SEMI-STABLE SUBCATEGORIES FOR EUCLIDEAN QUIVERS

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Dedicated to the memory of Dieter Happel

ABSTRACT. In this paper, we study the semi-stable subcategories of the category of representations of a Euclidean quiver, and the possible intersections of these subcategories. Contrary to the Dynkin case, we find out that the intersection of semi-stable subcategories may not be semi-stable. However, only a finite number of exceptions occur, and we give a description of these subcategories. Moreover, one can attach a simplicial complex to any acyclic quiver Q, and this simplicial complex allows one to completely determine the canonical decomposition of any dimension vector. This simplicial complex has a nice description in the Euclidean case: it is described using an arrangement of convex pieces of hyperplanes, each piece being indexed by a real Schur root or a set of quasi-simple objects.

1. INTRODUCTION

Semi-stable subcategories. In Mumford's geometric invariant theory, in order to form the quotient of a variety by a group action, one first replaces the variety by its semi-stable points. This important idea was interpreted in the setting of quiver representations by King [19].

Let Q be a (possibly disconnected) acyclic quiver, and k an algebraically closed field. In this paper, all quivers are acyclic. We write $\mathrm{rep}(Q)$ for the category of finite-dimensional representations of Q over k. For θ a $\mathbb{Z}$-linear functional on the Grothendieck group of $\mathrm{rep}(Q)$, referred to as a stability condition, King shows how to define a subcategory of θ-semi-stable objects in $\mathrm{rep}(Q)$. It is immediate from King's definition, which we shall recall below, that for any θ, the θ-semi-stable subcategory of $\mathrm{rep}(Q)$ is abelian and extension-closed. It is therefore natural to ask which abelian and extension-closed subcategories can arise as semi-stable subcategories. (All subcategories considered in this paper are full subcategories. Also, when we refer to abelian subcategories, we mean exact abelian subcategories, that is to say, subcategories that are abelian with respect to the abelian structure on the ambient category.)

In [16], two of the authors of the present paper showed that when Q is Dynkin, *any* abelian and extension-closed subcategory arises as the θ-semi-stable subcategory of a suitable choice of θ. In fact, somewhat more is known. For any acyclic quiver Q, the following conditions on an abelian extension-closed subcategory $\mathcal{A}$ of $\mathrm{rep}(Q)$ are equivalent:

- $\mathcal{A}$ admits a projective generator.
- $\mathcal{A}$ is generated by the elements of an exceptional sequence.
- $\mathcal{A}$ is equivalent to the category of representations of some quiver Q'.

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We refer to subcategories satisfying these equivalent conditions as finitely generated. In [16], it was shown for any acyclic quiver Q that any finitely generated, abelian and extension-closed subcategory of $\mathrm{rep}(Q)$ arises as the semi-stable subcategory for some stability condition. This resolves the Dynkin case because in that case, every abelian and extension-closed subcategory is finitely generated.

The next case one would hope to settle is the Euclidean case, which we resolve as follows:

Theorem 5.4. *For Q a Euclidean quiver, an abelian and extension-closed subcategory $\mathcal{B}$ of $\mathrm{rep}(Q)$ is the subcategory of θ-semi-stable representations for some θ if either:*

  (i) *$\mathcal{B}$ is finitely generated, or*
  (ii) *there exists some abelian, extension-closed, finitely generated subcategory $\mathcal{A}$ of $\mathrm{rep}(Q)$, equivalent to the representations of a Euclidean quiver (possibly disconnected), and $\mathcal{B}$ consists of all the regular objects of $\mathcal{A}$.*

In (ii) above, we refer to disconnected Euclidean quivers. A disconnected Euclidean quiver is defined to be a quiver with one Euclidean component and all other components (if any) Dynkin. A *regular object* is then a representation which is a direct sum of regular representations of the Euclidean component or a representation of a Dynkin component

The semi-stable subcategories of the second type can also be described more explicitly as follows:

Proposition 9.13. *The semi-stable subcategories in (ii) of Theorem 5.4 can also be described as those abelian extension-closed subcategories of the regular part of $\mathrm{rep}(Q)$, which contain infinitely many indecomposable objects from each tube.*

Equivalence of stability conditions and canonical decomposition of dimension vectors. There is a natural equivalence relation on stability conditions, which we call ss-equivalence, where θ and θ' are equivalent if they induce the same semi-stable subcategory. The geometry of this decomposition has been studied in [8, 9, 14, 5]. The space of stability conditions is naturally a free abelian group of finite rank, dual to the Grothendieck group, but in order to think about this equivalence, it is easier to work in the corresponding finite-dimensional vector space over $\mathbb{Q}$. It is also convenient to use the Euler form on the Grothendieck group to identify the Grothendieck group and its dual: we say that d_1 and d_2 are ss-equivalent if $\langle d_1, ? \rangle$ and $\langle d_2, ? \rangle$ are. (We recall the definition of the Euler form in Section 2.)

There is a collection $\mathcal{J}$ of convex codimension-one subsets ("pieces of hyperplanes") in $\mathbb{Q}^d$ such that d_1 and d_2 are ss-equivalent if and only if they lie in the same set of subsets from $\mathcal{J}$. For a dimension vector d, define

$$\Gamma_d = \{J \in \mathcal{J} \mid d \in J\}.$$

Then we have:

Theorem 7.7. *Let Q be a Euclidean quiver and d_1, d_2 two dimension vectors. Then d_1 and d_2 are ss-equivalent if and only if $\Gamma_{d_1} = \Gamma_{d_2}$.*

Somewhat surprisingly, the geometry of $\mathcal{J}$, which controls ss-equivalence, can also be used to describe canonical decompositions. Given a dimension vector for Q, i.e., a nonzero element $d \in \mathbb{N}^n$, where n is the number of vertices of Q, it was shown by Kac [18] that the dimension vectors of the indecomposable summands of

a generically-chosen representation of dimension vector d are well-defined. The expression of d as the sum of the dimension vectors of the indecomposable summands of a generically-chosen representation is called the canonical decomposition. If the same summands appear in the canonical decomposition of two dimension vectors, we say that the dimension vectors are cd-equivalent. We show that cd-equivalence can be characterized in terms of $\mathcal{J}$.

Specifically, let $\mathcal{L}$ be the set of all intersections of subsets of $\mathcal{J}$. We order $\mathcal{L}$ by inclusion. This is a lattice, which we call the intersection lattice associated to $\mathcal{J}$. For $L \in \mathcal{L}$, define the faces of L to be the connected components of the set of points which are in L but not in any smaller intersection. Define the faces of $\mathcal{L}$ to be the collection of all faces of all the elements of $\mathcal{L}$. Then we have the following theorem:

Theorem 8.2. *Two dimension vectors d_1, d_2 are cd-equivalent if and only if they lie in the same face of $\mathcal{L}$.*

Posets of subcategories. In Dynkin type, the semi-stable subcategories (or equivalently the abelian and extension-closed subcategories) form a lattice. It was shown in [16] that this poset is isomorphic to the lattice of noncrossing partitions associated to the Weyl group corresponding to Q. These lattices had already been studied by combinatorialists and group theorists and, especially relevant for our purposes, they play a central role in the construction of the dual Garside structure of Bessis [2] on the corresponding Artin group, which also leads to their use in constructing Eilenberg-Mac Lane spaces for the Artin groups [3, 4]. For these latter two uses, the lattice property of this partial order is essential.

