GRADED LEFT MODULAR LATTICES ARE SUPERSOLVABLE

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Abstract. We provide a direct proof that a finite graded lattice with a maximal chain of left modular elements is supersolvable. This result was established via a detour through EL-labellings in [MT] by combining results of McNamara [Mc] and Liu [Li]. As part of our proof, we show that the maximum graded quotient of the free product of a chain and a single-element lattice is finite and distributive.

Supersolvability for lattices was introduced by Stanley [St]. A finite lattice is supersolvable iff it has a maximal chain (called the M-chain) such that the sublattice generated by the M-chain and any other chain is distributive.

An element x of a lattice is left modular if it satisfies:

$$(y \vee x) \wedge z = y \vee (x \wedge z)$$

for all $y \leq z$. We say that a lattice is left modular if it has a maximal chain of left modular elements. Stanley [St] showed that the elements of the M-chain of a supersolvable lattice are left modular, and thus that supersolvable lattices are left modular.

We say that a lattice is graded if, whenever $x < y$ and there is a finite maximal chain between x and y, all the maximal chains between x and y have the same length. It is easy to see that supersolvable lattices are graded.

The main result of our paper is the converse of these two results:

Theorem 1. *If L is a finite, graded, left modular lattice, then L is supersolvable.*

This result was first proved in [MT], as an immediate consequence of results of Liu and McNamara. Liu [Li] showed that if a finite lattice is graded of rank n and left modular, then it has an EL-labelling of the edges of its Hasse diagram, such that the labels which appear on any maximal chain are the numbers 1 through n in some order. McNamara [Mc] showed that for graded lattices of rank n, having such a labelling is equivalent to being supersolvable. These two results together immediately yield that finite graded left modular lattices are supersolvable. However, since this proof involves considerations which seem to be extraneous to the character of the result, it seemed worth giving a more direct and purely lattice-theoretic proof.

On the way to our main result, we introduce the notion of the maximum graded quotient of a lattice. The maximum graded quotient of a lattice is a graded quotient through which any quotient to a graded lattice factors. The maximum graded quotient need not exist, but if it exists, it is unique. We calculate explicitly the maximum graded quotient of the free product of the $k+1$-element chain C_k with the single element lattice S and show that it is finite and distributive.

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The Maximum Graded Quotient of a Lattice

The maximum graded quotient of a lattice L is, by definition, a quotient $\phi : L \to H$ such that H is graded and for any quotient $\psi : L \to K$, where K is graded, ψ factors through ϕ.

Note that a lattice may have no maximum graded quotient. The lattice shown in Figure 1, for instance, has no maximum graded quotient.

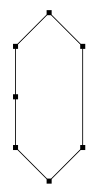

Figure 1

For L a lattice, we can define an equivalence relation $\sim$ on L where $x \sim y$ iff $\theta(x) = \theta(y)$ for all θ quotient maps from L to a graded lattice. It is straightforward to check that this is a lattice congruence. We then define $g(L) = L/\sim$. For the lattice in Figure 1, $g(L) = L$.

The following lemma is immediate.

Lemma 1. *If $g(L)$ is graded, then it is the unique maximum graded quotient of L. If $g(L)$ is not graded, then L has no maximum graded quotient.*

For $x \in L$, we will write $[x]$ for the class of x in $g(L)$. We write $a \lessdot b$ to indicate that either $a \lessdot b$ or $a = b$.

Lemma 2. *If $[x] \lessdot [y] \lessdot [z]$ (for instance, if $x \lessdot y \lessdot z$ in L), and $[x] \leq [u] \leq [v] \leq [z]$, such that $[u] \vee [y] = [z]$ and $[v] \wedge [y] = [x]$, then $[u] = [v]$.*

Proof. We consider separately graded quotients of $g(L)$ where $[y]$ is identified with $[x]$, where $[y]$ is identified with $[z]$, where $[y]$ is not identified with either $[x]$ or $[z]$, and where $[x]$, $[y]$, and $[z]$ are all identified. We see that in all these cases, $[u]$ and $[v]$ must be identified in the quotient. Since every graded quotient of L factors through $g(L)$, this implies that u and v are identified in any graded quotient of L, and therefore $[u] = [v]$.

