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# Quasiprojective varieties admitting Zariski dense entire holomorphic curves

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**Abstract.** Let $X$ be a complex quasiprojective variety. A result of Noguchi–Winkelmann–Yamanoi shows that if $X$ admits a Zariski dense entire curve, then its quasi-Albanese map is a fiber space. We show that the orbifold structure induced by a proper birationally equivalent map on the base is special in this case. As a consequence, if $X$ is of log-general type with $\bar{q}(X) \geq \dim X$, then any entire curve is contained in a proper subvariety in $X$.

**Keywords.** Entire curve, special variety, quasiprojective variety.

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## 1 Introduction and the statement of the main result

This paper deals with a question of Campana concerning the characterization of complex algebraic varieties that admit Zariski dense entire holomorphic curves. This problem for an algebraic surface not of log-general type nor a very general algebraic K3 surface was solved completely by [3, 4]. Campana in [5] introduced the notion of special varieties which is a practical way to extend the so far conjectural characterization to higher dimensions. Using a recent result of Noguchi–Winkelmann–Yamanoi [24], we verify one direction of this characterization here for all algebraic varieties whose quasi-Albanese map is generically finite.

Given a complex projective manifold $\bar{X}$ with a normal crossing divisor on it, we call the pair $X = (\bar{X}, D)$ a log-manifold. Recall that there is a locally free subsheaf of the holomorphic tangent sheaf of $\bar{X}$, called the log-tangent sheaf of $X$, which we denote by $\bar{T}_X$. It is the sheaf of holomorphic vector fields leaving $D$ invariant. Its dual $\bar{\Omega}_X = \bar{T}_X^{\vee}$ is called the log-cotangent sheaf of $X$ and $\bar{K}_X = \det \bar{\Omega}_X$ the log-canonical sheaf of $X$. Their sections are called logarithmic 1-forms, respectively logarithmic volume forms. Here and later we will consistently abuse notation and identify holomorphic vector bundles with their sheaves of sections. We will abuse the notation further at times and identify a line bundle with a divisor it corresponds to, for example in the identification

$\bar{K}_X = K_{\bar{X}}(D) = K_{\bar{X}} + D$. We first give some proper birational invariants of $\bar{X} \setminus D$, which we will also identify with $X$ by a standard abuse of notation.

**Definition 1.1.** With this setup, we define the *log-irregularity of* $X$ by

$$\bar{q}(X) = \dim H^0(\bar{\Omega}_X)$$

and we define the *log-Kodaira dimension of* $X$ by $\bar{\kappa}(X) = \kappa(\bar{K}_X)$ where the *Kodaira dimension* for an invertible sheaf $L$ is given by

$$\kappa(L) = \limsup_{m\to\infty} \frac{\log \dim H^0(L^{\otimes m})}{\log m}.$$

We also define, see [5, 16], the *essential* or the *core dimension of* $X$ by

$$\kappa'_+(X) = \max\{p \mid L \hookrightarrow \bar{\Omega}^p_X \text{ is an invertible subsheaf with } \kappa(L) = p,\ 0 \leq p \leq \dim X\}.$$

It is an easy fact that the Kodaira dimension of an invertible sheaf $L$ is invariant under positive tensor powers of $L$ and so the Kodaira dimension $\kappa$ makes sense for $\mathbb{Q}$-invertible sheaves of the form $L(A)$ where $A$ is a $\mathbb{Q}$-divisor. We recall also the fact that $\kappa(L) \in \{\infty, 0, \dots, \dim(X)\}$ for a $\mathbb{Q}$-invertible sheaf $L$ and that sections of powers of $L$, if they exist, define a dominant rational map $I_L$ to a projective variety of dimension $\kappa(L)$, called the Iitaka fibration of $L$. We usually allow $I_L$ to be defined on any smooth birational model of $\bar{X}$ and choose a model on which $I_L$ is a morphism. With such a choice, recall that the general (in fact generic) fibers $F$ of $I_L$ are connected and $\kappa(I|_F) = 0$, see for example [26].

Let $X_0$ be a quasiprojective variety and $\bar{X}_0$ be a projectivization. We recall that a log-resolution of $X_0$ is a birational morphism $r : \bar{X} \to \bar{X}_0$ where $(\bar{X}, D)$ is a log-manifold with $D = r^{-1}(\bar{X}_0 \backslash X_0)$. Such a resolution exists by the resolution of singularity theorem. The Hartog extension theorem allows us to define $\bar{q}$, $\bar{\kappa}$ and $\kappa'_+$ for $X_0$ by taking them to be those of a log-resolution. These are thus proper birational invariants of $X_0$. Here, proper birational maps between two quasi-projective varieties are just compositions of proper birational morphisms and their inverses. Another proper birational invariant is given by the (quasi-)Albanese map of $X_0$, which is an algebraic morphism

$$\alpha_{X_0} : X_0 \to \mathrm{Alb}(X_0)$$

defined for any log-resolution $(\bar{X}, D)$ by line integrals of logarithmic 1-forms on $\bar{X} \setminus D$ with a choice of base point outside $D$ where $\mathrm{Alb}(X)$ is a $\bar{q}(X_0)$ dimensional semi-Abelian variety called the quasi-Albanese variety of $X$, see for example [22]. Implicit here is the invariance of $\mathrm{Alb}(X)$ and the compatibility of the quasi-Albanese map among log-resolutions. We recall that given a compactification of $\mathrm{Alb}(X_0)$, there exist a log-resolution of $X_0$ (a compactification of $X_0$ by normal crossing divisors in the case $X_0$ is smooth) over which $\alpha_X$ extends to a morphism. We recall also that a semi-Abelian variety is a complex Abelian group $T$ that admits a semidirect product structure via a holomorphic exact sequence of groups

$$0 \to (\mathbb{C}^*)^k \to T \xrightarrow{\pi} A \to 0 \tag{1.1}$$

where $A$ is an abelian variety and $k \geq 0$. It follows that the algebro-geometric image of the Albanese map (or the Albanese image) of $X_0$, its dimension as well as the Albanese variety are proper birational invariants of $X_0$.

**Definition 1.2.** We say that $X_0$ is *special* if $\kappa'_+(X_0) = 0$ and that it is of *general type* (or if more precision is required, of *log-general type*) if $\bar{\kappa}(X_0) = \dim X_0$.

Recall that a holomorphic curve in $X_0$ is called algebraically degenerate if its image is not Zariski dense. Our main theorem in this paper is as follows.