For this reason, another motivation for our paper is to construct a potential replacement lattice in Euclidean type. The most obvious choice for a lattice associated to a general Q would be to take all the abelian and extension-closed subcategories. However, this lattice is very big and somehow non-combinatorial; already in Euclidean type, it contains the Boolean lattice on the $\mathbb{P}^1(k)$-many tubes. On the other hand, we could consider finitely generated abelian and extension-closed subcategories. It was shown in [16] (Euclidean type) and [15] (general acyclic Q) that this yields the natural generalization of the noncrossing partitions of the associated Weyl group. However, for Q non-Dynkin, this poset is typically not a lattice [11], which makes it unsuitable.

Therefore, in pursuit of a suitable lattice of subcategories associated to Q it seems that we need to consider a class of subcategories which are not all finitely generated, but not so broad as to include the full plethora of abelian extension-closed subcategories. The lattice property can be guaranteed in a natural way if we can verify that our class of subcategories is closed under intersections. It turns out that the semi-stable subcategories do not form a lattice, but it is quite easy to describe the subcategories that arise as intersections of semi-stable subcategories; this class of subcategories then (automatically) forms a lattice. Specifically, we show:

Theorem 10.7 (simplified form). *Let Q be a Euclidean quiver. There are finitely many subcategories of $\mathrm{rep}(Q)$ which arise as an intersection of semi-stable subcategories, and which are not themselves semi-stable subcategories for any stability condition. These subcategories are contained entirely in the regular representations of Q. Any such subcategory can be written as the intersection of two semi-stable subcategories.*

We hope to investigate the applicability of the lattice of these subcategories to the problem of constructing a dual Garside structure for Euclidean type Artin groups in subsequent work.

2. Some representation theory

Let $Q = (Q_0, Q_1)$ be an (acyclic) quiver with n vertices and let k be an algebraically closed field. For simplicity, we shall assume that $Q_0 = \{1, 2, \ldots, n\}$. Our main concern is to study the semi-stable subcategories of $\mathrm{rep}(Q)$ and their intersections, when Q is a *Euclidean quiver*. Unless otherwise specified, this includes the assumption that Q is connected. When Q is a Euclidean quiver, $\mathrm{rep}(Q)$ is of tame representation type and the representation theory of Q is well understood. For the basic results concerning the structure of $\mathrm{rep}(Q)$, the reader is referred to [1, 23]. We shall use many representation-theoretic results for $\mathrm{rep}(Q)$ and in particular, the structure of its Auslander-Reiten quiver. Let $\langle\ ,\ \rangle$ stand for the bilinear form defined on $\mathbb{Z}^n$ as follows. If $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ and $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$, then

$$\langle d, e \rangle = \sum_{i=1}^{n} d_i e_i \ - \sum_{\alpha: i \to j \in Q_1} d_i e_j.$$

This is known as the *Euler form* associated to Q. This is a (non-symmetric) bilinear form defined on the Grothendieck group $K_0(\mathrm{rep}(Q)) = \mathbb{Z}^n$ of $\mathrm{rep}(Q)$. For a representation M in $\mathrm{rep}(Q)$, we denote by d_M its dimension vector. A crucial property of the above bilinear form is the following.

Proposition 2.1. *Let $M, N \in \mathrm{rep}(Q)$. Then*

$$\langle d_M, d_N \rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

This result justifies the terminology *homological form* that is sometimes used for the Euler form. We shall call a nonzero element $d = (d_1, \ldots, d_n)$ in $\mathbb{N}^n$ a *dimension vector*, as it is the dimension vector of some representation in $\mathrm{rep}(Q)$. It is *sincere* if $d_i > 0$ for $i = 1, \ldots, n$; and called a *root* if there exists an indecomposable representation M with $d_M = d$. The previous statement is equivalent, for Q connected, to $\langle d, d \rangle \le 1$; see [18]. If d is a root with $\langle d, d \rangle = 1$, then d is called a *real root*, and otherwise, an *imaginary root*. If d is a real root, then there is a unique, up to isomorphism, indecomposable representation having d as a dimension vector; see [18]. In this case, we write $M(d)$ for a fixed indecomposable representation of this dimension vector.

A root is called a *Schur root* if there exists an indecomposable representation M with $d = d_M$ and M having a trivial endomorphism ring. When d is real, the latter condition is equivalent to $\mathrm{Ext}^1(M, M) = 0$.

We refer to an infinite component of the Auslander-Reiten quiver of $\mathrm{rep}(Q)$ which contains projective representations as a *preprojective component* (it is unique when Q is connected and non-Dynkin); *preinjective component* is defined similarly.

A real Schur root d is called *preprojective* (resp. *preinjective*) if $M(d)$ lies in a preprojective (resp. preinjective) component of the Auslander-Reiten quiver of $\mathrm{rep}(Q)$. Otherwise, it is called *regular*.

Suppose now that Q is a Euclidean quiver. An imaginary root d is also called *isotropic* since $\langle d, d \rangle = 0$. Recall that the quadratic form $q(x) := \langle x, x \rangle$ is positive-semidefinite, and its radical is of rank one and is generated by a dimension vector δ. We shall call δ the *null root* of Q. Any isotropic root is a positive multiple of δ.

Observe moreover that δ is a Schur root and any other positive multiple of δ is also a root, but not a Schur root. The *regular roots* are the dimension vectors of the indecomposable representations which lie in a stable tube. Finally, observe that if d is a root, then $\langle \delta, d \rangle$ is zero (resp. negative, positive) if and only if d is regular (resp. preprojective, preinjective).

3. Canonical decompositions, semi-invariants and semi-stable subcategories

In this section, Q is any acyclic quiver, unless otherwise indicated. For a dimension vector $d = (d_1, \ldots, d_n)$, define $\operatorname{rep}(Q, d)$ to be the product of affine spaces specifying matrix entries for each of the arrows of Q. An element in $\operatorname{rep}(Q, d)$ is canonically identified with the corresponding representation in $\operatorname{rep}(Q)$ of dimension vector d. The reductive algebraic group $\operatorname{GL}_d(k) := \prod_{i=1}^{n} \operatorname{GL}_{d_i}(k)$ acts on $\operatorname{rep}(Q, d)$ by simultaneous change of basis and has a natural subgroup $\operatorname{SL}_d(k) := \prod_{i=1}^{n} \operatorname{SL}_{d_i}(k)$ which is a reductive subgroup as well. A *semi-invariant* for $\operatorname{rep}(Q, d)$ is just a polynomial function in $k[\operatorname{rep}(Q, d)]$ which is stable under the action of $\operatorname{SL}_d(k)$.

Now any linear map in $\operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is called a *weight* or a *stability condition* for Q. For an element $g = (g_1, \ldots, g_n)$ in $\operatorname{Gl}_d(k)$, we set

$$w(g) = w(g_1, \ldots, g_n) = \prod_{i=1}^{n} \det(g_i)^{w(e_i)},$$

where for $1 \leq i \leq n$, the vector e_i is the dimension vector of the simple representation at vertex i. In this way, the set of all weights for Q corresponds to the set of all multiplicative characters for $\operatorname{GL}_d(k)$. Following King [19], an element $f \in k[\operatorname{rep}(Q, d)]$ is called a *semi-invariant of weight* w if $g(f) = w(g)f$ for any $g = (g_1, \ldots, g_n)$ in $\operatorname{Gl}_d(k)$. We write $\operatorname{SI}(Q, d)_w$ for the set of semi-invariants of weight w of $\operatorname{rep}(Q, d)$.

