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Cohen-Macaulay Dimension for Coherent Rings

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COHEN-MACAULAY DIMENSION FOR COHERENT RINGS

by

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COHEN-MACAULAY DIMENSION FOR COHERENT RINGS

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This dissertation presents a homological dimension notion of Cohen-Macaulay for non-Noetherian rings which reduces to the standard definition in the case that the ring is Noetherian, and is inspired by the homological notion of Cohen-Macaulay for local rings developed by Gerko in [13]. Under this notion, both coherent regular rings (as defined by Bertin in [3]) and coherent Gorenstein rings (as defined by Hummel and Marley in [20]) are Cohen-Macaulay.

This work is motivated by Glaz's question in [15] and [16] regarding whether a notion of Cohen-Macaulay exists for coherent rings which satisfies certain properties and agrees with the usual notion when the ring is Noetherian. Hamilton and Marley gave one answer in [18]; we develop an alternative approach using homological dimensions which seems to have more satisfactory properties. We explore properties of coherent Cohen-Macaulay rings, as well as their relationship to non-Noetherian Cohen-Macaulay rings as defined by Hamilton and Marley.

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Chapter 1

Introduction

Special classes of commutative Noetherian rings (e.g., rings which are regular, Gorenstein, or Cohen-Macaulay), the interactions between these special classes of rings, and the properties of modules over such rings form a rich theory in commutative algebra. In the case of local Noetherian rings, we have the following hierarchy of rings:

$$\text{regular} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay.}$$

The focus of this work is Cohen-Macaulay rings, in particular, on extending the theory of Cohen-Macaulay rings beyond the Noetherian setting in a way which preserves the relationships shown above for these three special classes of rings. This work is motivated by the following question asked by Glaz in [15], [16]:

Question. Is there a workable definition of Cohen-Macaulay for commutative rings which extends the usual definition in the Noetherian case, and such that every coherent regular ring is Cohen-Macaulay?

In the Noetherian case, these special types of rings are typically defined using classical ring-theoretic invariants (e.g., depth, dimension, embedding dimension, etc.).

However, these rings can also be classified using various homological dimensions. For instance, regularity is characterized by projective dimension: a local ring (R, m) is regular if and only if every finitely generated R-module has finite projective dimension, or equivalently, if and only if the residue field R/m has finite projective dimension. Gorenstein rings are characterized by G-dimension, as introduced by Auslander and Bridger [1]: a local ring (R, m) is Gorenstein if and only if every finitely generated R-module has finite G-dimension, or equivalently, if and only if the residue field R/m has finite G-dimension.

In [3], Bertin extends the homological characterization of regularity to quasi-local rings (i.e., rings which have a unique maximal ideal but are not necessarily Noetherian) by defining a ring to be regular if every finitely generated ideal has finite projective dimension. Similarly, Hummel and Marley develop a notion of Gorenstein for quasi-local rings by defining a ring to be Gorenstein if every finitely generated ideal has finite G-dimension [20]. These definitions reduce to the standard definitions if the ring is local and Noetherian, and much of the known behavior of regular and Gorenstein rings is enjoyed by these non-Noetherian analogs.

In [18], Hamilton and Marley provide an affirmative answer to Glaz's question. They define a notion of Cohen-Macaulay for non-Noetherian rings in terms of the Čech cohomology of sequences of ring elements. As with the above examples, their definition reduces to the standard definition in the case that the ring is Noetherian, and exhibits much hoped-for behavior. This thesis focuses on developing a homological dimension version of Cohen-Macaulay for non-Noetherian rings which exhibits and expands upon the properties enjoyed by the Hamilton-Marley definition.

We focus here on the class of coherent rings, i.e., rings in which every finitely generated ideal is finitely presented. For example, Noetherian rings and valuation domains are coherent, as well as polynomial rings in any number (possibly infinite) of variables with coefficients from a Noetherian ring or a valuation domain. Furthermore, quotients of coherent rings by finitely generated ideals are coherent. A module over a coherent ring is finitely presented if and only if it has a resolution consisting of finitely generated free modules in each degree (a property also enjoyed by finitely generated modules over Noetherian rings). Modules which have such a resolution are called $(FP)_\infty$-modules. Note that all finitely generated modules over a Noetherian ring are $(FP)_\infty$. Just as finitely generated modules play an important role in the theory of Noetherian rings, the class of $(FP)_\infty$- modules play a central role in this work. Chapter 2 gives more detail on properties of these $(FP)_\infty$ modules, as well as properties of coherent rings. A notion of depth over an arbitrary quasi-local ring as developed in [2], [19], and [22] is also discussed in this chapter.

In addition to satisfying the conditions put forth by Glaz, the following properties hold under the Hamilton-Marley notion of Cohen-Macaulay (notation: HMCM) put forth in [18]:

1. If a faithfully flat R-algebra S is HMCM, then R is as well.

2. If R is a zero-dimensional ring or a one-dimensional domain, then R is HMCM.

On the other hand, some properties of (local, Noetherian) Cohen-Macaulay rings fail to carry over with this definition; the specialization of a HMCM ring need not be HMCM, and it is unknown if localization at a prime ideal, or the addition of an indeterminate, preserves the HMCM-ness of a ring. Thus we aim to answer an augmented version of Glaz's question – is there a workable definition of Cohen-Macaulay which also satisfies these properties?

The homological classification of Gorenstein for arbitrary coherent rings given by Hummel and Marley in [20] not only answers the Gorenstein version of Glaz's question; this definition shares many properties which are held by the definition for local rings, closure under specialization, localization, and addition of an indeterminate among them. In addition, coherent Gorenstein rings are HMCM. With the success of Hummel and Marley's characterization of Gorenstein rings via homological tools, a homological dimension notion of Cohen-Macaulay would seem a good candidate to improve upon the Hamilton-Marley definition. Building from the notion of Gorenstein dimension, Gerko defines CM-dimension for finitely generated modules over local rings in [13]. If (R, m) is local, he proves R is Cohen-Macaulay if and only if $CM \dim_R M < \infty$ for all finitely generated R-modules M, or equivalently, if and only if $CM \dim_R R/m < \infty$. This homological device is what we aim to generalize to the case of an arbitrary coherent ring. Chapters 3 and 4 provide the necessary background in developing CM-dimension. In Chapter 3, we develop a theory of semi-dualizing modules for quasi-local rings; Chapter 4 discusses the G_K-class of a ring and the G_K-dimension of a module, where K is a semi-dualizing module. In Chapter 5, we prove the Auslander-Bridger formula for G_K-dimension:

Theorem. *(See Theorem 5.1.4.) Let (R, m) be a quasi-local ring, K a semi-dualizing module for R, and M a non-zero R-module of finite G_K-dimension. Then*

$$\operatorname{depth} M + G_K \dim_R M = \operatorname{depth} R.$$

Cohen-Macaulay dimension for finitely presented modules over quasi-local rings, as well as what it means for an arbitrary ring to be Cohen-Macaulay in the sense of Gerko (notation: GCM), is defined in Chapter 6. Here, we prove the Auslander-Bridger formula for CM-dimension:

Theorem. *(See Theorem 6.1.7.) Let (R, m) be a quasi-local coherent ring, and M*

an R-module of finite CM-dimension. Then

$$\operatorname{depth} M + CM \dim_R M = \operatorname{depth} R.$$

In addition to reducing to the standard definition in the local case, GCM rings satisfy the following properties:

Theorem. *(See Theorem 6.1.6 and Propositions 6.2.1 and 6.2.2.) Let (R, m) be a coherent ring. Then the following hold:*

1. *If R is regular, then R is GCM.*

2. *If R is Gorenstein, then R is GCM.*

3. *If R is GCM and $U \subseteq R$ is a multiplicatively closed set, then R_U is also GCM.*

4. *If R is GCM and $x \in m$ is R-regular, then $R/(x)$ is GCM.*

Lastly, we consider those GCM rings which are the homomorphic image of a Gorenstein ring:

Theorem. *(See Corollary 6.3.3, Theorem 6.4.1, and Propositions 6.3.5 and 6.3.8.) Let (S, n) be a quasi-local coherent Gorenstein ring of finite depth. Let $I \subseteq S$ be a finitely generated ideal, and set $R = S/I$.*

1. *If $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$, then R is GCM.*

2. *If $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$, then R is HMCM.*

3. *If R is 0-dimensional, then R is GCM.*

4. *If R is a 1-dimensional domain, then R is GCM.*

The final chapter describes possible future directions for this project.

Chapter 2

Preliminaries

2.1 $(FP)_n$-modules

Throughout, R will denote a commutative ring with identity. The term *local ring* will be used exclusively for a commutative Noetherian ring with a unique maximal ideal. The term *quasi-local* will be used when the ring is not necessarily Noetherian but has a unique maximal ideal.

The following definition is due to Bieri [4]:

Definition 2.1.1. An R-module M is said to be $(FP)_n^R$ for some $n \geq 0$ if there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_0, \ldots, F_n$ are finitely generated free R-modules. If the ring R in question is clear, then we write that M is $(FP)_n$ instead of $(FP)_n^R$. If M is $(FP)_n$ for all $n \geq 0$, then we say that M is $(FP)_\infty$.

Note then that M is $(FP)_0$ if and only if M is finitely generated, and that M is

$(FP)_1$ if and only if M is finitely presented.

Lemma 2.1.2. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then the following hold for any $n \geq 0$:*

1. *If L is $(FP)_n$ and M is $(FP)_{n+1}$, then N is $(FP)_{n+1}$.*

2. *If M and N are $(FP)_{n+1}$, then L is $(FP)_n$.*

3. *If L and N are $(FP)_n$, then M is $(FP)_n$.*

Consequently, if any two modules in a short exact sequence are $(FP)_\infty$, then so is the third.

Proof. See [4], Proposition 1.4, or [14], Theorem 2.1.2. □

Remark 2.1.3. If M is $(FP)_\infty$, then M has a resolution by finitely generated free R-modules.

Proof. Consider the exact sequence

$$0 \longrightarrow L_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where F_0 is a finitely generated free R-module. By Lemma 2.1.2, F_0, and hence L_1, is $(FP)_\infty$. This yields a short exact sequence

$$0 \longrightarrow L_2 \longrightarrow F_1 \longrightarrow L_1 \longrightarrow 0,$$

where F_1 is a finitely generated free R-module. Composing these sequences gives an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Continuing this process yields a resolution of M by finitely generated free R-modules.

□

Remark 2.1.4. From Lemma 2.1.2, if M has a resolution by $(FP)_\infty$- modules, then M is $(FP)_\infty$.

Remark 2.1.5. Let S be a flat R-algebra, M an R-module, and $n \geq 0$ an integer.

1. If M is $(FP)_n^R$, then $M \otimes_R S$ is $(FP)_n^S$.

2. If S is faithfully flat, then the converse to (1) holds.

Proof. The first item follows clearly from the definition of $(FP)_n$. We show the second using induction on n. Let $n \geq 0$, and suppose that $M \otimes_R S$ is $(FP)_n^S$. Suppose $n = 0$, i.e., that $M \otimes_R S$ is finitely generated. $M \otimes_R S$ is generated by finitely many elements of the form

$$\alpha_i = \sum_j x_{ij} \otimes s_{ij}.$$

Let N be the submodule of M generated by $\{x_{ij}\}_{i,j}$, and note that N is a finitely generated submodule of M. As S is faithfully flat, we have an exact sequence

$$0 \longrightarrow N \otimes_R S \longrightarrow M \otimes_R S \longrightarrow M/N \otimes_R S \longrightarrow 0.$$

Note however that $N \otimes_R S = M \otimes_R S$, since every generator of $M \otimes_R S$ is contained in $N \otimes_R S$. So $M/N \otimes_R S = 0$, and, as S is faithfully flat, $M/N = 0$. Thus $M = N$ is finitely generated.

Suppose now that $n > 0$. If $M \otimes_R S$ is $(FP)_n^S$, then in particular it is $(FP)_0^S$, and so the above argument shows that M is $(FP)_0^R$. Consider the exact sequences

$$(*) \quad 0 \longrightarrow K \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K \otimes_R S \longrightarrow R^m \otimes_R S \longrightarrow M \otimes_R S \longrightarrow 0.$$

Since $R^m \otimes_R S \cong S^m$ is $(FP)^S_\infty$, and $M \otimes_R S$ is $(FP)^S_n$, the above lemma gives that $K \otimes_R S$ is $(FP)^S_{n-1}$. Thus by induction K is $(FP)^R_{n-1}$. Applying Lemma 2.1.2 to $(*)$, we have that M is $(FP)^R_n$, as desired. □

Remark 2.1.6. Suppose that M is $(FP)^R_n$, and that $x \in R$ is a non-zero-divisor on R and on M. Then M/xM is $(FP)^{R/(x)}_n$.

Proof. See [20], Remark 2.5. □

Lemma 2.1.7. *Let S be a ring, and $I \subseteq S$ an ideal such that $S/I \in (FP)^S_\infty$. Then any $(FP)^S_\infty$ module which is annihilated by I is $(FP)_\infty$ as an S/I-module.*

Proof. Let M be an $(FP)^S_\infty$ module which is annihilated by I. We'll show by induction that if M is $(FP)_n$ as an S/I-module, then M is $(FP)_{n+1}$ as an S/I-module. Since M is $(FP)^S_\infty$, there exists an exact sequence

$$S^m \longrightarrow M \longrightarrow 0$$

for some m. Applying $- \otimes_S S/I$ yields the exact sequence

$$(S/I)^m \longrightarrow M \longrightarrow 0,$$

and thus M is finitely generated as an S/I-module.

Suppose that M is $(FP)^{S/I}_n$ for some $n \geq 0$. Consider the exact sequence

$$(*) \quad 0 \longrightarrow L \longrightarrow (S/I)^m \longrightarrow M \longrightarrow 0$$

of S/I-modules. Both M and $(S/I)^m$ are $(FP)_\infty$ as S-modules, and so by Lemma 2.1.2 L is also $(FP)_\infty$ as an S-module. Note also that I annihilates L. Applying Lemma 2.1.2 again, but by considering $(*)$ as a sequence of S/I-modules, we have that L is $(FP)_{n-1}^{S/I}$. Thus by induction L is $(FP)_n^{S/I}$, and hence M is $(FP)_{n+1}^{S/I}$, as desired. □

2.2 Coherent Rings and Modules

A ring R is called *coherent* if every finitely generated ideal is finitely presented. An R-module M is called *coherent* if M is finitely generated, and every finitely generated submodule is finitely presented. (So in particular, coherent modules are finitely presented.)

Coherent rings and modules were introduced in the literature by Chase in [8], and first named as such in Bourbaki [5]. Some important properties of coherent rings and modules are given below; see [14] for more details.

Proposition 2.2.1. *([14], Theorem 2.2.1) Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of R-modules.

1. *If M is coherent and L is finitely generated, then N is coherent.*

2. *If any two of the modules are coherent, then the third is as well.*

Corollary 2.2.2. *([14], Corollary 2.2.3) A finite direct sum of coherent modules is coherent.*

Proposition 2.2.3. *([14], Corollary 2.2.2) Let R be a ring, and $\phi : M \to N$ an R-module homomorphism of coherent R-modules. Then* $\ker\phi, \operatorname{im}\phi$, *and* $\operatorname{coker}\phi$ *are coherent R-modules.*

Proof. Note that M and N are finitely presented, and that $\operatorname{im}\phi$ is finitely generated. Consider the short exact sequence

$$0 \longrightarrow \operatorname{im}\phi \longrightarrow N \longrightarrow \operatorname{coker}\phi \longrightarrow 0.$$

By Proposition 2.2.1, $\operatorname{coker}\phi$ is coherent, and hence $\operatorname{im}\phi$ is as well. This same proposition applied to the short exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow M \longrightarrow \operatorname{im}\phi \longrightarrow 0$$

gives that $\ker\phi$ is coherent as well. □

Proposition 2.2.4. *([14], Theorem 2.3.2, parts (1) and (2)) A ring R is coherent if and only if every finitely presented R-module is a coherent module.*

Proof. Let M be a finitely presented R-module, and

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a presentation by finitely generated free R-modules. By Corollary 2.2.2, F_1 and F_0 are coherent modules. Thus by Proposition 2.2.3, M is a coherent module. □

Proposition 2.2.5. *([14], Theorem 2.5.2) Let R be a coherent ring, and M an R-module. Then M is $(FP)_\infty$ if and only if M is finitely presented.*

Proof. Since M is finitely presented, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Since K is the kernel of a homomorphism of coherent modules, it is itself coherent; in particular, there is some finitely generated free R-module F_2 such that $F_2 \to K \to 0$. Continuing this process, we have that M is $(FP)_\infty$. □

Proposition 2.2.6. *([14], Corollary 2.5.3) Let R be a coherent ring, and M and N finitely presented R-modules. Then $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}_i^R(M,N)$ are finitely presented for all $i \geq 0$.*

Proof. Let $\boldsymbol{F}$ be a resolution of M by finitely generated free R-modules, and apply $\operatorname{Hom}_R(-,N)$. If $F_i \cong R^{l_i}$, then $\operatorname{Hom}_R(F_i,N) \cong N^{l_i}$. By Proposition 2.2.4 and Corollary 2.2.2, $\operatorname{Hom}_R(F_i,N)$ is a coherent module for all i. By Proposition 2.2.3, all kernels and images of the complex $\operatorname{Hom}_R(\boldsymbol{F},N)$ are coherent, and in particular, finitely presented. Thus the homology module $\operatorname{Ext}^i_R(M,N)$ is finitely presented for all i.

