied.

Let

$$a_{0i} = (n-i)\lambda_0 - (n-i-1) \ \ (i = 1,\dots,n) \ \text{ at } \ \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t-\omega, & \text{if } \ \omega < +\infty. \end{cases}$$

For the formulation of the main results, we introduce the following notation.

$$\gamma_k = 1 - \sum_{j=0}^{n-1} \sigma_{kj}, \quad \mu_{kn} = \sum_{j=0}^{n-2} \sigma_{kj}(n-j-1),$$

$$C_k = \prod_{j=0}^{n-2} \left| \frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_{kj}}, \quad J_{kn}(t) = \int_{A_{kn}}^{t} p_k(\tau)|\pi_\omega(\tau)|^{\mu_{kn}}\,d\tau,$$

$$A_{kn} = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p_k(t)|\pi_\omega(t)|^{\mu_{kn}}\,dt = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p_k(t)|\pi_\omega(t)|^{\mu_{kn}}\,dt < +\infty, \end{cases}$$

where $k = \overline{1, m}$,

$$Y(t) = \left| \gamma_s C_s J_{sn}(t) \prod_{j=0}^{n-1} L_{sj}\big(\nu_j |\pi_\omega(t)|^{\frac{a_{0j+1}}{\lambda_0-1}}\big) \sum_{k\in\Gamma} \alpha_k c_{ks} \right|,$$

$$Y_j(t) = \nu_{n-1}|Y(t)|^{\frac{1}{\gamma_s}} \frac{[(\lambda_0-1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{k=j+1}^{n-1} a_{0k}}.$$

We say that a continuous and slowly varying as $y \to Y$ function $L : \Delta_Y \to ]0,+\infty[$ ($Y$ is equal to 0 or $\pm\infty$, $\Delta_Y$ – one-sided neighborhood of $Y$) satisfies the condition $S$ if for any continuously differentiable function $l : \Delta_Y \to ]0,+\infty[$ such that

$$\lim_{\substack{y\to Y \\ y\in\Delta_Y}} \frac{y\,l'(y)}{l(y)} = 0,$$

the asymptotic relation holds

$$L(yl(y)) = L(y)[1+o(1)] \ \text{ for } \ y \to Y \ \ (y \in \Delta_Y).$$

**Theorem 1.** *Let $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\}$ and for some $s \in \{1, \dots, m\}$ and not empty set $\Gamma \subset \{1, \dots, m\}$ complied inequality $\gamma_s \neq 0$. Suppose, moreover, that slowly varying components $L_{kj}(y)$ $\forall\, k \in \Gamma$ $(j = 0, \dots, n-1)$ of functions $\varphi_{kj}$ satisfy the condition $S$. Then for the existence of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution of* (1) *for which performed* (3)*, where $\sum\limits_{k\in\Gamma} \alpha_k c_{ks} \neq 0$ and* (4)*, it is necessary the inequalities* (5)*, the condition*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{sn}(t)}{J_{sn}(t)} = \frac{\gamma_s}{\lambda_0 - 1}, \tag{6}$$

*the inequalities*

$$\begin{gathered} \nu_j \nu_{j+1} a_{0j+1} (\lambda_0 - 1) \pi_\omega(t) > 0 \ \ (j = \overline{0, n-2}), \\ \nu_{n-1} \gamma_s J_{sn}(t) \Big( \sum_{k\in\Gamma} \alpha_k c_{ks} \Big) > 0 \ \ \text{at} \ \ t \in\, ]a, \omega[\,, \end{gathered} \tag{7}$$

*as well as the conditions*

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(Y_j(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(Y_j(t))} = 0 \ \ \text{at} \ \ k \in \{1, \dots, m\} \setminus \{s\}, \tag{8}$$

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(Y_j(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(Y_j(t))} = c_{ks} \ \ \text{at} \ \ k \in \Gamma \tag{9}$$

*to be satisfied. Moreover, each such solution as $t \uparrow \omega$ has the asymptotic representation*

$$y^{(j-1)}(t) = \nu_{n-1} \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-j}}{\prod\limits_{i=j}^{n-1} a_{0i}} \, |Y(t)|^{\frac{1}{\gamma_s}} [1 + o(1)] \ \ (j = \overline{1, \dots, n}), \tag{10}$$

*where*

$$L_{sj}(y^{(j)}) = |y^{(j)}|^{-\sigma_{sj}} \varphi_{sj}(y^{(j)}) \ \ (j = \overline{0, \dots, n-1}).$$

Let us introduce the following notation.

$$B_m = \frac{\sum\limits_{k\in\Gamma} \alpha_k c_{ks} \sigma_{km}}{\sum\limits_{k\in\Gamma} \alpha_k c_{ks}}\,. \tag{11}$$

**Theorem 2.** *Let the conditions of Theorem* 1 *be executed. Then, if in addition to* (5)*,* (6)*,* (7)*,* (8) *and* (9) *the algebraic respect to $\rho$ equation*

$$\sum_{m=0}^{n-1} B_m \prod_{i=m+1}^{n-1} a_{0i} \prod_{j=1}^{m} (a_{0j} + \rho) = \prod_{j=1}^{n} (a_{0j} + \rho) \tag{12}$$

*doesn't have roots with a zero real part, then the differential equation* (1) *has $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of the type* (10)*. Moreover, there is an $l$-parameter family of solutions with these representations when among the roots an algebraic equation* (12) *there are $l$ roots of real parts which have the opposite sign of $\beta(\lambda_0 - 1)$.*

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# Asymptotic Representations of Solutions of Second-Order Differential Equations with Rapidly Varying Nonlinearities

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi(y) \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a,\omega[ \to ]0,+\infty[$ is a continuous function, $\varphi : \Delta_{Y_0} \to ]0;+\infty[$ $(i = \overline{1,n})$ is a continuously differentiable function satisfying the conditions

$$\varphi'(y) \neq 0 \ \text{ at } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = 1, \tag{2}$$

where $\Delta_{Y_0}$ is some one-sided neighborhood of the points $Y_0$, $Y_0$ is equal to either 0 or $\pm\infty$.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1$$

and the conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

The function $\varphi$ in the equation (1) and its derivative of the first order are (see, Seneta E. [1, Ch. 3, § 3.4, pp. 91–92]) rapidly varying as $y \to Y_0$.

The most simple example of such a function is the function $\varphi(y) = e^{\sigma y}$ $(\sigma \neq 0)$ as $Y_0 = +\infty$. In case of such function $\varphi$ the asymptotic behaviour of solutions of the differential equation (1) was studied in [2–6].

Under conditions (2) in the monography by V. Maric [7, Ch. 3, § 3, pp. 90–99] for the case when $\alpha_0 = 1$, $\omega = +\infty$, $Y_0 = 0$ and $p$-regularly varying function as $t \to +\infty$, and in [8] for the general case, asymptotic representations for some classes of solutions of the differential equation (1) have been established. Thus in [8] a class of studied solutions was defined through the function $\varphi$.

Naturally, however, it is represented for the equation (1) to investigate the same class of solutions, which was studied earlier (see, for example, [9]) in case of regularly varying as $y \to Y_0$ nonlinearity $\varphi$.

**Definition.** A solution $y$ of the equation (1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on some interval $[t_0, \omega[ \subset [a, \omega[$ and satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y'(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{t\uparrow\omega} \frac{y'^2(t)}{y''(t)y(t)} = \lambda_0.$$

The aim of the paper is to derive necessary and sufficient conditions for the existence of $P_\omega(\Lambda_0)$-solutions of the equation (1) when $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$, and also to establish asymptotic formulas for such solutions and their derivatives of the first order.

Let

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0, y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

where $|y_0| < 1$, if $Y_0 = 0$, and $y_0 > 1$ ($y_0 < -1$), if $Y_0 = +\infty$ ($Y_0 = -\infty$).

We set

$$\nu_0 = \operatorname{sign} y_0, \quad \mu_0 = \operatorname{sign} \varphi'(y),$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty,\\ t-\omega, & \text{if } \omega < +\infty, \end{cases} \quad J(t) = \int\limits_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad \Phi(y) = \int\limits_B^y \frac{ds}{\varphi(s)},$$

$$q(t) = \frac{\alpha_0(\lambda_0-1)\pi_\omega^2(t)\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}\,,$$

$$H(t) = \frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\,,$$

where

$$A = \begin{cases} \omega, & \text{if } \int\limits_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau < +\infty,\\ a, & \text{if } \int\limits_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau = \pm\infty, \end{cases} \qquad B = \begin{cases} Y_0, & \text{if } \int\limits_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \text{const},\\ y_0, & \text{if } \int\limits_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \pm\infty. \end{cases}$$

With use of properties of rapidly varying functions (see, Bingham N. H., Goldie C. M., Teugels J. L. [10, Ch. 3, 3.10, pp. 174–178]) and the results from [11] on the existence of systems of quasilinear differential equations with vanishing solutions in singular point, the following two theorems are established.

**Theorem 1.** *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$. Then for the existence of $P_\omega(\Lambda_0)$-solutions of the equation* (1) *it is necessary that*

$$\alpha_0\nu_0\lambda_0 > 0, \quad \alpha_0\mu_0(\lambda_0-1)J(t) < 0 \ \text{ at } \ t \in ]a,\omega[\,, \tag{3}$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q(t) = \frac{\lambda_0}{\lambda_0-1}\,. \tag{4}$$

*Moreover, each solution of this kind admits the following asymptotic representation:*

$$y(t) = \Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\Big[1 + \frac{o(1)}{H(t)}\Big] \ \text{ at } \ t\uparrow\omega, \tag{5}$$

$$y'(t) = \frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,[1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{6}$$

**Theorem 2.** *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$, conditions* (3), (4) *be satisfied and there exist a final or equal to infinity*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\sqrt{\Big|\frac{y\varphi'(y)}{\varphi(y)}\Big|}\,.$$

*Then:*

1) *if*

$$(\lambda_0-1)J(t)<0 \ \ at \ \ t\in\,]a,\omega[ \ \ and \ \ \lim_{t\uparrow\omega}\Big[\frac{\lambda_0}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac12}=0,$$

*the differential equation* (1) *has a one-parametric family of $P_\omega(Y_0,\lambda_0)$-solutions with asymptotic representations* (5), (6), *and the derivative of such solutions admits the representation*

$$y'(t)=\frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,\big[1+|H(t)|^{-\frac12}o(1)\big] \ \ at \ \ t\uparrow\omega;$$

2) *if*

$$(\lambda_0-1)J(t)>0 \ \ at \ \ t\in\,]a,\omega[\,, \quad \lim_{t\uparrow\omega}\Big[\frac{\lambda_0}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac12}\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)}\bigg)^2=0$$

*and*

$$\lim_{t\uparrow\omega}\frac{\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)}}{|H(t)|^{\frac12}}=0, \quad \lim_{t\uparrow\omega}|H(t)|^{\frac12}\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)}\bigg)\,\frac{\big(\frac{y\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{y\varphi'(y)}{\varphi(y)}\big)^2}\bigg|_{y=\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}=0,$$

*where $t_0$ – some number from $[a,\omega[$, the differential equation* (1) *as $\omega=+\infty$ has a one-parametric family of $P_\omega(Y_0,\lambda_0)$-solutions admitting the asymptotic representations*

$$y(t)=\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\bigg[1+\bigg(H(t)\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)}\bigg)^{-1}o(1)\bigg] \ \ at \ \ t\uparrow\omega,$$

$$y'(t)=\frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,\bigg[1+\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac12}\,d\tau}{\pi_\omega(\tau)}\bigg)^{-1}o(1)\bigg] \ \ at \ \ t\uparrow\omega,$$

*and for $\omega<+\infty$, a two-parametric families of $P_\omega(Y_0,\lambda_0)$-solutions with such representations.*

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# The Infinite Version of Perron's Effect of Value Change in Characteristic Exponents in the Neighbourhood of Integer Points

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Just as in our previous report [1], we consider here both the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in R^n, \ \ t \geq t_0 \tag{1}$$

with bounded infinitely differentiable on the semi-axis $[t_0, +\infty)$ coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A) < 0$, and the nonlinear systems

$$\dot{y} = A(t)y + f(t, y), \ \ y \in R^n, \ \ t \geq t_0 \tag{2}$$

with infinitely differentiable in time $t$ and variables $y_1, \dots, y_n$ so-called $m$-perturbations $f(t, y)$. These perturbations have the order $m > 1$ of smallness in the neighbourhood of the origin and admissible growth outside of it, satisfying the inequality

$$\|f(t, y)\| \leq C_f \|y\|^m, \quad C_f = const > 0, \ \ y \in R^n, \ \ t \geq t_0. \tag{3}$$

The well-known (partial) Perron's effects of sign and value changes [1], [2, pp. 50–61] in characteristic exponents claimed the existence of such two-dimensional system (1) with specific characteristic exponents $\lambda_1(A) = \lambda_1 < \lambda_2(A) = \lambda_2 < 0$ and the 2-perturbation (3) $f(t, y)$ that all solutions $y(t, c)$, $c \in R^2$ of the two-dimensional perturbed system (2) turned out to be infinitely extendable to the right and had characteristic exponents

$$\lambda\big[y(\cdot, c)\big] = \begin{cases} \lambda_2 < 0, & c = (0, c_2) \neq 0, \\ \lambda_2 > 0, & c_1 \neq 0. \end{cases}$$

The equal to $\lambda_2$ coincidence of characteristic exponents of solutions $x(t, c)$ and $y(t, c)$, $c = (c_1, c_2)$ of systems (1) and (2), respectively, on the axis $c_1 = 0$ (for $c_2 \neq 0$) of the plane $R^2$ as well as the lack of arbitrariness in the parameters $\lambda_1 \leq \lambda_2 < 0$, $m > 1$, and in the set $\beta = \{\lambda[y(\cdot, c)] : \ 0 \neq c \in R^2\}$ just right stipulates its partiality.

To the construction of various complete analogues of Perron's effect of value change in characteristic exponents of differential systems is devoted a cycle of our works, including those written jointly with S. K. Korovin. In particular, in our previous report, for arbitrary parameters $m > 1$, $\lambda_1 \leq \lambda_2 < 0$ and for bounded closed from the above countable set

$$\beta \subset [\lambda_1, +\infty), \ \ \lambda_2 \leq \sup \beta \in \beta,$$

we have stated that there exist the two-dimensional linear system (1) with exponents $\lambda_1(A) = \lambda_1 \leq \lambda_2 = \lambda_2(A)$ and the nonlinear system (2) with $m$-perturbation (3) such that all its nontrivial solutions $y(t,c)$, $c \in R^2$, are infinitely extendable to the right, and their characteristic exponents form the set $\Lambda(A,f) = \beta$ which coincides for $p = 0 \in R^2$ with its limiting subset

$$\Lambda_p(A,f) \equiv \operatorname*{Lim}_{r \to +0} \Big\{ \lambda\big[y(\cdot,c)\big] : \ 0 < \|c-p\| \le r \Big\}, \ \ p \in R^2,$$

of characteristic exponents of nontrivial solutions of system (2) starting in any arbitrarily small neighbourhood of the point $p \in R^2$.

In this connection, there arises the problem on the existence of another, different from the origin $(0,0)$, points $p \in R^2$ of the space of initial solutions for which the equality

$$\Lambda(A,f) = \Lambda_p(A,f) = \beta \tag{4}$$

would be fulfilled for an infinite number of points $p = (p_1,p_2) \in R^2$ and for any bounded countable (not necessarily closed from the above) set $\beta$ of positive, in particular, numbers. Its solution is involved in the following theorem.

**Theorem.** *For any parameters $m > 1$, $\lambda_1 \le \lambda_2 < 0$ and for any finite or bounded countable set*

$$\beta \subset [\lambda_1,+\infty), \quad \beta \cap [\lambda_2,+\infty) \ne \varnothing,$$

*there exist:*

1) *the two-dimensional system* (1) *with characteristic exponents $\lambda_1(A) = \lambda_1 \le \lambda_2 = \lambda_2(A)$;*
2) *the infinitely differentiable with respect to the variables $t$, $y_1$, $y_2$, and satisfying the condition* (3) *perturbation $f : [1,+\infty) \times R^2 \to R^2$ of order $m > 1$ such that all nontrivial solutions of the nonlinear two-dimensional system* (2) *with linear approximation* (1) *are infinitely extendable to the right, and their characteristic exponents form the set $\Lambda(A,f) = \beta$ which takes at the points $p = (p_1,p_2) \in R^2$ with integer coordinates its limiting values*

$$\Lambda_p(A,f) = \begin{cases} \beta & \text{if } p_1 \in Z, \ \ p_2 = 0, \\ \beta \cap [\lambda_2,+\infty) & \text{if } p_1 \in Z, \ \ p_2 \in Z \setminus \{0\}. \end{cases} \tag{5}$$

Statement (5) and condition (4) result in the following

**Corollary 1.** *In the case of a finite or bounded countable set $\beta \subset (0,+\infty)$ of positive numbers, the representation*

$$\Lambda(A,f) = \Lambda_p(A,f), \ \ p_1 \in Z, \ \ p_2 \in Z$$

*is valid.*

When proving the theorem in the case, where

$$\beta \cap [\lambda_2,+\infty) \ne \beta,$$

we have obtained a stronger compared with the second statement in (5)

**Corollary 2.** *For the limiting at the point $p = (p_1,p_2) \in R^2$ set $\Lambda_p(A,f)$ of characteristic exponents of solutions of the perturbed system* (2)*, the representation*

$$\Lambda_p(A,f) = \beta \cap [\lambda_2 + \infty) \ne \beta, \ \ p_1 \in R, \ \ p_2 \in Z \setminus \{0\}$$

*is valid.*

The results obtained in the present report are published in [1]– [3].

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# On Limit Irreducibility Sets of Linear Differential Systems

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We consider the linear systems of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I = [0, +\infty), \tag{$1_A$}$$

with piecewise continuous bounded coefficients ($\|A(t)\| \le a$ for $t \in I$). Along with original systems (1) we will consider perturbed systems $(1_{A+Q})$ with piecewise continuous perturbations $Q$ defined on $I$ and satisfying either the condition

$$\|Q(t)\| \le C_Q e^{-\sigma t}, \quad \sigma > 0, \quad t \ge 0, \tag{2}$$

or the more general condition

$$\lambda[Q] \equiv \overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|Q(t)\| \le -\sigma < 0. \tag{3}$$

If $\sigma = 0$ in (2), (3), then we additionally suppose that $Q(t) \to 0$ as $t \to +\infty$.

Following Yu. S. Bogdanov [1], we say that systems $(1_A)$ and $(1_{A+Q})$ are asymptotically equivalent (Lyapunov's equivalent, reducible) if there exists a Lyapunov transformation

$$x = L(t)y, \quad \max\Big\{ \sup_{t\in I} \|L(t)\|, \ \sup_{t\in I} \|L^{-1}(t)\|, \ \sup_{t\in I} \|\dot{L}(t)\| \Big\} < +\infty,$$

reducing one of them to the other.

The sets $N_2(a,\sigma)$, $N_3(a,\sigma)$, $a \ge 0$, $\sigma \ge 0$, are said to be the irreducibility sets if they consist of all systems $(1_A)$ with the following properties [2]:

1) the norm of the coefficient matrix $A$ is less than or equal to $a$ on $I$;

2) for each system $(1_A) \in N_i(a,\sigma)$, $i = 2, 3$, there exists a system $(1_{A+Q})$ with the matrix $Q$ satisfying either the condition (2) or the more general condition (3), respectively, which cannot be reduced to system $(1_A)$.

If $Q$ satisfies (2) or (3) with $\sigma > 2a$, then $\| \int_t^{+\infty} Q(u)\, du\| \le Ce^{-\sigma_1 t}$ for some $C > 0$ and $\sigma_1 > 2a$, therefore [3,5] systems $(1_A)$ and $(1_{A+Q})$ are asymptotically equivalent, and, therefore, the sets $N_2(a,\sigma)$, $N_3(a,\sigma)$ are empty for all $\sigma > 2a$.

We have [6] the following

**Theorem 1.** *The following strict inclusions are valid for the irreducibility sets $N_2(a,\sigma)$ and $N_3(a,\sigma)$:*

$$N_i(a_1,\sigma) \subset N_i(a_2,\sigma) \ \ \forall\, 0 \le a_1 < a_2, \ \ \forall\, \sigma \in [0, 2a_2], \ \ i = 2, 3.$$

The limit irreducibility sets

$$N_i(\sigma) \equiv \operatorname*{Lim}_{a\to+\infty} N_i(a,\sigma), \quad i = 2,3,$$

were defined in [4]. The properties of these sets treated as functions of the parameter $\sigma$ are similar to the properties of the irreducibility sets $N_i(a,\sigma)$, $i = 2,3$. By Theorem 1, the limit irreducibility sets are defined as the union of appropriate irreducibility sets

$$\operatorname*{Lim}_{a\to+\infty} N_i(a,\sigma) = \bigcup_{a\geq 0} N_i(a,\sigma),$$

and, by virtue of their definition, they are related by the inclusions $N_2(\sigma) \subseteq N_3(\sigma)$ for all $\sigma \geq 0$. The following statements are valid [6].

**Theorem 2.** *The limit irreducibility sets $N_2(\sigma)$ and $N_3(\sigma)$ coincide for $\sigma = 0$ and do not coincide for any $\sigma > 0$, i.e., $N_3(\sigma) \setminus N_2(\sigma) \neq \varnothing$ for any $\sigma > 0$.*

**Theorem 3.** *The limit irreducibility sets $N_2(\sigma)$ and $N_3(\sigma)$ of linear differential $n$-dimensional systems $(1_A)$ satisfy the strict inclusions*

$$N_i(\sigma_2) \subset N_i(\sigma_1) \;\; \forall\, 0 \leq \sigma_1 < \sigma_2, \;\; i = 2,3.$$

**Theorem 4.** *The limit irreducibility sets satisfy the relations*

$$\operatorname*{Lim}_{\sigma\to\sigma_0+0} N_i(\sigma) \subset N_i(\sigma_0) \;\; \forall\, \sigma_0 \geq 0, \;\; i = 2,3,$$
$$\operatorname*{Lim}_{\sigma\to\sigma_0-0} N_2(\sigma) \supset N_2(\sigma_0) \;\; \forall\, \sigma_0 > 0,$$
$$\operatorname*{Lim}_{\sigma\to\sigma_0-0} N_3(a,\sigma) = N_3(a,\sigma_0) \;\; \forall\, \sigma_0 > 0.$$

**Theorem 5.** *The limit sets $N_2(\sigma)$ and $N_3(\sigma)$ are invariant under Lyapunov transformations.*

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# Investigation and Approximate Resolution of One Nonlinear Integro-Differential Parabolic Equation

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One type integro-differential nonlinear parabolic model is obtained at mathematical simulation of processes of electromagnetic field penetration in the substance. Based on Maxwell system [1] this model at first appeared in [2]. Many other processes are described by integro-differential system obtained in [2] (see, for example, [3] and references therein). A lot of scientific works are dedicated to investigation and numerical resolution of the initial-boundary value problems for these type models (see, for example, [3] and references therein). The existence, uniqueness and asymptotic behavior of the solution for such type equations and systems are studied in the works [2–6] and in a number of other works as well (for more detail citations see, for example, [3] and references therein).

The present work is dedicated to the investigation and approximate resolution of the initial-boundary value problem with first type boundary conditions for one generalization and one-dimensional variant of such model.

In the domain $Q = (0,1) \times (0,T)$, where $T$ is a positive constant, uniqueness and existence properties and semi-discrete and finite difference approximations are discussed for the following nonlinear integro-differential problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left[1 + \int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2 d\tau\right]^p \Big|\frac{\partial U}{\partial x}\Big|^{q-2} \frac{\partial U}{\partial x}\right\} = f(x,t), \tag{1}$$

$$U(0,t) = U(1,t) = 0, \tag{2}$$

$$U(x,0) = U_0(x), \tag{3}$$

where $p$, $q$ are constants and $f = f(x,t)$ and $U_0 = U_0(x)$ are given functions of their arguments.

Principal characteristic peculiarity of the equation (1) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral in time. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

Using one modification of compactness method developed in [7] (see also [8]) the following existence statement takes place [5].

**Theorem 1.** *If $0 < p \le 1$, $q \ge 2$, $f \in W_2^1(Q)$, $f(x,0) = 0$, $U_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution $U$ of problem (1)–(3) satisfying the following properties:*

$$U \in L_{pq+q}\big(0,T; \overset{\circ}{W}{}^1_{pq+q}(0,1)\big), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\frac{\partial}{\partial x}\Big(\Big|\frac{\partial U}{\partial x}\Big|^{\frac{q-2}{2}} \frac{\partial U}{\partial x}\Big) \in L_2(Q), \quad \sqrt{T-t}\,\frac{\partial}{\partial t}\Big(\Big|\frac{\partial U}{\partial x}\Big|^{\frac{q-2}{2}} \frac{\partial U}{\partial x}\Big) \in L_2(Q).$$

Here usual well-known spaces are used. The proof of the formulated theorem is divided into several steps applying Galerkin's method and the method of compactness. One of the basic step is to obtain necessary a priori estimates.

In [4], there is proposed the operational scheme with so called conditionally closed operators. That scheme is applied for investigation of problems of (1) types in this work, too.

In order to describe the space-discretization to problem (1)–(3), it is introduced a net whose mesh points are denoted by $x_i = ih$, $i = 0, 1, \dots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at $(x_i, t)$ is designed $u_i = u_i(t)$. The exact solution to the problem at $(x_i, t)$, denoted by $U_i = U_i(t)$, is assumed to exist and be smooth enough. From the boundary conditions (2) we have $u_0(t) = u_M(t) = 0$. At other points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximating the space derivatives by a forward and backward differences. We will use the following known notations.

$$u_{x,i}(t) = \frac{u_{i+1}(t) - u_i(t)}{h}, \quad u_{\overline{x},i}(t) = \frac{u_i(t) - u_{i-1}(t)}{h}.$$

Let's correspond to problem (1)–(3) the following semi-discrete scheme:

$$\frac{du_i}{dt} - \left\{ \left[ 1 + \int\limits_0^t \left( u_{\overline{x},i} \right)^2 d\tau \right]^p |u_{\overline{x},i}|^{q-2} u_{\overline{x},i} \right\}_{x,i} = f(x_i, t), \ \ i = 1, 2, \dots, M-1, \tag{4}$$

$$u_0(t) = u_M(t) = 0, \tag{5}$$

$$u_i(0) = U_{0,i}, \ \ i = 0, 1, \dots, M. \tag{6}$$

The (4)–(6) is a Cauchy problem for nonlinear system of ordinary integro-differential equations. Using multiplier $u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$, after simple transformations we obtain the inequality

$$\|u(t)\|^2 + \int\limits_0^t \|u_{\overline{x}}]|^q \, d\tau < C, \tag{7}$$

where $C$ is a positive constant which do not depend on $h$ and norms are defined as follows:

$$(u, v) = \sum_{i=1}^{M-1} u_i v_i h, \quad (u, v] = \sum_{i=1}^{M} u_i v_i h,$$

$$\|u\| = (u, u)^{1/2}, \quad \|u]| = (u, u]^{1/2}.$$

The a priori estimate (7) guarantees the global solvability of problem (4)-(6). It is not difficult to prove the uniqueness of the solution of problem (4)–(6), too.

The following statement takes place.

**Theorem 2.** *If $0 < p \le 1$, $q \ge 2$, and problem (1)–(3) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$ of problem (4)–(6) tends to $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\| < Ch.$$

In order to describe the finite difference method it is introduced a net whose mesh points are denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \dots, M$; $j = 0, 1, \dots, N$ with $h = \frac{1}{M}$, $\tau = \frac{T}{N}$. The initial line is denoted by $j = 0$. The discrete approximation at $(x_i, t_j)$ is designed by $u_i^j$ once again and the exact solution to problem (1)–(3) by $U_i^j$.

Using forward derivative formula for time variable and rectangle formula for integration, let us correspond to problem (1)–(3) the following finite difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}\left(u_{\overline{x},i}^k\right)^2\right]^p|u_{\overline{x},i}^{j+1}|^{q-2}u_{\overline{x},i}^{j+1}\right\}_{x,i}=f_i^j, \tag{8}$$

$$i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,$$

$$u_0^j=u_M^j=0,\ \ j=0,1,\dots,N, \tag{9}$$

$$u_i^0=U_{0,i},\ \ i=0,1,\dots,M. \tag{10}$$

So, system of nonlinear algebraic equations (8)–(10) is obtained. It is not difficult to get the inequality

$$\|u^n\|^2+\sum_{j=1}^n\|u_{\overline{x}}^j]|^q\tau<C,\ \ n=1,2,\dots,N, \tag{11}$$

where $C$ here and below is a positive constant independent of $\tau$ and $h$.

The a priori estimate (11) guarantees the stability of the scheme (8)–(10). Note that it is easy to prove the existence and uniqueness of a solution of the scheme (8)–(10), too.

The following statement takes place.

**Theorem 3.** *If $p=1$, $q\geq 2$, and problem (1)–(3) has a sufficiently smooth solution $U=U(x,t)$, then the solution $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $j=1,2,\dots,N$ of difference scheme (8)–(10) tends to the $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to 0$, $h\to 0$ and the following estimate is true*

$$\|u^j-U^j\|<C(\tau+h),\ \ j=1,2,\dots,N.$$

Note that for solving the difference scheme (8)–(10) Newton's iterative process is used. Various numerical experiments are done. These experiments agree with theoretical research.

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# Periodic Problem for the Nonlinear Telegraph Equation

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In the plane of independent variables $x$ and $t$ in the strip $\Omega := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ t \in \mathbb{R}\}$ consider the problem on finding a solution $U(x,t)$ to the nonlinear telegraph equation of the form

$$LU := U_{tt} - U_{xx} + 2aU_t + cU + g(U) = F(x,t), \ \ (x,t) \in \Omega, \tag{1}$$

satisfying the Poincare homogeneous boundary conditions

$$\gamma_1 U_x(0,t) + \gamma_2 U_t(0,t) + \gamma_3 U(0,t) = 0, \ \ t \in \mathbb{R}, \tag{2}$$

and the Dirichlet boundary condition

$$U(l,t) = 0, \ \ t \in \mathbb{R}, \tag{3}$$

respectively, for $x = 0$ and $x = l$, and also the condition of periodicity with respect to the variable $t$

$$U(x, t+T) = U(x,t), \ \ x \in [0,l], \ \ t \in \mathbb{R}, \tag{4}$$

with constant real coefficients $a$, $c$, $\gamma_i$, $i = 1,2,3$, with $\gamma_1\gamma_2 \neq 0$. Here $T := const > 0$; $F$ is a given, while $U$ is an unknown real $T$-periodic in time functions; $g : \mathbb{R} \to \mathbb{R}$ is a given continuous real nonlinear function.

**Remark 1.** Let $\Omega_T := \Omega \cap \{0 < t < T\}$, $f := F\big|_{\overline{\Omega}_T}$. It is easy to see that if $U \in C^2(\overline{\Omega})$ is a classical solution of the problem (1)–(4), then the function $u := U\big|_{\overline{\Omega}_T}$ is a classical solution of the following nonlocal problem

$$Lu = f(x,t), \ \ (x,t) \in \Omega_T, \tag{5}$$

$$\gamma_1 u_x(0,t) + \gamma_2 u_t(0,t) + \gamma_3 u(0,t) = 0, \ \ u(l,t) = 0, \ \ 0 \leq t \leq T, \tag{6}$$

$$(B_0 u)(x) = 0, \quad (B_0 u_t)(x) = 0, \ \ x \in [0,l], \tag{7}$$

where $(B_\beta w)(x) := w(x,0) - \exp(-\beta T) w(x,T)$, $\beta \in \mathbb{R}$, $x \in [0,l]$, and, vice versa, if $f \in C(\overline{\Omega}_T)$ and $u \in C^2(\overline{\Omega}_T)$ is a classical solution of the problem (5)–(7), then the function $U \in C^2(\overline{\Omega})$, being $T$-periodic continuation of the function $u$ from the domain $\Omega_T$ into the strip $\Omega$, will be a classic solution of the problem (1)–(4), if $f(x,0) = f(x,T)$, $x \in [0,l]$.

**Definition 1.** Let $f \in C(\overline{\Omega}_T)$ be a given function, and $\Gamma_1 : x = 0$, $0 \leq t \leq T$, $\Gamma_2 : x = l$, $0 \leq t \leq T$. The function $u$ is called a strong generalized solution of the problem (5)–(7) of the class $C$, if $u \in C(\overline{\Omega}_T)$ and there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2) := \big\{ w \in C^2(\overline{\Omega}_T) : (\gamma_1 w_x + \gamma_2 w_t + \gamma_3 w)|_{\Gamma_1} = 0, \ w|_{\Gamma_2} = 0 \big\}$ such that $u_n \to u$ and $Lu_n \to f$ in the space $C(\overline{\Omega}_T)$, while $B_0 u_n \to 0$ and $B_0 u_{nt} \to 0$ as $n \to \infty$ in the spaces $C^1([0,l])$ and $C([0,l])$, respectively.

**Remark 2.** It is obvious that a classical solution of the problem (5)–(7) from the space $C^2(\overline{\Omega}_T)$ is a strong generalized solution of this problem of the class $C$.

Consider the following conditions

$$G(s):=\int\limits_0^s g(s_1)\,ds_1\geq 0,\quad sg(s)-2G(s)\geq 0,\ \ s\in\mathbb{R}, \tag{8}$$

$$a>0,\ \ c\geq a^2,\ \ \gamma_1\gamma_2<0,\ \ \gamma_3\gamma_2^{-1}\geq a. \tag{9}$$

The following Theorem is valid.

**Theorem.** *Let $T=2l$, the conditions (8), (9) and $f\in C(\overline{\Omega}_{2l})$ be fulfilled. Then the problem (5)–(7) has at least one strong generalized solution $u$ of the class $C$ in the sense of Definition 1, which belongs to the space $C^1(\overline{\Omega}_{2l})$, besides, in the case $f\in C^1(\overline{\Omega}_{2l})$ this solution is a classical one.*

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# Stability of Linear Stochastic Difference Equations with Delay

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Stochastic difference equations were truly introduced in [3]. Stability of these equations is an important problem which has not been comprehensively studied yet. Some results can be found in [2, 4, 9–12]. Stochastic functional difference equations were introduced in [8] and studied further in [13]. Stability of difference equations with a random delay was studied in [6].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis satisfying usual assumptions (see e.g. [7]). In what follows we assume that $\mathcal{B}_i$, $i = 2, \dots, m$ are independent standard scalar Wiener processes; $E$ is the expectation with respect to the probability measure $P$; $|\cdot|$ is a fixed norm in $R^n$; $\|\cdot\|$ is the norm of an $n\times n$-matrix, which is consistent with the chosen vector norm in $R^n$; $N$ is the set of all natural numbers; $N_+ = \{0\}\cup N$; $Z$ is the set of all integers.

For given $1 \leq p < \infty$, $h > 0$ the number $c_p^h$ is the universal constant for which the following inequalities are satisfied

$$E\bigg|\int\limits_t^{t+h} \varphi(s)\,d\mathcal{B}(s)\bigg|^{2p} \leq c_p^h E\int\limits_t^{t+h} |\varphi(s)|^{2p}\,ds. \tag{1}$$

The inequalities should be valid for any $t \geq 0$, any $\mathcal{F}_t$-adapted stochastic process $\varphi$ and a standard scalar Wiener process $\mathcal{B}$. In [7, p. 39], these constants are defined (up to a change of the notation) as $c_p^h = p^p(2p-1)^p h^{p-1}$ for $p > 1$ and $c_1 = 1$ for $p = 1$. The Burkholder–Davis–Ghandy inequalities give the estimates which are independent of $h$ (see e.g. [7, p. 40] where $p$ should be replaced with $2p$).

