DEFINING AN m-CLUSTER CATEGORY

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Abstract. In this paper, starting with a simply laced root system, we define a triangulated category which we call the m-cluster category, and we show that it encodes the combinatorics of the m-clusters of Fomin and Reading in a fashion similar to the way the cluster category of Buan, Marsh, Reineke, Reiten, and Todorov encodes the combinatorics of the clusters of Fomin and Zelevinsky.

For Φ any root system, Fomin and Zelevinsky [FZ] define a *cluster complex* $\Delta(\Phi)$, a simplicial complex on $\Phi_{\geq -1}$, the *almost positive roots* of Φ. Its facets (maximal faces) are called *clusters*. In [BM+], starting in the more general context of a finite dimensional hereditary algebra H over a field K, Buan et al. define a *cluster category* $\mathcal{C}(H) = \mathcal{D}^b(H)/\tau^{-1}[1]$. ($\mathcal{D}^b(H)$ is the bounded derived category of representations of H; more will be said below about it, its shift functor [1], and its Auslander-Reiten translate τ.) The cluster category $\mathcal{C}(H)$ is a triangulated Krull-Schmidt category. We will be mainly interested in the case where H is a path algebra associated to the simply laced root system Φ, in which case we write $\mathcal{C}(\Phi)$ for $\mathcal{C}(H)$. There is a bijection V taking $\Phi_{\geq -1}$ to the indecomposables of $\mathcal{C}(\Phi)$. A *tilting set* in $\mathcal{C}(\Phi)$ is a maximal set $\mathcal{S}$ of indecomposables such that $\mathrm{Ext}^1_{\mathcal{C}(\Phi)}(X,Y) = 0$ for all $X, Y \in \mathcal{S}$. $\mathcal{C}(\Phi)$ encodes the combinatorics of $\Delta(\Phi)$ in the sense that the clusters of Φ correspond bijectively to the tilting sets of $\mathcal{C}(\Phi)$ under the map V.

Tilting sets in $\mathcal{C}(\Phi)$ always have cardinality n, the rank of Φ. An *almost tilting set* is a set $\mathcal{T}$ of $n-1$ indecomposables such that $\mathrm{Ext}^1_{\mathcal{C}(\Phi)}(X,Y) = 0$ for $X, Y \in \mathcal{T}$. A *complement* for $\mathcal{T}$ is an indecomposable M such that $\mathcal{T} \cup \{M\}$ is a tilting set. A tilting set always has exactly two complements. (This was shown from the cluster perspective in [FZ] and from the representation theoretic perspective in [BM+].) Given one complement, the other can be constructed by a procedure which we shall now describe.

Write $\mathrm{add}(\mathcal{T})$ for the full additive subcategory of $\mathcal{C}(\Phi)$ generated by the indecomposables in $\mathcal{T}$. A right $\mathrm{add}(\mathcal{T})$ approximation to an object M in $\mathcal{C}(\Phi)$ is an object B in $\mathrm{add}(\mathcal{T})$ and a morphism $B \to M$ such that any morphism from an object in $\mathrm{add}(\mathcal{T})$ to M factors through $B \to M$. Let $\mathcal{T}$ be an almost tilting set, and let M be one of the complements to $\mathcal{T}$. Take a minimal right $\mathrm{add}(\mathcal{T})$ approximation to M, call it B. Then complete the map $B \to M$ to a triangle:

$$M^* \to B \to M \to M^*[1].$$

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M^* will be the other complement [BM+]. If we take the minimal right $\mathrm{add}(\mathcal{T})$ approximation to M^*, call it B^*, and then complete to a triangle, we of course recover M. It follows (eventually) that if M and M^* are complements of some almost tilting set, then $\dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(M, M^*) = \dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(M^*, M) = 1$.

Conversely, given two indecomposables M, M^* such that $\dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(M, M^*) = \dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(M^*, M) = 1$, we can define B^* and B to be the corresponding extensions. Then there exists an almost tilting set $\mathcal{T}$ (and typically many of them) such that $\{M, M^*\}$ are the complements of $\mathcal{T}$. Any such almost tilting set $\mathcal{T}$ includes the indecomposables of B and B^*, and B and B^* are the minimal right $\mathrm{add}(\mathcal{T})$ approximations to M and M^*. It was conjectured in [BM+] and proved in [BMR] that the sets of indecomposables of B and B^* are disjoint, and that they encode the exchange relation in the cluster algebra corresponding to the cluster mutation between $\mathcal{T} \cup \{M\}$ and $\mathcal{T} \cup \{M^*\}$ (in a certain precise sense which we shall not get into here).

Fomin and Reading [FR] recently introduced a generalization of clusters known as m-clusters, for $m \in \mathbb{N}$. When $m = 1$, the classical clusters are recovered. The m-cluster complex $\Delta_m(\Phi)$ is a simplicial complex on a set of *coloured roots* $\Phi^m_{\geq -1}$. It has been studied further in [AT1, T, AT2]. The facets of $\Delta_m(\Phi)$ are known as m-clusters. The goal of this paper is to construct an m-cluster category which plays a similar role to the cluster category but with respect to the combinatorics of m-clusters. Specifically, we define a triangulated m-cluster category $\mathcal{C}_m(\Phi) = \mathcal{D}^b(\Phi)/\tau^{-1}[m]$. We define a bijection $W : \Phi^m_{\geq -1} \to \mathrm{ind}\, \mathcal{C}_m(\Phi)$. We define an *$m$-tilting set* in $\mathcal{C}_m(\Phi)$ to be a maximal set of indecomposables $\mathcal{S}$ satisfying $\mathrm{Ext}^i_{\mathcal{C}_m(\Phi)}(X, Y) = 0$ for all $X, Y \in \mathcal{S}$ and $i = 1 \ldots m$. Then we show:

Theorem 1. *The map W induces a bijection from m-clusters of Φ to m-tilting sets of $\mathcal{C}(\Phi)$.*

Like tilting sets in $\mathcal{C}(\Phi)$, m-tilting sets in $\mathcal{C}_m(\Phi)$ have cardinality n. (This follows from [FR, Theorem 2.9] together with our Theorem 1). We make the natural definition of almost m-tilting sets. Via [FR, Proposition 2.10] and Theorem 1, we know that an almost m-tilting set $\mathcal{T}$ has exactly $m + 1$ complements.

If M_0 is a complement of $\mathcal{T}$, take a minimal right $\mathrm{add}(\mathcal{T})$ approximation to M_0, say B_0, and then complete to a triangle.

$$M_1 \to B_0 \to M_0 \to M_1[1].$$

Applying the same process to M_1, define B_1 and M_2, and continue inductively.

Theorem 2. *Let $\mathcal{T}$ be an almost m-tilting set, and let M_0 be a complement for $\mathcal{T}$. Define the M_i as above. Then $M_{m+1} = M_0$, and the set of M_i for $i = 0, \ldots, m$ is the complete set of complements of $\mathcal{T}$. Also, B_i is the minimal left* $\mathrm{add}(\mathcal{T})$ *approximation to M_{i+1}.*

We can prove the analogue of part of Theorem 6.1 of [BMR] (Conjecture 9.3 of [BM+]):

Theorem 3. *If $\mathcal{T}$ is an almost m-tilting set, M_i its complements as above, and B_i the minimal right* $\mathrm{add}(\mathcal{T})$ *approximation to M_i, then the sets of indecomposables of B_i, $i = 0, \ldots, m$ are disjoint.*

We can also say something about $\mathrm{Ext}^i_{\mathcal{C}_m(\Phi)}(M_j, M_k)$:

Theorem 4. *If $\mathcal{T}$ is an almost m-tilting set, with complements M_i as above, then:*

(1) *For* $1 \leq i \leq m$, $\dim_K \operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(M_j, M_k) = \begin{cases} 1 & \textit{iff } k - j = i \bmod m+1 \\ 0 & \textit{otherwise.} \end{cases}$

There is also a converse to Theorem 4:

Theorem 5. *Given a set of M_i ($i = 0 \dots m$) satisfying (1), form B_i as the uniquely determined non-trivial extension of M_i by M_{i+1}. Then there exists an almost m-tilting set $\mathcal{T}$ such that $\{M_i\}$ are the set of complements for $\mathcal{T}$.*

The fact that the set of M_i determine the B_i, independent of what m-tilting object $\mathcal{S}$ the M_i are contained in, and that the indecomposables of the B_i are disjoint and are necessarily contained in $\mathcal{S}$ (which follows from Theorem 5), suggests that the B_i are encoding some kind of generalized exchange relation. On the other hand, the way the definition of the B_i involves all the m-clusters $\mathcal{T} \cup \{M_i\}$ simultaneously suggests that if such a generalized exchange relation exists at all, it might not involve only two m-clusters at a time.