Maximum graded quotient of $C_k * S$

Let C_k denote the lattice of length k, with elements $x_0 \lessdot \cdots \lessdot x_k$. Let S denote the one element lattice, with a single element y.

Lemma 3. *$C_k * S$ is a disjoint union of elements lying above x_0 and elements lying below y.*

Proof. This is an immediate application of the Splitting Theorem [Gr, Theorem VI.1.11], which says that the free product of two lattices A and B is the disjoint union of the dual ideal generated by A and the ideal generated by B.

We shall now proceed to consider these two subsets of $C_k * S$ in more detail.

Lemma 4. *The elements of $C_k * S$ lying below y are exactly y and $y \wedge x_i$ for $0 \leq i \leq k$.*

Proof. For $f \in C_k * S$, write $f^{(x)}$ for the smallest element of the C_k which lies above f. If there is no such element, set $f^{(x)} = \hat{1}$.

Lemma 5. *For $f \in C_k * S$, $f \wedge y = f^{(x)} \wedge y$.*

Proof. By definition, $f^{(x)} \geq f$, so $f^{(x)} \wedge y \geq f \wedge x$. We prove the other inequality by induction on the rank of a polynomial expression for f. The statement is clearly true for rank 1 polynomials. Suppose that $f = g \wedge h$. Then $f^{(x)} = g^{(x)} \wedge h^{(x)}$ [Gr, Theorem VI.1.10].

$$f^{(x)} \wedge y = g^{(x)} \wedge h^{(x)} \wedge y \leq g \wedge h \wedge y = f \wedge x.$$

Suppose that $f = g \vee h$. Then $f^{(x)} = g^{(x)} \vee h^{(x)}$ [Gr, Theorem VI.1.10]. Since $C_x \cup \{\hat{1}\}$ forms a chain, we may assume without loss of generality that $f^{(x)} = g^{(x)}$. Thus,

$$f^{(x)} \wedge y = g^{(x)} \wedge y \leq g \wedge y \leq (g \vee h) \wedge y = f \wedge y.$$

This completes the proof of the lemma.

Thus, if $z \not\leq x_0$, then $z \leq y$, and $z = z \wedge y = z^{(x)} \wedge y$, and we have written z in the form described in the statement of Lemma 4.

Lemma 6. *The elements of $g(C_k * S)$ which lie strictly above x_0 are generated by $x_1, \ldots, x_n$, $y \vee x_0$.*

Proof. We begin by showing that the elements of $C_k * S$ lying strictly above x_0 are generated by $x_1, \ldots, x_n, y \vee x_0, (y \wedge x_1) \vee x_0, \ldots, (y \wedge x_n) \vee x_0$. Let T_0 denote $\{x_0, \ldots, x_n, y\}$. Define T_i inductively as those elements of $C_k * S$ which can be formed as either a meet or a join of a pair of elements in T_{i-1}. Clearly, the union of the T_i is $C_k * S$. We wish to show by induction on i that any element of the T_i lying strictly above x_0 can be written as a polynomial in $x_1, \ldots, x_n, y \vee x_0, (y \wedge x_1) \vee x_0, \ldots, (y \wedge x_n) \vee x_0$. The statement is certainly true for $i = 0$. Suppose it is true for $i - 1$. The statement is also true for an element of T_i formed by a meet, since if the meet lies strictly above x_0, so did both the elements of T_{i-1}. Now consider the case of the join of two elements, a and b, from T_{i-1}. If both a and b lie strictly above x_0, the statement is true for $a \vee b$ by induction. If neither a nor b lies strictly above x_0, then (by Lemma 4) one of a or b must equal x_0, and $a \vee b$ is one of the generators which we are allowing. Now suppose that a lies strictly above x_0 and b does not. If $b = x_0$, then $a \vee b = a$, and we are done. Otherwise, by Lemma 4, b equals either y or $y \wedge x_i$. In either case, $a \vee b = a \vee (b \vee x_0)$, and $b \vee x_0$ is one of the allowed generators, so we are done. We have shown that every element of T_i lying above x_0 can be written in the desired form, and hence by induction that the same is true of any element of $C_k * S$ lying above x_0.