Below we consider the following stochastic difference equations:

(a) *The linear ordinary difference Itô equation*

$$x(s+1) = x(s) + A_1(s)x(s)h + \sum_{i=2}^{m} A_i(s)x(j)\big(\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)\big) \ \ (s\in N_+), \tag{2}$$

where $x(s)$ is a $\mathcal{F}_s$-measurable, $n$-dimensional random variable for any $s \in N_+$, $h$ is a positive number, $A_i(s)$ is an $n\times n$-matrix, whose entries are $\mathcal{F}_s$-measurable random variables for any $i = 1, \dots, m$, $s \in N_+$.

(b) *The linear difference Itô equation with delay*

$$\begin{aligned} x(s+1) = x(s) &+ \sum_{j=-\infty}^{s} A_1^2(s,j)x(j)h + \\ &+ \sum_{i=2}^{m}\sum_{j=-\infty}^{s} A_i^2(s,j)x(j)\big(\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)\big) \ \ (s\in N_+), \\ x(j) = \varphi(j) &\ \ (j<0), \end{aligned} \tag{3}$$

where $x(s)$ is a $\mathcal{F}_s$-measurable, $n$-dimensional random variable for any $s \in N_+$, $h$ is a positive number, $A_i^2(s,j)$ is an $n \times n$-matrix, whose entries are $\mathcal{F}_s$-measurable random variables for any $s \in N_+$, $j = -\infty, \dots, s$, $i = 1, \dots, m$, $\varphi(j)$ $(j < 0)$ is a $\mathcal{F}_0$-measurable random variable.

Note that the equation (2) is a particular case of the equation (3). Below we therefore formulate the definitions and results in terms of (3), only.

A solution of the equation (3) is a sequence of $n$-dimensional and $\mathcal{F}_s$-measurable random variables $x(s)$ $(s \in Z)$, which satisfies (3) $P$-almost everywhere. More precisely, $x(s)$ satisfies the difference equation for $s \in N_+$ and coincides with $\varphi(s)$ for $s < 0$. Thus the only degree of freedom of the solution of (3) is its initial value $x(0) = x_0$ at $s = 0$.

Note that for any $\mathcal{F}_0$-measurable initial value $x_0$, the solution of (3) always exists, and it is unique up to the natural $P$-equivalence. Moreover, this solution is a $\mathcal{F}_s$-adapted discrete stochastic process $x : Z \times \Omega \to R^n$. Restricted to the set $N_+$, this solution will be denoted by $x_\varphi(s, x_0)$, $s \in N_+$.

**Definition 1.** The trivial solution of the equation (3) is called $p$-stable with respect to the initial data ($\varphi$ and $x_0$) if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\mathbb{E}|x_0|^p + \operatorname*{vrai\,sup}_{j<0} \mathbb{E}|\varphi(j)|^p) < \eta$ implies $E|x(s,x_0)|^p \le \varepsilon$ for all $s \in N_+$.

If, in addition, $\mathbb{E}|x_\varphi(s,x_0)|^p \to 0$ as $s \to \infty$, then the trivial solution is called asymptotically $p$-stable.

The first result concerns asymptotic stability of the ordinary difference equation (2).

**Theorem 1.** *Assume that $A_i(s) = a_i$, $i = 1, \dots, m$ for $s \in N_+$.*

*If now*

$$-1 < a_1 h < 0, \;\; c_p^h \sum_{i=2}^m |a_i| < -a_1 h^{1/2},$$

*then the equation* (2) *is asymptotically* $2p$*-stable with respect to initial data.*

The second result applies to the vector equation (3). However, it does not guarantee asymptotic stability.

**Theorem 2.** *Assume that there exist positive numbers $a_i(s,j)$, $i = 1, \dots, n$, $s \in N_+$, $j = -\infty, \dots, s$ such that the coefficients in* (3) *satisfy*

$$\|A_i(s,j)\| \le a_i(s,j), \;\; i = 1, \dots, m, \;\; s \in N_+, \;\; j = -\infty, \dots, s$$

*$P$-almost everywhere,*

$$\sum_{\tau=0}^{\infty} \sum_{j=-\infty}^{-1} a_i(\tau, j) < \infty \;\; (i = 1, \dots, m)$$

*and*

$$\overline{c} \overset{def}{=} \sum_{\tau=0}^{\infty} \Big( \sum_{j=0}^{\tau} a_1(\tau,j) h + c_p^h \sum_{i=2}^{m} \sum_{j=0}^{\tau} a_i(\tau,j) h^{1/2} \Big) < 1.$$

*Then the trivial solution of the equation* (3) *is* $2p$*-stable with respect to the initial data.*

**The idea of the proofs.**

The proofs of the theorems are based on Azbelev's $W$-transform of the equations (2) and (3), respectively (see e. g. [1]). The transform is designed in a special manner with the help of the so-called "reference equation". Usually, the latter is an equation which already possesses the desired asymptotic properties, but which is simpler than the equation to be studied. The $W$-method works if the integral operator, which results from the substitution of the solutions of the reference equations into the given equation, is invertible.

Applying this idea, we first of all introduce two spaces of discrete stochastic processes:

1) $d^n$ is the linear space of all possible solutions of the difference equation (3);

2) $l^n$ is the linear space of all sequences of $m \times n$-matrices $H(s)(s \in N_+)$, with the entries being $\mathcal{F}_s$-measurable random variables.

We will also need the following operator equation constructed from the equation (3):

$$x(s+1) = x(s) + [(Vx)(s) + f(s)]Z(s) \ \ (s \in N_+), \tag{4}$$

where

$$\begin{aligned}
(Vx)(s) &= \Big( \sum_{j=0}^{s} A_1(s,j)x(j), \sum_{j=0}^{s} A_2(s,j)x(j), \dots, \sum_{j=0}^{s} A_m(s,j)x(j) \Big) \ \ (s \in N_+), \\
f(s) &= \big(f_1(s), f_2(s), \dots, f_m(s)\big) \ \ (s \in N_+), \\
Z(s) &= \Big(h, \big(\mathcal{B}_2((s+1)h) - \mathcal{B}_2(sh)\big), \dots, \big(\mathcal{B}_m((s+1)h) - \mathcal{B}_m(sh)\big)\Big) \ \ (s \in N_+).
\end{aligned}$$

Here $f \in l^n$. Let us note that the initial function $\varphi(s)$ from (3) is in this representation included in the equation (4) as a special case of $f$, see the formula (8) below and [1,5] for further details. This trick gives us opportunity to study stability with respect to $\varphi$ as a particular case of admissibility of pairs of spaces (see Definition 2 below).

It is easy to see that $V$ is a linear operator from $d^n$ to $l^n$.

The crucial step in the $W$-transform is the choice of "a reference equation", which has the same shape as the equation to be studied, but already has the desired asymptotic properties

$$x(s+1) = x(s) + \big[(Qx)(s) + g(s)\big]Z(s) \ \ (s \in N_+), \tag{5}$$

where $Q : d^n \to l^n$ is a linear operator and $g \in l^n$.

One usually assumes that for any admissible $x_0$ there exists a unique (up to the $P$-equivalence) solution $x$ of the equation (5). In this case, the solution $x_g(s, x_0)$ $(s \in N_+)$ of (5) satisfying $x_g(0, x_0) = x_0$ has the following canonical representation

$$x_g(s, x_0) = U(s)x_0 + (Wg)(s) \ \ (s \in N_+), \tag{6}$$

where $U(s)$ $(s \in N_+)$ is the fundamental matrix to (5) and $W : l^n \to d^n$ is a linear operator such that $(Wg)(0) = 0$ and $(Wg)(s)$ $(s \in N_+)$ is a solution of (5)).

We rewrite the equation (4) using the representation (6) for the reference equation (5) as follows

$$x(s+1) = x(s) + \big[(Qx)(s) + ((V-Q)x)(s) + f(s)\big]Z(s) \ \ (s \in N_+)$$

or alternatively,

$$x(s+1) = x(s) + U(s)x_0 + (W(V-Q)x)(s) + (Wf)(s) \ \ (s \in N_+).$$

Introducing the notation $W(V-Q) = \Theta$, we obtain the equation

$$((I-\Theta)x)(s) = U(s)x_0 + (Wf)(s) \ \ (s \in N_+).$$

To study asymptotic properties of a stochastic difference equation we need a notion of admissibility of a pair of spaces. In the sequel we will use the following spaces of random variables.

The space $k^n$ consists of all $n$-dimensional $\mathcal{F}_0$-measurable random variables and

$$k_p^n = \Big\{\alpha : \ \alpha \in k^n, \ \|\alpha\|_{k^n} \overset{def}{=} (E|\alpha|^p)^{1/p} < \infty\Big\}.$$

Given a sequence $\gamma(s)$ $(s \in N_+)$ of positive real numbers, we define two more spaces of discrete stochastic processes:

$$m_p^\gamma = \Big\{x : \ x \in d^n, \|x\|_{m_p^\gamma} \overset{def}{=} \sup_{s \in N_+} \big(E|\gamma(s)x(s)|^p\big)^{1/p} < \infty\Big\} \ \ (m_p^1 = m_p);$$

and

$$b^\gamma = \big\{f : \ f \in b, \ \gamma f \in b\big\}$$

which is endowed with the induced norm $\|f\|_{b^\gamma} = \|\gamma f\|_b$, where $b$ is a linear subspace of the space $l^n$ equipped with some norm $\|\cdot\|_b$.

**Definition 2.** We say that the pair $(m_p^\gamma, b^\gamma)$ is admissible for the system (4) if there exists a number $\overline{c} \in R_+^1$ such that for any $x_0 \in k_p^n$, $f \in b^\gamma$ we have that $x_f(\,\cdot\,, x_0) \in m_p^\gamma$ and

$$\big\|x_f(\,\cdot\,, x_0)\big\|_{m_p^\gamma} \leq \overline{c}\big(\|x_0\|_{k_p^n} + \|f\|_{b^\gamma}\big). \tag{7}$$

Now we make assumptions on the space $b$. Letting

$$f = \Big(\sum_{j=-\infty}^{-1} A_1^2(\,\cdot\,, j)\varphi(j), \dots, \sum_{j=-\infty}^{-1} A_m^2(\,\cdot\,, j)\varphi(j)\Big), \tag{8}$$

we assume that the coefficients of the system (3) satisfy the following condition:

- for any $\varphi$ such that $\sup\limits_{j<0} E|\varphi(j)|^p < \infty$ the stochastic process (8) belongs to the linear subspace $b$ of the space $l^n$, the norm in $b$ satisfies the estimate

  $$\|f\|_b \leq K \sup_{j<0} \big(E|\varphi(j)|^p\big)^{1/p}$$

  for some positive constant $K$.

The proofs of the above theorems are based on the following lemmas.

**Lemma 1.** *Let the pair $(m_p^\gamma, b^\gamma)$ be admissible for the reference equation* (5) *and the operator $\Theta$ act in the space $m_p^\gamma$. If the operator $(I - \Theta_l) : m_p^\gamma \to m_p^\gamma$ is continuously invertible, then the pair $(m_p^\gamma, b^\gamma)$ is admissible for the system* (4)*.*

**Lemma 2.** *If for the system* (4) *corresponding to the equation* (3) *the pair $(m_p, b)$ is admissible, then the trivial solution of* (3) *is $p$-stable with respect to the initial data.*

**Lemma 3.** *If for the system* (4) *corresponding to the equation* (3) *the pair $(m_p^\gamma, b^\gamma)$ is admissible for some sequence of numbers $\gamma(s)$ $(s \in N_+)$ satisfying $\gamma(s) \geq \delta > 0$ for all $s \in N_+$ $(\delta > 0)$, $\lim\limits_{s\to+\infty} \gamma(s) = +\infty$ , then the trivial solution of* (3) *is asymptotically $p$-stable with respect to the initial data.*

For the technical details of the proofs see the paper [5].

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# A Complete Description of The Largest Lyapunov Exponent of Linear Differential Systems with Parameter-Multiplier as Function of Parameter

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Consider the $n$-dimensional ($n \geq 2$) linear system of differential equations

$$\mathrm{d}x/\mathrm{d}t = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous on the half-line $t \geq 0$ coefficient matrix $A(\,\cdot\,)\colon [0,+\infty) \to \operatorname{End}\mathbb{R}^n$. Denote the class of all such systems by $\mathcal{M}_n^*$. We identify the system (1) and it's coefficient matrix and therefore write $A \in \mathcal{M}_n^*$. Along with the system (1) we consider the one-parameter family

$$\mathrm{d}x/\mathrm{d}t = \mu A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

of linear differential systems with a parameter-multiplier $\mu \in \mathbb{R}$. Denote by $\lambda_1(\mu A) \leq \dots \leq \lambda_n(\mu A)$ the Lyapunov exponents [1, p. 34], [2, p. 63] of the system (2).

V. I. Zubov in [3, p. 408; Problem 1] set the following problem: find out how the Lyapunov exponents of the systems (1) and (2) are related. For every $A \in \mathcal{M}_n^*$ we consider the exponent $\lambda_i(\mu A)$ as function of variable $\mu \in \mathbb{R}$ and call it the $i$-th Lyapunov exponent of the family (2). Emphasize that in [3] in the formulation of the problem it is not necessary that the coefficient matrix of (1) is bounded. Therefore the exponent $\lambda_i(\mu A)$, $i = \overline{1,n}$, can take improper values $-\infty$ and $+\infty$. Hence the function $\lambda_i(\mu A)$ is a mapping $\mathbb{R} \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, +\infty\}$.

Zubov problem is equivalent to the following: for every $i = \overline{1,n}$ give a complete description of the set of $i$-th Lyapunov exponents of families (2), i.e. the set $\mathcal{L}_i^n \stackrel{\text{def}}{=} \{\lambda_i(\mu A) \mid \ A \in \mathcal{M}_n^*\}$ of functions $\lambda_i(\mu A)\colon \mathbb{R} \to \overline{\mathbb{R}}$. In the present report this problem is solved for the largest Lyapunov exponent, i.e. a complete description of the set $\mathcal{L}_n^n$ is given for every integer $n \geq 2$.

Note that for parametric families of linear differential systems

$$\mathrm{d}x/\mathrm{d}t = A(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{3}$$

with continuous in the variables $t$, $\mu$ and bounded on the half-line $t \geq 0$ for every fixed $\mu \in \mathbb{R}$ coefficient matrix $A(t,\mu)\colon [0,+\infty) \times \mathbb{R} \to \operatorname{End}\mathbb{R}^n$, a similar problem is solved in [4]. It is proved that for every $i = \overline{1,n}$ a function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \mathbb{R}$ is the $i$-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$ and have an upper semicontinuous minorant. In the paper [4] it is proved that this result holds in a more general situation – for the Lyapunov exponents of families of morphisms of Millionshchikov bundles.

Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [5, p. 223–224] if for each $r \in \mathbb{R}$ the preimage of the interval $[r,+\infty)$ under the mapping $f$ is a $G_\delta$-set, i.e. can be represented as a countable intersection of open sets. Consider $\overline{\mathbb{R}}$ with a natural (order) topology, so that $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1,1]$. Choose such a homeomorphism $\ell\colon \overline{\mathbb{R}} \to [-1,1]$ in a standard way: $\ell(x) = \frac{x}{|x|+1}$ if $x \in \mathbb{R}$, and $\ell(x) = \operatorname{sgn}(x)$ if $x = \pm\infty$. Since the mapping $\ell$ performs an order-preserving homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$, we say that a function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the Baire class $\mathcal{K}$ if the composition $\ell \circ f$ belongs to the class $\mathcal{K}$. This definition is equivalent to the definition [6, p. 382, 401] of Baire classes of mappings between metric spaces.

Slightly modifying proofs of [4] one can get that similar result holds true for the generalized Lyapunov exponents which implies that for every $i = \overline{1,n}$ a function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \overline{\mathbb{R}}$ is the $i$-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) with not necessary bounded coefficients if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$.

Despite the fact that the dependence on the parameter in the families (2) is linear, the description of the largest Lyapunov exponents of families (2) is similar to the description of the largest Lyapunov exponents in the general case of families (3).

A partial solution to the Zubov problem was announced in report [7]. In the paper [8] it was proved that conditions 1) – 4) of the theorem below are necessary. In [8] it was also proved that conditions 1) – 3) are sufficient under the assumption that there exists such a real number $b$ that the inequality $f(\mu) \geq b\mu$ holds for all $\mu \in \mathbb{R}$. In the general case the theorem below gives a complete description of the set $\mathcal{L}_n^n$ for an arbitrary integer $n \geq 2$.

**Theorem.** *A function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the class $\mathcal{L}_n^n = \{\lambda_n(\mu A)\colon \mathbb{R} \to \overline{\mathbb{R}} \mid \ A \in \mathcal{M}_n^*\}$ for an arbitrary $n \geq 2$ if and only if it fits the next four conditions:*

1) *$f$ belongs to $({}^*, G_\delta)$ Baire class;*
2) *$f(0) = 0$;*
3) *$f$ is nonnegative on some real semiaxis;*
4) *if $f$ is not identically equal to $+\infty$ on any semiaxis, then there exists such a real number $b$ that the inequality $f(\mu) \geq b\mu$ holds for all $\mu \in \mathbb{R}$.*

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# On One Boundary Value Problem for Semilinear Equation with the Iterated Multidimensional Wave Operator in the Principal Part

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In the Euclidian space $\mathbb{R}^{n+1}$ of the variables $x_1, \dots, x_n$, $t$ we consider the semilinear equation of the type

$$L_\lambda u := \Box^2 u + \lambda |u|^\alpha \operatorname{sgn} u = F, \tag{1}$$

where $\lambda \neq 0$ and $\alpha > 0$ are given real numbers, $F$ is a given and $u$ is an unknown real functions,

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,, \ \ n \geq 2.$$

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T = \Omega \times (0,T)$, where $\Omega$ is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u(x_1, \dots, x_n, t)$ of that equation according to the boundary conditions

$$u\big|_{\partial D_T} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial D_T} = 0, \tag{2}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to $\partial D_T$.

Let

$$\overset{\circ}{C}{}^k(D_T, \partial D_T) := \Big\{ u \in C^k(\overline{D}_T) : \ u\big|_{\partial D_T} = \frac{\partial u}{\partial \nu}\Big|_{\partial D_T} = 0 \Big\}, \ \ k \geq 2.$$

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ as the completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} = \int\limits_{D_T} \bigg[ u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 + (\Box u)^2 \bigg]\, dx\, dt$$

of the classical space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$.

**Definition.** Let $\alpha < \frac{n+1}{n-1}$ and $F \in L_2(D_T)$. The function $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ is said to be a weak generalized solution of the problem (1), (2) if the integral equality

$$\int\limits_{D_T} \Box u \Box \varphi \, dx\, dt + \lambda \int\limits_{D_T} |u|^\alpha \operatorname{sgn} u \varphi \, dx\, dt = \int\limits_{D_T} F \varphi \, dx\, dt$$

is valid for any $\varphi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$.

It is not difficult to verify that if a weak generalized solution $u$ of the problem (1), (2) belongs to the class $\overset{\circ}{C}{}^4(D_T, \partial D_T)$, then it will also be a classical solution of that problem.

**Theorem.** *Let $\lambda > 0$, $\alpha < \frac{n+1}{n-1}$ . Then for any $F \in L_2(D_T)$ the problem* (1), (2) *has a unique weak generalized solution in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

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# On Boundary Value Problems with the Condition at Infinity for Systems of Higher Order Nonlinear Differential Equations

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In the interval $\mathbb{R}_+ = [0, +\infty[$, we consider the problem on the existence of a solution of the nonlinear differential system

$$u^{(m)} = f_1(t, v, \dots, v^{(n-1)}), \quad v^{(n)} = f_2(t, u, \dots, u^{(m-1)}), \tag{1}$$

satisfying the boundary conditions

$$\begin{gathered} u^{(i-1)}(0) = \varphi_i\big(v^{(n-1)}(0)\big) \ \ (i = 1, \dots, m), \quad v^{(k-1)}(0) = \psi_k\big(v^{(n-1)}(0)\big) \ \ (k = 1, \dots, n-1), \\ \liminf_{t \to +\infty} |v^{(n-1)}(t)| = 0. \end{gathered} \tag{2}$$

Here $m \geq 1$, $n \geq 2$, and $f_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $f_2 : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, m)$, $\psi_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, n-1)$ are continuous functions.

Problem (1), (2) is interesting because its different particular cases arise in the oscillation theory (see, e.g., [1, 2]). Nevertheless, in the general case this problem is not studied yet. We have established sufficient conditions for the solvability and unique solvability of that problem. In particular, the following theorems are proved.

**Theorem 1.** *Let there exist a positive constant $r$ and continuous functions $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $h_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ such that*

$$\begin{gathered} \big|f_1(t, x_1, \dots, x_n)\big| \leq h_0(t)\Big(1 + \sum_{i=1}^{n} |x_i|\Big) \ \ \text{for } t \in \mathbb{R}_+, \ (x_1, \dots, x_n) \in \mathbb{R}^n, \\ f_1(t, x_1, \dots, x_n) \operatorname{sgn}(x_1) \geq \sum_{k=1}^{n} h_k(t)|x_k| \ \ \text{for } t \in \mathbb{R}_+, \ x_i \operatorname{sgn}(x_1) \geq r \ (i=1, \dots, n-1), \ x_n x_1 > 0, \\ \big|f_2(t, x_1, \dots, x_m)\big| \leq h_0(t)\Big(1 + \sum_{i=1}^{m} |x_i|\Big) \ \ \text{for } t \in \mathbb{R}_+, \ (x_1, \dots, x_m) \in \mathbb{R}^m, \\ f_2(t, x_1, \dots, x_m) \operatorname{sgn}(x_1) \geq 0 \ \ \text{for } t \in \mathbb{R}_+, \ x_i \operatorname{sgn}(x_1) \geq r \ (i = 1, \dots, m), \end{gathered}$$

*and*

$$\liminf_{|x| \to +\infty} \varphi_i(x) \operatorname{sgn}(x) > r \ \ (i = 1, \dots, n), \quad \liminf_{|x| \to +\infty} \psi_k(x) \operatorname{sgn}(x) > r \ \ (k = 1, \dots, n-1).$$

*If, moreover,*

$$\int_0^\infty \Big(\sum_{k=1}^{m} t^{n-k} h_k(t)\Big)\, dt = +\infty \ \ (k = 1, \dots, n), \tag{3}$$

*then problem* (1), (2) *has at least one solution.*

**Theorem 2.** *Let the functions $f_i$ $(i=1,2)$ have continuous partial derivatives in the phase variables,*

$$f_i(t,0,\dots,0)=0 \ \textit{ for } \ t\in\mathbb{R}_+ \ \ (i=1,2),$$

*and let there exist continuous functions $h_k:\mathbb{R}_+\to\mathbb{R}_+$ $(k=0,\dots,n)$ such that*

$$h_k(t)\leq \frac{\partial f_1(t,x_1,\dots,x_n)}{\partial x_k}\leq h_0(t) \ \textit{ for } \ t\in\mathbb{R}_+, \ \ (x_1,\dots,x_n)\in\mathbb{R}^n \ \ (k=1,\dots,n),$$
$$0\leq \frac{\partial f_2(t,x_1,\dots,x_m)}{\partial x_k}\leq h_0(t) \ \textit{ for } \ t\in\mathbb{R}_+, \ \ (x_1,\dots,x_m)\in\mathbb{R}^m \ \ (k=1,\dots,m).$$

*Let, moreover, $h_k$ $(k=1,\dots,n)$ satisfy condition* (3), *while $\varphi_i$ $(i=1,\dots,m)$ and $\psi_k$ $(k=1,\dots,n-1)$ be nondecreasing functions such that*

$$\liminf_{|x|\to+\infty}\varphi_i(x)\operatorname{sgn}(x)>0 \ \ (i=1,\dots,m), \qquad \liminf_{|x|\to+\infty}\psi_k(x)\operatorname{sgn}(x)>0 \ \ (k=1,\dots,n-1).$$

*Then problem* (1), (2) *has one and only one solution.*


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# On One Boundary Value Problem with the Condition at Infinity, Arising in the Oscillation Theory

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In the infinite interval $\mathbb{R}_+ = [0, +\infty[$, we consider the $(n \geq 2)$-th order differential equation

$$u^{(n)}(t) = f\Big(t, u(t), \dots, u^{(n-1)}(t), u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\Big) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(0) = \varphi_i\big(u^{(n-1)}(0)\big) \ \ (i = 1, \dots, n-1), \quad \liminf_{t \to +\infty} |u^{(n-1)}(t)| < +\infty, \tag{2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^{2n} \to \mathbb{R}$, $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n-1)$ and $\tau_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ are continuous functions and

$$0 \leq \tau_k(t) < t \ \text{ for } \ t > 0, \quad \lim_{t \to +\infty} \tau_k(t) = +\infty \ \ (k = 1, \dots, n). \tag{3}$$

Problems of the type (1), (2) arise in the oscillation theory when studying the existence of proper oscillatory solutions of differential and functional differential equations having the property $B$ (see, e.g., [1–3]).

We have found conditions guaranteeing, respectively, the solvability and unique solvability of problem (1), (2). In particular, the following theorems are proved.

**Theorem 1.** *Let there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and a positive constant $\rho$ such that*

$$\begin{aligned} &\big|f(t, x_1, \dots, x_n, y_1, \dots, y_n)\big| \leq \\ &\qquad \leq g(t, y_1, \dots, y_n)\Big(1 + \sum_{k=1}^{n} |x_k|\Big) \ \ \textit{for} \ \ t \in \mathbb{R}_+, \ \ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}, \end{aligned} \tag{4}$$

$$f(t, x_1, \dots, x_n, y_1, \dots, y_n)x_1 \geq 0 \ \ \textit{for} \ \ t \in \mathbb{R}_+, \ \ x_k \operatorname{sgn}(x_1) \geq \rho, \ \ y_k \operatorname{sgn}(y_1) \geq \rho \ \ (k = 1, \dots, n) \tag{5}$$

*and*

$$\liminf_{|x| \to +\infty} \varphi_i(x) \operatorname{sgn}(x) > \rho \ \ (i = 1, \dots, n-1). \tag{6}$$

*Then problem* (1), (2) *has at least one solution.*

**Theorem 2.** *Let the function $f$ be nondecreasing and locally Lipschitz in the last $2n$ arguments and along with* (4), (5) *satisfy the condition*

$$\int_0^{+\infty} \big|f(t, t^{n-1}x, \dots, x, \tau^{n-1}(t)x, \dots, x)\big| \, dt = +\infty \ \ \textit{for} \ \ x \neq 0.$$

*If, moreover, $\varphi_i$ $(i = 1, \dots, n)$ are nondecreasing functions satisfying inequalities* (6)*, then problem* (1), (2) *has one and only one solution.*

As examples, we consider the differential equations

$$u^{(n)}(t)=\sum_{k=1}^{n}p_k(t)\big|u^{(k-1)}(\tau_k(t))\big|^{\lambda_k}u^{(k-1)}(t)+q(t), \tag{7}$$

$$\begin{aligned}u^{(n)}(t)=&\sum_{k=1}^{n}p_{1k}(t)\big|u^{(k-1)}(\tau_k(t))\big|^{\lambda_{1k}}\operatorname{sgn}\big(u^{(k-1)}(\tau_k(t))\big)+\\&+\sum_{k=1}^{n}p_{2k}(t)\big(1+|u^{(k-1)}(t)|\big)^{-\lambda_{2k}}u^{(k-1)}(t)+q(t)\end{aligned} \tag{8}$$

with the boundary conditions

$$u^{(i-1)}(0)=\alpha_i|u^{(n-1)}(0)|^{\mu_i}\operatorname{sgn}\big(u^{(n-1)}(0)\big)+\beta_i\ \ (i=1,\dots,n-1),\quad \liminf_{t\to+\infty}|u^{(n-1)}(t)|<+\infty, \tag{9}$$

where

$$\lambda_k>0,\ \ \lambda_{1k}\geq 1,\ \ 0\leq\lambda_{2k}\leq 1\ \ (k=1,\dots,n),$$
$$\alpha_i>0,\ \ \mu_i>0,\ \ \beta_i\in\mathbb{R}\ \ (i=1,\dots,n),$$

$p_k:\mathbb{R}_+\to\mathbb{R}_+$, $p_{ik}:\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,2;\ k=1,\dots,n)$, $q:\mathbb{R}_+\to\mathbb{R}$ are continuous functions, while $\tau_k:\mathbb{R}_+\to\mathbb{R}_+$ $(k=1,\dots,n)$ are functions satisfying conditions (3).

Theorems 1 and 2 imply the following proposition.

**Corollary 1.** *If*

$$|q(t)|\leq r\sum_{k=1}^{m}p_k(t)\ \ \textit{for}\ \ t\in\mathbb{R}_+,$$

*where $r=const>0$, then problem* (7), (9) *has at least one solution.*

**Corollary 2.** *If*

$$|q(t)|\leq r\sum_{k=1}^{n}\big(p_{1k}(t)+p_{2k}(t)\big)\ \ \textit{for}\ \ t\in\mathbb{R}_+$$

*and*

$$\int_0^{+\infty}\sum_{k=1}^{n}\big(p_{1k}(t)\tau^{(n-k)\lambda_{1k}}(t)+p_{2k}(t)t^{(1-\lambda_{2k})(n-k)}\big)\,dt=+\infty,\qquad \int_0^{+\infty}|q(t)|\,dt<+\infty,$$

*then problem* (8), (9) *has one and only one solution.*

## Acknowledgement

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# On One Two-Dimensional Nonlinear Integro-Differential Equation

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As it is known the magnetic field diffusion process in the medium can be modeled by Maxwell's system of partial differential equations [1]. Assume that coefficients of thermal heat capacity and electroconductivity of the substance depend on temperature. In this case, as it is shown in [2], the system of Maxwell's equation can be reduced to the following integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\Big(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\Big)\operatorname{rot} H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0,\infty)$.

In the work [3] some generalization of equations of type (1) is proposed. In particular, if the temperature is kept constant throughout the material, the same process of penetration of a magnetic field into a substance can be rewritten in the following integro-differential form [3]:

$$\frac{\partial H}{\partial t} = a\Big(\int\limits_0^t \int\limits_\Omega |\operatorname{rot} H|^2\,dx\,d\tau\Big)\Delta H, \tag{2}$$

where $x \in \Omega \subset R^3$.

Note that integro-differential parabolic models of (1) and (2) type are complex and still yield to the investigation only for special cases (see, for example, [2], [4]–[17] and references therein). Investigations mainly are done for one-dimensional case, i.e., when components of magnetic field $H$ depend on one space variable.

The existence of a weak solution to the first boundary value problem for the one component magnetic field and one dimensional spatial version for the case $a(S) = 1 + S$ and uniqueness results for some general cases of model (1) were proved in [2]. The same questions for model (2) has been discussed in [8].

The theorems and discussions of a large time behavior to the solutions of the initial-boundary value problems for the one-dimensional analog of (2) type models for the different cases of function $a = a(S)$ are studied in [4], [8]–[14], [16]. The multidimensional case for (1) type model is considered in [6]. The questions of numerical solution of corresponding initial-boundary value problems for (2) type models are discussed in [7], [12]–[17].

Purpose of this note is to study asymptotic behavior as $t \to \infty$ of a solution of the Dirichlet problem for model (2) in one component magnetic field and two-dimensional spatial case. Assume that the magnetic field has the following form $H = (0, 0, U)$ and $U = U(x, y, t)$. Then we have

$$\operatorname{rot} H = \Big(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}, 0\Big)$$

and equation (2) takes the following form

$$\frac{\partial U}{\partial t} = a(S)\Big(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\Big), \ \ (x,t) \in Q = \Omega \times (0,\infty), \tag{3}$$

where

$$S(t)=\int_0^t\int_\Omega\bigg[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial U}{\partial y}\Big)^2\bigg]\,dx\,dy\,d\tau, \tag{4}$$

and $\Omega=(0,1)\times(0,1)$.

In the domain $Q$, let us consider the following initial-boundary value problem for equation (3), (4):

$$U(x,y,t)=0,\quad (x,y)\in\partial\Omega,\ \ t\geq 0, \tag{5}$$

$$U(x,y,0)=U_0(x,y),\quad (x,y)\in\overline{\Omega}, \tag{6}$$

where $U_0=U_0(x,y)$ is a given function.

Recall the $L_2$-inner product and norm:

$$(u,v)=\int_\Omega u(x,y)v(x,y)\,dx\,dy,\quad \|u\|=(u,u)^{1/2}.$$

The following statements take place.

**Theorem 1.** *If $a(S)=(1+S)^p$, $p>0$; $U_0\in H_0^1(\Omega)$, then for the solution of problem* (3)–(6) *the following estimate is true*

$$\Big\|\frac{\partial U}{\partial x}\Big\|^2+\Big\|\frac{\partial U}{\partial y}\Big\|^2\leq C\exp(-2t).$$

Here and below we use usual Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ and constant $C$ which denotes various positive values independent of $t$.

Note that Theorem 1 gives exponential stabilization of the solution of problem (3)–(6) in the norm of the space $H^1(\Omega)$.

**Theorem 2.** *If $a(S)=(1+S)^p$, $p>0$; $U_0\in H_0^1(\Omega)\cap H^2(\Omega)$, then for the solution of problem* (3)–(6) *the following estimate is true*

$$\Big\|\frac{\partial U(x,t)}{\partial t}\Big\|\leq C\exp\Big(-\frac{t}{2}\Big).$$

The algorithm of an approximate solution is constructed by using of which numerous numerical experiments for problem (3)–(6) with different kind of initial-boundary value problems are carried out. Results of numerical experiments agree with the theoretical ones obtained in Theorems 1 and 2.

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# Existence and Asymptotic Behavior (as $t \to +\infty$) of Unboudedly Continuable to the Right Solutions of the Ordinary Differential Equation of the Second Order

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We consider the second order ordinary differential equation of the form:

$$F(t,y,y',y'') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} |y'|^{\beta_k} |y''|^{\gamma_k} = 0, \tag{1}$$

$n \in \mathbb{N}$, $n \geq 2$, $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $\sum\limits_{k=1}^{n} |\gamma_k| \neq 0$, $p_k \in \mathrm{C}([a;+\infty),\ a>0;\ \mathbb{R})$ $(k = \overline{1,n})$, $p_i(t) \neq 0$ $(i = \overline{1,s}$, $2 \leq s \leq n)$. We investigate the question of the existence and asymptotic behavior (as $t \to +\infty$) of unboudedly continuable to the right solutions ($R$-solutions) $y(t)$ of equation (1) and the derivatives $y'(t)$, $y''(t)$ of these solutions.

Earlier in [1] we have considered a similar question of the asymptotic behavior of solutions of equation of the form (1) when $\sum\limits_{k=1}^{n} |\gamma_k| = 0$.

The main result is obtained under the assumption that there exists a function $v \in \mathrm{C}^2([t_1;+\infty), t_1 > a; \mathbb{R})$ which possesses the following properties:

**(A)** $v(t) > 0$, $v''(t) \neq 0$ on $[t_1;+\infty)$, $v(+\infty)$ is equal to 0 or $+\infty$;

**(B)** $\lim\limits_{t\to+\infty} \dfrac{p_i(t) v^{\alpha_i}(t) |v'(t)|^{\beta_i} |v''(t)|^{\gamma_i}}{p_1(t) v^{\alpha_1}(t) |v'(t)|^{\beta_1} |v''(t)|^{\gamma_1}} = c_i$ $(0 \neq c_i \in \mathbb{R},\ i = \overline{1,s})$,

$$\lim_{t\to+\infty} \frac{p_j(t) v^{\alpha_j}(t) |v'(t)|^{\beta_j} |v''(t)|^{\gamma_j}}{p_1(t) v^{\alpha_1}(t) |v'(t)|^{\beta_1} |v''(t)|^{\gamma_1}} = 0 \ \ (j = \overline{s+1,n});$$

**(C)** $\exists \lim\limits_{t\to+\infty} \frac{v''(t)v(t)}{(v'(t))^2} = \mu$ $(0 \neq \mu \in \mathbb{R})$.

The following theorem is valid.