**Theorem 1.3.** *Let $X$ be a complex quasi-projective variety with $\bar{q}(X) \geq \dim X$. Then every entire holomorphic curve in $X$ is algebraically degenerate if $X$ is not special. In other words $X$ may admit a Zariski dense entire holomorphic curve only if $X$ is special.*

This answers the conjecture of Green–Griffiths in our special situation:

**Corollary 1.4.** *With the same hypothesis on $X$, let $f : \mathbb{C} \to X$ be holomorphic and nontrivial. If $X$ is of log-general type, then there is a proper subvariety of $X$ containing $f(\mathbb{C})$.*

We note that [23] has proved the same theorem with $\kappa'_+$ replaced by $\bar{\kappa}$ but with the additional hypothesis that the quasi-Albanese map of $X$ is proper and generically finite. However, without the properness condition for the quasi-Albanese map, the birational condition of $\bar{\kappa}(X) = 0$ is not implied by the condition that $X$ admits a Zariski dense holomorphic image of $\mathbb{C}$, see [7].

An important part of this paper is an adaptation to the context of special varieties of the results of Noguchi–Winkelmann–Yamanoi [22, 23, 24] concerning varieties that admit finite maps to semi-Abelian varieties. The key results on special varieties used for the main theorem here are worked out independently of previous sources. The second author has spoken about the result on surfaces at a workshop at the Fields Institute in 2008 that claimed the connection with the characterization by special varieties. One direction of this characterization is fully worked out here for all dimensions.

## 2 Preliminaries on special varieties

Throughout this section, let $X$ be a complex projective manifold and $\mathrm{Div}'(X)$ the set of codimension-one subvarieties of $X$. An orbifold structure on $X$ is a $\mathbb{Q}$-divisor of the form

$$A = \sum_i (1 - 1/m_i) D_i$$

where the $D_i$ are distinct elements of $\mathrm{Div}'(X)$ and $1 \le m_i \in \mathbb{Q} \cup \{\infty\}$ for all $i$. We denote $X$ with its orbifold structure by $X \backslash A$ and we set $K_{X\backslash A} := K_X(A)$ to be the orbifold canonical $\mathbb{Q}$-bundle. We set $m(A \cap D_i) = m(D_i \cap A) = m_i$ and call it the multiplicity of the orbifold $X \backslash A$ (or simply, the orbifold multiplicity) at $D_i$. We note that the coefficient of $D_i$ in $A$ satisfies

$$0 \le 1 - \frac{1}{m(A \cap D_i)} \le 1$$

and it vanishes, respectively equals one, precisely when the corresponding orbifold multiplicity is one, respectively equals $\infty$. Note that when $A$ is a (reduced) normal crossing divisor, the orbifold $X \backslash A$ is nothing but a log-manifold $(X, A)$ whose birational geometry is dictated precisely by the proper birational geometry of the complement of $A$ in $X$. More generally, when $A_{\mathrm{red}}$ is normal crossing, one can make good geometric sense of the orbifold $X \backslash A$ via the usual branched covering trick (see [16], see also [6] for a variant approach) and we will call such an orbifold smooth.

We now define the Kodaira dimension of a rational map from an orbifold following [16], cf. also [5]. Let $f : X \dashrightarrow Y$ be a rational map between complex projective manifolds and let $w$ be a rational section of $K_Y$. If $f$ is dominant, then $f^*w$ defines a rational section of $\Omega_X^m$ with $m = \dim Y$ and hence determines in the standard way a saturated rank-one subsheaf $L$ of $\Omega_X^m$ which is easily seen to be unique in the birational equivalence class of $f$ (it is even unchanged after composing with a dominant map from $Y$ to a variety of the same dimension as $Y$). We recall that a saturated subsheaf of a locally free sheaf $\mathcal{S}$ is one that is not contained in any larger subsheaf of the same rank and that it is reflexive. It follows that such a subsheaf, if it is rank-one, is locally free (see for example [18]). Hence, we can even define $L$ without the dominant condition on $f$ by setting $m = \dim f(X)$ and replacing $Y$ by a desingularization of the algebraic image of $f$ in general. Now $f$ gives rise to an orbifold rational map in the category of orbifolds if an orbifold structure $A$ is imposed on $X$. We denote this orbifold map by $f^\partial$ and the orbifold $X \backslash A$ by $X^\partial$ if $A$ is implicit.

**Definition 2.1.** Let $f : X \dashrightarrow Y$ be a rational map giving rise to an invertible sheaf $L$ on $X$ as defined above. Let $A$ be an orbifold structure on $X$ giving rise to an orbifold rational map that we denote by

$$f^{\partial} = f|_{X\setminus A} : X\setminus A \dashrightarrow Y.$$

Define the *vertical part of $A$ with respect to $f$* by

$$A \cap f = \sum \Big\{ \Big(1 - \frac{1}{m(D \cap A)}\Big) D \;\Big|\; f(D) \neq f(X),\ D \in \mathrm{Div}'(X) \Big\}.$$

We set $L_{f^{\partial}} = L \otimes \mathcal{O}(A \cap f)$, which is a $\mathbb{Q}$-invertible sheaf, and we define the *Kodaira dimension* of the orbifold rational map $f^{\partial}$ by

$$\kappa(f, A) = \kappa(f^{\partial}) := \kappa(L_{f^{\partial}}).$$

Recall that a dominant rational map is called almost holomorphic if it is well defined along its general fibers (i.e., the general fibers do not intersect with the indeterminacy locus). More specifically, the restriction of the second projection to the exceptional locus of the first projection of the graph of the map is not dominant.

Such a map is called an almost holomorphic fibration if the general fibers are connected. Recall also that a fibration is a proper surjective morphism with connected fibers while a fiber space is a dominant morphism whose general fibers are connected. A dominant rational map is called a rational fibration if it becomes a fibration after resolving its indeterminacies.

**Definition 2.2.** Notation as above, we call the orbifold rational map $f^{\partial}$ to be *(base-wise) of general type* (or simply to be *base-general(-typical)*) if

$$\kappa(f^{\partial}) = \dim(f) > 0$$

where $\dim(f)$ is given by the dimension of the algebraic image of $f$. We call the orbifold $X\setminus A$ *special* if it admits no base-general orbifold rational map and to be *general-typical* or *of general type* if the identity map restricted to the orbifold is base general. If $f$ is a rational fibration, we say that $f^{\partial} = f|_{X\setminus A}$ is *base-special* if $X\setminus A$ has no orbifold rational map that is base-general and that factors through $f$. We will also consider the obvious generalization of this notion to dominant rational maps via Stein factorization. If $f$ is an almost holomorphic fibration, we say that $f^{\partial}$ is *special* (respectively *general-typical*) if its general fiber endowed with the orbifold structure given by the restriction of $A$ are special orbifolds (respectively orbifolds of general type). It should be clear that orbifold structures under generic restrictions make sense, see Lemma 2.6).

In the case $A$ is reduced and normal crossing, it is easily seen that these notions are, in the obvious manner, proper birational invariants of the open subset $X\backslash A$ and of the restriction of $f$ to it. Hence, these notions make sense for quasiprojective varieties and mappings from them and we will so understand them in this context.