It follows from Lemma 4 that $y \wedge x_n \lessdot y$. Dually, $y \lessdot y \vee x_0$. Observe that $y \wedge x_n < (y \wedge x_n) \vee x_0 < (y \vee x_0) \wedge x_n < y \vee x_0$ in $C_k * S$. Thus, by Lemma 2, $[(y \wedge x_n) \vee x_0] = [(y \vee x_0) \wedge x_n]$.

We now proceed to show that

$$[(y \wedge x_i) \vee x_0] = [(y \vee x_0) \wedge x_i]$$

for all $1 \leq i \leq n$. The proof is by downward induction; we have already finished the base case, when $i = n$. So suppose the result holds for $i + 1$. In L,

$$y \wedge x_i \lessdot y \wedge x_{i+1} \lessdot (y \wedge x_{i+1}) \vee x_0 < (y \vee x_0) \wedge x_{i+1},$$

but when we pass to $g(L)$ the final inequality becomes an equality by the induction hypothesis. Since in L we also have that

$$y \wedge x_i < (y \wedge x_i) \vee x_0 < (y \vee x_0) \wedge x_i < (y \vee x_0) \wedge x_{i+1},$$

we can apply Lemma 2 to conclude that $[(y \wedge x_i) \vee x_0] = [(y \vee x_0) \wedge x_i]$ as desired.

We have already shown that the elements of L lying above x_0 are generated by the x_i, $y \vee x_0$, and the $(y \wedge x_i) \vee x_0$, for $i \geq 1$. It follows that the elements of $g(L)$ above $[x_0]$ are generated by the $[x_i]$, $[y \vee x_0]$, and the $[(y \wedge x_i) \vee x_0]$. But $[(y \wedge x_i) \vee x_0] = [(y \vee x_0) \wedge x_i] = [y \vee x_0] \wedge [x_i]$, and so the $[(y \wedge x_i) \vee x_0]$ are redundant, proving the lemma.

Proposition 1. *The lattice $g(C_k * S)$ is as shown in Figure 2.*

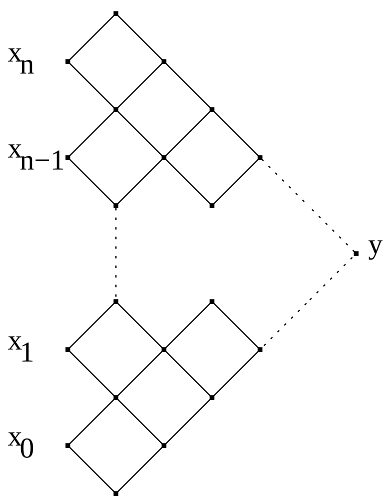

Figure 2

Proof. Observe that by Lemma 6, the elements of $g(C_k * S)$ lying strictly over x_0 are isomorphic to a quotient of $g(C_{k-1} * S)$. Now applying Lemma 4 inductively, we see that every element of $g(C_k * S)$ can be written as $(y \vee x_i) \ \wedge x_j$ for $j \geq i$. It follows that $g(C_k * S)$ is a quotient of the lattice from Figure 2, but since the lattice from Figure 2 is graded, it must coincide with $g(L)$.

Left Modular Lattices

In this section, we recall a few results about left modular elements and left modular lattices from [Li] and [MT].

Lemma 7 [Li]. *Suppose $u \lessdot v$ are left modular in L. Let $z \in L$. Then:*
(i) $u \vee z \leq v \vee z$.
(ii) $u \wedge z \leq v \wedge z$.

Proof. We prove (i). Suppose otherwise, so that there is some element y such that $u \vee z < y < v \vee z$. Now observe that $((u \vee z) \vee v) \wedge y = y$. However, $v \wedge y = u$. So $(u \vee z) \vee (v \wedge u) = u \vee z$, contradicting the left modularity of u. This proves (i). Now (ii) follows by duality.

