

# AN ANALOGUE OF DISTRIBUTIVITY FOR UNGRADED LATTICES

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**ABSTRACT.** In this paper, we define a property, trimness, for lattices. Trimness is a not-necessarily-graded generalization of distributivity; in particular, if a lattice is trim and graded, it is distributive. Trimness is preserved under taking intervals and suitable sublattices. Trim lattices satisfy a weakened form of modularity. The order complex of a trim lattice is contractible or homotopic to a sphere; the latter holds exactly if the maximum element of the lattice is a join of atoms.

Other than distributive lattices, the main examples of trim lattices are the Tamari lattices and various generalizations of them. We show that the Cambrian lattices in types  $A$  and  $B$  defined by Reading are trim, and we conjecture that all Cambrian lattices are trim.

## 1. INTRODUCTION

Some of the first examples of lattices which anyone encounters are the finite distributive lattices. Supersolvable lattices are a generalization of them introduced by Stanley [St] in 1972. Lattices of both these types are necessarily graded. Left modular lattices were introduced by Blass and Sagan [BS] as a further generalization of supersolvable lattices. In [MT], combining results from [Mc] and [Li], McNamara and the author showed that left modularity for lattices can be thought of as “supersolvability without gradedness,” in the sense that supersolvable lattices are left modular (as was shown in [St]), and all graded left modular lattices are supersolvable [MT]. Thus, we have the following diagram:

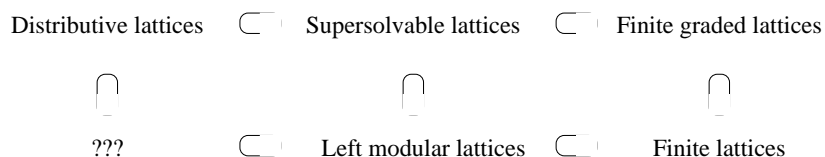


Figure 1

This paper is an attempt to provide something to fit in place of the ??? in the diagram, that is to say, a not-necessarily-graded generalization of distributivity.

We begin with some necessary definitions. All our lattices are assumed to be finite. An element  $x$  of a lattice  $L$  is said to be *left modular* if for any  $y < z$  in  $L$ ,

$$(y \vee x) \wedge z = y \vee (x \wedge z).$$

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A set of pairwise comparable elements is called a *chain*. A *maximal chain* (also called an unrefinable chain) is one which is maximal with respect to inclusion. A lattice is called *graded* if every maximal chain in a given interval is of the same cardinality. A lattice is said to be *left modular* if it has a maximal chain of elements all of which are left modular.

An element in a lattice is called *join-irreducible* if it cannot be written as the join of two strictly smaller elements. ( $\hat{0}$ , the minimum element of a lattice, does not count as a join-irreducible.) Dually, an element is called *meet-irreducible* if it cannot be written as the meet of two strictly larger elements. (Similarly,  $\hat{1}$ , the maximum element of a lattice, does not count as a meet-irreducible.) If a lattice has a maximal chain of  $n + 1$  elements, then it must have at least  $n$  join-irreducibles and at least  $n$  meet-irreducibles.

**Definition.** We say that a lattice is *trim* if it has a maximal chain of  $n + 1$  left modular elements, exactly  $n$  join-irreducibles, and exactly  $n$  meet-irreducibles (that is to say, the minimum possible number of each).

Distributive lattices are example of trim lattices, though not typical examples, since distributive lattices are graded and trim lattices need not be. However, if a lattice is graded and trim, then it is distributive (Theorem 2). This is a special case of a theorem of Markowsky [Ma].

In this paper, we investigate some of the properties of trim lattices, and show that they are in many respects similar to distributive lattices. We show that if  $L$  is trim, then so are its intervals (Theorem 1), and so are its sublattices which contain its left modular chain (Theorem 3). We also show that if  $G$  is a group which acts on a trim lattice  $L$  by lattice automorphisms, then  $L^G$ , the sublattice of  $L$  consisting of elements fixed by  $G$ , is again trim (Theorem 4).

We show that a trim lattice satisfies the level condition of [BS] (Theorem 5). Left modular lattices satisfying this condition are known as LL-lattices [BS]; thus, trim lattices are LL-lattices.

One consequence of Theorem 5 is that in a trim lattice, if  $w$  is covered by  $y$  and  $z$ , then  $y \vee z$  covers at least one of  $y$  and  $z$  (Theorem 6). This is a weakened form of modularity. Another consequence of Theorem 5 is that the order complex of a trim lattice is either contractible or homotopic to a sphere, and the latter holds exactly when the maximum element of the lattice is a join of atoms (Theorem 7). (Note that since intervals of trim lattices are trim, Theorem 7 can also be applied to any interval in a trim lattice, thus showing that the order complex of any interval is again either homotopic to a sphere or contractible.)

In [Re], Reading introduced a family of *Cambrian lattices* for each finite reflection group. (Recall that finite reflection groups consist of four infinite families,  $A_n$  ( $n \geq 1$ ),  $B_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 4$ ),  $I_2(n)$  ( $n = 5$  or  $n \geq 7$ ), and seven exceptional groups,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ ,  $G_2$ ,  $H_3$ ,  $H_4$ .) The Cambrian lattices in type  $A$  include the classical Tamari lattice (which goes back to [Ta]; a more recent reference is [BW]) and in type  $B$  include the type  $B$  Tamari lattice (also studied in [Th]). We show that all the Cambrian lattices in types  $A$  and  $B$  are trim (Theorems 8 and 9). The Cambrian lattices in the other types are not yet well understood, but we offer the following conjecture:

**Conjecture 1.** *All Cambrian lattices are trim.*

## 2. LEFT MODULAR LATTICES

The proofs in this paper depend on the theory of left modular lattices. The study of such lattices was initiated by Blass and Sagan [BS], and continued in Liu [Li], Liu and Sagan [LS], and McNamara and Thomas [MT]. We will begin with a review of the properties of left modular lattices. More details on all of these properties can be found in [MT].

**Proposition 1** ([MT]). *If  $L$  has a left modular maximal chain  $\hat{0} = x_0 < x_1 < \dots < \hat{1}$ , then any interval  $[y, z]$  also has a left modular maximal chain. More precisely, the elements  $y \vee x_i \wedge z$  form a left modular maximal chain in  $[y, z]$ . (Note that the  $y \vee x_i \wedge z$  will not all be distinct.)*

We define three edge-labellings of a left modular lattice  $L$  with left modular maximal chain  $\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$ , which we refer to as the labelling induced from join-irreducibles, that induced from meet-irreducibles, and that induced from the left modular chain. (Note that these labellings all depend on the prior choice of a left modular maximal chain.)

If  $v$  is a join-irreducible of  $L$ , we label it by the natural number

$$\delta(v) = \min(\{i \mid v \leq x_i\}).$$

Now, for any  $y < z$ , define

$$\gamma_1(y < z) = \min(\{\delta(v) \mid v \text{ join-irreducible, } v \leq z, v \not\leq y\}).$$

The labelling induced from meet-irreducibles is defined similarly. If  $v$  is a meet-irreducible, we set

$$\epsilon(v) = \max(\{i \mid x_i \leq v\}) + 1.$$

(Except for the +1, this is just the dual of the definition of  $\delta$ .) Now, as we did for the labelling induced from join-irreducibles, for  $y < z$ , we define

$$\gamma_2(y < z) = \max(\{\epsilon(v) \mid v \text{ meet-irreducible, } v \geq y, v \not\geq z\}).$$

Thirdly, the labelling induced from the left modular chain is defined as follows:

$$\gamma_3(y < z) = \min(\{i \mid y \vee x_i \wedge z = z\}).$$

**Proposition 2** ([Li]). *For any left modular lattice with a specified left modular maximal chain, the three labellings  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$  coincide.*

The fact that  $\gamma_1$  and  $\gamma_3$  coincide is proved in [Li]; the dual of that result shows that  $\gamma_2$  and  $\gamma_3$  coincide. Since the three labellings coincide, we will drop the subscripts and denote the labelling by  $\gamma$ .

A labelling of the edges of the Hasse diagram of a poset is called an EL-labelling [B1] if it satisfies the following two properties:

(i) In any interval, there is a unique maximal chain which has the property that the labels on the chain strictly increase as you read up the chain. (This chain is called the “increasing chain”.)

(ii) In any interval, the label word obtained by reading up the increasing chain lexicographically precedes the word obtained by reading up any other maximal chain in the interval.

(In our context, the labellings of the edges of a Hasse diagram will always be positive integers with the usual order. In general, the labels may be drawn from

any poset; this introduces some additional technicalities which we shall not need to refer to.)