This notion of being special corresponds to the same "geometric" notion introduced by Campana in [6] in the case $A_{\text{red}}$ is normal crossing and to the notion given in Section 1 in the case $A$ is reduced and normal crossing by virtue of the following two lemmas respectively, see [16]. The first of these lemmas is self-evident (with the help of the existence of Diagram 2.1 in Lemma 2.13 of [16] as a convenient but not absolutely necessary shortcut).

**Lemma 2.3.** *Let $L$ be a saturated line subsheaf of $\Omega^i_X$ and $X\backslash A$ be a log-manifold. Then the saturation of $L$ in $\Omega^i(X, \log A)$ is $L(A')$ where $A'$ consists of components $D$ of $A$ whose normal bundles $N_D$ over their smooth loci satisfy $N^*_D \wedge L = 0$ in $\Omega^{i+1}_X|_D$. Hence given a dominant map $f^\partial : X\backslash A \dashrightarrow Y$, $L_{f^\partial}$ is the saturation of $L_f$ in $\Omega^r(X, \log A)$, $r = \dim Y$.* □

**Lemma 2.4** (Bogomolov, Castelnuovo–DeFranchis). *Let $L$ be a saturated line subsheaf of $\Omega^p(X, \log A)$ where $A$ is a normal crossing divisor in $X$. Then*

(I) $\kappa(L) \leq p$.

(II) *If $\kappa(L) = p$, then the Iitaka fibration $I_L$ of $L$ defines an almost holomorphic fibration to a projective base $B$ of dimension $p$ and $I^*_L K_B$ saturates to $L$ in $\Omega^p(X, \log A)$. In particular, $L = L_{I^\partial_L}$.* □

Now let $f : X \to Y$ be a fibration with $X$ and $Y$ projective and smooth and let $A$ be an orbifold structure on $X$. Then the induced orbifold fibration $f^\partial = f|_{X\backslash A}$ imposes an orbifold structure on $Y$ as follows. Given $D \in \operatorname{Div}'(Y)$, we may write $f^*D = \sum_i m_i D_i$ for $m_i \in \mathbb{N}$ and $D_i \in \operatorname{Div}'(X)$. Then we define the (minimum) multiplicity of $f^\partial$ over $D$ by

$$m(D, f^\partial) = \min\{m_i\, m(D_i \cap A) \mid f(D_i) = D\}.$$

**Definition 2.5.** With the notation as given above, the $\mathbb{Q}$-divisor on $Y$

$$D(f^\partial) = D(f|_{X\backslash A}) = D(f, A) := \sum \left\{ \left(1 - \frac{1}{m(D, f^\partial)}\right) D \;\middle|\; D \in \operatorname{Div}'(Y) \right\}$$

defines the *orbifold base* $Y\backslash D(f^\partial)$ of $f^\partial = f|_{X\backslash A}$.

It is immediate that $D(f^\partial)$ is supported on the union of $f((A \cap f)_{\text{red}})$ with the divisorial part $\Delta(f)$ of the discriminant locus of $f$. Note that replacing $f$ by its composition with a birational morphism $r : \tilde{X} \to X$ and imposing the $\infty$

multiplicity along the exceptional divisor of $r$ while keeping the other orbifold multiplicities the same does not change $D(f^{\partial})$. Hence, although the definition of $D(f^{\partial})$ would no longer make sense if we allow $f$ to be meromorphic, we can deal with the problem in a consistent way (though not always the best way) by resolving the singularities of $f$ and imposing the $\infty$ multiplicity along the exceptional divisor of of the resolution. In the case $A$ is reduced, the same can be achieved by imposing only the $\infty$ multiplicity along the exceptional divisor of $r$ that maps to $A$, that is, $r^{\partial}$ gives a proper birational morphism to $X\backslash A$. This is always adopted in the case $A$ is reduced. The following two lemmas (Lemmas 3.5 and 3.4 of [16]) are essentially immediate consequences of the definition.

**Lemma 2.6.** *With the notation as above, let $g : Y \to T$ be a fibration and $h = g \circ f$. Let $i : X_t \hookrightarrow X$, respectively $j : Y_t \hookrightarrow Y$, be the inclusion of the fiber of $h$, respectively $g$, above a general point $t \in T$. Then $D(f^{\partial})_t := j^* D(f^{\partial})$ and $A_t := i^* A$ are orbifold structures on the nonsingular fibers $Y_t$ and $X_t$ respectively and $D(f^{\partial})_t = D(f_t^{\partial})$ where $f_t^{\partial} = f_t|_{X_t\backslash A_t}$, that is*

$$D(f|_{X\backslash A})|_{Y_t} = D(f_t|_{X_t\backslash A_t}).$$

*Hence $(Y^{\partial})_t := Y_t\backslash D(f^{\partial})_t$ and the definition of $f_t^{\partial}$ make sense, and we have the identity $(Y^{\partial})_t = Y_t\backslash D(f_t^{\partial}) =: (Y_t)^{\partial}$.*

*Proof.* The lemma follows from the fact that $h$, respectively $g$, and its restriction to the divisor $R = (A + f^*\Delta(f))_{\mathrm{red}}$ in $X$, respectively the divisorial part of $f(R)_{\mathrm{red}}$, are generically of maximal rank when restricted to their fibers above $t$ (by Sard's theorem). □

Hence, the definition of $D(f^{\partial})$ is well behaved under generic restrictions.

**Lemma 2.7.** *Let $f^{\partial}$, $g$, $h$ and $A$ be as above, let $B = D(f^{\partial}) = D(f, A)$, $g^{\partial} = g|_{Y\backslash B}$ and $h^{\partial} = h|_{X\backslash A}$. Then $D(g^{\partial}) \geq D(h^{\partial})$, i.e., $D(g, D(f, A)) \geq D(g\circ f, A)$. If the exceptional part of $A$ with respect to $f$ is reduced or if $A$ and $B$ are reduced and $f^{\partial} : X\backslash A \to Y\backslash B$ is proper and birational, then equality holds.* □

The following theorem is the key fact about special orbifolds used to establish our main theorem. It will be used in the next section.

**Theorem 2.8.** *Let $X^{\partial}$ be a (smooth) orbifold, $f^{\partial} : X^{\partial} \to T$ be a special orbifold fibration and $h^{\partial} : X^{\partial} \dashrightarrow Z$ be a base-general orbifold rational map. Then $h^{\partial} = k \circ f^{\partial}$ for a rational map $k : T \dashrightarrow Z$ and $k^{\partial} := k|_{T\backslash D(f^{\partial})}$ is base-general. In particular, if $T\backslash D(f^{\partial})$ is special, then $f^{\partial}$ is base special and hence $X^{\partial}$ is special.*

This is Proposition 6.5 of [16], see also [17] and Chapter 8 of [6]. In view of its importance here, we reproduce a proof below adapted to our situation.

Recall that a $\mathbb{Q}$-divisor is called big if it has maximal Kodaira dimension. We first quote two elementary and well-known lemmas concerning the Kodaira dimension.