Finally, we apply our results to derive a combinatorial conclusion about $\Delta_m(\Phi)$. By definition, the vertices of Δ_m are partitioned into m classes (referred to as "colours"); we show:

Theorem 6. *If $\mathcal{T}$ is a codimension 1 face of $\Delta_m(\Phi)$, then there is at least one vertex of each colour which, together with $\mathcal{T}$, forms a facet of $\Delta_m(\Phi)$. (Since there are $m+1$ such vertices in total and m colours, it follows that there are two such vertices of one colour and one of each of the others.)*

We conclude the paper by sketching some of the details in type A, where the combinatorics of m-clusters are well-understood and easy to work with. For further consideration of the type A situation, the reader is directed to [BM].

As was already mentioned, the cluster category of [BM+] is constructed starting with an arbitrary finite dimensional hereditary algebra H over a field K; their results mentioned above apply in that generality. We are hopeful that many of our results could be generalized to the broader context they consider.

Clusters

We begin with a quick introduction to the combinatorics of clusters. Our presentation is based on [FZ] and [FR].

Let Φ be a crystallographic root system of rank n. (In fact, the assumption that Φ is crystallographic is not essential [FR], but since we will shortly be assuming that Φ is not merely crystallographic but also simply laced, there is no advantage to considering the slightly more general situation.)

Label the vertices of the Dynkin diagram for Φ by the numbers from 1 to n. Let W be the Weyl group corresponding to Φ. Let the simple roots of Φ be $\Pi = \{\alpha_1, \dots, \alpha_n\}$, and let s_i be the reflection in W corresponding to α_i.

The ground set for the *cluster complex* $\Delta(\Phi)$ is the set of *almost positive roots*, $\Phi_{\geq -1}$, which are, by definition, the positive roots Φ^+ together with the negative simple roots $-\Pi$.

Since the Dynkin diagram for Φ is a tree, it is a bipartite graph. Let I_+, I_- be a decomposition of $[n]$ corresponding to the bipartition. (I_+ and I_- are determined up to interchanging $+$ and $-$. We fix this choice once and for all.)

For $\epsilon \in \{+,-\}$, define the bijection $\tau_\epsilon : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\epsilon(\beta) = \begin{cases} \beta & \text{if } \beta = -\alpha_i \text{ for some } i \in I_{-\epsilon} \\ \left(\prod_{i\in I_\epsilon} s_i\right)\beta & \text{otherwise} \end{cases}$$

Now set $R = \tau_-\tau_+$. R is in some sense a deformation of the Coxeter element of W. (We will give a more representation-theoretic interpretation for R in Lemma 1 below.)

The crucial fact about R is that every root in $\Phi_{\geq -1}$ has at least one negative simple root in its R-orbit. For that reason, the following suffices to define a relation called compatibility.

(i) $-\alpha_i$ is compatible with β iff α_i does not appear when we write β as a sum of simple roots. (This is called the *simple root expansion* for β.)

(ii) α and β are compatible iff $R(\alpha)$ and $R(\beta)$ are compatible.

This relation is well-defined (not *a priori* obvious, since a root may have two negative simples in its R-orbit) and symmetric [FZ, §§3.1-2].

In fact, there is more information associated to a pair of almost positive roots than mere compatibility or incompatibility. The compatibility degree $(\alpha \parallel \beta)$ can be defined by saying that:

(i) $(\beta \parallel -\alpha_i)$ is the coefficient of α_i in the expansion of β if β is positive and 0 if β is negative.

(ii) $(R(\beta) \parallel R(\alpha)) = (\beta \parallel \alpha)$.

Compatibility degree is well-defined, and, if Φ is simply laced, it is also symmetric [FZ, §3.1]. Two roots are compatible iff their compatibility degree is zero.

The *cluster complex* $\Delta(\Phi)$ is defined to be the simplicial complex whose faces are the sets of almost positive roots which are pairwise compatible. The facets (maximal faces) of $\Delta(\Phi)$ are all of the same cardinality, n, the rank of Φ. They are called *clusters*.

Derived Category

Fix Q an orientation of the Dynkin diagram for Φ. The representations of Q are denoted $\mathcal{L}(Q)$. The bounded derived category $\mathcal{D}^b(Q)$ is a triangulated category, and it comes with a $\mathbb{Z}$ grading and a shift functor [1] which takes $\mathcal{D}^b(Q)_i$ to $\mathcal{D}^b(Q)_{i-1}$. $\mathcal{D}^b(Q)_i$ is just a copy of $\mathcal{L}(Q)$. We refer to this grading as the *coarse grading* on $\mathcal{D}^b(Q)$, and denote the degree function with respect to this grading by d_C.

To give a more concrete description of $\mathcal{D}^b(Q)$, we we will define an infinite quiver $\mathbb{Z}Q^{op}$. Its vertex set consists of $[n] \times \mathbb{Z}$. For each edge from v_i to v_j in Q, $\mathbb{Z}Q^{op}$ has an edge from (j,p) to (i,p) and one from (i,p) to $(j,p-1)$, for all $p \in \mathbb{Z}$. This means that one way of thinking of $\mathbb{Z}Q^{op}$ is as $\mathbb{Z}$ many copies of Q^{op} (Q with its orientation reversed) with some edges added connecting copy i to copy $i-1$.

It turns out that $\mathbb{Z}Q^{op}$ is the Auslander-Reiten quiver for $\mathcal{D}^b(Q)$, so in particular the indecomposables of $\mathcal{D}^b(Q)$ can be identified with the vertices of $\mathbb{Z}Q^{op}$.

If Q and Q' are two different orientations of the Dynkin diagram for Φ, then $\mathcal{D}^b(Q)$ and $\mathcal{D}^b(Q')$ are isomorphic as triangulated categories, but their coarse gradings disagree. We will generally therefore forget the orientation (and the grading it induces), and write $\mathcal{D}^b(\Phi)$.

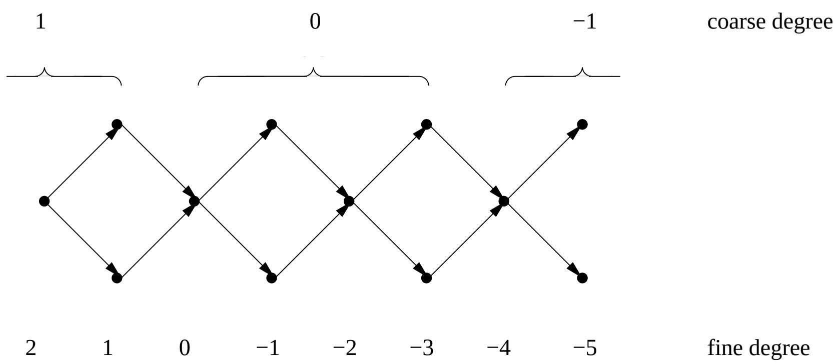
Since $\mathcal{D}^b(\Phi)$ does not depend on the choice of an orientation, we may fix a convenient orientation if we like. Let Q_{bip} denote the bipartite orientation of the Dynkin diagram of Φ in which the arrows go from roots in I_+ and towards roots in

I_-. We want to fix a grading on the vertices of $\mathbb{Z}Q_{bip}^{op}$, which we shall call the *fine grading*, and denote it d_F. Vertices in $\mathbb{Z}Q_{bip}^{op}$ are indexed by (i,k) with $i \in [n]$ and $k \in \mathbb{Z}$. We say that a vertex (i,k) is in fine degree $2k$ if $i \in I_-$ and $2k-1$ if $i \in I_+$. It follows that all the arrows in $\mathbb{Z}Q_{bip}^{op}$ diminish fine degree by 1.

The coarse and fine gradings of $\mathcal{D}^b(Q_{bip})$ are related: $d_C(M) = \lceil d_F(M)/h \rceil$, where h is the Coxeter number for Φ. (The Coxeter number is the order of the Coxeter element, and can be computed from the fact that $|\Phi| = nh$.)

The copy of the indecomposables of $\mathcal{L}(Q_{bip})$ which sits in coarse degree 0, consists of the vertices of $\mathbb{Z}Q_{bip}^{op}$ in fine degree between 0 and $-h+1$. The indecomposables of $\mathcal{L}(Q_{bip})$ which are projective are exactly those in fine degree 0 and -1.