**Theorem 1.** *Let there exist a function $v \in \mathrm{C}^2([t_1;+\infty), t_1 > a; \mathbb{R})$ which possesses the properties* **(A)**–**(C)**. *Then for the $R$-solution $y(t)$ of the differential equation* (1) *with the asymptotic representation*

$$y^{(k)}(t) \sim v^{(k)}(t) \ \ (k = \overline{0,2}) \tag{2}$$

*to exist it is necessary, and if the roots $\lambda_1$, $\lambda_2$ of the algebraic equation*

$$\lambda^2 + \left(1 + \frac{m \sum\limits_{i=1}^{s} (\beta_i + \gamma_i) c_i}{\sum\limits_{i=1}^{s} \gamma_i c_i}\right)\lambda + \frac{m \sum\limits_{i=1}^{s} (\alpha_i + \beta_i + \gamma_i) c_i}{\sum\limits_{i=1}^{s} \gamma_i c_i} = 0$$

*have the property* $\operatorname{Re}\lambda_k \neq 0$ $(k=1,2)$, *then it is also sufficient that* $\sum\limits_{i=1}^{s} c_i = 0$.

*Moreover, if in some suburb of* $+\infty$ $\operatorname{sign}(\operatorname{Re}\lambda_1) \neq \operatorname{sign}(\operatorname{Re}\lambda_2)$, *then there exists a one-parametric set of* $R$*-solutions with the asymptotic representation* (2)*; if* $\operatorname{sign}(\operatorname{Re}\lambda_1) = \operatorname{sign}(\operatorname{Re}\lambda_2) \neq \operatorname{sign}(v'(t))$, *then there exists a two-parametric set of* $R$*-solutions with the asymptotic representation* (2).

This result is obtained using the results from [2,3].

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# Asymptotic Representations of Solutions of Differential Equations with Regularly Varying Nonlinearities

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We consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[ \to ]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to ]0; +\infty[$ is a continuous and regularly varying as $y^{(j)} \to Y_j$ function of order $\sigma_j$, $j = \overline{0, n-1}$, where $\Delta Y_j$ is some one-sided neighborhood of the point $Y_0$, $Y_0$ is equal to either 0 or $\pm\infty$[1].

The set of solutions of equation (1), that is defined in some neighborhood of $+\infty$, consists of monotonous functions and their derivatives of orders till $n-1$ and falls into two classes:

1) solutions, for each of them

$$\lim_{t \to +\infty} y^{(k-1)}(t) = \begin{cases} \text{or} \quad \pm\infty, \\ \text{or} \ \ 0 \end{cases} \qquad (k = \overline{1, n});$$

2) solutions, for each of them there exists $k \in \{1, \dots, n\}$ such that

$$y(t) = t^{k-1} \, [c + o(1)] \ \ (c \neq 0) \ \text{ as } \ t \to +\infty.$$

From the first class of solutions a sufficiently wide subclass of solutions of the equation (1) was picked out in the works of Evtukhov V. M. and Samoĭlenko A. M. [1], Klopot A. M. [2]. Asymptotic representations for this class of solutions as $t \to +\infty$ were established and necessary and sufficient conditions for the existence of these solutions were derived there.

The aim of the paper is to derive necessary and sufficient conditions for the existence of solutions of the equation (1) and more particular case, each of that for some $k \in \{1, \dots, n\}$ admits representations

$$y(t) = t^{k-1} \, [c_0 + o(1)], \quad y^{(k-1)} = c_0 + o(1) \ \ (c_0 \neq 0) \ \text{ as } \ t \to +\infty.$$

Moreover, we establish asymptotic formulas as $t \to +\infty$ for their derivatives of orders till $n-1$ and solve a question of quantity of these solutions.

Let us introduce notation for signs of numbers from neighborhoods of $\Delta Y_j$ $(j = \overline{0, n-1})$.

$$\mu_j = \begin{cases} 1, & \text{if } \ Y_j = +\infty, ; \text{ or } \ Y_j = 0 \ \text{ and } \ \Delta(Y_j) \ \text{ is a right neighborhood of the point } 0, \\ -1, & \text{if } \ Y_j = -\infty, \ \text{ or } \ Y_j = 0 \ \text{ and } \ \Delta(Y_j^0) \ \text{ is a left neighborhood of the point } 0. \end{cases}$$

[1]When $Y_j = \pm\infty$, here and in the sequel all signs in the neighborhood of the point $\Delta Y_j$ are assumed to have the uniform sign.

**Theorem 1.** *For the existence of solutions of the equation* (1)*, that admit the representation*

$$y^{(n-1)} = c + o(1) \ \ (c \neq 0) \ \ \textit{as} \ \ t \to +\infty,$$

*it is necessary and sufficient that* $c \in \Delta Y_{n-1}$ *and conditions be satisfied*

$$Y_{j-1} = \begin{cases} +\infty, & \textit{if} \ \mu_{n-1} > 0, \\ -\infty, & \textit{if} \ \mu_{n-1} < 0, \end{cases} \quad \textit{when} \ \ j = \overline{1, n-1},$$

$$\int\limits_{t_0}^{+\infty} p(\tau)\varphi_0(\mu_0\tau^{n-1})\varphi_1(\mu_1\tau^{n-2})\cdots\varphi_{n-2}(\mu_{n-2}\tau)\,d\tau < +\infty,$$

*where* $t_0 \geq a$ *is chosen so that* $\frac{ct^{n-k}}{(n-k)!} \in \Delta Y_{k-1}$ $(k = \overline{1, n-1})$ *for* $t \geq t_0$.

*Moreover, when these conditions are implemented, there exists an n-parameter family of such solutions and each of them admits the following asymptotic representations as* $t \to +\infty$*:*

$$y^{(j-1)}(t) = \frac{ct^{n-j}}{(n-j)!}\,[1 + o(1)] \ \ (j = \overline{1, n-1}),$$

$$y^{(n-1)}(t) = c + \alpha M(c)\varphi_{n-1}(c)\int\limits_{+\infty}^{t} p(\tau)\varphi_0(\mu_0\tau^{n-1})\varphi_1(\mu_1\tau^{n-2})\cdots\varphi_{n-2}(\mu_{n-2}\tau)\,d\tau\cdot[1 + o(1)],$$

*where*

$$M(c) = \prod_{k=1}^{n-1}\Big|\frac{c}{(n-k)!}\Big|^{\sigma_{k-1}}.$$

Let us introduce the notation needed in the forthcoming theorem.

$$I(t) = \alpha\varphi_{n-2}(c)M(c)\int\limits_{B}^{t} p(\tau)\varphi_0(\mu_{n-2}\tau^{n-2})\varphi_1(\mu_{n-2}\tau^{n-3})\cdots\varphi_{n-3}(\mu_{n-3}\tau)\,d\tau,$$

$$M(c) = \prod_{k=1}^{n-2}\Big|\frac{c}{(n-k-1)!}\Big|^{\sigma_{k-1}}, \ \ c \in \Delta Y_{n-2}, \quad \Phi(z) = \int\limits_{A}^{z}\frac{ds}{\varphi_{n-1}(s)}\,,$$

$$A = 0 \ \text{ and } \ B = +\infty, \ \text{ if } \int\limits_{y_{n-1}}^{Y_{n-1}}\frac{ds}{\varphi_{n-1}(s)} < +\infty \ \ (\sigma_{n-1} < 1, \ \ y_{n-1} \in \Delta Y_{n-1}),$$

$$A = y_{n-1} \ \text{ and } \ B = t_0, \ \text{ if } \int\limits_{y_{n-1}}^{Y_{n-1}}\frac{ds}{\varphi_{n-1}(s)} = \pm\infty \ \ (\sigma_{n-1} > 1, \ \ y_{n-1} \in \Delta Y_{n-1}).$$

**Theorem 2.** *Let* $\sigma_{n-1} \neq 1$*. For the existence of solutions of the equation* (1)*, that admit the representation*

$$y^{(n-2)}(t) = c + o(1) \ \ (c \neq 0) \ \ \textit{when} \ \ t \to +\infty,$$

*it is necessary and sufficient that* $c \in \Delta Y_{n-2}$ *and conditions be satisfied*

$$Y_{n-1} = 0, \quad Y_{j-1} = \begin{cases} +\infty, & \textit{if} \ \mu_{n-2} > 0, \\ -\infty, & \textit{if} \ \mu_{n-2} < 0, \end{cases} \quad \textit{when} \ \ j = \overline{1, n-2},$$

$$\int\limits_{t_0}^{+\infty} \Phi^{-1}(I(\tau))\,d\tau < +\infty,$$

*where $t_0 \ge a$ is chosen so that*

$$\frac{ct^{n-k-1}}{(n-k-1)!} \in \Delta Y_{k-1} \ \ (k=\overline{1,n-2}) \ \ \text{for } \ t \ge t_0,$$

*$\Phi^{-1}$ is an inverse function for $\Phi$.*

*Moreover, when these conditions are implemented, there exists an $n$-parameter family of such solutions, if $\sigma_{n-1} < 1$, and $(n-1)$-parameter family of such solutions, if $\sigma_{n-1} < 1$, and each of them admits the following asymptotic representations as $t \to +\infty$:*

$$\begin{aligned}
y^{(j-1)}(t) &= \frac{ct^{n-j-1}}{(n-j-1)!}\,[1+o(1)] \ \ (j=\overline{1,n-2}),\\
y^{(n-2)}(t) &= c + \int\limits_{+\infty}^{t} \Phi^{-1}(I(s))ds\,[1+o(1)],\\
y^{(n-1)}(t) &= \Phi^{-1}(I(t))\,[1+o(1)].
\end{aligned}$$

Let us consider a particular type of equation (2)

$$y^{(n)} = \alpha p(t)\varphi_0(y)\varphi_1(y')\cdots\varphi_{n-k}(y^{(n-k)}), \tag{2}$$

where $n \ge 2$, $\alpha \in \{-1,1\}$, $k \in \{1,\dots,n\}$, $p : [a,+\infty[\to]0,+\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to ]0;+\infty[$ is a continuous and regularly varying as $y^{(j)} \to Y_j$ function of order $\sigma_j$, $j=\overline{0,n-k}$, where $\Delta Y_j$ is some one-sided neighborhood of the point $Y_j$, $Y_j$ is equal to either 0 or $\pm\infty$.

**Theorem 3.** *For the existence of solutions of the equation (2), that admit the representation as $i \in \{1,\dots,k\}$:*

$$y^{(n-k)}(t) = \frac{ct^{i-1}}{(i-1)!}\,[1+o(1)]\ ; \ \ (c \ne 0) \ \ \text{as } \ t \to +\infty,$$

*it is necessary and sufficient that $c \in \Delta Y_{n-k}$ and conditions be satisfied*

$$Y_{j-1} = \begin{cases} +\infty, & \text{if } \mu_{n-k} > 0,\\ -\infty, & \text{if } \mu_{n-k} < 0, \end{cases} \quad \text{when } \ j=\overline{1,n-k}, \ \ \text{if } \ i=1;$$

$$Y_{j-1} = \begin{cases} +\infty, & \text{if } \mu_{n-k} > 0,\\ -\infty, & \text{if } \mu_{n-k} < 0, \end{cases} \quad \text{when } \ j=\overline{1,n-k+1}, \ \ \text{if } \ i>1;$$

$$\int\limits_{t_0}^{+\infty}\int\limits_{t_{k-i}}^{+\infty}\cdots\int\limits_{t_1}^{+\infty} p(\tau)\varphi_0(\mu_0\tau^{n-k+i-1})\cdots\varphi_{n-k}(\mu_{n-k}\tau^{i-1})\,dt_1\cdots dt_{k-i}\,d\tau < +\infty,$$

*where $t_0 \ge a$ is chosen so that*

$$\frac{ct^{n-k+i-j}}{(n-k+i-j)!} \in \Delta Y_{j-1} \ \ (j=\overline{1,n-k-1}) \ \ \text{for } \ t \ge t_0.$$

*Moreover, when these conditions are implemented, there exists an $n$-parameter family of such solutions and each of them admits the following asymptotic representations as $t \to +\infty$:*

$$\begin{aligned}
y^{(j-1)}(t) &= \frac{ct^{n-k+i-j}}{(n-k+i-j)!}\,[1+o(1)] \ \ (j=\overline{1,n-k+i-1}),\\
y^{(n-k+i-1)}(t) &= c + \alpha M(c)W_{k-i+1}(t)[1+o(1)],\\
y^{(j)}(t) &= \alpha M(c)W_{n-j}(t)[1+o(1)] \ \ (j=\overline{n-k+i,n-1}),
\end{aligned}$$

*where*

$$M(c) = \prod_{j=1}^{n-k+1} \Big| \frac{c}{(n-k+i-j)!} \Big|^{\sigma_{k-1}}, \;\; c \in \Delta Y_{n-k}, \quad W_j(t) = \int\limits_{+\infty}^{t} W_{j-1}(s)\, ds \;\; (j = \overline{1, k-i+1}),$$

$$W_0(t) = p(t)\varphi_0(\mu_0 t^{n-k+i-1})\varphi_1(\mu_1 t^{n-k+i-2}) \cdots \varphi_{n-k}(\mu_{n-k} t^{i-1}).$$

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# Precise Asymptotic Behavior of Regularly Varying Solutions of Second Order Half-Linear Differential Equations

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We consider the second order half-linear differential equation

$$(|x'|^{\alpha}\operatorname{sgn} x')' + q(t)|x|^{\alpha}\operatorname{sgn} x = 0, \tag{A}$$

under the assumption that:

(a) $\alpha$ is a positive constant;

(b) $q(t)$ is a continuous and integrable function on $[a,\infty)$, $a>0$.

Let $c$ be a constant such that

$$c \in (-\infty, E(\alpha)), \ \text{ where } \ E(\alpha) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}},$$

and let $\lambda_1$, $\lambda_2$ ($\lambda_1 < \lambda_2$) denote the real roots of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + c = 0. \tag{1}$$

It is known [2] that equation (A) possesses regularly varying solutions $x_i(t)$ such that

$$x_i \in \mathrm{RV}\,(\lambda_i^{\frac{1}{\alpha}*}), \ \ i=1,2,$$

if and only if

$$\lim_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)ds = c,$$

where use is made of the asterisk notation

$$u^{\gamma*} = |u|^{\gamma}\operatorname{sgn} u, \ \ \gamma > 0, \quad u \in \mathbf{R}.$$

A question arises: Is it possible to determine precisely the asymptotic behavior at infinity of the solutions of (A) mentioned above? It is natural to expect that the behavior of solutions would depend heavily on the rate of decay of the function

$$Q_c(t) = t^{\alpha} \int_t^{\infty} q(s)\,ds - c$$

as $t \to \infty$. The purpose of this report is to confirm the truth of this expectation by presenting some of the results, obtained in our recent paper [3], which provide explicit asymptotic formulas for regularly varying solutions of (A).

For conciseness of presentation we assume throughout that $c$ is a nonzero constant in $(-\infty, E(\alpha))$, in which case the real roots $\lambda_i$, $i = 1, 2$, of (1) satisfy

$$0 < \lambda_1 < \lambda_2 \ \text{ if } \ c > 0, \quad \lambda_1 < 0 < \lambda_2 \ \text{ if } \ c < 0.$$

and

$$\lambda_1 < \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} < \lambda_2 \ \text{ regardless of the sign of } \ c.$$

First we prove the following theorems which describe how the asymptotic behavior of the regularly varying solutions $x_i(t)$, $i = 1, 2$, of (A) is affected by the function $Q_c(t)$ decaying to zero as $t \to \infty$.

**Theorem 1.** *Suppose that there exists a positive continuous function $\phi(t)$ on $[0, \infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$|Q_c(t)| \le \phi(t) \ \text{ for all large } t.$$

*Then, equation* (A) *possesses a regularly varying solution $x_1 \in \mathrm{RV}\,(\lambda_1^{\frac{1}{\alpha}*})$ which is expressed in the form*

$$x_1(t) = \exp\bigg\{\int\limits_T^t \Big(\frac{\lambda_1 + v_1(s) + Q_c(s)}{s^{\alpha}}\Big)^{\frac{1}{\alpha}*}\,ds\bigg\}, \quad t \ge T, \tag{2}$$

*for some $T > a$, where $v_1(t)$ satisfies*

$$v_1(t) = O(\phi(t)) \ \text{ as } \ t \to \infty. \tag{3}$$

**Theorem 2.** *Suppose that there exists a continuous slowly varying function $\psi(t)$ on $[0, \infty)$ which tends to 0 as $t \to \infty$ and satisfies*

$$|Q_c(t)| \le \psi(t) \ \text{ for all large } t.$$

*Then, equation* (A) *possesses a regularly varying solution $x_2 \in \mathrm{RV}\,(\lambda_2^{\frac{1}{\alpha}*})$ which is expressed in the form*

$$x_2(t) = \exp\bigg\{\int\limits_T^t \Big(\frac{\lambda_2 + v_2(s) + Q_c(s)}{s^{\alpha}}\Big)^{\frac{1}{\alpha}*}\,ds\bigg\}, \quad t \ge T, \tag{4}$$

*for some $T > a$, where $v_2(t)$ satisfies*

$$v_2(t) = O(\psi(t)) \ \text{ as } \ t \to \infty. \tag{5}$$

In the proofs of these theorems it is crucial to determine the functions $v_i(t)$ in (2) and (4) so as to satisfy (3) and (5), respectively. This can be done by deriving the integral equations for $v_i(t)$, $i = 1, 2$, via the generalized Riccati equation associated with (A) and solving them by means of the contraction mapping principle.

It is expected that the accurate asymptotic formulas for solutions $x_i(t)$, $i = 1, 2$, could be obtained from their representations (2) and (4) provided some stronger decay conditions are imposed on $Q_c(t)$. That this is indeed the case is illustrated by the following theorems.

**Theorem 3.** *Let $\phi(t)$ be a positive continuously differentiable function on $[0, \infty)$ which decreases to 0 as $t \to \infty$, has the property that $t|\phi'(t)|$ is decreasing and satisfies*

$$\int\limits_a^{\infty} \frac{\phi(t)}{t}\,dt = \infty, \quad \int\limits_a^{\infty} \frac{\phi(t)^2}{t}\,dt < \infty.$$

*Suppose that $Q_c(t)$ is one-signed and satisfies*

$$|Q_c(t)| = \phi(t) + O(\phi(t)^2), \ \ t \to \infty.$$

*Then, equation* (A) *possesses a regularly varying solution $x(t)$ of index $\lambda_1^{\frac{1}{\alpha}*}$ such that*

$$x(t) \sim k_1 t^{\lambda_1^{\frac{1}{\alpha}*}} \exp\bigg\{\frac{\lambda_1^{\frac{1}{\alpha}*}}{\lambda_1(\alpha-\mu_1)}\,\operatorname{sgn} Q_c \int\limits_a^t \frac{\phi(s)}{s}\,ds\bigg\}, \ \ t \to \infty,$$

*for some constant $k_1 > 0$, where $\mu_1 = (\alpha+1)\lambda_1^{\frac{1}{\alpha}*}$.*

**Theorem 4.** *Let $\psi(t)$ be a positive continuously differentiable slowly varying function on $[0,\infty)$ which decreases to $0$ as $t \to \infty$, has the property that $t|\psi'(t)|$ is slowly varying and satisfies*

$$\int\limits_a^\infty \frac{\psi(t)}{t}\,dt = \infty, \quad \int\limits_a^\infty \frac{\psi(t)^2}{t}\,dt < \infty.$$

*Suppose that $Q_c(t)$ is one-signed and satisfies*

$$|Q_c(t)| = \psi(t) + O(\psi(t)^2), \quad t \to \infty.$$

*Then, equation* (A) *possesses a regularly varying solution $x(t)$ of index $\lambda_2^{\frac{1}{\alpha}}$ such that*

$$x(t) \sim k_2 t^{\lambda_2^{\frac{1}{\alpha}}} \exp\bigg\{\frac{\lambda_1^{\frac{1}{\alpha}-1}}{\alpha-\mu_2}\,\operatorname{sgn} Q_c \int\limits_a^t \frac{\psi(s)}{s}\,ds\bigg\}, \ \ t \to \infty,$$

*for some constant $k_2 > 0$, where $\mu_2 = (\alpha+1)\lambda_2^{\frac{1}{\alpha}}$.*

**(NB)** For the almost complete exposition of theory of regular variation and its applications we refer to the treatise of Bingham et al. [1]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [4].

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# Sign-Constant Periodic Solutions to Second-Order Differential Equations with a Sub-Linear Non-Linearity

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We are interested in the question on the existence and uniqueness of a **non-trivial non-negative** (resp. **positive**) solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u + (-1)^i q(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega).} \tag{$1_i$}$$

Here, $p \in L([0,\omega])$, $q : [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $i \in \{1,2\}$. Under a *solution* to problem (1), as usually, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere and verifies periodic conditions. A solution $u$ to problem (1) is referred as a *sign-constant solution* if there exists $i \in \{0,1\}$ such that $(-1)^i u(t) \geq 0$ for $t \in [0,\omega]$, and a *sign-changing solution* otherwise.

**Definition 1.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \;\text{ for a.e. }\; t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \;\text{ for }\; t \in [0,\omega] \quad \big(\text{resp. }\; u(t) \leq 0 \;\text{ for }\; t \in [0,\omega]\,\big)$$

is fulfilled.

**Definition 2.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a nontrivial sign-constant solution.

**Theorem $1_1$.** *Let $p \in \mathcal{V}^-(\omega)$,*

$$\left.\begin{array}{l} q(t,x) \leq q_0(t,x) \;\textit{ for a.e. }\; t \in [0,\omega] \;\textit{ and all }\; x \geq x_0, \\ x_0 \geq 0, \quad q_0 : [0,\omega] \times [x_0,+\infty[ \,\to \mathbb{R} \;\textit{ is a Carathéodory function}, \\ q_0(t,\cdot) : [x_0,+\infty[ \,\to \mathbb{R} \;\textit{ is non-decreasing for a.e. }\; t \in [0,\omega], \\ \displaystyle\lim_{x\to+\infty} \frac{1}{x} \int_0^\omega |q_0(s,x)|\,\mathrm{d}s = 0, \end{array}\right\} \tag{$H_1$}$$

*and there exist a function* $\alpha \in AC^1([0,\omega])$ *satisfying*

$$\alpha''(t) \geq p(t)\alpha(t) - q(t,\alpha(t)) \ \text{ for a.e. } t \in [0,\omega], \quad \alpha(0) = \alpha(\omega), \quad \alpha'(0) \geq \alpha'(\omega).$$

*Then problem* $(1_1)$ *has at least one solution* $u$ *such that*

$$u(t) \geq \alpha(t) \ \text{ for } t \in [0,\omega].$$

**Theorem $1_2$.** *Let* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ *and*

$$\left.\begin{array}{l} |q(t,x)| \leq q_0(t,x) \ \text{ for a.e. } t \in [0,\omega] \ \text{ and all } x \geq x_0, \\ x_0 > 0, \quad q_0 : [0,\omega] \times [x_0,+\infty[ \to [0,+\infty[ \ \text{ is a Carathéodory function}, \\ q_0(t,\cdot) : [x_0,+\infty[ \to [0,+\infty[ \ \text{ is non-decreasing for a.e. } t \in [0,\omega], \\ \displaystyle\lim_{x\to+\infty} \frac{1}{x} \int_0^\omega q_0(s,x)\,\mathrm{d}s = 0. \end{array}\right\} \tag{$H_2$}$$

*Let, moreover,*

$$q(t,0) \leq 0 \ \text{ for a.e. } t \in [0,\omega] \tag{2}$$

*and there exist a function* $\beta \in AC^1([0,\omega])$ *satisfying*

$$\begin{gathered} \beta(t) > 0 \ \text{ for } t \in [0,\omega], \\ \beta''(t) \leq p(t)\beta(t) + q(t,\beta(t)) \ \text{ for a.e. } t \in [0,\omega], \quad \beta(0) = \beta(\omega), \quad \beta'(0) \leq \beta'(\omega). \end{gathered} \tag{3}$$

*Then problem* $(1_2)$ *has at least one solution* $u$ *such that*

$$u(t) \geq 0 \ \text{ for } t \in [0,\omega], \ \ u \not\equiv 0, \tag{4}$$

*and, moreover,*

$$u(t_u) \geq \beta(t_u) \ \text{ for some } t_u \in [0,\omega]. \tag{5}$$

The following example shows that, under the assumptions of Theorem $1_2$, problem $(1_2)$ may have a solution $u$ satisfying (4) and (5), which is not positive.

**Example 1.** Consider the problem

$$u'' = -u + 3(1-\sin t)\sqrt{|u|}\,\operatorname{sgn} u; \quad u(0) = u(2\pi), \ \ u'(0) = u'(2\pi). \tag{6}$$

Clearly, problem (6) is a particular case of $(1_2)$, where $\omega := 2\pi$, $p(t) := -1$ for $t \in [0,2\pi]$, and

$$q(t,x) := 3(1-\sin t)\sqrt{|x|}\,\operatorname{sgn} x \ \text{ for } t \in [0,2\pi], \ x \in \mathbb{R}.$$

It is not difficult to verify that $p \notin \mathcal{V}^-(2\pi) \cup \mathcal{V}_0(2\pi) \cup \mathcal{V}^+(2\pi)$, hypothesis $(H_2)$ holds with $q_0(t,x) := 3(1-\sin t)\sqrt{x}$, and condition (2) is fulfilled. Moreover, one can show that there exists a function $\beta \in AC^1([0,2\pi])$ satisfying condition (3) and

$$0 < \beta(t) \leq 1 \ \text{ for } t \in [0,2\pi].$$

On the other hand, the function

$$u(t) := (1+\sin t)^2 \ \text{ for } t \in [0,2\pi]$$

is a solution to problem (6), which satisfies conditions (4) and (5), however, it is not positive.

Now we present efficient conditions guaranteeing the existence of a non-trivial sign-constant (resp. positive) solution to problem $(1_i)$. Introduce the assumption:

$$\left.\begin{array}{l} q(t,x)\geq xg(t,x) \ \text{ for a.e. } \ t\in[0,\omega] \ \text{ and all } \ x\in\,]0,\delta[\,,\\ 0<\delta\leq+\infty, \quad g:[0,\omega]\times]0,\delta[\,\to\mathbb{R} \ \text{ is a locally Carathéodory function,}\\ g(t,\cdot):]0,\delta[\,\to\mathbb{R} \ \text{ is non-increasing for a.e. } \ t\in[0,\omega]. \end{array}\right\} \tag{$G$}$$

**Corollary $1_1$.** *Let $p\in\mathcal{V}^-(\omega)$, hypotheses $(H_1)$ and $(G)$ be satisfied, and*

$$\lim_{x\to\delta-} g(t,x)\leq 0 \ \textit{ for a.e. } \ t\in[0,\omega], \quad \lim_{x\to 0+}\int\limits_0^\omega g(s,x)\,\mathrm{d}s=+\infty. \tag{7}$$

*Then problem $(1_1)$ has at least one* ***positive*** *solution.*

**Corollary $1_2$.** *Let $p\notin\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$, $q(\cdot,0)\equiv 0$, hypotheses $(H_2)$ and $(G)$ be satisfied, and*

$$\lim_{x\to 0+}\int\limits_E g(s,x)\,\mathrm{d}s=+\infty \ \textit{ for every } \ E\subseteq[0,\omega], \ \ \mathrm{meas}\,E>0. \tag{8}$$

*Then problem $(1_2)$ has at least one* ***non-trivial non-negative*** *solution.*

*If, in addition, $p\in\mathcal{V}^+(\omega)$ and*

$$q(t,x)\geq 0 \ \textit{ for a.e. } \ t\in[0,\omega] \ \textit{ and all } \ x\geq 0, \tag{9}$$

*then problem $(1_2)$ has at least one* ***positive*** *solution and, moreover, any solution to this problem is either positive or non-positive.*

If $p\in\mathcal{V}^+(\omega)$ in Corollary $1_2$, then assumption (8) can be relaxed to

$$\lim_{x\to 0+}\int\limits_0^\omega g(s,x)\,\mathrm{d}s=+\infty. \tag{10}$$

**Corollary $2_2$.** *Let $p\in\mathcal{V}^+(\omega)$, $q(\,\cdot\,,0)\equiv 0$, and hypotheses $(H_2)$ and $(G)$ be satisfied. Let, moreover, condition (10) hold and*

$$\lim_{x\to\delta-} g(t,x)\geq 0 \ \textit{ for a.e. } \ t\in[0,\omega].$$

*Then problem $(1_2)$ has at least one* ***non-trivial non-negative*** *solution.*

*If, in addition, (9) holds, then problem $(1_2)$ has at least one* ***positive*** *solution and, moreover, any solution to this problem is either positive or non-positive.*

The next statements show that, under the hypothesis

$$\left.\begin{array}{l} \text{for every } \ b>a>0 \ \text{ there exists } \ h_{ab}\in L([0,\omega]) \ \text{ such that}\\ h_{ab}(t)\geq 0 \ \text{ for a.e. } \ t\in[0,\omega], \ \ h_{ab}\not\equiv 0,\\ q(t,x)\geq h_{ab}(t) \ \text{ for a.e. } \ t\in[0,\omega] \ \text{ and all } \ x\in[a,b], \end{array}\right\} \tag{$N$}$$

the assumptions $p\in\mathcal{V}^-(\omega)$ and $p\notin\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$ in the above-stated results are necessary for the existence of a positive solution to problem $(1_1)$ and a non-trivial non-negative solution to problem $(1_2)$, respectively.

**Proposition $1_1$.** *Let hypothesis $(N)$ hold and problem $(1_1)$ possess a positive solution. Then $p\in\mathcal{V}^-(\omega)$.*

**Proposition $1_2$.** *Let hypothesis* $(N)$ *hold and problem* $(1_2)$ *possess a non-trivial non-negative solution. Then* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.

It worth mentioning that some uniqueness type results for problem $(1_i)$ can be also proved. However, we omit here their formulation instead of which we present consequences of the general results for the following particular case of $(1_i)$:

$$\boxed{u'' = p(t)u + (-1)^i h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),} \tag{$11_i$}$$

where $p, h \in L([0,\omega])$ and $\lambda \in ]0,1[$. Observe that if $u$ is a solution to problem $(11_i)$, then the function $-u$ is its solution, as well.

**Definition 3.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}_1(\omega)$ if for any $a \in [0,\omega[$, the solution $u$ to the initial value problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \;\; u'(a) = 1$$

has at most one zero in the interval $]a, a+\omega[$, where $\widetilde{p}$ is the $\omega$-periodic extension of the function $p$ to the whole real axis.

**Remark 1.** One can show that $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \subset \mathcal{D}_1(\omega)$.

**Corollary $3_1$.** *Let* $\lambda \in ]0,1[$ *and*

$$h(t) \geq 0 \;\; \textit{for a.e.} \;\; t \in [0,\omega], \quad h \not\equiv 0. \tag{12}$$

*Then the following assertions hold:*

(i) *Problem* $(11_1)$ *has a positive* (*resp. negative*) *solution if and only if* $p \in \mathcal{V}^-(\omega)$.

(ii) *If* $p \in \mathcal{V}^-(\omega)$, *then problem* $(11_1)$ *has exactly three sign-constant solutions* (*positive, negative, and trivial*).

**Corollary $3_2$.** *Let* $\lambda \in ]0,1[$ *and*

$$h(t) > 0 \;\; \textit{for a.e.} \;\; t \in [0,\omega]. \tag{13}$$

*Then the following assertions hold:*

(i) *If* $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, *then problem* $(11_2)$ *possesses only the trivial solution.*

(ii) *If* $p \in \mathcal{D}_1(\omega) \setminus [\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)]$, *then problem* $(11_2)$ *possesses at least three sign-constant solutions* (*non-trivial non-negative, non-trivial non-positive, and trivial*) *and no sign-changing solutions.*

(iii) *If* $p \notin \mathcal{D}_1(\omega)$, *then problem* $(11_2)$ *has at least three sign-constant solutions* (*non-trivial non-negative, non-trivial non-positive, and trivial*).

In the next statement, assumption (13) appearing in Corollary $3_2$ is relaxed to (12).

**Corollary $4_2$.** *Let* $\lambda \in ]0,1[$ *and condition* (12) *be fulfilled. Then the following assertions hold:*

(i) *If* $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, *then problem* $(11_2)$ *possesses only the trivial solution.*

(ii) *If* $p \in \mathcal{V}^+(\omega)$, *then problem* $(11_2)$ *has exactly three solutions* (*positive, negative, and trivial*).

(iii) *If* $p \in \mathcal{D}_1(\omega) \setminus [\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)]$, *then problem* $(11_2)$ *has no sign-changing solutions.*

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# On Integral Conditions Determining Some $\Gamma$-Ultimate Classes of Perturbations

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ := [0, +\infty[\,, \tag{1}$$

with a piecewise continuous bounded coefficient matrix $A$ and with the Cauchy matrix $X_A$. Together with system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+, \tag{2}$$

with a piecewise continuous bounded perturbation matrix $Q$. For the higher exponent of system (2), we use the notation $\lambda_n(A+Q)$. By $\mathbb{R}^{n\times n}$ we denote the set of all real $n \times n$-matrices with the spectral norm $\|\cdot\|$. By $\mathrm{PC}_n(\mathbb{R}^+)$ we denote the linear space of all piecewise continuous matrix functions $S : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. The space of bounded elements of $\mathrm{PC}_n(\mathbb{R}^+)$ is denoted by $\mathrm{KC}_n(\mathbb{R}^+)$. Lyapunov exponent of $\beta \in \mathrm{PC}_1(\mathbb{R}^+)$ is denoted by $\lambda[\beta]$. We say that a function $\gamma \in \mathrm{PC}_1(\mathbb{R}^+)$ is strictly positive iff the condition $\inf\limits_{t\in J} \gamma(t) > 0$ holds for every finite interval $J \subset \mathbb{R}^+$.

Let $\mathfrak{M}$ be a class of perturbations. It is well known that the number $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : Q \in \mathfrak{M}\}$ is an important asymptotic characteristics for system (1) [1, p. 157], [2, p. 39]. Many authors investigated how to find $\Lambda(\mathfrak{M})$ for various $\mathfrak{M}$ (see, e.g. [3]– [13]). In numerous cases, an algorithm similar to the algorithm for the computation of the sigma-exponent [3] can be constructed for $\Lambda(\mathfrak{M})$. In some other cases [4], [5], [10]– [13], the result is similar to the formula

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{m\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT-T)\|$$

for the computation of the central exponent [1, p. 99], [10].

Let $\mathbb{T}$ be the set of all sequences $\tau : \mathbb{N}_0 \to \mathbb{R}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, monotonically increasing to $+\infty$. For arbitrary $\tau \in \mathbb{T}$, let

$$\Omega(A, \tau) = \overline{\lim_{k\to\infty}} \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln \|X_A(t_{i+1}, t_i)\|,$$

where $t_i := \tau(i), i \in \mathbb{N}_0$.

**Definition 1.** A class of perturbations $\mathfrak{M}$ is called $\Gamma$-ultimate if there exists a set $\Gamma \subset \mathbb{T}$ such that the relation

$$\Lambda(\mathfrak{M}) = \sup_{\tau\in\Gamma} \Omega(A, \tau)$$

is valid for every system (1).

In [14] we give sufficient conditions for $\mathfrak{M}$ to be $\Gamma$-ultimate when $\mathfrak{M}$ is defined by some conditions of the form $\|Q(t)\| \le N\beta(t)$, where $N > 0$ and $\beta$ is taken from a certain family $\mathcal{K} \subset \mathrm{KC}_1(\mathbb{R}^+)$.

In the report we present an analogous condition for classes of perturbations $\mathfrak{N}_n[\mathcal{P}] \subset \mathrm{KC}_n(\mathbb{R}^+)$ defined by integral conditions. More precisely, by $\mathfrak{N}_n[\mathcal{P}]$ we denote the set of perturbations $Q \in \mathrm{KC}_n(\mathbb{R}^+)$ such that $Q$ satisfies the condition

$$\lim_{t\to+\infty} t^{-1}\int\limits_0^t p(s)\|Q(s)\|\,ds = 0$$

for some $p \in \mathcal{P}$, where $\mathcal{P} \subset \mathrm{PC}_1(\mathbb{R}^+)$ is a given set of nonnegative functions. In what follows, we refer to $\mathcal{P}$ as a collection of weights.