To see that $\operatorname{Tor}_i^R(M,N)$ is finitely presented, note that if $F_i \cong R^{l_i}$, then $F_i \otimes_R N \cong N^{l_i}$, a coherent module. Then $\mathrm{H}_i(\boldsymbol{F} \otimes_R N)$ is finitely presented by the same argument as above. □

Proposition 2.2.7. *([14], Theorem 2.3.2, parts (1) and (7)) A ring R is coherent if both of the following hold:*

1. *$(0:r)$ is finitely presented for all $r \in R$;*
2. *the intersection of two finitely generated ideals of R is a finitely generated ideal of R.*

Proposition 2.2.8. *([14], Theorem 2.4.1) Let R be a ring and $I \subseteq R$ an ideal. If R is coherent and I is finitely generated, then R/I is a coherent ring.*

Proposition 2.2.9. *([14], Theorem 2.4.2) Let R be a ring and $U \subseteq R$ a multiplicatively closed set. If R is coherent, then R_U is coherent.*

Proposition 2.2.10. *([14], Theorem 2.4.4) Let $R \to S$ be a faithfully flat extension of rings and M an R-module. If $M \otimes_R S$ is coherent as an S-module, then M is coherent as an R-module.*

2.3 Depth in Quasi-local Rings

We will make use of the notions of grade and depth as extended to quasi-local rings by Hochster [19].

Definition 2.3.1. Let M be an R-module, and I an ideal of R such that $IM \neq M$. Define the *classical grade* of M with respect to I, denoted $\mathrm{Grade}(I, M)$, to be the supremum of the lengths of regular sequences on M contained in I. Define the *grade* of M with respect to I, denoted $\mathrm{grade}(I, M)$, to be the supremum of $\mathrm{Grade}(IS, M \otimes_R S)$ over all faithfully flat extensions S of R. If $IM = M$, set $\mathrm{Grade}(I, M) = \mathrm{grade}(I, M) = \infty$.

We write $\mathrm{Grade}\, I$ and $\mathrm{grade}\, I$ (or occasionally $\mathrm{Grade}_R\, I$ and $\mathrm{grade}_R\, I$) for $\mathrm{Grade}(I, R)$ and $\mathrm{grade}(I, R)$, respectively. Note that if R is Noetherian and M is finitely generated, then $\mathrm{grade}(I, M) = \mathrm{Grade}(I, M)$.

If (R, m) is quasi-local, then we define the *classical depth* of M and *depth* of M, respectively, as follows:

$$\mathrm{Depth}\, M = \mathrm{Grade}(m, M) \quad \text{and} \quad \mathrm{depth}\, M = \mathrm{grade}(m, M).$$

Proposition 2.3.2. *Let M be an R-module, and I an ideal of R such that $IM \neq M$.*

1. $\operatorname{grade}(I, M) = \sup\{\operatorname{grade}(J, M) \mid J \subseteq I, J$ *finitely generated* $\}$.
2. $\operatorname{grade}(I, M) = \operatorname{grade}(IS, M \otimes_R S)$ *for any faithfully flat R-algebra S.*
3. $\operatorname{grade}(I, M) = \operatorname{grade}(\sqrt{I}, M)$.
4. *If $x \in I$ is M-regular, then* $\operatorname{grade}(I, M) = \operatorname{grade}(I, M/xM) + 1$.
5. *Suppose $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules such that $IL \neq L$ and $IN \neq N$. If* $\operatorname{grade}(I, M) > \operatorname{grade}(I, N)$, *then* $\operatorname{grade}(I, L) = \operatorname{grade}(I, N) + 1$.
6. *If I is generated by n elements, then* $\operatorname{grade}(I, M) \leq n$.

Proof. For parts $1, 3, 4, 5$, and 6, see Theorems $11, 12, 16, 20$, and 13 of Chapter 5 of [22], respectively. For part 2, see Corollary 7.1.4 of [14]. □

Proposition 2.3.3. *Let $I \subseteq J$ be ideals of R, and M a nonzero R-module such that $IM = 0$. Then* $\operatorname{grade}_R(J, M) = \operatorname{grade}_{R/I}(J/I, M)$. *In particular, if (R, m) is quasi-local and $IM = 0$, then* $\operatorname{depth}_R M = \operatorname{depth}_{R/I} M$.

Proof. By part (1) of Proposition 2.3.2 and Proposition 2.7 of [18], $\operatorname{grade}(J, M) = \sup\{k \geq 0 \mid \check{\mathrm{H}}^i_{\boldsymbol{x}}(M) = 0 \ \forall i < k$ and $\boldsymbol{x} = x_1, \ldots, x_n \in J\}$, where $\check{\mathrm{H}}^i_{\boldsymbol{x}}(M)$ is the i^{th} Čech cohomology of M with respect to $\boldsymbol{x}$. Change of rings for Čech cohomology (see Section 5.1 of [6]) gives that $\operatorname{grade}_R(J, M) = \operatorname{grade}_{R/I}(J/I, M)$. □

Lemma 2.3.4. *Let M be an R-module and I an ideal such that $IM \neq M$. Suppose that R/I is $(FP)_n$. The following are equivalent:*

1. $\operatorname{grade}(I, M) \geq n$

2. $\operatorname{Ext}^i_R(R/I, M) = 0$ *for* $0 \leq i < n$.

In particular, if I *is finitely generated, then* $\operatorname{grade}(I, M) = 0$ *if and only if* $\operatorname{Hom}_R(R/I, M) \neq 0$.

Proof. See [14], Theorem 7.1.2 or Theorem 7.1.8. □

Chapter 3

Semi-dualizing Modules

3.1 Definitions and Properties

Throughout this section, let (R, m) be a quasi-local ring.

See [11], [17], [12], and [10] for the development of semi-dualizing modules (also called suitable modules in some of those sources). Here, we extend this notion to finitely presented modules over a quasi-local ring.

Definition 3.1.1. An R-module K is called *semi-dualizing* if all of the following conditions hold:

1. K is $(FP)^R_\infty$;
2. the homothety map $\Theta_R : R \to \mathrm{Hom}_R(K, K)$ given by $r \mapsto \mu_r$ is an isomorphism, where μ_r is given my multiplication by r;
3. $\mathrm{Ext}^i_R(K, K) = 0$ for all $i > 0$.

A semi-dualizing module K is called *dualizing* if K has finite injective dimension.

We remark that R is a semi-dualizing R-module. Also, if R is a (Noetherian) local Cohen-Macaulay ring which is the quotient of a Gorenstein ring, then the canonical module ω_R is a semi-dualizing (in fact, a dualizing) R-module.

The following theorem allows us to relax condition (2) in the definition of a semi-dualizing module. This result is well-known in certain cases, e.g., see Proposition 1.1.9 of [9], though we haven't seen it stated in the following generality.

Theorem 3.1.2. *Let R be a ring, and K an R-module. Let $(-)^\vee$ denote the functor $\mathrm{Hom}_R(-, K)$. If M is a finitely generated R-module and $M \cong M^{\vee\vee}$, then the natural map $ev_M : M \to M^{\vee\vee}$ given by $m \mapsto ev_m$ is an isomorphism.*

Proof. Claim: For any R-module M, $ev_{M^\vee} : M^\vee \to M^{\vee\vee\vee}$ is a split injection.

Define $\phi : M^{\vee\vee\vee} \to M^\vee$ by $\phi(c)(m) = c(ev_M(m))$ for all $m \in M$ and $c \in M^{\vee\vee\vee}$. We'll show that $\phi \circ ev_{M^\vee}$ is the identity map on $M^\vee$. Let $f \in M^\vee$, and let $u \in M$. We have

$$\begin{aligned} \phi(ev_{M^\vee}(f))(u) &= ev_{M^\vee}(f)(ev_M(u)) \\ &= ev_M(u)(f) \\ &= f(u), \end{aligned}$$

and hence $ev_{M^\vee}$ is split injective.

Let M be a finitely generated R-module, and let $f : M \to M^{\vee\vee}$ be an isomorphism. By the above claim with $M^\vee$ in place of M, we have that $ev_{M^{\vee\vee}} : M^{\vee\vee} \to M^{\vee\vee\vee\vee}$ is a split injective map. Consider the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{ev_M} & M^{\vee\vee} \\ {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle f^{\vee\vee}}\downarrow{\scriptstyle \cong} \\ M^{\vee\vee} & \xrightarrow{ev_{M^{\vee\vee}}} & M^{\vee\vee\vee\vee} \end{array}$$

Since $ev_{M^{\vee\vee}}$ is split injective, ev_M is a split injection as well. Let $\beta : M^{\vee\vee} \to M$ be a splitting map, so that $\beta \circ ev_M$ is the identity on M. Since β is clearly surjective, we need to show that β is injective, so that β, and hence ev_M, is an isomorphism. It suffices to show that $\beta \circ f : M \to M$ is one-to-one. But $\beta \circ f$ is onto, as both β and f are onto, and thus since M is finitely generated, $\beta \circ f$ is an isomorphism by Theorem 2.4 of [21]. □

Remark 3.1.3. With notation as above, if $M = R$, note that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{ev_R} & \mathrm{Hom}_R(\mathrm{Hom}_R(R,K),K) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle g} \\ R & \xrightarrow{\Theta_R} & \mathrm{Hom}_R(K,K), \end{array}$$

where $g(\phi) = \phi\beta$,

$$\beta : \quad K \longrightarrow \mathrm{Hom}_R(R,K)$$

$$k \longrightarrow f_k,$$

and $f_k(1) = k$. For $r \in R$ and $k \in K$,

$$\begin{aligned} g(ev_R(r))(k) &= ev_r(r)(\beta(k)) \\ &= ev_R(r)(f_k) \\ &= f_k(r) \\ &= rk \\ &= \mu_r(k) \\ &= \Theta(r)(k). \end{aligned}$$

Hence ev_R is an isomorphism if and only if Θ_R is.

The following theorem is due to Gruson; see Theorem 4.1 of [24].

Theorem 3.1.4. *Let E be a finitely generated faithful module over the commutative ring R. Then every R-module admits a finite filtration of submodules whose factors are quotients of direct sums of copies of E.*

Proposition 3.1.5. *Let K be a semi-dualizing R-module and let $x \in R$. Then x is a non-zero-divisor on R if and only if x is a non-zero-divisor on K.*

Proof. Since $R \cong \operatorname{Hom}_R(K, K)$, we have $\operatorname{Ann}_R K \subseteq \operatorname{Ann}_R R = 0$, and hence K is a faithful R-module. By Gruson's Theorem, there exists a finite filtration

$$0 = q_t \subset q_{t-1} \subset \cdots \subset q_0 = R/(x)$$

such that for each i there exists a surjection

$$\oplus_{\alpha \in \Lambda_i} K \longrightarrow q_i/q_{i+1} \longrightarrow 0$$

(where Λ_i is some index set, possibly infinite). Applying $\mathrm{Hom}_R(-, K)$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(q_i/q_{i+1}, K) \longrightarrow \mathrm{Hom}_R(\oplus_{\alpha\in\Lambda_i} K, K) \cong \prod_{\alpha\in\Lambda_i} \mathrm{Hom}_R(K, K) \cong \prod_{\alpha\in\Lambda_i} R.$$

As x is a non-zero-divisor on R, x is also a non-zero-divisor on $\prod_{\alpha\in\Lambda_i} R$. Since $x \cdot q_i/q_{i+1} = 0$, $x \cdot \mathrm{Hom}_R(q_i/q_{i+1}, K) = 0$. But $\mathrm{Hom}_R(q_i/q_{i+1}, K)$ is isomorphic to a submodule of $\prod_{\alpha\in\Lambda_i} R$, and thus $\mathrm{Hom}_R(q_i/q_{i+1}, K) = 0$ for all i. Consider the exact sequence

$$0 \longrightarrow q_t \longrightarrow q_{t-1} \longrightarrow q_{t-1}/q_t \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(-, K)$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(q_{t-1}/q_t, K) \longrightarrow \mathrm{Hom}_R(q_{t-1}, K) \longrightarrow \mathrm{Hom}_R(q_t, K).$$

Using the fact that $\mathrm{Hom}_R(q_{t-1}/q_t, K) = 0$ and that $q_t = 0$, the above sequence gives $\mathrm{Hom}_R(q_{t-1}, K) = 0$. Applying this same argument to the exact sequence

$$0 \longrightarrow q_{i+1} \longrightarrow q_i \longrightarrow q_i/q_{i+1} \longrightarrow 0,$$

we get that $\mathrm{Hom}_R(q_i, K) = 0$ for all i. Taking $i = 0$, we get $\mathrm{Hom}_R(R/(x), K) = 0$, and hence x is a non-zero-divisor on K.

Now, suppose that x is a non-zero-divisor on K. If $xy = 0$ for some $y \in R$, then $(xy)K = x(yK) = 0$ implies that $yK = 0$. But the map $R \to \mathrm{Hom}_R(K, K)$ given by $r \mapsto \mu_r$ is one-to-one, hence $yK = 0$ implies that $y = 0$. Thus x is a non-zero-divisor on R. □

3.2 Change of Rings Results

The following result is well-known (see, e.g, Chapter 18, Lemma 2 of [21]).

Lemma 3.2.1. *Let R be a ring, L be an R-module, and $x \in R$ such that x is a non-zero-divisor on R and on L. Let N be an $R/(x)$-module. Then*

$$\mathrm{Ext}^i_{R/(x)}(L/xL, N) \cong \mathrm{Ext}^i_R(L, N)$$

for all i.

Proposition 3.2.2. *Let K be a semi-dualizing R-module, and let $x \in R$ be a non-zero-divisor. Then K/xK is semi-dualizing for $R/(x)$.*

Proof. By Proposition 3.1.5, x is an non-zero-divisor on K. As K is $(FP)^R_\infty$, by Remark 2.1.6 we have that K/xK is $(FP)^{R/(x)}_\infty$. Consider the exact sequence

$$0 \longrightarrow K \xrightarrow{x} K \longrightarrow K/xK \longrightarrow 0.$$

Since $\mathrm{Ext}^1_R(K, K) = 0$, applying $\mathrm{Hom}_R(K, -)$ yields an exact sequence

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_R(K, K) \xrightarrow{x} \mathrm{Hom}_R(K, K) \longrightarrow \mathrm{Hom}_R(K, K/xK) \longrightarrow 0.$$

So we have

$$\begin{aligned} R/(x) &\cong R \otimes_R R/(x) \\ &\cong \mathrm{Hom}_R(K, K) \otimes_R R/(x) \\ &\cong \mathrm{Hom}_R(K, K/xK) \\ &\cong \mathrm{Hom}_{R/(x)}(K/xK, K/xK) \end{aligned}$$

As K is finitely generated, Theorem 3.1.2 (and the subsequent remark) gives that the homothety map

$$R/(x) \to \mathrm{Hom}_{R/(x)}(K/xK, K/xK)$$

is an isomorphism. Continuing the sequence $(*)$, we have that $\mathrm{Ext}^i_R(K, K/xK) = 0$ for all $i > 0$, and so by Lemma 3.2.1,

$$\mathrm{Ext}^i_{R/(x)}(K/xK, K/xK) \cong \mathrm{Ext}^i_R(K, K/xK) = 0$$

for all $i > 0$. Hence K/xK is a semi-dualizing $R/(x)$-module. □

The following result is well-known in the Noetherian case; for convenience, we provide a proof in the coherent case.

Lemma 3.2.3. *Let M, N be R-modules, with M an $(FP)_\infty$-module, and let S be a flat R-algebra. Then*

$$\mathrm{Ext}^i_R(M, N) \otimes_R S \cong \mathrm{Ext}^i_S(M \otimes_R S, N \otimes_R S)$$

for all i.