Lemma 8 [MT]. *Let x be left modular, and $y < z$. Then $y \vee x \wedge z$ is left modular in $[y, z]$.*

Proof. Let $s < t$ in $[y, z]$.

$$(s \vee (y \vee x \wedge z)) \wedge t = (s \vee x \wedge z) \wedge t = s \vee x \wedge t = s \vee (y \vee x \wedge t) = s \vee ((y \vee x \wedge z) \wedge t).$$

Lemma 9 [MT]. *If L is a finite lattice with a maximal left modular chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = \hat{1}$, and $y \leq z$, then the set of all $\{y \vee x_i \wedge z\}$ forms a maximal left modular chain in $[y, z]$.*

Proof. The fact that the $\{y \vee x_i \wedge z\}$ form a maximal chain in $[y, z]$ follows from Lemma 7; the fact that they are left modular, from Lemma 8.

Modularity

For $y \leq z$, let us write $M(x, y, z)$ for the statement:

$$M(x, y, z): \; (y \vee x) \wedge z = y \vee (x \wedge z).$$

(Equivalently, for $y \leq z$, $M(x, y, z)$ asserts that the sublattice generated by x, y, and z is modular.)

Standard notation is to write xMz for the statement that $M(x, y, z)$ holds for all $y \leq z$. In this case (x, z) is called a *modular pair*. An element x is said to be modular if for any z both xMz and zMx are modular pairs. As we have already seen, an element x is *left modular* if it satisfies half the condition of being modular, namely that xMz for all z.

Let L be a finite graded left modular lattice, with maximal left modular chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = \hat{1}$. By definition, for any $y \leq z$, we have $M(x_i, y, z)$. We also have the following lemma:

Lemma 10. *In a finite graded left modular lattice L, with maximal chain $\{x_i\}$, for any $w \in L$ and $i < j$, we have $M(w, x_i, x_j)$.*

Proof. Consider the sublattice K of L generated by the x_i and w. First, we show that K is graded. Let $y < z \in K$. By Lemma 9, we know that the $y \vee x_i \wedge z$ form a maximal chain in L. These are all elements of K, so there is a maximal chain between y and z having the same length as in L. It follows that the covering relations in K are a subset of the covering relations in L, and hence that K is graded (with the same rank function as L).

Since K is generated by the x_i and w, K is a quotient of $C_r * S$. Further, since K is graded, it is a quotient of $g(C_r * S)$. Since $g(C_r * S)$ is distributive, the modular equality is always satisfied in it, and therefore also in K. So $M(w, x_i, x_j)$ holds in K, and therefore in L.

Graded Left Modular Lattices are Supersolvable

In this section, we prove Theorem 1, that finite graded left modular lattices are supersolvable. To do this, we have to show that the sublattice generated by the left modular chain and another chain is distributive.

The proof mimics the proof of Proposition 2.1 of [St], which shows that if L is a finite lattice with a maximal chain of modular elements, then this chain is an M-chain, and hence L is supersolvable. The proof from [St] is based on Birkhoff's proof in [Bi] that a modular lattice generated by two chains is distributive.

We recall briefly the way Birkhoff's proof works. Let L be a finite modular lattice, and let $\hat{0} = x_0 < \cdots < x_r = \hat{1}$ and $\hat{0} = y_0 < \cdots < y_s = \hat{1}$ be two chains both of which include $\hat{0}$ and $\hat{1}$. Let M denote the set of meets of $\{x_i \vee y_j\}$. We observe that any element of M can be written as

$$\bigvee_{i=1}^{t} a_i \wedge b_i$$

where the a_i form a decreasing sequence of elements from the x_i, and the b_i form an increasing sequence of elements from the y_i. Let J denote the set of joins of $\{x_i \wedge y_j\}$. It is clear that J is closed under joins and M is closed under meets. The following two identities are established for all a_i an decreasing sequence of x_i and b_i an increasing sequence of y_i, and all t:

$$\mathbf{P}_t\colon (b_1 \vee a_1) \wedge (b_2 \vee a_2) \wedge \cdots \wedge (b_t \vee a_t) = b_1 \vee (a_1 \wedge b_2) \vee \cdots \vee (a_{t-1} \wedge b_t) \vee a_t$$
$$\mathbf{Q}_t\colon (a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \cdots \vee (a_t \wedge b_t) = a_1 \wedge (b_1 \vee a_2) \wedge \cdots \wedge (b_{t-1} \vee a_t) \wedge b_t$$