Here is an example for A_3. Here $h = 4$, and I_+ consists of the two outside nodes while I_- is the middle node.



We can define an automorphism τ of $\mathbb{Z}Q^{op}$ which takes (i,p) to $(i,p+1)$ for all $i \in \{1,\ldots,n\}$ and $p \in \mathbb{Z}$. This automorphism corresponds to an auto-equivalence of $\mathcal{D}^b(\Phi)$, also denoted τ, which is the Auslander-Reiten translate for $\mathcal{D}^b(\Phi)$. The functor τ respects the fine degree, increasing it by 2. The shift functor [1] also respects the fine degree, decreasing it by h.

Factor categories of the derived category

Let $\mathcal{D}^b(H)$ be the bounded derived category of modules over a hereditary algebra H, finite dimensional over a field K. We quote some general results from [BM+] about the factor of $\mathcal{D}^b(H)$ by a suitable automorphism.

Let G be an automorphism of $\mathcal{D}^b(H)$, satisfying conditions (g1) and (g2) of [BM+]:

(g1): For each $U \in \operatorname{ind}\mathcal{D}^b(H)$, only a finite number of G^nU lie in $\operatorname{ind} H$ for $n \in \mathbb{Z}$.

(g2): There is some $N \in \mathbb{N}$ such that $\{U[n] \mid U \in \operatorname{ind} H, n \in [-N,N]\}$ contains a system of representatives of the orbits of G on $\mathcal{D}^b(H)$.

$\mathcal{D}^b(H)/G$ denotes the corresponding factor category: the objects of $\mathcal{D}^b(H)/G$ are by definition G-orbits in $\mathcal{D}^b(H)$. Let X and Y be objects of $\mathcal{D}^b(H)$, and let $\tilde{X}$ and $\tilde{Y}$ be the corresponding objects in $\mathcal{D}^b(H)/G$. Then the morphisms in $\mathcal{D}^b(H)/G$ are given by:

$$\mathrm{Hom}_{\mathcal{D}^b(H)/G}(\tilde{X},\tilde{Y}) = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(H)}(G^iX, Y).$$

From [K] we know that $\mathcal{D}^b(H)/G$ is a triangulated category, and the canonical map from $\mathcal{D}^b(H)$ to $\mathcal{D}^b(H)/G$ is a triangle functor. It is shown in [BM+, Proposition 1.2] that $\mathcal{D}^b(H)/G$ is a Krull-Schmidt category.

G defines an automorphism ϕ of the AR quiver $\Gamma(\mathcal{D}^b(H))$; $\mathcal{D}^b(H)/G$ has almost split triangles, and the AR quiver $\Gamma(\mathcal{D}^b(H)/G)$ is $\Gamma(\mathcal{D}^b(H))/\phi$ [BM+, Proposition 1.3]

The shift [1] on $\mathcal{D}^b(H)$ passes to $\mathcal{D}^b(H)/G$; we use the same notation. Define $\mathrm{Ext}^i_{\mathcal{D}^b(H)/G}(\tilde{X},\tilde{Y}) = \mathrm{Hom}_{\mathcal{D}^b(H)/G}(\tilde{X},\tilde{Y}[i])$.

It is also straightforward to show [BM+, Proposition 1.4] that Serre duality in $\mathcal{D}^b(H)$ passes to $\mathcal{D}^b(H)/G$, so $\mathrm{Ext}^1_{\mathcal{D}^b(H)/G}(\tilde{X},\tilde{Y})$ is dual to $\mathrm{Hom}_{\mathcal{D}^b(H)/G}(\tilde{Y},\tau\tilde{X})$.

Cluster category

The cluster category is defined by $\mathcal{C}(H) = \mathcal{D}^b(H)/\tau^{-1}[1]$. Since $\tau^{-1}[1]$ is an automorphism satisfying conditions (g1) and (g2), $\mathcal{C}(H)$ is a triangulated Krull-Schmidt category.

We will mainly be concerned here with $\mathcal{C}(\Phi) = \mathcal{D}^b(\Phi)/\tau^{-1}[1]$. In [BM+], the connection between $\mathcal{C}(\Phi)$ and $\Delta(\Phi)$ is made via a reformulation of clusters in terms of *decorated representations* due to Marsh, Reineke, and Zelevinsky [MRZ]. We proceed in a different fashion, the basis of which is our Lemma 1 below.

Fix $Q = Q_{bip}$. Identify the indecomposables of $\mathcal{D}^b(\Phi)$ with the vertices of $\mathbb{Z}Q^{op}$. It is clear that the vertices of $\mathbb{Z}Q^{op}$ satisfying $2 \geq d_F(M) \geq -h+1$ are a fundamental domain for $\tau^{-1}[1]$. The representations of Q in coarse degree 0 correspond to indecomposables with fine degree between 0 and $-h+1$, so the fundamental domain we have identified for $\tau^{-1}[1]$ consists of the indecomposable representations of Q in coarse degree zero together with n extra indecomposables which correspond to the injective representations in coarse degree 1.

We wish to put these indecomposables in bijection with $\Phi_{\geq -1}$. Given a representation V of Q, its dimension is by definition $\dim(V) = \sum_i \dim_K(V_i)\alpha_i$. By Gabriel's Theorem, dim is a bijection from indecomposable representations of Q to Φ^+. We write $V(\beta)$ for the indecomposable representation in coarse degree zero whose dimension is β. We write P_i for the projective representation corresponding to vertex v_i, and we write I_i for the injective representation correspoding to vertex v_i. Observe that $\tau P_i = I_i[-1]$. We define $V(-\alpha_i)$ to be $I_i[-1]$.

Lemma 1. $V(R(\alpha)) = V(\alpha)[1]$.

Proof. On the representations of Q which do not lie in fine degree 0 or -1 (i.e. the indecomposable representations of Q which are not projective), $\tau_-\tau_+$ acts like a product of the corresponding reflection functors, and the product of the reflection functors coincides with τ [BB]. So $\tau_-\tau_+(V(\beta)) = \tau(V(\beta)) = V(\beta)[1]$, as desired.

Now consider the case that $V(\alpha)$ is projective. If $V(\alpha) = P_i$ is simple projective, then $i \in I_+$ and $\alpha = \alpha_i$, so $\tau_-\tau_+(\alpha) = -\alpha_i$. If $V(\alpha) = P_i$ is non-simple projective, then $i \in I_i$ and $\alpha = \alpha_i+$(the sum of the adjacent roots). Thus, again, $\tau_-\tau_+(\alpha) = -\alpha_i$. In both these cases, $V(R(\alpha)) = V(-\alpha_i) = I_i[-1] = \tau P_i = \tau V(\alpha) = V(\alpha)[1]$, as desired.

Finally we consider the case where $\alpha = -\alpha_i$. For $i \in I_+$, we know that $V(-\alpha_i)$ sits in fine degree 2. Now $\tau_-\tau_+(-\alpha_i) = \tau_+(\alpha_i) = \alpha_i+$(the sum of the roots adjacent to α_i). We recognize this as $\dim I_i$: in other words, $V(R(\alpha)) = I_i = V(\alpha)[1]$, as desired. For $i \in I_-$, the object $V(-\alpha_i)$ sits in fine degree 1. In this case

$\tau_-\tau_+(-\alpha_i) = \alpha_i$. Now $V(\alpha_i) = I_i$, so again $V(R(\alpha)) = \tau V(\alpha) = V(\alpha)[1]$. This completes the proof.

The connection between representation theory and clusters now appears strongly:

Proposition 1 [BM+]. $\dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(V(\beta), V(\alpha)) = (\beta \parallel \alpha)$

Proof. We check the two defining properties of compatibility degree given above.

$$\text{(i)} \quad \begin{aligned} \dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(V(\beta), V(-\alpha_i)) &= \dim_K \mathrm{Ext}^1_{\mathcal{C}(\Phi)}(V(\beta), I_i[-1]) \\ &= \dim_K \mathrm{Hom}_{\mathcal{C}(\Phi)}(I_i[-1], \tau V(\beta)) \\ &= \dim_K \mathrm{Hom}_{\mathcal{C}(\Phi)}(\tau^{-1} I_i[-1], V(\beta)) \\ &= \dim_K \mathrm{Hom}_{\mathcal{L}(\Phi)}(P_i, V(\beta)) \end{aligned}$$

(The first equality is because $V(-\alpha_i) = I_i[-1]$. The second equality is by Serre duality. The third follows because τ is an automorphism; the fourth from the fact that $\tau^{-1}I_i[-1] = P_i$.) Now $\dim_K \mathrm{Hom}_{\mathcal{L}(Q)}(P_i, V(\beta))$ is the coefficient of α_i in the simple root expansion of β, proving condition (i).