For each $\tau \in \mathbb{T}$ and $N > 0$, define the function $K_N^\tau : \mathbb{R}^+ \to \mathbb{R}$ by $K_N^\tau(s) = e^{N(s-t_k)}$ for $s \in ]t_k, t_{k+1}]$, $k \in \mathbb{N}$, and $K_N^\tau(s) = 0$ for $s \le t_0$, where $t_k := \tau(k)$, $k \in \mathbb{N}_0$, are the elements of the sequence $\tau$. Let us also put

$$\gamma(\beta,\tau) = \overline{\lim_{k\to\infty}}\,\frac{1}{t_{k+1}}\sum_{i=m_0}^{k}\ln\frac{2}{\sin\varphi_i},\quad \varphi_i = \min\bigg\{\frac{\pi}{2}\,, e^{-2N_A}\int\limits_{t_i-1}^{t_i}\beta(s)ds\bigg\},\ \ i \ge m_\tau,$$

where $\tau \in \mathbb{T}$, $\beta \in \mathrm{KC}_1(\mathbb{R}^+)$, $m_\tau := \min\{i \in \mathbb{N} :\ t_i \ge 1\} \ge 1$, and $m_0 \ge m_\tau$ is such that $\varphi_i > 0$ for all $i \ge m_0$. If the inequality $\varphi_i \le 0$ holds for arbitrarily large $i \in \mathbb{N}$, we put $\gamma(\beta,\tau) = +\infty$.

Finally, by $\mathbb{T}_0$ we denote the subset of $\mathbb{T}$ that consists of sequences satisfying the condition $\lim\limits_{k\to+\infty} t_k^{-1}t_{k+1} = 1$ of slow growth [15] and the condition $\lim\limits_{k\to+\infty}(t_{k+1}-t_k) = +\infty$.

**Theorem 1.** *Let $\mathcal{P}$ be a collection of weights. If there exists a set $\Gamma \subset \mathbb{T}_0$ such that the equality $\inf\limits_{\beta\in\mathfrak{N}_1[\mathcal{P}]}\gamma(\beta,\tau) = 0$ holds for any $\tau \in \Gamma$, and for any $p \in \mathcal{P}$ and $M > 0$ there exists a sequence $\tau \in \Gamma$ such that $K_M^\tau \le Cp$ with some $C = C(p,M,\tau) > 0$, then $\mathfrak{N}_n[\mathcal{P}]$ is $\Gamma$-ultimate.*

Let $\mathfrak{M}_0[\theta]$ be the set of all perturbations satisfying the estimate $\|Q(t)\| \le N_Q e^{-\sigma\theta(t)}$, where $N_Q \ge 0$, $\sigma > 0$ are numbers depending on $Q$ and $\theta : \mathbb{R}^+ \to ]0,+\infty[$ is a fixed piecewise continuous function increasing to $+\infty$ such that $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\theta(t) < +\infty$. It was proved in [4], [5] that

$$\Lambda(\mathfrak{M}_0[\theta]) = \lim_{\delta\to+0}\Omega(A,\eta(\theta,\delta)), \tag{3}$$

where the sequence $\eta(\theta,\delta) \in \mathbb{T}$ is defined by the recursion formula

$$T_{k+1}(\delta) = T_k(\delta) + \delta\theta(T_k(\delta)),\ \ k \in \mathbb{N}_0, \tag{4}$$

with arbitrary initial condition $T_0(\delta) \ge 0$. The sequence $\eta(\theta,\delta)$ is called the $\delta$-characteristic sequence of $\theta$. This notion was introduced in [4], [5]. It should be stressed that relation (3) is not valid if $\theta$ is not monotonic and $\eta$ is given by (4).

In [14] we define an implicit $\delta$-characteristic sequence of $\theta$ by the recurrence relation

$$t_{k+1} = t_k + \delta\theta(t_{k+1}) \tag{5}$$

for continuous non-monotonic functions. It occurs that in general settings of $\theta \in \mathrm{PC}_1(\mathbb{R}^+)$ the appropriate definition can be given in the form

$$\delta\theta(t_{k+1}-0) \ge t_{k+1} - t_k \ge \delta\theta(t_{k+1}+0). \tag{6}$$

Obviously, (6) is equivalent to (5) if $\theta$ is continuous. If condition (6) does not define the value of $t_{k+1}$ uniquely , we consider the set $S_k$ of all values satisfying (6) and take the minimal element. It can be proved that the required minimal value exists if $S_k$ is not empty.

We denote the set of all implicit $\delta$-characteristic sequences of $\theta$ by $\mathbb{X}(\theta)$. The element of $\mathbb{X}(\theta)$ corresponding to certain values of $\delta$ and $t_0$ is denoted by $\xi(\theta,\delta,t_0)$. It can be easily proved that $\mathbb{X}(\theta) \subset \mathbb{T}_0$ if $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\theta(t) = 0$ and $\theta(t) \to +\infty$ as $t \to +\infty$.

**Definition 2.** A collection of weights $\mathcal{P}$ is said to be radical if for any $\varepsilon \in ]0,1]$ and $p \in \mathcal{P}$ there exist a weight $p_\varepsilon \in \mathcal{P}$ and a number $R_p(\varepsilon) > 0$ such that $p_\varepsilon < R_p(\varepsilon)p^\varepsilon$.

**Definition 3.** A function $q \in \mathrm{PC}_1(\mathbb{R}^+)$ is said to be moderately discontinuous if $q$ is strictly positive and there exists a number $c_q > 0$ such that $q(t^* + 0) \geq c_q q(t^* - 0)$ for any discontinuity point $t^*$ of $q$.

**Theorem 2.** *Suppose that $\mathcal{P}$ is radical and each $p \in \mathcal{P}$ is left-continuous, moderately discontinuous, and bounded away from zero by some constant $C_p > 1$. If for any $p \in \mathcal{P}$ the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$ hold, then $\mathfrak{N}_n[\mathcal{P}]$ is $\Gamma_{\mathcal{P}}$-ultimate with $\Gamma_{\mathcal{P}} = \{\xi(\ln p, \delta, t_p) :\ p \in \mathcal{P},$ $;\delta \in ]0,1]\} \subset \mathbb{T}_0$, where the mapping $\mathcal{P} \ni p \mapsto t_p \in \mathbb{R}^+$ is arbitrary.*

**Corollary 1.** *If $\mathcal{P}$ is radical and each $p \in \mathcal{P}$ satisfy the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$, then $\mathfrak{N}_n[\mathcal{P}]$ is $\Gamma$-ultimate for some appropriate $\Gamma \subset \mathbb{T}_0$.*

**Remark.** It can be easily observed from the proof that the inequality

$$\Lambda(\mathfrak{N}_n[\mathcal{P}]) \leq \sup_{\tau \in \Gamma_{\mathcal{P}}} \Omega(A, \tau)$$

follows from the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$, whereas the rest of conditions of Theorem 2 is used only to prove the opposite relation. So we are motivated to consider some radicalization operation on weight collections.

**Corollary 2.** *Any collection of weights $\mathcal{P}$ such that each $p \in \mathcal{P}$ satisfies the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$ may be extended to a colection $\bar{\mathcal{P}}$ such that $\mathfrak{N}_n[\bar{\mathcal{P}}]$ is $\Gamma$-ultimate for some appropriate $\Gamma \subset \mathbb{T}_0$.*

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# An Optimal Control Problem for a Class of Functional Differential Equations with Continuous and Discrete Times

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## 1 Introduction

Here we continue the study of functional differential systems that cover many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference), see [5, 3] and references therein. First we recall the description of a class of continuous-discrete functional differential equations with linear Volterra operators and appropriate spaces where those are considered. On the basis of the representation of general solution to the system with the use of the Cauchy operator we consider an optimal control problem and propose sufficient and necessary conditions for its solvability in the terms of programming control.

## 2 A class of Continuous-Discrete Functional Differential Systems

Fix a segment $[0,T] \subset R$. By $L^n = L^n[0,T]$ we denote the space of summable functions $v : [0,T] \to R^n$ under the norm $\|v\|_{L^n} = \int\limits_0^T |v(s)|_n \, ds$, where $|\cdot|_n$ stands for the norm of $R^n$; $L_2^n = L_2^n[0,T]$ is the space of square summable functions $u : [0,T] \to R^r$ with the inner product $(u,v) = \int\limits_0^T u^{\perp}(s)v(s)\,ds$, where $\perp$ stands for transposition.

The space $AC^n = AC^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ with the norm
$$\|x\|_{AC^n} = \|\dot{x}\|_{L^n} + |x(0)|_n.$$

Let us fix a set $J = \{t_0, t_1, \dots, t_\mu\}$, $0 = t_0 < t_1 < \dots < t_\mu = T$.

$FD^\nu(\mu) = FD^\nu\{t_0, t_1, \dots, t_\mu\}$ denotes the space of functions $z : J \to R^\nu$ under the norm
$$\|z\|_{FD^\nu(\mu)} = \sum_{i=0}^{\mu} |z(t_i)|_\nu.$$

We consider the system under control
$$\begin{aligned} \dot{x} &= \mathcal{T}_{11}x + \mathcal{T}_{12}z + Fu + f, \\ z &= \mathcal{T}_{21}x + \mathcal{T}_{22}z + g, \end{aligned} \tag{1}$$
where the linear operators $\mathcal{T}_{ij}$, $i,j = 1,2$, are defined as follows.

**1.**
$$\mathcal{T}_{11} : AC^n \to L^n; \tag{$\mathcal{T}_{11}$}$$
$$(\mathcal{T}_{11}x)(t) = \int\limits_0^t K^1(t,s)\dot{x}(s)\,ds + A^1(t)x(0), \quad t \in [0,T].$$

Here the kernel $K^1(t,s)$ with its elements $k^1_{ij}(t,s)$ satisfies *the condition* $\mathcal{K}$: $k^1_{ij}(t,s)$, $i,j=1,\dots,n$, are measurable on the set $0\le s\le t\le T$ and there exists a summable nonnegative function $\kappa(\,\cdot\,)\in L^1[0,T]$ such that $|k^1_{ij}(t,s)|\le\kappa(t)$, $t\in[0,T]$, $i,j=1,\dots,n$; $(n\times n)$-matrix $A^1$ has elements summable on $[0,T]$.

**2.**

$$\mathcal{T}_{12}: FD^\nu(\mu)\to L^n;\quad (\mathcal{T}_{12}z)(t)=\sum_{\{j:\,t_j\le t\}} B^1_j(t)z(t_j),\ \ t\in[0,T], \tag{$\mathcal{T}_{12}$}$$

where elements of matrices $B^1_j$, $j=0,\dots,\mu$, are summable on $[0,T]$.

**3.**

$$\mathcal{T}_{21}: AC^n\to FD^\nu(\mu); \tag{$\mathcal{T}_{21}$}$$

$$(\mathcal{T}_{21}x)(t_i)=\int\limits_0^{t_i} K^2_i(s)\dot{x}(s)\,ds+A^2_i x(0),\ \ i=0,1,\dots,\mu,$$

with measurable and essentially bounded on $[0,T]$ elements of matrices $K^2_i$ and constant $(\nu\times n)$-matrices $A^2_i$, $i=0,1,\dots,\mu$.

**4.**

$$\mathcal{T}_{22}: FD^\nu(\mu)\to FD^\nu(\mu);\quad (\mathcal{T}_{22}z)(t_i)=\sum_{j=0}^{i-1} B^2_{ij}z(t_j),\ \ i=1,\dots,\mu, \tag{$\mathcal{T}_{22}$}$$

with constant $(\nu\times\nu)$-matrices $B^2_{ij}$.

In what follows we shall use some results from [6, 2] concerning the equation

$$\dot{x}=\mathcal{T}_{11}x+f \tag{2}$$

and the results of [1] concerning the equation

$$z=\mathcal{T}_{22}z+g. \tag{3}$$

The general solution of (2) has the form

$$x(t)=X(t)\alpha+\int\limits_0^t C_1(t,s)f(s)\,ds,$$

with arbitrary $\alpha\in R^n$, where $X(\,\cdot\,)$ is the fundamental matrix, $C_1(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

As for equation (3), it has the immediate analogs of the above terms. Thus, the general solution of (3) has the representation

$$z(t_i)=Z(t_i)\beta+(C_2g)(t_i),\ \ i=1,\dots,\mu,$$

with arbitrary $\beta\in R^\nu$, where $Z(\,\cdot\,)$ is the fundamental matrix, $C_2(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

## 3 An Optimal Control Problem for a Continuous-Discrete Functional Differential System

Let us fix the initial state of the system (1):

$$x(0)=\alpha,\quad z(0)=\beta. \tag{4}$$

Next we assume that the constraints with respect to the control are formed as a system of linear inequalities:

$$Gu(t) \leqslant \gamma, \quad t \in [0,T], \tag{5}$$

where $G$ is a given $(N \times r)$-matrix; also it is assumed that the set of all solutions to the system $Gv \leqslant \gamma$ (that is the set of admissible control values) is nonempty and bounded in $R^r$. Let us denote this set by $\mathcal{V}$.

As for the aim of control, it is defined with the use of a linear bounded functional $\Lambda : AC^n \times FD^\nu(\mu) \times L_2^r \to R$,

$$\Lambda(x,z,u) = l_1 x + l_2 z + \lambda u,$$

where $l_1 : AC^n \to R$, $l_2 : FD^\nu(\mu) \to R$, $\lambda : L_2^r \to R$ are linear bounded functionals.

We need to find an admissible control $u : [0,T] \to R^r$ under which the corresponding trajectory of (1) with conditions (2) brings a minimal value to the objective functional $\Lambda$. Thus we consider the optimal control problem

$$\Lambda(x,z,u) \longrightarrow \min \text{with constraints (1), (4), (5).} \tag{6}$$

Let us recall the general form of $l_1 :\ l_1 x = \psi_1 x(0) + \int\limits_0^T \varphi_1(s)\dot{x}(s)\,ds$ and $\lambda :\ \lambda u = \int\limits_0^T \lambda(s)u(s)\,ds$. Here $\psi_1$ is a constant $(1 \times n)$-vector, $\varphi_1(s)$ is a $(1 \times n)$-vector with elements bounded in essence, $\lambda^\perp(\,\cdot\,) \in L_2^r$. As for $l_2$, we put $l_2 z = \sum\limits_{i=0}^{\mu} q_i z(t_i)$ with given $(1 \times \mu)$-vectors $q_i$, $i = 0, \dots, \mu$.

**Lemma 1.** *The operator $\mathcal{T} : AC^n \to L^n$, $\mathcal{T} = \mathcal{T}_{11} + \mathcal{T}_{12} C_2 \mathcal{T}_{21}$ can be represented in the form*

$$(\mathcal{T}x)(t) = \int\limits_0^t K(t,s)\dot{x}(s)\,ds + A(t)x(0), \quad t \in [0,T],$$

*where the kernel $K(t,s)$ satisfies the condition $\mathcal{K}$, the columns of the matrix $A(\,\cdot\,)$ belongs to the space $L^n$.*

**Remark 1.** The kernel $K(t,s)$ and the matrix $A$ can be effectively constructed.

**Lemma 2.** *The functional $l : AC^n \to R$, $l = l_1 + l_2 C_2 \mathcal{T}_{21}$ can be represented in the form*

$$lx = \psi x(0) + \int\limits_0^T \varphi(s)\dot{x}(s)\,ds,$$

*where $\psi$ is a constant $(1 \times n)$-vector, $\varphi(s)$ is $(1 \times n)$-vector with essentially bounded elements.*

**Remark 2.** The vectors $\psi$ and $\varphi(s)$ can be effectively constructed.

Below we shall use the kernel $K(t,s)$ and the function $\varphi(s)$ to formulate the main result.

Now denote by $\vartheta : [0,T] \to (R^n)^*$ the solution to the integral equation

$$\vartheta(t) = \int\limits_t^T \vartheta(\tau)K(\tau,t)\,d\tau - \int\limits_t^T \varphi(\tau)K(\tau,t)\,d\tau, \quad t \in [0,T]. \tag{7}$$

The unique solvability of this equation is established in [6]. As for properties of the solution that are generated by properties of the kernel $K(t,s)$ as a function of the second argument, those are studied in [7], where in particular some conditions are formulated under which the function $\vartheta(\,\cdot\,)$ iherits the corresponding properties of $K(t,\,\cdot\,)$ (being of bounded variation, continuous, absolutely continuous). Define the functional $H : [0,T] \times (L^n)^* \times (L^n)^* \times R^r \to R$ by the equality

$$H\big(t, v(\,\cdot\,), w(\,\cdot\,), u\big) = F^*(v - w)(t) \cdot u - \lambda(t) \cdot u.$$

Here the symbol $*$ stands for adjoint spaces and operators.

**Theorem.** *The control $\overline{u}(t)$ solves problem* (6) *if and only if the equality*

$$H\big(t,\vartheta(\,\cdot\,),\varphi(\,\cdot\,),\overline{u}(t)\big)=\max_{u\in\mathcal{V}}H\big(t,\vartheta(\,\cdot\,),\varphi(\,\cdot\,),u\big)$$

*holds almost everywhere on* $[0,T]$.

**Remark 3.** In the case where the matrix $C(t,s)$ of the system $\dot{x}\;=\;\mathcal{T}x\,+\,f$ is known the function $\vartheta(t)$ can be written in the following explicit form:

$$\vartheta(t)=\int\limits_t^T\varphi(\tau)C_\tau'(\tau,t)\,d\tau.$$

Let us give three explicit forms of the functional $H$, which correspond to the following cases of $F$.

***Case 1.*** $(Fu)(t)=F(t)u(t)$. For such a case we have

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=(v(t)-w(t))\cdot F(t)\cdot u-\lambda(t)\cdot u.$$

Here the columns of $(n\times r)$-matrix $F(\,\cdot\,)$ are from $L_2^n$.

***Case 2.*** $(Fu)(t)=\int\limits_0^t F(t,\tau)u(\tau)\,d\tau$. For this case, $H$ has the representation

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=\int\limits_t^T(v(s)-w(s))\cdot F(s,t)\,ds\cdot u-\lambda(t)\cdot u.$$

Here the kernel $F(t,\tau)$ provides the continuous action of the integral operator $F$ from $L_2^r$ into $L^n$.

***Case 3.*** $(Fu)(t)=\begin{cases}F(t)u(t-\Delta) & \text{if } t\in[\Delta,T],\\ 0 & \text{otherwise,}\end{cases}$ where $\Delta$, $0<\Delta<T$, is a constant delay. In such a case the functional $H$ is defined by the equality

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=\chi_{[0,T-\Delta]}(t)\big(v(t+\Delta)-w(t+\Delta)\big)\cdot F(t+\Delta)\cdot u-\lambda(t)\cdot u,$$

$\chi_{[0,T-\Delta]}(\,\cdot\,)$ is the characteristic function of the segment $[0,T-\Delta]$.

It should be noted that an approach to derivation of the maximum principle on the base of the variational derivatives, covering nonlinear systems with aftereffect, is thoroughly treated in [4]. Our approach is based on the use of the Cauchy matrix of the linear system and allows one to formulate the maximum principle in the terms of control only. In this case the role of the adjoint equation is played by equation (7) whose form is unified and common for all possible kinds of aftereffect in the frame of problem (6). The case of a functional differential system with continuous time only is considered in [8].

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# Oscillation Criteria for Certain System of Non-Linear Ordinary Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$ , we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $p, g : \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions.

By a solution of system (1) on the interval $J \subseteq [0, +\infty[$ we understand a pair $(u, v)$ of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in $J$ and satisfy equalities (1) almost everywhere in $J$.

It was proved by Mirzov in [10] that all non-extendable solutions of system (1) are defined on the whole interval $[0, +\infty[$ . Therefore, when we are speaking about a solution of system (1), we assume that it is defined on $[0, +\infty[$ .

**Definition 1.** A solution $(u, v)$ of system (1) is called *non-trivial* if $u \not\equiv 0$ on any neighborhood of $+\infty$. We say that a non-trivial solution $(u, v)$ of system (1) is *oscillatory* if the function $u$ has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

In [10, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1), if the additional assumption

$$g(t) \geq 0 \ \text{ for a.e. } \ t \geq 0 \tag{2}$$

is satisfied. Especially, under assumption (2), if system (1) has an oscillatory solution, then any other its non-trivial solution is also oscillatory.

On the other hand, it is clear that if $g \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions of system (1) are non-oscillatory. That is why it is natural to assume that inequality (2) is satisfied and

$$\operatorname{meas}\big\{\tau \geq t : \ g(\tau) > 0\big\} > 0 \ \text{ for } \ t \geq 0. \tag{3}$$

**Definition 2.** We say that system (1) is *oscillatory* if all its non-trivial solutions are oscillatory.

Oscillation theory for ordinary differential equations and their systems is a widely studied and well-developed topic of the qualitative theory of differential equations. As for the results which are closely related to those of this section, we should mention [2, 4, 5, 6, 7, 8, 9, 11, 12, 13]. Some criteria established in these papers for the second order linear differential equations or for two-dimensional systems of linear differential equations are generalized to the considered system (1) below.

Many results (see, e.g., survey given in [2]) have been obtained in oscillation theory of so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1} \operatorname{sgn} u'\big)' + p(t)|u|^{q-1} \operatorname{sgn} u = 0 \tag{4}$$

(alternatively this equation is referred as "equation with the scalar $q$-Laplacian"). Equation (4) is usually considered under the assumptions $q > 1$, $p, r : [0, +\infty[ \to \mathbb{R}$ are continuous and $r$ is positive. One can see that equation (4) is a particular case of system (1). Indeed, if the function $u$, with properties $u \in C^1$ and $r|u'|^{q-1} \operatorname{sgn} u' \in C^1$, is a solution of equation (4), then the vector function $(u, r|u'|^{q-1} \operatorname{sgn} u')$ is a solution of system (1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q - 1$.

Moreover, the equation

$$u'' + \frac{1}{\alpha}\,p(t)|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn} u = 0 \tag{5}$$

is also studied in the existing literature under the assumptions $\alpha \in ]0,1]$ and $p : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. It is mentioned in [6] that if $u$ is a so-called proper solution of (5) then it is also a solution of system (1) with $g \equiv 1$ and vice versa. Some oscillations and non-oscillations criteria for equation (5) can be found, e.g., in [6, 7].

Finally, we mention the paper [1], where a certain analogy of Hartman–Wintner's theorem is established (origin one can find in [3, 14]), which allows us to derive oscillation criteria of Hille's type for system (1).

In what follows, we assume that the coefficient $g$ is non-integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,ds = +\infty. \tag{6}$$

Let

$$f(t) := \int\limits_0^{t} g(t)\,ds \ \text{ for } \ t \geq 0.$$

In view of assumptions (2), (3), and (6), we have

$$\lim_{t\to+\infty} f(t) = +\infty$$

and there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \ \text{ for } \ t > 0.$$

We put

$$c_\alpha(t) := \frac{\alpha}{f^{\alpha}(t)} \int\limits_0^{t} \frac{g(s)}{f^{1-\alpha}(s)} \bigg( \int\limits_0^{s} (\xi)p(\xi)\,d\xi \bigg)\,ds \ \text{ for } \ t > 0.$$

Now, we formulate an analogue (in a suitable form for us) of the Hartman–Wintner's theorem for the system (1) established in [1].

**Theorem 3** ([1, Corollary 2.5 (with $\nu = 1-\alpha$)]). *Let conditions* (2), (3), *and* (6) *hold, and either*

$$\lim_{t\to+\infty} c_\alpha(t) = +\infty,$$

*or*

$$-\infty < \liminf_{t\to+\infty} c_\alpha(t) < \limsup_{t\to+\infty} c_\alpha(t).$$

*Then system* (1) *is oscillatory.*

One can see that two cases are not covered by Theorem 3, namely, the function $c_\alpha(t)$ has a finite limit and $\liminf\limits_{t\to+\infty} c_\alpha(t) = -\infty$. Our aim is to find oscillation criteria for system (1) in the first mentioned case. Consequently, in what follows, we assume that

$$\lim_{t\to+\infty} c_\alpha(t) =: c_\alpha^* \in \mathbb{R}. \tag{7}$$

Now we formulate main results.

**Theorem 4.** *Let* (7) *and the inequality*

$$\limsup_{t\to+\infty} \frac{f^\alpha(t)}{\ln f(t)}\,(c^*_\alpha - c_\alpha(t)) > \Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

*hold. Then system* (1) *is oscillatory.*

For better formulation of the next statement we introduce the following notations.

$$Q(t;\alpha) := f^\alpha(t)\bigg(c^*_\alpha - \int\limits_0^t p(s)(s)\,ds\bigg) \ \text{ for } \ t>0,$$

where the number $c^*_\alpha$ is given by (7). Moreover, we denote lower and upper limits of the function $Q(\,\cdot\,;\alpha)$ as follows

$$Q_*(\alpha) := \liminf_{t\to+\infty} Q(t;\alpha), \quad Q^*(\alpha) := \limsup_{t\to+\infty} Q(t;\alpha).$$

Oscillation criteria from the next theorem coincide with the well-known Hille's results for the second order linear differential equations established in [4].

**Theorem 5.** *Let* (7) *hold. Let, moreover, either*

$$Q_*(\alpha) > \frac{1}{\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha},$$

*or*

$$Q^*(\alpha) > 1.$$

*Then system* (1) *is oscillatory.*

**Remark 6.** Presented statements generalize results stated in [2, 4, 5, 6, 7, 8, 9, 11, 13] concerning system (1) as well as equations (4) and (5). In particular, if we put $\alpha=1$, then we obtain oscillatory criteria for linear system of differential equations presented in [13]. Moreover, the results of [6] obtained for equation (5) are in a compliance with those above, where we put $g\equiv 1$. Observe also that Theorem 5 extends oscillation criteria for equation (5) stated in [7], where the coefficient $p$ is suppose to be non-negative. In the monograph [2], it is noted that the assumption $p(t)\ge 0$ for $t$ large enough can be easily relaxed to $\int\limits_0^t p(s)ds > 0$ for large $t$. It is worth mentioning here that we do not require any assumption of this kind.

Finally we show an example, where we can not apply oscillatory criteria from the above mentioned papers, but we can use Theorem 4 succesfully.

**Example 7.** Let $\alpha=2$, $g(t)\equiv 1$, and

$$p(t) := t\cos\Big(\frac{t^2}{2}\Big) + \frac{1}{(t+1)^3} \ \text{ for } \ t\ge 0.$$

It is clear that the function $p$ and its integral

$$\int\limits_0^t p(s)\,ds = \sin\Big(\frac{t^2}{2}\Big) - \frac{1}{2(t+1)^2} + \frac{1}{2} \ \text{ for } \ t\ge 0$$

change their sign in any neighbourhood of $+\infty$. Therefore neither of the results mentioned in Remark 6 can be applied.

On the other hand, we have

$$\begin{aligned} c_2(t) &= \frac{2}{t^2}\int\limits_0^t s\Big(\int\limits_0^s \Big(\xi\cos\frac{\xi^2}{2}+\frac{1}{(\xi+1)^3}\Big)\,d\xi\Big)\,ds \\ &= \frac{1}{2}-\frac{2\cos\frac{t^2}{2}}{t^2}+\frac{3}{t^2}-\frac{\ln(t+1)}{t^2}-\frac{1}{t^2(t+1)} \ \text{ for } \ t>0 \end{aligned}$$

and thus, the function $c_2(\,\cdot\,)$ has the finite limit

$$c_\alpha^* = \lim_{t\to+\infty} c_2(t) = \frac{1}{2}\,.$$

Moreover,

$$\limsup_{t\to+\infty}\frac{t^2}{\ln t}\,(c_\alpha^*-c_2(t)) = \limsup_{t\to+\infty}\Big(\frac{2\cos\frac{t^2}{2}-3}{\ln t}+\frac{\ln(t+1)}{\ln t}+\frac{1}{(t+1)\ln t}\Big) = 1.$$

Consequently, according to Theorem 4, system (1) is oscillatory.

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# The Nonlinear Kneser Problem for Singular in Phase Variables Two-Dimensional Differential Systems

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Let $a > 0$, $\mathbb{R}_- = ]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, and $\mathbb{R}_{0+} = ]0, +\infty[$. On a positive semi-axis $\mathbb{R}_{0+}$, we consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \;\; (i = 1, 2) \tag{1}$$

with the boundary condition

$$\varphi(u_1) = c, \tag{2}$$

where $c$ is a positive constant, $f_i : \mathbb{R}_{0+} \times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ are continuous functions, and $\varphi : C([0, a]; \mathbb{R}_+) \to \mathbb{R}_+$ is a continuous nondecreasing functional.

A continuously differentiable vector function $(u_1, u_2) : \mathbb{R}_{0+} \to \mathbb{R}^2_{0+}$, satisfying system (1) in $\mathbb{R}_{0+}$, is said to be **a positive solution** of that system.

If the component $u_i$ of a positive solution $(u_1, u_2)$ at the point 0 has the right-hand limit

$$u_i(0+) = \lim_{t>0,\, t\to 0} u_i(t),$$

then we put $u_i(0) = u_i(0+)$.

A positive solution $(u_1, u_2)$ of system (1) is said to be **a positive solution of problem** $(1), (2)$ if there exists $u_1(0+)$ and equality (2) is satisfied.

A positive solution $(u_1, u_2)$ of system (1) is said to be **a vanishing at infinity positive solution** if

$$\lim_{t\to+\infty} u_i(t) = 0 \;\; (i = 1, 2).$$

If

$$f_1(t, x, y) \equiv -y, \quad f_2(t, x, y) \equiv -f(t, x, -y),$$

then the differential system (1) is equivalent to the differential equation

$$u'' = f(t, u, u'), \tag{3}$$

and condition (2) is equivalent to the condition

$$\varphi(u) = c, \tag{4}$$

respectively. Consequently, problem $(1), (2)$ has a positive solution if and only if problem $(3), (4)$ has a so-called Kneser solution, i.e. a solution satisfying the inequalities

$$u(t) > 0, \quad u'(t) < 0 \;\text{ for }\; t \in \mathbb{R}_{0+}.$$

Problem $(1), (2)$, as problem $(3), (4)$, is said to be the nonlinear Kneser problem. These problems are investigated in detail in the case where the functions $f_i$ $(i = 1, 2)$ and $f$ have no singularities in phase variables (see, e.g., [1–6], and the references therein).

In [7], for the singular in a phase variable equation (3), sufficient conditions for the existence of a Kneser solution satisfying the condition (4) are established. Theorems below are generalizations of the above mentioned results for system (1).

Below everywhere it is assumed that the functions $f_i$ $(i=1,2)$ on the set $\mathbb{R}_{0+}\times\mathbb{R}^2_{0+}$ admit the estimates

$$g_{10}(t)\le -x^{\lambda_1}y^{-\mu_1}f_1(t,x,y)\le g_1(t),$$
$$g_{20}(t)\le -x^{\lambda_2}y^{-\mu_2}f_2(t,x,y)\le g_2(t),$$

where $\lambda_i$ and $\mu_i$ $(i=1,2)$ are nonnegative constants, and $g_{i0}:\mathbb{R}_{0+}\to\mathbb{R}_{0+}$, $g_i:\mathbb{R}_{0+}\to\mathbb{R}_{0+}$ $(i=1,2)$ are continuous functions. If $\lambda_i>0$ for some $i\in\{1,2\}$, then

$$\lim_{x\to 0} f_i(t,x,y)=+\infty \ \text{ for } \ t>0, \ \ y>0.$$

And if $\mu_2>0$, then

$$\lim_{y\to 0} f_2(t,x,y)=+\infty \ \text{ for } \ t>0, \ \ x>0.$$

Consequently, in both cases system (1) has the singularity in at least one phase variable.

We use the following notation and definitions.

$$\nu_0=\frac{\mu_1}{1+\mu_2}\,, \quad \nu=1+\lambda_1+\lambda_2\nu_0.$$

$C([0,a];\mathbb{R})$ is the Banach space of continuous functions $u:[0,a]\to\mathbb{R}$ with the norm

$$\|u\|_C=\max\big\{|u(t)|:\ 0\le t\le a\big\},$$

$C([0,a];\mathbb{R}_+)=\big\{u\in C([0,a];\mathbb{R}):\ u(t)\ge 0 \text{ for } 0\le t\le a\big\}$.

A functional $\varphi: C([0,a];\mathbb{R}_+)\to\mathbb{R}_+$ is said to be **nondecreasing** if for any $u\in C([0,a];\mathbb{R}_+)$ and $u_0\in C([0,a];\mathbb{R}_+)$ the inequality

$$\varphi(u+u_0)\ge\varphi(u)$$

holds.

**Theorem 1.** *If*

$$\int\limits_t^{+\infty} g_{20}(s)\,ds<+\infty, \quad w_0(t)\equiv\int\limits_t^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0}ds<+\infty \ \textit{ for } \ t>0,$$

*and*

$$w(t)\equiv\int\limits_t^{+\infty} w_0^{-\frac{\lambda_2}{\nu}}(s)g_2(s)\,ds<+\infty, \quad \int\limits_t^{+\infty} g_1(s)w^{\nu_0}(s)\,ds<+\infty \ \textit{ for } \ t>0,$$

*then system* (1) *has at least one vanishing at infinity positive solution.*

**Corollary 1.** *Let*

$$\liminf_{t\to+\infty}(t^{1-\alpha}g_{10}(t))>0, \quad \limsup_{t\to+\infty}(t^{1-\alpha}g_1(t))<+\infty, \tag{5}$$

$$\liminf_{t\to+\infty}(t^{\beta}g_{20}(t))>0, \quad \limsup_{t\to+\infty}(t^{\beta}g_2(t))<+\infty, \tag{6}$$

*where $\alpha$ and $\beta$ are nonnegative constants. Then for the existence of at least one vanishing at infinity positive solution of system* (1) *it is necessary and sufficient that*

$$\beta>\frac{1+\mu_2}{\mu_1}\,\alpha+1.$$

If
$$\int\limits_t^{+\infty} g_2(s)\,ds < +\infty \ \text{ for } \ t>0, \quad \int\limits_0^{+\infty} g_1(s)\bigg(\int\limits_s^{+\infty} g_2(\tau)\,d\tau\bigg)\,ds < +\infty, \tag{7}$$
then on the set $\mathbb{R}_+ \times \mathbb{R}_{0+}$ we put
$$v_0(t,x) = \bigg[x^\nu + \nu(1+\mu_2)^{\nu_0}\int\limits_t^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0} ds\bigg]^{\frac{1}{\nu}},$$
$$v_1(t,x) = \bigg[x^{1+\lambda_1} + (1+\lambda_1)\int\limits_t^{+\infty} \nu^{\mu_1}(s,x)g_1(s)\,ds\bigg]^{\frac{1}{1+\lambda_1}},$$
where
$$v(t,x) = \bigg[(1+\mu_2)\int\limits_t^{+\infty} \nu_0^{-\lambda_2}(s,x)g_2(s)\,ds\bigg]^{\frac{1}{1+\mu_2}} \ \text{ for } \ t>0, \ \ x>0.$$

**Theorem 2.** *Let either*
$$\int\limits_t^{+\infty} g_{10}(s)\,ds = +\infty \ \textit{ for } \ t>0,$$
*or*
$$\int\limits_t^{+\infty} g_{20}(s)\,ds < +\infty \ \textit{ for } \ t>0, \quad \int\limits_0^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0} ds < +\infty$$
*and*
$$\varphi(v_0(\,\cdot\,;0)) > c.$$
*Then problem* (1), (2) *has no solution.*

**Theorem 3.** *Let along with* (7) *the conditions*
$$\lim_{x\to+\infty} \varphi(x) = +\infty$$
*and*
$$\inf\big\{\varphi(v_1(\,\cdot\,;x)) : \ x>0\big\} < c$$
*be satisfied. Then problem* (1), (2) *has at least one positive solution.*

Theorems 2 and 3 yield the following propositions.