One has a direct sum decomposition, also called a *weight space decomposition*,

$$\operatorname{SI}(Q, d) = \bigoplus_{w \text{ weight}} \operatorname{SI}(Q, d)_w.$$

We refer the reader to [9] for fundamental results on semi-invariants of quivers. Let us just recall the results we need. Given two representations M, N, we have an exact sequence

$$(1) \qquad 0 \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(P_0, N) \xrightarrow{f_N^M} \operatorname{Hom}(P_1, N) \to \operatorname{Ext}^1(M, N) \to 0$$

where $0 \to P_1 \to P_0 \to M \to 0$ is a fixed projective resolution of M. We see that the k-linear map f_N^M corresponds to a square matrix if and only if $\langle d_M, d_N \rangle = 0$. In this case, we define $C^M(N)$ to be the determinant of f_N^M. Then $C^M(-)$ is a semi-invariant of weight $\langle d_M, - \rangle$ for $\operatorname{rep}(Q, d_N)$. The following result was proven in [8] for the general case and in [22] for the characteristic zero case.

Proposition 3.1. *Let w be a weight and d a dimension vector. Then the space $\operatorname{SI}(Q, d)_w$ is generated over k by the semi-invariants $C^M(-)$ where $\langle d_M, d \rangle = 0$ and $w = \langle d_M, - \rangle$.*

Now, let us introduce the main object of study of this paper. Let w be a weight. A representation M in $\operatorname{rep}(Q)$ is said to be *semi-stable with respect to* w if there exist $m > 0$ and $f \in \operatorname{SI}(Q, d_M)_{mw}$ such that $f(M) \neq 0$. By Proposition 3.1, a

weight associated to a semi-stable nonzero representation needs to be of the form $\langle d, -\rangle$ for a dimension vector d. Given a dimension vector d, write $\mathrm{rep}(Q)_d$ for the full subcategory of $\mathrm{rep}(Q)$ of the semi-stable representations with respect to the weight $\langle d, -\rangle$. We warn the reader here not to confuse $\mathrm{rep}(Q)_d$ with $\mathrm{rep}(Q, d)$. These subcategories will be referred to as the *semi-stable subcategories* of $\mathrm{rep}(Q)$. The following observation will be handy in the sequel.

Corollary 3.2. *Let $M \in \mathrm{rep}(Q)$ and d be a dimension vector. Then $M \in \mathrm{rep}(Q)_d$ if and only if there exist a positive integer m and a representation V in $\mathrm{rep}(Q, md)$ with $C^V(M) \neq 0$.*

Proof. Assume that $M \in \mathrm{rep}(Q)_d$. Then, there exist $m > 0$ and $f \in \mathrm{SI}(Q, d_M)_{\langle md, -\rangle}$ such that $f(M) \neq 0$. By Proposition 3.1, there exists a representation $V \in \mathrm{rep}(Q)$ with $C^V(M) \neq 0$ and $\langle md, -\rangle = \langle d_V, -\rangle$. Since $\langle -, -\rangle$ is non-degenerate, we have $d_V = md$ and this proves the necessity. The sufficiency follows from the definitions of $\mathrm{rep}(Q)_d$ and $\mathrm{SI}(Q, d_M)_{\langle md, -\rangle}$. $\qquad\square$

Now, consider $\mathrm{K}_0(\mathrm{rep}(Q)) \otimes \mathbb{Q}$ and let Δ be the $n - 1$ simplex in the projective space $\mathbb{P}^{n-1}(\mathrm{K}_0(\mathrm{rep}(Q)) \otimes \mathbb{Q})$ generated by the elements $[1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0 : 1]$. For convenience, if d is a (nonzero) dimension vector, we also write d for the corresponding element in Δ. Conversely, if $[a_1 : \ldots : a_n] \in \Delta$, then there exists a unique dimension vector $d = (d_1, \ldots, d_n)$ with $\gcd(d_1, \ldots, d_n) = 1$ and $[a_1 : \ldots : a_n] = [d_1 : \ldots, d_n]$. Given a dimension vector d, we define H_d to be the set of all elements f in $\mathbb{P}^{n-1}(\mathrm{K}_0(\mathrm{rep}(Q)) \otimes \mathbb{Q})$ with $\langle f, d \rangle = 0$. We denote by H_d^Δ the intersection of H_d with Δ. When Q is a Euclidean quiver, a distinguished such set is H_δ^Δ defined by the equation $\langle -, \delta \rangle = 0$ (or equivalently, of equation $\langle \delta, -\rangle = 0$). As observed above, the roots in H_δ^Δ are precisely the regular roots.

Now, let us recall the concept of canonical decomposition defined by Kac [17] of a dimension vector; see also [9, 18]. Let d be a dimension vector. Suppose that there exists an open set $\mathcal{U}$ in $\mathrm{rep}(Q, d)$ and dimension vectors $d(1), \ldots, d(r)$ satisfying the following property: for each $M \in \mathcal{U}$, there exists a decomposition

$$M \cong M_1 \oplus \cdots \oplus M_r$$

where each M_i is indecomposable of dimension vector $d(i)$. (Note that the isomorphism class of M_i may vary depending on the choice of $M \in \mathcal{U}$ — all that is fixed is the dimension vector.) In this case, we write

$$d = d(1) \oplus \cdots \oplus d(r)$$

and call this expression the *canonical decomposition* of d. The canonical decomposition always exists and is unique; see [17]. For more details concerning the canonical decomposition, we refer the reader to [9]. Let us recall two fundamental results due to Kac and Schofield. Given two roots d_1, d_2, we set

$$\mathrm{ext}(d_1, d_2) = \min(\{\dim_k \mathrm{Ext}^1(M_1, M_2) \mid d_{M_1} = d_1, d_{M_2} = d_2\}).$$

The following result is due to Kac [18].

Proposition 3.3 (Kac). *A decomposition*

$$d = d(1) + \cdots + d(m),$$

where the $d(i)$ are dimension vectors corresponds to the canonical decomposition for d if and only if the $d(i)$ are all Schur roots and $\mathrm{ext}(d(i), d(j)) = 0$ for $i \neq j$.

Actually, using the previous notations, we have a much stronger statement; see [18]. There exists an open set $\mathcal{U}'$ in $\mathrm{rep}(Q, d)$ such that for any $M \in \mathcal{U}'$,

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_r$$

where each M_i is a Schur representation of dimension vector $d(i)$ and we have $\mathrm{Ext}^1(M_i, M_j) = 0$ if $i \neq j$. A representation M which decomposes as a direct sum of pairwise Ext-orthogonal Schur representations will be called a *general representation of dimension vector d*. Hence, given a general representation M of dimension vector d, one can recover the canonical decomposition of d by looking at the dimension vectors of the indecomposable summands of M.

A dimension vector d whose canonical decomposition only involves real Schur roots is called *prehomogeneous*. In such a case, there is a unique, up to isomorphism, general representation of dimension vector d. A fixed such representation will be denoted by $M(d)$ in the sequel. Note that $\mathrm{Ext}^1(M(d), M(d)) = 0$. When d is a real Schur root, this notation agrees with our previous notation. In the sequel, for a dimension vector d, we will denote by $\hat{d}$ the sum of the real Schur roots appearing in its canonical decomposition.

The following result, due to Schofield [21], shows how the canonical decomposition behaves when we multiply a dimension vector by a positive integer. For a Schur root d and a positive integer r, we use the following notation

$$d^r = \begin{cases} d \oplus \cdots \oplus d\,(r\,\text{copies}), & \text{if } d \text{ is real or isotropic} \\ rd\,, & \text{otherwise.} \end{cases}$$

Proposition 3.4 (Schofield)**.** *Let d be a dimension vector with canonical decomposition $d = d(1) \oplus \cdots \oplus d(r)$. For m a positive integer, md has canonical decomposition $md = d(1)^m \oplus \cdots \oplus d(r)^m$.*

Observe that when Q is of wild type and s is a Schur imaginary non-isotropic root, then all the positive multiples of s are also Schur imaginary roots corresponding to the same point in Δ. When Q is a Dynkin or a Euclidean quiver, two distinct Schur roots correspond to different points in Δ.