Proof. First, note that $\mathrm{Hom}_R(R^n, N) \otimes_R S \cong \mathrm{Hom}_S(R^n \otimes_R S, N \otimes_R S)$ for any n. Let $\boldsymbol{F}$ be a finite free resolution for M. Then $\boldsymbol{F} \otimes_R S$ is a finite free resolution for $M \otimes_R S$. Further,

$$\mathrm{Hom}_R(\boldsymbol{F}, N) \otimes_R S \cong \mathrm{Hom}_S(\boldsymbol{F} \otimes_R S, N \otimes_R S).$$

Taking homology, we get

$$\begin{aligned}\operatorname{Ext}^i_R(M,N)\otimes_R S &\cong \mathrm{H}^i(\operatorname{Hom}_R(\boldsymbol{F},N))\otimes_R S\\ &\cong \mathrm{H}^i(\operatorname{Hom}_R(\boldsymbol{F},N)\otimes_R S)\\ &\cong \mathrm{H}^i(\operatorname{Hom}_S(\boldsymbol{F}\otimes_R S, N\otimes_R S))\\ &\cong \operatorname{Ext}^i_S(M\otimes_R S, N\otimes_R S).\end{aligned}$$

□

Proposition 3.2.4. *Let S be a quasi-local flat R-algebra.*

1. *If K is semi-dualizing for R, then $K\otimes_R S$ is semi-dualizing for S.*

2. *If S is faithfully flat, then the converse of (1) holds.*

In particular, for any multiplicatively closed subset $U\subseteq R$ and any semi-dualizing R-module K, we have that K_U is semi-dualizing for R_U.

Proof. By Remark 2.1.5, if K is $(FP)^R_\infty$ and $R\to S$ is flat, then $K\otimes_R S$ is $(FP)^S_\infty$. Consider the commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & R\otimes_R S & \xrightarrow{\phi\otimes 1} & \operatorname{Hom}_R(K,K)\otimes_R S & \longrightarrow & T & \longrightarrow & 0\\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & S & \xrightarrow{\psi} & \operatorname{Hom}_S(K\otimes_R S, K\otimes_R S) & \longrightarrow & U & \longrightarrow & 0\end{array}$$

with exact rows, where ϕ and ψ are the homothety maps, and α and β are the canonical maps. The diagram commutes, as

$$\psi\alpha(r\otimes s)=\psi(rs)=\mu_{rs}=\mu_r\otimes\mu_s=\beta(\mu_r\otimes s)=\beta(\phi\otimes 1)(r\otimes s).$$

First, consider the case where K is a semi-dualizing R-module. As S is flat and ϕ is an isomorphism, we have $A = T = 0$. Note also that ϕ, α, and β are isomorphisms (where this last fact follows from the Lemma 3.2.3). Hence ψ is an isomorphism.

Finally, by Lemma 3.2.3, we have

$$(\#) \quad \operatorname{Ext}^i_R(K,K) \otimes_R S \cong \operatorname{Ext}^i_S(K \otimes_R S, K \otimes_R S)$$

for all $i > 0$. As K is semi-dualizing, $\operatorname{Ext}^i_R(K,K) = 0$, and thus $\operatorname{Ext}^i_S(K \otimes_R S, K \otimes_R S) = 0$. Hence $K \otimes_R S$ is semi-dualizing for S.

Suppose that $K \otimes_R S$ is semi-dualizing and S is faithfully flat. Note then that $K \otimes_R S$ is $(FP)^S_\infty$, and hence K is $(FP)^R_\infty$, by Remark 2.1.5. Also, as α, β, and ψ are isomorphisms, so is $\phi \otimes 1$. Again, as S is faithfully flat, this implies that ϕ is also an isomorphism. Finally, the isomorphism (#) holds, and so $\operatorname{Ext}^i_S(K \otimes_R S, K \otimes_R S) = 0$ implies that $\operatorname{Ext}^i_R(K,K) = 0$ for all $i > 0$. Hence K is semi-dualizing for R. □

Proposition 3.2.5. *Let K be a semi-dualizing R-module. Then* $\operatorname{depth} R = \operatorname{depth} K$.

Proof. Let m denote the maximal ideal of R. Let $n \geq 0$ and suppose first that $\operatorname{depth} R \geq n$. We'll use induction on n to show that $\operatorname{depth} K \geq n$. If $n = 0$ there is nothing to show. If $n = 1$, then there exists a faithfully flat extension $R \to S$ and some $x \in mS$ which is a non-zero-divisor on S. By Proposition 3.2.4, $K \otimes_R S$ is semi-dualizing for S. Hence x is a non-zero-divisor on $K \otimes_R S$, by Proposition 3.1.5, and thus $\operatorname{depth} K \geq 1$.

Suppose $n > 1$. Since $\operatorname{depth} R > 0$ and $\operatorname{depth} K > 0$, there exist finitely generated ideals $I_1, I_2 \subseteq m$ such that $\operatorname{grade}(I_1, R) > 0$ and $\operatorname{grade}(I_2, K) > 0$. So we have

$$\operatorname{Hom}_R(R/I_1, R) = 0 = \operatorname{Hom}_R(R/I_2, K).$$

Consider the finitely generated ideal $I = I_1 + I_2$. Note that

$$\mathrm{Hom}_R(R/I, R) = 0 = \mathrm{Hom}_R(R/I, K),$$

and so

$$0 = \mathrm{Hom}_R(R/I, R) \oplus \mathrm{Hom}_R(R/I, K) \cong \mathrm{Hom}_R(R/I, R \oplus K).$$

Hence $\operatorname{grade}(I, R \oplus K) > 0$. Thus there exists a flat quasi-local R-algebra S such that mS is the maximal ideal of S and IS contains an element x which is regular on $(R \oplus K) \otimes_R S \cong S \oplus (K \otimes_R S)$. Hence x is regular on both S and $K \otimes_R S$.

Now, as $K' = K \otimes_R S$ is a semi-dualizing module for S, K'/xK' is a semi-dualizing module for S/xS by Proposition 3.2.2. Since $\operatorname{depth}_S S/xS = \operatorname{depth}_S S - 1 = \operatorname{depth}_R R - 1 \geq n - 1$, we have by the induction hypothesis that $\operatorname{depth}_R K - 1 = \operatorname{depth}_S K' - 1 = \operatorname{depth}_S K'/xK' \geq n - 1$. Hence, $\operatorname{depth}_R K \geq n$.

A similar argument shows that if $\operatorname{depth} K \geq n$ then $\operatorname{depth} R \geq n$. Thus, $\operatorname{depth} K = \operatorname{depth} R$. □

Chapter 4

G_K-dimension

4.1 The G_K-class of R

Definition 4.1.1. Let K be an R-module. An R-module M is called K*-reflexive* if the canonical (evaluation) map

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, K), K)$$

is an isomorphism.

Note that by Theorem 3.1.2, a finitely generated R-module M is K-reflexive if there exists a map

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, K), K))$$

which is an isomorphism.

Note. For the remainder of this chapter, let R be a commutative ring and K a semi-dualizing R-module. For any R-module A, let $A^\vee$ denote the K-dual module $\mathrm{Hom}_R(A, K)$.

In the manner of Auslander and Bridger [1], Golod [17], and Hummel and Marley [20], we make the following definition:

Definition 4.1.2. Let K be a semi-dualizing module. An R-module M is said to be a member of the *G_K-class of R*, denoted $G_K(R)$, if all of the following hold:

1. both M and $\mathrm{Hom}_R(M, K)$ are $(FP)_\infty$;
2. M is K-reflexive;
3. $\mathrm{Ext}^i_R(M, K) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, K), K) = 0$ for $i > 0$.

Given a semi-dualizing R-module K, note that both R and K are elements of $G_K(R)$. Also, $G_K(R)$ is closed under taking (finite) direct sums, summands, and K-duals. In particular, every finitely generated projective R-module is an element of the G_K-class of R. Finally, we remark that the G_R-class of R is the restricted G-class of R, as defined by Hummel and Marley in [20].

In the manner of Gerko in [13], we make the following defintions:

Definition 4.1.3. Let K be semi-dualizing for R, and let M be an $(FP)^R_\infty$-module. A complex $\boldsymbol{G}$ is called a *G_K-resolution* of M if

1. each $G_i \in G_K(R)$,
2. $G_i = 0$ for all $i < 0$,
3. $H_i(\boldsymbol{G}) = 0$ for $i \neq 0$, and
4. $H_0(\boldsymbol{G}) \cong M$.

The *length* of a resolution $\boldsymbol{G}$ is $\sup\{i \,|\, G_i \neq 0\}$. The *$n^{th}$ syzygy of $\boldsymbol{G}$* is the kernel of the map $G_{n-1} \to G_{n-2}$ (where the first syzygy is the kernel of $G_0 \to M$ and the 0^{th} syzygy is M). The *G_K-dimension of M*, denoted $G_K \dim_R M$, is the infimum of the lengths of all G_K-resolutions of M. Note that any $(FP)_\infty$-module has a G_K resolution consisting of finitely generated free R-modules.

Remark 4.1.4. By Lemma 2.1.2, any syzygy of a G_K-resolution is $(FP)_\infty$.

Proposition 4.1.5. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of $(FP)^R_\infty$*-modules. Suppose that* $N \in G_K(R)$*. Then* $L \in G_K(R)$ *if and only if* $M \in G_K(R)$*.*

Proof. See Lemma 1.1.10(a) of [9], replacing the R-dual of a module with its K-dual. We add that Lemma 2.1.2 applied to the short exact sequence

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow L^\vee \longrightarrow 0$$

shows that $L^\vee$ is $(FP)^R_\infty$ if and only if $M^\vee$ is $(FP)^R_\infty$. □

Lemma 4.1.6. *Let* M *be an* $(FP)^R_\infty$*-module of finite* G_K*-dimension such that* $\operatorname{Ext}^i_R(M,K) = 0$ *for all* $i > 0$*. Then* $M \in G_K(R)$*.*

Proof. See Lemma 1.2.6 of [9], replacing the R-dual of a module with its K-dual. Since $G_K \dim_R M < \infty$, suppose that

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

is a G_K-resolution of M. As $\operatorname{Ext}^i_R(M,K) = 0$ for all $i > 0$,

$$0 \longrightarrow M^\vee \longrightarrow G_0^\vee \longrightarrow \cdots \longrightarrow G_n^\vee \longrightarrow 0$$

is exact. As $G_i^\vee$ is $(FP)_\infty^R$ for all i, $M^\vee$ is $(FP)_\infty^R$ as well, by Lemma 2.1.2. □

Lemma 4.1.7. *Let $M \in G_K(R)$, and let $x \in R$ be R-regular. Then x is also a non-zero-divisor on M.*

Proof. Note by Proposition 3.1.5, x is also a non-zero-divisor on K. As $M \in G_K(R)$, $M^\vee$ is $(FP)_\infty^R$. In particular, we have an exact sequence

$$R^l \longrightarrow M^\vee \longrightarrow 0$$

for some l. Applying $\mathrm{Hom}_R(-, K)$ yields

$$0 \longrightarrow M^{\vee\vee} \longrightarrow \mathrm{Hom}_R(R^l, K),$$

and hence $M \cong M^{\vee\vee}$ is isomorphic to a submodule of K^l. Hence x is a non-zero-divisor on M. □

Since the G_K-class is closed under taking K-duals, x is also a non-zero-divisor on $M^\vee$, under the hypotheses of Lemma 4.1.7.

Lemma 4.1.8. *Let $M \in G_K(R)$, and let $x \in R$ be R-regular. Then M/xM is in $G_{K/xK}(R/(x))$.*

Proof. The proof of Lemma 1.3.5 of [9] is easily adapted. Note that x is also a non-zero-divisor on M, $M^\vee$, and K, by Lemma 4.1.7 and Proposition 3.1.5. So M/xM and $M^\vee/xM^\vee$ are both $(FP)_\infty^{R/(x)}$, and K/xK is semi-dualizing for $R/(x)$. For an $R/(x)$-module N, let $N^{\vee_x}$ denote the K/xK-dual module $\mathrm{Hom}_{R/(x)}(N, K/xK)$. By Lemma 3.2.1, we have

$$\mathrm{Ext}^i_R(M, K/xK) \cong \mathrm{Ext}^i_{R/(x)}(M/xM, K/xK)$$

and

$$\mathrm{Ext}^i_R(M^\vee, K/xK) \cong \mathrm{Ext}^i_{R/(x)}(M^\vee/xM^\vee, K/xK)$$

for all i. In particular we have

$$(M/xM)^{\vee_x} = \mathrm{Hom}_{R/(x)}(M/xM, K/xK) \cong \mathrm{Hom}_R(M, K/xK).$$

Applying $\mathrm{Hom}_R(M, -)$ to the short exact sequence

$$0 \longrightarrow K \xrightarrow{x} K \longrightarrow K/xK \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, K) \xrightarrow{x} \mathrm{Hom}_R(M, K) \longrightarrow \mathrm{Hom}_R(M, K/xK) \longrightarrow 0,$$

and thus

$$(M/xM)^{\vee_x} \cong \mathrm{Hom}_R(M, K/xK) \cong M^\vee/xM^\vee.$$

□

Proposition 4.1.9. *Let S be a flat R-algebra and M an R-module.*

1. *If $M \in G_K(R)$, then $M \otimes_R S \in G_{K\otimes_R S}(S)$.*

2. *If S is faithfully flat, then the converse of (1) holds.*

Proof. By Proposition 3.2.4, $K \otimes_R S$ is semi-dualizing for S. For an S-module N, let N^+ denote $\mathrm{Hom}_S(N, K \otimes_R S)$.

By Remark 2.1.5, if M is $(FP)^R_\infty$, then $M \otimes_R S$ is $(FP)^S_\infty$; the converse holds if S

is faithfully flat. So in the proof of both (1) and (2), by Lemma 3.2.3 we have

$$\operatorname{Ext}^i_R(M,K)\otimes_R S \cong \operatorname{Ext}^i_S(M\otimes_R S, K\otimes_R S)$$

for all $i \geq 0$. In particular,

$$M^\vee \otimes_R S \cong (M\otimes_R S)^+.$$

Thus in the proof of both (1) and (2), we have that $M^\vee$ is $(FP)^R_\infty$ and $(M\otimes_R S)^+$ is $(FP)^S_\infty$. Again by Lemma 3.2.3, we have isomorphisms

$$\operatorname{Ext}^i_R(M^\vee,K)\otimes_R S \cong \operatorname{Ext}^i_S(M^\vee\otimes_R S, K\otimes_R S) \cong \operatorname{Ext}^i_S((M\otimes_R S)^+, K\otimes_R S)$$

for all $i \geq 0$. So if $\operatorname{Ext}^i_R(M,K) = \operatorname{Ext}^i_R(M^\vee,K) = 0$ for all $i > 0$, then $\operatorname{Ext}^i_S(M\otimes_R S, K\otimes_R S) = \operatorname{Ext}^i_S((M\otimes_R S)^+, K\otimes_R S) = 0$ for all $i > 0$, and the converse holds if S is faithfully flat. Let $\rho : M \to M^{\vee\vee}$ and $\varphi : M\otimes_R S \to (M\otimes_R S)^{++}$ be the canonical maps. Let A and B be the kernel and cokernel of ρ, and A' and B' the kernel and cokernel of φ. As S is flat, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & A\otimes_R S & \longrightarrow & M\otimes_R S & \xrightarrow{\rho\otimes 1} & M^{\vee\vee}\otimes_R S & \longrightarrow & B\otimes_R S & \longrightarrow 0 \\
& \downarrow g & & \downarrow = & & \downarrow \cong & & \downarrow h & \\
0 \longrightarrow & A' & \longrightarrow & M\otimes_R S & \xrightarrow{\varphi} & (M\otimes_R S)^{++} & \longrightarrow & B' & \longrightarrow 0.
\end{array}$$

Note that both g and h are isomorphisms, by the Five Lemma. If $M \in G_K(R)$, then ρ is an isomorphism. Then $\rho\otimes 1$ is also an isomorphism, and hence φ is as well. Conversely, if φ is an isomorphism and S is faithfully flat, then $A = B = 0$, and so ρ is an isomorphism. $\square$

Proposition 4.1.10. *Suppose that M and $M^\vee$ are $(FP)^R_\infty$. The following are equivalent:*

1. *M is in $G_K(R)$.*

2. *M_p is in $G_{K_p}(R_p)$ for all prime ideals p of R.*

3. *M_m is in $G_{K_m}(R_m)$ for all maximal ideals m of R.*

Proof. By Proposition 4.1.9, it's enough to show that (3) implies (1). Note that by Proposition 3.2.4, K_m is semi-dualizing for R_m for all maximal ideals m of R. For an R_m-module N, let $N^{\vee_m}$ denote the K_m-dual of N. As M and $M^\vee$ are $(FP)^R_\infty$, we have

$$(*) \quad \operatorname{Ext}^i_{R_m}(M_m, K_m) \cong \operatorname{Ext}^i_R(M, K) \otimes_R R_m$$

and

$$(**) \quad \operatorname{Ext}^i_{R_m}(M^\vee \otimes_R R_m, K_m) \cong \operatorname{Ext}^i_R(M^\vee, K) \otimes_R R_m$$

for all i and all maximal ideals m. Note that $i = 0$ gives isomorphisms

$$(M_m)^{\vee_m} \cong (M^\vee)_m \quad \text{and} \quad (M_m)^{\vee_m\vee_m} \cong (M^{\vee\vee})_m.$$

Let $\rho : M \to M^{\vee\vee}$ be the canonical homomorphism, let $A = \ker \rho$, and $B = \operatorname{coker} \rho$. For any maximal ideal m, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\rho} & M^{\vee\vee} & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_m & \longrightarrow & M_m & \xrightarrow{\psi} & (M_m)^{\vee_m\vee_m} & \longrightarrow & B_m & \longrightarrow & 0
\end{array}$$

where ψ is the canonical map. Since $M_m \in G_{R_m}(K_m)$ for all maximal ideals m, we have that $A_m = B_m = 0$ for all maximal ideals m, and hence $A = B = 0$. Thus M is

K-reflexive. Finally, by (*) and (**), note that for all $i > 0$ and all maximal ideals m we have

$$\operatorname{Ext}^i_R(M, K) \otimes_R R_m = 0 \text{ and } \operatorname{Ext}^i_R(M^\vee, K) \otimes_R R_m = 0$$

and hence $\operatorname{Ext}^i_R(M, K) = \operatorname{Ext}^i_R(M^\vee, K) = 0$ for $i > 0$. □

4.2 G_K-dimension

In this section, we summarize the basic properties of G_K-dimension, which are similar to results for G-dimension of a finitely generated module over a Noetherian ring. We refer the reader to Chapter 1 of [9] for a thorough discussion of G-dimension over Noetherian rings, and to Section 3 of [20] for a discussion of G-dimension over an arbitrary commutative ring. See Section 1.3 of [13] for a discussion of G_K-dimension over Noetherian rings.