If a partially ordered set admits an EL-labelling then its order complex is shellable, and is therefore homotopic to a wedge of spheres, one for each maximal chain such that the labels weakly decrease as you read up the chain. (Such chains are called “decreasing chains”.) The dimension of the sphere corresponding to a given decreasing chain is two less than the length of the chain.

**Proposition 3** ([Li]). *For a left modular lattice  $L$ , the edge-labelling of  $L$  already described is an EL-labelling.*

In fact, we can say more about the labelling of a left modular lattice. In [MT], we defined *interpolating labellings* to be EL-labellings such that in addition, if  $v < u < w$  is a maximal chain which is not increasing, and the corresponding increasing chain is  $v = y_0 < y_1 < \dots < y_r = w$ , then the label of  $v < u$  coincides with the label of  $y_{r-1} < y_r$ , and the label of  $u < w$  coincides with the label of  $y_0 < y_1$ . We showed the following proposition:

**Proposition 4** ([MT]). *If  $L$  is a left modular lattice, then the labelling defined above is interpolating. Conversely, if a lattice  $L$  admits an interpolating labelling, then the elements of the increasing chain from  $\hat{0}$  to  $\hat{1}$  are left modular, and therefore  $L$  is left modular.*

We need one more result from [MT] about labellings of intervals. Let  $[y, z]$  be an interval in a left modular lattice  $L$ . Since the  $y \vee x_i \wedge z$  form a left modular chain in  $[y, z]$ , the above construction can be applied to yield an EL-labelling. The restriction of the labelling of  $L$  to  $[y, z]$  also yields an EL-labelling. These two labellings typically do not coincide for the trivial reason that their label sets differ. However, we have the following proposition:

**Proposition 5** ([MT]). *Let  $[y, z]$  be an interval in a lattice  $L$  with left modular maximal chain  $\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$ . The labelling of  $L$  restricted to  $[y, z]$  agrees (up to an order-preserving relabelling) with the labelling which  $[y, z]$  has as a lattice with left modular chain  $y \vee x_i \wedge z$ .*

(When we speak of an order-preserving relabelling, we mean that one label set has been replaced by a different label set, but the relative orders of the labels have been preserved.)

We record here one additional lemma about left modular lattices which we shall need.

**Lemma 1.** *Let  $L$  be a lattice with left modular maximal chain  $\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$ , and let  $y$  and  $z$  be two join-irreducibles with  $\delta(y) = \delta(z)$ . Then  $y$  and  $z$  are incomparable.*

*Proof.* Suppose on the contrary that  $y < z$ . Let  $j = \delta(y) = \delta(z)$ . Observe that  $x_{j-1} \vee y = x_j \geq z$ , so

$$(z \wedge x_{j-1}) \vee y = z \wedge (x_{j-1} \vee y) = z.$$

However,  $z \wedge x_{j-1}$  and  $y$  are both strictly less than  $z$ , so  $z$  is not join-irreducible, contrary to our assumption.  $\square$

## 3. TRIM LATTICES

We now proceed to our study of trim lattices. Let  $L$  be a trim lattice, with a specified left modular chain  $\hat{0} = x_0 \triangleleft x_1 \cdots \triangleleft x_n = \hat{1}$ .

**Lemma 2.** *If  $L$  is a trim lattice, it has exactly one join-irreducible and one meet-irreducible labelled  $i$  for  $1 \leq i \leq n$ .*

*Proof.* Since  $x_i$  is the join of the join-irreducibles labelled at most  $i$ , while  $x_{i-1}$  is the join of the join-irreducibles labelled at most  $i-1$ , there must be at least one join-irreducible labelled  $i$ . By trimness, there is exactly one. The dual argument proves the statement for meet-irreducibles.  $\square$

**Theorem 1.** *If  $L$  is trim, so is any interval of  $L$ .*

*Proof.* It is sufficient to show that if  $x \in L$ , then the interval  $[\hat{0}, x]$  is trim, since the dual result follows, and the trimness of  $[y, x]$  can be proved by showing the trimness of  $[\hat{0}, x]$ , and then applying the dual result to the trim lattice  $[\hat{0}, x]$ .

By Proposition 1,  $[\hat{0}, x]$  is left modular. Let the length of the left modular maximal chain in  $[\hat{0}, x]$  be  $m$ . We must show that there are exactly  $m$  join-irreducibles and  $m$  meet-irreducibles in  $[\hat{0}, x]$ .

We consider  $[\hat{0}, x]$  labelled by the labelling induced from  $L$ . The join-irreducibles of  $[\hat{0}, x]$  are exactly the join-irreducibles of  $L$  that lie in  $[\hat{0}, x]$ , and they have the same labels that they do in  $L$ , so their labels are all different. Since, by Proposition 5, the labelling induced from  $L$  agrees (up to an order-preserving relabelling) with the labelling of  $[\hat{0}, x]$  induced from its left modular chain, the induced labelling uses  $m$  different labels. Thus,  $[\hat{0}, x]$  has  $m$  join-irreducibles, as desired.

Let  $a$  be a label that does not appear on a join-irreducible of  $[\hat{0}, x]$  (and which therefore doesn't appear in  $[\hat{0}, x]$  at all). Since the labelling on  $[\hat{0}, x]$  can also be considered as being induced by its meet-irreducibles, there is no meet-irreducible of  $[\hat{0}, x]$  labelled  $a$ .

Let  $b$  be a label that appears on a join-irreducible of  $[\hat{0}, x]$ . Let  $y$  be the join-irreducible of  $L$  with that label (which is also a join-irreducible of  $[\hat{0}, x]$ ). Let  $z$  be the meet-irreducible of  $L$  with label  $b$ . Let  $\bar{z} = z \wedge x$ . Since  $\bar{z} \leq z$ , and  $y \not\leq z$ ,  $y \not\leq \bar{z}$ . So  $y \vee \bar{z} \neq \bar{z}$ . Let the increasing chain from  $\bar{z}$  to  $y \vee \bar{z}$  be  $\bar{z} = t_0 \triangleleft t_1 \triangleleft \cdots \triangleleft t_r = \bar{z} \vee y$ . Since all the  $t_i \leq x$ , it follows that  $t_1 \not\leq z$  (otherwise  $t_1 \leq x \wedge z = t_0$ , a contradiction). Thus, by the meet-irreducible labelling,  $\gamma(t_0, t_1) \geq b$ . By the join-irreducible labelling,  $\gamma(t_{r-1}, t_r) \leq b$ . Since the labels on the chain are increasing, the chain consists of a single covering relation, which is labelled by  $b$ .

By the meet-irreducible labelling for  $[\hat{0}, x]$ , it follows that  $\bar{z}$  lies below some meet-irreducible of  $[\hat{0}, x]$  labelled  $b$ . But any element at the bottom of an edge labelled  $b$  in  $[\hat{0}, x]$  lies below  $x$  and below  $z$ , thus below  $\bar{z}$ . So  $\bar{z}$  must be a meet-irreducible labelled  $b$  in  $[\hat{0}, x]$ . Since any other meet-irreducible labelled  $b$  in  $[\hat{0}, x]$  would have to lie below  $\bar{z}$ ,  $\bar{z}$  is the only one, since two meet-irreducibles with the same label in a left modular lattice must be incomparable, by the dual of Lemma 1. Thus there is exactly one meet-irreducible labelled  $b$ , as desired.  $\square$

**Theorem 2** ([Ma]). *If  $L$  is trim and graded, it is distributive.*

*Remark.* A lattice with a chain of length  $n$  (i.e. with  $n+1$  elements) and which has exactly  $n$  join-irreducibles and  $n$  meet-irreducibles is called *extremal*. Extremal

lattices were introduced by Markowsky in [Ma]. There, he showed that graded extremal lattices are distributive. Since trim lattices are by definition extremal, Theorem 2 follows. However, in the interests of self-containedness, we give a different proof.

It is worth noting that there are extremal lattices which are not trim. Markowsky shows that any finite lattice can be embedded as an interval of an extremal lattice, while Theorem 1 tells us that the intervals of trim lattices are trim. These two results imply that not all extremal lattices are trim. In particular, [Ma] gives an example of an extremal lattice with 39 elements containing  $M_3$  (see below) as an interval; since  $M_3$  is not trim, we know that this example is not trim.

*Proof.* To show that a lattice is distributive, it suffices to show that it has no sublattice  $M_3$  or  $N_5$  (see Figure 2) [Gr, Theorem II.1]. This will follow from the following two lemmas.