For any acyclic quiver Q, we will give Δ the structure of a simplicial complex. First, for a Schur root s, we denote by $[s]$ the set of all Schur roots that are equal to s as elements of Δ. The cardinality of $[s]$ is greater than one if and only if s is imaginary and non-isotropic. The vertices of the simplicial complex are the $[s]$ for s a Schur root. There exists a $(m-1)$-simplex generated by the vertices $[s_1], \ldots, [s_m]$ if and only if there exists a dimension vector d such that the canonical decomposition of d involves exactly the Schur roots $s_1, \ldots, s_m$ (with some multiplicities). The following Lemma says in particular that this construction is well-defined and makes Δ into a simplicial complex.

Lemma 3.5. *For an acyclic quiver Q, the above-defined structure makes Δ into a simplicial complex of dimension $n - 1$.*

Proof. To show that the above construction is well-defined, suppose that the canonical decomposition of a dimension vector d is $d = d(1) \oplus \cdots \oplus d(s) \oplus \cdots \oplus d(r)$, where for $1 \leq j \leq s$, the $d(j)$ are imaginary non-isotropic, and for $s + 1 \leq j \leq r$, the $d(j)$ are either real or isotropic. We have to show that for s non-zero, and for any $1 \leq j \leq s$ with $d(j)' \in [d(j)]$,

$$d(1)' \oplus \cdots \oplus d(s)' \oplus d(s+1) \oplus \cdots \oplus d(r)$$

is the canonical decomposition of $f := d(1)' + \cdots + d(s)' + d(s+1) + \cdots + d(r)$. Let the canonical decomposition of f be

$$f = f(1) \oplus \cdots \oplus f(t).$$

There are positive integers $p_1, \ldots, p_s, \ r_1, \ldots, r_s$ such that $p_j d(j) = r_j d(j)'$ for $1 \le j \le s$. Let ℓ be the least common multiple of the p_j and r_j. From Proposition 3.4, we know that the canonical decomposition of ℓf is

$$\ell f = f(1)^{\ell} \oplus \cdots \oplus f(t)^{\ell}.$$

Observe now that

$$\begin{aligned}
\ell f &= \ell d'(1) + \cdots + \ell d'(s) + \ell d(s+1) + \cdots + \ell d(r) \\
&= t_1 d(1) + \cdots + t_s d(s) + \ell d(s+1) + \cdots + \ell d(r)
\end{aligned}$$

for some positive integers t_j. Using the canonical decomposition of d, we see that there are indecomposable Schur representations $M_1, \ldots, M_r$, that are pairwise Ext-orthogonal such that $d_{M_j} = d(j)$. Using these Schur representations, we see that the representations

$$M_1^{t_1}, \ldots, M_s^{t_s}, M_{s+1}, \ldots, M_r$$

are pairwise Ext-orthogonal. Moreover, for $s+1 \le j \le r$, M_j is Ext-orthogonal to itself, since M_j is exceptional. Hence, using the definition of ext, we see that the Schur roots

$$\{t_1 d(1) = \ell d(1)', \ldots, t_s d(s) = \ell d(s)', d(s+1), \ldots, d(r)\}$$

are ext-orthogonal. Hence, from Proposition 3.3, the canonical decomposition of ℓf is also given by

$$(d(1)')^{\ell} \oplus \cdots \oplus (d(s)')^{\ell} \oplus d(s+1)^{\ell} \oplus \cdots \oplus d(r)^{\ell}.$$

By uniqueness of the canonical decomposition, we have

$$f(1)^{\ell} \oplus \cdots \oplus f(t)^{\ell} = (d(1)')^{\ell} \oplus \cdots \oplus (d(s)')^{\ell} \oplus d(s+1)^{\ell} \oplus \cdots \oplus d(r)^{\ell},$$

from which it follows that $t = r$ and

$$\{f(j) \mid 1 \le j \le t\} = \{d(j) \mid s+1 \le j \le r\} \cup \{d'(j) | 1 \le j \le s\}.$$

This proves the fact that the above construction is well-defined.

Let us now show that the construction makes Δ into a simplicial complex. From Proposition 3.3, we see that if $d = d(1) \oplus \cdots \oplus d(r)$ is the canonical decomposition of d, then $d - d(1) = d(2) \oplus \cdots \oplus d(r)$ is the canonical decomposition for $d - d(1)$. This shows that the facets of a simplex are also simplices. Using the uniqueness of the canonical decomposition of a dimension vector, we see that if two simplices intersect in their relative interior, then they coincide. Therefore, the intersection of two simplices is clearly a simplex. □

The above simplicial complex will be called the *canonical decomposition simplicial complex* or *cd-simplicial complex* of Δ, or of Q. This simplicial complex is such that every point in Δ lies in the relative interior of exactly one simplex. (The relative interior of a 0-simplex is itself.) If Q is Dynkin or Euclidean, then every vertex of the cd-simplicial complex corresponds to a unique Schur root and hence, one can recover the canonical decomposition of a dimension vector d if one knows the simplex in whose relative interior d lies.

4. Exceptional objects and exceptional sequences

A representation M in $\mathrm{rep}(Q)$ with $\mathrm{Ext}^1(M, M) = 0$ is said to be *rigid*. If it is also indecomposable, it is called *exceptional*. It is well known (see for example [9]) that a representation M is rigid if and only if its orbit in $\mathrm{rep}(Q, d_M)$ is open. A sequence of exceptional representations $(E_1, \ldots, E_r)$ is called an *exceptional sequence* if for all $i < j$, we have $\mathrm{Hom}(E_i, E_j) = 0 = \mathrm{Ext}^1(E_i, E_j)$. Note that some authors define an exceptional sequence using the "upper triangular" convention rather than the "lower triangular" convention that is used here.

The maximum length of an exceptional sequence in $\mathrm{rep}(Q)$ is the number of vertices in Q. Any exceptional sequence can be extended to one of maximal length; see [6].

Given any full subcategory $\mathcal{V}$ of $\mathrm{rep}(Q)$, which may consist of a single object, we define $\mathcal{V}^\perp$ to be the full subcategory of $\mathrm{rep}(Q)$ generated by the representations M for which $\mathrm{Hom}(V, M) = \mathrm{Ext}^1(V, M) = 0$ for every $V \in \mathcal{V}$. We call $\mathcal{V}^\perp$ the *right orthogonal category* to $\mathcal{V}$. One also has the dual notion of *left orthogonal category* to $\mathcal{V}$, denoted $^\perp\mathcal{V}$. We denote by $\mathcal{C}(\mathcal{V})$ the smallest abelian extension-closed subcategory of $\mathrm{rep}(Q)$ generated by the objects in $\mathcal{V}$. If E is an exceptional sequence, then $\mathcal{C}(E)$ is the smallest abelian extension-closed subcategory of $\mathrm{rep}(Q)$ generated by the objects of that sequence. We need the following well known fact. It was first proven by Geigle and Lenzing in [12], and later by Schofield in [21].

Proposition 4.1. *Let X be an exceptional representation in $\mathrm{rep}(Q)$, where Q is an acyclic quiver. Then the category $X^\perp$ (or $^\perp X$) is equivalent to $\mathrm{rep}(Q')$ where Q' is an acyclic quiver having $|Q_0| - 1$ vertices.*

As an easy consequence, we have the following.