Proposition 4.2.1. *Let M be an R-module which is $(FP)^R_\infty$ and $n \geq 0$ an integer. The following are equivalent:*

1. $G_K \dim_R M \leq n$.

2. $G_K \dim_R M < \infty$ *and* $\operatorname{Ext}^i_R(M, K) = 0$ *for all* $i > n$.

3. *The* n^{th} *syzygy of any* G_K*-resolution of* M *is in* $G_K(R)$.

Proof. The proofs for the corresponding results for G-dimension (Theorem 1.2.7 of [9] in the Noetherian case, and Proposition 3.5 of [20] in the case of an arbitrary commutative ring) are easily adapted. □

Corollary 4.2.2. *Let M be a non-zero $(FP)^R_\infty$-module of finite G_K-dimension. Then*

$$G_K \dim_R M = \sup\{i \geq 0 \mid \operatorname{Ext}^i_R(M, K) \neq 0\}.$$

Proposition 4.2.3. *Suppose that*

$$0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$$

is an exact sequence of $(FP)^R_\infty$-modules, where $G_K \dim_R M > 0$ and $G \in G_K(R)$. Then $G_K \dim_R L = G_K \dim_R M - 1$.

Proof. Note that as $G \in G_K(R)$, every G_K-resolution of L yields a G_K-resolution of M of length one more. Hence if $G_K \dim_R M = \infty$, then $G_K \dim_R L = \infty$ as well. Otherwise, suppose that $n = G_K \dim_R M$ is positive and finite. Then we have that $G_K \dim_R L \geq n-1$. Let $\boldsymbol{F}$ be a resolution of L consisting of finitely generated free R-modules. The composition $\boldsymbol{F} \to G$ gives a G_K-resolution of M, so by Proposition 4.2.1, the $(n-1)^{st}$ syzygy of $\boldsymbol{F}$ is an element of $G_K(R)$. Hence $G_K \dim_R L \leq n-1$. $\square$

Remark 4.2.4. In the notation of the above proposition, if $G_K \dim_R M = 0$, then $G_K \dim_R L = 0$, by Proposition 4.1.5.

Proposition 4.2.5. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of $(FP)^R_\infty$-modules. Then the following hold

1. *If $G_K \dim_R L \leq n$ and $G_K \dim_R N \leq n$, then $G_K \dim_R M \leq n$.*

2. *If $G_K \dim_R M \leq n$ and $G_K \dim_R N \leq n$, then $G_K \dim_R L \leq n$.*

3. If $G_K \dim_R L \leq n$ *and* $G_K \dim_R M \leq n$*, then* $G_K \dim_R N \leq n+1$*.*

In particular, if any two of the modules has finite G_K*-dimension, then so does the third.*

Proof. We'll prove (3); parts (1) and (2) are proved similarly. Let $\boldsymbol{F}$ and $\boldsymbol{F'}$ be free resolutions of L and N, respectively, which consist of finitely generated free R-modules. By the Horseshoe Lemma (Proposition 6.24 of [23]), there exists a free resolution $\boldsymbol{F''}$ of M consisting of finitely generated free R-modules and chain maps $\boldsymbol{F} \to \boldsymbol{F''}$ and $\boldsymbol{F''} \to \boldsymbol{F'}$ such that

$$0 \longrightarrow \boldsymbol{F} \longrightarrow \boldsymbol{F''} \longrightarrow \boldsymbol{F'} \longrightarrow 0$$

is an exact sequence of complexes. Let K_n, K'_n, and K''_n denote the n^{th} syzygies of $\boldsymbol{F}, \boldsymbol{F'}$, and $\boldsymbol{F''}$, respectively. Then the sequence

$$0 \longrightarrow K_n \longrightarrow K''_n \longrightarrow K'_n \longrightarrow 0$$

is exact. Since the G_K-dimension of L and M is at most n, we have that K_n and K''_n are elements of $G_K(R)$, by Proposition 4.2.1. Hence $G_K \dim_R K'_n \leq 1$, and thus $G_K \dim_R N \leq n+1$. □

4.3 Change of Rings results for G_K-dimension

Proposition 4.3.1. *Suppose that* M *is* $(FP)^R_\infty$*. Then*

1. $G_K \dim_R M \geq G_{K \otimes_R S} \dim_S M \otimes_R S$ *for all flat* R*-algebras* S*.*

2. $G_K \dim_R M = G_{K \otimes_R S} \dim_S M \otimes_R S$ *for all faithfully flat* R*-algebras* S*.*

3. If in addition $M^\vee$ is $(FP)^R_\infty$, then

$$G_K \dim_R M = \sup\{G_{K_m} \dim_{R_m} M_m \,|\, m \text{ is a maximal ideal of } R\}.$$

Proof. Part (1) follows from Proposition 4.1.9. For (2), suppose that $G_{K\otimes_R S} \dim_S M \otimes_R S = n$. We may assume n is finite, by (1). Let $\boldsymbol{G}$ be a G_K-resolution of M, and L the n^{th} syzygy of $\boldsymbol{G}$. Since $\boldsymbol{G} \otimes_R S$ is a $G_{K\otimes_R S}$-resolution of $M \otimes_R S$, with n^{th} syzygy $L \otimes_R S$, by Proposition 4.2.1, we have that $L \otimes_R S \in G_{K\otimes_R S}(S)$. By Proposition 4.1.9, $L \in G_K(R)$, and thus $G_K \dim_R M \leq n$. Part (3) is proved similarly, using Proposition 4.1.10. □

Lemma 4.3.2. *Let M be an R-module such that $G_K \dim_R M \leq n$. Then $\operatorname{Ext}^n_R(M, K)$ is finitely presented.*

Proof. By Corollary 4.2.2, we have $\operatorname{Ext}^i_R(M, K) = 0$ for $i > G_K \dim_R M$, so we may assume that $G_K \dim_R M = n$. If $n = 0$, the result is clear, as $M \in G_K(R)$. Suppose that $n > 0$. Let

$$(*) \quad 0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$$

be exact, where $G \in G_K(R)$. By Proposition 4.2.3, $G_K \dim_R L = G_K \dim_R M - 1$, and so by induction, $\operatorname{Ext}^{n-1}_R(L, K)$ is finitely presented. Applying $\operatorname{Hom}_R(-, K)$ to $(*)$ yields an exact sequence

$$\operatorname{Ext}^{n-1}_R(G, K) \longrightarrow \operatorname{Ext}^{n-1}_R(L, K) \longrightarrow \operatorname{Ext}^n_R(M, K) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(G, K)$ is $(FP)^R_\infty$, and $\operatorname{Ext}^i_R(G, K) = 0$ for $i > 0$, the kernel of

$$\operatorname{Ext}^{n-1}_R(L, K) \to \operatorname{Ext}^n_R(M, K)$$

is finitely generated. Since $\operatorname{Ext}_R^{n-1}(L, K)$ is finitely presented, $\operatorname{Ext}_R^n(M, K)$ is also finitely presented, by Lemma 2.1.2. □

Let $J(R)$ denote the Jacobson radical of R.

Proposition 4.3.3. *Let M be an R-module which is $(FP)_\infty^R$. Let $x \in J(R)$ be a non-zero-divisor on M and R. Suppose that $M/xM \in G_{K/xK}(R/(x))$. Then:*

1. *If $G_K \dim_R M < \infty$, then $M \in G_K(R)$.*
2. *If R is coherent, then $M \in G_K(R)$.*

Proof. For (1), let $n = G_K \dim_R M$, and assume that $n > 0$. As x is a non-zero-divisor on M, R, and K, by Lemma 3.2.1, we have

$$\operatorname{Ext}_R^i(M, K/xK) \cong \operatorname{Ext}_{R/(x)}^i(M/xM, K/xK) = 0$$

for all $i > 0$. Also, $\operatorname{Ext}_R^n(M, K)$ is non-zero and finitely generated, by Corollary 4.2.2 and Lemma 4.3.2. Applying $\operatorname{Hom}_R(M, -)$ to the short exact sequence

$$0 \longrightarrow K \xrightarrow{x} K \longrightarrow K/xK \longrightarrow 0$$

yields the exact sequence

$$\cdots \longrightarrow \operatorname{Ext}_R^n(M, K) \xrightarrow{x} \operatorname{Ext}_R^n(M, K) \longrightarrow 0,$$

which implies by Nakayama's Lemma that $\operatorname{Ext}_R^n(M, K) = 0$, a contradiction. Hence $n = 0$, i.e., $M \in G_K(R)$.

For the proof of (2), we adapt the argument given in Lemma 1.4.4 of [9]. Let $(-)^{\vee_x}$ denote the functor $\operatorname{Hom}_{R/(x)}(-, K/xK)$.

Note that M is $(FP)^R_\infty$ by assumption, and that $M^\vee$ is as well, by Proposition 2.2.6. As in the proof of part (1), we have a long exact sequence

$$(\#) \qquad \ldots \longrightarrow \operatorname{Ext}^i_R(M,K) \xrightarrow{x} \operatorname{Ext}^i_R(M,K) \longrightarrow \operatorname{Ext}^i_R(M,K/xK) \longrightarrow \ldots.$$

By Lemma 3.2.1,

$$\operatorname{Ext}^i_R(M,K/xK) \cong \operatorname{Ext}^i_{R/(x)}(M/xM,K/xK) = 0$$

for all $i > 0$, and so we have surjections

$$\operatorname{Ext}^i_R(M,K) \xrightarrow{x} \operatorname{Ext}^i_R(M,K) \longrightarrow 0.$$

Since R is coherent, Proposition 2.2.6 implies that $\operatorname{Ext}^i_R(M,K)$ is finitely presented for all i, and so by Nakayama's Lemma, $\operatorname{Ext}^i_R(M,K) = 0$ for all $i > 0$. The vanishing of $\operatorname{Ext}^1_R(M,K)$ and (#) also yields a short exact sequence

$$0 \longrightarrow M^\vee \xrightarrow{x} M^\vee \longrightarrow (M/xM)^{\vee_x} \longrightarrow 0,$$

and hence we have that

$$(M/xM)^{\vee_x} \cong M^\vee/xM^\vee.$$

To see that $\operatorname{Ext}^i_R(M^\vee,K) = 0$ for all $i > 0$, apply the functor $\operatorname{Hom}_R(M^\vee,-)$ to the short exact sequence

$$0 \longrightarrow K \xrightarrow{x} K \longrightarrow K/xK \longrightarrow 0.$$

This yields a long exact sequence

$$(*) \quad \dots \longrightarrow \operatorname{Ext}^i_R(M^\vee, K) \xrightarrow{x} \operatorname{Ext}^i_R(M^\vee, K) \longrightarrow \operatorname{Ext}^i_R(M^\vee, K/xK) \longrightarrow \dots.$$

Since x is also $M^\vee$ (and $M^{\vee\vee}$)-regular, we have by Lemma 3.2.1 that

$$\operatorname{Ext}^i_R(M^\vee, K/xK) \cong \operatorname{Ext}^i_{R/(x)}(M^\vee/xM^\vee, K/xK) \cong \operatorname{Ext}^i_{R/(x)}((M/xM)^{\vee_x}, K/xK) = 0$$

for all $i > 0$. So Nakayama's Lemma and $(*)$ show that $\operatorname{Ext}^i_R(M^\vee, K) = 0$ for all $i > 0$.

It remains to show that M is K-reflexive. Let $\delta_M : M \to M^{\vee\vee}$ and $\delta_{M/xM} : M/xM \to (M/xM)^{\vee_x\vee_x}$ denote the canonical maps. Consider the commutative diagram

$$\begin{array}{ccc} M \otimes_R R/(x) & \xrightarrow{\delta_{M\otimes 1}} & M^{\vee\vee} \otimes_R R/(x) \\ \cong \downarrow & & \downarrow \\ M/xM & \underset{\delta_{M/xM}}{\xrightarrow{\cong}} & (M/xM)^{\vee_x\vee_x}. \end{array}$$

Notice that

$$(M/xM)^{\vee_x\vee_x} \cong (M^\vee/xM^\vee)^{\vee_x} \cong M^{\vee\vee}/xM^{\vee\vee}.$$

Hence $\delta_M \otimes 1$ is an isomorphism.

Consider the exact sequence

$$(**) \quad 0 \longrightarrow L \longrightarrow M \xrightarrow{\delta_M} M^{\vee\vee} \longrightarrow C \longrightarrow 0.$$

Tensoring with $R/(x)$ yields an exact sequence

$$M \otimes_R R/(x) \xrightarrow{\delta_M \otimes 1} M^{\vee\vee} \otimes_R R/(x) \longrightarrow C \otimes_R R/(x) \longrightarrow 0.$$

Since $\delta_M \otimes 1$ is an isomorphism, we have that $C/xC = 0$. So Proposition 2.2.3 and Nakayama's Lemma imply that $C = 0$. Applying $- \otimes_R R/(x)$ to $(**)$ also yields an exact sequence

$$\operatorname{Tor}_1^R(M^{\vee\vee}, R/(x)) \longrightarrow L \otimes_R R/(x) \longrightarrow M \otimes_R R/(x) \xrightarrow{\delta_M \otimes 1} M^{\vee\vee} \otimes_R R/(x) \longrightarrow 0.$$

Note that $\operatorname{Tor}_1^R(M^{\vee\vee}, R/(x)) = 0$, as x is both R- and $M^{\vee\vee}$-regular. Hence we have an exact sequence

$$0 \longrightarrow L/xL \longrightarrow M \otimes_R R/(x) \xrightarrow[\cong]{\delta_M \otimes 1} M^{\vee\vee} \otimes_R R/(x),$$

which, as above, implies that $L = 0$. Thus δ_M is an isomorphism, and $M \in G_K(R)$.

$\square$

Proposition 4.3.4. *Let M be an R-module which is $(FP)_\infty^R$, and let $x \in R$ be a non-zero-divisor on M and R. Then:*

1. *$G_{K/xK} \dim_{R/(x)} M/xM \leq G_K \dim_R M$.*

2. *If $x \in J(R)$ and either M has finite G_K-dimension or R is coherent, then*

$$G_{K/xK} \dim_{R/(x)} M/xM = G_K \dim_R M.$$

Proof. For (1), we may assume that $G_K \dim_R M = n < \infty$. If $n = 0$, then the result follows by Lemma 4.1.8. If $n > 0$, then there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where $G \in G_K(R)$. Note that L is $(FP)^R_\infty$, and that $G_K \dim_R L = n-1$, by Proposition 4.2.3. Note also that x is G-regular (and hence L-regular), by Lemma 4.1.7. As $\mathrm{Tor}_1^R(M, R/(x)) = 0$, tensoring with $R/(x)$ yields an exact sequence

$$0 \longrightarrow L/xL \longrightarrow G/xG \longrightarrow M/xM \longrightarrow 0.$$

By the induction hypothesis, $G_{K/xK} \dim_{R/(x)} L/xL \leq n-1$. Also, $G/xG \in G_{K/xK}(R/(x))$, and so by Proposition 4.2.5, $G_{K/xK} \dim_{R/(x)} M/xM \leq n$.