**Lemma 2.9** (Kodaira, [15]). *Let $H$ and $L$ be invertible sheaves on $X$ with $H$ ample. Then $L$ is big if and only if there exists a positive integer $m$ such that* $\dim H^0(L^m \otimes H^{-1}) \neq 0$. □

**Lemma 2.10** (Easy addition law, [12]). *Let $f : X \to Y$ be a fibration with general fiber $F$ and $L$ be an invertible sheaf on $X$. Then*

$$\kappa(L) \leq \kappa(L|_F) + \dim(Y). \qquad \square$$

The following is a simplified version of Lemma 5.7 of [16].

**Lemma 2.11.** *Consider the following commutative diagram of rational maps between complex projective manifolds:*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, "g"] & Z \\ T & Y \arrow[l, "u"] \arrow[u, "w"'] \end{tikzcd}$$

*where $f$ is a fibration, $g$ and $h$ are dominant rational maps and $u$ and $w$ are morphisms, necessarily surjective. Let $A$ be an orbifold structure on $X$. Let $i$ and $j$ be the inclusion of the general fibers $X_t := f^{-1}(t)$ and $Y_t := u^{-1}(t)$ over $T$. Let $g_t = g \circ i$ and $h_t = h \circ i$. Assume that $w \circ j$ is generically finite so that $L_{g_t} = L_{h_t}$. Then $i^*A$ is an orbifold structure on $X_t$ and we have with $p = \dim Z = \dim Y$, $q = \dim Y_t$ that*

$$\kappa(h^\partial) - (p-q) \leq \kappa(i^* L_{h^\partial}) \leq \kappa(L_{h_t^\partial}) =: \kappa(h_t^\partial).$$

*In particular, if $h^\partial$ is base-general, then so are $h_t^\partial$ and $g_t^\partial$ if* $\dim Y_t > 0$.

*Proof.* It will be clear from our proof that we may assume w.l.o.g., by taking repeated hyperplane sections of $T$ if necessary, that $w$ is generically finite. So we assume this from the start. Note then that $\dim(T) = p - q$ so that the first inequality above follows from the easy addition law of Kodaira dimension. To obtain the second inequality and thus the lemma, the easily verified fact that

$$i^*(A \cap h) = (i^*A) \cap (h \circ i) = (i^*A) \cap h_t = (i^*A) \cap g_t$$

allows us to deduce it from an inclusion of $i^*L_h$ in $L_{h_t} = L_{g_t}$ that can be seen as follows. The conormal short exact sequence on $X_t$

$$0 \to N^*_{X_t} \to \Omega_X|_{X_t} \to \Omega_{X_t} \to 0$$

gives rise to a natural sheaf morphism from $i^*\Omega^p_X = \Omega^p_X|_{X_t}$ to the factor $\Omega^q_{X_t} \otimes \Lambda^{p-q} N^*_{X_t}$ in its quotient filtration. Now, over the Zariski open set $U$ of $X_t$ where $g_t$ and $h_t$ are defined, $i^*g^* = g^*_t$ gives a map from the same short exact sequence on $Y_t$ to that of $X_t$. So it does the same for the corresponding natural sheaf morphism on $Y_t$ to that of $X_t$. Thus we obtain a commutative diagram over $U$:

$$\begin{array}{ccccc} i^*h^*K_Z & \longrightarrow & \Omega^p_X|_{X_t} & \longrightarrow & \Omega^q_{X_t} \otimes \Lambda^{p-q} N^*_{X_t} \\ \big\downarrow{\scriptstyle \delta} & & \uparrow & & \uparrow \\ i^*g^*K_Y & \xrightarrow{\sim} & g^*_t(K_Y|_{Y_y}) & \longrightarrow & g^*_t(K_{Y_t} \otimes \det N^*_{Y_t}) \end{array}$$

where $\delta$ is induced from the inclusion $w^*K_Z \hookrightarrow K_Y$. As both $\det N^*Y_t$ and $\Lambda^{p-q} N^*_{X_t} = \det N^*_{X_t}$ are trivial invertible sheaves by construction, we see that $g^*_t K_{Y_t}$ has the same image in $\Omega^q_{X_t}$ as that of $i^*h^*K_Z$ over a Zariski open subset of $X_t$. As the former saturates to $L_{g_t}$ in $\Omega^q_{X_t}$, we see that $i^*L_h \hookrightarrow L_{g_t}$ as required. □

*Proof of Theorem 2.8.* Let $g_0 = (f, h) : X \dashrightarrow T \times Z$ and $Y_0$ be its image. Let $r : Y \to Y_0$ be a resolution of singularities of $Y_0$ and let $g = g_0 \circ r^{-1}$, which is a rational map in general. Let $u$ and $w$ be r composed with the projections of $T \times Z$ to $T$ and $Z$ respectively. Then we are in the situation of Lemma 2.11 with $h^\partial$ base-general. We first note that the general fibers of $u$ are connected by construction, being images of the fibers of the special fibration $f$. As the general fibers of $f$ are special, our lemma implies that $Y_t$ for the general $t \in T$ are points. It follows that $u$ is birational so that $Y_0$ form the graph of a rational map $k : T \dashrightarrow Z$ and $h = k \circ f$. The theorem now follows directly from the following elementary lemma, which is a simplification of Proposition 3.19 of [16]. □

**Lemma 2.12.** *Let $f : X \to T$ be a fibration, $k : T \dashrightarrow Z$ a rational map and $h = k \circ f$. Let $A$ be an orbifold structure on $X$ inducing the orbifold maps $f^\partial = f|_{X \setminus A}$ and $h^\partial = h|_{X \setminus A}$. Let $B = D(f^\partial)$ be the orbifold structure on $T$ imposed by $f^\partial$ and let $k^\partial = k|_{T \setminus B}$ be the induced orbifold map from $T$. Then*

$$\kappa(h^\partial) \leq \kappa(k^\partial).$$

*In particular, if $h^\partial$ is base-general, then so is $k^\partial$.*

*Proof.* We only prove the lemma in the case we are using in the paper where $A$ and $B$ are reduced normal crossing divisors; more specifically, in the case $A$ is reduced normal crossing, $f^\partial$ is a special fibration and and $B = D(f^\partial)$ is just the standard boundary divisor of the compactification of a semi-Abelian variety. In this case, both $X\backslash A$ and $Y\backslash B$ are log-manifold and so $L_{f^\partial}$ and $L_{k^\partial}$ can be considered respectively as invertible subsheaves of $\Omega(X, \log A)$ and $\Omega(T, \log B)$ by Lemma 2.3. We note that, outside the exceptional divisor $E(f)$ of $f$, $f$ is a log-morphism (i.e., $f^{-1}(B) \subset A$) and so gives an inclusion of sheaves $f^*\Omega(T, \log B) \hookrightarrow \Omega(X, \log A)$ there and it is actually a vector bundle inclusion on a Zariski open subset of $f^{-1}(B)$ ([12]). Thus, we have an inclusion

$$f^* L_{k^\partial} \hookrightarrow L_{h^\partial}$$

outside $E(f)$ that is an equality on a Zariski open subset of $f^{-1}(B)$. This equality extends to the open subset outside $A\,(\supset f^{-1}(B))$ where $f$ is smooth. But by our definition of the multiplicity that gives the orbifold base, this open subset surjects to the complement of a codimension two subset of $T\backslash B$. Hence $H^0(L^l_{h^\partial}) \hookrightarrow H^0(f^* L^l_{k^\partial}) = H^0(L^l_{k^\partial})$ for all positive integer $l$ by the Hartog extension theorem. □

We remark that in our case at hand, $Y\backslash B$, being a semi-Abelian variety with a smooth equivariant compactification $Y$ (see the next section for the definition and basic facts), has trivial log-cotangent sheaf. Hence $\kappa(L_{k^\partial}) \leq 0$ and $Y\backslash B$ is a special orbifold.