Proposition 4.2. *Let $E = (X_1, \ldots, X_r)$ be an exceptional sequence in $\mathrm{rep}(Q)$ where Q is an acyclic quiver. Then $\mathcal{C}(E)^\perp$ (or $^\perp\mathcal{C}(E)$) is equivalent to $\mathrm{rep}(Q')$ where Q' is an acyclic quiver having $|Q_0| - r$ vertices.*

The following lemma is probably well known. We include a proof for completeness.

Lemma 4.3. *Let Q be a Euclidean quiver and V, X be indecomposable representations with X non-exceptional.*
(1) If V is preprojective, then $\mathrm{Hom}(V, X) \neq 0$.
(2) If V is preinjective, then $\mathrm{Ext}^1(V, X) \neq 0$.

Proof. We only prove the second part. Suppose that V is preinjective. There exists a non-negative integer r with $\tau^{-r}V$ injective indecomposable. Hence, by the Auslander-Reiten formula,

$$\mathrm{Ext}^1(V, X) \cong D\mathrm{Hom}(X, \tau V) \cong D\mathrm{Hom}(\tau^{-r-1}X, \tau^{-r}V).$$

Being non-exceptional, X lies in a stable tube of the Auslander-Reiten quiver of $\mathrm{rep}(Q)$. We also have that all the representations in the cyclic τ-orbit of X are sincere, since they are non-exceptional. Hence, $\tau^{-r-1}X$ is sincere. Since $\tau^{-r}V$ is injective indecomposable, we clearly have $\mathrm{Hom}(\tau^{-r-1}X, \tau^{-r}V) \neq 0$. $\square$

Proposition 4.4. *Let $\mathcal{C}$ be an abelian extension-closed subcategory of $\mathrm{rep}(Q)$. The following are equivalent.*
(a) $\mathcal{C}$ is generated by an exceptional sequence,

(b) $\mathcal{C}$ *is equivalent to* $\mathrm{rep}(Q')$ *for an acyclic quiver* Q',
(c) $\mathcal{C} = V^{\perp}$ *for some rigid representation* V *of* $\mathrm{rep}(Q)$.

Proof. Suppose that $\mathcal{C}$ is generated by an exceptional sequence $E = (X_{r+1}, \ldots, X_n)$. Then, as was already remarked, E can be completed to a full exceptional sequence

$$E' = (X_1, \ldots, X_r, X_{r+1}, \ldots, X_n).$$

Then $\mathcal{C} = \mathcal{C}(E'')^{\perp}$ where $E'' = (X_1, \ldots, X_r)$ is an exceptional sequence. By Proposition 4.2, we get that $\cap_{i=1}^r X_i^{\perp} = \mathcal{C}(E'')^{\perp}$ is equivalent to $\mathrm{rep}(Q')$ for an acyclic quiver Q'. This proves that (a) implies (b). The fact that (b) implies (a) follows from the fact that the module category of any triangular algebra is generated by an exceptional sequence, and an exceptional sequence for $\mathrm{rep}(Q')$ is also an exceptional sequence for $\mathrm{rep}(Q)$, since $\mathrm{rep}(Q')$ is an exact subcategory of $\mathrm{rep}(Q)$.

Suppose now that $\mathcal{C}$ is generated by an exceptional sequence $E = (X_{r+1}, \ldots, X_n)$. As done in the first part of the proof, E can be completed to a full exceptional sequence $E' = (X_1, \ldots, X_r, X_{r+1}, \ldots, X_n)$. Then $E'' = (X_1, \ldots, X_r)$ is an exceptional sequence and $\mathcal{C}(E'')$ is equivalent to $\mathrm{rep}(Q')$ for an acyclic quiver Q'. Take V a minimal projective generator of $\mathcal{C}(E'')$. Then V is rigid and $\mathcal{C}(V) = \mathcal{C}(E'')$ with $V^{\perp} = \mathcal{C}(E'')^{\perp} = \mathcal{C}$. This proves that (a) implies (c). Finally, assume that $\mathcal{C} = V^{\perp}$ for a rigid representation V. Then V is a partial tilting module and by Bongartz's lemma, it can be completed to a tilting module $V \oplus V'$. Then $\mathrm{End}(V \oplus V')$ is a tilted algebra and hence has no oriented cycles in its quiver. In particular, the indecomposable summands of V form an exceptional sequence $E'' = (Y_1, \ldots, Y_s)$ that can be completed to a full exceptional sequence $E' = (Y_1, \ldots, Y_s, Y_{s+1}, \ldots, Y_n)$. Then $V^{\perp} = \mathcal{C}(E)$, where $E = (Y_{s+1}, \ldots, Y_n)$. This proves that (c) implies (a). $\square$

We can refine the previous proposition in the case that Q is Euclidean. Recall our definition that a *possibly disconnected Euclidean* quiver is a quiver which has one Euclidean component and any number of Dynkin components. Similarly a *possibly disconnected Dynkin* quiver is a quiver whose connected components are all of Dynkin type.

Lemma 4.5. *Let* Q *be a Euclidean quiver and* V *be a rigid representation. Then* $V^{\perp}$ *is equivalent to the representations of a (possibly disconnected) Euclidean quiver if* V *is regular; otherwise,* $V^{\perp}$ *is equivalent to the representations of a (possibly disconnected) Dynkin quiver.*

Proof. If V is regular, then $V^{\perp}$ will include all the objects from the non-homogeneous tubes, and in particular, it will include representations whose dimension vector is the unique imaginary Schur root. Since we know $V^{\perp}$ is equivalent to the representations of some quiver, it must be equivalent to the representations of a Euclidean quiver. (It clearly cannot be equivalent to the representations of a wild quiver, since it is contained in $\mathrm{rep}(Q)$.)

If V is not regular, then let Y be a non-regular indecomposable summand of V. Applying Lemma 4.3, we see that $Y^{\perp}$ contains no non-exceptional indecomposables. Thus $V^{\perp} \subseteq Y^{\perp}$ must be of finite representation type. $\square$

5. Description of semi-stable subcategories — proof of Theorem 5.4

In this section, we prove Theorem 5.4. We begin with Q any acyclic quiver, and then specialize to the Euclidean case. The following proposition is well known, but we include a proof of our own.

Proposition 5.1. *Suppose that d is a prehomogeneous dimension vector and set $V = M(d)$. Then $\mathrm{rep}(Q)_d = \{M \in \mathrm{rep}(Q) \mid C^V(M) \neq 0\} = V^\perp$.*

Proof. Let $M \in \mathrm{rep}(Q)_d$. By Corollary 3.2, there exist $m > 0$ and $V' \in \mathrm{rep}(Q, md)$ with $C^{V'}(M)$ nonzero. Now, $C^-(M)$ is a semi-invariant on $\mathrm{rep}(Q, md)$. Since V is rigid, V^m is also rigid and its orbit $\mathcal{O}(V^m)$ is open in $\mathrm{rep}(Q, md)$. If $C^{V^m}(M) = 0$, then $C^-(M)$ vanishes on the open set $\mathcal{O}(V^m)$, which gives $C^-(M) = 0$, contradicting that $C^{V'}(M) \neq 0$. This shows that $C^{V^m}(M) \neq 0$ and hence, that $C^V(M) \neq 0$. Conversely, it is clear that if $M \in \mathrm{rep}(Q)$ with $C^V(M) \neq 0$, then $M \in \mathrm{rep}(Q)_d$. This proves that

$$\mathrm{rep}(Q)_d = \{M \in \mathrm{rep}(Q) \mid C^V(M) \neq 0\}.$$

The fact that $\{M \in \mathrm{rep}(Q) \mid C^V(M) \neq 0\} = V^\perp$ follows from the exact sequence (1) and the definition of $C^V(M)$. $\qquad\square$

Let us denote by $\mathcal{R}\mathrm{eg}$ the full additive subcategory of $\mathrm{rep}(Q)$ generated by the indecomposable regular representations. When Q is a Euclidean quiver, the canonical decomposition of a dimension vector only involves real Schur roots and possibly the null root δ. The following result complements the previous one in the Euclidean case.