For (2), it's enough to show that $G_K \dim_R M \leq G_{K/xK} \dim_{R/(x)} M/xM$. As above, we may assume that $G_{K/xK} \dim_{R/(x)} M/xM = n < \infty$. If $n = 0$, i.e., if $M/xM \in G_{K/xK}(R/(x))$, then the result follows from Proposition 4.3.3. Otherwise, $n \geq 1$ and there exists a short exact sequence

$$(*) \qquad 0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where $G \in G_K(R)$. As in part (1), tensoring with $R/(x)$ yields an exact sequence

$$0 \longrightarrow L/xL \longrightarrow G/xG \longrightarrow M/xM \longrightarrow 0,$$

where $G/xG \in G_{K/xK}(R/(x))$. By Proposition 4.2.3, $G_{K/xK} \dim_{R/(x)} L/xL = n-1$. Thus by induction, $G_K \dim_R L \leq n-1$, and so applying Proposition 4.2.5 to (*), we have that $G_K \dim_R M \leq n$, as desired. □

Lemma 4.3.5. *Let n be a nonnegative integer and consider an exact sequence of*

R-modules

$$0 \longrightarrow M \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \longrightarrow 0.$$

Suppose that for each $1 \leq i \leq n$ *we have* $\operatorname{Ext}^j_R(A_i, K) = 0$ *for all* $1 \leq j \leq i$. *Then*

$$0 \longrightarrow A_n^\vee \longrightarrow A_{n-1}^\vee \longrightarrow \cdots \longrightarrow A_0^\vee \longrightarrow M^\vee \longrightarrow 0$$

is exact.

Proof. See Lemma 3.18 of [20], substituting the semi-dualizing module K for R. □

Theorem 4.3.6. *Let* M *be a non-zero* R*-module such that* $G_K \dim_R M < \infty$*, and suppose that* x *is a non-zero-divisor on* R *such that* $xM = 0$. *Then*

$$G_{K/xK} \dim_{R/(x)} M = G_K \dim_R M - 1.$$

Proof. We adapt the argument given in the proof of Theorem 3.19 in [20].

We'll proceed by induction on $n = G_K \dim_R M < \infty$. Since x is R-regular, it is also a non-zero-divisor on K and on all elements of $G_K(R)$. Since $xM = 0$, we have that $n \geq 1$. Again, let $(-)^{\vee_x}$ denote the K/xK dual functor $\operatorname{Hom}_{R/(x)}(-, K/xK)$.

Suppose that $n = 1$. Then there exists a short exact sequence

$$(*) \quad 0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

where $G_1, G_0 \in G_K(R)$. Note that

$$M^{\vee_x} = \operatorname{Hom}_{R/(x)}(M, K/xK) \cong \operatorname{Ext}^1_R(M, K),$$

and

$$\operatorname{Ext}^i_{R/(x)}(M, K/xK) \cong \operatorname{Ext}^{i+1}_R(M, K) = 0$$

for all $i > 0$, by Lemma 2 of Chapter 18 of [21]. Apply $\operatorname{Hom}_R(-, K)$ (and the above isomorphisms) to $(*)$ to get the exact sequence

$$0 \longrightarrow M^\vee \longrightarrow G_0^\vee \longrightarrow G_1^\vee \longrightarrow M^{\vee_x} \longrightarrow 0.$$

Since x is K-regular, but $xM = 0$, we have that $M^\vee = 0$, and thus, since $G_K(R)$ is closed under taking K-duals, $G_K \dim_R M^{\vee_x} \leq 1$. By Proposition 4.2.1, we have that

$$\operatorname{Ext}^i_{R/(x)}(M^{\vee_x}, K/xK) \cong \operatorname{Ext}^{i+1}_R(M^{\vee_x}, K) = 0$$

for all $i + 1 > 1$, i.e., for all $i > 0$.

Note that $\operatorname{Tor}^R_j(G_i, R/(x)) = 0$ for $j > 0$, and that $\operatorname{Tor}^R_1(M, R/(x)) \cong M$. Thus applying $- \otimes_R R/(x)$ to $(*)$ yields an exact sequence

$$0 \longrightarrow M \longrightarrow G_1/xG_1 \longrightarrow G_0/xG_0 \longrightarrow M \longrightarrow 0.$$

Note that by Lemma 4.1.8, $G_i/xG_i \in G_{K/xK}(R/(x))$ for $i = 1, 2$. Let L denote the kernel of the map $G_0/xG_0 \to M$. Hence we have short exact sequences

$$0 \longrightarrow M \longrightarrow G_1/xG_1 \longrightarrow L \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow G_0/xG_0 \longrightarrow M \longrightarrow 0.$$

M is $(FP)^R_\infty$, and hence is finitely generated as an $R/(x)$-module. Applying Lemma

2.1.2 to the first short exact sequence, we have that L is $(FP)_1^{R/(x)}$. Thus, applying the same lemma to the second short exact sequence, we have that M is $(FP)_2^{R/(x)}$. Continuing this argument, M is $(FP)_\infty^{R/(x)}$.

Consider the exact sequence

$$(**) \qquad 0 \longrightarrow G_0^\vee \longrightarrow G_1^\vee \longrightarrow M^{\vee_x} \longrightarrow 0.$$

Appyling $- \otimes_R R/(x)$, and a similar argument as given above shows that $M^{\vee_x}$ is also $(FP)_\infty^{R/(x)}$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & G_1/xG_1 & \longrightarrow & G_0/xG_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & \cong \downarrow \phi & & \rho \downarrow \cong & & & & \\ 0 & \longrightarrow & M^{\vee_x\vee_x} & \longrightarrow & (G_0/xG_0)^{\vee_x\vee_x} & \longrightarrow & (G_1/xG_1)^{\vee_x\vee_x} & \longrightarrow & M^{\vee_x\vee_x} & \longrightarrow & 0. \end{array}$$

Note that the bottom row is exact by Lemma 4.3.5 (applied twice). Here, ϕ and ρ denote the natural maps. Both are isomorphisms, as $G_i/xG_i \in G_{K/xK}(R/(x))$ for all i, and hence $M \cong M^{\vee_x\vee_x}$. Thus $M \in G_{K/xK}(R/(x))$.

Suppose now that $n > 1$. Then there exists a surjection

$$G \xrightarrow{\phi} M \longrightarrow 0$$

where $G \in G_K(R)$. Since $xM = 0$, tensoring with $R/(x)$ yields an exact sequence of $R/(x)$ modules

$$(\#) \qquad 0 \longrightarrow L \longrightarrow G/xG \xrightarrow{\overline{\phi}} M \longrightarrow 0,$$

where $L = \ker \overline{\phi}$. The exactness of the sequence

$$0 \longrightarrow G \xrightarrow{x} G \longrightarrow G/xG \longrightarrow 0$$

implies that $G_K \dim_R G/xG = 1$, and thus by Proposition 4.2.5, we have that $G_K \dim_R L < \infty$. Applying $\mathrm{Hom}_R(-, K)$ to (#) yields a long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}_R^{i-1}(G/xG, K) \longrightarrow \mathrm{Ext}_R^{i-1}(L, K) \longrightarrow \mathrm{Ext}_R^{i}(M, K) \longrightarrow \ldots.$$

By Proposition 4.2.1, $\mathrm{Ext}_R^{i-1}(G/xG, K) = 0$ for all $i - 1 > 1$, and $\mathrm{Ext}_R^{i}(M, K) = 0$ for all $i > n$. Since $n > 1$, both these Ext modules vanish for all $i > n$, and hence $Ext_R^{i-1}(L, K) = 0$ for all $i - 1 > n$, i.e., for all $i > n - 1$. Thus Proposition 4.2.1 implies that $G_K \dim_R L \leq n - 1$. Since x annihilates L, by the induction hypothesis,

$$G_{K/xK} \dim_{R/(x)} L = n - 2.$$

As $G/xG \in G_{K/xK}(R/(x))$, by Proposition 4.2.3 we have

$$G_{K/xK} \dim_{R/(x)} M = G_{K/xK} \dim_{R/(x)} L + 1 = n - 1,$$

as desired. □

Proposition 4.3.7. *Let R be coherent, M a non-zero finitely presented R-module, and $x \in J(R)$ a non-zero-divisor on R such that $xM = 0$. If $G_{K/xK} \dim_{R/(x)} M < \infty$, then*

$$G_{K/xK} \dim_{R/(x)} M = G_K \dim_R M - 1.$$

Proof. The proof in the Noetherian case ([9], Lemma 1.5.2) is easily adapted to the case that R is coherent. $\square$

Summarizing our results in the coherent case, we have:

Corollary 4.3.8. *Let R be coherent, M a nonzero finitely presented R-module, and $x \in J(R)$ a non-zero-divisor on R.*

1. *If $xM = 0$, then $G_{K/xK} \dim_{R/(x)} M = G_K \dim_R M - 1$.*

2. *If x is a non-zero-divisor on M, then $G_K \dim_R M/xM = G_K \dim_R M + 1$.*

Chapter 5

The Auslander-Bridger Formula for G_K-dimension

5.1 The Auslander-Bridger Formula for G_K-dimension

Throughout this section, let (R, m) be a quasi-local ring and K a semi-dualizing R-module. Recall that for an R-module M, $M^\vee$ denotes the K-dual module $\mathrm{Hom}_R(M, K)$. Here, we prove the Auslander-Bridger formula for modules of finite G_K-dimension.

Lemma 5.1.1. *Suppose that* $\operatorname{depth} R = 0$*, and let M be a finitely presented R-module. Then $M = 0$ if and only if $M^\vee = 0$.*

Proof. We adapt the proof of Lemma 4.1 of [20], replacing the R-dual of a module with its K-dual.

Note that by Proposition 3.2.5, $\operatorname{depth} K = 0$ as well. Let M be generated by n elements, and suppose that $M^\vee = 0$. We'll show that $M = 0$ using induction on n. If $n = 1$, then $M \cong R/I$ for some finitely generated ideal $I \subseteq R$. For all finitely generated ideals $J \subsetneq R$, we have

$$\operatorname{grade}_R(J, K) \leq \operatorname{depth}_R K = 0,$$

and hence $\operatorname{Hom}_R(R/J, K) \neq 0$, by Lemma 2.3.4. Since

$$\operatorname{Hom}_R(R/I, K) \cong M^\vee = 0,$$

we have that $I = R$, and thus $M = 0$.

Suppose that $n > 1$. Then there exists a submodule $M' \subseteq M$ which is generated by $n-1$ elements and such that M/M' is cyclic. We have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0,$$

and hence M/M' is finitely presented, by Lemma 2.1.2. Applying $\operatorname{Hom}_R(-, K)$ yields an exact sequence

$$0 \longrightarrow (M/M')^\vee \longrightarrow M^\vee = 0,$$

and hence $(M/M')^\vee = 0$. So $M/M' = 0$ and hence $M = M'$ is generated by $n-1$ elements. The induction hypothesis implies that $M = 0$. □

Lemma 5.1.2. *Suppose that* $\operatorname{depth} R = 0$*, and let* M *be a non-zero* R*-module of finite* G_K*-dimension. Then* $G_K \dim_R M = 0$*, and* $\operatorname{depth} M = 0$.

Proof. We adapt the proof of Lemma 4.2 of [20], replacing the R-dual of a module with its K-dual.

We'll proceed by induction on n to show that if $G_K \dim_R M \leq n$, then $G_K \dim_R M =$

0. First suppose that $n = 1$. Then there exists a short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

where $G_1, G_0 \in G_K(R)$. Applying $\mathrm{Hom}_R(-, K)$ yields an exact sequence

$$(*) \qquad \ldots \longrightarrow \mathrm{Ext}_R^{i-1}(G_1, K) \longrightarrow \mathrm{Ext}_R^i(M, K) \longrightarrow \mathrm{Ext}_R^i(G_0, K) \longrightarrow \ldots.$$

In particular, we have an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow G_0^\vee \longrightarrow G_1^\vee \longrightarrow \mathrm{Ext}_R^1(M, K) \longrightarrow \mathrm{Ext}_R^1(G_0, K) = 0.$$

Applying $\mathrm{Hom}_R(-, K)$ again yields the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathrm{Ext}_R^1(M, K))^\vee & \longrightarrow & G_1^{\vee\vee} & \longrightarrow & G_0^{\vee\vee} \\ & & & & \cong\downarrow & & \cong\downarrow \\ & & 0 & \longrightarrow & G_1 & \longrightarrow & G_0. \end{array}$$

Hence $(\mathrm{Ext}_R^1(M, K))^\vee = 0$. Since Lemma 4.3.2 implies that $\mathrm{Ext}_R^1(M, K)$ is finitely presented, we have that $\mathrm{Ext}_R^1(M, K) = 0$, by Lemma 5.1.1. Note that $\mathrm{Ext}_R^i(G_j, K) = 0$ for all $i > 0$ and $j = 1, 2$, and so from $(*)$ we have that $\mathrm{Ext}_R^i(M, K) = 0$ for all $i > 1$. Thus by Lemma 4.1.6, $G_K \dim_R M = 0$.

If $G_K \dim_R M \leq n$ for some $n > 1$, then we have an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow G_{n-1} \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

where each $G_i \in G_K(R)$. Let L_{n-1} denote the $(n-1)^{th}$ syzygy of this G_K-resolution.

We have a short exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow L_{n-1} \longrightarrow 0.$$

Hence $G_K \dim_R L_{n-1} \leq 1$, and so the previous case implies that $L_{n-1} \in G_K(R)$. So

$$0 \longrightarrow L_{n-1} \longrightarrow G_{n-2} \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

is a G_K-resolution of R of length $n-1$, and hence $G_K \dim_R M \leq n-1$.

It remains to show that if $M \in G_K(R)$, then $\operatorname{depth} M = 0$. Suppose that $\operatorname{depth} M > 0$. Resetting notation (since by Proposition 4.1.9, $M \in G_K(R)$ if and only if $M \otimes_R S \in G_{K \otimes_R S}(S)$ for a faithfully flat extension $R \to S$), we may assume that there exist some $x \in m$ which is M-regular. Hence we have exact sequences

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

and

$$(\#) \quad 0 \longrightarrow (M/xM)^\vee \longrightarrow M^\vee \xrightarrow{x} M^\vee \longrightarrow \operatorname{Ext}^1_R(M/xM, K) \longrightarrow \operatorname{Ext}^1_R(M, K) = 0.$$

We have an exact sequence

$$0 \longrightarrow xM^\vee \longrightarrow M^\vee \longrightarrow \operatorname{Ext}^1_R(M/xM, K) \longrightarrow 0,$$

where $M^\vee$ is $(FP)_\infty$ and $xM^\vee$ is finitely generated. Hence $\operatorname{Ext}^1_R(M/xM, K)$ is finitely presented. Applying $\operatorname{Hom}_R(-, K)$ to $(\#)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathrm{Ext}^1_R(M/xM,K))^\vee & \longrightarrow & M^{\vee\vee} & \xrightarrow{x} & M^{\vee\vee} \\ & & & & \downarrow\cong & & \downarrow\cong \\ & & 0 & \longrightarrow & M & \xrightarrow{x} & M. \end{array}$$

Since x is M-regular, we have that $(\mathrm{Ext}^1_R(M/xM,K))^\vee = 0$, and hence $\mathrm{Ext}^1_R(M/xM,K) = 0$, by Lemma 5.1.1. So from (#) we have a surjection

$$M^\vee \xrightarrow{x} M^\vee \longrightarrow 0.$$

Hence by Nakayama's Lemma, $M^\vee = 0$. Thus Lemma 5.1.1 implies that $M = 0$, a contradiction. Hence $\operatorname{depth} M = 0$. □

Lemma 5.1.3. *Let M be a non-zero module in $G_K(R)$. Then* $\operatorname{depth} M = \operatorname{depth} R$.

Proof. We adapt the proof of Lemma 4.3 of [20], replacing the R-dual of a module with its K-dual; note that by Proposition 3.2.5, $\operatorname{depth} K = \operatorname{depth} R$.

Let $n = \operatorname{depth} R$. First, suppose that $n < \infty$; we'll proceed by induction on n. If $n = 0$, then the result holds by Lemma 5.1.2. If $n \geq 1$, then, passing to $R[x]_{m[x]}$ if necessary, and resetting notation, there exists $x \in m$ such that x is R-regular. Since $M \in G_K(R)$, x is also M-regular, and $M/xM \in G_{K/xK}(R/(x))$, by Lemmas 4.1.7 and 4.1.8.

By part (4) of Proposition 2.3.2 and Proposition 2.3.3, we have that

$$\operatorname{depth}_{R/(x)} R/(x) = n - 1.$$

Similarly, $\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M - 1$. By induction, $\operatorname{depth}_{R/(x)} M/xM = n - 1$, and hence $\operatorname{depth}_R M = n$.