(ii) The invariance under R follows by Lemma 1 from fact that [1] is an auto-equivalence of $\mathcal{C}(\Phi)$.

Thus, the roots in a cluster correspond to a maximal collection of irreducible modules in $\mathcal{C}(\Phi)$ such that all the $\mathrm{Ext}^1_{\mathcal{C}(\Phi)}$'s between them vanish. This is exactly the definition of a *tilting set* for $\mathcal{C}$, so we have seen that tilting sets for $\mathcal{C}$ are in one-one correspondence with clusters for Φ.

m-Clusters

The m-clusters are a simplicial complex whose ground set, denoted $\Phi^m_{\geq -1}$, consists of the negative simple roots $-\Pi$ together with m copies of Φ^+. These m copies are referred to as having m different "colours" 1 through m. To keep track of the roots of different colour, we use superscripts. So β^i is the root β with colour i. Negative simple roots are considered to have colour 1.

Fomin and Reading define an m-ified rotation on $\Phi^m_{\geq -1}$:

$$R_m(\alpha^k) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi^+ \text{ and } k < m \\ R(\alpha)^1 & \text{otherwise} \end{cases}$$

Again, the crucial fact is that every root has at least one negative simple in its R_m-orbit.

We now follow [FR] in defining a relation called compatibility. (Strictly speaking, perhaps, we should call this m-compatibility, but no ambiguity will result, because this is a relation on $\Phi^m_{\geq -1}$, not $\Phi_{\geq -1}$.)

(i) $-\alpha_i$ is compatible with all negative simple roots and any positive root (of whatever colour) that does not use α_i in its simple root expansion.

(ii) α and β are compatible iff $R_m(\alpha)$ and $R_m(\beta)$ are compatible.

Because of the crucial fact mentioned above, this is sufficient to define compatibility, but not to prove that such a relation exists. This is verified in [FR], where the relation is also shown to be symmetric. As part of that proof, a more explicit definition of compatibility for $\Phi^m_{\geq -1}$ is given, relating it to compatibility for $\Phi_{\geq -1}$. We shall not need that definition here.

The m-cluster complex $\Delta_m(\Phi)$ is the simplicial complex on $\Phi^m_{\geq -1}$ whose faces are sets of pairwise compatible roots. The facets of the complex are called m-clusters.

m-Cluster Category

We define the m-cluster category to be $\mathcal{C}_m(\Phi) = \mathcal{D}^b(\Phi)/\tau^{-1}[m]$. This category is discussed in [K, Section 8.3], where it is shown to be triangulated. It is also being studied at present by A. Wraalsen.

We now identify the indecomposables of $\mathcal{C}_m(\Phi)$ with $\Phi^m_{\geq -1}$, as follows. For β^j a positive root in $\Phi^m_{\geq -1}$, let $W(\beta^j) = V(\beta)[j-1]$. Let $W(-\alpha_i) = I_i[-1]$. Observe that the set of $W(\beta^k)$ which we have identified are a fundamental domain with respect to $F = \tau^{-1}[m]$, and therefore they correspond in a 1-1 fashion to the indecomposables of $\mathcal{C}_m(\Phi)$.

We now prove the m-ified analogue of Lemma 1.

Lemma 2. $W(R_m(\beta^k)) = W(\beta^k)[1]$.

Proof. There are three cases to consider: firstly when $\beta^k = -\alpha_i$, secondly when β is a positive root and $k < m$, and thirdly when β is a positive root and $k = m$.

In the first case, $\beta^k = -\alpha_i$, and $R_m(-\alpha_i) = R(-\alpha_i)^1$. In this case $W(-\alpha_i) = I_i[-1]$, and by the proof of Lemma 1, $W(R(-\alpha_i)^1) = W(I_i)$, which proves the claim in this case.

In the second case (β a positive root and $k < m$), $R_m(\beta^k) = R_m(\beta^{k+1})$, and the desired result follows by the definition of W.

In the third case, $R_m(\beta^m) = R(\beta)^1$. By the proof of Lemma 1, $W(R_m(\beta^m)) = \tau(V(\beta)) = V(\beta)[m] = W(\beta^m)[1]$, as desired.

We now prove an m-ified analogue of Proposition 1. Here, we consider only compatibility, not compatibility degree, as [FR] does not define compatibility degree in the m-cluster context. (We shall give a definition of an m-compatibility degree below.)

Proposition 2. *A pair of coloured roots β^j and γ^k are compatible in $\Phi^m_{\geq -1}$ iff $\mathrm{Ext}^i_{\mathcal{C}_m(\Phi)}(W(\beta^j), W(\gamma^k)) = 0$ for $i = 1 \dots m$.*

Proof. The proof is analogous to the proof of Proposition 1: We check the two conditions of the definition of compatibility in $\Phi^m_{\geq -1}$.

(i)
$$\begin{aligned}
\dim_K \mathrm{Ext}^j_{\mathcal{C}_m(\Phi)}(W(\beta^k), W(-\alpha_i)) &= \dim_K \mathrm{Ext}^j_{\mathcal{C}_m(\Phi)}(W(\beta^k), I_i[-1]) \\
&= \dim_K \mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(W(\beta^k), I_i[j-2]) \\
&= \dim_K \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(I_i[j-2], \tau W(\beta^k)) \\
&= \dim_K \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(\tau^{-1} I_i[j-2], W(\beta^k)) \\
&= \dim_K \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(P_i[j-1], W(\beta^k)).
\end{aligned}$$

(The second equality is from the definition of Ext^i; the third is from Serre duality; the fourth follows because τ is an auto-equivalence.) If $k \neq j$, then $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(P_i[j-1], W(\beta^k))$ is zero. If $k = j$, it is the coefficient of α_i in the root expansion of β. This implies that β^k is incompatible with $-\alpha_i$ iff α_i appears with positive coefficient in the simple root expansion of β.

(ii) This follows from Lemma 2 and the fact that [1] is an auto-equivalence of $\mathcal{C}_m(\Phi)$.

We define an *m-tilting set* in $\mathcal{C}_m(\Phi)$ to be a maximal set of indecomposables $\mathcal{S}$ satisfying $\mathrm{Ext}^i(X, Y) = 0$ for all $X, Y \in \mathcal{S}$ and $i = 1 \dots m$. The following theorem is an immediate consequence of Proposition 2:

Theorem 1. *The map W induces a bijection from m-clusters of Φ to m-tilting sets of $\mathcal{C}(\Phi)$.*

m-Compatibility degree

As already remarked, no analogue of compatibility degree is defined for m-clusters. However, it is easy to make such a definition.

First, we prove the following analogue of Proposition 1.7(b) of [BM+]:

Lemma 3. *If $X, Y \in \mathcal{C}_m(\Phi)$, then $\dim_K \operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(X,Y) = \dim_K \operatorname{Ext}^{m+1-i}_{\mathcal{C}_m(\Phi)}(Y,X)$.*

Proof. This is essentially (a slightly naive version of) the Calabi-Yau condition of dimension $m+1$, proved for $\mathcal{C}_m(\Phi)$ by Keller in [K, Section 8.3]. Observe that $\dim_K \operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(X,Y) = \dim_K \operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(X, Y[i-1]) = \dim_K \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(\tau^{-1}Y[i-1], X) = \dim_K \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(Y[i-1-m], X) = \dim_K \operatorname{Ext}^{m+1-i}_{\mathcal{C}_m(\Phi)}(Y,X)$. The second equality follows by Serre duality, and the third because we are in $\mathcal{D}^b(\Phi)/\tau^{-1}[m]$.

Define the m-compatibility degree between two coloured roots α^j and β^k by

$$(\alpha^j \parallel \beta^k) = \sum_{i=1}^{m} \dim_K \operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(W(\alpha^j), W(\beta^k)).$$

Lemma 4. *(a) The m-compatibility degree $(\alpha^j \parallel \beta^k)$ satisfies*

(i) $(R_m(\alpha^j) \parallel R_m(\beta^k)) = (\alpha^j \parallel \beta^k)$

(ii) $(-\alpha_i^1 \parallel \beta^k)$ is the coefficient of α_i in the simple root expansion of β.

(b) Properties (i) and (ii) suffice to determine the m-compatibility degree of any two coloured roots.

(c) m-compatibility degree is symmetric.