Now consider the set of all lattice paths from $(r, 0)$ to $(0, s)$ which move up and to the left. We order lattice paths by inclusion on the region below and to the left of them. Associate to a path the join of all $x_i \wedge y_j$ with (i, j) on or below the path. We can see from $\mathbf{P}$ and $\mathbf{Q}$ that this is a lattice quotient. Thus, L is a quotient of a distributive lattice, and hence distributive.

The point at which modularity is used in Birkhoff's proof is in establishing $\mathbf{P}_t$ and $\mathbf{Q}_t$. Stanley noticed that it was sufficient to assume only that the x_i were modular. In fact, still less is sufficient.

Lemma 11. *$\mathbf{P}_t$ and $\mathbf{Q}_t$ hold in any graded lattice such that the x_i form a maximal chain of left modular elements.*

Proof. We prove $\mathbf{P}_t$ and $\mathbf{Q}_t$ by simultaneous induction on t. $\mathbf{P}_1$ and $\mathbf{Q}_1$ are tautologous. Assume that $\mathbf{P}_{t-1}$ and $\mathbf{Q}_{t-1}$ hold. We now prove $\mathbf{Q}_t$. Recall that the a_i are a decreasing sequence from the x_i, and the b_i an increasing sequence from the

y_i.

$$
\begin{aligned}
&(a_1 \wedge b_1) \vee \cdots \vee (a_{t-1} \wedge b_{t-1}) \vee (a_t \wedge b_t) \\
&\quad = \big((a_1 \wedge b_1) \vee \cdots \vee (a_{t-1} \wedge b_{t-1}) \vee a_t\big) \wedge b_t\big) \\
&\qquad \text{by } M(a_t, (a_1 \wedge b_1) \vee \cdots \vee (a_{t-1} \wedge b_{t-1}), b_t) \\
&\quad = \big[\big(a_1 \wedge (b_1 \vee a_2) \wedge \cdots \wedge (b_{t-2} \vee a_{t-1}) \wedge b_{t-1}\big) \vee a_t\big] \wedge b_t \\
&\qquad \text{by } \mathbf{Q}_{t-1} \\
&\quad = a_1 \wedge \big[\big((b_1 \vee a_2) \wedge \cdots \wedge (b_{t-2} \vee a_{t-1}) \wedge b_{t-1}\big) \vee a_t\big] \wedge b_t \\
&\qquad \text{by } M((b_1 \vee a_2) \wedge \cdots \wedge (b_{t-2} \vee a_{t-1}) \wedge b_{t-1}, a_t, a_1) \text{ (Lemma 10)} \\
&\quad = a_1 \wedge \big[\big((b_1 \vee a_2) \wedge \cdots \wedge (b_{t-2} \vee a_{t-1}) \wedge (b_{t-1} \vee \hat{0})\big) \vee a_t\big] \wedge b_t \\
&\quad = a_1 \wedge \big[\big(b_1 \vee (a_2 \wedge b_2) \cdots \vee (a_{t-1} \wedge b_{t-1})\big) \vee a_t\big] \wedge b_t \\
&\qquad \text{by } \mathbf{P}_{t-1} \\
&\quad = a_1 \wedge [(b_1 \vee a_2) \wedge \cdots \wedge (b_{t-1} \vee a_t)] \wedge b_t \\
&\qquad \text{by } \mathbf{P}_{t-1}.
\end{aligned}
$$

This proves $\mathbf{Q}_t$. The dual argument holds for $\mathbf{P}_t$, which completes the induction step, and the proof of the lemma

This shows that Birkhoff's proof can be adapted to our situation, proving Theorem 1.

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