Suppose now that $\operatorname{depth}_R R = \infty$. We'll show by induction on l that $\operatorname{depth}_R M \geq l$. The result is clear for $l = 0$, so suppose that $l \geq 1$ and that for all quasi-local rings S of infinite depth, all semi-dualizing S-modules K', and all modules $N \in G_{K'}(S)$, we have $\operatorname{depth}_S N \geq l$. Resetting notation if necessary, we may assume that there exists $x \in m$ which is R-, and hence M-regular. As above, $M/xM \in G_{K/xK}(R/(x))$ with $\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M - 1$. Moreover, $\operatorname{depth}_{R/(x)} R/(x) = \infty$, and so by induction,

$$\operatorname{depth}_{R/(x)} M/xM \geq l.$$

Hence $\operatorname{depth}_R M = \operatorname{depth}_{R/(x)} M/xM + 1 \geq l + 1$, as desired. □

We now prove a generalization of the Auslander-Bridger formula for G_K-dimension over quasi-local rings.

Theorem 5.1.4. *Let (R, m) be a quasi-local ring, K as semi-dualizing module for R, and M a non-zero R-module of finite G_K-dimension. Then*

$$\operatorname{depth} M + G_K \dim_R M = \operatorname{depth} R.$$

Proof. We adapt the proof of Theorem 4.4 of of [20], replacing the R-dual of a module with its K-dual.

First, consider the case where $\operatorname{depth}_R R = \infty$. If $G_K \dim_R M = \infty$, then the equality holds, so suppose that $G_K \dim_R < \infty$. If $G_K \dim_R M = 0$, then by Lemma 5.1.3, $\operatorname{depth}_R M = \infty$, and the equality holds. Otherwise, we induct on $n = G_K \dim_R M > 0$. There exists a short exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where $G \in G_K(R)$ and $G_K \dim_R L = n-1$. By the induction hypothesis, $\operatorname{depth}_R L = \operatorname{depth}_R R = \infty$, and so by part (5) of Proposition 2.3.2, we have that $\operatorname{depth}_R M = \infty$.

Now, suppose that $\operatorname{depth}_R R < \infty$; we'll induct on the depth of R. If $\operatorname{depth} R = 0$, then by Lemma 5.1.2, we have that $M \in G_K(R)$ and $\operatorname{depth}_R M = 0$, and hence the equality holds. Otherwise, suppose that $\operatorname{depth}_R R = l > 0$, and that in rings of depth less than l, the Auslander-Bridger formula holds for modules of finite G_K-dimension.

Suppose that $\operatorname{depth}_R M > 0$.

Claim: We may assume that there exists $x \in m$ such that x is both M- and R-regular.

As $\operatorname{depth} R > 0$ and $\operatorname{depth} M > 0$, there exist finitely generated ideals I and J such that $\operatorname{Hom}_R(R/I, R) = 0$ and $\operatorname{Hom}_R(R/J, M) = 0$. Let $U = I + J$, which is finitely generated as well; note that $\operatorname{grade}(U, R), \operatorname{grade}(U, M) > 0$. Applying $\operatorname{Hom}_R(R/U, -)$ to the split exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus R \longrightarrow R \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/U, M) \longrightarrow \operatorname{Hom}_R(R/U, M \oplus R) \longrightarrow \operatorname{Hom}_R(R/U, R).$$

We have that $\operatorname{Hom}_R(R/U, M) = \operatorname{Hom}_R(R/U, R) = 0$, by Lemma 2.3.4, and hence $\operatorname{Hom}_R(R/U, M \oplus R) = 0$. So $\operatorname{depth}_R M \oplus R > 0$; passing to a faithfully flat extension and resetting notation, if necessary, gives the required non-zero-divisor. This proves the claim.

Thus we have by part (2) of Proposition 4.3.4

$$\begin{aligned} G_{K/xK}\dim_{R/(x)} M/xM &= G_K\dim_R M, \\ \operatorname{depth}_{R/(x)} M/xM &= \operatorname{depth}_R M - 1, \\ \text{and} \quad \operatorname{depth}_{R/(x)} R/(x) &= \operatorname{depth}_R R - 1. \end{aligned}$$

Hence by the induction hypothesis (on $\operatorname{depth} R$),

$$G_{K/xK}\dim_{R/(x)} M/xM + \operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_{R/(x)} R/(x).$$

Thus

$$G_K\dim_R M + \operatorname{depth}_R M = \operatorname{depth}_R R,$$

as desired.

Now, suppose that $\operatorname{depth}_R M = 0$. Note that $M \notin G_K(R)$, as $\operatorname{depth}_R M < \operatorname{depth}_R R$. So there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$$

where $G \in G_K(R)$ and $G_K\dim_R L = G_K\dim_R M - 1$. Since

$$\operatorname{depth}_R G = \operatorname{depth}_R R > \operatorname{depth}_R M,$$

by part (5) of Proposition 2.3.2, we have that

$$\operatorname{depth}_R L = \operatorname{depth}_R M + 1 = 1.$$

The previous case shows that the Auslander-Bridger formula holds for modules of positive depth, and hence

$$\begin{aligned}\operatorname{depth}_R R &= \operatorname{depth}_R L + G_K \dim_R L \\ &= (\operatorname{depth}_R M + 1) + (G_K \dim_R M - 1) \\ &= \operatorname{depth}_R M + G_K \dim_R M.\end{aligned}$$

□

Chapter 6

Cohen-Macaulay Dimension

Throughout this chapter, we let CM denote Cohen-Macaulay in the standard sense for Noetherian rings, and HMCM denote Cohen-Macaulay as defined by Hamilton and Marley in [18].

6.1 CM-dimension

In [13], A.A. Gerko defines the following homological dimension; in the context of [13], all rings are commutative, local, and Noetherian, and all modules are finitely generated.

Definition 6.1.1. ([13], Definition 4.2′) Let (R, m) be a local ring, and let M be a finitely generated R-module. The *CM-dimension* of an R-module M is given by the following:

$$CM \dim_R M = \inf_{S,K}\{G_K \dim_S(M \otimes_R S)\},$$

where S ranges over all faithfully flat extensions of R, and K over all semi-dualizing S-modules.

Gerko's CM-dimension classifies (local, Noetherian) CM rings:

Theorem 6.1.2. *([13], Theorem 4.10) If a local ring* (R, m, k) *is CM, then for any finitely generated* R*-module* M *we have* $CM \dim_R M < \infty$*. Conversely, if* $CM \dim_R k < \infty$*, then* R *is a CM ring.*

In this spirit we make the following definitions:

Definition 6.1.3. Let (R, m) be a quasi-local coherent ring, and let M be an $(FP)_\infty$ R-module. The *CM-dimension* of M is given by the following:

$$CM \dim_R M = \inf_{S,K} \{G_K \dim_S M \otimes_R S\},$$

where (S, n) ranges over all coherent, quasi-local faithfully flat extensions of R, and K over all semi-dualizing S-modules.

Definition 6.1.4. Let (R, m) be a quasi-local coherent ring. R is called *GCM* (that is, Cohen-Macaulay in the sense of Gerko) if for all $(FP)_\infty$ R-modules M, $CM \dim_R M < \infty$. In general, a coherent ring R is called *GCM* if R_m is GCM for every maximal ideal m of R.

Proposition 2.2.5 implies that over a coherent ring, the class of $(FP)_\infty$-modules is equivalent to the class of finitely presented modules; thus a quasi-local coherent ring R is GCM if all finitely presented R-modules have finite CM-dimension.

These definitions provide an affirmative answer to Glaz's question:

Question 6.1.5. ([16], [15]) Is there a workable definition of Cohen-Macaulay for commutative rings which extends the usual definition in the Noetherian case, and such that every coherent regular ring is Cohen-Macaulay?

Proposition 5.3 of [20] shows that every coherent regular ring is Gorenstein. Thus it suffices to show that every coherent Gorenstein ring is GCM.

Given a local ring R, a finitely generated R-module M is called *maximal Cohen-Macaulay (MCM)* if $\operatorname{depth} M = \dim M = \dim R$.

Theorem 6.1.6. *1. If R is a a coherent Gorenstein ring, then R is GCM.*

2. A Noetherian ring R is GCM if and only if R is Cohen-Macaulay (in the usual sense).

Proof. For (1), let R be a coherent Gorenstein ring. Let m be a maximal ideal of R, and M an R_m module which is $(FP)_\infty^{R_m}$. As R_m is Gorenstein, every finitely presented R_m-module has finite G-dimension, by Proposition 5.5 of [20]. Taking $K = R_m$, we have

$$CM \dim_{R_m} M \leq G_{R_m} \dim_{R_m} M < \infty.$$

Thus R_m, and hence R, is GCM.

For (2), let R be a Noetherian ring. First, suppose that R is GCM. The argument in the proof of Theorem 4.10 of [13] shows that R is CM.

Now, suppose that R is CM. Since R_m is CM for all maximal ideals m, we may assume that (R, m) is local. Moreover, since $R \to \hat{R}$ is faithfully flat, and $\hat{R}$ is a CM local ring, we may assume that R is complete. By Corollary 3.3.8 of [7], there exists a canonical module for R; let ω denote the canonical module. Note that ω is MCM, hence finitely generated, and that by Theorem 3.3.10 of [7] $\operatorname{Ext}_R^i(\omega, \omega) = 0$ for all $i > 0$. Moreover, $\operatorname{Hom}_R(\omega, \omega) \cong R$, by Theorem 3.3.4 of [7]. Thus ω is semi-dualizing for R.

First, we'll show that an MCM R-module is an element of the G_ω-class of R. Suppose that M is an MCM module. By Theorem 3.3.10(d) of [7], $\operatorname{Ext}_R^i(M, \omega) = 0$

for all $i > 0$, and the natural map

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega), \omega)$$

is an isomorphism. Moreover, $\operatorname{Hom}_R(M, \omega)$ is also an MCM module, and hence $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, \omega), \omega) = 0$ for all $i > 0$. Hence $M \in G_\omega(R)$.

Let M be a finitely generated R-module. It suffices to show that M has finite G_ω-dimension. We show by induction on $\operatorname{depth} R - \operatorname{depth} M$ that $G_\omega \dim_R M < \infty$. If $\operatorname{depth} M = \operatorname{depth} R$, then M is MCM, and hence has finite G_ω-dimension. Otherwise, $\operatorname{depth} M < \operatorname{depth} R$. As M is finitely generated, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

Since $\operatorname{depth} R^n = \operatorname{depth} R$, by Proposition 2.3.2 we have that $\operatorname{depth} L = \operatorname{depth} M + 1$. As $\operatorname{depth} R - \operatorname{depth} L < \operatorname{depth} R - \operatorname{depth} M$, by the induction hypothesis we have that $G_\omega \dim_R L < \infty$. Since $R^n \in G_\omega(R)$, we have by Proposition 4.2.5 that $G_\omega \dim_R M < \infty$. ∎

The Auslander-Bridger formula holds for modules of finite CM-dimension (and hence for all $(FP)_\infty$ modules over a GCM ring):

Theorem 6.1.7. *Let (R, m) be a quasi-local coherent ring, and M an R-module of finite CM-dimension. Then*

$$\operatorname{depth} M + CM \dim_R M = \operatorname{depth} R.$$

Proof. Since $CM \dim_R M < \infty$, there exists a faithfully flat R-algebra (S, n) and a semi-dualizing S-module K such that $CM \dim_R M = G_K \dim_S M \otimes_R S < \infty$. By the Auslander-Bridger formula for G_K-dimension over S (Theorem 5.1.4), as well as properties of grade given in Proposition 2.3.2, we have

$$\begin{aligned} CM\dim_R M &= G_K \dim_S M \otimes_R S \\ &= \operatorname{depth}_S S - \operatorname{depth}_S M \otimes_R S \\ &= \operatorname{depth}_R R - \operatorname{depth}_R M, \end{aligned}$$

and hence $\operatorname{depth} M + CM\dim_R M = \operatorname{depth} R$. □

It is unknown if there is a "two out of three" lemma for finite CM-dimension along a short exact sequence; however, we are able to show this holds in the following special case:

Lemma 6.1.8. *Let (R, m) be a quasi-local coherent ring, and let A, B be finitely presented R-modules. If there exists a short exact sequence*

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$$

where F is a finitely generated free R-module, then $CM\dim_R A < \infty$ if and only if $CM\dim_R B < \infty$. In particular, if $CM\dim_R A \leq n$, then $CM\dim_R B \leq n+1$; if $CM\dim_R B \leq n$, then $CM\dim_R A \leq n$.

Proof. We can write $F = R^l$ for some l. Suppose that $CM\dim_R A = n < \infty$. Then there exists a faithfully flat extension $R \to S$ and a semi-dualizing S-module K such that

$$G_K \dim_R A \otimes_R S = n.$$

Notice that $F \otimes_R S \cong S^l$, and that $G_K \dim_S S^l = 0$, as a finitely generated free module is an element of the $G_K(S)$ class. Since S is faithfully flat, we have an exact sequence

$$0 \longrightarrow A \otimes_R S \longrightarrow F \otimes_R S \longrightarrow B \otimes_R S \longrightarrow 0,$$

where $G_K \dim_S A \otimes_R S = n$ and $G_K \dim_S F \otimes_R S = 0$. So by Proposition 4.2.5, $G_K \dim_S B \otimes_R S \leq n + 1$, and hence $CM \dim_R B \leq n + 1$, as desired. The proof is similar if one assumes that $CM \dim_R B$ is finite. □

6.2 Change of Rings Results

As in the local case, the GCM property localizes. It is unknown if the HMCM property localizes.

Proposition 6.2.1. *If R is a coherent GCM ring and $U \subseteq R$ is a multiplicatively closed set, then R_U is also GCM.*

Proof. First, note that localizing preserves coherence. Suppose that p_U is a maximal ideal of R_U, where p is a prime ideal of R. We need to show that every $(FP)_\infty$-module over $(R_U)_{p_U} \cong R_p$ has finite CM dimension.

Notice that $p \subseteq m$ for some maximal ideal m of R, and that R_m is GCM by definition. Let M be a finitely presented R_p-module; then there exists a finitely presented R-module N such that $N_p = M$. As N_m is finitely presented as an R_m-module, we have

$$CM \dim_{R_m} N_m < \infty,$$

and so there exists a faithfully flat extension $R_m \to S$ and a semi-dualizing S-module K such that

$$CM \dim_{R_m} N_m = G_K \dim_S N_m \otimes_{R_m} S < \infty.$$

Choose $q \in \operatorname{Spec} S$ such that q is minimal over $p_m S$. Notice that $R_p \cong (R_m)_{p_m} \to S_q$ is faithfully flat, and that K_q is semi-dualizing for S_q. Since

$$M \otimes_{R_p} S_q \cong N_p \otimes_{R_p} S_q \cong (N_m \otimes_{R_m} S)_q,$$

by Proposition 4.12 we have

$$G_{K_q} \dim_{S_q} M \otimes_{R_p} S_q \leq G_K \dim_S N_m \otimes_{R_m} S < \infty,$$

and hence $CM \dim_{R_p} M < \infty$, as desired.

□

If (R, m) is local and $x \in m$ is R-regular, then R is CM if and only if $R/(x)$ is CM; see Theorem 2.1.3 of [7]. The forward implication holds for the GCM property. The HMCM property need not specialize; see Example 4.9 of [18].