(d) α^j, β^k are compatible iff $(\alpha^j \parallel \beta^k) = 0$.

Proof. Part (a) follows from Lemma 2. Part (b) follows from the fact that any coloured root in $\Phi^m_{\geq -1}$ has a negative simple root in its m-orbit. Part (c) follows from Lemma 3. Part (d) is clear.

Combinatorics of m-clusters

The following result is proved in [FR] on a type-by-type basis, with a computer check for the exceptionals. We will give a type-free proof.

Proposition 3 [FR, Theorem 2.7]. *If α and β are roots of Φ contained in a parabolic root system Ψ within Φ, then $\alpha^{(i)}$ and $\beta^{(j)}$ are compatible in $\Phi^m_{\geq -1}$ iff they are compatible in $\Psi^m_{\geq -1}$.*

Proof. Let $\bar{X}$ and $\bar{Y}$ be indecomposables of $\mathcal{C}_m(\Phi)$ corresponding to $\alpha^{(i)}$ and $\beta^{(j)}$ respectively. Let X and Y be corresponding indecomposables of $\mathcal{D}^b(\Phi)$, chosen so that $2 \geq d_F(X), d_F(Y) \geq -mh+1$. Without loss of generality, we may assume that $d_F(X) \geq d_F(Y)$.

Suppose that $\alpha^{(i)}$ and $\beta^{(j)}$ are not compatible in $\Phi^m_{\geq -1}$. So there is some $1 \leq k \leq m$ with $\operatorname{Ext}^k_{\mathcal{C}_m(\Phi)}(\bar{X}, \bar{Y}) \neq 0$. This asserts that there is a non-zero morphism in $\mathcal{D}^b(\Phi)$ from some G^pX to $Y[k]$, where $G = \tau^{-1}[m]$ and p is some integer. Since $d_F(X) \geq d_F(Y)$, so $d_F(X) - d_F(Y[k]) \geq h$, it follows that p must be strictly positive. On the other hand, $d_F(Y[k]) > d_F(G^pX)$ for $p \geq 2$. So $p = 1$, and $\mathrm{Hom}_{\mathcal{D}^b(\Phi)}(\tau^{-1}X[m], Y[k]) \neq 0$. By Serre duality, $\mathrm{Ext}^1_{\mathcal{D}^b(\Phi)}(Y[k], X[m]) \neq 0$, so $\mathrm{Ext}^{m+1-k}_{\mathcal{D}^b(\Phi)}(Y, X) \neq 0$. The crucial point here is that we know this statement on the level of the derived category, rather than just the m-cluster category.

Let Q' be the subquiver of Q corresponding to Ψ. There is a natural inclusion of $\mathcal{L}(Q')$ into $\mathcal{L}(Q)$ as a full subcategory, which extends to an inclusion of $\mathcal{D}^b(Q')$ into $\mathcal{D}^b(Q)$ as a full triangulated subcategory, where the inclusion respects the coarse grading. X and Y represent $\alpha^{(i)}$ and $\beta^{(j)}$ respectively in both $\mathcal{C}_m(\Phi)$ and $\mathcal{C}_m(\Psi)$. Thus, the non-vanishing Ext that we have shown exists in $\mathcal{D}^b(\Phi)$ also exists in $\mathcal{D}^b(\Psi)$, and testifies that $\alpha^{(i)}$ and $\beta^{(j)}$ are not compatible in $\Psi^m_{\geq -1}$ either.

The converse is proved similarly.

The following two results are proved by inductive arguments in [FR], relying on the above proposition.

Proposition 4 [FR, Theorem 2.9]. *All the facets of $\Delta_m(\Phi)$ are of size n.*

Proposition 5 [FR, Proposition 2.10]. *Given a set $\mathcal{T}$ of $n-1$ pairwise compatible roots from $\Phi^m_{\geq -1}$, there are exactly $m+1$ roots not in $\mathcal{T}$ which are compatible with all the roots of $\mathcal{T}$. (In other words, every codimension 1 face of $\Delta_m(\Phi)$ is contained in exactly $m+1$ facets.)*

These two propositions can be rephrased in our terms as follows:

Proposition $4'$**.** *Any m-tilting set in $\mathcal{C}_m(\Phi)$ has n elements, where n is the rank of Φ.*

An *almost m-tilting set* is a set $\mathcal{T}$ of $n-1$ indecomposables in $\mathcal{C}_m(\Phi)$ such that $\mathrm{Ext}^i(X, Y) = 0$ for all $X, Y \in \mathcal{T}$ and $i = 1 \dots m$.

Proposition $5'$**.** *Any almost m-tilting set is contained in exactly $m+1$ m-tilting sets.*

The proofs of these propositions go through exactly as in [FR]. We include the proofs for completeness.

Proof of Proposition $4'$*.* The proof is by induction on n. The statement is clear when $n = 1$. Let $\mathcal{S}$ be a tilting set. Pick X an indecomposable in $\mathcal{S}$. Applying τ if necessary, we may assume that X is of the form $I_i[-1]$ for some i. Let Q' be the quiver Q with the vertex i removed, and let Ψ be the associated subroot system. For each indecomposable $Y \in \mathcal{S} \setminus \{X\}$, choose a representative $\hat{Y}$ in $\mathcal{D}^b(Q)$ with fine degree between 2 and $-hm+1$ (in other words, $\hat{Y}$ is either of the form $I_j[-1]$ for $j \neq i$ or in $\mathcal{L}(Q)[k]$ for some $0 \leq k \leq m-1$.

Since Y is compatible with X, $\mathrm{Ext}^j_{\mathcal{C}_m(\Phi)}(I_i[-1], Y) = 0$ for all $1 \leq j \leq m$. By Serre duality, this is equivalent to the condition that $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(P_i[m-j], Y) = 0$ for all $1 \leq j \leq m$, or, in other words, that, if $\hat{Y} \in \mathcal{L}(Q)[k]$, that in fact $\hat{Y} \in \mathcal{L}(Q')[k]$. Thus, by Proposition 3, the images of the $\hat{Y}$ form a tilting set in $\mathcal{C}_m(\Psi)$, so $\mathcal{S} \setminus \{X\}$ contains $n-1$ indecomposables by induction, and thus $\mathcal{S}$ contains n indecomposables.

Proof of Proposition $5'$*.* The proof is, again, by induction on n. The base case, when $n = 1$, is clear. For the induction step, let $\mathcal{T}$ be an almost tilting set. As before, we choose an indecomposable in X in $\mathcal{T}$, which we may assume is of the form $I_i[-1]$, and then we observe that $\mathcal{T} \setminus \{X\}$ consists of an almost tilting set for a root system of rank $n-1$, and the $m+1$ complements for that almost-tilting set are precisely the complements of X.

Iyama [I] has announced some results which are similar in spirit to Proposition $5'$, and in some respects more general, but the exact connection between his results and Proposition $5'$ is not yet clear.

Technicalities

We collect here a couple of simple results to do with $\mathcal{C}_m(\Phi)$. Recall that h is the Coxeter number for Φ.

In $\mathcal{C}_m(\Phi)$, it is natural to consider fine degree cyclically, that is to say, modulo $N = mh + 2$. Write $[i, j]_N$ for the set $\{i, i+1, \ldots, j\}$ of remainders modulo N

Lemma 5. *For any indecomposable X of $\mathcal{C}_m$, if $\mathrm{Ext}^i(X, Y) \neq 0$ then $d_F(Y) \in [d_F(X) + (i-1)h + 2, d_F(X) + ih]_N$.*

Proof. By applying τ, we can assume that X is projective. The result is clear in this case.

The following lemma describes a fact which is particular to m-clusters with $m \geq 2$. The subsequent proofs which are not essentially restatements of proofs from [BM+] generally rely on this fact.

Lemma 6. *Provided $m \geq 2$, if X and Y are indecomposables and there is a nonzero map from X to Y and from Y to X, then X and Y are isomorphic, and these maps are isomorphisms.*

Proof. Because $m \geq 2$, when we consider the ranges of fine degrees in which $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(X, \cdot)$ and $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(\cdot, X)$ are supported, any Y admitting morphisms to and from X would have to be in the same fine degree as X. But since all morphisms except multiples of the identity morphism connect indecomposables in different degrees, the claim follows.

Constructing complements

If $\mathcal{T}$ is an almost m-tilting set, an indecomposable $M \in \mathcal{C}_m(\Phi)$ is called a complement for $\mathcal{T}$ if $\mathcal{T} \cup \{M\}$ is an m-tilting set.