We now address the very important question of when is the base Kodaira dimension of an orbifold fibration equal to the Kodaira dimension of its orbifold base. The question was posed by Campana in [5] for which he gave a partial answer in the case the base has positive Kodaira dimension. We have also given a partial answer in Lemmas 2.2 and 2.4 of [16] which showed at the same time the equivalence of our approach to that of Campana's. It is this latter that we give below but restricted here for simplicity to the context of log-manifold.

**Lemma 2.13.** *Let $f : X \to Y$ be a fibration where $X$ and $Y$ are complex projective manifolds, $A$ be a normal crossing divisor on $X$ and $f^\partial = f|_{X\backslash A}$. Then $\kappa(f^\partial) \leq \kappa(Y\backslash D(f^\partial))$. Also, one can find a commutative diagram of morphisms between complex projective manifolds*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{2.1}$$

*with $u, v$ birational and onto such that $E(f' \circ v^{-1}) = \emptyset$ and $A' = v^{-1}(A)$ is normal crossing. Let $f'^{\partial} = f'|_{X' \setminus A'}$. Then $v$ induces a proper birational morphism $X \setminus A \to X' \setminus A'$ and $\kappa(f^{\partial}) = \kappa(f'^{\partial})$. If $m$ is divisible by the multiplicities of $D(f'^{\partial})$, then*

$$H^0(X, L^m_{f'^{\partial}}) = H^0(Y', K^m_{Y' \setminus D(f'^{\partial})}) \quad \textit{and} \quad \kappa(f'^{\partial}) = \kappa(Y' \setminus D(f'^{\partial})).$$

*Proof.* The first statement follows from Lemma 2.12 by letting $k$ be the identity map there.

The construction of a birationally equivalent fibration as given by the commutative diagram with the above property is achieved by resolving the singularities of the flattening of $f$, which exists by [25, 8], and in such way that the inverse image of $A$ is normal crossing, which is possible by [10]. As $u$ is birational, $v^* L_{f^{\partial}} \hookrightarrow L_{f'|_{X' \setminus A'}}$ by Lemma 2.3 and hence $\kappa(L_{f^{\partial}}) \leq \kappa(L_{f'|_{X' \setminus A'}})$. The reverse inequality follows from Lemma 2.12.

For the last statement, we have (with $r = \dim Y'$) as before the inclusion

$$f'^* K_{Y'}(D(f'^{\partial}))^m \hookrightarrow L^m_{f'^{\partial}} \left( \hookrightarrow (\Omega^r_{X'}(\log A'))^{\otimes m} \right) \tag{2.2}$$

outside $O \cup E(f')$ where $O$ is a subset of $X'$ of codimension two or higher contained above the discriminant $\Delta(f')$ of $f'$ and $m$ is a positive multiple of all relevant multiplicities. Moreover, this inclusion is an isomorphism on an open subset of $X'$ that surjects to the complement of a subset of $Y'$ of codimension two or higher. Hence $H^0(Y', K_{Y'}(D(f'^{\partial}))^m) \hookrightarrow H^0(X, L^m_{f^{\partial}}) = H^0(X', L^m_{f'^{\partial}})$ by the Hartog extension theorem applied to $X$ and the reverse inclusion by the Hartog extension theorem applied to $Y'$. □

We give below generalizations to the relative setting of Lemma 2.13 and Theorem 2.8. They are used to extend our main theorem but are otherwise not needed for its proof.

**Lemma 2.14.** *With the setup as in Lemma 2.13 and with all elements of the commutative diagram* (2.1) *as given there, let $g' : Y' \dashrightarrow Z'$ be a dominant rational map. Then*

$$\kappa(g', D(f'|_{X' \setminus A'})) = \kappa(g' \circ f', A').$$

*Proof.* The proof is the same as that of Lemma 2.13, replacing $K_{Y'}(D(f'^{\partial}))$ by $L_{g'|_{Y' \setminus D(f'^{\partial})}}$. □

**Proposition 2.15.** *Let the setup be as in Lemma 2.6, i.e., let $f : X \to Y$ and $g : Y \to T$ be fibrations with orbifold structures $A$ on $X$ and $D(f, A)$ on $Y$ and*

*$t$ be a general point on $T$. Assume $A$ is a normal crossing divisor. If $f^\partial$ and $Y_t^\partial = Y_t \backslash D(f|_{X_t^\partial})$ are special, then so is $X_t^\partial := X_t \backslash A_t$. If $T \backslash D(g \circ f, A)$ is special, then $f^\partial$ is base-special if so is $f_t^\partial$.*

*Proof.* The last statement follows by noting that $f^\partial$ is base-special if so is $f'^\partial$, in the notation of Lemma 2.13, and that $f'^\partial$ is base-special by Lemma 2.14 and so Theorem 2.8 applies. □

## 3 Structure of the quasi-Albanese map

Let $X$ be a complex quasi-projective manifold, $T$ be a semi-Abelian variety and $u : X \to T$ be an algebraic morphism. Let $\bar{T}$ be a smooth equivariant compactification of $T$, i.e., $\bar{T}$ is a smooth compact algebraic variety containing $T$ as an open subvariety such that the natural $T$-action on itself by (left) translation extends to an algebraic $T$-action on $\bar{T}$ – an example being the compactification of $T$ in the exact sequence (1.1) via the compactification $(\mathbb{C}^*)^k \subset (\mathbb{P}^1)^k$. Then one can observe, see [21], that $\bar{T} \setminus T$ is a normal crossing divisor and that the sheaf of logarithmic one-forms $\bar{\Omega}_T$ is a trivial bundle over $\bar{T}$.

By the resolution of singularity theorem (see [1, 10]), there is a compactification $\bar{X}$ of $X$ with normal crossing boundary divisor $A$ such that $u$ extends to a morphism $\bar{u} : \bar{X} \to \bar{T}$.

**Definition 3.1.** We call $\bar{u}$ as above a *natural compactification* of $u$. We will set $\bar{u}^\partial = \bar{u}|_{\bar{T} \backslash T}$ and, in the case $u$ is a fiber space (i.e., $\bar{u}$ is a fibration), we set $D(u) := D(\bar{u}^\partial)|_T$.