**Corollary 2.** *Let*
$$\int\limits_{t_0}^{+\infty} g_{10}(s)\,ds = +\infty,$$
*where $t_0 > 0$. Then for the existence of at least one positive solution of problem* (1), (2) *for every sufficiently large $c$, it is necessary and sufficient that*
$$\int\limits_t^{+\infty} g_{20}(s)\,ds < +\infty \ \textit{ for } \ t>0, \quad \int\limits_0^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0} ds < +\infty.$$

**Corollary 3.** *Let conditions* (5) *and* (6) *hold, where* $\alpha$ *and* $\beta$ *are nonnegative constants. Then for the existence of at least one positive solution of problem* (1), (2) *for every sufficiently large* $c$*, it is necessary and sufficient that*

$$\beta > \frac{1+\mu_2}{\mu_1}\,\alpha + 1.$$

Finally we note that the proofs of the above formulated theorems are based on the results of [8].

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# Stability of Trivial Invariant Torus of Dynamical System

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## 1 Introduction and Preliminaries

We consider an autonomous system of differential equations

$$\dot{x} = F(x), \quad x \in \mathbb{R}^k \tag{1}$$

that possesses $m$-dimensional invariant toroidal manifold $\mathcal{T}_m$. For a comprehensive description of the dynamics in the vicinity of invariant toroidal manifold it is convenient to introduce so-called local coordinates $(\varphi_1, \dots, \varphi_m, h_1, \dots, h_n)$, $n = k - m$, where $\varphi = (\varphi_1, \dots, \varphi_m)$ is a point on torus $\mathcal{T}_m$ and $h = (h_1, \dots, h_n)$ is from Euclidean space in transversal direction to the torus. The change of variables is performed in such a way that the invariant toroidal manifold gets a representation $h = 0, \varphi \in \mathcal{T}_m$ in new coordinates. System (1) transforms into

$$\dot{\varphi} = a(\varphi, h), \quad \dot{h} = f(\varphi, h) \tag{2}$$

with $f(\varphi, 0) \equiv 0$. The last condition guarantees the existence of invariant toroidal set $h = 0, \varphi \in \mathcal{T}_m$ that is called trivial.

Problems of the existence, stability and an approximate construction of non-trivial invariant toroidal manifolds for system (2) are treated carefully in [10]. The central object of investigation is a so-called linear extension of dynamical system on torus

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = A(\varphi)h + f(\varphi), \tag{3}$$

where $a \in C_{Lip}(\mathcal{T}_m)$ is an $m$-dimensional vector function, $A, f \in C(\mathcal{T}_m)$ are $n \times n$ square matrix and $n$-dimensional vector function respectively; $C(\mathcal{T}_m)$ stands for a space of continuous $2\pi$-periodic with respect to each of the variables $\varphi_j$, $j = 1, \dots, m$ functions defined on the surface of the torus $\mathcal{T}_m$. The main ingredient in the investigation of the existence and stability analysis of non-trivial invariant tori of system (3) is Green function introduced in [8]. The existence of such a function is sufficient for the existence of non-trivial invariant torus for system (3). Later a numerous of works by different authors have developed and extended this approach to a broad classes of equations including impulsive [4, 3], stochastic [11] and infinite-dimensional [7] and equations with delay [9]. This method of investigation got a *Green-Samoilenko function method* name [7].

A deep connection of the existence of invariant tori and quadratic functions was explored in [1]. A Lyapunov-like approach was proposed for stability analysis of invariant tori and their robustness properties characterization. A question of the preservation of invariant tori under perturbations of the right-hand side was also considered. It has been proven that a sufficiently small perturbations do not ruin the invariant torus, which enables it to become a convenient object for investigations of quasi-periodic motions of dynamical system. As it is widely known, quasi-periodic solution may be easily transformed into a periodic one by a small perturbation of right-hand side. The existence of invariant tori that is a carrier of quasi-periodic trajectories ensures the existence of multi-frequency oscillations in the system. It makes this theory well-adapted for the applications in electronics and radiophysics with complex oscillatory processes of several frequencies.

## 2 Motivation

In this paper we are interested in stability analysis of trivial invariant torus of the system

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = A(\varphi, h)h, \tag{4}$$

where $\varphi \in \mathcal{T}_m$, $h \in \mathbb{R}^n$.

We begin with a simple example that demonstrates that the existing theorems for stability analysis of invariant tori are too restrictive and set too severe constraints on the system. On the other hand, we propose relaxed conditions that are applicable to a wide class of equations and provide a deeper understanding of the processes in a vicinity of invariant set.

**Example 1.** Consider system (4) with $a(\varphi) = \begin{pmatrix} -\sin^2 \frac{\varphi_1}{2} \\ \omega \end{pmatrix}$ and $A(\varphi, h) = -1 + \lambda \sin \varphi_1$, where $\lambda > 0$ is an arbitrary fixed constant value from $\mathbb{R}$.

System from the example may be analyzed in two steps:

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = -1 \cdot h \implies \dot{\varphi} = a(\varphi), \quad \dot{h} = (-1 + \lambda \sin \varphi_1)h, \tag{5}$$

where $\lambda \cos \varphi$ is considered as a perturbation term. The fundamental matrix $\Omega^t_\tau(\varphi)$ of the system $\dot{h} = -h$ has a form $\Omega^t_\tau(\varphi) = e^{-(t-\tau)}$. It means that system $\dot{\varphi} = a(\varphi)$, $\dot{h} = -h$ has an exponentially stable trivial invariant torus $h = 0$, $\varphi \in \mathcal{T}_m$. The previously known perturbation theorems guarantee stability of trivial torus of system (5) in the case of a sufficiently small perturbation term, e.g. there exists $\delta > 0$ such that for any perturbation with $\|\lambda \cos \phi\| \leq \delta$ system (5) has an exponentially stable trivial invariant toroidal manifold. In other words, a stability of manifold is guaranteed only for a sufficiently small constant $\lambda$. However a numerical simulations provides an intuition that the trivial torus is actually asymptotically stable even for large enough values of the constant parameter $\lambda$. Indeed, for the cases of $\lambda = 1$, $\lambda = 10$, and $\lambda = 100$ a qualitative behavior of solutions to system (5) coincide and all the trajectories tend to the invariant set as time $t \to \infty$. This fact originate a hypothesis that a smallness of a perturbation term is too severe constraint and can be relaxed. This is the main motivation for this research.

Further propositions deeply rely on the results from [5, 6].

## 3 Results

Denoting $A(\varphi, h) := A(\varphi) + A_1(\varphi)$, system (4) can be represented in the following form

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dh}{dt} = [A(\varphi) + A_1(\varphi, h)]h, \tag{6}$$

where $A_1$ is a perturbation term from $C(\mathcal{T}_m, \mathbb{R}^n)$, $\|h\| \leq d \in \mathbb{R}_+$. Let $\mathcal{H}^t_\tau(\varphi)$ be a fundamental matrix of the unperturbed system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dh}{dt} = A(\varphi)h,$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter and turns into an identical matrix when $t = \tau$, e.g. $\mathcal{H}^\tau_\tau(\varphi) \equiv I$.

**Definition 1** ([2])**.** A point $\varphi$ is called wandering if there exist its neighbourhood $U(\varphi)$ and a positive number $T > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = 0 \ \text{ for } \ t \geq T.$$

Let $W$ be a set of all wandering points of dynamical system and $\Omega = \mathcal{T}_m \setminus W$ be a set of nonwandering points. From the compactness of a torus it follows that the set $\Omega$ is nonempty and compact. Since function $A_1(\varphi, h)$ is continuous on a compact set, there exists

$$\sup_{\varphi\in\Omega, \|h\|\le d} A_1(\varphi, h) = \widetilde{a}_1.$$

The following proposition sets constraints on the perturbation term in order to guarantee the exponential stability of the trivial invariant torus $h = 0, \varphi \in \mathcal{T}_m$. These constraints are relaxed comparing to the previously known [1, 10] and demand the perturbation to be small in non-wandering set of dynamical system $\Omega$, but not on the whole surface of the torus $\mathcal{T}_m$.

**Theorem 1.** *Let the fundamental matrix $\mathcal{H}_\tau^t(\varphi)$ satisfy the estimate*

$$\| \mathcal{H}_\tau^t(\varphi)\| \le K e^{-\gamma(t-\tau)} \ \text{ for } \ t \ge \tau$$

*with some $K \ge 1$, $\gamma > 0$. Then if the following condition holds*

$$K\widetilde{a}_1 < \gamma,$$

*then system (6) has an exponentially stable trivial invariant toroidal manifold.*

**Example 2** (revisited)**.** The dynamical system on two-dimensional torus

$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -\sin^2 \dfrac{\varphi_1}{2} \\ \omega \end{pmatrix}$$

has a very simple structure of limit sets and recurrent trajectories. In particular a non-wandering set $\Omega$ consists of only one meridian $\varphi_1 = 0$:

$$\Omega = \big\{\varphi \in \mathcal{T}_2 : \varphi_1 = 0, \varphi_2 \in \mathcal{T}_1\big\}.$$

A point that is starting on meridian spinning with constant velocity $\omega$. All other trajectories tend to $\Omega$ by spirals. The estimate for the perturbation term is

$$\sup_{\varphi\in\Omega,\ \|x\|\le d} \lambda \sin \varphi_1 = \lambda \sin 0 = 0.$$

It means that the system from the example and the perturbation term satisfy the conditions of Theorem 1 and the trivial invariant tori of system (4) with $a(\varphi) = \begin{pmatrix} -\sin^2 \frac{\varphi_1}{2} \\ \omega \end{pmatrix}$ and $A(\varphi, h) = -1 + \lambda \sin \varphi_1$ is exponentially stable for an arbitrary fixed constant $\lambda$.

# 4 Discussion

We have proved that it is sufficient for a perturbation term to be small only in a non-wandering set $\Omega$ in order to preserve an exponential stability of a trivial invariant torus of a perturbed system. New theorem allows to investigate qualitative behavior of solutions of a class of nonlinear systems that have a simple structure of limit sets and recurrent trajectories. The constraints of Theorem 1 are less restrictive than of the previously known ones. However it is worth to note that if the first equation of the unperturbed system is $\dot{\varphi} = \omega = const$, that is very frequent in applications, then its non-wandering set $\Omega$ coincides with a whole torus and Theorem 1 has no advantages compared to results from [1, 10].

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# Global Attractors for Some Class of Discontinuous Dynamical Systems

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An autonomous evolution system is called discontinuous (or impulsive) dynamical system (DS) if its trajectories have jumps at moments of intersection with certain surface of the phase space [8]. Some aspects of qualitative behavior of impulsive finite-dimensional DS have been studied in recent years [8, 5, 4]. For infinite-dimensional dissipative DS one of the most important problems is investigation of global attractor [9]. This approach has been applied to impulsive DS in [1, 2, 7]. In this paper we propose a new concept of global attractor for discontinuous DS, which is based on the definition of uniform attractor for non-autonomous DS [3], in particular, for systems with impulsive perturbation at fixed moment of time [6]. Using this concept, we investigate asymptotic behavior of a wide class of dissipative infinite-dimensional impulsive DS, generated by parabolic equation with impulsive perturbations at non-fixed moments of time. We consider existence and non-existence results in the case of linear equation and we also give effective sufficient conditions for existence of global attractor in the case of weakly nonlinear parabolic equation.

Let $(X,\rho)$ be a metric space, $P(X)$ $(\beta(X))$ be a set of all nonempty (nonempty bounded) subsets of $X$,

$$\operatorname{dist}_X(A,B) := \sup_{y\in A}\inf_{z\in B}\rho(y,z).$$

A pair $(X,G)$ is called dynamical system (DS) if

$$\forall\, x\in X \ \ G(0,x)=x, \quad G(t+s,x)=G(t,G(s,x)) \ \ \forall\, t,s\geq 0.$$

We assume no conditions of continuity for the map $x\to G(t,x)$.

**Definition.** A set $\Theta\subset X$ is called global attractor of DS $(X,G)$ if

1) $\Theta$ is a compact set;

2) $\Theta$ is uniformly attracting set, i.e.,

$$\forall\, B\in\beta(X) \ \ \operatorname{dist}_X(G(t,B),\Theta)\to 0, \ \ t\to\infty;$$

3) $\Theta$ is minimal among closed sets satisfying 2).

**Theorem 1.** *Suppose DS $(X,G)$ satisfies dissipativity condition:*

$$\exists\, B_0\in\beta(X) \ \ \forall\, B\in\beta(X) \ \ \exists\, T=T(B) \ \ \forall\, t\geq T \ \ G(t,B)\subset B_0.$$

*Then DS $(X,G)$ has global attractor if and only if*

$$\forall\,\{x_n\}\in\beta(X) \ \ \forall\,\{t_n\nearrow\infty\} \ \ \textit{sequence} \ \ \{G(t_n,x_n)\} \ \ \textit{is precompact}.$$

Now we construct DS, generated by impulsive system. It is called impulsive DS and consists of classical (continuous) DS $(X,V)$, a closed set $M\subset X$ (impulsive set) and a map $I: M\to X$ (impulsive map). The phase point $x(t)$ moves along trajectories of DS $(X,V)$ and when it reaches the set $M$ at the moment $\tau$, it jumps to a new position $Ix(\tau)$.

We assume the following conditions hold:

$$M\cap I(M)=\varnothing,\ \ \forall\,x\in M\ \ \exists\,\tau=\tau(x)>0\ \ \forall\,t\in(0,\tau)\ \ V(t,x)\notin M.$$

We define $\forall\,x\in M\ \ Ix=x^+$, $\forall\,x\in X\ \ M^+(x)=(\bigcup\limits_{t>0}V(t,x))\cap M$.

It follows from continuity of $V$ that if $M^+(x)\neq\varnothing$, then there exists $s:=\phi(x)>0$ such that

$$\forall\,t\in(0,s)\ \ V(t,x)\notin M,\ \ V(s,x)\in M. \tag{1}$$

So, for fixed $x\in X$ we have:

- if $M^+(x)=\varnothing$, then $\widetilde{V}(t,x)=V(t,x)\ \ \forall\,t\geq 0$;
- if $M^+(x)\neq\varnothing$, then for $s_0=\phi(x)$, $x_1=V(s_0,x)$

$$\widetilde{V}(t,x)=\begin{cases}V(t,x), & 0\leq t<s_0,\\ x_1^+, & t=s_0;\end{cases}$$

- if $M^+(x_1^+)=\varnothing$, then $\widetilde{V}(t,x)=V(t-s_0,x_1^+)\ \forall\,t\geq s_0$;
- if $M^+(x_1^+)\neq\varnothing$, then for $s_1=\phi(x_1^+)$, $x_2=V(s_1,x_1^+)$

$$\widetilde{V}(t,x)=\begin{cases}V(t-s_0,x_1^+), & s_0\leq t<s_0+s_1,\\ x_2^+, & t=s_0+s_1,\end{cases}$$

and so on. As a result, we obtain finite or infinite number of impulsive points $\{x_n^+\}_{n\geq 1}$ and corresponding moments of time $\{s_n\}_{n\geq 0}$ such that

$$V(s_0,x)=x_1,\quad V(s_n,x_n^+)=x_{n+1},\ \ n\geq 1.$$

Let us assume the following condition holds:

$$\forall\,x\in X\ \ \widetilde{V}(t,x)\ \text{ is well-defined on }\ [0,+\infty).$$

This condition means that either the number of impulsive points is finite or $\sum\limits_{n=0}^{\infty}s_n=\infty$. Then [5] the map $\widetilde{V}:R_+\times X\mapsto X$ satisfies semigroup property and we have impulsive DS $(X,\widetilde{V})$.

We study global attractors of impulsive DS $(X,\widetilde{V})$ in the following two cases [8]:

(a) $X$ is a Banach space, $M=\{x\in X\mid\ \|x\|=a\}$, $Ix=(1+\mu)x$, where $a>0$, $\mu>0$;

(b) $X$ is a Hilbert space, $\{\psi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $H$, $M=\{x\in X\mid\ (\psi_1,x)=a\}$, and for $x=\sum\limits_{k=1}^{\infty}c_k\psi_k$, $Ix=(1+\mu)c_1\psi_1+\sum_{k=2}^{\infty}c_k\psi_k$.

At first we illustrate some interesting properties in linear case: in bounded domain $\Omega\subset R^p$, $p\geq 1$ we consider the linear problem

$$\begin{cases}\dfrac{\partial y}{\partial t}=\Delta y, & (t,x)\in(0,\infty)\times\Omega,\\ y\big|_{\partial\Omega}=0.\end{cases} \tag{2}$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be a complete and orthonormal in $L^2(\Omega)$ family of eigenfunctions of $-\Delta$, i.e., $-\Delta\psi_i=\lambda_i\psi_i$, $\psi_i\in H_0^1(\Omega)$, $0<\lambda_1\leq\lambda_2\leq\cdots$, $\lambda_i\to\infty$, $i\to\infty$.

The problem (2) in the phase space $X = L^2(\Omega)$ with a norm $\|\cdot\|$ and a scalar product $(\cdot,\cdot)$ generates classical DS $(X,V)$, where

$$y(t) = V\Big(t, \sum_{i=1}^{\infty} c_i\psi_i\Big) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t}\psi_i.$$

As $\forall\, t \ge 0$ $\|y(t)\| \le e^{-\lambda_1 t}\|y_0\|$, then DS $(X,V)$ has a trivial global attractor $\Theta = \{0\}$.

Now let us consider impulsive DS $(X,\widetilde{V})$, where

$$M = \big\{y \in X \mid \ \|y\| = \varepsilon\big\}, \ \ Iy = (1+\mu)y, \ \ \varepsilon > 0, \ \ \mu > 0. \tag{3}$$

**Lemma 1.** *For every $\varepsilon > 0$, $\mu > 0$ the problem* (2), (3) *generates dissipative impulsive DS $(X,\widetilde{V})$, which does not possess global attractor.*

Let us consider impulsive DS $(X,\widetilde{V})$, where

$$M = \{y \in X \mid \ (y,\psi_1) = a\}, \ \ I : M \mapsto X, \tag{4}$$

$$\text{for } \ y = \sum_{i=1}^{\infty} c_i\psi_i, \ \ Iy = (\mu+1)c_1\psi_1 + \sum_{i=2}^{\infty} c_i\psi_i, \ \ a > 0, \ \ \mu > 0.$$

**Lemma 2.** *For every $a > 0$, $\mu > 0$ the problem* (2), (4) *generates dissipative impulsive DS $(X,\widetilde{V})$, which has global attractor*

$$\Theta = \bigcup_{t\in[0,\ln(1+\mu)]} \big\{(1+\mu)ae^{-t}\psi_1\big\} \cup \{0\}. \tag{5}$$

From (5) we can see that $\Theta \cap M \neq \varnothing$ and $\forall\, t > 0$ $\widetilde{V}(t,\Theta) \not\subset \Theta$. But invariance property is true in the following form:

$$\forall\, t > 0 \ \ \widetilde{V}(t, \Theta\setminus M) \subset \Theta \setminus M. \tag{6}$$

The main result of the work is to prove that the statements of Lemma 2 remain true in nonlinear case.

In bounded domain $\Omega \subset R^p$, $p \ge 1$ we consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y - \varepsilon f(y), \ \ (t,x) \in (0,\infty)\times\Omega, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{7}$$

where $\varepsilon > 0$ is a small parameter, $f \in C^1(R)$, $f(0) = 0$,

$$\exists\, C > 0 \ \ \forall\, y \in R, \ \ f'(y) \ge -C, \ \ |f(y)| \le C. \tag{8}$$

Under conditions (8) for arbitrary $y_0 \in X = L^2(\Omega)$ the problem (7) has a unique solution $y_\varepsilon \in C([0,+\infty);X)$, $y_\varepsilon(0) = y_0$.

**Theorem 2.** *For every $a > 0$, $\mu > 0$ and for sufficiently small $\varepsilon > 0$ impulsive problem* (7), (4) *generates impulsive DS $(X,\widetilde{V}_\varepsilon)$, which has global attractor $\Theta_\varepsilon$ and, moreover,*

$$\operatorname{dist}(\Theta_\varepsilon,\Theta) \to 0, \ \ \varepsilon \to 0, \tag{9}$$

$$\forall\, t > 0 \ \ \widetilde{V}_\varepsilon(t, \Theta_\varepsilon \setminus M) \subset \Theta_\varepsilon \setminus M. \tag{10}$$

*Proof.* For every solution of (7) $y_\varepsilon(\cdot)$ we have

$$\forall\, t \ge 0 \ \ (y_\varepsilon(t),\psi_1) = e^{-\lambda_1 t}(y_\varepsilon(0),\psi_1) - \varepsilon \int_0^t e^{-\lambda_1(t-p)}(f(y_\varepsilon(p)),\psi_1)\,dp. \tag{11}$$

Equality (11) allows us to estimate the moments of impulsive perturbation of every trajectory of (7), (4) with the help of Implicit Function Theorem. Then we prove existence of global attractor and limit equality (9). To prove invariance property (10) we consider function $x \mapsto \phi(x)$, defined in (1), and we show its continuity on $X \setminus M$. $\square$

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# How to Construct Solutions of State-Dependent Impulsive Boundary Value Problems

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## 1 Formulation of the Problem

We consider the nonlinear system of differential equations

$$u'(t) = f(t, u(t)), \ \text{ a.e. } t \in [a, b] \subset \mathbb{R}, \tag{1}$$

with continuous $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Equation (1) is subject to the *state-dependent* impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)) \ \text{ for such } t \text{ that } \ g(t, u(t-)) = 0. \tag{2}$$

Here $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous, and the impulse instants $t \in (a, b)$ in (2) are unknown. These instants are called state-dependent because they depend on a solution $u$ through the equation $g(t, u(t-)) = 0$. Impulsive problem (1), (2) is investigated together with the linear boundary condition

$$Au(a) + Cu(b) = d, \tag{3}$$

where $d$ is a constant vector, and $A$, $C$ are constant (possibly singular) matrices satisfying the condition $\operatorname{rank}[A, C] = n$.

A left-continuous vector-function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (1)–(3) if there exist $p \in \mathbb{N}$ and $t_i \in (a, b)$, $i = 1, \dots, p$, such that:

- $a < t_1 < t_2 < \dots < t_p < b$,
- the restrictions $u|_{[a,t_1]}, u|_{(t_1,t_2]}, \dots, u|_{(t_p,b]}$ have continuous derivatives,
- $u$ satisfies (1) for $t \in [a, b]$, $t \neq t_i$, $i = 1, \dots, p$,
- $u$ satisfies (2) for $t = t_i$, i.e. $u(t_i+) - u(t_i) = \gamma(u(t_i))$, $g(t_i, u(t_i)) = 0$, $i = 1, \dots, p$,
- $u$ fufils the boundary conditions (3).

The set

$$G = \big\{(t, x) \in [a, b] \times \mathbb{R}^n : \ g(t, x) = 0\big\} \tag{4}$$

is called a *barrier*.

Wee see that if $u$ satisfies condition (2) for $t = t_i \in (a, b)$, then $u$ has an intersection point $(t_i, u(t_i))$ with the barrier $G$, and in addition, $u$ has a jump of the size $\gamma(u(t_i))$ at the point $t_i$.

Most of the results in the literature devoted to boundary value problems concern fixed-times impulses. A reason for the lack of results for state-dependent impulsive boundary value problems lies in the fact that state-dependent impulses significantly change properties of boundary value problems. In the book [2], state dependent impulsive boundary value problems with barriers given explicitely in the form $t = g(x)$ are investigated. The existence results in [2] are reached by means of fixed point theorems or topological degree methods. But there are no constructive numerical results for state-dependent impulsive boundary value problems in the literature. This is our main motivation for the investigation of problem (1)–(3).

We focus our attention to the case where $p=1$, that is $u$ has a unique intersection point with the barrier $G$, and then we use the technique suggested in [3], which makes it possible to discuss the solvability of problem (1)–(3) as well as to find approximate solutions. This approach is based on a construction of two simple parametrized model problems (5), (6) and (7), (8). We give conditions which guarantee that if the parameters $\tau$, $\xi$, $\lambda$, $\eta$ belong to some bounded sets, then solutions of these parametrized model problems can be obtained as limits of uniformly convergent sequences of successive approximations (10) and (12). Equations in the parametrized model problems contain functional perturbation terms which essentially depend on the parameters and which together with the original boundary conditions (3) and the barrier (4) generate the system of algebraic *determining* equations (14). Numerical values of the parameters should be found from (14) in the bounded sets mentioned above where the uniform convergence is guaranteed. A solution of problem (1)–(3) is then constructed (see (13)) by means of such solutions of problems (5), (6) and (7), (8) which have the values of parameters satisfying (14). Consequently, the infinite-dimensional problem (1)–(3) is reduced to the finite-dimensional algebraic system (14).

In practice, we investigate system (14), where explicitly determined successive approximations are written instead of their limits (cf. (16)). Then the solvability of (14) can be checked more easily and we get approximate solutions of problem (1)–(3) and error estimates using for example Maple 14. By our knowledge this is the first numerical-analytic method for this type of impulsive problems. This method can be applied on problems with linear as well as with nonlinear boundary conditions.

## 2 Construction of Solutions

Choose a compact convex set $\Omega_a\subset\mathbb{R}^n$ and put $\Omega_b=\{x+\gamma(x):x\in\Omega_a\}$. Consider a scalar parameter $\tau\in(a,b)$ together with vector parameters $\xi,\lambda\in\Omega_a$ and $\eta\in\Omega_b$. Instead of the impulsive boundary value problem (1)–(3) we study two auxiliary parametrized boundary value problems on the intervals $[a,\tau]$ and $[\tau,b]$, respectively:

$$x'(t)=f(t,x(t))+\frac{1}{\tau-a}\Big(\lambda-\xi-\int\limits_a^\tau f(s,x(s))\,\mathrm{d}s\Big), \tag{5}$$

$$x(a)=\xi,\quad x(\tau)=\lambda, \tag{6}$$

and

$$y'(t)=f(t,y(t))+\frac{1}{b-\tau}\Big(\eta-(\lambda+\gamma(\lambda))-\int\limits_\tau^b f(s,y(s))\,\mathrm{d}s\Big), \tag{7}$$

$$y(\tau)=\lambda+\gamma(\lambda),\quad y(b)=\eta. \tag{8}$$

I. Let us connect problem (5), (6) with the parametrized sequence of functions

$$x_0(t;\tau,\xi,\lambda)=\Big(1-\frac{t-a}{\tau-a}\Big)\xi+\frac{t-a}{\tau-a}\,\lambda,\ \ t\in[a,\tau], \tag{9}$$

$$\begin{aligned}x_m(t;\tau,\xi,\lambda)&=\xi+\int\limits_a^t f\big(s,x_{m-1}(s;\tau,\xi,\lambda)\big)\,\mathrm{d}s-\\&-\frac{t-a}{\tau-a}\int\limits_a^\tau f\big(s,x_{m-1}(s;\tau,\xi,\lambda)\big)\,\mathrm{d}s+\frac{t-a}{\tau-a}\,(\lambda-\xi),\ \ t\in[a,\tau],\ \ m\in\mathbb{N}.\end{aligned} \tag{10}$$

II. Let us connect problem (7), (8) with the parametrized sequence of functions

$$y_0(t;\tau,\lambda,\eta)=\Big(1-\frac{t-\tau}{b-\tau}\Big)(\lambda+\gamma(\lambda))+\frac{t-\tau}{b-\tau}\,\eta,\ \ t\in[\tau,b], \tag{11}$$

$$
\begin{aligned}
y_m(t;\tau,\lambda,\eta) = (\lambda+\gamma(\lambda)) + \int\limits_{\tau}^{t} f\big(s, y_{m-1}(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s - \\
- \frac{t-\tau}{b-\tau} \int\limits_{\tau}^{b} f\big(s, y_{m-1}(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s + \\
+ \frac{t-\tau}{b-\tau}\big(\eta - (\lambda+\gamma(\lambda))\big), \ \ t\in[\tau,b], \ \ m\in\mathbb{N}.
\end{aligned} \tag{12}
$$

Choose $\rho \in \mathbb{R}^n$ and assume that $\mathcal{O}_a \subset \mathbb{R}^n$ and $\mathcal{O}_b \subset \mathbb{R}^n$ are componentwise neighbourhoods of $\Omega_a$ and $\Omega_b$, respectively. We have proved that if $f$ fulfils the Lipschitz conditions $|f(t,x) - f(t,y)| \le K|x-y|$ on $\mathcal{O}_a$ and $\mathcal{O}_b$ with a sufficiently large vector $\rho$ and with a sufficiently small matrix $K$, then

$$
\lim_{m\to\infty} x_m(t;\tau,\xi,\lambda) = x_\infty(t;\tau,\xi,\lambda) \ \text{ uniformly on } \ [a,\tau],
$$

and

$$
\lim_{m\to\infty} y_m(t;\tau,\lambda,\eta) = y_\infty(t;\tau,\lambda,\eta) \ \text{ uniformly on } \ [\tau,b].
$$

More precisely, on $\mathcal{O}_a$:

$$
\rho \ge \frac{b-a}{4}\,\delta_{\mathcal{O}_a}(f), \quad r(K) < \frac{10}{3(b-a)}\,,
$$

where

$$
\delta_{\mathcal{O}_a}(f) =: \max_{[a,b]\times\mathcal{O}_a} f(t,x) - \min_{[a,b]\times\mathcal{O}_a} f(t,x),
$$

and $r(K)$ is the spectral radius of $K$. Similarly, on $\mathcal{O}_b$.

Further, we have proved that for each $\tau\in(a,b)$, $\xi,\lambda\in\Omega_a$, the vector function $x_\infty(t;\tau,\xi,\lambda)$ is a unique solution of problem (5), (6) and that for each $\tau\in(a,b)$, $\lambda\in\Omega_a$, $\eta\in\Omega_b$, the vector function $y_\infty(t;\tau,\lambda,\eta)$ is a unique solution of problem (7), (8).

Finally, we have found such values of the parameters $\tau$, $\xi$, $\lambda$, $\eta$ that the solution $u$ of (1)–(3) can be written in the form

$$
u(t) = \begin{cases} x_\infty(t;\tau,\xi,\lambda) & \text{if } \ t\in[a,\tau], \\ y_\infty(t;\tau,\lambda,\eta) & \text{if } \ t\in(\tau,b]. \end{cases} \tag{13}
$$

It turned out that such parameters $\tau,\xi,\lambda,\eta$ fulfil the system of algebraic “determining” equations

$$
\begin{cases}
\lambda - \xi - \displaystyle\int\limits_{a}^{\tau} f\big(s, x_\infty(s;\tau,\xi,\lambda)\big)\,\mathrm{d}s = 0, \\
\big(\eta - (\lambda+\gamma(\lambda))\big) - \displaystyle\int\limits_{\tau}^{b} f\big(s, y_\infty(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s = 0, \\
A\xi + C\eta = d, \\
g(\tau,\lambda) = 0,
\end{cases} \tag{14}
$$

and in addition,

$$
g\big(t, y_\infty(t;\tau,\lambda,\eta)\big) \ne 0, \ \ t\in(\tau,b]. \tag{15}
$$

The solvability of the determining system (14) can be established by studying its approximate version

$$\begin{cases} \lambda - \xi - \displaystyle\int_a^\tau f\big(s, x_m(s; \tau, \xi, \lambda)\big)\,\mathrm{d}s = 0, \\ \big(\eta - (\lambda + \gamma(\lambda))\big) - \displaystyle\int_\tau^b f\big(s, y_m(s; \tau, \lambda, \eta)\big)\,\mathrm{d}s = 0, \\ A\tau + C\eta = d, \\ g(\tau, \lambda) = 0, \end{cases} \tag{16}$$

with

$$g\big(t, y_m(t; \tau, \lambda, \eta)\big) \neq 0, \quad t \in (\tau, b], \tag{17}$$

which can be constructed explicitely for a fixed $m$. System (16) can be solved, for example, by Maple 14. If the quartet $(\widehat{\tau}, \widehat{\xi}, \widehat{\lambda}, \widehat{\eta}) \in (a, b) \times \Omega_a \times \Omega_a \times \Omega_b$ is a root of system (16) and inequality (17) holds, then $\widehat{\xi}$ is an approximation of the inital value $u(a)$ of the solution $u$ of problem (1)–(3), $\widehat{\tau}$ is an approximation of the impulse point $\tau$ of $u$, $\widehat{\lambda}$ is an approximation of $u(\tau)$ and $\widehat{\lambda} + \gamma(\widehat{\lambda})$ is an approximation of $u(\tau+)$.

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# On the Existence of a Special Type Integral Manifold of a Quasilinear Differential System

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Let
$$G(\varepsilon_0) = \big\{t, \varepsilon : \ t \in \mathbf{R}, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 \in \mathbf{R}^+\big\}.$$

**Definition 1.** We say that a function $f(t, \varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$ if:

1) $f : G(\varepsilon_0) \to \mathbf{C}$;

2) $f(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect to $t$;

3) $d^k f(t, \varepsilon)/dt^k = \varepsilon^k f_k^*(t, \varepsilon)$ $(0 \le k \le m)$,
$$\|f\|_m \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t, \varepsilon)| < +\infty.$$

**Definition 2.** We say that a function $f(t, \varepsilon, \theta)$ belongs to the class $F^\theta_{m,\infty}(\varepsilon_0)$ $(m \in \mathbf{N} \cup \{0\})$ if this function can be represented as
$$f(t, \varepsilon, \theta) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) e^{in\theta},$$

and

1) $f_n(t, \varepsilon) \in S_m(\varepsilon_0)$, $\theta \in \mathbf{R}$;

2)
$$\|f_0\|_m + \sum_{\substack{n=-\infty \\ (n\ne 0)}}^{\infty} |n|^l \cdot \|f_n\|_m < +\infty \ \ (l = 0, 1, 2, \dots).$$

If the function $f(t, \varepsilon, \theta)$ is real, then $f_{-n}(t, \varepsilon) = \overline{f_n(t, \varepsilon)}$.
We denote
$$\|f\|_{m,\theta} = \sum_{n=-\infty}^{\infty} \|f_n\|_m.$$

For any $f \in F^\theta_{m,\infty}(\varepsilon_0)$, we introduce the linear operators:
$$\Gamma_n[f(t, \varepsilon, \theta)] = \frac{1}{2\pi} \int\limits_0^{2\pi} f(t, \varepsilon, \theta) e^{-in\theta}\, d\theta, \ \ n \in \mathbf{Z},$$

in particular,

$$\Gamma_0[f(t,\varepsilon,\theta)] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\,d\theta,$$
$$I[f(t,\varepsilon,\theta)] = \sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty} \frac{\Gamma_n[f(t,\varepsilon,\theta)]}{in}\,e^{in\theta}.$$

We note some properties of the functions of the class $F^{\theta}_{m,\infty}(\varepsilon_0)$. Let $u(t,\varepsilon,\theta)$, $v(t,\varepsilon,\theta)$ belongs to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$, and $k = const$.

1) $u(t,\varepsilon,\theta)$ is $2\pi$-periodic with respect to $\theta$.

2)
$$\frac{\partial^l u(t,\varepsilon,\theta)}{\partial\theta^l} \in F^{\theta}_{m,\infty}(\varepsilon_0) \;\; (l = 0,1,2,\dots),$$
$$\frac{\partial^k u(t,\varepsilon,\theta)}{\partial t^k} \in F^{\theta}_{k-1,\infty}(\varepsilon_0) \;\; (k = 1,\dots,m).$$

3) $\Gamma_n[u(t,\varepsilon,\theta)] \in S_m(\varepsilon_0)$ $(n \in \mathbf{Z})$.

4)
$$I[u(t,\varepsilon,\theta)] \in F^{\theta}_{m,\infty}(\varepsilon_0), \quad I\Big[\frac{\partial u(t,\varepsilon,\theta)}{\partial\theta}\Big] = u(t,\varepsilon,\theta) - \Gamma_0[u(t,\varepsilon,\theta)] \in F^{\theta}_{m,\infty}(\varepsilon_0).$$

5) $\|ku\|_{m,\theta} = |k|\cdot\|u\|_{m,\theta}$.

6) $\|u+v\|_{m,\theta} \le \|u\|_{m,\theta} + \|v\|_{m,\theta}$.