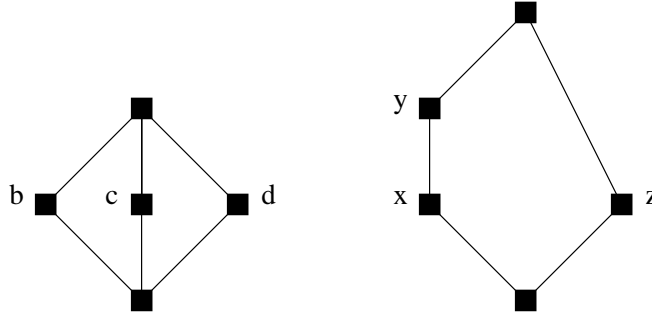


Figure 2:  $M_3$  and  $N_5$

**Lemma 3.** *If  $L$  is trim, then  $L$  contains no sublattice isomorphic to  $M_3$ .*

*Proof.* We are going to assume that  $L$  contains a sublattice isomorphic to  $M_3$  and obtain a contradiction. The property of being trim passes to intervals, by Theorem 1, so we can reduce to the case where the minimum and maximum elements of the copy of  $M_3$  are  $\hat{0}$  and  $\hat{1}$ . Let the elements of the copy of  $M_3$  be identified as in Figure 2.

As always, let  $n$  be the maximum label on the increasing chain from  $\hat{0}$  to  $\hat{1}$ . Let  $B, C, D$  denote the set of labels on the increasing chains from  $\hat{0}$  to  $b, c, d$  respectively. Suppose some two of them, say  $B$  and  $C$ , both contain  $n$ . Then  $b$  and  $c$  both lie over some join-irreducible labelled  $n$ . Since  $b \wedge c = \hat{0}$ , there is more than one join-irreducible labelled  $n$ , contradicting Lemma 2.

On the other hand, suppose that some two of  $B, C, D$ , say  $B$  and  $C$ , do not contain  $n$ . Since  $n$  is the maximum label on the increasing chain from  $\hat{0}$  to  $\hat{1}$ , we can see from the labelling induced from meet-irreducibles that every maximal chain from  $\hat{0}$  to  $\hat{1}$  contains an edge labelled  $n$ . Thus, it occurs on both the increasing chain from  $b$  to  $\hat{1}$  and on the increasing chain from  $c$  to  $\hat{1}$ . We now apply the dual of the previous argument to obtain a contradiction in this case also.

Since either two of  $B, C, D$  contain  $n$  or two do not, we are done.  $\square$

**Lemma 4.** *Let  $L$  be a graded trim lattice. Then  $L$  contains no  $N_5$ .*

*Proof.* As in the proof of the previous lemma, we may assume that the minimum and maximum elements of the  $N_5$  are  $\hat{0}$  and  $\hat{1}$ . Let the other elements be identified

as in Figure 2. Let  $B$  be the set of labels on the increasing chain from  $x$  to  $y$ . Suppose that the increasing chain from  $\hat{0}$  to  $z$  has a label drawn from  $B$ , say  $b$ . Then  $z$  and  $y$  both lie over join-irreducibles with label  $b$ . By the assumption that  $L$  is trim, there is only one join-irreducible labelled  $b$ , so  $z$  and  $y$  both lie over it, which contradicts the assumption that  $z \wedge y = \hat{0}$ .

Dually, no label from  $B$  can occur on the increasing chain from  $z$  to  $\hat{1}$ . However, since we are assuming that  $L$  is graded, the set of labels appearing on every maximal chain from  $\hat{0}$  to  $\hat{1}$  is the same, and we have a contradiction.  $\square$

This completes the proof of Theorem 2.  $\square$

**Theorem 3.** *If  $L$  is trim, and  $K$  is a sublattice of  $L$  containing the left modular chain of  $L$ , then  $K$  is trim.*

*Proof.* It is clear that the left modular chain in  $L$  is still left modular in  $K$ , so  $K$  is left modular.

Suppose  $K$  is not trim. Therefore,  $K$  has either two join-irreducibles with the same label, or two meet-irreducibles with the same label. Dualizing if necessary, we may assume that  $K$  has two join-irreducibles with the same label, say  $y$  and  $z$ , with label  $b$ . The fact that  $y$  and  $z$  are labelled  $b$  means that  $y$  and  $z$  lie below  $x_b$  but not below  $x_{b-1}$ . This implies that, in  $L$ , each can be written as a join of join-irreducibles with labels no more than  $b$ , and including at least one join-irreducible of  $L$  labelled  $b$ . Let  $j$  be the unique join-irreducible of  $L$  labelled  $b$ . So both  $y$  and  $z$  lie over  $j$ . It follows that  $p = y \wedge z$  also lies over  $j$ . So  $p$  lies below  $x_b$  but not below  $x_{b-1}$ . It follows that, in  $K$ ,  $p$  lies over some join-irreducible labelled by  $b$ . But this implies that there are two join-irreducibles labelled by  $b$  in  $K$  which are comparable, and that is impossible, by Lemma 1.  $\square$

#### 4. THE SUBLATTICE FIXED UNDER A GROUP OF AUTOMORPHISMS

The goal of this section is to show that if  $L$  is a trim lattice, and a group  $G$  acts on  $L$  by lattice automorphisms, then  $L^G$ , the sublattice of  $L$  consisting of elements of  $L$  fixed by  $G$ , is a trim lattice.

*Example.* To orient oneself in this section, it is useful to consider the case where  $L$  is the Boolean lattice of all subsets of  $[n]$ , and  $G = \{1, \sigma\}$  where  $\sigma$  acts by interchanging 1 and  $n$ .

The first important thing to notice about this example is that the maximal chains in  $L^G$  are shorter than the maximal chains in  $L$ . The second thing to notice is that if we make a reasonable-seeming choice of left modular chain by setting  $x_i = [i]$ , only the top and bottom elements of our chosen left modular chain actually lie in  $L^G$ . Inspired by this example, before we try to show that  $L^G$  is trim, we will find some more left modular elements in  $L$ .

For  $L$  a trim lattice, we follow the terminology suggested by Drew Armstrong and say that the *spine* of  $L$  consists of those elements of  $L$  which lie on some chain of maximum length in  $L$ .

**Lemma 5.** *If  $L$  is a trim lattice, then all the elements of the spine of  $L$  are left modular.*

*Proof.* This proof was suggested to me by Peter McNamara [Mc2]. Suppose  $z$  is in the spine of  $L$ . Let the labels which occur on the increasing chain from  $\hat{0}$  to  $z$  be

$C$ , and let the labels which occur on the label from  $z$  to  $\hat{1}$  be  $D$ . Since  $z$  is in the spine,  $C \cup D = [n]$ , where  $n$  is the length of the left modular maximal chain in  $L$ .

Now suppose, for the sake of contradiction, that  $z$  is not left modular. It follows that there are some elements  $p < q$  in  $L$  such that  $p \vee (z \wedge q) \neq (p \vee z) \wedge q$ . Since the modular inequality tells us that  $p \vee (z \wedge q) \leq (p \vee z) \wedge q$ , it is in fact true that

$$p \vee (z \wedge q) < (p \vee z) \wedge q.$$

Now set  $x = p \vee (z \wedge q)$ ,  $y = (p \vee z) \wedge q$ . Note that  $x \vee (z \wedge y) = x$ , while  $(x \vee z) \wedge y = y$ . Thus,  $x, y, z$  generate a sublattice of  $L$  of the following form:

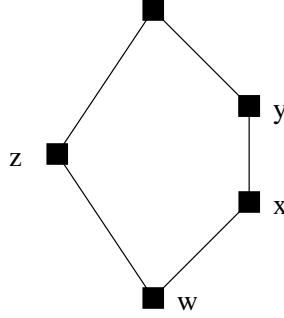


Figure 3: Sublattice of  $L$  generated by  $x, y, z$

Let  $b$  be a label on the increasing chain from  $x$  to  $y$ . Suppose  $b \in C$ . Since there is a unique join-irreducible labelled by  $b$ , say  $j$ , it follows that  $y$  and  $z$  both lie over  $j$ . But this means that  $w$  lies over  $j$ , and therefore  $x$  lies over  $j$ , so  $j$  cannot appear as a label on the increasing chain from  $x$  to  $y$ , which contradicts our assumption. On the other hand, if  $b \notin C$ , then  $b \in D$ , and we can apply the dual argument to yield a contradiction.  $\square$

The following lemma was suggested to me by Drew Armstrong [Ar], who observed it to hold in the Tamari lattice.

**Lemma 6.** *The spine of a trim lattice  $L$  is a distributive sublattice of  $L$ .*

*Proof.* The main difficulty is to show that the spine is closed under lattice operations. Suppose  $y$  and  $z$  are in the spine. We will show that  $y \wedge z$  is also in the spine.