We note that $D(u)$ is a $\mathbb{Q}$-divisor on $T$ that is independent of the natural compactification $\bar{u}$ of $u$ chosen since two such compactifications are always dominated by a third such compactification. By the same token, the notions of being special, being general-typical, being base-special and being base-general(-typical) are well defined for $u$ (independent of the chosen natural compactifications).

**Definition 3.2** ([24]). Let $D$ be a $\mathbb{Q}$-divisor in $T$. We define $\mathrm{St}(D)$ to be the identity component of $\{a \in T : a + D = D\}$ which is easily verified to be a semi-Abelian subvariety of $T$.

If $D = \sum a_i D_i$ for some reduced hypersurfaces $D_i$ and rational numbers $a_i$ and if $\bar{D}_i$ is the Zariski closure of $D_i$ in $\bar{T}$, then (by abuse of language) we call $\sum a_i \bar{D}_i$ the "Zariski closure of $D$ in $\bar{T}$" and denote it by $\bar{D}$.

**Proposition 3.3.** *Let $X$ be a quasi projective variety with quasi-Albanese map $\alpha : X \to T$. Assume that $\alpha$ has generically finite fibers and that there exists a non-degenerate entire curve $f : \mathbb{C} \to X$. Then $\alpha$ is generically bijective.*

The proof is based on the main theorem of [23]:

**Theorem 3.4** ([23]). *Let $X$ be a normal complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Let $f : X \to T$ be a finite morphism to a semi-Abelian variety. Then $f$ is an étale covering morphism.*

*Proof of Proposition 3.3.* Due to Noguchi's logarithmic version of the Bloch–Ochiai theorem ([19]) the image $\alpha(X) \subset T$ must be Zariski dense.

Let $\hat{\alpha} : \bar{X} \to T$ be a partial compactification, i.e., we choose an open embedding $i : X \to \hat{X}$ such that $\alpha$ extends to a proper morphism $\hat{\alpha} : \hat{X} \to T$. Let $\hat{X} \to \bar{X} \to T$ be the Stein factorization. We obtain a generically injective dominant morphism $j : X \to \bar{X}$ such that there exists a finite morphism $\bar{\alpha} : \bar{X} \to T$. Note that $(j \circ f) : \mathbb{C} \to \bar{X}$ is a non-degenerate entire curve taking values in $\bar{X}$. Hence Theorem 3.4 implies that $\bar{\alpha} : \bar{T} \to T$ is étale. By the universal property of the quasi-Albanese it follows that $\bar{\alpha}$ is biholomorphic. As a consequence, $\alpha$ is generically bijective. □

**Corollary 3.5.** *Let $X$ be a smooth complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Then the (quasi-)Albanese map of $X$ is a fiber space.*

*Proof.* This follows by applying Theorem 3.4 to the Stein factorization of a natural compactification of $f$. □

**Lemma 3.6.** *Let $D$ be an effective $\mathbb{Q}$-divisor in a semi-Abelian variety $T$, let $\bar{T}$ be a smooth equivariant compactification of $T$ and let $\bar{D}$ be the Zariski closure of $D$ in $\bar{T}$. Assume that* $\mathrm{St}(D) = \{0\}$. *Then $\bar{D}$ is big.*

*Proof.* This is [24, Proposition 3.9]. It is stated there only for $\mathbb{Z}$-divisors, but the generalization to $\mathbb{Q}$-divisors is immediate. □

**Lemma 3.7.** *Let $A$ be a semi-abelian variety and let*

$$B_0 = \{e\} \subset B_1 \subset \cdots \subset B_n$$

*be a sequence of semi-abelian subvarieties of $A$. For each $k \in \{0, \dots, n\}$ define $A_k := A/B_k$ and let $D_k$ be a $\mathbb{Q}$-divisor on $A_k$ such that $(A_k \setminus D_k)$ is an orbifold. Let $\pi_k : A_k \to A_{k+1}$ be the natural projection map. Assume that the following* *conditions are fulfilled:*

(i) $(A_n \setminus D_n)$ *is special.*

(ii) *For every* $k$ *the* $\mathbb{Q}$*-divisor* $D_k$ *is invariant under the natural* $B_{k+1}/B_k$*-action on* $A_k = A/B_k$.

(iii) $D_k - \pi_k^* D_{k+1}$ *is effective for every* $k$.

*Then both* $(A_0 \setminus D_0)$ *and every general fiber of* $(A_0 \setminus D_0) \to (A_n \setminus D_n)$ *are special.*

*Proof.* By construction the $\mathbb{Q}$-divisor $D_{n-1}$ does not intersect the general fiber of $\pi_{n-1} : A_{n-1} \to A_n$. Hence a general fiber of

$$\pi_{n-1} : (A_{n-1} \setminus D_{n-1}) \to (A_n \setminus D_n)$$

is special.

If $n = 2$, the statement thus follows from Theorem 2.8.

In general we argue by induction over $n$. We fix a general point $p \in A_n$ and let $(C_k \setminus \Delta_k)$ denote the fiber over $p$ of the natural projection

$$(A_k \setminus D_k) \to (A_n \setminus D_n).$$

Using the induction hypothesis we may deduce that $(C_0 \setminus \Delta_0)$ is special. Combined with Theorem 2.8 and the fact that the general fiber of

$$\pi_{n-1} : (A_{n-1} \setminus D_{n-1}) \to (A_n \setminus D_n)$$

is special, the desired statement follows. □

**Proposition 3.8.** *Let* $X$ *be a complex quasi-projective manifold and* $f : X \to T_0$ *be a generically bijective morphism to a semi-Abelian variety* $T_0$. *Then either* $X$ *is special or there exists a morphism of semi-abelian varieties* $p : T_0 \to T_1$ *with* $\dim(T_1) > 0$ *such that if* $(T_1 \setminus D)$ *is the orbifold base of the composite map*

$$p \circ f : X \to T_1,$$

*then the Zariski closure of* $D$ *is a big* $\mathbb{Q}$*-divisor on every smooth equivariant compactification of* $T_1$.

