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Study of Temporal Point Process as a Renewal Process with the Distribution of Interevent Time is Exponential Family

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ABSTRACT

Study of inter events time is a part of temporal point process modeling. In this study, temporal point process is consider as a renewal process which is the inter event times have independent identity distributions. The conditional intensity of the process coincides with so called hazard rate. In this paper, we analyze hazard rate with assume that the distribution of inter events time of hazard rate is exponential family, that are exponential, Pareto, and Rayleigh. To estimate the parameter of hazard rate, Maximum Likelihood Estimates (MLE) method is used. The simulation of Conditional Intensity value is given at the end of the paper. The result shows that all of the distribution gives the hazard rate value are different.

Keywords: Point Process, Renewal Process, Conditional Intensity, exponentially family distributed.

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1 Introduction

A conditional intensity (CI) completely characterizes the corresponding point process. If the conditional intensity depends only on the elapsed time since the last occurrence, then the corresponding point process is a renewal process, and the conditional intensity coincides with so-called hazard rate (Jones, 1995). Hazard rate is the key to the theory of likelihood of point process. In point process model, the conditional intensity function is defined as an instance of emergence events changing opportunities (Daley and Vere Jones, 2003). It plays a very important as proposed by Yoshihiko in 1999 (Yoshihiko, 1999), because the conditional intensity function is characterized by corresponding point process. For example, if the conditional intensity depends only on time or event now, it's like a form of conditional intensity characterizes the non stationary Poisson process. Whereas for the conditional intensity depends only on the difference in the time since the last event occurrence time, then this is a renewal process. The study of point process and its application have been studied by Vere Jones in 1995 (Jones, 1995), Darwis et al. in 2009 (Darwis, Sunusi, Mangku and Gunawan, 2009), Darwis et al. in 2008 (Darwis, Mangku, Gunawan, Sunusi and Wahyuningsih, 2008). Conditional intensity estimation of temporal point process have been studied by Sunusi et al. in 2010 (Sunusi, Darwis, Triyoso and Mangku, 2010). In this research, the estimation of parameter of conditional intensity of point process is considered as a renewal process. Some of the exponential family distribution as inter events time distribution are considered, such as exponential, Pareto, and Rayleigh.

2 Conditional Intensity Function of Renewal Process

To declare a Conditional Intensity Function $\lambda(t_i|\mathcal{H}_{t_i})$ as a renewal process, it will show the relationship of the time between events in determining the conditional intensity function. We know that,(Ogata,1999)

$$P(\text{no event in} \quad (t_{i-1}, t_i]) = \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(s|\mathcal{H}_s)ds\right). \tag{2.1}$$

Probability that at least one event after the last event t_{i-1} and t_i is a cumulative distribution function of interevent time distribution as defined by:

$$F(t_i|\mathcal{H}_{t_i}) = \int_{t_{i-1}}^{t_i} f(s|\mathcal{H}_s)ds, \tag{2.2}$$

so that, probability that there is no event in the interval is:

$$P(\text{no event in} \quad (t_{i-1}, t_i]) = 1 - F(t_i|\mathcal{H}_{t_i}). \tag{2.3}$$

Using (2.1) and (2.3), we have

$$\exp\left(-\int_{t_{i-1}}^{t_i} \lambda(s|\mathcal{H}_s)ds\right) = 1 - F(t_i|\mathcal{H}_{t_i}), \tag{2.4}$$

so that

$$\begin{aligned} f(t_i|\mathcal{H}_{t_i}) &= \lambda(t_i|\mathcal{H}_{t_i}) \exp\left(-\int_{t_{i-1}}^{t_i} \lambda(s|\mathcal{H}_s)ds\right) \\ &= \lambda(s|\mathcal{H}_s)(1 - F(t_i|\mathcal{H}_{t_i})). \end{aligned} \tag{2.5}$$

Thus, we have

$$\lambda(t_i|\mathcal{H}_{t_i}) = \frac{f(t_i|\mathcal{H}_{t_i})}{1 - F(t_i|\mathcal{H}_{t_i})}. \tag{2.6}$$