Suppose M is a complement for an almost m-tilting set $\mathcal{T}$. We would like to construct the other complements. Our procedure here is to mimic Section 6 of [BM+].

We begin by recalling the theory of approximation, which originates in [AS]. Let $\mathcal{E}$ be a category and χ a full additive subcategory, closed under summands and isomorphisms. For Y an object of χ, a morphism $Y \to E$ is a called a *right χ approximation* to E if the induced map $\mathrm{Hom}_{\mathcal{E}}(X, Y) \to \mathrm{Hom}_{\mathcal{E}}(X, E)$ is surjective for X any object of χ.

If $\mathcal{E} = \mathrm{mod}\, H$ for H a finite-dimensional algebra and χ has a finite number of non-isomorphic indecomposables, then right approximations always exist [AS, Proposition 4.2]. We will want to take $\mathcal{E} = \mathcal{C}_m(\Phi)$; this does not introduce any additional complications. If E is an indecomposable of $\mathcal{C}_m(\Phi)$, after applying τ we may assume that E is an injective representation of Q in coarse degree zero. Then the only indecomposables of $\mathcal{C}_m(\Phi)$ which have non-zero homomorphisms into E are representations of Q in coarse degree zero, and the result from [AS] applies.

A map $f : F \to E$ is called *right minimal* if any map $g : F \to F$ which satisfies $fg = f$ must be an isomorphism. If $f : F \to E$ is not right minimal, then there is a direct summand F' of F on which f is zero. Further, there is a decomposition such that $F \cong F' \coprod F''$ such that $f|_{F'} = 0$ and $f'' = f|_{F''}$ is right minimal [AS, Proposition 1.2]. If $f : F \to E$ is a right approximation to E, so is $f'' : F'' \to E$. Thus $f'' : F'' \to E$ is simultaneously a right approximation and right minimal. Such a morphism is called a right minimal approximation. There are dual notions of left approximation and left minimality.

We write $\operatorname{add}(\mathcal{T})$ for the full additive subcategory of $\mathcal{C}_m(\Phi)$ generated by the indecomposables in $\mathcal{T}$. We write T for the direct sum of the indecomposables from $\mathcal{T}$.

Let M_0 be a complement for the almost-tilting set $\mathcal{T}$. Let $B_0 \to M_0$ be a minimal right $\operatorname{add}(\mathcal{T})$ approximation in $\mathcal{C}_m(\Phi)$. Complete this map to a triangle.

$$M_1 \to B_0 \to M_0 \to M_1[1]. \tag{2}$$

We wish to show that M_1 is also a complement of $\mathcal{T}$. The next four lemmas are analogues of Lemmas 6.3, 6.4, 6.5, 6.6 of [BM+], and have essentially the same proofs.

Lemma 7. $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(M_1, T) = 0 = \operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(T, M_1)$ *for* $1 \leq i \leq m$.

Proof. Apply $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(T, \cdot)$ to the triangle (1). We obtain a long exact sequence:

$$\begin{aligned}
&\to \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(T, B_0) \to \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(T, M_0)\\
\to \operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(T, M_1) &\to \operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(T, B_0) \to \operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(T, M_0)\\
\to \operatorname{Ext}^2_{\mathcal{C}_m(\Phi)}(T, M_1) &\to \operatorname{Ext}^2_{\mathcal{C}_m(\Phi)}(T, B_0) \to \operatorname{Ext}^2_{\mathcal{C}_m(\Phi)}(T, M_0)\\
&\dots\\
\to \operatorname{Ext}^m_{\mathcal{C}_m(\Phi)}(T, M_1) &\to \operatorname{Ext}^m_{\mathcal{C}_m(\Phi)}(T, B_0) \to
\end{aligned}$$

Recall that $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(T, B_0) = 0$ and $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(T, M_0) = 0$ for $1 \leq i \leq m$. From this, $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(T, M_1) = 0$ for $2 \leq i \leq m$ is immediate. Now remark that, since B_0 is a minimal right approximation of M_0, the map from $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(T, B_0)$ to $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(T, M_0)$ is surjective. Thus, $\operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(T, M_1) = 0$.

The vanishing of $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(T, M_1)$ now follows by the Calabi-Yau condition (Lemma 3).

Lemma 8. *The map* $M_1 \to B_0$ *is a minimal left* $\operatorname{add}(\mathcal{T})$ *approximation.*

Proof. Apply $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(\cdot, T)$ to the triangle (1), to get the long exact sequence

$$\to \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(B_0, T) \to \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(M_1, T) \to \operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(M_0, T) \to .$$

Since M_0 is a complement for $\mathcal{T}$, $\operatorname{Ext}^1_{\mathcal{C}_m(\Phi)}(M_0, T) = 0$. It therefore follows that the map $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(B_0, T) \to \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(M_0, T)$ is surjective, so B_0 is a left approximation. It remains to check that it is minimal. Suppose, on the contrary, that there is a direct summand B_0' of B_0 such that $0 \to B_0'$ splits off. But then there would be a B_0' summand of M_0. Since M_0 is indecomposable, M_0 would have to be isomorphic to B_0'. But then M_0 would be contained in $\operatorname{add}(\mathcal{T})$, which contradicts the assumption that M_0 is a complement to $\mathcal{T}$.

Lemma 9. *M_1 is indecomposable.*

Proof. The proof is exactly as in [BM+]. Suppose $M_1 \cong U' \coprod U''$. Then take minimal left $\mathrm{add}(\mathcal{T})$ approximations $U' \to B_0'$ and $U'' \to B_0''$, and complete to triangles:

$$U' \to B_0' \to M_0' \to U'[1]$$
$$U'' \to B_0'' \to M_0'' \to U''[1]$$

The direct sum of these two triangles is the triangle (2). Since M_0 is indecomposable, one of M_0', M_0'' is zero. Without loss of generality $M_0' = 0$. But then $B_0' \to 0$ is a direct summand of $B_0 \to M_0$, contradicting the minimality of $B_0 \to M_0$.

Lemma 10. *M_1 is not in* $\mathrm{add}(\mathcal{T})$*.*

Proof. Again, the proof is exactly as in [BM+]. If M_1 were in $\mathrm{add}(\mathcal{T})$, then $M_1 \to B_0$ (being a left approximation) would be an isomorphism, and thus M_0 would be zero, a contradiction.

To show that $\mathcal{T} \cup \{M_1\}$ is an m-tilting set, it only remains for us to show that $\mathrm{Ext}^i_{\mathcal{C}(\Phi)}(M_1, M_1) = 0$ for $1 \le i \le m$. This is the content of Lemma 6.7 in [BM+], but since we are in the Dynkin case, this is true for any indecomposable representation. Thus M_1 is a complement for $\mathcal{T}$.

Repeat this procedure to define $M_2, M_3, \dots$. By induction, each of these is a complement for $\mathcal{T}$. Suppose that M_0 is in fine degree d. It follows that the fine degree of M_1 is no more than h greater than d. Similarly, the fine degree of M_2 is no more than h greater than the fine degree of M_1. It follows that $M_0, \dots, M_m$ must all be distinct.

By Proposition $5'$, this is the complete list. This completes the proof of the theorem:

Theorem 2. *Let $\mathcal{T}$ be an almost m-tilting set, and let M_0 be a complement for $\mathcal{T}$. Define the M_i as above. Then $M_{m+1} = M_0$, and the set of M_i for $i = 0, \dots, m$ is the complete set of complements of $\mathcal{T}$. Also, B_i is the left* $\mathrm{add}(\mathcal{T})$ *approximation to M_{i+1}.*

We now prove:

Theorem 3. *If $\mathcal{T}$ is an almost m-tilting set, M_i its complements, and B_i the right* $\mathrm{add}(\mathcal{T})$ *approximation to M_i, then the sets of indecomposables of B_i, $i = 0, \dots, m$ are disjoint.*

Proof. We prove this theorem under the assumption that $m > 1$. (The $m = 1$ case is dealt with in [BMR].) The proof is simpler in our situation, because Lemma 6 is available.

The fine degrees of indecomposables in B_i must lie in $(d(M_i), d(M_{i+1}))_N$. These intervals are all disjoint. Thus the indecomposables of each of the different B_i are distinct.

Complete sets of complements

We say that a set of indecomposables is a complete set of complements if it consists of the $m+1$ indecomposables which are the complements to some almost

m-tilting set. We have already seen that the elements of a complete set of indecomposables can be assigned an order so that M_{i+1} is obtained as the third term of the triangle whose other two terms are M_i and its minimal right $\mathrm{add}(\mathcal{T})$ approximation.