*Proof.* We define recursively semi-abelian quotient varieties $A_k$ of $T_0$, $\mathbb{Q}$-divisors $D_k$ on $A_k$ and morphisms $f_k : X \to A_k$ as follows: We start with $A_0 = T_0$, $f_0 = f$ and let $D_0$ be the $\mathbb{Q}$-divisor such that $(A_0 \setminus D_0)$ is the orbifold base of $f$. Given $A_k$, $f_k$ and $D_k$, we let $B_{k+1}$ denote the connected component of the identity of the stabilizer group $\mathrm{St}(D_k)$ and define $A_{k+1} = A_k/B_k$. The morphism $f_{k+1}$ is defined as the composition of $f_k$ with the natural projection map from $A_k$ to $A_{k+1}$ and $D_{k+1}$ is chosen such that $(A_{k+1} \setminus D_{k+1})$ is the orbifold base of $f_{k+1}$. We continue until $D_n$ is empty or $\mathrm{St}(D_n)$ is finite. In the first case, i.e., if $D_n$ is empty, we will show that $X$ is special using Lemma 3.7. We start by observing that condition (i) of Lemma 3.7 is fulfilled. Condition (ii) is evident from the construction, while (iii) follows using the fact that the projection maps between semi-abelian varieties are smooth. Therefore in the first case (i.e., if $D_n = \{\}$), Lemma 3.7 implies that the general fibers of $q : (A_0 \setminus D_0) \to (A_n \setminus \{\})$ are special. Because $f : X \to A_0 = T_0$ is generically bijective, a general fiber of $q$ is also a general fiber of $f : X \to A_n$. Since $(A_n \setminus \{\})$ is special, Theorem 2.8 now implies that $X$ is special.

In the second case (i.e., $\mathrm{St}(D_n) = \{e\}$) the divisor $D_n$ is big on every smooth equivariant compactification of $A_n$ due to Lemma 3.6. □

## 4 Implication of Zariski dense entire curves

We first recall some relevant definitions and facts from Nevanlinna theory. We will follow Section 2 of [24]. Let $T$ be a compact complex manifold, $\omega$ be a real smooth $(1,1)$-form and $\gamma : \mathbb{C} \to T$ be a holomorphic map. Then the order function of $\gamma$ with respect to $\omega$ is defined by

$$T_\gamma(r;\omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega \quad (r > 1). \tag{4.1}$$

If $T$ is Kähler and $\omega$, $\omega'$ are $d$-closed real (1,1)-forms in the same cohomology class $[\omega]$, then

$$T_\gamma(r;\omega) = T_\gamma(r;\omega') + O(1).$$

Hence we may set, up to $O(1)$-terms,

$$T_\gamma(r;[\omega]) = T_\gamma(r;\omega). \tag{4.2}$$

Let $L \to T$ be a hermitian line bundle. As its Chern class is a real $(1,1)$-class, we may set

$$T_\gamma(r;L) = T_\gamma(r;c_1(L)).$$

We will denote by $\mathcal{O}_T(D)$ the line bundle determined by a divisor $D$ on $T$ via a standard abuse of notation and set $T_\gamma(r;D) = T_\gamma(r;\mathcal{O}_T(D))$.

Let $E = \sum_{z\in\mathbb{C}} (\mathrm{ord}_z E) z$ be an effective divisor on $\mathbb{C}$ and $l \in \mathbb{N} \cup \{\infty\}$. Then the above sum is a sum over a discrete subset of $\mathbb{C}$. Hence the sum is finite when restricted to the disk $\mathbb{D}_t$ of radius $t > 0$ and so

$$n(t; E) = \deg_{\mathbb{D}_t} E := \sum_{z \in \mathbb{D}_t} \operatorname{ord}_z E$$

makes sense. We set $n_l(t; E) = \sum_{z \in \mathbb{D}_t} \min(\operatorname{ord}_z E, l)$ Then the counting functions of $E$ with, respectively without, truncation to order $l$ are given by

$$N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt,$$

respectively $N(r; E) = N_\infty(r; E)$.

If $Z$ is an arbitrary subvariety of a compact complex manifold $T$ and $\gamma : \mathbb{C} \to T$ is an entire curve with $\gamma(\mathbb{C}) \not\subset Z$, then $\gamma^* Z$ is a divisor on $\mathbb{C}$ because every proper analytic subvariety of $\mathbb{C}$ is a divisor. Hence it still makes sense to talk about counting functions $N_l(r, \gamma^* Z)$. The following is a standard result in Nevanlinna theory, called "First Main Theorem":

**Theorem 4.1** ([20]). *Let $D$ be an effective divisor on a compact complex manifold $T$ and let $\gamma : \mathbb{C} \to T$ be an entire curve. Then*

$$N(r; \gamma^* D) \le T_\gamma(r; D) + O(1).$$

By definition, for a strictly effective divisor $D$ on $T$ we have $T_\gamma(r; D) \ge 0$ if the image of $\gamma$ is not in $D_{\text{red}}$. This fact along with Kodaira's lemma (Lemma 2.9) and linearity of $T_\gamma$ with respect to to the second variable yields easily the following (which is Lemma 2.3 of [24]):

**Lemma 4.2.** *Suppose that $T$ is a complex projective manifold with an ample divisor $H$ and a big divisor $A$. Then*

$$T_\gamma(r; H) = O(T_\gamma(r; A)).$$

We will need the following "Second Main Theorem" of Noguchi–Winkelmann–Yamanoi.

For our purposes it is essential that this is what is known in Nevanlinna theory as a "Second main theorem with truncation level one".

**Theorem 4.3** ([24]). *Let $T$ be a semi-abelian variety and $\gamma : \mathbb{C} \to T$ be a holomorphic map with Zariski dense image. Let $E$ be a divisor on $T$. Then there exists an equivariant smooth compactification $\bar{T}$ of $T$ such that*

$$T_\gamma(r; \bar{E}) \le N_1(r; \gamma^* E) + \epsilon T_\gamma(r; \bar{E}) \,||_\epsilon \quad \textit{for all } \epsilon > 0 \tag{4.3}$$

*where $\bar{E}$ denotes the closure of $E$ in $\bar{T}$.*

**Theorem 4.4** ([24]). *Let $T$ be a semi-abelian variety and $\gamma : \mathbb{C} \to T$ a holomorphic map with Zariski dense image. Let $Z$ be an algebraic subvariety of codimension at least two in $T$. Then*

$$N_1(r; \gamma^* Z) \le \epsilon T_\gamma(r; H) \,||_\epsilon \quad \textit{for all } \epsilon > 0 \tag{4.4}$$

*for every equivariant smooth compactification $\bar{T}$ of $T$ and every ample divisor $H$ on $\bar{T}$.*

(As usual in Nevanlinna theory, the notion "$||_\epsilon$" stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure that depend on $\epsilon$.)

We will need the following consequence of the above theorems.

**Corollary 4.5.** *Let $X$ be a smooth complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Let $h : X \to T$ be a fibration to a semi-abelian variety $T$. Let $(T \setminus D(h))$ be the orbifold base of this morphism $h$. Then there is a smooth equivariant compactification $T \hookrightarrow \bar{T}$ such that the closure of $D(h)$ in $\bar{T}$ is not big.*

*Proof.* Assume the converse. Let $\gamma_0 : \mathbb{C} \to X$ be an entire curve with Zariski dense image in $T$. Let $\gamma = h \circ \gamma_0$. As $\gamma$ has Zariski dense image in $T$, Theorem 4.3 gives

$$T_\gamma(r; \bar{D}) \le N_1(r; \gamma^* D) + \epsilon T_\gamma(r; \bar{D}) ||_\epsilon \quad \text{for all } \epsilon > 0 \tag{4.5}$$

for a suitably chosen smooth equivariant compactification $T \hookrightarrow \bar{T}$. From the definition of $D = D(h)$ we see that the effective divisor $D_0 = h^*(D)$ can be decomposed into $D_0 = D_1 + D_2$ such that $D_1$ is nowhere-reduced and $h(|D_2|)$ is of codimension at least two in $T$.