7)
$$\|u\|_{m,\theta} = \sum_{k=0}^{m} \Big\|\frac{1}{\varepsilon^k}\,\frac{\partial^k u}{\partial t^k}\Big\|_{0,\theta}.$$

8)
$$\|uv\|_{m,\theta} \le 2^m \|u\|_{m,\theta}\cdot\|v\|_{m,\theta}.$$

9) If $u$, $v$ are real, then $u(t,\varepsilon,\theta+v(t,\varepsilon,\theta)) \in F^{\theta}_{m,\infty}(\varepsilon_0)$.

10) The chains of includes are true:
$$F^{\theta}_{0,\infty}(\varepsilon_0) \supset F^{\theta}_{1,\infty}(\varepsilon_0) \supset \cdots \supset F^{\theta}_{m,\infty}(\varepsilon_0), \quad S_0(\varepsilon_0) \supset S_1(\varepsilon_0) \supset \cdots \supset S_m(\varepsilon_0).$$

**Definition 3.** We say that a vector $f = \operatorname{colon}(f_1,\dots,f_N)$ belongs to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$) if $f_j \in F^{\theta}_{m,\infty}(\varepsilon_0)$ (relatively, $f_j \in S_m(\varepsilon_0)$) $(j = 1,\dots,N)$.

**Definition 4.** We say that a matrix $(a_{jk})_{j,k=\overline{1,N}}$ belongs to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$) if $a_{j,k} \in F^{\theta}_{m,\infty}(\varepsilon_0)$ (relatively, $a_{j,k} \in S_m(\varepsilon_0)$) $(j,k = 1,\dots,N)$.

Consider the system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon)+\mu P(t,\varepsilon,\theta))x+f(t,\varepsilon,\theta),\\ \frac{d\theta}{dt} &= \omega(t,\varepsilon)+\mu a(t,\varepsilon,\theta), \end{aligned} \tag{1}$$

where $(t,\varepsilon)\in G(\varepsilon_0)$, $x=\operatorname{colon}(x_1,\dots,x_N)$, $\Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_N)\in S_m(\varepsilon_0)$, $P=(p_{jk})_{j,k=\overline{1,N}}\in F^{\theta}_{m,\infty}(\varepsilon_0)$, scalar real functions $\omega\in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\omega>0$, $a\in F^{\theta}_{m,\infty}(\varepsilon_0)$, $\mu\in(0,\mu_0)\subset\mathbf{R}^+$.

We study the problem of the conditions of existence of integral manifold $x(t,\varepsilon,\theta,\mu)$ of the system (1), belongs to the class $F^{\theta}_{m^*,\infty}(\varepsilon_0)$, where $m^*\le m$.

**Lemma 1.** *There exists $\mu_1\in(0,\mu_0)$ such that for all $\mu\in(0,\mu_1)$ there exists the real reversible transformation*

$$\theta=\varphi+\mu v(t,\varepsilon,\varphi,\mu), \tag{2}$$

*where $v\in F^{\varphi}_{m,\infty}(\varepsilon_0)$, reducing the system (1) to the kind*

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon)+\mu Q(t,\varepsilon,\varphi,\mu))x+g(t,\varepsilon,\varphi,\mu),\\ \frac{d\varphi}{dt} &= \omega(t,\varepsilon)+\mu b(t,\varepsilon,\mu)+\mu\varepsilon\beta(t,\varepsilon,\varphi,\mu), \end{aligned} \tag{3}$$

*where $Q=P(t,\varepsilon,\varphi+\mu v(t,\varepsilon,\varphi,\mu))\in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $g=f(t,\varepsilon,\varphi+\mu v(t,\varepsilon,\varphi,\mu))\in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $b\in S_m(\varepsilon_0)$, $\beta\in F^{\varphi}_{m-1,\infty}(\varepsilon_0)$.*

**Lemma 2.** *There exists $\mu_2\in(0,\mu_1)$ such that for all $\mu\in(0,\mu_2)$ there exists the chain of reversible transformations of kind*

$$\varphi=\psi_1+\mu\varepsilon w_1(t,\varepsilon,\psi_1,\mu), \tag{4}$$

$$\psi_1=\psi_2+\mu\varepsilon^2 w_2(t,\varepsilon,\psi_2,\mu), \tag{5}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\psi_{m_1-2}=\psi_{m_1-1}+\mu\varepsilon^{m_1-1}w_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \tag{6}$$

*where $m_1<m$, $w_k\in F^{\psi_k}_{m-k,\infty}(\varepsilon_0)$ $(k=1,\dots,m_1-1)$, reducing the system (3) to the kind:*

$$\begin{aligned} \frac{dx}{dt} &= \big(\Lambda(t,\varepsilon)+\mu R_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu)\big)x+h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu),\\ \frac{d\psi_{m_1-1}}{dt} &= \omega(t,\varepsilon)+\mu b(t,\varepsilon,\mu)+\mu\sum_{l=1}^{m_1-1}\varepsilon^k\beta_k(t,\varepsilon,\mu)+\mu\varepsilon^{m_1}\widetilde{\beta}_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \end{aligned} \tag{7}$$

*where $R_{m_1-1}\in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $h_{m_1-1}\in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $\beta_k\in S_{m-k}(\varepsilon_0)$, $\widetilde{\beta}_{m_1-1}\in F^{\psi_{m_1-1}}_{m-m_1,\infty}(\varepsilon_0)$ $(k=1,\dots,m_1-1)$.*

**Theorem.** *Let the elements $\lambda_j(t,\varepsilon)$ $(j=1,\dots,N)$ of matrix $\Lambda(t,\varepsilon)$ in system (1) be such that*

$$\inf_{G(\varepsilon_0)}|\operatorname{Re}\lambda_j(t,\varepsilon)|\ge\gamma>0\ \ (j=1,\dots,N).$$

*Then there exists $\mu^*\in(0,\mu_0)$ such that for all $\mu\in(0,\mu^*)$ the system (1) has the integral manifold $\widetilde{x}(t,\varepsilon,\theta,\mu)\in F^{\theta}_{m_1,\infty}(\varepsilon_0)$, where $2m_1\le m$ $(m_1\in\mathbf{N}\cup\{0\})$.*

*Proof.* Based on Lemmas 1, 2, we reduce the system (1) to the kind (7). We denote

$$\psi = \psi_{m_1-1},$$
$$R(t,\varepsilon,\psi,\mu) = R_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \quad h(t,\varepsilon,\psi,\mu) = h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu),$$
$$\omega_1(t,\varepsilon,\mu) = \omega(t,\varepsilon_0) + \mu b(t,\varepsilon,\mu) + \sum_{l=1}^{m_1-1} \varepsilon_k\beta_k(t,\varepsilon,\mu) + \varepsilon^{m_1}\widetilde{\beta}_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu).$$

Based on condition of Theorem and property 10) of the functions of class $F^{\theta}_{m,\infty}(\varepsilon_0)$, we can state that $R(t,\varepsilon,\psi,\mu)$, $h(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$, $\omega_1(t,\varepsilon,\mu) \in S_{m_1}(\varepsilon_0)$. Then we write the system (7) in kind

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu R(t,\varepsilon,\psi,\mu))x + h(t,\varepsilon,\psi,\mu), \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{8}$$

With the system (8) consider the system

$$\begin{aligned} \frac{dx_0}{dt} &= \Lambda(t,\varepsilon)x_0 + h(t,\varepsilon,\psi,\mu), \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{9}$$

Based on the results [1] and condition of Theorem, we can state that the system (9) has the integral manifold $x_0(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$. And there exists $K \in (0,+\infty)$ such that

$$\|x_0\|_{m_1,\psi} \le K\|h\|_{m_1,\psi}. \tag{10}$$

We seek the integral manifold of system (8) by the method of succesive approximations, defining as an initial approximation $x_0$, and the subsequents approximations defining from the systems:

$$\begin{aligned} \frac{dx_{s+1}}{dt} &= \Lambda(t,\varepsilon)x_{s+1} + h(t,\varepsilon,\psi,\mu) + \mu R(t,\varepsilon,\psi,\mu)x_s, \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu), \quad s = 0,1,2,\dots . \end{aligned} \tag{11}$$

Based on inequality (10) and using the ordinary technicue of the contraction mapping principle [2], it is easy to show that there exists $\mu_3 \in (0,\mu_0)$ such that for all $\mu \in (0,\mu_3)$ all approximations $x_s$ belong to the class $F^{\psi}_{m_1,\infty}(\varepsilon_0)$, and process (11) converges by the norm $\|\cdot\|_{m,\psi}$ to integral manifold $x(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$ of the system (8).

Based on the reversibility of the transformations (2), (4)–(6), we can state the existence of the integral manifold $\widetilde{x}(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m,\infty}(\varepsilon_0)$ of the system (1) for sufficiently small values $\mu$. □

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# On the Solvability of One Class of Boundary Value Problems

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The questions of determining the conditions of solvability and finding the solutions for various types of boundary value problems remain actual for a long period of time. A vast number of scientific works are devoted to the investigation of different aspects of the question under consideration. The Noetherian boundary value problems have been considered and studied in [8]. The works [1] and [4] are devoted to the study of autonomous boundary value problems.

The weakly nonlinear boundary value problems have been considered in [1, 7]. The conditions for the solvability of boundary value problems with perturbation for systems of linear differential equations of the first order have been studied in [7, 8]. The conditions of solvability of degenerated boundary value problems, bifurcations and branching of their solutions are considered in [8]. In [6], the author considers weakly perturbed boundary value problems for systems of linear differential equations of the second order for which the conditions of solvability are found.

We study a linear inhomogeneous boundary value problem with perturbation

$$(P(t)x')' - Q(t)x - \varepsilon Q_1(t)x = f(t), \ \ t \in [a,b], \tag{1}$$

$$lx(\,\cdot\,,\varepsilon) = \alpha + \varepsilon l_1 x(\,\cdot\,,\varepsilon). \tag{2}$$

Here, $[a,b]$ is a segment on which we consider the linear boundary value problem with perturbations (1), (2), $x = x(t,\varepsilon)$ – is a twice continuously differentiable unknown vector-function $x''(\,\cdot\,,\varepsilon) \in C^2([a,b]\times(0,\varepsilon_0])$. $P(t)$, $Q(t)$, $Q_1(t)$ are square matrices of dimension $n$. Elements of the matrix $P(t)$ are real, continuously differentiable on the segment $[a,b]$ functions $P(t) \in C^1([a,b])$; Elements of the matrices $Q(t)$ and $Q_1(t)$ are continuous on the segment $[a,b]$: $Q(t), Q_1(t) \in C([a,b])$. The matrix $P(t)$ is nondegenerated $\det P(t) \neq 0$. The function $f(t)$ is a continuous $n$-dimensional on the segment $[a,b]$ vector-function $f(t) \in C([a,b])$. $l$, $l_1$ are linear bounded $m$-dimensional vector-functionals defined on the space $n$-dimensional piecewise continuous vector functions $l$, $l_1 : C([a,b]) \to R^m$. $\alpha$ is an $m$-dimensional real vector $\alpha \in R^m$; $\varepsilon$ is a small nonnegative parameter.

To the boundary value problem with perturbation (1), (2) we put into correspondence the generating boundary value problem

$$\big(P(t)x'\big)' - Q(t)x = f(t), \ \ t \in [a,\ b], \tag{3}$$

$$lx(\,\cdot\,,\varepsilon) = \alpha. \tag{4}$$

The system (3) of differential equations of second order has a general solution of the type $x(t) = X(t)c + \overline{x}(t)$, $c \in R^{2n}$, where $X(t)$ is an $(n \times 2n)$-dimensions fundamental matrix of the homogeneous $(f(t) = 0)$system of second order (3) which consists of $2n$ linear independent solutions of that homogeneous system $(f(t) = 0)$ (3); The vector-function $\overline{x}(t) = \int\limits_a^b K(t,s)P^{-1}(s)f(s)\,ds$ is a partial solution of the system of differential equations (3); $K(t,s)$ is the Cauchy $(n\times n)$-dimensional matrix [?, ?]. $D$ is a rectangula,$(m \times 2n)$-dimensional matrix formed under the action of the $m$-dimensional functional $l$ onto the fundamental matrix $X(t)$, $\operatorname{rank} D = n_1$, $n_1 < \min(2n,m)$. The matrix $D^*$ is transposed to the matrix $D$. The $(2n \times m)$-dimensional matrix $D^+$ is Moore–Penrose pseudo-inverse to the matrix $D$ [2, 5, 6, 8]. By $P_D$ we denote the $(2n \times 2n)$-dimensional matrix-orthoprojector $P_D : R^{2n} \to N(D)$, $N(D) = P_D R^{2n}$. The matrix $N(D)$ is the null-space of the matrix $D$: $\dim N(D) = 2n - \operatorname{rank} D = 2n - n_1 = r$. By $P_{D^*}$ we denoted the $(m \times m)$-measurable matrix-orthoprojector $P_{D^*} : R^m \to N(D^*)$, $N(D^*) = P_{D^*} R^m$. The matrix $N(D^*)$ is the null-space of the matrix $D^*$: $\dim N(D^*) = 2n - \operatorname{rank} D^* = 2n - n_1 = r$. Thus the matrix $N(D)$ is of dimension $r$: $\dim N(D) = 2n - \operatorname{rank} D = 2n - n_1 = r$, and the matrix $N(D^*)$ is of dimension $d$: $\dim N(D^*) = m - \operatorname{rank} D = m - n_1 = d$. Consequently, $\operatorname{rank} P_D = r$, $\operatorname{rank} P_{D^*} = d$, this implies that the matrix $P_D$ consists of $r$ linearly independent columns, and the matrix $P_{D^*}$ consists of $d$ linearly independent columns. Thus the $(2n \times 2n)$-dimensional matrix $P_D$ can be replaced by the $(2n \times r)$-dimensional matrix $P_{D_r}$ which consists of $r$ linearly independent columns of the matrix $P_D$; the $(m\times m)$-dimensional matrix $P_{D^*}$ can be replaced by $(d\times m)$-dimensional matrix $P_{D_d^*}$ which consists of $d$ linearly independent series of the matrix $P_{D^*}$ [3, 5].

For the generating boundary value problem (3), (4) the theorem below is fulfilled [5].

**Theorem 1** (Critical case)**.** *Let the condition* $\operatorname{rank} D = n_1 < \min\{2n, m\}$ *be fulfilled. Then the homogeneous* ($f(t) = 0$, $\alpha = 0$) *boundary value problem* (3), (4) *has* $r$, ($r = 2n - n_1$) *and only* $r$ *linearly independent solutions. The inhomogeneous boundary value problem* (3), (4) *is solvable if and only if the vector-function* $f(t) \in C([a, b])$ *and the constant vector* $\alpha \in R^m$ *satisfy the condition of solvability*

$$P_{D_d^*}\big[\alpha - l\overline{x}(\,\cdot\,)\big] = 0 \ \ (d = m - n_1). \tag{5}$$

*If these conditions are fulfilled, the boundary value problem* (3), (4) *has an $r$-parametric set of solutions* $x(t, c_r) = X_r(t)c_r + (G[f])(t) + X(t)D^+\alpha$, $t \in [a, b]$, $\forall\, c_r \in R^r$, *where* $X_r(t)$ *is the* $(n\times n)$*-matrix whose columns consist of a full system of $r$ linearly independent solutions of the homogeneous system of second order* (3)*:* $X_r(t) = X(t)P_{D_r}$*;* $P_{D_r}$ *is the* $(2n\times r)$*-dimensional matrix-orthoprojector consisting of $r$ linearly independent columns of the matrix* $P_D$*;* $c_r$ *is an arbitrary vector column from the space* $R^r$*;* $(G[f])(t)$, $t \in [a, b]$ *is the Greens generalized operator acting onto an arbitrary vector-function* $f(t) \in C([a, b])$*:*

$$(G[f])(t) \overset{def}{=} \int\limits_a^b K(t,s)P^{-1}(s)f(s)\,ds - X(t)D^+ l \int\limits_a^b K(\,\cdot\,,s)P^{-1}(s)f(s)\,ds.$$

We have to define whether there exist the conditions under fulfillment of which the boundary value problem with perturbation (1), (2) will be solvable under the condition that its generating boundary value problem (3), (4) has no solutions. We consider the case, where the generating boundary value problem (3), (4) has no solutions for arbitrary inhomogeneities $f(t) \in C([a, b])$ and $\alpha \in R^m$; this implies that for the above problem the critical case($\operatorname{rank} D = n_1 < n$) is valid, and respectively, for arbitrary inhomogeneities $f(t) \in C([a, b])$, $\alpha \in R^m$, for the generating boundary value problem (3), (4) the solvability criterion (5) fails to be fulfilled. For the boundary value problem (1), (2) using the $(d\times r)$-measurable matrix $B_0 := P_{D_d^*}\big\{l_1X_r(\,\cdot\,) - l\int\limits_a^b K(\,\cdot\,,s)P^{-1}(s)Q_1(s)X_r(s)\,ds\big\}$, the conditions of solvability of the problem under consideration and the condition of uniqueness of its solution, having the form of converging Laurent series $x(\,\cdot\,,\varepsilon) = \sum\limits_{k=-1}^{\infty} \varepsilon^k x_k(t)$, are found. Here, $P_{B_0}$ is the $(r \times r)$-dimensional matrix-orthoprojector, $P_{B_0}$: $R^r \to N(B_0)$; $B_0^*$ is the $(r \times d)$-dimensional matrix, transposed to the matrix $B_0$, $P_{B_0^*}$ is the $(d \times d)$-dimensional matrix-orthoprojector, $P_{B_0^*}$: $R^d \to N(B_0^*)$; $B_0^+$ is the $(r \times d)$-dimensional matrix, pseudo-inverse due to Moore–Penrose to the matrix $B_0$ [6]. In the case, where the condition $P_{B_0^*} = 0$ is not fulfilled, for determination of conditions of solvability of the problem under consideration, the $(d \times r)$-measurable matrix $B_1$: $B_1 := P_{D_d^*}\big\{l_1G_1(\,\cdot\,) - l\int\limits_a^b K(\,\cdot\,,s)P^{-1}(s)Q_1(s)G_1(s)\,ds\big\}$ has been constructed, where $G_1(t)$ is the $(n \times r)$-dimensional matrix of the type $G_1(t) = (G[Q_1(s)X_r(s)])(t) + X(t)D^+l_1X_r(\,\cdot\,)$. Here, $B_1^*$ is the $(r \times d)$-dimensional matrix, transposed to the matrix $B_1$; $P_{B_1^*}$ is the $(d \times d)$-dimensional matrix-orthoprojector, $P_{B_1^*}$: $R^d \to N(B_1^*)$. In the case, where the conditions $P_{B_0^*} = 0$,

$P_{B_1^*}P_{B_0^*} = 0$ for the problem (1), (2) are not fulfilled, to find the conditions of solvability of that problem, the $(d \times r)$-dimensional matrix $\overline{B_1} := -P_{B_0^*}B_1P_{B_0}$ has been constructed. The following theorem is valid.

**Theorem 2.** *Let the generating boundary value problem* (3), (4) *for arbitrary inhomogeneities* $f(t) \in C([a,b])$ *and* $\alpha \in R^m$ *have no solutions. For the boundary value problem* (1), (2) *the conditions* $P_{B_0^*} \neq 0$, $P_{B_1^*}P_{B_0^*} \neq 0$*) are fulfilled.*

*Then the boundary value problem with perturbation* (1), (2) *is solvable if the condition* $P_{\overline{B}_1^*}P_{B_0^*} = 0$ *is fulfilled, and in this case, for a sufficiently small fixed* $\varepsilon \in (0, \varepsilon_0]$ *it has a solution in a form of a part of converging Laurent's series* $x(\,\cdot\,,\varepsilon) = \sum\limits_{k=-3}^{\infty} \varepsilon^k x_k(t)$, *the coefficients* $x_k$, $k \geq -3$ *of Laurent's series are sought from the corresponding boundary value problems constructed after substitution of the Laurent's series into the problem* (1), (2) *and equating the corresponding coefficients for each from powers* $\varepsilon$.

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# To a Question on the Stability of Linear Hybrid Functional Differential Systems with Aftereffect

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## 1 Introduction

The recent general theory of functional differential equations [2]–[5] allowed us to give a clear and concise description of their basic properties including the properties of solution stability. At the same time broad classes of linear hybrid functional differential systems with aftereffect (LHFDSA) arising in many applications are not formally covered by the developed theory and remain out of view of specialists using functional differential and difference systems with aftereffect for simulation of real processes. Below we suggest hybrid functional differential analogues of fundamental assertions of the theory of functional differential equations for problems of stability.

## 2 The $W$-method of N. V. Azbelev

First, let us consider the case when one of the equations is a linear differential one and is defined on a set of discrete points, and the other one is a linear functional differential equation with aftereffect (LFDEA) on a semiaxis. For this case we describe the W-method scheme of N. V. Azbelev.

Let us denote the infinite matrix with the columns $y(-1), y(0), y(1), \dots, y(N), \dots$ of size $n$, by $y = \{y(-1), y(0), y(1), \dots, y(N), \dots\}$ and the infinite matrix with columns $g(0), g(1), \dots, g(N), \dots$ the of size $n$, by $g = \{g(0), g(1), \dots, g(N), \dots\}$.

Each infinite matrix

$$y = \big\{y(-1), y(0), y(1), \dots, y(N), \dots\big\}$$

can be associated with the vector function

$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + y(1)\chi_{[1,2)}(t) + \dots + y(N)\chi_{[N,N+1)}(t) + \cdots .$$

Similarly, each of the infinite matrices $g = \{g(0), g(1), \dots, g(N), \dots\}$ can be associated with the vector function

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + \dots + g(N)\chi_{[N,N+1)}(t) + \cdots .$$

Let us denote the vector function $y(t) = y([t])$, $t \in [-1, \infty)$, by $y(t) = y[t]$ and the vector function $g(t) = g([t])$, $t \in [0, \infty)$, by $g[t]$.

The set of vector functions $y[\,\cdot\,]$ is denoted by $\ell_0$. The set of vector functions $g[\,\cdot\,]$ is denoted by $\ell$. Let $(\Delta y)(t) = y(t) - y(t-1) = y[t] - y[t-1]$ at $t \geq 1$, and $(\Delta y)(t) = y(t) = y[t] = y(0)$ at $t \in [0, 1)$.

The abstract hybrid functional differential system takes the form

$$\begin{aligned} \mathcal{L}_{11}x + \mathcal{L}_{12}y &= \dot{x} - F_{11}x - F_{12}y = f, \\ \mathcal{L}_{21}x + \mathcal{L}_{22}y &= \Delta y - F_{21}x - F_{22}y = g. \end{aligned} \tag{1}$$

Here and below $\mathbb{R}^n$ is the space of vectors $\alpha = \operatorname{col}\{\alpha^1, \dots, \alpha^n\}$ with real components and the norm $\|\alpha\|_{\mathbb{R}^n}$. Assume the space $L$ of locally summable $f : [0, \infty) \to \mathbb{R}^n$ with seminorms

$\|f\|_{L[0,T]} = \int\limits_0^T \|f(t)\|_{\mathbb{R}^n}\,dt$ for all the $T > 0$ and the space $D$ of locally absolutely continuous functions $x : [0,\infty) \to \mathbb{R}^n$ with seminorms

$$\|x\|_{D[0,T]} = \|\dot{x}\|_{L[0,T]} + \|x(0)\|_{\mathbb{R}^n}$$

for all the $T > 0$.

Also assume the space $\ell$ of vector functions

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + \cdots + g(N)\chi_{[N,N+1)}(t) + \cdots$$

with the seminorms $\|g\|_{\ell_T} = \sum\limits_{i=0}^{T} \|g_i\|_{\mathbb{R}^n}$ for all the $T \geq 0$ and the space $\ell_0$ of vector functions

$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + y(1)\chi_{[1,2)}(t) + \cdots + y(N)\chi_{[N,N+1)}(t) + \cdots$$

with the seminorms $\|y\|_{\ell_{0T}} = \sum\limits_{i=-1}^{T} \|y_i\|_{\mathbb{R}^n}$ for all the $T \geq -1$.

The operators $\mathcal{L}_{11}, F_{11} : D \to L$, $\mathcal{L}_{12}, F_{12} : \ell_0 \to L$, $\mathcal{L}_{21}, F_{21} : D \to \ell$, $\mathcal{L}_{22}, F_{22} : \ell_0 \to \ell$ are assumed to be continuous linear and Volterra.

Let $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$. Then (1) can be written as $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$. Suppose that for any $x(0) \in \mathbb{R}^n$ and $y(-1) \in \mathbb{R}^n$ the Cauchy problem for the "model" system $\dot{x} = F_{11}^0 x + F_{12}^0 z + z$, $\Delta y = F_{21}^0 z + F_{22}^0 y + u$, where the operators $F_{11}^0 : D \to L$, $F_{12}^0 : \ell_0 \to L$, $F_{12}^0 : \ell_0 \to L$, $F_{21}^0 : D \to \ell$, $F_{22}^0 : \ell_0 \to \ell$ are assumed to be continuous linear and Volterra. Then the model system can be written as $\mathcal{L}_0\{x,y\} = \operatorname{col}\{z,u\}$. Suppose its solution can be represented as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ y(-1) \end{pmatrix} + \begin{pmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}.$$

Here $\mathcal{W} : L \times \ell \to D \times \ell_0$ is the continuous Volterra operator, the Cauchy operator for the system, $\mathcal{W} = \begin{pmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{pmatrix}$, $U : \mathrm{R}^n \times \mathbb{R}^n \to D \times \ell_0$ is the fundamental matrix for the system $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$.

If the elements $\operatorname{col}\{x,y\} : [0,\infty) \times [-1,\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ forming the Banach space $\mathbf{D} \times \mathbf{M}_0 \cong (\mathbf{B} \times \mathbb{R}^n) \times (\mathbf{M} \times \mathbb{R}^n)$ (space $\mathbf{D} \subset D$, space $\mathbf{M}_0 \cong \mathbf{M} \oplus \mathbb{R}^n \subset \ell_0$, space $\mathbf{B} \subset L$, space $\mathbf{M} \subset \ell$, $\mathbf{B}, \mathbf{M}$ are the Banach spaces) have certain specific properties, such as

$$\sup_{t\geq 0} \|x(t)\|_{\mathbb{R}^n} + \sup_{k=-1,0,1,\dots} \|y(k)\|_{\mathbb{R}^n} < \infty,$$

and the Cauchy problem is uniquely solvable for the equation $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$ with the bounded linear operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$, then the solutions of this problem have the same asymptotic properties. This follows from the theorem given below [6] (see [2, Theorem 2.1.1] and [1, Theorem 1]).

**Theorem.** *Assume $\mathcal{W} : \mathbf{B} \times \mathbf{M} \to \mathbf{D} \times \mathbf{M}_0$ is the bounded Cauchy operator of the Cauchy problem for the model equation $\mathcal{L}_0\{x,y\} = \operatorname{col}\{f,g\}$, $\operatorname{col}\{x(0), y(-1)\} = \operatorname{col}\{0,0\}$ and $U$ is the fundamental matrix of the model equation $\mathcal{L}_0\{x,y\} = \operatorname{col}\{0,0\}$. Here $\mathcal{L}_0 : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$. Assume the linear operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is bounded, $C$ is the Cauchy operator of the Cauchy problem $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$, $\operatorname{col}\{x(0), y(-1)\} = \operatorname{col}\{0,0\}$ and $X$ is the fundamental matrix of the equation $\mathcal{L}\{x,y\} = \operatorname{col}\{0,0\}$. Then for the equality*

$$\mathcal{W}\{\mathbf{B},\mathbf{M}\} + U\{\mathbb{R}^n,\mathbb{R}^n\} = C\{\mathbf{B},\mathbf{M}\} + X\{\mathbb{R}^n,\mathbb{R}^n\} \tag{2}$$

*to hold true it is necessary and sufficient that the operator $\mathcal{L}\mathcal{W}$ (the operator $\mathcal{W}\mathcal{L}$) have a bounded inverse*

$$(\mathcal{L}\mathcal{W})^{-1} : \mathbf{B}\times\mathbf{M} \to \mathbf{B}\times\mathbf{M} \quad \big( (\mathcal{W}\mathcal{L})^{-1} : (\mathbf{D}\times\mathbf{M}_0)^0 \to (\mathbf{D}\times\mathbf{M}_0)^0 \big),$$

*where* $(\mathbf{D}\times\mathbf{M}_0)^0 = \{\mathrm{col}\{x,y\} \in \mathbf{D}\times\mathbf{M}_0 : \ \mathrm{col}\{x(0), y(-1)\} = \mathrm{col}\{0,0\}\}$.

**Corollary** ([1], [2, pp. 36, 48])**.** *If the operator $\mathcal{L} : \mathbf{D}\times\mathbf{M}_0 \to \mathbf{B}\times\mathbf{M}$ is bounded and $\|(\mathcal{L}-\mathcal{L}_0)\mathcal{W}\|_{\mathbf{B}\times\mathbf{M}\to\mathbf{B}\times\mathbf{M}} < 1$ is true or $\|\mathcal{W}(\mathcal{L}-\mathcal{L}_0)\|_{(\mathbf{D}\times\mathbf{M}_0)^0\to(\mathbf{D}\times\mathbf{M}_0)^0} < 1$ is true, then Equality* (2) *holds true as well.*

In the case when (2) holds true (when the solution spaces of the model equation and equation under study coincide), we say that the equation $\mathcal{L}\{x,y\} = \mathrm{col}\{f,g\}$ has the property $\mathbf{D}\times\mathbf{M}_0$, or, in short, the equation is $\mathbf{D}\times\mathbf{M}_0$-stable.

Assume the model equation [1]–[5] and Banach space $\mathbf{B}$ with the elements of the space $L$ ($\mathbf{B}\subset L$, this embedding is continuous) are selected so that the solutions of this equation possess asymptotic properties we are interested in.

We introduce the Banach space $D(\mathcal{L}_{11}, \mathbf{B})$ with the norm

$$\|x\|_{D(\mathcal{L}_{11},\mathbf{B})} = \|\mathcal{L}_{11}x\|_{\mathbf{B}} + \|x(0)\|_{\mathbb{R}^n}.$$

Assume that the operator $\mathcal{W}_{11}$ acts continuously from the space $\mathbf{B}$ into the space $\mathbf{B}$, and the operator $U_{11}$ acts from space $\mathbb{R}^n$ into the space $\mathbf{B}$. This condition is equivalent to the fact [1]–[5] that the space $D(\mathcal{L}_{11}, \mathbf{B})$ is linearly isomorphic to the Sobolev space with the norm

$$\|x\|_{W^{(1)}_{\mathbf{B}}[0,\infty)} = \|\dot{x}\|_{\mathbf{B}} + \|x\|_{\mathbf{B}}.$$

Hereinafter this space is referred to as $W_{\mathbf{B}}$ ($W_{\mathbf{B}}\subset D$, this embedding is continuous).

The equation $\mathcal{L}_{11}x = z$ with the operator $\mathcal{L}_{11} : W_{\mathbf{B}} \to \mathbf{B}$ is $D(\mathcal{L}_{11},\mathbf{B})$-stable if and only if it is strongly $\mathbf{B}$-stable. $\mathcal{L}_{11}x = z$ is strongly $\mathbf{B}$- stable if for any $z\in\mathbf{B}$ each solution $x$ of this equation has the property $x\in\mathbf{B}$ and $\dot{x}\in\mathbf{B}$ ([2, Ch. IV, § 4.6], [5]).

## 3 Reduction of LFDEA on the Semiaxis

Let us consider the scheme from Clause 2 for two equations (1). The operators $\mathcal{L}_{11} : D \to L$, $\mathcal{L}_{12} : \ell_0 \to L$, $\mathcal{L}_{21} : D \to \ell$, $\mathcal{L}_{22} : \ell_0 \to \ell$ are considered as reduction to pairs $(\mathbf{W_B}, \mathbf{B})$, $(\mathbf{M_0}, \mathbf{B})$, $(\mathbf{W_B}, \mathbf{M})$, $(\mathbf{M_0}, \mathbf{M})$. These operators are assumed to be Volterra linear and bounded operators.

Assume that the general solution of the equation $\mathcal{L}_{22}y = g$ for $g\in M$is the space of $M_0$ and is represented by the Cauchy formula

$$y[t] = Y_{22}[t]y(-1) + \sum_{s=0}^{[t]} C_{22}[t,s]g[s].$$

Let

$$(C_{22}g)[t] = \sum_{s=0}^{[t]} C_{22}[t,s]g[s], \quad (Y_{22}y(-1))[t] = Y_{22}[t]y(-1).$$

Then every solution $y$ of the second equation in (1) has the form

$$y = -C_{22}\mathcal{L}_{21}x + Y_{22}y(-1) + C_{22}g.$$

Substituting the first equation into (1), we obtain

$$\begin{gathered}
\mathcal{L}_{11}x + \mathcal{L}_{12}y = \mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x + \mathcal{L}_{12}Y_{22}y(-1) + \mathcal{L}_{12}C_{22}g = f,\\
\mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x = f_1 = f - \mathcal{L}_{12}Y_{22}y(-1) - \mathcal{L}_{12}C_{22}g.
\end{gathered}$$

Let $\mathcal{L} = \mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}$, then the first equation in (1) takes the form of $\mathcal{L}x = f_1$. Suppose the Volterra operator $\mathcal{L} : (\mathbf{W_B})^0 \to B$ is Volterra invertible, that is (when the Cauchy problem for $\mathcal{L}x = f_1$ possesses the following property: at any $f_1\in\mathbf{B}$ its solutions are $x\in\mathbf{W_B}$). Thus, we solved the problem, when for Equation (1) at any $\{f,g\}\in\mathbf{B}\times\mathbf{M}$ its solutions are $\{x,y\}\in\mathbf{W_B}\times\mathbf{M}$.

# 4 Reduction to a Linear Difference Equation (LDE) on a Discrete Set of Points

Let us use the ability of the hybrid system to be reduced to a LDE defined on a discrete set of points. For Equation (1) we use the designations given in Clauses 2 and 3.

Assume the general solution of the equation $\mathcal{L}_{11}x = f$ for $f \in L$ is a member of the space $D$ and is represented by the Cauchy formula

$$x(t) = X_{11}(t)x(0) + \int\limits_0^t C_{11}(t,s)f(s)\,ds.$$

Let $(C_{11}f)(t) = \int\limits_0^t C_{11}(t,s)f(s)\,ds$, $(X_{11}x(0))(t) = X_{11}(t)x(0)$, then for $x \in D$ the representation $x = X_{11}x(0) + C_{11}f$ holds true.

The first variable $x$ can be estimated out of the first equation in (1)

$$x = -C_{11}\mathcal{L}_{12}y + X_{11}x(0) + C_{11}f.$$

We use this substitution in the second equation of (1), we obtain

$$\begin{gathered}\mathcal{L}_{21}x + \mathcal{L}_{22}y = -\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y + \mathcal{L}_{21}X_{11}x(0) + \mathcal{L}_{21}C_{11}f + \mathcal{L}_{22}y = g,\\ -\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y + \mathcal{L}_{22}y = g_1 = g - \mathcal{L}_{21}X_{11}x(0) - \mathcal{L}_{21}C_{11}f.\end{gathered}$$

Let $\mathcal{L} = \mathcal{L}_{22} - \mathcal{L}_{21}C_{11}\mathcal{L}_{12}$, then the second equation in (1) takes the form $\mathcal{L}y = g_1$. Suppose the Volterra operator $\mathcal{L} : (\mathbf{M_0})^0 \to \mathbf{M}$ is Volterra invertible (when the Cauchy problem for $\mathcal{L}y = g_1$ at any $g_1 \in \mathbf{M}$ its solutions are $x \in \mathbf{M}_0$). Thus, we solved the problem, when at any $\{f,g\} \in \mathbf{B} \times \mathbf{M}$ for (1) its solutions are $\{x,y\} \in \mathbf{D} \times \mathbf{M_0}$.

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# On Conjugacy of Second-Order Half-Linear Differential Equations on the Real Axis

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On the real axis, we consider the equation

$$\boxed{\big(|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0,} \tag{1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a locally integrable function and $\alpha > 0$.