Choose a left modular maximal chain  $\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$  in  $L$ . Let the set of labels below  $y$  be  $A$ , below  $z$  be  $B$ , above  $y$  be  $C$ , above  $z$  be  $D$ . So  $A \cup C = B \cup D = [n]$ . Let  $P = A \cap B = \{p_1, \dots, p_r\}$ , with  $p_1 < \dots < p_r$ . Let  $w_i$  be the join of the join-irreducibles indexed by  $\{p_1, \dots, p_i\}$ . The  $w_i$  are all distinct, since  $w_i$  lies below  $x_{p_i}$  but not below  $x_{p_i-1}$ . Thus, they form a chain of length  $r$  from  $\hat{0}$  to  $y \wedge z$ . Since there are only  $r$  labels available for this chain (namely, the labels in  $P$ ), and each label occurs at most once, this chain must be maximal.

Now let  $Q = C \cup D = \{q_1, \dots, q_{n-r}\}$ , with  $q_1 < \dots < q_{n-r}$ . Similarly to the above, the meet-irreducibles lying above  $y \wedge z$  are exactly those indexed by  $Q$ . Let  $v_i$  be the meet of the meet-irreducibles indexed by  $\{q_i, \dots, q_{n-r}\}$ . By a dual argument, the  $v_i$  are all distinct, and therefore form a chain of length  $n - r$  from  $y \wedge z$  to  $\hat{1}$ , which is necessarily maximal. Thus  $y \wedge z$  is in the spine of  $L$ , and dually the same is true for  $y \vee z$ .



We have shown that the spine is a sublattice of  $L$ . By Lemma 3, it contains no sublattice isomorphic to  $M_3$ , and by Lemma 5 it contains no sublattice isomorphic to  $N_5$ , so it is distributive.  $\square$

We are now ready to prove the main theorem of this section:

**Theorem 4.** *If  $L$  is a trim lattice and  $G$  is a group which acts on  $L$  by lattice automorphisms, then  $L^G$  is also trim.*

*Proof.* First, we want to show that  $L^G$  is left modular. Let  $S$  be the spine of  $L$ . Now the elements of  $S^G$  are left modular in  $L^G$ , since they are left modular in  $L$ . To show that  $L^G$  is left modular, it remains to show that  $S^G$  contains a maximal chain in  $L^G$ . It will suffice to show that if  $y < z$  in  $S^G$ , then  $y < z$  in  $L^G$ .

Fix  $y < z$  in  $S^G$ , and pick a maximal chain in  $S$ ,  $y = t_0 < t_1 < \dots < t_r = z$ . Let  $j$  be the (unique) irreducible of  $L$  which lies below  $t_1$  but not below  $y$ . Let the  $G$ -orbit of  $j$  be  $\{j = j_1, \dots, j_k\}$ . Let  $v_i = y \vee j_i$ . Because  $G$  acts by lattice automorphisms, for every  $i$ ,  $y < v_i$ , and  $v_i \in S$ . Let  $w$  be the join of the  $v_i$ . Observe that  $w \in S^G$ . But  $w \leq z$ , so, since  $y < z$  in  $S^G$ ,  $z = w$ .

The  $v_i$  are all distinct, and since  $S$  is distributive, the length of any maximal chain in  $S$  from  $y$  to  $z$  is of length  $k$ . This means that the only join-irreducibles lying below  $z$  but not below  $y$  are the  $j_i$ .

Now suppose that there is some  $u$  in  $L^G$  such that  $y < u < z$ . There must be some join-irreducible below  $z$  but not below  $y$  which is also below  $u$ , but since  $u \in L^G$ , all the  $j_i$  must lie below  $u$ , which would force  $u = z$ , a contradiction.

This implies that the maximal chains in  $S^G$  are left modular maximal chains in  $L^G$  as desired.

Now we want to show that  $L^G$  is trim. Let  $T$  be the set of elements of  $L^G$  formed by taking the join of the join-irreducibles in some  $G$ -orbit. Clearly, any element of  $L^G$  can be written as a join of elements from  $T$ , so  $T$  contains all the join-irreducibles of  $L^G$ . However, we showed above that if  $y < z$  in  $S^G$ , then there is exactly one  $G$ -orbit of irreducibles below  $z$  but not below  $y$ . So there are chains in  $L^G$  whose length is the number of  $G$ -orbits of irreducibles, which implies that there are at least that number of join-irreducibles in  $L^G$ , so all the elements of  $T$  are join-irreducibles in  $L^G$ , and in particular,  $L^G$  has the correct number of join-irreducibles to be trim. Dually,  $L^G$  has the correct number of meet-irreducibles, and it is therefore trim.  $\square$

## 5. THE LEVEL CONDITION AND ITS CONSEQUENCES

**Theorem 5.** *If  $L$  is a trim lattice then it satisfies the level condition of [BS]:*

*If  $a$  and  $b_1, \dots, b_k$  are atoms, and  $\delta(a) < \delta(b_1) < \dots < \delta(b_k)$ , then  $a \not\leq b_1 \vee \dots \vee b_k$ .*

*Proof.* Suppose otherwise. The proof is by induction on  $k$ . The statement is clearly true when  $k = 1$ . Suppose it is true for  $k - 1$ . Consider a set of atoms  $b_1, \dots, b_k$  as in the statement of the theorem. Let  $y = b_1 \vee \dots \vee b_{k-1}$ , and  $z = b_2 \vee \dots \vee b_k$ . Since, by assumption, the statement is true for  $\{b_2, \dots, b_k\}$ ,  $b_1 \not\leq z$ . Since  $b_1, \dots, b_{k-1}$  all lie below  $x_{\delta(b_{k-1})}$  while  $b_k$  does not,  $b_k \not\leq y$ .

Suppose there is some atom  $a$  with  $\delta(a) < \delta(b_1)$ , such that  $a < y \vee z$ . Choose such an  $a$  with  $\delta(a)$  as small as possible. Thus, we may assume that  $a$  is the first element on the left modular chain from  $\hat{0}$  to  $y \vee z$ . It follows that  $a$  appears on every maximal chain from  $\hat{0}$  to  $y \vee z$ . By the induction assumption,  $a$  lies below

neither  $y$  nor  $z$ . Thus,  $a$  appears as a label on the increasing chain from  $y$  to  $y \vee z$ , and also on the increasing chain from  $z$  to  $y \vee z$ . Since the interval from  $\hat{0}$  to  $y \vee z$  is trim, there is some meet-irreducible in it labelled  $a$ , and both  $y$  and  $z$  lie below it. But this contradicts the fact that  $y \vee z$  is the top of the interval. Thus there can be no such atom  $a$ .  $\square$

Recall that a lattice is said to be upper semimodular if, given three elements such that  $y$  and  $z$  both cover  $w$ , then  $y \vee z$  covers  $y$  and  $z$ . Lower semimodularity is the dual condition. A lattice is said to be modular if it is both upper and lower semimodular. Distributive lattices are examples of modular lattices.

Modularity implies gradedness, so we cannot hope that trim lattices will be modular. The following theorem shows that trim lattices possess a weakened form of upper semi-modularity. The dual statement, which is also true, gives an analogue of lower semi-modularity.

**Theorem 6.** *Let  $L$  be a trim lattice. Let  $y$  and  $z$  cover  $w$ , and suppose that  $\gamma(w \triangleleft y) < \gamma(w \triangleleft z)$ . Then  $z \triangleleft y \vee z$ .*

*Proof.* By Theorem 1, we can reduce to the case where  $w = \hat{0}$ ,  $y$  and  $z$  are atoms, and  $y \vee z = \hat{1}$ . As usual, let  $\hat{0} = x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_r = \hat{1}$  be the left modular chain. By Theorem 5,  $y$  is the join-irreducible of  $L$  with the smallest label, so  $y = x_1$ . The left modular maximal chain from  $z$  to  $\hat{1}$  consists of  $z \vee x_i$ . But  $x_1 = y$ , so the first element of this chain above  $z$  is  $y \vee z$ . Thus  $y \vee z$  covers  $z$ .  $\square$

We will call a lattice *nuclear* if  $\hat{1}$  is the join of the atoms of  $L$ . (In [Re], the term ‘‘atomic interval’’ is used for an interval in which the join of the atoms is the top of the interval. Because this might cause confusion with the standard use of atomic to describe a lattice in which every element can be written as a join of atoms, we prefer to use a different term.)

**Theorem 7.** *If  $L$  is trim and nuclear then its order complex is homotopic to a sphere, whose dimension is 2 less than the number of atoms of  $L$ . If  $L$  is trim but not nuclear, then its order complex is contractible.*

*Remark.* Note that since all intervals in a trim lattice  $L$  are trim by Theorem 1, this theorem also applies to any interval in a trim lattice.