Therefore

$$\begin{aligned} N(r; \gamma^*(D)) &\ge 2N_1(r; \gamma_0^* D_1) + N_1(\gamma_0^* D_2) \\ &\ge 2N_1(r; \gamma^* D) - N_1(\gamma_0^* D_2) \end{aligned} \tag{4.6}$$

with

$$N_1(\gamma_0^* D_2) \le \epsilon T(r; \gamma^* D) ||_\epsilon \quad \text{for all } \epsilon > 0 \tag{4.7}$$

due to Theorem 4.4. Also, since $\bar{D}$ is big, we have by Lemma 4.2 that $T_\gamma(r) = O(T_\gamma(r; \bar{D})$, that is, there exists a constant $C > 0$ (depending on the ample divisor used to define $T_\gamma(r)$) such that

$$T_\gamma(r) \le C T_\gamma(r; \bar{D}).$$

Combined with the First Main Theorem (Theorem 4.1) this gives

$$\begin{aligned} T_\gamma(r;\bar{D}) + O(1) &\geq N(r;\gamma^* D) \\ &\geq 2N_1(r;\gamma^* D) - N_1(\gamma_0^* D_2) \\ &\geq (2-(2+C)\epsilon)T_\gamma(r;\bar{D})\ ||_\epsilon, \end{aligned}$$

valid for all $\epsilon > 0$. This gives a contradiction for $\epsilon > 0$ sufficiently small. □

*Proof of Theorem 1.3.* Assume that there exists a non-degenerate entire curve $\gamma : \mathbb{C} \to X$. Let $\alpha : X \to T_0$ be the quasi-Albanese map of $X$. Noguchi's logarithmic version of the Bloch–Ochiai theorem ([19]) implies that $\alpha(X)$ is dense in $T_0$. Using the assumption $\bar{q}(X) \geq \dim X$ it follows that $\alpha$ has generically finite fibers. Thus $\alpha$ is generically bijective by Proposition 3.3. Due to Proposition 3.8 either $X$ is special or there exists a morphism of semi-abelian varieties $p : T_0 \to T_1$ such that if $(T_1 \setminus D)$ is the orbifold base of the composite map

$$p \circ f : X \to T_1,$$

then the Zariski closure of $D$ is a big $\mathbb{Q}$-divisor on every smooth equivariant compactification of $T_1$. But the second case leads to a contradiction with Corollary 4.5. Thus $X$ is special. □

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## Bibliography

[1] E. Bierstone and P. Milman, Canonical desingularization in characteristic zero by blowing up the maximal strata of a local invariant, *Invent. Math.* **128** (1997), 207–302; *Invent. Math.* **103** (1991), no. 1, 69–99.

[2] C. Birkenhake and H. Lange, *Complex Abelian Varieties*, Second Edition, Grundlehren der Mathematischen Wissenschaften 302, Springer-Verlag, Berlin, 2004.

[3] G. Buzzard and S. Lu, Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$, *Invent. Math.* **139** (2000), 617–659.

[4] G. Buzzard and S. Lu, Double sections, dominating maps, and the Jacobian fibration, *Amer. J. Math.* **122** (2000), no. 5, 1061–1084.

[5] F. Campana, Orbifolds, special varieties and classification theory, *Ann. Inst. Fourier* **54** (2004), no. 3, 499–630 (see also F. Campana, Special varieties and classification theory, preprint, arXiv:math.AG/0110051).

[6] F. Campana, Orbifoldes géométriques spéciales et classification biméromorphe des varieétés kählériennes compactes, preprint, arXiv:math.AG/0705.0737, 2008.

[7] G. Dethloff and S. Lu, Logarithmic surfaces and hyperbolicity, *Ann. Inst. Fourier* **57** (2007), 1575–1610.

[8] L. Gruson and M. Raynaud, Critères de platitude et de projectivitè, *Invent. Math.* **13** (1971), 1–89.

[9] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[10] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. Math.* **79** (1964), 109–326.

[11] H. Hironaka, Flattening theorem in complex-analytic geometry, *Amer. J. Math.* **97** (1975), 503–547.

[12] S. Iitaka, *Algebraic Geometry*, Graduate Texts in Mathematics 76, Springer-Verlag, New York, 1982.

[13] Y. Kawamata, Characterization of Abelian varieties, *Comp. Math.* **43** (1981), 253–276.

[14] Y. Kawamata and E. Viehweg, On a characterization of Abelian varieties, *Comp. Math.* **41** (1980), 355–359.

[15] K. Kodaira, Pluricanonical systems on algebraic surfaces of general type, *J. Math. Soc. Japan* **20** (1968), 170–192.

[16] S. Lu, A refined Kodaira dimension and its canonical fibration, preprint, arXiv: math.AG/0211029, 2002.

[17] S. Lu, The Kobayashi pseudometric on algebraic manifold and a canonical fibration, preprint, arXiv:math.AG/0206170, and MPIM online preprint, 2002.

[18] Y. Miyaoka and T. Peternell, *Geometry of Higher Dimensional Algebraic Varieties*, DMV Seminar 26, Birkhäuser-Verlag, Basel, 1997.

[19] J. Noguchi, Holomorphic curves in algebraic varieties, *Hiroshima Math. J.* **7** (1977), 833–853; Supplement, *ibid.* **10** (1980), 229–231.

[20] J. Noguchi and T. Ochiai, *Geometric Function Theory in Several Complex Variables*, Japanese edition, Iwanami, Tokyo, 1984; English Translation: Transl. Math. Mono. 80, Amer. Math. Soc., Providence, Rhode Island, 1990.

[21] J. Noguchi and J. Winkelmann, Bounds for curves in Abelian varieties, *J. reine angew. Math.* **572** (2003), 27–47.

[22] J. Noguchi, J. Winkelmann and K. Yamanoi, The second main theorem for holomorphic curves into semi-abelian varieties, *Acta Math.* **188** (2002), 129–161.

[23] J. Noguchi, J. Winkelmann and K. Yamanoi, Degeneracy of holomorphic curves into algebraic varieties, *J. Math. Pures Appl.* **88** (2007), no. 3, 293–306.

[24] J. Noguchi, J. Winkelmann and K. Yamanoi, The second main theorem for holomorphic curves into semi-abelian varieties II, *Forum Math.* **20** (2008), no. 3, 469–503.

[25] M. Raynaud, Flat modules in algebraic geometry, *Comp. Math.* **24** (1972), 11–31.

[26] K. Ueno, *Classification of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Mathematics 439, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

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