A function $u : I \to \mathbb{R}$ is said to be *a solution to equation* (1) *on the interval* $I \subseteq \mathbb{R}$ if it is continuously differentiable on $I$, $|u'|^\alpha \operatorname{sgn} u'$ is absolutely continuous on every compact subinterval of $I$, and $u$ satisfies equality (1) almost everywhere on $I$. In [6, Lemma 2.1], Mirzov proved that every solution to equation (1) is extendable to the whole real axis. Therefore, speaking about a solution to equation (1), we assume that it is defined on $\mathbb{R}$. Moreover, for any $a \in \mathbb{R}$, the initial value problem

$$\big(|u'|^\alpha \operatorname{sgn} u'\big)' + p(t)|u|^\alpha \operatorname{sgn} u = 0; \quad u(a) = 0, \ \ u'(a) = 0$$

has only the solution $u \equiv 0$ (see [6, Lemma 1.1]). Hence, a solution $u$ to equation (1) is said to be *non-trivial*, if $u \not\equiv 0$ on $\mathbb{R}$.

**Definition 1.** We say that equation (1) is *conjugate on* $\mathbb{R}$ if it has a non-trivial solution with at least two zeros, and *disconjugate on* $\mathbb{R}$ otherwise.

It is clear that in the case $\alpha = 1$, equation (1) reduces to the linear equation

$$u'' + p(t)u = 0. \tag{2}$$

As it is mentioned in [4], a history of the problem of conjugacy of (2) began in the paper by Hawking and Penrose [3]. In [8], Tipler presented an interesting relevance of the study of conjugacy of (2) to the general relativity and improved Hawking–Penrose's criterion, showing that (2) is conjugate on $\mathbb{R}$ if the inequality

$$\liminf_{\substack{t \to +\infty \\ \tau \to -\infty}} \int_\tau^t p(s)\,\mathrm{d}s > 0 \tag{3}$$

holds. Later, Peña [7] proved that the same condition is sufficient also for the conjugacy of half-linear equation (1).

The study of conjugacy of (1) on $\mathbb{R}$ is closely related to the question of oscillation of (1) on the whole real axis. It is known that Sturms's separation theorem holds for equation (1) (see [6, Theorem 1.1]). Therefore, if equation (1) possesses a non-trivial solution with a sequence of zeros tending to $+\infty$ (resp. $-\infty$), then any other its non-trivial solution has also a sequence of zeros tending to $+\infty$ (resp. $-\infty$).

**Definition 2.** Equation (1) is said to be *oscillatory in the neighbourhood of* $+\infty$ (resp. *in the neighbourhood of* $-\infty$) if every its non-trivial solution has a sequence of zeros tending to $+\infty$ (resp. to $-\infty$). We say that equation (1) is *oscillatory on* $\mathbb{R}$ if it is oscillatory in the neighbourhood of either $+\infty$ or $-\infty$, and *non-oscillatory on* $\mathbb{R}$ otherwise.

Clearly, if equation (1) is oscillatory on $\mathbb{R}$, then it is conjugate on $\mathbb{R}$, as well. It is known that oscillations of (1) in the neighbourhood of $+\infty$ (resp. $-\infty$) can be described by means of behaviour of the Hartman–Wintner type expression

$$\frac{1}{|t|}\int_0^t\bigg(\int_0^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \tag{4}$$

in the neighbourhood of $+\infty$ (resp. $-\infty$), see [5, Theorem 12.3]. However, expression (4) is useful also in the study of conjugacy of (1) on $\mathbb{R}$. In particular, efficient conjugacy and disconjugacy criteria for linear equation (2) formulated by means of expression (4) are given in [4]. Abd-Alla and Abu-Risha [1] observed that for the study of conjugacy on whole real axis, it is more convenient to consider a Hartman–Wintner type expression in a certain symmetric form, where all values of the function $p$ are involved simultaneously. They proved in [1], among other things, that equation (1) with a continuous $p$ is conjugate on $\mathbb{R}$ provided that $p \not\equiv 0$ and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\bigg(\int_{-s}^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \geq 0, \tag{5}$$

which obviously improves Peňa's criterion (3). Below, we generalise and supplement criterion (5) and present further statements, which can be applied in the cases not covered by Theorems 3 and 5.

For any $\nu < 1$, we put

$$c(t;\nu) := \frac{1-\nu}{(1+t)^{1-\nu}}\int_0^t \frac{1}{(1+s)^{\nu}}\bigg(\int_{-s}^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \text{ for } \ t \geq 0.$$

We start with a Hartman–Wintner type result, which guarantees that equation (1) is oscillatory on $\mathbb{R}$ (not only conjugate).

**Theorem 3.** *Let $\nu < 1$ be such that either*

$$\lim_{t\to+\infty} c(t;\nu) = +\infty,$$

*or*

$$-\infty < \liminf_{t\to+\infty} c(t;\nu) < \limsup_{t\to+\infty} c(t;\nu).$$

*Then equation* (1) *is oscillatory on $\mathbb{R}$ and consequently, conjugate on $\mathbb{R}$.*

**Remark 4.** Having $\nu_1, \nu_2 < 1$, one can show that there exists a finite limit $\lim\limits_{t\to+\infty} c(t;\nu_2)$ if and only if there exists a finite limit $\lim\limits_{t\to+\infty} c(t;\nu_1)$, in which case both limits are equal.

In view of Remark 4, Theorem 3 cannot be applied, in particular, if the function $c(\,\cdot\,;1-\alpha)$ has a finite limit as $t \to +\infty$. A conjugacy criterion covering this case is given in the following statement.

**Theorem 5.** *Let $p \not\equiv 0$ and*

$$0 \leq \lim_{t\to+\infty} c(t;1-\alpha) < +\infty.$$

*Then equation* (1) *is conjugate on $\mathbb{R}$.*

Theorems 3 and 5 yield

**Corollary 6.** *Let $p \not\equiv 0$ and $\nu < 1$ be such that*

$$\liminf_{t\to+\infty} c(t;\nu) > -\infty, \quad \limsup_{t\to+\infty} c(t;\nu) \geq 0.$$

*Then equation* (1) *is conjugate on* $\mathbb{R}$.

Corollary 6 generalises several conjugacy criteria known in the existing literature. In particular, [2, Theorem 2.2] can be derived from Corollary 6. Moreover, conjugacy criterion (5) given in [1, Theorem 2.2] follows immediately from Corollary 6 with $\nu := 0$. Corollary 6 also yields the following half-linear extension of [4, Theorem 1].

**Corollary 7.** *Let $p \not\equiv 0$ and the function*

$$M : t \longmapsto \frac{1}{|t|} \int\limits_0^t \bigg( \int\limits_0^s p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s$$

*have finite limits as $t \to \pm\infty$. If*

$$\lim_{t\to+\infty} M(t) + \lim_{t\to-\infty} M(t) \geq 0,$$

*then equation* (1) *is conjugate on* $\mathbb{R}$.

According to the above said, we conclude that neither of Theorems 3 and 5 can be applied in the following two cases:

$$\lim_{t\to+\infty} c(t;1-\alpha) =: c(+\infty) \in\, ]-\infty, 0[ \tag{6}$$

and

$$\liminf_{t\to+\infty} c(t;\nu) = -\infty \ \text{ for every } \ \nu < 1. \tag{7}$$

## The case (6)

In the first statement, we require that the function $c(\,\cdot\,;1-\alpha)$ is at some point far enough from its limit $c(+\infty)$.

**Theorem 8.** *Let* (6) *hold and*

$$\sup\bigg\{ \frac{(1+t)^{\alpha}}{\ln(1+t)} \big[c(+\infty) - c(t;1-\alpha)\big] :\ t > 0 \bigg\} > 2\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}. \tag{8}$$

*Then equation* (1) *is conjugate on* $\mathbb{R}$.

**Remark 9.** One can show that if (8) is replaced by

$$\limsup_{t\to+\infty} \frac{(1+t)^{\alpha}}{\ln(1+t)} \big[c(+\infty) - c(t;1-\alpha)\big] > 2\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}, \tag{9}$$

then we can claim in Theorem 8 that equation (1) is even oscillatory on $\mathbb{R}$.

Now we put

$$Q_\alpha(t) := \frac{(1+t)^{1+\alpha}}{t} \bigg[ c(+\infty) - \int\limits_{-t}^{t} p(s)\,\mathrm{d}s \bigg], \quad H_\alpha(t) := \frac{1}{t} \int\limits_{-t}^{t} (1+|s|)^{1+\alpha} p(s)\,\mathrm{d}s \ \text{ for } \ t > 0.$$

**Theorem 10.** *Let* (6) *hold and*

$$\sup\big\{Q_\alpha(t)+H_\alpha(t):\ t>0\big\}>2.$$

*Then equation* (1) *is conjugate on* $\mathbb{R}$.

**Remark 11.** One can show that if

$$\limsup_{t\to+\infty}\big(Q_\alpha(t)+H_\alpha(t)\big)>2,$$

then we can claim in Theorem 10 that equation (1) is even oscillatory on $\mathbb{R}$.

## The case (7)

First note that, in condition (7), the assumption that $\liminf\limits_{\nu\to+\infty} c(t;\nu)=-\infty$ for **every** $\nu<1$ is, in fact, not too restrictive. Indeed, let $\liminf\limits_{t\to+\infty} c(t;\nu_1)=-\infty$ for some $\nu_1<1$. Then Remark 4 yields that for any $\nu<1$, the function $c(\,\cdot\,;\nu)$ does not possess any finite limit. Consequently, if there exists $\nu_2<1$ such that $\liminf\limits_{t\to+\infty} c(t;\nu_2)>-\infty$, then equation (1) is oscillatory on $\mathbb{R}$ as it follows from Theorem 3.

**Proposition 12.** *Let condition* (7) *hold and there exist a number* $\kappa>\alpha$ *such that*

$$\limsup_{t\to+\infty}\frac{1}{t^\kappa}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s>-\infty. \tag{10}$$

*Then equation* (1) *is oscillatory on* $\mathbb{R}$ *and consequently, conjugate on* $\mathbb{R}$.

Finally, we give a statement which can be applied in the case, when condition (7) holds, but (10) is violated for every $\kappa>\alpha$, i. e.,

$$\lim_{t\to+\infty}\frac{1}{t^\kappa}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s=-\infty\ \text{ for every }\ \kappa>\alpha$$

(it may happen as can be justified by an example).

**Theorem 13.** *Let there exist a number* $\kappa>\alpha$ *such that*

$$\sup\left\{\frac{1}{t^{\kappa-\alpha}}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s:\ t>0\right\}>\frac{2}{\kappa-\alpha}\Big(\frac{\kappa}{1+\alpha}\Big)^{1+\alpha}.$$

*Then equation* (1) *is conjugate on* $\mathbb{R}$.

**Remark 14.** Observe that Theorem 13 does not require assumption (7), it is a general statement applicable without regard to behaviour of the function $c(\,\cdot\,;\nu)$.

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# Leray–Schauder Degree Method in Periodic Problem for the Generalized Basset Fractional Differential Equation

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Let $T>0$, $J=[0,T]$ and $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$.

In the literature [1, 2] the fractional differential equation

$$u'(t)=a\,{}^c\!D^\alpha u(t)+bu(t)+g(t),\ \ a\in\mathbb{R}\setminus\{0\},\ \ \alpha\in(0,1),$$

is called the Basset fractional differential equation.

We investigate the generalized Basset fractional differential equation

$$u'(t)=a(t)\,{}^c\!D^\alpha u(t)+f\big(t,u(t),{}^c\!D^\beta u(t)\big), \tag{1}$$

where $0<\beta<\alpha<1$, $a\in C(J)$, $f\in C(J\times\mathbb{R}^2)$ and ${}^c\!D$ stands for the Caputo fractional derivative. Further conditions on $a$ and $f$ will be given later.

Together with (1) we consider the periodic condition

$$u(0)=u(T). \tag{2}$$

We recall that the Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma>0$ of a function $x:J\to\mathbb{R}$ is defined as [1, 2]

$$I^\gamma x(t)=\int\limits_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s,$$

and the Caputo fractional derivative ${}^c\!D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ as

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\Big(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}\,s^k\Big)\,\mathrm{d}s,$$

where $\Gamma$ is the Euler gamma function and $n=[\gamma]+1$, $[\gamma]$ means the integral part of $\gamma$.

In particular,

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s,\ \ \gamma\in(0,1).$$

If $x\in C^1(J)$, then

$$ {}^c\!D^\gamma x(t)=\int\limits_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,x'(s)\,\mathrm{d}s,\ \ \gamma\in(0,1).$$

It is well known that $I^\gamma:C(J)\to C(J)$ for $\gamma\in(0,1)$ and $I^\gamma I^\delta x(t)=I^{\gamma+\delta}x(t)$ for $\gamma,\delta\in(0,\infty)$, $x\in C(J)$.

We say that $u: J \to \mathbb{R}$ is *a solution of problem* (1), (2) if $u \in C^1(J)$, $u$ satisfies (2) and (1) holds for $t \in J$.

The solvability of the periodic problem

$$u'(t) = a{}^c\!D^\alpha u(t) + f(t, u(t)), \quad u(0) = u(T),$$

where $a$ is a positive constant and $\alpha \in (0,1)$, is discussed in [3].

In order to give the existence result for problem (1), (2), we introduce operators $\mathcal{H}: C(J) \times \mathbb{R} \times [0,1] \to C(J)$ and $\mathcal{S}: C(J) \times \mathbb{R} \times [0,1] \to C(J) \times \mathbb{R}$,

$$\mathcal{H}(x, \mu, \lambda)(t) = (1-\lambda)\mu + \lambda\Big(a(t)x(t) + f\big(t, \mu + I^\alpha x(t), I^{\alpha-\beta} x(t)\big)\Big)$$

and

$$\mathcal{S}(x, \mu, \lambda) = \Big(I^{1-\alpha}\mathcal{H}(x, \mu, \lambda)(t), \mu + I^\alpha x(t)\big|_{t=T}\Big).$$

The following result gives the property of $\mathcal{S}$ and the relation between solutions of the periodic problem (1), (2) and fixed points of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$.

**Lemma 1.** *$\mathcal{S}$ is a completely continuous operator. If $(x, \mu)$ is a fixed point of $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$, then*

$$u(t) = \mu + I^\alpha x(t) \ \text{ for } t \in J$$

*is a solution of the periodic problem* (1), (2) *and $\mu = u(0)$.*

**Lemma 2.** *Let the conditions*

($H_1$) *$a(t) \geq 0$ for $t \in J$, $a \not\equiv 0$;*

($H_2$) *there exist positive constants $c$, $k$ and $l$ such that*

$$\begin{gathered} f(t, x, y \operatorname{sign} x) \operatorname{sign} x > 0 \ \text{ for } t \in J, \ |x| \geq c, \ y \in [0, \infty), \\ |f(t, x, y)| \leq k\big(|x| + |y|\big) + l \ \text{ for } t \in J, \ x, y \in \mathbb{R}, \end{gathered} \tag{3}$$

*hold. Then there exists a positive constant $S$ such that the estimate*

$$\|x\| < S, \quad |\mu| < S$$

*is fulfilled for all fixed points $(x, \mu)$ of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, \lambda)$ with $\lambda \in [0,1]$.*

**Remark 1.** Inequality (3) of ($H_2$) can be written in the following equivalent form

$$\begin{gathered} f(t, x, y) > 0 \ \text{ for } t \in J, \ x \geq c, \ y \in [0, \infty), \\ f(t, x, y) < 0 \ \text{ for } t \in J, \ x \leq -c, \ y \in (-\infty, 0]. \end{gathered}$$

**Theorem 1.** *Let ($H_1$) and ($H_2$) hold. Then the periodic problem* (1), (2) *has at least one solution.*

*Proof.* Keeping in mind Lemma 1, we need to prove that there exists a fixed point of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$.

Let $S > 0$ be from Lemma 2 and let

$$\Omega = \Big\{(x, \mu) \in C(J) \times \mathbb{R} : \ \|x\| < S, \ |\mu| < S\Big\}.$$

Then Lemma 2 guarantees that

$$\mathcal{S}(x, \mu, \lambda) \neq (x, \mu) \ \text{ for } (x, \mu) \in \partial\Omega \text{ and } \lambda \in [0,1].$$

Since $\mathcal{S}(-x,-\mu,0)=-\mathcal{S}(x,\mu,0)$ for $(x,\mu)\in C(J)\times\mathbb{R}$, $\mathcal{S}(\,\cdot\,,\,\cdot\,,0)$ is an odd operator. By Lemma 1, the restriction of $\mathcal{S}$ to $\overline{\Omega}\times[0,1]$ is a compact operator. Therefore, the Borsuk antipodal theorem and the homotopy property give [4]

$$\deg(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,0),\Omega,0)\neq 0,$$
$$\deg\big(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big)=\deg\big(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big),$$

where "deg" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on $C(J)\times\mathbb{R}$. Consequently, $\deg(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,1),\Omega,0)\neq 0$, which implies the existence of a fixed point of $\mathcal{S}(\,\cdot\,,\,\cdot\,,1)$. $\square$

**Example 1.** Let $\varphi,\psi,\gamma\in C(J)$ and $\varphi(t)\geq\varepsilon>0$, $\psi\geq 0$ on $J$. Then the function $f(t,x,y)=\varphi(t)(x+\sin y)+\psi(t)y+\gamma(t)$ satisfies condition $(H_2)$ for $c=\|\gamma\|/\varepsilon+1$, $k=\|\varphi\|+\|\psi\|$ and $l=\|\varphi\|+\|\gamma\|$. If $a\in C(J)$, $a\geq 0$ on $J$ and $a\neq 0$, then Theorem 1 guarantees that the periodic problem

$$\left.\begin{aligned} u'=a(t)^{c}\mathcal{D}^{\alpha}u+\varphi(t)\big(u+\sin({}^{c}\mathcal{D}^{\beta}u)\big)+\psi(t)^{c}\mathcal{D}^{\beta}u+\gamma(t),\\ u(0)=u(T)\end{aligned}\right\}$$

has at least one solution.

If $f(t,x,y)$ in (1) is independent of the variable $y$, that is, $f(t,x,y)=f(t,x)$, then Theorem 1 gives the following result for the periodic problem

$$\left.\begin{aligned} u'(t)=a(t)^{c}\mathcal{D}^{\alpha}u(t)+f(t,u(t)),\\ u(0)=u(T).\end{aligned}\right\}\tag{4}$$

**Corollary 1.** *Let $(H_1)$ hold and let $f\in C(J\times\mathbb{R})$ and there exist positive constants $c$, $k$ and $l$ such that*

$$f(t,x)<0\ \text{ for } (t,x)\in J\times(-\infty,-c],\quad f(t,x)>0\ \text{ for } (t,x)\in J\times[c,\infty),$$
$$|f(t,x)|\leq k|x|+l\ \text{ for } (t,x)\in J\times\mathbb{R}.$$

*Then the periodic problem* (4) *has at least one solution.*

The following result gives the existence of a unique solution of problem (4).

**Theorem 2.** *Let the conditions of Corollary* 1 *be satisfies. In addition, suppose that $f(t,x)$ is increasing in $x$ for all $t\in J$ and for any $\ell>0$ there exists $L_\ell>0$ such that*

$$|f(t,x)-f(t,y)|\leq L_\ell|x-y|\ \text{ for } t\in J,\ x,y\in[-\ell,\ell].$$

*Then the periodic problem* (4) *has a unique solution.*

**Example 2.** Let $a,\varphi,\gamma\in C(J)$, $a\geq 0$, $\varphi(t)\geq\varepsilon>0$ on $J$ and $a\neq 0$. Then the periodic problem

$$\left.\begin{aligned} u'=a(t)^{c}\mathcal{D}^{\alpha}u+\varphi(t)u+\gamma(t),\\ u(0)=u(T)\end{aligned}\right\}$$

has a unique solution.

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# On Solution of the Initial Value Problem for One Neutral Stochastic Differential Equation of Reaction-Diffusion Type in Hilbert Space

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The paper is devoted to the Cauchy problem for one equation of reaction-diffusion type – for a neutral stochastic integro-differential equation in Hilbert space $H = L_2^\rho(\mathbb{R}^d)$ (the space with an inner product $(f,g)_{L_2^\rho(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} f(x)g(x)\rho(x)\,dx$ and a corresponding norm $\|f\|_{L_2^\rho(\mathbb{R}^d)} = \sqrt{\int\limits_{\mathbb{R}^d} \|f(x)\|^2\rho(x)\,dx}$) of the form

$$\begin{gathered}
d\bigg(u(t,x) + \int\limits_{\mathbb{R}^d} b(t,x,\xi)u(\alpha(t))\,d\xi\bigg) = \\
= \big(\Delta_x u(t,x) + f(t,u(\alpha(t)),x)\big)\,dt + \sigma\big(t,u(\alpha(t)),x\big)\,dW(t,x), \;\; 0 < t \le T, \;\; x \in \mathbb{R}^d, \\
u(t,x) = \phi(t,x), \;\; -r \le t \le 0, \;\; x \in \mathbb{R}^d, \;\; r > 0,
\end{gathered} \tag{1}$$

namely, to the investigation of existence and uniqueness of its solution. Here, $d \in \{1,2,\dots\}$ – an arbitrary positive integer, $T > 0$ – a fixed real, $\Delta_x = \sum\limits_{j=1}^d \partial_{x_j}^2$, $d \in \{1,2,\dots\}$, – $d$-measurable Laplacian, $\partial_{x_j}^2 \equiv \frac{\partial^2}{\partial x_j^2}$, $j \in \{1,\dots,d\}$, $W(t,x)$ – $L_2^\rho(\mathbb{R}^d)$-valued Wiener process, $\{f,\sigma\}\colon [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $b\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ – some given functions, $\phi\colon [-r,0]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$ – an initial data and $\alpha\colon [0,T] \to [-r,\infty)$ – a delay function to be specified later. It is known [5, p. 242–244] that $\Delta_x$ is an (infinitesimal) generator of an analytic ($C_0$-)semigroup of operators $\{S(t),\ t \ge 0\}$ that generates the solution $u(t,x) = (S(t)g(\,\cdot\,))(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t,x-\xi)g(\,\cdot\,)\,d\xi$ of a homogenous Cauchy problem for a heat-equation

$$\begin{gathered}
\partial_t u(t,x) = \Delta_x u(t,x), \;\; t > 0, \;\; x \in \mathbb{R}^d, \\
u(0,x) = g(x), \;\; x \in \mathbb{R}^d.
\end{gathered} \tag{2}$$

Throughout the paper we assume the following:

**1)** $(\Omega,\mathcal{F},\mathbf{P})$ – a complete probability space, equipped with a normal filtration $\{\mathcal{F}_t,\ t \ge 0\}$ that generates $L_2^\rho(\mathbb{R}^d)$-valued nuclear $Q$-Wiener process $W(t,x) = \sum\limits_{n=1}^\infty \sqrt{\lambda_n}\, e_n(x)\beta_n(t)$, where $\{\beta_n(t),\ n \in \{1,2,\dots\}\}$ – one-dimensional independent Brownian motions, $\lambda_n > 0$, $n \in$

$\{1,2,\dots\}$, and $\sum\limits_{n=1}^{\infty}\lambda_n<\infty$, $\{e_n(x),\ n\in\{1,2,\dots\}\}$ – a complete orthonormal system in $L_2^\rho(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}}\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^d}\|e_n(x)\|\le 1$;

**2)** $\alpha\colon[0,T]\to[-r,\infty)$ – an increasing continuously differentiable function such that $\alpha(t^*)=0$, $0<\alpha'(t)\le 1$ and $\frac{1}{\alpha'(t)}\le c$, $c>0$;

**3)** $\{f,\sigma\}\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ – measurable in all their arguments functions;

**4)** an initial data function $\phi\colon[-r,0]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is $\mathcal{F}_0$-measurable and such that $\mathbf{E}\sup\limits_{-r\le t\le 0}\|\phi(t)\|^2_{L_2^\rho(\mathbb{R}^d)}<\infty$;

**5)** for $\rho\in L_1(\mathbb{R}^d)$ there exists $C_\rho(T)>0$ such that $\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\rho(\xi)\,d\xi\le C_\rho(T)\rho(x)$, $0\le t\le T$, $x\in\mathbb{R}^d$.

Our result-theorem is devoted to the existence and uniqueness for $0\le t\le T$ of so-called ***mild solution*** of (1), defined below, in the space $\mathfrak{B}_{2,T,\rho}$. Here $\mathfrak{B}_{2,T,\rho}$ is the Banach space of all $L_2^\rho(\mathbb{R}^d)$-valued $\mathcal{F}_t$-measurable for almost all $0\le t\le T$ stochastic random processes $\Phi\colon[0,T]\times\Omega\to L_2^\rho(\mathbb{R}^d)$ that are continuous in $t$ for almost all $\omega\in\Omega$, with the norm $\|\Phi\|_{\mathfrak{B}_{2,T,\rho}}=\sqrt{\mathbf{E}\sup\limits_{0\le t\le T}\|\Phi(t)\|^2_{L_2^\rho(\mathbb{R}^d)}}$.

**Definition.** A continuous stochastic random process $u\colon[-r,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is called a ***mild solution*** of (1) if it

1) is $\mathcal{F}_t$-measurable for almost all $-r\le t\le T$;

2) satisfies an integral equation of the form

$$
\begin{aligned}
u(t,x)={}&\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\Big(\phi(0)+\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r)\,d\zeta\Big)\,d\xi-\int\limits_{\mathbb{R}^d}b(t,x,\xi)u(\alpha(t))\,d\xi-\\
&-\int\limits_0^t\Big(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\Big(\int\limits_{\mathbb{R}^d}b(s,\xi,\zeta)u(\alpha(s))\,d\zeta\Big)\,d\xi\Big)\,ds+\\
&+\int\limits_0^t\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(s,u(\alpha(s)),\xi)\,d\xi\,ds+\\
&+\int\limits_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Big(\int\limits_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\sigma(s,u(\alpha(s)),\xi)e_n(\xi)\,d\xi\Big)\,d\beta_n(s), \qquad (3)\\
&\qquad\qquad 0\le t\le T,\ \ x\in\mathbb{R}^d,
\end{aligned}
$$

$$u(t,x)=\phi(t,x),\ \ -r\le t\le 0,\ \ x\in\mathbb{R}^d,\ \ r>0; \qquad (4)$$

3) satisfies the condition $\mathbf{E}\int\limits_0^T\|u(t)\|^2_{L_2^\rho(\mathbb{R}^d)}\,dt<\infty$.

**Theorem** (existence and uniqueness of a mild solution in the space $\mathfrak{B}_{2,T,\rho}$). *Let there exist $L \geq 0$ such that for $\{f, \sigma\}$ the following conditions of linear-growth and Lipschitz by the second argument are fulfilled:*

$$f^2(t,u,x) + \sigma^2(t,u,x) \leq L^2(1+u^2),$$
$$\big(f(t,u,x) - f(t,v,x)\big)^2 + \big(\sigma(t,u,x) - \sigma(t,v,x)\big)^2 \leq L^2(u-v)^2,$$
$$0 \leq t \leq T, \ \ \{u,v\} \subset \mathbb{R}, \ \ x \in \mathbb{R}^d,$$

*the function $b$ is such that* $\sup\limits_{0\leq t\leq T} \frac{\|b(t,x,\cdot)\|}{\sqrt{\rho(\cdot)}} \in L_2(\mathbb{R}^d)$, $0 \leq t \leq T$, $x \in \mathbb{R}^d$, *and*

$$\sup_{0\leq t\leq T} \int\limits_{\mathbb{R}^d} \bigg( \int\limits_{\mathbb{R}^d} \frac{\|b(t,x,\xi)\|^2}{\rho(\xi)}\, d\xi \bigg) \rho(x)\, dx < \infty,$$

*and for $\partial_x b$ there exists a majorizing function $\varphi\colon [0,T] \times \mathbb{R}^d \to [0,\infty)$ such that $\|\partial_x b(t,x,\xi)\| \leq \varphi(t,\xi)$, $0 \leq t \leq T$, $\{x,\xi\} \subset \mathbb{R}^d$, with $\sup\limits_{0\leq t\leq T} \frac{\varphi(t,\cdot)}{\sqrt{\rho(\cdot)}} \in L_2(\mathbb{R}^d)$. Then the problem* (1) *has a unique for $0 \leq t \leq T$ mild solution $u \in \mathfrak{B}_{2,T,\rho}$ if*

$$\sup_{0\leq t\leq T} \int\limits_{\mathbb{R}^d} \bigg( \int\limits_{\mathbb{R}^d} \frac{\|b(t,x,\xi)\|^2}{\rho(\xi)}\, d\xi \bigg) \rho(x)\, dx < \frac{1}{4}\,.$$

*Proof.* The method of the proof is taken from [2], where authors have proved uniqueness of a fixed point for a certain operator with the help of the classical theorem of Banach on a contractive mapping. Our goal is to check execution of conditions of this theorem for the operator $\Psi\colon \mathfrak{B}_{2,T,\rho} \to \mathfrak{B}_{2,T,\rho}$, whose action is given by the rule

$$\begin{aligned}
\big(\Psi(t)u(\cdot)\big)(x) &= \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x-\xi) \bigg( \phi(0) + \int\limits_{\mathbb{R}^d} b(0,\xi,\zeta)\phi(-r)\, d\zeta \bigg) d\xi - \int\limits_{\mathbb{R}^d} b(t,x,\xi) u(\alpha(t))\, d\xi - \\
&\quad - \int\limits_0^t \bigg( \Delta_x \int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \bigg( \int\limits_{\mathbb{R}^d} b(s,\xi,\zeta) u(\alpha(s))\, d\zeta \bigg) d\xi \bigg) ds + \\
&\quad + \int\limits_0^t \int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) f(s, u(\alpha(s)), \xi)\, d\xi\, ds + \\
&\quad + \int\limits_0^t \sum_{n=1}^{\infty} \sqrt{\lambda_n} \bigg( \int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \sigma(s, u(\alpha(s)), \xi) e_n(\xi)\, d\xi \bigg) d\beta_n(s) = \\
&= \sum_{i=0}^{4} I_i(t)(x), \ \ 0 \leq t \leq T, \ \ x \in \mathbb{R}^d.
\end{aligned}$$

Due to it, we need to prove that $\Psi(u) \in \mathfrak{B}_{2,T,\rho}$ for all $u \in \mathfrak{B}_{2,\mathrm{T},\rho}$ and to find out a condition of contraction. In order to prove the first item, we need to show that $\|I_j(s)\|^2_{\mathfrak{B}_{2,t,\rho}} = \mathbf{E} \sup\limits_{0\leq s\leq t} \|I_j(s)\|^2_{L_2^{\rho}(\mathbb{R}^d)}$, $j \in \{0,\dots,4\}$: indeed, a chain of computations, involving application of the inequality of Cauchy-Schwartz and the theorem of Fubini, yields

$$\|I_0(s)\|^2_{\mathfrak{B}_{2,t,\rho}} = \mathbf{E} \sup_{0\leq s\leq t} \int\limits_{\mathbb{R}^d} \bigg( \int\limits_{\mathbb{R}^d} \mathscr{K}(s, x-\xi) \bigg( \phi(0,\xi) + \int\limits_{\mathbb{R}^d} b(0,\xi,\zeta) \phi(-r,\zeta)\, d\zeta \bigg) d\xi \bigg)^2 \rho(x)\, dx \leq$$

$$\leq 2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s,x-\xi)}\,\sqrt{\mathscr{K}(s,x-\xi)}\phi(0,\xi)\,d\xi\bigg)^2\rho(x)\,dx+$$

$$+2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s,x-\xi)}\,\sqrt{\mathscr{K}(s,x-\xi)}\bigg(\int\limits_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-r,\zeta)\,d\zeta\bigg)\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq 2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s,x-\xi)\rho(\xi)\,d\xi\bigg)\phi^2(0,x)\,dx+$$

$$+2\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s,x-\xi)\rho(\xi)\,d\xi\bigg)\bigg(\int\limits_{\mathbb{R}^d}\|b(0,x,\zeta)\|\|\phi(-r,\zeta)\|\,d\zeta\bigg)^2dx\leq$$

$$\leq 2C_\rho(T)\mathbf{E}\int\limits_{\mathbb{R}^d}\phi^2(0,x)\rho(x)\,dx+2C_\rho(T)\mathbf{E}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(0,x,\zeta)\|}{\sqrt{\rho(\zeta)}}\,\|\phi(-r,\zeta)\|\sqrt{\rho(\zeta)}\,d\zeta\bigg)^2\rho(x)\,dx\leq$$

$$\leq 2C_\rho(T)\mathbf{E}\|\phi(0)\|^2_{L_2^\rho(\mathbb{R}^d)}+2C_\rho(T)\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(0,x,\zeta)\|^2}{\rho(\zeta)}\,d\zeta\bigg)\rho(x)\,dx\bigg)\mathbf{E}\|\phi(-r)\|^2_{L_2^\rho(\mathbb{R}^d)}<\infty, \tag{5}$$

$$\|I_1(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}b(s,x,\xi)u(\alpha(s),\xi)\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\|b(s,x,\xi)\|\|u(\alpha(s),\xi)\|\,d\xi\bigg)^2\rho(x)\,dx=$$

$$=\mathbf{E}\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|}{\sqrt{\rho(\xi)}}\,\|u(\alpha(s),\xi)\|\sqrt{\rho(\xi)}\,d\xi\bigg)^2\rho(x)\,dx\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\cdot\mathbf{E}\sup_{0\leq s\leq t}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}=$$

$$=\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{0\leq s\leq t^*}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{t^*\leq s\leq t}\|u(\alpha(s))\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{-r\leq s\leq 0}\|\phi(s)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\leq s\leq\alpha(t)}\|u(s)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)\leq$$

$$\leq\bigg(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\bigg)\rho(x)\,dx\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{-r\leq s\leq 0}\|\phi(s)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\leq s\leq t}\|u(s)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)<\infty, \tag{6}$$

$$\|I_2(s)\|^2_{\mathfrak{B}_{2,t,\rho}} =$$

$$= \mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\Big(\sum_{i=1}^d\int\limits_0^s\Big(\partial^2_{x_i}\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\Big(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\Big)\,d\xi\Big)\,d\tau\Big)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\Big(\partial^2_{x_i}\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\Big(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\Big)\,d\xi\Big)\,d\tau\bigg)^2\rho(x)\,dx =$$

$$= d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\bigg(\Big(\Big(\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\Big)\int\limits_{\partial B}\partial_{x_i}\mathscr{K}(s-\tau,x-\xi)\cos(\upsilon,\xi_i)\,dS_\xi+$$

$$+\Big(\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\alpha(\tau),\zeta)\,d\zeta\Big)\int\limits_{\overline{B}}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\,d\xi+$$

$$+\int\limits_{\mathbb{R}^d}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\Big(\int\limits_{\mathbb{R}^d}\big(b(\tau,\xi,\zeta)-b(\tau,x,\zeta)\big)u(\alpha(\tau),\zeta)\,d\zeta\Big)\,d\xi\Big)\,d\tau\bigg)^2\rho(x)\,dx =$$

$$= d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\partial^2_{x_i}\mathscr{K}(s-\tau,x-\xi)\Big(\int\limits_{\mathbb{R}^d}\big(b(\tau,\xi,\zeta)-b(\tau,x,\zeta)\big)u(\alpha(\tau),\zeta)\,d\zeta\Big)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\frac{C}{(s-\tau)^\mu\|x-\xi\|^{d+2-2\mu}}\times$$

$$\times\Big(\sup_{y\in\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\big|\partial_y b(\tau,y,\zeta)\|x-\xi\|u(\alpha(\tau),\zeta)\big|\,d\zeta\Big)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_{\mathbb{R}^d}\frac{C}{\|x-\xi\|^{d+1-2\mu}}\times$$

$$\times\bigg(\int\limits_0^s\frac{\int_{\mathbb{R}^d}\frac{\varphi(\tau,\zeta)}{\sqrt{\rho(\zeta)}}\|u(\alpha(\tau),\zeta)\|\sqrt{\rho(\zeta)}\,d\zeta}{(s-\tau)^{\frac{\mu}{2}}}\,\frac{1}{(s-\tau)^{\frac{\mu}{2}}}\,d\tau\bigg)\,d\xi\bigg)^2\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\bigg(\sup_{0\le s\le t}\int\limits_0^s\frac{d\tau}{(s-\tau)^\mu}\bigg)\times$$

$$\times\bigg(\mathbf{E}\sup_{0\le s\le t}\int\limits_0^s\frac{1}{(s-\tau)^\mu}\Big(\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\Big)\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)\rho(x)\,dx \le$$

$$\le d\int\limits_{\mathbb{R}^d}\sum_{i=1}^d\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\bigg(\sup_{0\le s\le t}\int\limits_0^s\frac{d\tau}{(s-\tau)^\mu}\bigg)^2\times$$

$$\times\Big(\mathbf{E}\sup_{0\le\tau\le s}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\Big)\Big(\sup_{0\le\tau\le s}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\Big)\rho(x)\,dx \le$$

$$\le d^2C^2\bigg(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\bigg)^2\,\frac{t^{2-2\mu}}{(1-\mu)^2}\,\Big(\int\limits_{\mathbb{R}^d}\rho(x)\,dx\Big)\Big(\sup_{0\le\tau\le t}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\Big)\times$$

$$\times\Big(\mathbf{E}\sup_{-r\le\tau\le0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+\mathbf{E}\sup_{0\le\tau\le t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\Big)<\infty,\ \ \frac12<\mu<1,\ \ C>0, \tag{7}$$

$$\|I_3(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=\mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)f(\tau,u(\alpha(\tau),\xi),\xi)\,d\xi\,d\tau\bigg)^2\rho(x)\,dx\le$$

$$\le t\,\mathbf{E}\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\bigg(\int\limits_0^s\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,f\big(\tau,u(\alpha(\tau),\xi),\xi\big)\,d\xi\bigg)^2d\tau\bigg)\rho(x)\,dx\le$$

$$\le t\mathbf{E}\sup_{0\le s\le t}\int\limits_0^s\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\rho(\xi)\,d\xi\bigg)f^2\big(\tau,u(\alpha(\tau),x),x\big)\,dx\,d\tau\le$$

$$\le C_\rho(T)t\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}f^2\big(\tau,u(\alpha(\tau),x),x\big)\rho(x)\,dx\,d\tau\le$$

$$\le L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+\mathbf{E}\int\limits_0^t\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)=$$

$$=L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+\mathbf{E}\int\limits_0^{t^*}\big\|u(\alpha(\tau))\big\|^2_{L_2^\rho(\mathbb{R}^d)}\,\frac{1}{\alpha'(\tau)}\,\alpha'(\tau)\,d\tau+$$

$$+\mathbf{E}\int\limits_{t^*}^t\big\|u(\alpha(\tau))\big\|^2_{L_2^\rho(\mathbb{R}^d)}\,\frac{1}{\alpha'(\tau)}\alpha'(\tau)\,d\tau\bigg)\le$$

$$\le L^2C_\rho(T)t\bigg(t\int\limits_{\mathbb{R}^d}\rho(x)\,dx+c\mathbf{E}\int\limits_{-r}^0\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau+c\mathbf{E}\int\limits_0^{\alpha(t)}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\,d\tau\bigg)\le$$

$$\le L^2C_\rho(T)t^2\bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx+c\mathbf{E}\sup_{-r\le\tau\le0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+c\mathbf{E}\sup_{0\le\tau\le t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\bigg)<\infty, \tag{8}$$

$$\|I_4(s)\|^2_{\mathfrak{B}_{2,t,\rho}}=$$

$$=\int\limits_{\mathbb{R}^d}\mathbf{E}\sup_{0\le s\le t}\bigg(\int\limits_0^s\sum_{n=1}^\infty\sqrt{\lambda_n}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\sigma(\tau,u(\alpha(\tau),\xi),\xi)e_n(\xi)\,d\xi\bigg)\,d\beta_n(\tau)\bigg)^2\rho(x)\,dx\le$$

$$\le4\sum_{n=1}^\infty\lambda_n\times$$

$$\times\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\sqrt{\mathscr{K}(s-\tau,x-\xi)}\,\sqrt{\mathscr{K}(s-\tau,x-\xi)}\sigma(\tau,u(\alpha(\tau),\xi),\xi)e_n(\xi)\,d\xi\bigg)^2\rho(x)\,dx\bigg)\,d\tau\le$$

$$\le4\sum_{n=1}^\infty\lambda_n\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\bigg(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\rho(\xi)\,d\xi\bigg)\sigma^2(\tau,u(\alpha(\tau),x),x)e_n^2(x)\,dx\bigg)\,d\tau\le$$

$$\le4C_\rho(T)\sum_{n=1}^\infty\lambda_n\cdot\mathbf{E}\int\limits_0^t\bigg(\int\limits_{\mathbb{R}^d}\sigma^2(\tau,u(\alpha(\tau),x),x)e_n^2(x)\rho(x)\,dx\bigg)\,d\tau\le$$

$$\leq 4L^2 C_\rho(T)\Big(\sum_{n=1}^{\infty}\lambda_n\Big)t\Big(\int\limits_{\mathbb{R}^d}\rho(x)\,dx + c\mathbf{E}\sup_{-r\leq\tau\leq 0}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)} + c\mathbf{E}\sup_{0\leq\tau\leq t}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\Big) < \infty. \quad (9)$$