*Proof.* Observe that  $x_1$  is the join-irreducible labelled 1, and is an atom. Any maximal chain in  $L$  has an edge labelled by 1; in a decreasing chain, this must be the last edge. The bottom of such an edge is a meet-irreducible labelled 1; thus, there is at most one edge labelled 1 descending from  $\hat{1}$ .

Suppose  $L$  is nuclear. We prove the statement of the theorem by induction on the number of atoms of  $L$ . If  $L$  has only one atom, the statement is obvious. Suppose the statement holds for nuclear trim lattices with  $r - 1$  atoms. Let  $a_1, \dots, a_r$  be the atoms of  $L$ , in increasing order by their labels. Let  $z = a_2 \vee \cdots \vee a_r$ . The interval  $[\hat{0}, z]$  is a nuclear trim lattice with  $r - 1$  atoms, so by induction it has a unique decreasing chain from  $\hat{0}$  to  $z$ . This chain corresponds to a sphere of dimension  $r - 3$ , so it is of length  $r - 1$ . Now consider the increasing chain from  $z$  to  $\hat{1} = z \vee a_1$ . The top of this chain is labelled with the label of  $a_1$ , which is 1, and all the other labels must be strictly greater than 1. Since the chain is increasing, this means that the chain is of length 1. Thus, the decreasing chain from  $\hat{0}$  to  $z$  extends uniquely to a decreasing chain from  $\hat{0}$  to  $\hat{1}$ . By the remarks in the first

paragraph, any decreasing chain from  $\hat{0}$  to  $\hat{1}$  passes through  $z$ . Since there is only one decreasing chain from  $\hat{0}$  to  $z$ , the decreasing chain from  $\hat{0}$  to  $\hat{1}$  which we have found is unique, and it is clearly of length  $r$ , which implies that the order complex of  $L$  is homotopic to a sphere of dimension  $r - 2$ , as desired.

For the second statement, it is well-known that if  $L$  is any finite non-nuclear lattice, then its order complex is contractible. This follows from the Crosscut Theorem; see, for example [B3].  $\square$

One of the reasons to be interested in statements about the homotopy types of order complexes of intervals is that for  $x < y$  in any poset, the Möbius function  $\mu(x, y)$  is the reduced Euler characteristic of the order complex of the interval  $[x, y]$ . Thus, from Theorem 7 combined with Theorem 1, we can deduce that the Möbius function of any interval in a trim lattice is either 0, 1, or  $-1$ . That  $\mu(\hat{0}, \hat{1})$  is 0, 1 or  $-1$  for a trim lattice also follows easily from results in [BS].

## 6. CAMBRIAN LATTICES

Let  $W$  be a finite subgroup of the orthogonal transformations of  $E = \mathbb{R}^n$ , generated by reflections. Such a group is called a *finite reflection group*. It has an associated finite root system  $\Phi \subset E$ , which is partitioned into positive and negative roots, denoted  $\Phi^+$  and  $\Phi^-$ . The elements of  $W$  permute  $\Phi$ .

For any element  $w$  of  $W$ , let the *inversion set* of  $w$  be defined by:

$$I(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}.$$

If we order the elements of  $W$  by inclusion of inversion sets, we obtain a partially ordered set structure called *weak order* on  $W$ . Weak order on  $W$  is a lattice. A general reference for weak order on finite reflection groups is [B2].

A lattice homomorphism is a map of lattices which preserves lattice operations. A quotient lattice of a lattice  $L$  is the image of a homomorphism from  $L$ . The fibers of a lattice homomorphism from  $L$  are necessarily intervals in  $L$ .

Associated to any finite reflection group  $W$  is a graph called its Coxeter diagram, which we denote  $G$ . Let  $\bar{G}$  be an orientation of  $G$  (that is to say, for each edge of  $G$ , we designate one end of the edge as the source and the other as the target). Associated to  $\bar{G}$  is a *Cambrian lattice*  $C(\bar{G})$ , which is a quotient of weak order on  $W$ . We shall not give the general definition here, restricting our attention to reflection groups of types  $A$  and  $B$ , where (in contrast to the other types) explicit descriptions of the Cambrian lattices are known. The general definition and the explicit description in types  $A$  and  $B$  are due to Reading [Re]. In what follows, we will review these descriptions, and then show that the Cambrian lattices in types  $A$  and  $B$  are trim, and consequently that the results of the first half of this paper apply to them (and their intervals). The result of Theorem 7 applied to Cambrian lattices was already proved in [Re]. Theorems 4 and 5 are new.

**Type A Cambrian Lattices.** In type  $A_{n-1}$ , the reflection group  $W$  is isomorphic to  $S_n$ . Let  $e_1, \dots, e_n$  be a basis for  $\mathbb{R}^n$ . A permutation  $\pi \in S_n$  acts on  $\mathbb{R}^n$  by taking  $e_i$  to  $e_{\pi(i)}$ . The roots are the vectors  $e_j - e_i$  for  $i \neq j$ . The positive roots are  $e_j - e_i$  for  $j > i$ .

For  $\pi \in S_n$ ,  $e_j - e_i$  is an inversion of  $\pi$  for  $j > i$  if  $j$  precedes  $i$  in the word  $\pi_1, \dots, \pi_n$ . As already mentioned, weak order on  $S_n$  is the inclusion order on inversion sets.

The Coxeter diagram  $G$  consists of a path of  $n - 1$  nodes, labelled  $s_1, \dots, s_{n-1}$ . Let  $\bar{G}$  be an orientation of this diagram. We write  $s_{i-1} \rightarrow s_i$  and  $s_{i-1} \leftarrow s_i$  to represent the two possible orientations of the edge between  $s_{i-1}$  and  $s_i$ . Define two complementary subsets of  $[2, n - 1]$  by  $D = \{i \mid s_{i-1} \rightarrow s_i\}$ ,  $U = \{i \mid s_{i-1} \leftarrow s_i\}$ .

For our purposes, a *pattern* is a permutation of  $[k]$ . A permutation  $\pi$  contains a given pattern  $\sigma$  if there are some  $i_1 < \dots < i_k$  such that  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$  are in the same relative order as  $\sigma_1, \dots, \sigma_k$ . If we put a bar over an element of a pattern (as, for example, in  $\bar{2}31$ ), then to say that  $\pi$  contains that pattern means that  $\pi$  contains an instance of the pattern in which the element of  $\pi$  that corresponds to the barred element of the pattern belongs to  $U$ . Similarly, if we underline an element of the pattern, we mean that the corresponding element of  $\pi$  must belong to  $D$ .

Let  $\mathcal{B}$  be the set of permutations in  $S_n$  avoiding  $\bar{2}31$  and  $31\bar{2}$ . Let  $\mathcal{T}$  be the set of permutations in  $S_n$  avoiding  $\bar{2}13$  and  $13\bar{2}$ .

**Proposition 6** ([Re]). *There is a quotient of weak order on  $S_n$  the minimal elements of whose fibers are  $\mathcal{B}$  and the maximal elements of whose fibers are  $\mathcal{T}$ . This is by definition the Cambrian lattice  $C(\bar{G})$ .  $\mathcal{B}$  and  $\mathcal{T}$  are sublattices of weak order on  $S_n$ , each also isomorphic to  $C(\bar{G})$ .*

Note that in the case that all the edges of  $\bar{G}$  are oriented  $s_{i-1} \rightarrow s_i$ ,  $\mathcal{B}$  consists of all those permutations avoiding  $312$ , while  $\mathcal{T}$  consists of those permutations avoiding  $132$ . In this case,  $C(\bar{G})$  is a Tamari lattice, and the map from  $S_n$  to  $C(\bar{G})$  is the well-known quotient map from weak order on  $S_n$  to the Tamari lattice. (See, for instance, [BW] for more details.)

Here is an example, showing weak order on  $S_3$ , an oriented Dynkin diagram, and the induced Cambrian lattice.