Proposition 5.2. *Let Q be a Euclidean quiver. Suppose that d is a dimension vector with canonical decomposition $d = \hat{d} \oplus \delta^s$ with $s \neq 0$. Set $V = M(\hat{d})$. Then*

$$\mathrm{rep}(Q)_d = \{M \in \mathrm{rep}(Q) \mid C^V(M) \neq 0\} \cap \mathcal{R}\mathrm{eg} = V^\perp \cap \mathcal{R}\mathrm{eg}.$$

Proof. Let $M \in \mathrm{rep}(Q)_d$ be indecomposable. By Corollary 3.2, there exists $V' \in \mathrm{rep}(Q, md)$, for some $m > 0$, with $C^{V'}(M) \neq 0$. By Proposition 3.4, the canonical decomposition of md is

$$d(1)^m \oplus \cdots \oplus d(r)^m \oplus \delta^{ms}.$$

Let $\mathcal{U}$ be an open set in $\mathrm{rep}(Q, md)$ defined by the latter canonical decomposition. Since the set

$$\{N \in \mathrm{rep}(Q, md) \mid C^N(M) \neq 0\}$$

is open and non-empty, it intersects with $\mathcal{U}$. Hence, we can assume that $V' \in \mathcal{U}$. In particular, V' decomposes as $V' \cong V^m \oplus V''$ where V'' is a direct sum of ms indecomposable representations of dimension vector δ. Therefore, $C^{V'}(M) \neq 0$ yields $C^V(M) \neq 0$ and $C^{V''}(M) \neq 0$. The first condition is equivalent to $M \in V^\perp$, and the second yields that $\langle \delta, d_M \rangle = 0$, which tells us that M is in $\mathcal{R}\mathrm{eg}$. Conversely, assume that M is an indecomposable representation both in $V^\perp$ and $\mathcal{R}\mathrm{eg}$. First, $M \in V^\perp$ is clearly equivalent to $C^V(M) \neq 0$. Since M is regular, M lies in a stable tube. Hence, there exists a quasi-simple representation W lying in a homogeneous tube (and hence having dimension vector δ) with $\mathrm{Hom}(W, M) = \mathrm{Ext}^1(W, M) = 0$. We therefore have $C^{V \oplus W^s}(M) \neq 0$, which shows that $M \in \mathrm{rep}(Q)_d$. $\qquad\square$

We also need the following simple lemma about canonical decompositions.

Lemma 5.3. *Let Q be a Euclidean quiver. Let d be a dimension vector which has the null root in its canonical decomposition. Then all the other summands of the canonical decomposition are regular.*

Proof. Suppose that some real Schur root α appears in the canonical decomposition of d. Seeking a contradiction, suppose that $M(\alpha)$ is preinjective. We know by Proposition 3.3 that there exists an indecomposable with dimension vector δ, say V, such that $\mathrm{Ext}^1(V, M(\alpha)) = 0 = \mathrm{Ext}^1(M(\alpha), V)$. On the other hand, Lemma 4.3 applied to $M(\alpha)$ and V implies that $\mathrm{Ext}^1(M(\alpha), V) \neq 0$. A similar contradiction follows dually if $M(\alpha)$ is preprojective. Therefore $M(\alpha)$ is regular. $\square$

We can now prove our first theorem from the introduction. Recall that for a possibly disconnected Euclidean quiver Q, the indecomposable *regular representations* are defined to be the regular representations of the Euclidean component together with the indecomposable representations of the Dynkin components.

Theorem 5.4. *For Q a Euclidean quiver, an abelian and extension-closed subcategory $\mathcal{B}$ of $\mathrm{rep}(Q)$ is the subcategory of θ-semi-stable representations for some θ if either:*

- *(i) $\mathcal{B}$ is finitely generated, or*
- *(ii) there exists some abelian, extension-closed, finitely generated subcategory $\mathcal{A}$ of $\mathrm{rep}(Q)$, equivalent to the representations of a Euclidean quiver (possibly disconnected), and $\mathcal{B}$ consists of all the regular objects of $\mathcal{A}$.*

Proof. Let d be a dimension vector. Suppose the canonical decomposition of d does not include the null root. Then by Proposition 5.1, we know that $\mathrm{rep}(Q)_d$ is of the form $V^\perp$ for some rigid representation V. By Proposition 4.4, this implies that it is finitely generated.

Suppose the canonical decomposition of d does include the null root. By Proposition 5.2, we know that the semi-stable category corresponding to d is of the form $V^\perp \cap \mathcal{R}\mathrm{eg}$. By Lemma 5.3, we can take V to be regular. By Lemma 4.5, $V^\perp$ is equivalent to the representations of a Euclidean quiver. The subcategory is therefore of type (ii).

Conversely, suppose that we have a finitely generated abelian and extension-closed subcategory $\mathcal{B}$ of $\mathrm{rep}(Q)$. It is generated by an exceptional sequence, say $(X_{r+1}, \ldots, X_n)$ which can be extended to a full exceptional sequence $(X_1, \ldots, X_n)$. Let $\mathcal{F}$ be the subcategory generated by $(X_1, \ldots, X_r)$, and let P be a projective generator of $\mathcal{F}$. Then $\mathcal{B} = P^\perp$. Since P is partial tilting, it is the general representation of dimension d_P. Therefore the semi-stable subcategory associated to d_P is $\mathcal{B}$.

Finally, suppose that we have an abelian category $\mathcal{B}$ as in (ii), which consists of the regular objects in some finitely generated abelian and extension-closed subcategory $\mathcal{A}$, with $\mathcal{A}$ equivalent to the representations of some Euclidean quiver. As in the previous case, we know that $\mathcal{A}$ can be written as $P^\perp$. Since $\mathcal{A}$ is of Euclidean type, P must be regular by Lemma 4.5. By Proposition 3.3, we know that a general representation of dimension vector $\delta + d_P$ will be isomorphic to a direct sum of P and an indecomposable representation of dimension vector δ. Therefore the semi-stable subcategory associated to $\delta + d_P$ will be $\mathcal{B}$. $\square$

6. Canonical decomposition on the regular hyperplane

In this section, Q is assumed to be a (connected) Euclidean quiver. Let us introduce some notations. A representation is called *quasi-simple* if it is a simple object in the full subcategory of $\mathrm{rep}(Q)$ consisting of the regular representations. A regular real Schur root which corresponds to a quasi-simple representation will

be called a *quasi-simple root*. We define $H^{ss}_\delta \subseteq H^\Delta_\delta$ to be the convex set in Δ generated by the quasi-simple roots and δ. We call it the *regular hyperplane* of Q or of Δ. Our goal in this section is to describe the geometry of H^{ss}_δ and the structure of the canonical decomposition for $d \in H^{ss}_\delta$.