Furthermore, as we know that renewal process is a point process with probability of one event is occurred in time depend on the last event, but it is not depend on the time of event before. In other words, the renewal process is a process that is based on the distribution of time between the incidences but not related to the history. So that equation (2.6)) can be written as:

$$\lambda(t_i|\mathcal{H}_{t_i}) = \frac{f(t_i - t_{i-1})}{1 - F(t_i - t_{i-1})} = \frac{f(\tau_i)}{1 - F(\tau_i)}, \tag{2.7}$$

so we have the likelihood of the process is :

$$\begin{aligned} L &= \left(\prod_{i=1}^{N} f(t_i|\mathcal{H}_{t_i})\right) P(N_{(t_N,T]} = 0) \\ &= \left(\prod_{i=1}^{N} f(\tau_i)\right)(1 - F(\tau_T)) \end{aligned} \tag{2.8}$$

3 Main Result: Conditional Intensity Estimation

3.1 Interevents time (τ) is exponentially distributed

Let the probability density function of τ is :(Klungman, Panjer and Wilmot, 2002)

$$f(\tau_i) = \lambda e^{-\lambda(\tau_i)}; \quad \tau_i \geq 0, \tag{3.1}$$

and the cumulative distribution function of τ is :

$$F(\tau_i) = 1 - e^{-\lambda(\tau_i)}; \quad \tau_i \geq 0. \tag{3.2}$$

We have the CI function as follows:

$$\lambda(t_i|\mathcal{H}_{t_i}) = \frac{f(t_i)}{1 - F(\tau_i)} = \frac{\lambda e^{-\lambda(\tau_i)}}{1 - (1 - e^{-\lambda(\tau_i)})} = \lambda. \tag{3.3}$$

The parameter of λ is estimated using likelihood function of temporal point process, so that we have:

$$\hat{\lambda} = \frac{n}{T}. \tag{3.4}$$

3.2 Interevent time (τ) is Pareto distributed

Let the probability density function of τ is :(Klungman et al., 2002)

$$f(\tau_i) = \frac{\alpha k^{\alpha}}{(\tau_i)^{\alpha+1}}; \quad k \leq \tau_i \leq \infty; \alpha, k > 0, \tag{3.5}$$

and the cumulative distribution function of τ is :

$$F(\tau_i) = 1 - \left(\frac{k}{\tau_i}\right)^{\alpha}; \quad k \leq \tau_i \leq \infty; \alpha, k > 0. \tag{3.6}$$

We have the CI function as follows:

$$\lambda(t_i|\mathcal{H}_{t_i}) = \frac{f(t_i)}{1 - F(\tau_i)} = \frac{\frac{\alpha k^{\alpha}}{(\tau_i)^{\alpha+1}}}{1 - \left(1 - \left(\frac{k}{\tau_i}\right)\right)} = \frac{\alpha}{\tau_i}.$$

The parameter of λ is estimated by,

$$\begin{aligned} L &= \left(\prod_{i=1}^{N} f(t_i|\mathcal{H}_{t_i})\right) P(N_{(t_N,T]}=0) \\ &= \left(\prod_{i=1}^{N} f((\tau_i)\right) (1-F(\tau_T)) \\ &= \left(\prod_{i=1}^{n} \frac{\alpha k^{\alpha}}{(\tau_i)^{\alpha+1}}\right) \left(\frac{k}{\tau_T}\right)^{\alpha} \\ &= \frac{\alpha^n k^{(n+1)\alpha}}{(\tau_1)^{\alpha+1}(\tau_2)^{\alpha+1}\cdots(\tau_n)^{\alpha+1}(\tau_T)^{\alpha}}, \end{aligned} \tag{3.7}$$

so that

$$\begin{aligned} \ln L &= \ln(\alpha^n k^{(n+1)\alpha}) - \ln((\tau_1)^{\alpha+1}(\tau_2)^{\alpha+1})\cdots(\tau_n)^{\alpha+1}(\tau_T)^{\alpha} \\ &= n\ln\alpha + (n+1)\alpha\ln k - \left((\alpha+1)\sum_{i=1}^{n}\ln\tau_i + \alpha\sum\ln\tau_T\right). \end{aligned} \tag{3.8}$$