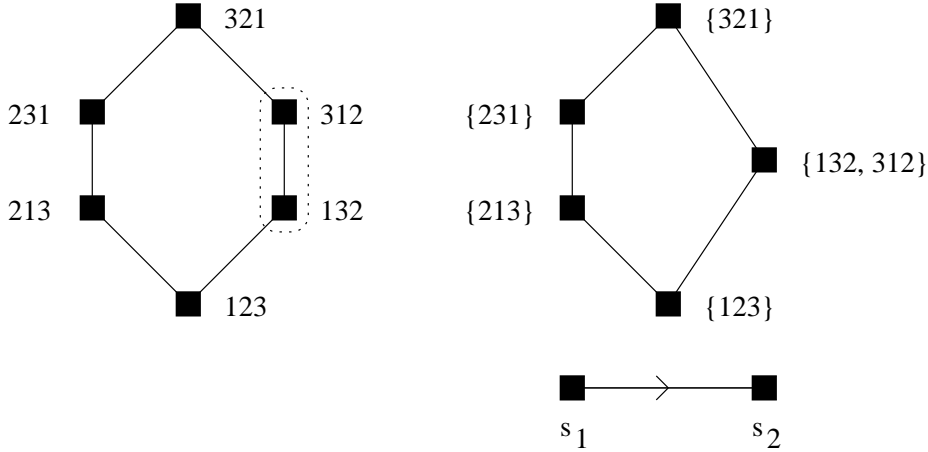


Figure 4

**Type  $B$  Cambrian lattices.** We now consider the type  $B_n$  Cambrian lattices. Here,  $W$  is isomorphic to  $B_n$ , the group of signed permutations of  $[n]$ , that is, permutations of  $\{-n, \dots, -1, 1, \dots, n\}$  which are fixed under the involution interchanging positive and negative numbers. We think of  $\pi \in B_n$  as acting on  $\mathbb{R}^n$  by taking  $e_i$  to  $e_{\pi(i)}$ , where we let  $e_{-i} = -e_i$ . The roots of  $B_n$  are  $\pm e_i \pm e_j$  for  $i \neq j$ , together with  $\pm e_i$ . The positive roots are those of the form  $e_j - e_i$  for  $j > i$ ,  $e_j + e_i$ , and  $e_i$ .

The Coxeter diagram  $G$  consists of a path of  $n$  vertices, labelled  $s_0, \dots, s_{n-1}$ , where the edge from  $s_0$  to  $s_1$  is the special edge labelled 4. (For those unfamiliar with Coxeter diagrams, this encodes certain information about the relations among the generators of the group which correspond to these nodes, but this is not essential for our purposes.) Let  $\bar{G}$  be an orientation for  $G$ . We define two complementary subsets of  $\{-n+1, \dots, -1, 1, \dots, n-1\}$ : for  $1 \leq i \leq n-1$ , if  $s_{i-1} \rightarrow s_i$ , then  $i \in D$  and  $-i \in U$ , and vice versa if  $s_{i-1} \leftarrow s_i$ .

The one-line notation for  $\pi \in B_n$  is  $\pi_{-n} \dots \pi_{-1} \pi_1 \dots \pi_n$ . For  $0 < i < j$ ,  $e_j - e_i$  is an inversion of  $\pi$  if  $j$  precedes  $i$  in the one-line notation for  $\pi$ ,  $e_i$  is an inversion of  $\pi$  if  $i$  precedes  $-i$  in the one-line notation for  $\pi$ , and  $e_j + e_i$  is an inversion for  $\pi$  if  $i$  precedes  $-j$  (or equivalently  $j$  precedes  $-i$ ) in the one-line notation for  $\pi$ . Weak order on  $B_n$  is defined (as always) by inclusion of inversion sets.

As in type  $A$ , we say that  $\pi$  contains a pattern  $\sigma \in S_k$  iff there are some  $i_1 < \dots < i_k$  such that the relative order of  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$  is the same as that of  $\sigma_1, \dots, \sigma_k$  — but we allow  $i_1, \dots, i_k$  to be chosen from  $[n] \cup -[n]$ . The meaning of overlines and underlines are the same as in type  $A$ . (Note: sometimes  $B_n$  is considered as a set of permutations on  $\{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ , i.e.  $\bar{k}$  is used where we would write  $-k$ . There is a possibility of confusion for the reader here. When we write  $\bar{k}$  we will never mean  $-k$ ; we will always mean that  $k \in U$ .)

The definition of the Cambrian lattice  $C(\bar{G})$  is very similar to the definition in type  $A$ . Let  $\mathcal{B}$  be the set of permutations in  $B_n$  avoiding  $\bar{2}31$  and  $31\bar{2}$ . (Note that, because of the symmetry of elements of  $B_n$ , it is actually sufficient to check that an element of  $B_n$  avoids one of these patterns — the avoidance of the other pattern comes for free.) Let  $\mathcal{T}$  be the set of permutations in  $B_n$  avoiding  $\bar{2}13$  and  $13\bar{2}$ . (As for  $\mathcal{B}$ , we only need to check one of these conditions.) An analogue of Proposition 6 holds in type  $B_n$ :

**Proposition 7** ([Re]). *There is a quotient of weak order on  $B_n$  the minimal elements of whose fibers are  $\mathcal{B}$  and the maximal elements of whose fibers are  $\mathcal{T}$ . This is by definition the Cambrian lattice  $C(\bar{G})$ .  $\mathcal{B}$  and  $\mathcal{T}$  are sublattices of weak order on  $B_n$ , each also isomorphic to  $C(\bar{G})$ .*

**Trimness of Cambrian lattices.** The story in type  $B$  is in some respects simpler than in type  $A$ , so we begin with the following theorem:

**Theorem 8.** *The type  $B$  Cambrian lattices are trim.*

*Proof.* First, we must understand the join-irreducibles of  $C(\bar{G})$ .

**Lemma 7.** *There are  $n^2$  join- and meet-irreducibles of  $C(\bar{G})$ .*

*Proof.* A join-irreducible  $\pi$  of  $\mathcal{B}$  is necessarily a join-irreducible of  $B_n$ , because  $C(\bar{G})$  is a quotient of  $B_n$ . Let the unique element which lies immediately below  $\pi$  in  $B_n$  be  $\sigma$ . Let the adjacent transposition relating  $\pi$  and  $\sigma$  interchange  $x$  and  $y$ , (and also  $-x$  and  $-y$ ), with  $y > 0$ , and  $|y| \geq |x|$ . (So  $x$  and  $y$  appear together, in the order  $xy$  in  $\sigma$ , and in the order  $yx$  in  $\pi$ .) We wish to show that  $x$  and  $y$  determine  $\pi$ .

We consider first the case where  $0 < x < y$ . Thus  $\pi$  looks like either

$$\dots(-x)(-y)\dots yx\dots \text{ or } \dots yx\dots(-x)(-y)\dots$$

The fact that  $\pi$  is join-irreducible in  $B_n$  means that each of the three segments into which  $\pi$  is divided by  $yx$  and  $(-x)(-y)$  must be increasing. This immediately

rules out the second of the two possibilities displayed above. Again using the fact that each of the three segments of  $\pi$  is increasing, to show that  $\pi$  is determined by  $x$  and  $y$ , it suffices to show that for any  $z$  other than  $x$ ,  $y$ ,  $-x$ , or  $-y$ , we can determine which segment it belongs to. If  $z > y$ , then  $z$  must occur in the rightmost segment. If  $0 < z < x$ , then  $z$  must lie in the middle segment. If  $x < z < y$ , then  $z$  cannot lie in the leftmost segment, and which of the other two segments it lies in is determined by the fact that  $\pi \in \mathcal{B}$ , and thus that exactly one of  $zyx$  or  $yxz$  is a forbidden configuration (depending on whether  $z \in U$  or  $z \in D$ ). This determines the position of all  $z > 0$ , and by the symmetry of  $\pi$ , it also determines the positions of all  $z < 0$ . Thus we see that  $\pi$  is determined by  $x$  and  $y$ .

The cases where  $x = -y$  and where  $-y < x < 0 < y$  are very similar. Thus, for every pair  $x, y$  with  $-y \leq x < y$ , there is exactly one join-irreducible in  $\mathcal{B}$ , and thus there are  $n^2$  in total. Using a dual argument, there are exactly  $n^2$  meet-irreducibles of  $\mathcal{T}$ . Using Proposition 7 which says that  $\mathcal{B}$  and  $\mathcal{T}$  are both isomorphic to  $C(\bar{G})$ , we see that  $C(\bar{G})$  has exactly  $n^2$  join- and meet-irreducibles.  $\square$

Our next step will be to identify a maximal chain of length  $n^2$  in  $C(\bar{G})$ .

We let  $s_i$  (the labels of the nodes of the Coxeter diagram) also denote the corresponding reflection in  $B_n$ : for  $i > 0$ ,  $s_i$  interchanges  $i$  and  $i+1$ , while  $s_0$  interchanges 1 and  $-1$ .

Write out a word in which each  $s_i$  occurs once, and such that for any edge  $s_i \rightarrow s_j$ ,  $s_i$  occurs to the right of  $s_j$  in the word. Let  $c$  be the product of the  $s_i$  in this order. It is a Coxeter element, and one convinces oneself easily that it takes  $-n$  to the smallest element of  $D$ , each element of  $D$  to the next largest one, the largest element of  $D$  to  $n$ , and by symmetry  $n$  to the largest element of  $U$ , etc.