The next statement is the analogue of Proposition 6.14 of [BM+]. The proof is a version of their proof, simpler because we are in the Dynkin case.

Proposition 6. $\dim_K \mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1}) = 1$.

Proof. Applying $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_i, \cdot)$ to the triangle

$$M_{i+1} \to B_i \to M_i \to M_{i+1}[1],$$

we obtain

$$\begin{aligned}&\to \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_i, B_i) \to \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_i, M_i) \to \mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1})\\ &\to \mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, B_i).\end{aligned}$$

Now $\mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, B_i) = 0$ because B is composed of indecomposables from $\mathcal{T}$, so the dimension of $\mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1})$ is at most the dimension of $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_i, M_i)$, which is 1. We know that $\mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1}) \neq 0$, because we have already constructed a nontrivial extension. So $\dim_K \mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1}) = 1$.

Theorem 4. *If $\mathcal{T}$ is an almost m-tilting set, with complements M_i as above, then:*

$$\textit{For } 1 \leq i \leq m,\ \dim_K \mathrm{Ext}^i_{\mathcal{C}_m(\Phi)}(M_j, M_k) = \begin{cases} 1 & \textit{iff } k - j = i \bmod m+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Without loss of generality, assume that $j = 0$. We will induct on k. The case $k = 1$ follows from Proposition 6, once we observe (by Lemma 5 above) that since M_0 and M_1 are both indecomposable, at most one of the $\mathrm{Ext}^i_{\mathcal{C}_m(\Phi)}(M_0, M_1)$ can be nonvanishing.

So assume that the statement holds for $k-1$. Now apply $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_0, \cdot)$ to

$$M_k \to B_{k-1} \to M_{k-1} \to M_k[1],$$

obtaining

$$\begin{aligned}0 = \mathrm{Ext}^{k-1}_{\mathcal{C}_m(\Phi)}(M_0, B_{k-1}) &\to \mathrm{Ext}^{k-1}_{\mathcal{C}_m(\Phi)}(M_0, M_{k-1})\\ &\to \mathrm{Ext}^{k}_{\mathcal{C}_m(\Phi)}(M_0, M_k) \to \mathrm{Ext}^{k}_{\mathcal{C}_m(\Phi)}(M_0, B_{k-1}) = 0,\end{aligned}$$

where the ends vanish because B_{k-1} is composed of irreducibles from $\mathcal{T}$. This shows that $\dim_K \mathrm{Ext}^k_{\mathcal{C}_m(\Phi)}(M_0, M_k) = 1$, and then the vanishing of the other Ext^i follows from Lemma 5 as above.

One way to think about all these extensions at the same time is to observe that the nonvanishing of $\mathrm{Ext}^1_{\mathcal{C}_m(\Phi)}(M_i, M_{i+1})$ gives rise to a map $f_i : M_i \to M_{i+1}[1]$. Putting all of these together, we get a sequence of maps

$$M_0 \to M_1[1] \to M_2[2] \to \cdots \to M_m[m] \to M_0[m+1] = \tau M_0[1].$$

Proposition 7. *The composition $f_m \circ \cdots \circ f_0 : M_0 \to \tau M_0[1]$ is nonzero.*

Proof. By the proof of Theorem 4, we know that the map from M_0 to $M_m[m]$ is nonzero. Now apply $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_0, \cdot)$ to

$$M_0 \to B_m \to M_m \to M_0[1].$$

We get

$$\begin{aligned} 0 \to \mathrm{Ext}^m_{\mathcal{C}_m(\Phi)}(M_0, B_m) \to & \mathrm{Ext}^m_{\mathcal{C}_m(\Phi)}(M_0, M_m) \\ & \to \mathrm{Ext}^{m+1}_{\mathcal{C}_m(\Phi)}(M_0, M_0) = \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_0, \tau M_0[1]), \end{aligned}$$

so the nonzero map from $M_0 \to M_m[m]$ corresponds to the nonzero class in $\mathrm{Ext}^m_{\mathcal{C}_m(\Phi)}(M_0, M_m)$ which injects into $\mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_0, \tau M_0[1])$, implying that the map from M_0 to $\tau M_0[1]$ is nonzero.

Corollary. *The non-zero class in $\mathrm{Ext}^{j-i}_{\mathcal{C}_m(\Phi)}(M_j, M_i) = \mathrm{Hom}_{\mathcal{C}_m(\Phi)}(M_j, M_i[j-i])$ is given by $f_{j-1} \circ \ldots f_i$.*

Notice that by applying τ if necessary, we can make M_0 a projective representation in coarse degree zero, and then $\tau M_0[1]$ is the corresponding injective representation in coarse degree zero, and all the $M_i[i]$ are also representations in coarse degree zero.

Theorem 5 is a converse to Theorem 4:

Theorem 5. *Given a set of M_i ($i = 0 \ldots m$) satisfying the conclusion of Theorem 4, form B_i as the uniquely determined non-trivial extension of M_i by M_{i+1}. Then there exists an almost m-tilting set $\mathcal{T}$ such that $\{M_i\}$ are the set of complements for $\mathcal{T}$.*

Proof. Let V be the direct sum of the B_i. Let f_i be the induced map $M_i \to M_{i+1}[1]$. The f_i are nonzero. In fact, we can say more:

Lemma 11. *The composition $f_j \circ \cdots \circ f_{i+1} \circ f_i \neq 0$ (where we understand the subscripts cyclically, and we do not take a composition of more than $m+1$ functions).*

Remark. In proving the analogue of this lemma for classical clusters, there is only a single composition to consider, and it is nonvanishing by Serre duality. It is not clear to us that there is any equally general argument for m-clusters, so, if one attempts to generalize this theory outside the Dynkin context, one may need to include the result of this lemma in the hypotheses of the generalization of Theorem 5. However, since at present we are only considering the Dynkin case, no further assumptions are needed.

Proof. Without loss of generality, we may assume that $i = 0$. By applying automorphisms of $\mathcal{C}_m(\Phi)$ if necessary (and possibly interchanging I_+ and I_-), we may assume that M_0 is a simple projective representation, corresponding to vertex v_k of the quiver Q. Write X_t for the representation of Q corresponding to $M_t[t]$. The fact that $\dim_K \mathrm{Ext}^t_{\mathcal{C}_m(\Phi)}(M_0, M_t) = 1$ implies that the coefficient of α_k in $\dim(X_t)$ is 1 for all $1 \leq t \leq m$.

Write $\langle \cdot, \cdot \rangle$ for the Euler form on $\mathbb{Z}\Pi$ (the root lattice). If the homomorphism from X_t to X_{t+1} kills the copy of M_0 inside X_t, then it induces a map from X_t/M_0 to X_{t+1}. So $\langle \dim(X_t) - \dim(M_0), \dim(X_{t+1}) \rangle \geq 1$. Thus $\langle \dim(X_t), \dim(X_{t+1}) \rangle \geq 2$, contradicting the facts which we know, that $\dim_K \operatorname{Hom}_{\mathcal{L}(Q)}(X_t, X_{t+1}) = 1$ and also $\dim_K \operatorname{Ext}^1_{\mathcal{L}(Q)}(X_t, X_{t+1}) = 0$.

This implies that the map from M_0 to $\tau M_0[1]$ is nonzero, proving the lemma.

Lemma 12. $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(V \coprod M_j, V \coprod M_j) = 0$ *for* $i = 1 \dots m$.

Proof. The vanishing of $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(M_j, B_k)$ now follows from applying $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(M_j, \cdot)$ to

$$M_{k+1} \to B_k \to M_k \to M_{k+1}[1].$$

The vanishing of $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(B_k, M_j)$ now follows from the Calabi-Yau condition (Lemma 3). Since we are in the Dynkin case, $\operatorname{Ext}^i(M_j, M_j) = 0$ is automatic. Finally, we apply $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(B_k, \cdot)$ to

$$M_{j+1} \to B_j \to M_j \to M_{j+1}[1]$$

to deduce that $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(B_k, B_j) = 0$.

Thus, the indecomposables of V are a partial m-tilting set, which can be extended to an m-tilting set $\mathcal{S}$. We wish to show that $\mathcal{S}$ necessarily contains some M_j, and that the set of all the M_i is a complete set of complements for $\mathcal{S} \setminus \{M_j\}$.