Thus, we have:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{T}\ln\tau_i - (n+1)\ln k}. \tag{3.9}$$

Hence, the value of k in that equation is a minimum of τ_i. Furthermore, the estimator of CI is follows:

$$\lambda(t_i|\mathcal{H}_{t_i} = \frac{\alpha}{\tau_i} = \frac{n}{\tau_i(\sum_{i=1}^{T}\ln\tau_i - (n+1)\ln k))}. \tag{3.10}$$

3.3 Interevent time (τ) is Rayleigh distributed

Let the probability density function of τ is :(Osaki, 1992)

$$f(\tau_i) = \frac{\tau_i}{\eta^2} e^{\frac{-\tau_i^2}{2\eta^2}} \quad ;\tau_i \geq 0; \eta > 0. \tag{3.11}$$

We have the CI function as follows :

$$\lambda(t_i|\mathcal{H}_{t_i}) = \frac{f(\tau_i)}{1-F(\tau_i)} = \frac{\frac{\tau_i}{\eta^2}\exp(-\frac{\tau_i^2}{2\eta^2})}{1-\left(1-\exp(-\frac{\tau_i^2}{2\eta^2})\right)} = \frac{\tau_i}{\eta^2} \tag{3.12}$$

The parameter of λ is estimated by:

$$\begin{aligned} L &= \left(\prod_{i=1}^{n} f(\tau_i)\right)(1-F(\tau_T)) \\ &= \left(\prod_{i=1}^{n}\frac{\tau_i}{\eta^{2n}}\right)\exp\left[-\sum_{i=1}^{T}\frac{\tau_i^2}{2\eta^2}\right] \end{aligned} \tag{3.13}$$

so that,

$$\ln L = \ln \prod_{i=1}^{n} \tau_i - 2n \ln \eta - \sum_{i=1}^{T} \frac{\tau_i^2}{2\eta^2}. \tag{3.14}$$

Thus, we have:

$$\hat{\eta} = \sqrt{\frac{\sum_{i=1}^{T} \tau_i^2}{2n}}. \tag{3.15}$$

Furthermore, the parameter estimator of CI is:

$$\lambda(t_i|\mathcal{H}_{t_i} = \frac{\tau_i}{\eta^2} = \frac{2n\tau_i}{\sum_{i=1}^{T} \tau_i^2}. \tag{3.16}$$

4 Case Study

The data is used in this study is a secondary data. The data is taken from www.who.intinfluenzaH5N1_avian_influenza_update_20121217b.pdf. The period of observation is 8 years (2005-2012).

Table 1: The Summary of CI value (hazard rate) with Distribution of Inter Event Time is Exponential Family

No	Interval(τ_i)	$\lambda_{Exp}(t_i\|\mathcal{H}_{t_i})$	$\lambda_{Pareto}(t_i\|\mathcal{H}_{t_i})$	$\lambda_{Rayleigh}(t_i\|\mathcal{H}_{t_i})$
1	(0,1]	0.0073	0.0250	0.0313
2	(1,2]	0.0074	0.0120	0.0625
3	(2,3]	0.0075	0.1410	0.0055
...	...	...	...	...
30	(29,30]	0.0094	0.0850	0.0091
31	(30,31]	0.0094	0.0036	0.0036
...	...	...	...	...
60	(59,60]	0.0129	0.0180	0.0420
61	(60,61]	0.0132	0.0330	0.0237
...	...	...	...	...
90	(89,90]	0.0213	0.0600	0.0128
91	(90,91]	0.0217	0.1060	0.0073
...	...	...	...	...
134	(133,134]	0.3333	0.0070	0.1114
135	(134,135]	0.5000	0.0120	0.0657

Table 1 shows that the CI value for each distribution of inter events time are different. The value of CI for inter events time has exponentially distribution is increase tend to 1. Otherwise, it does not apply to the inter events time have Pareto and Rayleigh distributions too.

5 Conclusions

The estimator of Conditional Intensity with inter events time has Pareto distributed is depend on how long until the next occurrence. It applies to Rayleigh also. But it does not apply to the inter events time is exponentially distributed. The CI function with the inter events time is exponentially distributed is more simple than Pareto and Rayleigh.

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