We know that  $c^n = -1$ . (This holds for any Coxeter element in type  $B$ . It is also easy to see from our explicit description.) For  $0 \leq i \leq n^2$ , let  $x_i$  denote the element of  $B_n$  which consists of the product of the rightmost  $i$  simple reflections in  $c^n$  (where we think of  $c$  as being written as a word of length  $n$  as above). Since the minimum length of an expression for  $-1$  as a product of simple reflections in  $B_n$  has length  $n^2$ , and our expression for  $c^n$  has exactly this length, it follows that  $x_i < x_{i+1}$  in  $B_n$ .

**Lemma 8.** *The  $x_i$  are contained in  $\mathcal{T} \cap \mathcal{B}$  (so in particular, each determines a different element of  $C(\bar{G})$ ).*

*Proof.* In order to prove this, we will need to give a description of inversion sets of elements of  $\mathcal{T}$  and  $\mathcal{B}$  in terms of their allowed intersections with irreducible rank 2 root systems contained in our  $B_n$  root system  $\Phi$ .

These are the types of rank 2 root systems contained in  $\Phi$ :

(i) The type  $B_2$  root system corresponding to positive roots  $e_i, e_j + e_i, e_j, e_j - e_i$ , ( $i < j$ ). To read off which of these elements lie in the inversion set of  $\pi \in B_n$ , we need only consider the relative positions of  $i, j, -i, -j$ .

(ii) The type  $A_2$  root system corresponding to positive roots  $e_j - e_i, e_k - e_i, e_k - e_j$  for  $i < j < k$ . To read off which of these elements lie in the inversion set of  $\pi$ , we need to look at the relative positions of  $i, j, k$ .

(iii) The type  $A_2$  root system corresponding to positive roots  $e_j - e_i, e_j + e_k, e_i + e_k$  (for  $i < j$ ). To read off which of these elements lie in the inversion set of  $\pi$ , we need to look at the relative positions of  $i, j, -k$ .

The inversion set for any element of  $B_n$  intersected with any of these rank 2 root systems is an initial or final subset of the list of inversions (in the order in which they are listed above). This can be seen by inspection in our case; a similar statement holds for all finite reflection groups, see [B2].

**Lemma 9.** *For  $R$  a rank 2 root system contained in  $\Phi$ , there is an order on its roots (either the one given above or its reverse) which we call the  $\bar{G}$ -order such that:*

(i) *the inversion set of an element of  $\mathcal{B}$  intersected with  $R$  is either an initial subset with respect to the order, or consists of exactly the final element.*

(ii) *the inversion set of an element of  $\mathcal{T}$  intersected with  $R$  is either an initial subset or consists of all the elements except the first.*

*Proof.* This essentially follows by inspection, considering the three possible types of root systems contained in  $B_n$ . Suppose that the rank 2 root system is of type  $B_2$ . The possible relative positions for  $i, j, -i, -j$  (ignoring all other symbols) are as follows:

$$\begin{array}{llll} & ji(-i)(-j), & I = \{e_i, e_j + e_i, e_j, e_j - e_i\} & \\ ij(-j)(-i), & I = \{e_i, e_j + e_i, e_j\} & j(-i)i(-j), & I = \{e_j + e_i, e_j, e_j - e_i\} \\ i(-j)j(-i), & I = \{e_i, e_j + e_i\} & (-i)j(-j)i, & I = \{e_j, e_j - e_i\} \\ (-j)i(-i)j, & I = \{e_i\} & (-i)(-j)ji, & I = \{e_j - e_i\} \\ & (-j)(-i)ij, & I = \emptyset & \end{array}$$

Observe that if  $i \in D$  then  $ij(-j)(-i)$  and  $i(-j)j(-i)$  are impossible for an element of  $\mathcal{B}$ , while if  $i \in U$  then  $j(-i)i(-j)$  and  $(-i)j(-j)i$  are impossible for an element of  $\mathcal{B}$ . Thus, if  $i \in D$ , part (i) of the lemma is satisfied if we set the  $\bar{G}$  order to be  $e_j - e_i, e_j, e_j + e_i, e_i$ , while if  $i \in U$ , part (i) of the lemma is satisfied if we set  $\bar{G}$ -order to be the reverse order. It is straightforward to check that the same order also satisfies part (ii) of the lemma.

The other two types of root systems are handled similarly, proving the lemma.  $\square$

We now prove a converse to Lemma 9. First, we introduce some notation. We say that a subset of a rank 2 root system is *initial* if it is initial with respect to the  $\bar{G}$ -order. We say that the subset is *last* if it consists of only the final element (with respect to the  $\bar{G}$ -order). We say that a subset is *all but first* if it consists of all the elements except the first. We will say that a set of roots has  $\mathcal{B}$ -good intersection with a rank two root system if its intersection is initial or last, and  $\mathcal{T}$ -good intersection if its intersection is initial or all but first. Thus, Lemma 9 says that if  $w \in \mathcal{B}$  then  $I(w)$  has  $\mathcal{B}$ -good intersection with every rank 2 root system in  $\Phi$ , and similarly with  $\mathcal{T}$  replacing  $\mathcal{B}$ . The following lemma is a converse.

**Lemma 10.** *If a set of roots has  $\mathcal{B}$ -good intersection with every rank 2 root system, then the set of roots is the inversion set of an element of  $\mathcal{B}$ . Similarly, if it has  $\mathcal{T}$ -good intersection with every rank 2 root system, it is the inversion set of an element of  $\mathcal{T}$ .*

*Proof.* We prove the first statement. Given a set of roots  $I$  whose intersection with any rank 2 root system is either initial or final, it is the inversion set of a unique element  $\pi$  of  $B_n$  [B2]. We must show that  $\pi$  contains neither a  $\bar{2}31$  nor a  $31\bar{2}$ . Suppose it does, and suppose first that this pattern involves three elements of distinct absolute values. If these are all the same sign (which we may assume to be positive) then we have found  $i, j, k$  such that  $e_i - e_j, e_i - e_k, e_j - e_k$  has an illegal

intersection with  $I$ . Similarly, if the pattern involves elements not all of the same sign then we are in the other type of  $A_2$  root system, while if the pattern involves two elements of the same absolute value, then we are in a similar situation with respect to a  $B_2$  root system. Thus,  $\pi$  contains no  $\bar{2}31$  or  $31\bar{2}$ , and therefore is an element of  $\mathcal{B}$ .

The second statement follows from a similar argument, and the lemma is proved.  $\square$

Now that we understand the possible inversion sets of element of  $\mathcal{B}$  and  $\mathcal{T}$ , we can return to the proof of Lemma 8. We now proceed to show that the inversion set of  $x_i$  intersected with any rank two root system is initial. Consider, for example, a root system of type  $B_2$ . We must determine whether, in our word for  $c^n$ , the inversions  $e_i, e_i + e_j, e_j, e_j - e_i$  appear in that order or the reverse order. By inspection (recalling our explicit description of  $c$ ), we see that they occur in the forward order if  $i \in U$ , and in the backward order if  $i \in D$ . But now observe that this order on the roots is exactly the order provided by Lemma 9, as desired. The other types of root systems are dealt with similarly. This completes the proof of Lemma 8.  $\square$

We wish to show that the  $x_i$  are left modular. By Proposition 4, it is sufficient to exhibit an interpolating labelling for  $C(\bar{G})$  such that the  $x_i$  form the increasing chain from  $\hat{0}$  to  $\hat{1}$ .

We now introduce some notation related to  $C(\bar{G})$ . For  $x \in B_n$ , we write  $[x]$  for the fibre of the quotient map to  $C(\bar{G})$  which includes  $x$ . We also write  $p^\uparrow(x)$  for the top element of  $[x]$ , and  $p^\downarrow(x)$  for the bottom element of  $[x]$ .

We define an edge-labelling for  $C(\bar{G})$  as follows. First, observe that the edges in the Hasse diagram of weak order on  $B_n$  have a natural labelling by positive roots: we label the edge  $x < y$  by  $I(y) \setminus I(x)$ . We now use this labelling to define a labelling for  $C(\bar{G})$ .

Suppose  $[x] < [y]$  in  $C(\bar{G})$ , with  $x \in \mathcal{T}$ . Then  $x$  is covered by an element of  $[y]$ , say  $y'$ . Then set  $\gamma([x] < [y]) = I(y') \setminus I(x)$ .