Suppose $M_k \notin \mathcal{S}$; we wish to show that there is necessarily some j such that M_j is in $\mathcal{S}$. Without loss of generality, we may assume that $k = 0$. Since $M_0 \cup \mathcal{S}$ is not m-tilting, there is some $N \in \mathcal{S}$ such that $\operatorname{Ext}^i(M_0, N) \neq 0$ for some $1 \leq i \leq m$.

Now consider:

$$M_1 \to B_0 \to M_0 \to M_1[1].$$

Apply $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(\cdot, N)$. Since $\operatorname{Ext}^i_{\mathcal{C}_m(\Phi)}(B_0, N) = 0 = \operatorname{Ext}^{i-1}_{\mathcal{C}_m(\Phi)}(B_0, N)$, we deduce that $\operatorname{Ext}^{i-1}(M_1, N) \neq 0$. Repeating this procedure, we eventually deduce that $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(M_i, N) \neq 0$.

Now, start with

$$M_0 \to B_m \to M_m \to M_0[1].$$

Apply $\operatorname{Hom}_{\mathcal{C}_m(\Phi)}(\cdot, N)$. We deduce that $\operatorname{Ext}^{i+1}(M_m, N) \neq 0$. Repeating this procedure, we eventually deduce that $\operatorname{Ext}^{m+1}(M_i, N) \neq 0$. But $\operatorname{Ext}^{m+1}_{\mathcal{C}_m(\Phi)}(M_i, N) = \operatorname{Hom}_{\mathcal{C}_m(\Phi)}(N, M_i)$. So we have concluded that there are nonzero maps from N to M_i and vice versa. By Lemma 6, it follows that $N = M_i$, so $\mathcal{S}$ contains M_i.

Further, by the proof of Lemma 12, if M_j has a non-vanishing $\operatorname{Ext}^k_{\mathcal{C}_m(\Phi)}$ (for some $1 \leq k \leq m$) with an indecomposable of $\mathcal{S}$, that indecomposable must be M_i, so if we write $\mathcal{T}$ for $\mathcal{S} \setminus \{M_i\}$, there are no non-vanishing $\operatorname{Ext}^k_{\mathcal{C}_m(\Phi)}$ among $\mathcal{T} \cup \{M_j\}$, so $\mathcal{T} \cup \{M_j\}$ is an m-tilting set, and thus $\{M_j\}$ for $0 \leq j \leq m$ form a complete set of complements for $\mathcal{T}$, as desired. This completes the proof of Theorem 5.

Combinatorial Consequences

One of the goals of this paper is to establish the beginning of an algebraic approach to m-clusters. The other goal is to draw some directly combinatorial conclusions about m-clusters which do not seem amenable to a more direct argument.

Theorem 6. *Let $\mathcal{T}$ be a codimension 1 face of $\Delta_m(\Phi)$. Then there is a complement to $\mathcal{T}$ of each colour.*

Proof. Let $\{M_i\}$ be the complements to $\mathcal{T}$. By Lemma 5, if M_i has colour j, then M_{i+1} has colour j or $j+1$ (interpreted cyclically). Since after $m+1$ steps we must make it all the way around the circle, there must be an M_i of each colour.

This theorem has some further consequences related to the topology of $\Delta_m(\Phi)$ which will be worked out in a subsequent paper.

Interpretation in type A

As described in [FZ], clusters in type A_n correspond to triangulations of a convex $(n+3)$-gon, where by triangulation we mean a subdivision into triangles by straight lines connecting vertices of the $(n+3)$-gon which do not cross in the polygon's interior. More precisely, there is an almost positive root associated to each chord of the $(n+3)$-gon. Two roots are compatible iff their corresponding chords do not cross in the interior of the polygon. Thus, clusters correspond to maximal sets of non-crossing chords, and it is easy to see that maximal sets of non-crossing chords are nothing but triangulations.

Now we describe the generalization to m-clusters from [FR]. Let $N = m(n+1)+2$. We say that an $(m+2)$-angulation of a convex N-gon is a subdivision of the N-gon by straight lines connecting vertices, which do not cross in the interior of the N-gon, such that each of the regions defined by the subdivision is an $(m+2)$-gon.

Notice that only certain, *allowable* chords of the N-gon can ever appear as a chord in an $(m+2)$-angulation: for a chord to be allowable, the two regions into which the chord subdivides the N-gon must both be polygons whose number of vertices is congruent to 2 modulo m.

There is a labelling of allowable chords of the N-gon by $\Phi^m_{\geq -1}$. The description from [FR] is as follows. Label the Dynkin diagram by $\{1,\dots,n\}$ in the standard (linear) fashion, and let I_+ be the odd indices. Label the vertices of the N-gon $1,\dots,N$ in counter-clockwise order. Label the chord connecting $(i-1)m+1$ to $(n+1-i)m+2$ by $-\alpha_{2i-1}$. Label the chord connecting $im+1$ to $(n+1-i)m+2$ by $-\alpha_{2i}$. These chords form the *snake*. Now, for any positive root $\beta = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$, there are m chords which cross the chords corresponding to $-\alpha_i,\dots,-\alpha_j$ and no other chords from the snake. These chords will consist of an initial chord C together with its rotations $R^i_m(C)$ for $1 \leq i \leq m-1$. Label $R^i_m(C)$ by β^{i+1}.

For an allowable chord C, we shall denote the corresponding indecomposable in $\mathcal{C}_m(\Phi)$ by M_C. As in the classical cluster case, two roots are compatible iff the corresponding chords do not cross in the interior of the N-gon. We shall also need the fact that R_m acts on the N-gon by a rotation moving each vertex to its clockwise neighbour.

We know that two chords C, D are non-crossing iff the corresponding indecomposables M_C, M_D have $\mathrm{Ext}^i(M_C, M_D) = 0$ for $1 \leq i \leq m$. Let C be an allowable chord with endpoints $c_1 < c_2$, D an allowable chord with endpoints $d_1 < d_2$. Suppose that C and D are crossing, so either $c_1 < d_1 < c_2 < d_2$ or $d_1 < c_1 < d_2 < c_2$. Observe that the clockwise distance from c_1 to d_1 and the clockwise distance from c_2 to d_2 must be equal mod m. Denote this distance (expressed as a number between 1 and m) by $d(C, D)$. Note that $d(C, D) + d(D, C) = m + 1$.

Proposition 8. *C and D are crossing with $d(C,D) = j$ iff $\mathrm{Ext}^j_{\mathcal{C}_m(\Phi)}(M_D, M_C) \neq 0$.*

Proof. Since both conditions are preserved under the action of R_m, it suffices to check that the conditions are equivalent if $M_D = W(-\alpha_i)$. Now it is a straightforward application of the definition of the correspondence between chords and $\Phi^m_{\geq -1}$.

An almost tilting set corresponds to a set of chords which is an $(m+2)$-angulation with one chord missing. This missing chord leaves a "hole" which is a $(2m+2)$-gon, and the set of complements are the $m+1$ diameters of this hole. By Proposition 8, we know that if we number the diameters in counter-clockwise order, we will have found our standard order on a complete set of complements.

By Theorem 5, the B_i are irreducibles corresponding to chords which must be present in any $(m+2)$-angulation for which the M_i are a complete set of complements. This shows that the B_i must be (among) the chords of the circumference of the hole. In fact, they are all the chords on the circumference of the hole (excluding any edges of the N-gon, which have no roots associated to them).

Proposition 9. *For $m \geq 2$, B_i is the sum of the irreducibles corresponding to the allowable chords joining an endpoint of M_i to an endpoint of M_{i+1}. (There will be at most two of these, and fewer if some of the endpoints are adjacent vertices of the N-gon.)*

Proof. Again, the proof that the indecomposables of the B_i must be *among* the indecomposables corresponding to these representations is very easy: B_i is determined once we know M_i and M_{i+1}, so it must correspond to chords whose corresponding roots are guaranteed to appear in $\mathcal{T}$ once we know that M_i and M_{i+1} are among a set of complements for $\mathcal{T}$. There are at most two such chords: the chords connecting the endpoints of M_i to M_{i+1} proceeding counter-clockwise.

If there are no such chords (because the segments connecting the endpoints are on tbe boundary of the polygon), we are done. (In this case, the chord corresponding to M_i is the rotate of M_{i+1}, so $M_{i+1}[1] = M_i$, and the extension B_i is zero, as predicted.)

In general, it suffices to consider the case where $M_i = W(-\alpha_j)$ and $M_{i+1} = W(\beta^m)$, where α_i appears in the simple root expansion of β. Now there are two cases, depending on whether $j \in I_-$ or $j \in I_+$, both of which are straightforward.

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