Proposition 6.2.2. *Let (R, m) be a quasi-local coherent ring, and let $x \in m$ be a non-zero-divisor on R. If R is GCM, then $R/(x)$ is GCM.*

Proof. Note that $R/(x)$ is coherent and quasi-local with maximal ideal $\overline{m} = m/(x)$. Let N be a finitely presented $R/(x)$-module; we'll show that $CM \dim_{R/(x)} N < \infty$. As N is finitely presented as an $R/(x)$-module, there exists an exact sequence

$$0 \longrightarrow A \longrightarrow (R/(x))^n \longrightarrow N \longrightarrow 0,$$

where A is finitely generated. Consider the following diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & \ker\gamma & \longrightarrow & R^n & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle \cdot x} & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & R^n & \xrightarrow{f} & N & \longrightarrow & 0 \\
 & & {\scriptstyle \gamma}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle =}\downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & (R/(x))^n & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0. & &
\end{array}
$$

Here, $L = \ker f$ and $\gamma : L \to A$ is induced by the diagram. Note that $\ker\gamma \hookrightarrow R^n$, and that the Snake Lemma gives an exact sequence

$$0 \longrightarrow \ker\gamma \longrightarrow R^n \longrightarrow 0.$$

Hence $\ker\gamma \cong R^n$, and thus we have an exact sequence

$$0 \longrightarrow R^n \longrightarrow L \longrightarrow A \longrightarrow 0.$$

As A and R^n are $(FP)_0$, Lemma 2.1.2 implies that L is also $(FP)_1$. By the same lemma and the exact sequence

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow N \longrightarrow 0,$$

we obtain that N is $(FP)_1$. Hence, as R is GCM, $CM\dim_R N < \infty$. So there exists a faithfully flat extension $R \to S$ and a semi-dualizing S-module K such that

$$CM\dim_R N = G_K \dim_S N \otimes_R S < \infty.$$

Note that $R/(x) \to S/xS$ is faithfully flat, and that

$$N \otimes_R S \cong N \otimes_{R/(x)} R/(x) \otimes_R S \cong N \otimes_{R/(x)} S/xS.$$

Since $x \in m$ is regular on S and K, with $x(N \otimes_{R/(x)} S/xS) = 0$, by Theorem 4.3.6 we have

$$\begin{aligned} G_{K/xK} \dim_{S/xS} N \otimes_{R/(x)} S/xS &= G_K \dim_S N \otimes_{R/(x)} S/xS - 1 \\ &= G_K \dim_S N \otimes_R S - 1 \\ &< \infty. \end{aligned}$$

Thus $CM \dim_{R/(x)} N < \infty$, and hence $R/(x)$ is GCM. □

6.3 GCM rings

If a ring R is GCM, is there a faithfully flat extension S of R and a semi-dualizing module K of S such that $G_K \dim_R M \otimes_R S < \infty$ for all finitely presented R-modules M? If not, what conditions on R guarantee the existence of such a pair (S, K)? The answer to the first question is unknown; in Corollary 6.3.3, we give a partial answer to the second question.

Given a local ring (R, m) and an ideal $I \subseteq R$, I is called *perfect* if the projective dimension of R/I is equal to the grade of I. If R is a regular ring, then R/I is CM if and only if I is perfect. The ideal I is called *G-perfect* if $G \dim_R R/I = \operatorname{grade} I$. If R is a Gorenstein ring, then R/I is CM if and only if I is G-perfect. The following results are motivated by results on G-perfect ideals in [17].

Proposition 6.3.1. *Let (S, n) be a quasi-local ring, and $I \subseteq S$ an ideal such that*

$S/I \in (FP)^S_\infty$. If $G_S \dim_S S/I = \operatorname{grade} I < \infty$, then for $t = \operatorname{grade} I$, $\operatorname{Ext}^t_S(S/I, S)$ is a semi-dualizing module for S/I.

Proof. We'll proceed by induction on t. First suppose that $t = 0$ and that $S/I \in G_S(S)$. Let $K = \operatorname{Hom}_S(S/I, S)$. Since $S/I \in G_S(S)$, both S/I and K are $(FP)_\infty$ as S-modules; as $IK = 0$, Lemma 2.1.7 gives that K is also $(FP)_\infty$ as an S/I-module. Moreover, S/I is reflexive as an S-module, and so

$$\begin{aligned} S/I &\cong \operatorname{Hom}_S(\operatorname{Hom}_S(S/I, S), S) \\ &\cong \operatorname{Hom}_S(\operatorname{Hom}_S(S/I, S) \otimes_{S/I} S/I, S) \\ &\cong \operatorname{Hom}_S(\operatorname{Hom}_S(S/I, S), \operatorname{Hom}_S(S/I, S)) \\ &\cong \operatorname{Hom}_{S/I}(\operatorname{Hom}_S(S/I, S), \operatorname{Hom}_S(S/I, S)) \\ &= \operatorname{Hom}_{S/I}(K, K). \end{aligned}$$

Hence by Theorem 3.1.2, the homothety map $S/I \to \operatorname{Hom}_{S/I}(K, K)$ is an isomorphism.

It remains to show that $\operatorname{Ext}^i_{S/I}(K, K) = 0$ for all $i > 0$. Let $\boldsymbol{E}$ be an injective S-resolution for S. Applying the left exact functor $\operatorname{Hom}_S(S/I, -)$ to $\boldsymbol{E}$, we see that

$$\mathrm{H}^0(\operatorname{Hom}_S(S/I, \boldsymbol{E})) = \operatorname{Hom}_S(S/I, S) = K.$$

As $G_S \dim_S S/I = 0$, we have

$$\mathrm{H}^i(\operatorname{Hom}_S(S/I, \boldsymbol{E})) = \operatorname{Ext}^i_S(S/I, S) = 0$$

for all $i > 0$, and hence $\operatorname{Hom}_S(S/I, \boldsymbol{E})$ is an injective S/I-resolution for K. Thus for all $i > 0$ we have

$$\begin{aligned}
\operatorname{Ext}^i_{S/I}(K,K) &= \mathrm{H}^i(\operatorname{Hom}_{S/I}(K, \operatorname{Hom}_S(S/I, \boldsymbol{E}))) \\
&\cong \mathrm{H}^i(\operatorname{Hom}_S(K \otimes_{S/I} S/I, \boldsymbol{E})) \\
&\cong \mathrm{H}^i(\operatorname{Hom}_S(\operatorname{Hom}_S(S/I, S), \boldsymbol{E})) \\
&= \operatorname{Ext}^i_S(\operatorname{Hom}_S(S/I, S), S) \\
&= 0,
\end{aligned}$$

where the last equality holds since $S/I \in G_S(S)$. Thus K is semi-dualizing for S/I.

Suppose now that $t > 0$, and as above let $K = \operatorname{Ext}^t_S(S/I, S)$. Then there exists a quasi-local faithfully flat extension $S \to T$ and some $x \in IT$ such that x is a non-zero-divisor on T. Note by Remark 2.1.5 that $S/I \otimes_S T \cong T/IT$ is $(FP)_\infty$ as a T-module. Moreover, $\operatorname{grade}_T IT = \operatorname{grade}_S I$, and $G_T \dim_T T/IT = G_S \dim_S S/I$, by Corollary 4.3.1. Since $S/I \to T/IT$ is also a faithfully flat extension, by Proposition 3.2.4

$$\operatorname{Ext}^t_T(T/IT, T) \cong \operatorname{Ext}^t_S(S/IS, S) \otimes_S T$$

is semi-dualizing for T/IT if and only if $\operatorname{Ext}^t_S(S/I, S)$ is semi-dualizing for S. Resetting notation, we may assume that S is as in the hypothesis, and that there exists some $x \in I$ which is a non-zero-divisor on S.

Suppose that $x \in I$ is a non-zero-divisor on S; let $\overline{S} = S/(x)$ and $\overline{I} = I/(x)$. We'll first show that $\overline{S}/\overline{I} \cong S/I$ is $(FP)^S_\infty$ as an $\overline{S}$-module. Since S/I is $(FP)_\infty$ by assumption, and $x(S/I) = 0$, by Lemma 2.1.7, it's enough to show that $\overline{S}$ is $(FP)^S_\infty$. This follows from the short exact sequence

$$(*) \qquad 0 \longrightarrow S \xrightarrow{x} S \longrightarrow \overline{S} \longrightarrow 0.$$

We claim that $G_{\overline{S}} \dim_{\overline{S}} S/I = \operatorname{grade}_{\overline{S}} \overline{I} = t-1$. By Theorem 4.3.6, we have

$$G_{\overline{S}} \dim_{\overline{S}} S/I = G_S \dim_S S/I - 1 = t - 1.$$

By part 4 of Proposition 2.3.2

$$\operatorname{grade}_{\overline{S}} I = \operatorname{grade}_S I - 1.$$

By Proposition 2.3.3,

$$\operatorname{grade}_{\overline{S}} I = \operatorname{grade}_{\overline{S}} \overline{I}.$$

Hence $\operatorname{grade}_{\overline{S}} \overline{I} = t-1$, completing the claim. By induction, $\operatorname{Ext}^{t-1}_{\overline{S}}(S/I, \overline{S})$ is semi-dualizing for $\overline{S}/\overline{I} \cong S/I$. Thus by Lemma 18.2 of [21], we have that

$$K = \operatorname{Ext}^t_S(S/I, S) \cong \operatorname{Ext}^{t-1}_{\overline{S}}(S/I, \overline{S})$$

is semi-dualizing for S/I.

□

Theorem 6.3.2. *Let (S, n) be a quasi-local ring, and let $I \subseteq S$ be an ideal such that*

$$G_S \dim_S S/I = \operatorname{grade}_S I < \infty.$$

Let $t = \operatorname{grade}_S I$, and consider the semi-dualizing S/I-module $K = \operatorname{Ext}^t_S(S/I, S)$. For any S/I-module M such that $G_S \dim_S M < \infty$, we have

$$G_K \dim_{S/I} M = G_S \dim_S M - t.$$

In particular, $G_K \dim_{S/I} M$ is finite.

Note that under these hypothesis, Proposition 6.3.1 shows that K is in fact semidualizing for S/I.

Proof. Let M be an S/I-module such that $G_S \dim_S M < \infty$. Note that both S/I and M are $(FP)^S_\infty$, and that M is $(FP)^{S/I}_\infty$, by Lemma 2.1.7. We first show that it suffices to prove that $G_K \dim_{S/I} M < \infty$.

Note that for any S/I-module A, Proposition 2.3.3 implies that $\operatorname{depth}_{S/I} A = \operatorname{depth}_S A$. If $G_K \dim_{S/I} M < \infty$, then by the Auslander-Bridger formula for both G_K-dimension and G-dimension over S/I and S, respectively (Theorem 5.1.4), we have

$$\begin{aligned}
G_K \dim_{S/I} M &= \operatorname{depth}_{S/I} S/I - \operatorname{depth}_{S/I} M \\
&= \operatorname{depth}_S S/I - \operatorname{depth}_S M \\
&= \operatorname{depth}_S S/I - (\operatorname{depth}_S S - G_S \dim_S M) \\
&= G_S \dim_S M - (\operatorname{depth}_S S - \operatorname{depth}_S S/I) \\
&= G_S \dim_S M - G_S \dim_S S/I \\
&= G_S \dim_S M - \operatorname{grade}_S I \\
&= G_S \dim_S M - t.
\end{aligned}$$

We'll show that $G_K \dim_{S/I} M < \infty$ by induction on $t = \operatorname{grade} I$. First, consider the case that $t = 0$, i.e., that $K = \operatorname{Hom}_S(S/I, S)$ and $S/I \in G_S(S)$. Let G be an S/I-module; we'll show that $G \in G_S(S)$ if and only if $G \in G_K(S/I)$. Let $(-)^*$ denote the functor $\operatorname{Hom}_S(-, S)$, and let $(-)^\vee$ denote the functor $\operatorname{Hom}_{S/I}(-, K)$.

Notice that

$$G^\vee = \operatorname{Hom}_{S/I}(G, \operatorname{Hom}_S(S/I, S)) \cong \operatorname{Hom}_S(G \otimes_{S/I} S/I, S) \cong \operatorname{Hom}_S(G, S) = G^*.$$

Consequently, $G^{\vee\vee} \cong G^{**}$. Hence $G \cong G^{**}$ if and only if $G \cong G^{\vee\vee}$.

If G is $(FP)_{\infty}^{S}$, then since $IG = 0$ and S/I is $(FP)_{\infty}^{S}$, Lemma 2.1.7 implies that G is $(FP)_{\infty}^{S/I}$, and hence G has a resolution by finitely generated free S/I-modules. Since S/I, and therefore any finitely generated free S/I-module, is $(FP)_{\infty}^{S}$, G has a resolution by $(FP)_{\infty}^{S}$ modules, i.e., G is $(FP)_{\infty}^{S}$. Hence G is $(FP)_{\infty}^{S}$ if and only if G is $(FP)_{\infty}^{S/I}$. Consequently, $G^{*} \cong G^{\vee}$ is $(FP)_{\infty}^{S}$ if and only if $G^{\vee} \cong G^{*}$ is $(FP)_{\infty}^{S/I}$.

Let $\boldsymbol{E}$ be an injective S-resolution for S. Then, as $\operatorname{Ext}_{S}^{i}(S/I, S) = 0$ for all $i > 0$, $\operatorname{Hom}_{S}(S/I, \boldsymbol{E})$ is an injective S/I-resolution for K, and hence

$$\begin{aligned} \operatorname{Ext}_{S/I}^{i}(G, K) &= \mathrm{H}^{i}(\operatorname{Hom}_{S/I}(G, \operatorname{Hom}_{S}(S/I, \boldsymbol{E}))) \\ &\cong \mathrm{H}^{i}(\operatorname{Hom}_{S}(G \otimes_{S/I} S/I, \boldsymbol{E})) \\ &\cong \mathrm{H}^{i}(\operatorname{Hom}_{S}(G, \boldsymbol{E})) \\ &= \operatorname{Ext}_{S}^{i}(G, S) \end{aligned}$$

for all i. Similarly $\operatorname{Ext}_{S/I}^{i}(G^{\vee}, K) \cong \operatorname{Ext}_{S}^{i}(G^{*}, S)$ for for all i, and hence we obtain that $G \in G_{S}(S)$ if and only if $G \in G_{K}(S/I)$.

Let $n = G_{S} \dim_{S} M$. If $n = 0$, i.e., if $M \in G_{S}(S)$, then by the above argument, $G_{K} \dim_{S/I} M = 0$. Otherwise, suppose that $n > 0$, and let $\boldsymbol{G} \to M \to 0$ be a G_{K}-resolution for M. For all i, $G_{i} \in G_{K}(S/I)$, and hence $G_{i} \in G_{S}(S)$. Since $G_{S} \dim_{S} M = n$, by Proposition 4.2.1 the n^{th} syzygy module L_{n} is an element of $G_{S}(S)$, and hence also an element of $G_{K}(S/I)$. Thus $G_{K} \dim_{S/I} M \leq n$.

Now, suppose that $t = \operatorname{grade} I = G_{S} \dim_{S} S/I > 0$. Then there exists a quasi-local faithfully flat extension $S \to T$ such that IT contains a non-zero-divisor on T. Note that $\operatorname{grade}_{T} IT = \operatorname{grade}_{S} I$, and that

$$G_{T} \dim_{T} T/IT = G_{S} \dim_{S} S/I \text{ and } G_{T} \dim_{T} M \otimes_{S} T = G_{S} \dim_{S} M,$$

by Proposition 4.3.1. Moreover, since $S/I \to T/IT$ is also faithfully flat, we have by Proposition 3.2.4 that

$$K \otimes_{S/I} T/IT = \operatorname{Ext}^t_S(S/I, S) \otimes_{S/I} T/IT \cong \operatorname{Ext}^t_T(T/IT, T)$$

is semi-dualizing for T/IT, and by Proposition 4.3.1 that

$$G_{K\otimes_{S/I}T/IT} \dim_{T/IT} M \otimes_{S/I} T/IT = G_K \dim_{S/I} M.$$

So, resetting notation, we may assume that there exists some $x \in I$ which is S-regular. Let $\overline{S} = S/(x)$, and let $\overline{I} = I/(x)$. Since x is a non-zero-divisor on S which annihilates S/I, by Theorem 4.3.6 we have that

$$G_{\overline{S}} \dim_{\overline{S}} S/I = t - 1.$$

Also, $\operatorname{grade}_{\overline{S}} \overline{I} = t-1$ by Propositions 2.3.2 and 2.3.3. Thus $G_{\overline{S}} \dim_{\overline{S}} S/I = \operatorname{grade}_{\overline{S}} \overline{I}$. Finally, note that since $xM = 0$, by Theorem 4.3.6 we have

$$G_{\overline{S}} \dim_{\overline{S}} M = G_S \dim_S M - 1 < \infty.$$

By Proposition 6.3.1, $\widehat{K} = \operatorname{Ext}^{t-1}_{\overline{S}}(S/I, \overline{S})$ is semi-dualizing for S/I. But

$$\widehat{K} = \operatorname{Ext}^{t-1}_{\overline{S}}(S/I, \overline{S}) \cong \operatorname{Ext}^t_S(S/I, S) = K$$

by Lemma 18.2 of [21], and hence by induction

$$G_K \dim_{S/I} M = G_{\widehat{K}} \dim_{S/I} M < \infty.$$

□

Corollary 6.3.3. *Let (S, n) be a quasi-local coherent Gorenstein ring of finite depth. Let $I \subseteq S$ be a finitely generated ideal, and set $R = S/I$. Suppose that* $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$, *and let* $t = \operatorname{grade} I$. *Then $K = \operatorname{Ext}^t_S(R, S)$ is semi-dualizing for R, and for any finitely presented R-module M, we have $G_K \dim_R M < \infty$. In particular, R is GCM.*

Proof. Since I is finitely generated and S is coherent, R is $(FP)^S_\infty$. Since S is Gorenstein, we have that $G_S \dim_S R < \infty$, and so the Auslander-Bridger formula on G_S-dimension (Theorem 5.1.4) implies that

$$\begin{aligned} G_S \dim_S R &= \operatorname{depth}_S S - \operatorname{depth}_S R \\ &= \operatorname{depth}_S S - (\operatorname{depth}_S S - \operatorname{grade}_S I) \\ &= \operatorname{grade}_S I. \end{aligned}$$

Note that as S is Gorenstein, $G_S \dim_S M < \infty$ for any finitely presented R-module M. Proposition 6.3.1 implies that K is semi-dualizing for R, and Theorem 6.3.2 implies that $G_K \dim_R M = G_S \dim_S M - t < \infty$. Hence we have $CM \dim_R M \leq G_K \dim_R M < \infty$ for any finitely presented R-module M, and so R is GCM. □

Proposition 6.3.4. *Let S be a quasi-local coherent ring, and I a finitely generated ideal of S with* $\operatorname{grade} I = 0$. *Set $R = S/I$. If* $\operatorname{depth} R = 0$ *and R is GCM, then* $\operatorname{depth} S = 0$.