**Lemma 11.** *If  $x < y$  and  $[x] \neq [y]$ , then  $\gamma([x] < [y]) = I(y) \setminus I(x)$ .*

*Proof.* Let  $x' = p^\uparrow(x)$ , and let  $y'$  be the element of  $[y]$  covering  $x'$ . Now  $y' \wedge y$  is in  $[y]$  but lies over  $x$ , so must equal  $y$ , which implies that  $y \leq y'$ . Since  $y$  does not lie under  $x'$ , but  $x$  does,  $I(y) \setminus I(x) = I(y') \setminus I(x')$ , as desired.  $\square$

We now prove an easy lemma which will be useful for computations in  $C(\bar{G})$ .

**Lemma 12.** *If  $x, y \in \mathcal{T}$ , then  $I(x \vee y) = I(x) \cup I(y)$ . If  $x, y \in \mathcal{B}$ , then  $I(x \wedge y) = I(x) \cap I(y)$ .*

*Proof.* We prove the first statement. Observe that  $I(x) \cup I(y)$  has  $\mathcal{T}$ -good intersection with every rank 2 root system, and therefore, by Lemma 10, defines an element of  $\mathcal{T}$ . Now it is clear that this element must be the join of  $x$  and  $y$ .

The argument for the second statement is similar.  $\square$

**Lemma 13.** *The labelling  $\gamma$  defined above is an interpolating labelling for  $C(\bar{G})$ .*

*Proof.* The first necessity for showing that a labelling is interpolating is to show that it is an EL-labelling. Let  $[v] < [w]$  in  $C(\bar{G})$ , with  $v$  and  $w$  in  $\mathcal{T}$ . Let  $\alpha = \min(I(w) \setminus I(v))$ .



We begin by showing that there is a  $z$  such that  $[v] \triangleleft [z] \leq [w]$ , with  $\gamma(v \triangleleft z) = \alpha$ . Let  $x$  be the element of  $\mathcal{B} \cap \mathcal{T}$  whose inversion set consists of all roots up to and including  $\alpha$  in  $\bar{G}$ -order. Let  $z = v \vee x$ . By Lemma 26,  $I(z) = I(v) \cup I(x) = I(v) \cap \{\alpha\}$ . Thus  $\gamma([v] \triangleleft [z]) = \alpha$ , and clearly  $[v] \triangleleft [z]$  is the only edge proceeding up from  $[v]$  labelled by  $\alpha$ .

Next we show that every maximal chain from  $[v]$  to  $[w]$  has an edge labelled by  $\alpha$ . Given a maximal chain, let  $[q]$  be the first element of the chain lying over  $z$ , and let  $[r]$  be the element lying below  $[q]$  in the chain. Let  $r \in \mathcal{T}$ . Let  $q'$  be the element of  $[q]$  covering  $r$ . Then  $q'$  lies over  $z$  but  $r$  does not, so  $I(q') \setminus I(r) = \{\alpha\}$ , and the edge  $[r] \triangleleft [q]$  is labelled  $\alpha$ .

So  $\alpha$  is the minimum possible label to occur on any edge of any maximal chain from  $[v]$  to  $[w]$ , and it must occur on every chain. Thus, the first step in any increasing chain from  $[v]$  to  $[w]$  must be labelled  $\alpha$ , so any increasing chain must begin  $[v] \triangleleft [z]$ . Now, by induction, there is a unique increasing chain from  $[v]$  to  $[w]$ .

Now we must show that the labelling  $\gamma$  is interpolating. So suppose that we have chain of length two which isn't increasing, say  $[v] \triangleleft [u] \triangleleft [w]$ . Let us assume that  $v \in \mathcal{T}$ . Let  $\alpha = \gamma([v] \triangleleft [u])$  and  $\beta = \gamma([u] \triangleleft [w])$ . Let  $[v] = [y_0] \triangleleft [y_1] \triangleleft \cdots \triangleleft [y_r] = [w]$  be the increasing chain from  $[v]$  to  $[w]$ .

Since  $\gamma([y_0] \triangleleft [y_1])$  is the minimum label on any chain in the interval, by what we have just shown it must occur on every chain from  $[v]$  to  $[w]$ . It cannot be that the edge  $[v] \triangleleft [u]$  has this label, so  $\gamma([y_0] \triangleleft [y_1]) = \gamma([u] \triangleleft [v])$ , one of the two conditions necessary for  $\gamma$  to be interpolating.

In weak order on  $B_n$ , we know that there are two edges rising from  $v$ , labelled by  $\alpha$  and  $\beta$ . These correspond to simple reflections  $s_\alpha$  and  $s_\beta$  (i.e. the tops of these edges are  $vs_\alpha$  and  $vs_\beta$  where  $s_\alpha$  and  $s_\beta$  are simple reflections.) Let  $V$  be the subgroup of  $W$  generated by  $s_\alpha$  and  $s_\beta$ . Then  $v$  is the unique minimum-length representative of its left coset  $vV$  in  $W$ . This coset appears in weak order on  $B_n$  as an interval with minimum element  $v$ . (For more details, see [Hu, Section 1.10].) This interval of  $B_n$  is isomorphic to weak order on  $V$ , which is a rank 2 reflection group. Therefore, this interval consists of a two incomparable chains  $vs_\alpha = c_1 \triangleleft c_2 \triangleleft \cdots \triangleleft c_k$ ,  $vs_\beta = d_1 \triangleleft d_2 \triangleleft \cdots \triangleleft d_k$ , together with a minimum element  $v$  and a maximum element, which we will call  $q$ .

Observe that  $c_1 = vs_\alpha \in [y_1]$  and  $d_1 = vs_\beta \in [u]$ . Thus, their join,  $vs_\alpha \vee vs_\beta$ , which equals  $q$ , lies in  $[w]$ . Since  $[q] > [d_1]$ , but  $c_k \not> d_1$ ,  $[c_k] \neq [q]$ . Since  $[y_1] = [c_1] \leq [c_k] \triangleleft [q] = [w]$ ,  $[c_k] = [y_{r-1}]$ .

Observe that the edge (in weak order on  $B_n$ ) from  $c_k$  to  $q$  is labelled by  $\beta$ . Thus, by Lemma 11, since  $c_k \in [y_{r-1}]$  and  $q \in [w]$ ,  $\gamma([y_{r-1}] \triangleleft [w]) = \beta$ , and we have shown that  $\gamma$  is interpolating.  $\square$

It is clear that the  $x_i$  form the increasing chain from  $\hat{0}$  to  $\hat{1}$  in  $C(\bar{G})$ , and thus they are left modular. We conclude that Cambrian lattices of type  $B$  are trim.  $\square$

The following theorem is an easy corollary of Theorem 8.

**Theorem 9.** *The type A Cambrian lattices are trim.*

*Proof.* Let  $\bar{G}$  be an oriented type A Coxeter diagram. Let  $\bar{G}'$  be the type B Coxeter diagram obtained by affixing an extra edge labelled 4 to  $\bar{G}$ , oriented arbitrarily. It is straightforward to see, either by the explicit description of Cambrian lattices in types A and B, or from general theory, that  $C(\bar{G})$  is a lower interval in  $C(\bar{G}')$ .

(The top of the interval is the equivalence class of the longest word for the type  $A$  Coxeter group.) It now follows by Theorem 1 that  $C(\bar{G})$  is trim.  $\square$

**Conjectural description of other Cambrian lattices.** Let  $W$  be a finite reflection group which contains  $-1$ . Let  $G$  be its Dynkin diagram, and  $\bar{G}$  an orientation. As in type  $B$ , we can order the nodes of the diagram in accordance with the orientation of the edges, and then take the product of the simple reflections in that order, obtaining a Coxeter element  $c$ . If  $h$  is the Coxeter number for  $W$  then, since  $-1 \in W$ ,  $h$  will be even, and  $c^{h/2} = -1$ , [Hu, Corollary 3.20].

Linearly order the roots of  $W$  in the order in which they appear as inversions in the word for  $c^{h/2}$ . Let  $x_i$  be the element of  $W$  whose inversion set consists of the first  $i$  roots, in this order.

Now, take the minimal quotient of  $W$  such that the  $x_i$  are all left modular. Call this the pre-Cambrian lattice associated to  $\bar{G}$ .

**Conjecture 2.** *The bottom elements of the fibres of this quotient will be exactly those elements whose inversion sets have  $\mathcal{B}$ -good intersection with all rank 2 sub-root systems, where the order on the sub-root system comes from the linear order on the positive roots. (And similarly for the top elements of the fibres.)*

**Conjecture 3.** *The pre-Cambrian lattice associated to  $\bar{G}$  coincides with the Cambrian lattice  $C(\bar{G})$ .*

Note that we have already showed that these conjectures hold in type  $B$ .

Since every root system embeds in one whose reflection group contains  $-1$ , and the Cambrian lattice associated to the smaller root system appears as a lower interval in the Cambrian lattice associated to the larger root system, it would follow from Conjectures 2 and 3 that all Cambrian lattices are trim.

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