Let $N \geq 0$ be the number of non-homogeneous tubes in the Auslander-Reiten quiver of $\mathrm{rep}(Q)$, and r_i, $1 \leq i \leq N$, be the ranks of these tubes. Let the quasi-simple roots in tube i be denoted $\beta_{i1}, \ldots, \beta_{ir_i}$. It is well known (see [23]) that

$$\sum_{i=1}^{N}(r_i - 1) = n - 2$$

if Q has more than 2 vertices. Hence there are exactly $n - 2 + N$ quasi-simple roots. If d_1, d_2 are roots corresponding to the indecomposable representations M_1, M_2, respectively, and M_2 is the Auslander-Reiten translate of M_1, then we write $\tau(d_1) = d_2$ and say that d_2 is the *Auslander-Reiten translate* of d_1. If C denotes the Coxeter transformation, this just means that $d_2 = C(d_1)$.

Lemma 6.1. *There are exactly $N - 1$ linear dependencies among the β_{ij}: exactly those arising from the fact that the sum of the quasi-simple roots of any tube equals the null root.*

Proof. It is well known that each tube contains representations whose dimension vector is the null root and that these representations have a filtration by quasi-simples in which each quasi-simple in the tube appears once, establishing that the dependencies mentioned in the statement of the lemma do actually arise. We need therefore only verify that there are no additional dependencies. Let $\sum c_{ij}\beta_{ij} = 0$ for some constants c_{ij}. By pairing $\sum c_{ij}\beta_{ij}$ with successive quasi-simples at the bottom of the i-th tube, we find that c_{ij} cannot depend on j. But then the linear dependency is one of the form we have already noted. $\square$

To describe the facets of H^{ss}_δ, we introduce some notation. If $|Q_0| > 2$, write R for the set of N-tuples $(a_1, \ldots, a_N)$ with $1 \leq a_i \leq r_i$. For $(a_1, \ldots, a_N) \in R$, write $F_{(a_1, \ldots, a_N)}$ for the convex hull of all the quasi-simple roots except for β_{ia_i} for $1 \leq i \leq N$.

Proposition 6.2. *Suppose that $|Q_0| > 2$. The boundary facets of H^{ss}_δ are exactly the F_I for $I \in R$.*

Proof. Think of Δ embedded inside affine $n - 1$-dimensional space, with δ at the origin. For $1 \leq i \leq N$, write P_i for the convex hull of the the quasi-simples from the i-th tube. It is clear that P_i is a simplex containing the origin (δ). By Lemma 6.1, the subspaces in which each of the P_i lie are complementary. Therefore, any convex combination of the quasi-simples can be expressed uniquely as a convex combination of vectors $v_i \in P_i$. It follows that codimension one facets of H^{ss}_δ are each formed by taking the convex hull of one facet from each P_i. The description of the facets given in the statement of the proposition follows. $\square$

If Σ_1 and Σ_2 are two simplicial complexes on disjoint vertex sets V_1, V_2, then the *simplicial join* of Σ_1 and Σ_2 is defined by saying that a set $F \subseteq V_1 \cup V_2$ is a face if and only if $F \cap V_1$ is a face of Σ_1 and $F \cap V_2$ is a face of Σ_2.
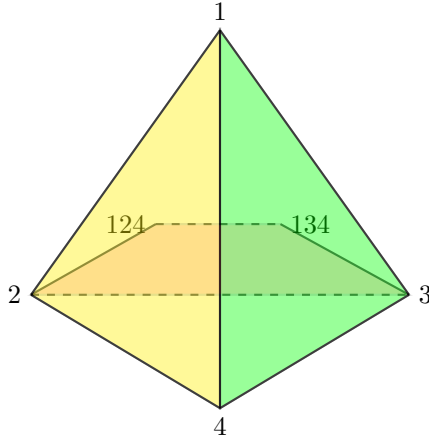
Corollary 6.3. *Suppose that $|Q_0| > 2$. The combinatorial structure of the boundary of H^{ss}_δ can be described as the simplicial join of the boundary of N simplices,*

one for each non-homogeneous tube, where the i-th simplex has as vertices the quasi-simples from the i-th tube

Example 6.4. Let Q be the quiver below:

$$1 \longrightarrow 3$$
$$\downarrow \qquad \downarrow$$
$$2 \longrightarrow 4$$
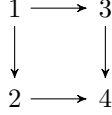
The following is an image of Δ, with H_δ^{ss} indicated:

Observe that H_δ^{ss} is not equal to $H_\delta \cap \Delta$ in this case (contrary to what one might have expected from smaller examples).

Corollary 6.5. *All the regular real Schur roots lie on the boundary of H_δ^{ss}.*

Proof. It is well known that regular real Schur roots correspond to representations which admit a filtration by quasi-simples from some tube, and which does not include all the quasi-simples from that tube. It therefore follows that each real Schur root lies on the boundary of some P_i (defined in the proof of Proposition 6.2), and thus on the boundary of H_δ^{ss}. $\qquad \square$

For Q the Kronecker quiver, that is, when $|Q_0| = 2$, we set R be a set with one element $\emptyset$ and we set $F_\emptyset = \emptyset$.

For each facet F_I of H_δ^{ss}, denote by C_I the $(n-2)$-simplex generated by the quasi-simple roots in F_I and the null root δ. We call C_I the *cone* corresponding to the facet F_I. Note that in the Kronecker case, there is a unique cone $C_\emptyset$, and it is equal to $\{\delta\} = H_\delta^{ss}$.

Proposition 6.6. *Let d be a dimension vector lying in H_δ^{ss}.*

(1) *If d lies on a facet F_I of H_δ^{ss}, then the canonical decomposition of d only involves regular real Schur roots in F_I.*

(2) *If d lies in the relative interior of H_δ^{ss}, say in C_I, then the canonical decomposition of d involves the null root and possibly some regular real Schur roots in F_I.*

Proof. From what we just proved, we have that δ lies in the relative interior of H_δ^{ss}. Moreover, all the quasi-simple roots lie on the boundary of H_δ^{ss}.

Suppose first that d is a dimension vector lying on the facet F_I of H_δ^{ss}. The additive abelian subcategory of $\mathrm{rep}(Q)$ generated by the quasi-simple representations corresponding to the roots in F_I is equivalent to the category of representations of a quiver which is a union of quivers of type $\mathbb{A}$. This observation together with Proposition 3.3 gives that the canonical decomposition of d only involves real Schur roots lying on F_I. Suppose now that d lies in the relative interior of H_δ^{ss}, say in C_I. In Δ, we can decompose d as $d = d_1 + r\delta$ where $d_1 \in F_I$ and $r \in \mathbb{Q}$. Hence, for some positive integers s, t, u, we have $sd = td_1 + u\delta$. Let $td_1 = d(1) \oplus \cdots \oplus d(m)$ be the canonical decomposition of td_1 which, by the above argument, only involve real Schur roots in F_I. By Proposition 3.3, it is clear that

$$sd = d(1) \oplus \cdots \oplus d(m) \oplus \delta^u$$

is the canonical decomposition of sd since $\mathrm{ext}(d(i), \delta) = \mathrm{ext}(\delta, d(i)) = 0$ for all i. By Proposition 3.4, the canonical decomposition of d involves the real Schur roots $d(i)$ and the null root δ. $\qquad\square$

7. Ss-equivalence and proof of Theorem 7.7

In this section, we again suppose that Q is a Euclidean quiver with n vertices. We prove Theorem 7.7, which describes when two dimension vectors determine the same semi-stable subcategories.

Recall that the quadratic form $q(x) = \langle x, x \rangle$ is positive semi-definite and the radical is of rank one, generated by the dimension vector δ. Recall also that the convex set H_δ^{ss} is defined as the convex hull of the quasi-simple roots, if Q is of Euclidean type and contains more than 2 vertices, or $\{\delta\}$, if Q is the Kronecker quiver.