Thus estimates (5)–(9) imply that for $u \in \mathfrak{B}_{2,T,\rho}$, $\|\Psi(u)\|^2_{\mathfrak{B}_{2,T,\rho}} \leq 5\sum\limits_{i=0}^{4}\|I_i(t)\|^2_{\mathfrak{B}_{2,T,\rho}} < \infty$, – that is $\Psi$ is well defined. The second step is to prove that the operator under consideration has a unique fixed point. Indeed, taking into account estimates (6)–(9), for any $\{u,v\}\subset\mathfrak{B}_{2,t,\rho}$ we conclude

$$\|\Psi(u)-\Psi(v)\|^2_{\mathfrak{B}_{2,t,\rho}} = \Big\|\sum_{i=1}^4 I_i(s)(u) - \sum_{i=1}^4 I_i(s)(v)\Big\|^2_{\mathfrak{B}_{2,t,\rho}} =$$

$$= \Big\|\sum_{i=1}^4\big(I_i(s)(u)-I_i(s)(v)\big)\Big\|^2_{\mathfrak{B}_{2,t,\rho}} \leq 4\sum_{i=1}^4\big\|I_i(s)(u)-I_i(s)(v)\big\|^2_{\mathfrak{B}_{2,t,\rho}} \leq$$

$$\leq 4\Big(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d}\frac{\|b(s,x,\xi)\|^2}{\rho(\xi)}\,d\xi\Big)\rho(x)\,dx +$$

$$+ d^2C^2\Big(\int\limits_{\mathbb{R}^d}\frac{d\xi}{\|x-\xi\|^{d+1-2\mu}}\Big)^2\frac{t^{2-2\mu}}{(1-\mu)^2}\Big(\int\limits_{\mathbb{R}^d}\rho(x)\,dx\Big)\Big(\sup_{0\leq\tau\leq t}\int\limits_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta)}{\rho(\zeta)}\,d\zeta\Big)+$$

$$+ L^2C_\rho(T)ct^2 + 4L^2C_\rho(T)\Big(\sum_{n=1}^\infty\lambda_n\Big)ct\Big)\|u-v\|^2_{\mathfrak{B}_{2,t,\rho}} = \gamma(t)\|u-v\|^2_{\mathfrak{B}_{2,t,\rho}}.$$

Because of the assumption of the theorem, the first term of $\gamma$ is less than one. Therefore, by choosing small $0\leq t_1\leq T$, we conclude that $0\leq\gamma(t_1)\leq 1$. It means that $\Psi$, defined in the Banach space $\mathfrak{B}_{2,t_1,\rho}$, is contractive, and therefore, by the theorem of Banach on a contractive mapping, has a unique fixed point – the solution $u\in\mathfrak{B}_{2,t_1,\rho}$ of the equation $\Psi(u)=u$ that can be obviously presented in the form (3) and satisfies (4), that is a mild solution in $\mathfrak{B}_{2,t_1,\rho}$ of (1) on the interval $[0,t_1]$. This procedure can be repeated finitely many steps on other sufficiently small intervals $[t_1,t_2],[t_2,t_3],\dots,[t_{n-2},t_{n-1}],[t_{n-1},T]$ – components of the entire interval $[0,T]$ – and, as a result, we get the solution as a union of the solutions on these intervals. $\square$

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# On the Representations of Sensitivity Coefficients for Nonlinear Delay Functional Differential Equations with the Discontinuous Initial Condition

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Let $\mathbb{R}^n$ be the $n$-dimensional vector space; suppose that $O \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^r$ are open sets. Let $0 < \tau_1 < \tau_2$ and $a < t_{01} < t_{02} < t_1 < b$ be given numbers with $t_{02} + \tau_2 < t_1$; let the $n$-dimensional function $f(t, x, y, u)$ be continuous on $I \times O^2 \times U$ and continuously differentiable with respect to $(x, y, u)$, where $I = [a, b]$. Furthermore, $\Phi$ is the set of continuous initial functions $\varphi : I_1 \to O$, where $I_1 = [\widehat{\tau}, b], \widehat{\tau} = a - \tau_2$ and $\Omega$ is the set of measurable control functions $u : I \to U$ with $\operatorname{cl} u(I)$ is compact set and $\operatorname{cl} u(I) \subset U$.

To each initial data $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda = (t_{01}, t_{02}) \times (\tau_1, \tau_2) \times O \times \Phi \times \Omega$ we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau), u(t)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x(t_0) = x_0. \tag{2}$$

**Definition 1.** Let $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to $\mu$ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed initial data. Introduce the following notations: $\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi(t), \delta u(t)) \in \Lambda - \mu_0 = \{\delta\mu = \mu - \mu_0 : \; \mu \in \Lambda\}$, $\delta\mu$ is called variation of the initial data $\mu_0$ and $\Lambda - \mu_0$ is called the set of variations. Next,

$$\|\delta\mu\| = |\delta t_0| + |\delta\tau| + |\delta x_0| + \|\delta\varphi\| + \|\delta u\|,$$

where

$$\|\delta\varphi\| = \sup\big\{|\delta\varphi(t)| : \; t \in I_1\big\}, \quad \|\delta u\| = \sup\big\{|\delta u(t)| : \; t \in I\big\}.$$

Let the solution $x(t; \mu_0)$ is defined on $[\widehat{\tau}, t_1]$. Then there exists number $\varepsilon_1 > 0$ such that for any $\delta\mu \in \Lambda_{\varepsilon_1} = \{\delta\mu \in \Lambda - \mu_0 : \; \|\delta\mu\| \leq \varepsilon_1\}$ there exists solution $x(t; \mu_0 + \delta\mu)$ defined on the interval $[\widehat{\tau}, t_1]$, [1].

**Theorem 1.** *Let the solution $x(t; \mu_0)$ be defined on $[\widehat{\tau}, t_1]$ and let the function $\varphi_0(t)$ be absolutely continuous. Moreover, let there exist the finite limits*

$$\lim_{w \to w_0} f(w, u_0(t)) = f_0^-, \;\; w = (t, x, y) \in (t_{01}, t_{00}] \times O^2$$

*and*

$$\lim_{(w_1,w_2)\to(w_{11},w_{12})} \big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1^-,\ \ w_1,w_2\in(t_{00},t_{00}+\tau_0]\times O^2,$$

*where*

$$w_0=\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0)\big),$$
$$w_{11}=\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00}\big),\ \ w_{12}=\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00})\big).$$

*Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that on the interval $[t_1-\delta,t_1]\subset(t_{00}+\tau_0,t_1]$ for arbitrary $\delta\mu\in\Lambda^-_{\varepsilon_2}=\{\delta\mu\in\Lambda_{\varepsilon_2}:\ \delta t_0\le 0,\ \delta\tau\le 0\}$ we have*

$$x(t;\mu_0+\delta\mu)=x(t;\mu_0)+\delta x^-(t;\delta\mu)+o(t;\delta\mu),$$

*where*

$$\delta x^-(t;\delta\mu)=Y(t_{00};t)\delta x_0-\big[Y(t_{00};t)f_0^-+Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^-\delta\tau+\gamma(t;\delta\mu)$$

*and*

$$\gamma(t;\delta\mu)=Y(t_{00};t)\delta x_0-\bigg[\int\limits_{t_{00}}^{t}Y(s;t)f_{0y}[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau+$$
$$+\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds+\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\delta u(s)\,ds. \tag{3}$$

*Here*

$$\dot x_0(t)=\dot\varphi_0(t),\ \ t\in(t_{00}-\tau_0,t_{00}),\quad f_{0y}[s]=f_y\big(s,x_0(s),x_0(s-\tau_0),u_0(s)\big);$$

*$Y(\xi;t)$ is the $n\times n$-matrix function satisfying the linear functional differential equation with advanced argument*

$$Y_s(s;t)=-Y(s;t)f_{0x}[s]-Y(s+\tau_0;t)f_{0y}[s+\tau_0],\ \ s\in[t_{00},t]$$

*and the condition*

$$Y(s;t)=\begin{cases} E & \textit{for } s=t,\\ \Theta & \textit{for } s>t,\end{cases}$$

*$E$ is the identity matrix and $\Theta$ is the zero matrix.*

**Theorem 2.** *Let the solution $x(t;\mu_0)$ be defined on $[\widehat\tau,t_1]$ and let the function $\varphi_0(t)$ be absolutely continuous. Moreover, let there exist the finite limits*

$$\lim_{w\to w_0} f(w,u_0(t))=f_0^+,\ \ w\in[t_{00},t_{00}+\tau_0)\times O^2$$

*and*

$$\lim_{(w_1,w_2)\to(w_{11},w_{12})}\big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1^+,\ \ w_1,w_2\in[t_{00}+\tau_0,t_1)\times O^2.$$

*Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that on the interval $[t_1-\delta,t_1]$ for arbitrary $\delta\mu\in\Lambda^+_{\varepsilon_2}=\{\delta\mu\in\Lambda_{\varepsilon_2}:\ \delta t_0\ge 0,\ \delta\tau\ge 0\}$ we have*

$$x(t;\mu_0+\delta\mu)=x(t;\mu_0)+\delta x^+(t;\delta\mu)+o(t;\delta\mu),$$

*where*

$$\delta x^+(t;\delta\mu)=-\big[Y(t_{00};t)f_0^++Y(t_{00}+\tau_0;t)f_1^+\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^+\delta\tau+\gamma(t;\delta\mu).$$

**Theorem 3.** *Let the assumptions of Theorems* 1 *and* 2 *be fulfilled. Moreover, let*

$$f_0^- = f_0^+ := f_0, \;\; f_1^- = f_1^+ := f_1.$$

*Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta > 0$ *such that on the interval* $[t_1 - \delta, t_1]$ *for arbitrary* $\delta\mu \in \Lambda_{\varepsilon_2}$ *we have*

$$x(t;\mu_0 + \delta\mu) = x(t;\mu_0) + \delta x(t;\delta\mu) + o(t;\delta\mu), \tag{4}$$

*where*

$$\delta x(t;\delta\mu) = -\big[Y(t_{00};t)f_0 + Y(t_{00}+\tau_0;t)f_1\big]\delta t_0 - Y(t_{00}+\tau_0;t)f_1\delta\tau + \gamma(t;\delta\mu). \tag{5}$$

# Some Comments

Theorems 1 and 2 correspond to cases when the variations at the point $t_{00}$ are performed on the left and on the right, respectively. Theorem 3 corresponds to the case when at the point $t_{00}$ two-sided variation is performed. The function $\delta x(t;\delta\mu)$ in the formula (4) is called the coefficient of sensitivity. The expression (5) is called representation of the sensitivity coefficient. The summand

$$-\bigg[Y(t_{00}+\tau_0;t)f_1 + \int\limits_{t_{00}}^{t} Y(s:t)f_{0y}[s]\dot{x}_0(s-\tau_0)\,ds\bigg]\delta\tau$$

in formula (5) (see (3)) is the effect of perturbation of the delay $\tau_0$.The expression

$$-\big[Y(t_{00};t)f_0 + Y(t_{00}+\tau_0;t)f_1\big]\delta t_0$$

in formula (5) (see again (3)) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment $t_{00}$. The expression

$$Y(t_{00};t)\delta x_0 + \int\limits_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds + \int\limits_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)\,ds$$

in formula (5) (see (3)) is the effect of perturbations of initial vector $x_{00}$, initial function $\varphi_0(t)$ and control function $u_0(t)$. It is clear that (5) can be rewrite in the form

$$\delta x(t;\delta\mu) = \delta x_1(t;\delta\mu) + \delta x_2(t;\delta\mu),$$

where

$$\begin{aligned} \delta x_1(t;\delta\mu) = {} & Y(t_{00};t)[\delta x_0 - f_0\delta t_0] + \\ & + \int\limits_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds + \int\limits_{t_{00}}^{t} Y(s;t)\big[f_{0u}[s]\delta u(s) - f_{0y}[s]\dot{x}_0(s-\tau_0)\delta\tau\big]\,ds \end{aligned}$$

and

$$\delta x_2(t;\delta\mu) = -Y(t_{00}+\tau_0;t)f_1[\delta\tau + \delta t_0].$$

On the basis of the Cauchy formula on representation of solutions of the linear delay functional differential equation we get that the function $\delta x_1(t;\delta\mu)$ on the interval $[t_{00}, t_1]$ satisfies the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + f_{0y}[t]\delta x(t-\tau_0) - f_{0y}[t]\dot{x}_0(t-\tau_0)\delta\tau + f_{0u}[t]\delta u(t)$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \;\; t \in [\widehat{\tau}, t_{00}), \;\; \delta x(t_{00}) = \delta x_0 - f_0\delta t_0,$$

and $\delta x_2(t;\delta\mu)$ on the interval $[t_{00}+\tau_0,t_1]$ satisfies the equation

$$\dot{\delta x}(t)=f_{0x}[t]\delta x(t)+f_{0y}[t]\delta x(t-\tau_0)$$

with the initial condition

$$\delta x(t)=0, t\in[t_{00},t_{00}+\tau_0), \delta x(t_{00}+\tau_0)=-f_1(\delta\tau+\delta t_0).$$

Thus, if $\delta x_1(t;\delta\mu)$ and $\delta x_2(t;\delta\mu)$ are solutions of the above considered linear differential equations with the corresponding initial conditions, then their sum will the coefficient of sensitivity on the interval $[t_1-\delta,t_1]$. Sensitivity analysis for various classes of functional differential equations are considered in [2–4].

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# Asymptotic Behavior of Positive Solutions of Second Order Half-Linear Differential Equations with Deviating Arguments of Mixed Type

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We consider the second order half-linear differential equations

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{m} q_i(t)\varphi(x(g_i(t))) \pm \sum_{j=1}^{n} r_j(t)\varphi(x(h_j(t))) = 0, \;\; t \geqq a, \tag{A$_\pm$}$$

$$(\varphi(\xi) = |\xi|^{\alpha-1}\xi = |\xi|^{\alpha}\operatorname{sgn}\xi, \;\; \alpha > 0, \;\; \xi \in \mathbb{R}, \;\; \text{Doubles sign correspondence})$$

for which the following conditions are always assumed to hold:

(a) $p, q_i, r_j : [a,\infty) \to (0,\infty)$, $a \geqq 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ are continuous functions;

(b) $p(t)$ satisfies

$$\int_a^{\infty} \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty,$$

and $\pi(t)$ is defined by

$$\pi(t) = \int_t^{\infty} \frac{ds}{p(s)^{\frac{1}{\alpha}}}. \tag{1}$$

By a positive solution on an interval $J$ of the differential equation (A$_\pm$) we mean a function $x : J \to (0,\infty)$ which is continuously differentiable on $J$ together with $p(t)\varphi(x'(t))$ and satisfies (A$_\pm$) there.

Since the publication of the book ( [8]) of Marić in the year 2000, the class of regularly varying functions in the sense of Karamata ( [4]) is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x''(t) = q(t)x(t), \;\; q(t) > 0.$$

The study of asymptotic analysis of nonoscillatory solutions of functional differential equation with deviating arguments in the framework of regularly varying functions was first attempted by Kusano and Marić ( [5,6]). They established a sharp condition for the existence of a slowly varying solution of the second order functional differential equation with retarded argument of the form

$$x''(t) = q(t)x(g(t))$$

and the following functional differential equation with both retarded and advanced arguments of the form

$$x''(t) \pm q(t)x(g(t)) \pm r(t)x(h(t)) = 0,$$

where $q, r : [a,\infty) \to (0,\infty)$ are continuous functions, $g$, $h$ are continuous and increasing with $g(t) < t$, $h(t) > t$ for $t \geqq a$ and $\lim\limits_{t\to\infty} g(t) = \infty$.

## The Definitions and Properties of Regularly Varying Functions

**Definition 1.** A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be a regularly varying of index $\rho$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \ \text{ for any } \ \lambda > 0, \ \ \rho \in \mathbb{R}.$$

**Proposition 1** (Representation Theorem)**.** *A measurable function $f : [a, \infty) \to (0, \infty)$ is regularly varying of index $\rho$ if and only if it can be written in the form*

$$f(t) = c(t) \exp\bigg\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \bigg\}, \ \ t \geqq t_0,$$

*for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \ \ \textit{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index $\rho$ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV}=\mathrm{RV}(0)$ is referred to as a slowly varying function. If $f \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho L(t)$ for some $L \in$SV. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ \ (m_i \in \mathbb{R}), \quad \exp\Big\{ \prod_{i=1}^{N} (\log_i t)^{n_i} \Big\} \ \ (0 < n_i < 1), \quad \exp\Big\{ \frac{\log t}{\log_2 t} \Big\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \dots, N$, also belong to the set of slowly varying functions.

**Proposition 2.** *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^\gamma L(t) = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels ( [1]).

## The Definitions and Properties of Generalized Regularly Varying Functions

**Definition 2.** A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be slowly varying with respect to $1/\pi(t)$ if the function $f \circ (1/\pi(t))^{-1}$ is slowly varying in the sense of Karamata, where the function $\pi(t)$ is defined by (1) and $(1/\pi(t))^{-1}$ denotes the inverse function of $1/\pi(t)$. The totality of slowly varying functions with respect to $1/\pi(t)$ is denoted by $\mathrm{SV}_{\frac{1}{\pi}}$.

**Definition 3.** A measurable function $g : [a, \infty) \to (0, \infty)$ is said to be regularly varying function of index $\rho$ with respect to $1/\pi(t)$ if the function $g \circ (1/\pi(t))^{-1}$ is regularly varying of index $\rho$ in the sense of Karamata. The set of all regularly varying functions of index $\rho$ with respect to $1/\pi(t)$ is denoted by $\mathrm{RV}_{\frac{1}{\pi}}(\rho)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Proposition 1.

**Proposition 3.**

(i) *A function $f(t)$ is slowly varying with respect to $1/\pi(t)$ if and only if it can be expressed in the form*

$$f(t) = c(t)\exp\bigg\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}\pi(s)}\,ds\bigg\},\ \ t \geqq t_0 \tag{2}$$

*for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \text{and} \ \ \lim_{t\to\infty} \delta(t) = 0.$$

(ii) *A function $g(t)$ is regularly varying of index $\rho$ with respect to $1/\pi(t)$ if and only if it has the representation*

$$g(t) = c(t)\exp\bigg\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}\pi(s)}\,ds\bigg\},\ \ t \geqq t_0 \tag{3}$$

*for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \text{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

If the function $c(t)$ in (2) (or (3)) is identically a constant on $[t_0,\infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regularly varying of index $\rho$) with respect to $1/\pi(t)$. The totality of such functions is denoted by n-$\mathrm{SV}_{\frac{1}{\pi}}$ (or n-$\mathrm{RV}_{\frac{1}{\pi}}(\rho)$).

It is easy to see that if $g \in \mathrm{RV}_{\frac{1}{\pi}}(\rho)$ (or n-$\mathrm{RV}_{\frac{1}{\pi}}(\rho)$), then $g(t) = (1/\pi(t))^{\rho}L(t)$ for some $L \in \mathrm{SV}_{\frac{1}{\pi}}$ (or n-$\mathrm{SV}_{\frac{1}{\pi}}$).

**Proposition 4.** *Let $L \in SV_{\frac{1}{\pi}}$. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty}\Big(\frac{1}{\pi(t)}\Big)^{\gamma} L(t) = \infty \ \ \text{and} \ \ \lim_{t\to\infty}\Big(\frac{1}{\pi(t)}\Big)^{-\gamma} L(t) = 0.$$

## Main Result

In our previous paper ( [3,7]) we have studied the problem of nonoscillation and asymptotic analysis of the half-linear differential equation involving nonlinear Sturm–Liouville type differential operator of the type

$$(p(t)\varphi(x'(t)))' \pm q(t)\varphi(x(t)) = 0 \tag{B$_\pm$}$$

and the half-linear functional differential equation with deviating arguments of the mixed type

$$(\varphi(x'(t)))' \pm q(t)\varphi(x(g(t))) \pm r(t)\varphi(x(h(t))) = 0, \tag{C$_\pm$}$$

where the functions $p(t)$, $q(t)$, $r(t)$, $g(t)$ and $h(t)$ are just as in the above equations.

**Theorem A** (J. Jaroš, T. Kusano and T. Tanigawa, [3]). *Suppose that* (1) *holds. The equation* (B$_\pm$) *have a normalized slowly varying solution with respect to $1/\pi(t)$ and a normalized regularly varying solution of index $-1$ with respect to $1/\pi(t)$ if and only if*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_{t}^{\infty} \pi(s)^{\alpha+1} q(s)\,ds = 0.$$

**Theorem B** (J. Manojlović and T. Tanigawa, [7]). *Suppose that*

$$\lim_{t\to\infty}\frac{g(t)}{t}=1 \ \ \textit{and} \ \ \lim_{t\to\infty}\frac{h(t)}{t}=1$$

*hold. Then, the equation* ($C_\pm$) *have a slowly varying solution and a regularly varying solution of index* 1 *if and only if*

$$\lim_{t\to\infty} t^\alpha\int\limits_t^\infty q(s)\,ds=\lim_{t\to\infty} t^\alpha\int\limits_t^\infty r(s)\,ds=0.$$

Aim of this talk is to establish a sharp condition of the existence of a normalized slowly varying solution with respect to $1/\pi(t)$ and a normalized regularly varying solution of index $-1$ with respect to $1/\pi(t)$ of the equation ($A_\pm$). Our main result is the following

**Theorem.** *Suppose that*

$$\lim_{t\to\infty}\frac{\pi(g_i(t))}{\pi(t)}=1 \ \ \textit{for} \ \ i=1,2,\dots,m$$

*and*

$$\lim_{t\to\infty}\frac{\pi(h_j(t))}{\pi(t)}=1 \ \ \textit{for} \ \ j=1,2,\dots,n$$

*hold. The equation* ($A_\pm$) *possesses a normalized slowly varying solution with respect to* $1/\pi(t)$ *and a normalized regularly varying solution of index* $-1$ *with respect to* $1/\pi(t)$ *if and only if*

$$\lim_{t\to\infty}\frac{1}{\pi(t)}\int\limits_t^\infty \pi(s)^{\alpha+1}q_i(s)\,ds=0 \ \ \textit{for} \ \ i=1,2,\dots,m$$

*and*

$$\lim_{t\to\infty}\frac{1}{\pi(t)}\int\limits_t^\infty \pi(s)^{\alpha+1}r_j(s)\,ds=0 \ \ \textit{for} \ \ j=1,2,\dots,n.$$

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# On Asymptotics of Solutions for Sufficiently Non-Linear Differential Equations of the Second Order

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_2(y'), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\,\to\,]0,+\infty[$ is a continuous function, $\varphi_i : \Delta(Y_i^0) \to\,]0,+\infty[$ $(i=1,2)$ are twice continuously differentiable functions, where $\Delta(Y_i^0)$ is some one-sided neighborhood of the point $Y_i^0$, $Y_i^0$ equals either zero or $\pm\infty$. For these functions the following conditions are satisfied:

$$\lim_{\substack{z\to Y_1^0\\ z\in\Delta(Y_1^0)}} \frac{z\varphi_1'(z)}{\varphi_1(z)} = \lambda \quad (\lambda\in\mathbb{R}), \tag{2}$$

$$\varphi_2'(z)\neq 0 \ \text{ for } \ z\in\Delta(Y_2^0), \qquad \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \varphi_2(z) = \Phi_2^0, \ \ \Phi_2^0\in\{0,+\infty\}, \qquad \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \frac{\varphi_2''(z)\varphi_2(z)}{[\varphi_2'(z)]^2} = 1. \tag{3}$$

Conditions (2), (3) define that the function $\varphi_1(z)$ is regularly or slowly varying as $z\to Y_1^0$, and $\varphi_2(z)$ is rapidly varying as $z\to Y_2^0$ (see, E. Seneta [1]).

For power-functions and regularly varying nonlinearities $\varphi_i$ $(i=1,2)$, the asymptotics of solutions for (1) are investigated in [2–10].

For equations of the type (1), in [11] the following class of monotonous solutions was introduced.

A solution $y$ of the equation (1) is called $P_\omega(\Lambda_0)$-solution, where $-\infty\le\Lambda_0\le+\infty$, if it is defined on some interval $[t_0,\omega[\,\subset[a,\omega[$ and satisfies the conditions

$$\lim_{t\uparrow\omega} y(t) = Y_1^0, \quad \lim_{t\uparrow\omega}\varphi_2(y'(t)) = \Phi_2^0, \quad \lim_{t\uparrow\omega}\frac{\varphi_2'(y'(t))}{\varphi_2(y'(t))}\,\frac{y''(t)y(t)}{y'(t)} = \Lambda_0.$$

Earlier, in case $\Lambda_0\in\mathbb{R}\setminus\{0\}$, the asymptotics of $P_\omega(\Lambda_0)$-solutions of (1) were established in [11].

Present work is devoted to the establishment of asymptotics, as well as sufficient and necessary conditions for the existence of $P_\omega(\Lambda_0)$-solutions of (1), when $\Lambda_0=0$. In order to formulate the main result, we introduce auxiliary definitions and notations.

We determine that slowly varying function $\theta : \Delta(U^0)\to\,]0,+\infty[$, $U^0\in\{0,\pm\infty\}$ satisfies the condition $S$ if for any continuously differentialble function $l : \Delta(U^0)\to\,]0,+\infty[$ such that

$$\lim_{\substack{z\to U^0\\ z\in\Delta(U^0)}} \frac{z\,l'(z)}{l(z)} = 0,$$

the following asymptotic representation is valid

$$\theta(zl(z)) = \theta(z)[1+o(1)] \ \text{ when } \ z\to U^0 \ \ (z\in\Delta(U^0)).$$

We introduce numbers

$$\mu_i^0 = \begin{cases} 1, & \text{if } Y_i^0=+\infty, \ \text{ or } \ Y_i^0=0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is right neighborhood of } 0,\\ -1, & \text{if } Y_i^0=-\infty, \ \text{ or } \ Y_i^0=0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is left neighborhood of } 0 \end{cases} \qquad (i=1,2).$$

These numbers define the signs of $P_\omega(0)$-solutions of (1) and their derivatives in some left neighborhood of $\omega$.

We also define the functions

$$\pi_\omega(t)=\begin{cases} t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty, \end{cases} \qquad J(t)=\int_A^t p(\tau)\varphi_1\big(\mu_1^0|\pi_\omega(\tau)|\big)\,d\tau,$$

where $A\in\{\omega,a\}$ and it is chosen so that the integral $J$ tends either to zero or to $\infty$ as $t\uparrow\omega$.

In addition, we introduce the numbers

$$A_1^*=\begin{cases} 1, & \text{if } \omega=\infty,\\ -1, & \text{if } \omega<\infty, \end{cases} \qquad A_2^*=\begin{cases} 1, & \text{if } A=a,\\ -1, & \text{if } A=\omega. \end{cases}$$

Since the function $\varphi_1(z)$ is regularly varying of the $\lambda$-order as $z\to Y_1^0$, the following representation is valid:

$$\varphi_1(z)=|z|^\lambda\theta_1(z),$$

where the function $\theta_1(z)$ is slowly varying as $z\to Y_1^0$.

**Theorem 1.** *Let the function $\theta_1(z)$ satisfy the condition $S$. Then for the existence of $P_\omega(0)$-solutions of* (1) *it is necessary and sufficient that*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)}=0 \tag{4}$$

*and the following conditions to be satisfied*

$$\begin{gathered} A_1^*>0 \ \text{ when } \ Y_1^0=\pm\infty, \quad A_1^*<0 \ \text{ when } \ Y_1^0=0,\\ A_2^*>0 \ \text{ when } \ \Phi_2^0=0, \quad A_2^*<0 \ \text{ when } \ \Phi_2^0=\pm\infty, \end{gathered} \tag{5}$$

$$\mu_1^0\mu_2^0A_1^*>0 \quad \text{and} \quad \alpha_0\mu_2^0A_2^*>0. \tag{6}$$

*Moreover, each solution of that kind admits the following asymptotic representation as $t\uparrow\omega$:*

$$\begin{aligned} \frac{y(t)}{y'(t)}&=\pi_\omega(t)[1+o(1)],\\ \frac{1}{|y'|^\lambda\varphi_2'(y'(t))}&=-\alpha_0J(t)[1+o(1)], \end{aligned}$$

*and there exists a one-parametric family of these solutions if there is only one positive number among $A_1^*$, $A_2^*$, and a two-parametric family of these solutions if both numbers $A_1^*$, $A_2^*$ are positive.*

**Theorem 2.** *Let the functions $\theta_1(z)$, $|\psi^{-1}(z)|$ satisfy the condition $S$. Then each $P_\omega(0)$-solution of the differential equation* (1) *(in case of its existence) admits the following asymptotic representations as $t\uparrow\omega$:*

$$\begin{aligned} y(t)&=\mu_1^0\big|\pi_\omega(t)\psi^{-1}\big(\mu_2^0|J(t)|\big)\big|\,[1+o(1)],\\ \frac{1}{\varphi_2'(y'(t))}&=-\mu_2^0|J(t)|\,\big|\psi^{-1}\big(\mu_2^0|J(t)|\big)\big|^\lambda[1+o(1)]. \end{aligned}$$

These results could be illustrated for the equation

$$y''=\alpha_0p(t)|y|^\lambda|\ln|y||^\gamma e^{-\sigma|y'|^\delta}|y'|^{1-\delta}, \tag{7}$$

where $\alpha_0\in\{1,-1\}$, $\delta,\sigma\in\mathbb{R}\setminus\{0\}$, $\lambda,\gamma\in\mathbb{R}$, $p:[a,\omega[\,\to\,]0,+\infty[$ is a continuous function.

For this equation $\varphi_1(z) = |z|^\lambda \ln^\gamma |z|$, $\varphi_2(z) = e^{-\sigma|z|^\delta}|z|^{1-\delta}$. The function $\varphi_1(z)$ is regularly varying of the $\lambda$-order as $z \to Y_2^0$. For $\delta > 0$, the function $\varphi_2(z)$ is rapidly varying as $z \to \pm\infty$, and for $\delta < 0$, the function $\varphi_2(z)$ is rapidly varying as $z \to 0$.

For (7), the $P_\omega(0)$-solution is

$$\lim_{t\uparrow\omega} \frac{yy''(t)}{|y'(t)|^{2-\delta}} = 0.$$

For the existence of $P_\omega(0)$-solution for the equation (7), it is necessary and sufficient the conditions (4)–(6) to be satisfied. Moreover, each solution of that kind admits the following asymptotic representation as $t \uparrow \omega$

$$y(t) = \mu_1^0 |\pi_\omega(t)| \left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right|^{\frac{1}{\delta}} [1 + o(1)],$$

$$y'(t) = \mu_2^0 \left|\frac{1}{\sigma} \ln|\sigma\delta J(t)| + \frac{\lambda}{\sigma\delta} \ln\left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right| + o(1)\right|^{\frac{1}{\delta}},$$

and there exists a one-parametric family of such solutions if there is only one positive number among $A_1^*$, $A_2^*$, and there exists a two-parametric family of such solutions if both numbers $A_1^*$, $A_2^*$ are positive.

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