Proof. First note that as R is GCM and $\operatorname{depth} R = 0$, by the Auslander-Bridger formula for CM-dimension (Theorem 6.1.7), we have that $\operatorname{depth} M = 0$ for any finitely presented R-module M.

By way of contradiction, assume $\operatorname{depth} S > 0$. Then there exists a finitely generated ideal J such that $(0 : J) = \operatorname{Hom}_S(S/J, S) = 0$. Without loss of generality, we can assume J contains I.

As $\operatorname{grade} I = 0$, $\operatorname{Hom}_S(S/I, S) \neq 0$, and thus there exists a nonzero $y \in S$ such that $yI = 0$.

We claim that $((0 : y) : J)$ is contained in $(0 : y)$: Suppose $z \in ((0 : y) : J)$. Then $zyJ = 0$. Since $(0 : J) = 0$, we get $zy = 0$. Thus, $z \in (0 : y)$. This implies that $\operatorname{Hom}_S(S/J, S/(0 : y)) = 0$. As S is coherent, $(0 : y)$ is finitely generated, by Propostion 2.2.7. As both J and $(0 : y)$ contain I, we obtain that $\operatorname{Hom}_R(R/J', R/(0 : y)') = 0$, where $J' = J/I$ and $(0 : y)' = (0 : y)/I$. But this implies that $\operatorname{depth} R/(0 : y)' > 0$, contadicting the fact that the depth of any finitely presented R-module is zero. $\square$

Note that under the hypotheses of Proposition 6.3.4, we have

$$\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I,$$

i.e., we have a partial converse to Corollary 6.3.3.

A local 0-dimensional ring is CM, in the usual sense, while an arbitrary 0-dimensional ring is HMCM. We now show that this holds for certain rings in the GCM case.

Proposition 6.3.5. *Let (S, n) be a quasi-local coherent Gorenstein ring of finite depth. Let $I \subseteq S$ be a finitely generated ideal, and set $R = S/I$. If R is 0-dimensional, then R is GCM.*

Proof. By Corollary 6.3.3, it's enough to show that

$$\operatorname{depth}_S R = \operatorname{depth}_S S - \operatorname{grade}_S I.$$

Let $m = n/I$ denote the maximal ideal of R. First, note that since $\dim R = 0$, we have

$$\sqrt{(0)} = \bigcap_{p \in \operatorname{Spec} R} p = m,$$

i.e., every element of m is nilpotent. In particular, given $x \in n$, $\overline{x} \in m$, and hence is nilpotent in R. So $x \in \sqrt{I}$. Thus $\sqrt{I} = n$, and so

$$\operatorname{grade} I = \operatorname{grade} \sqrt{I} = \operatorname{grade} n = \operatorname{depth} S.$$

So $\operatorname{depth} S - \operatorname{grade} I = 0$, and thus it is enough to show that $\operatorname{depth} R = 0$.

Note that $\operatorname{depth}_S R = \operatorname{grade}_S(I, R)$. Since $\operatorname{Hom}_S(S/I, R) \neq 0$, we have that $\operatorname{depth}_S R = 0$, as desired. □

The example of a GCM ring given in Proposition 6.3.8 requires the following result:

Theorem 6.3.6. *Let (R, m) be a quasi-local ring, and I an ideal of R such that R/I is $(FP)^R_\infty$. Suppose that $G_R \dim_R R/I < \infty$, and that $\operatorname{Ext}^i_R(R/I, R)$ is $(FP)^R_\infty$ for all i. Suppose that $x \in m$ is a non-zero-divisor on R/I. Then* $\operatorname{grade}(I, x) = \operatorname{grade} I + 1$.

Proof. Let $g = \operatorname{grade} I$. As I is finitely generated, $g < \infty$, by Proposition 2.3.2. Thus there exists a faithfully flat quasi-local extension $(R, m) \to (T, n)$ such that IT contains a regular sequence of length g on T. Note that T/IT is $(FP)^T_\infty$, as R/I is $(FP)^R_\infty$, and that $\operatorname{Ext}^i_T(T/IT, T) \cong \operatorname{Ext}^i_R(R/I, R) \otimes_R T$ is $(FP)^T_\infty$ as well. Also, $G_T \dim_T T/IT = G_R \dim_R R/I < \infty$, by Proposition 4.3.1. Finally, note that that $x \in mT \subseteq n$ is a non-zero-divisor on T/IT, and that

$$\operatorname{grade} I = \operatorname{grade} IT \quad \text{and} \quad \operatorname{grade}(I, x) = \operatorname{grade}(I, x)T.$$

Thus, without loss of generality, we may assume that I contains a regular sequence $\boldsymbol{u} = u_1, \ldots, u_g$. Let $\overline{R} = R/(\boldsymbol{u})$, and $\overline{I} = I/(\boldsymbol{u})$. Note that $R/I \cong \overline{R}/\overline{I}$ is $(FP)_\infty^{\overline{R}}$, by Lemma 2.1.7. Moreover,

$$G_{\overline{R}} \dim_{\overline{R}} \overline{R}/\overline{I} = G_R \dim_R R/I - g < \infty,$$

by Theorem 4.3.6. By Propositions 2.3.2(4) and 2.3.3, we have

$$\operatorname{grade}_{\overline{R}} \overline{I} = \operatorname{grade}_R I - g = 0$$

and

$$\operatorname{grade}_{\overline{R}}(\overline{I}, \overline{x}) = \operatorname{grade}_{\overline{R}}(I, x)/(\boldsymbol{u}) = \operatorname{grade}_R(I, x) - g.$$

Finally, note that for all i, $\operatorname{Ext}_{\overline{R}}^{i-g}(\overline{R}/\overline{I}, \overline{R}) \cong \operatorname{Ext}_R^i(R/I, R)$ is $(FP)_\infty^{\overline{R}}$, since $(\boldsymbol{u})$ annihilates $\operatorname{Ext}_{\overline{R}}^{i-g}(\overline{R}/\overline{I}, \overline{R})$. Thus to show that $\operatorname{grade}(I, x) = \operatorname{grade} I + 1$, it suffices to show that $\operatorname{grade}(\overline{I}, \overline{x}) = 1$. Replacing R with $\overline{R}$, I with $\overline{I}$ and x with $\overline{x}$, it's enough to prove the result in the case that $\operatorname{grade} I = 0$.

Consider the exact sequence

$$0 \longrightarrow R/I \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, R)$ yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/(I, x), R) \longrightarrow \operatorname{Hom}_R(R/I, R) \xrightarrow{x} \operatorname{Hom}_R(R/I, R) \longrightarrow \operatorname{Ext}_R^1(R/(I, x), R).$$

As $\operatorname{grade} I = 0$, $\operatorname{Hom}_R(R/I, R)$ is nonzero. Also, $\operatorname{Hom}_R(R/I, R)$ is $(FP)_\infty^R$ by assumption. If $\operatorname{Ext}_R^1(R/(I, x), R) = 0$, then Nakayama's lemma would imply that $\operatorname{Hom}_R(R/I, R) = 0$. Hence $\operatorname{Ext}_R^1(R/(I, x), R) \neq 0$, and so, as $R/(I, x)$ is $(FP)_\infty^{R/(x)}$, Lemma 2.3.4 implies that $\operatorname{grade}(I, x) \leq 1$.

Suppose that $\operatorname{grade}(I, x) = 0$. Then $(I, x) \subseteq (0 :_R y)$ for some $y \neq 0$. Let p be a prime ideal minimal over $(0 :_R y)$. By Theorem 2.8 of [18], $\operatorname{depth} R_p = 0$. Note that R_p/I_p is $(FP)_\infty^{R_p}$. Moreover $G_{R_p} \dim_{R_p} R_p/I_p \leq G_R \dim_R R/I < \infty$. The Auslander-Bridger formula for G-dimension over R_p gives

$$\operatorname{depth} R_p/I_p + G_{R_p} \dim_{R_p} R_p/I_p = \operatorname{depth} R_p.$$

Hence $\operatorname{depth} R_p/I_p = 0$, contradicting the fact that $x \in pR_p$ is a non-zero-divisor on R_p/I_p. □

Corollary 6.3.7. *Let (R, m) be a quasi-local coherent Gorenstein ring and $I \subseteq R$ a finitely generated ideal. Suppose that $x \in m$ is a non-zero-divisor on R/I. Then* $\operatorname{grade}(I, x) = \operatorname{grade} I + 1$.

A local 1-dimensional domain is CM, in the usual sense; an arbitrary 1-dimensional domain is HMCM. Again, we show that this holds in the GCM case for rings of a special type.

Proposition 6.3.8. *Let (S, n) be a quasi-local coherent Gorenstein ring of finite depth. Let $I \subseteq S$ be a finitely generated ideal, and set $R = S/I$. If R is a 1-dimensional domain, then R is GCM.*

Proof. By Corollary 6.3.3, it's enough to show that

$$\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I.$$

Since R is a 1-dimensional domain, the only prime ideals of R are (0) and n/I. In particular, $I \subsetneq n$, and there are no prime ideals properly contained between I and n.

Let $x \in n$, with $x \notin I$. As $\overline{x}$ is R-regular, we have that $\operatorname{depth} R \geq 1$, and $\sqrt{(\overline{x})} = n/I$. Moreover, $\sqrt{(I,x)} = n$. By Proposition 2.3.2

$$\operatorname{depth} R = \operatorname{grade} n/I = \operatorname{grade}(\overline{x}).$$

But $\operatorname{grade}(\overline{x}) \leq 1$, as $(\overline{x})$ is principal, and hence $\operatorname{depth} R = 1$.

We also have

$$\operatorname{depth} S = \operatorname{grade} n = \operatorname{grade}(I, x).$$

By Corollary 6.3.7, $\operatorname{grade}(I, x) = \operatorname{grade} I + 1$. Thus $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$, as desired. □

6.4 GCM and HMCM rings

Let R be a ring, and $\boldsymbol{x} = x_1, \ldots, x_n$ a finite sequence of elements of R. Given an R-module M, we let $\check{\mathrm{H}}^i_{\boldsymbol{x}}(M)$ denote the i^{th} Čech cohomology of M with respect to $\boldsymbol{x}$, and let $\mathrm{H}^i_{(\boldsymbol{x})}(M)$ denote the i^{th} local cohomology of M of M with respect to $(\boldsymbol{x})$. The sequence $\boldsymbol{x}$ is called *weakly proregular* if for all R-modules M and all $i \geq 0$, the natural map

$$\check{\mathrm{H}}^i_{\boldsymbol{x}}(M) \to \mathrm{H}^i_{(\boldsymbol{x})}(M)$$

is an isomorphism. A sequence $\boldsymbol{x}$ of length n is called a *parameter sequence* if $\boldsymbol{x}$ is weakly proregular, $(\boldsymbol{x})R \neq R$, and $\mathrm{H}^n_{(\boldsymbol{x})}(R)_p \neq 0$ for all primes containing $(\boldsymbol{x})$. A sequence $\boldsymbol{x}$ of length n is called a *strong parameter sequence* if $x_1, \ldots, x_i$ is a parameter sequence for all $1 \leq i \leq n$. Under the notion developed by Hamilton and Marley in [18], a ring is called HMCM if every strong parameter sequence on R is a regular sequence.

Theorem 6.4.1. *Let (S, n) be a quasi-local coherent Gorenstein ring of finite depth, and $I \subseteq S$ a finitely generated ideal. Set $R = S/I$. If* $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$, *then R is a GCM ring which is also HMCM.*

Proof. First, note that R is GCM by Corollary 6.3.3. In order to see that R is HMCM, we adapt the proof of Proposition 5.6 of [20].

Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a strong parameter sequence of R. Then $\mathrm{H}^n_{(\boldsymbol{x})}(R)_m \neq 0$, and hence $\check{\mathrm{H}}^n_{\boldsymbol{x}}(R) \cong \mathrm{H}^n_{(\boldsymbol{x})}(R) \neq 0$. As $(\boldsymbol{x})$ is finitely generated, $R/(\boldsymbol{x})^t$ is finitely presented for all t. By Corollary 6.3.3,

$$G_K \dim_R R/(\boldsymbol{x})^t < \infty$$

for all t, where $K = \operatorname{Ext}^l_R(R, S)$ and $l = \operatorname{grade} I$.

We'll show by induction on n that $\boldsymbol{x}$ is a regular sequence on R. Let $\boldsymbol{x}'$ denote the truncated sequence $x_1, \ldots, x_{n-1}$. By way of a contradiction, suppose that $\boldsymbol{x}'$ is a regular sequence on R but that x_n is a zero-divisor on $R/(\boldsymbol{x}')$. Then, by Lemma 2.8 of [18], there exists a prime p such that $(\boldsymbol{x}) \subseteq p$ and $\operatorname{depth} R_p/(\boldsymbol{x}')R_p = 0$. Hence $\operatorname{depth} R_p = n - 1$. By Proposition 4.1.10,

$$G_{K_p} \dim_{R_p} R_p/(\boldsymbol{x})^t R_p \leq G_K \dim_R R/(\boldsymbol{x})^t < \infty$$

for all t, and hence

$$G_{K_p} \dim_{R_p} R_p/(\boldsymbol{x})^t R_p \leq \operatorname{depth} R_p = n - 1$$

by the Auslander-Bridger formula for G_{K_p}-dimension over R_p. Hence by Proposition

4.2.1, $\operatorname{Ext}^n_{R_p}(R_p/(\boldsymbol{x})^t R_p, K_p) = 0$ for all t. Hence

$$\mathrm{H}^n_{(\boldsymbol{x})R_p}(K_p) = \varinjlim_t \operatorname{Ext}^n_{R_p}(R_p/(\boldsymbol{x})^t R_p, K_p) = 0.$$

As $(\boldsymbol{x})R_p$ is weakly proregular for R_p (by Remark 2.2 of [18]), we have that

$$\check{\mathrm{H}}^n_{\boldsymbol{x}}(R_p) \otimes_{R_p} K_p \cong \check{\mathrm{H}}^n_{\boldsymbol{x}}(K_p) \cong \mathrm{H}^n_{(\boldsymbol{x})R_p}(K_p) = 0.$$

Recall that semi-dualizing modules are faithful, as $\operatorname{Hom}_R(K, K) \cong R$. By Gruson's Theorem ([24], Corollary 4.3), we have that $\check{\mathrm{H}}^n_{\boldsymbol{x}}(R)_p = 0$, contradicting the fact that $\boldsymbol{x}$ is a parameter sequence. Hence $\boldsymbol{x}$ is R-regular, and R thus is HMCM. □

Chapter 7

Open Questions

Question 7.0.2. Let R be a quasi-local coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of finitely generated R-modules. Is it the case that if two of the modules have finite CM-dimension, then the third does as well? (C.f. Lemma 6.1.8.)

Question 7.0.3. Does the converse of Proposition 6.2.2 hold? I.e., if (R, m) is a quasi-local coherent ring and $x \in m$ is an R-regular element such that $R/(x)$ is GCM, must R be GCM?

Question 7.0.4. Let R be a coherent GCM ring and X an indeterminate. Suppose that $R[X]$ is coherent. Is $R[X]$ GCM?

Question 7.0.5. Does the converse of Corollary 6.3.3 hold? I.e., if S is a quasi-local coherent Gorenstein ring of finite depth, $I \subseteq S$ is a finitely generated ideal, and $R = S/I$ is GCM, must $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$?

Question 7.0.6. If S, T are quasi-local coherent Gorenstein rings and $I \subseteq S$ and $J \subseteq T$ are finitely generated ideals such that $R = S/I \cong T/J$, is it the case that $G_S \dim R = \operatorname{grade} I$ if and only if $G_T \dim R = \operatorname{grade} J$? Alternatively, is $\operatorname{depth} S - \operatorname{depth} R = \operatorname{grade} I$ if and only if $\operatorname{depth} T - \operatorname{depth} R = \operatorname{grade} J$?

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