Let X be an exceptional representation in $\mathrm{rep}(Q)$ and from Proposition 4.1, let Q_X be the quiver with $n - 1$ vertices for which ${}^\perp X \cong \mathrm{rep}(Q_X)$. Note that Q_X is a possibly disconnected Euclidean or Dynkin quiver. Let

$$F_{d_X} : {}^\perp X \to \mathrm{rep}(Q_X)$$

be an exact functor which is an equivalence. Denote by G_{d_X} a quasi-inverse functor. Let $S_1^X, S_2^X, \ldots, S_{n-1}^X$ be the non-isomorphic simple objects in ${}^\perp X$, called the *relative simples* in ${}^\perp X$. It is clear that $K_0({}^\perp X)$ (which is isomorphic to $K_0(\mathrm{rep}(Q_X))$) is the subgroup of $K_0(\mathrm{rep}(Q))$ generated by the classes $[S_1^X], \ldots, [S_{n-1}^X]$ in $K_0(\mathrm{rep}(Q))$. Put

$$\varphi_X : K_0({}^\perp X) \to K_0(\mathrm{rep}(Q_X))$$

for the canonical isomorphism. Let $\langle \ , \ \rangle_X$ be the Euler form for $\mathrm{rep}(Q_X)$. Since F_{d_X} is exact, $\varphi_X, \varphi_X^{-1}$ are isometries, that is, for $a, b \in K_0(\mathrm{rep}(Q_X))$, we have

$$\langle a, b \rangle_X = \langle \varphi_X^{-1}(a), \varphi_X^{-1}(b) \rangle.$$

Given an exceptional representation X, let $H_{d_X}^{ss}$ be the subset in Δ consisting of (scalar multiples of) dimension vectors f for which there exists a representation $M \in {}^\perp X$ with $d_M = f$. The following lemma is evident.

Lemma 7.1. *Let X be an exceptional representation. Then $H_{d_X}^{ss} \subseteq H_{d_X}^\Delta$ is the convex hull in $H_{d_X}^\Delta$ generated by the dimension vectors of the relative simples in ${}^\perp X$.*

Now, if $\delta \in H_{d_X}^{ss}$, then $\varphi_X(\delta)$ is clearly the null root for the quiver Q_X of $^\perp X$. Observe that even when Q is connected, Q_X may be disconnected (a union of quivers of Dynkin type and a quiver of Euclidean type), thus, $\varphi_X(\delta)$ may be non-sincere. Observe also that it is possible to have $f \in H_{d_X}^{ss}$ and a representation $M \in \mathrm{rep}(Q, f)$ with M not isomorphic to any representation in $^\perp X$. However, this does not happen for general representations.

Lemma 7.2. *Let X be an exceptional representation and $f \in H_{d_X}^{ss}$. Then there is a general representation of dimension vector f in $^\perp X$. Moreover, M is a general representation of dimension vector $\varphi_X(f)$ in $\mathrm{rep}(Q_X)$ if and only if $G_{d_X}(M)$ is a general representation of dimension vector f in $^\perp X$. In particular, $\varphi_X, \varphi_X^{-1}$ preserve the canonical decomposition.*

Proof. Since $f \in H_{d_X}^{ss}$, there exists a representation of dimension vector f in $^\perp X$. Let $\mathcal{U}$ be an open set of $\mathrm{rep}(Q, f)$ as in the definition of the canonical decomposition of f. A representation N in $\mathrm{rep}(Q, f)$ lies in $^\perp X$ if and only if the determinant $C^N(M)$ does not vanish. This defines an open set $\mathcal{U}'$ in $\mathrm{rep}(Q, f)$. The first statement follows from the fact the the intersection $\mathcal{U} \cap \mathcal{U}'$ is non-empty and $\mathcal{U}$ only contains general representations. Being exact equivalences, it is clear that F_{d_X}, G_{d_X} preserve the general representations. $\square$

If X is exceptional, then the cd-simplicial complex of Δ restricts to a simplicial complex in $H_{d_X}^{ss}$. From Lemma 7.2, this simplicial complex on $H_{d_X}^{ss}$ corresponds, under φ_X, to the cd-simplicial complex for $\mathrm{rep}(Q_X)$.

Let $\mathcal{J} = \mathcal{J}_Q = \{C_I\}_{I \in R} \cup \{H_\alpha^{ss}\}_\alpha$ where α runs through the set of all real Schur roots. Let $\mathcal{L} = \mathcal{L}_Q$ be the set of all subsets of Δ that can be expressed as intersections of some subset of $\mathcal{J}$. This is called the *intersection lattice* of $\mathcal{J}$. For $L \in \mathcal{L}$, define the faces of L to be the connected components of the set of points which are in L but not in any smaller intersection. Define the faces of $\mathcal{L}$ to be the collection of all faces of all the elements of $\mathcal{L}$. Given a dimension vector d, $\Gamma_d \subseteq \mathcal{J}$ consists of all the elements in $\mathcal{J}$ containing d.

Lemma 7.3. *Let Q be a Euclidean quiver, and let α be a real Schur root. Then $d \in H_\alpha^{ss}$ if and only if $M(\alpha) \in \mathrm{rep}(Q)_d$.*

Proof. If $M(\alpha) \in \mathrm{rep}(Q)_d$ then there is some representation V with dimension vector md such that $C^V(M(\alpha)) \neq 0$. This implies that $V \in {}^\perp M(\alpha)$, so $d_V \in H_\alpha^{ss}$.

Conversely, if $d \in H_\alpha^{ss}$ then a general representation of $Q_{M(\alpha)}$ of dimension vector $\varphi_{M(\alpha)}(d)$ is a general representation of Q of dimension vector d, and this gives us a supply of representations of dimension vector d which lie in $^\perp M(\alpha)$. Any such representation V will satisfy $C^V(M(\alpha)) \neq 0$, showing that $M(\alpha) \in \mathrm{rep}(Q)_d$. $\square$

For $I \in R$, we denote by $\mathcal{W}_I$ the abelian extension-closed subcategory generated by the indecomposable representations of dimension vectors in F_I. There is unique quasi-simple in each non-homogeneous tube not contained in $\mathcal{W}_I$.

An indecomposable representation lying in a non-homogeneous tube and having dimension vector the null root will be called a *singular-isotropic representation*. For each quasi-simple of a non-homogeneous tube, there is a unique singular-isotropic representation which admits a monomorphism from it.

For Q having more than 2 vertices and $I \in R$, define Z_I to be the direct sum of the singular-isotropic representations corresponding to the quasi-simples not in $\mathcal{W}_I$. We have the following simple lemma:

Lemma 7.4. *For Q having more than 2 vertices, $^\perp Z_I$ consists of the additive hull of $\mathcal{W}_I$ together with the homogeneous tubes.*

Proof. Lemma 4.3 tells us that $^\perp Z_I$ is contained in $\mathcal{R}\text{eg}$. It is clear that the homogeneous tubes lie in $^\perp Z_I$. The rest is just a simple check within each of the non-homogeneous tubes. $\square$

We can now prove an analogue of Lemma 7.3 for the C_I.

Lemma 7.5. *Let Q be a Euclidean quiver having n vertices. If $n > 2$, then for $I \in R$, we have $d \in C_I$ if and only if $Z_I \in \text{rep}(Q)_d$. If $n = 2$, then $d \in C_I = H_\delta^{ss}$ if and only if $\text{rep}(Q)_d = \mathcal{R}\text{eg}$.*

Proof. The case $n = 2$ is trivial. If $Z_I \in \text{rep}(Q)_d$, then there is some representation V with dimension vector md such that $C^V(Z_I) \neq 0$, and indeed, a Zariski open subset of the representations with dimension md have this property. By Lemma 7.4, this implies that the canonical decomposition of md must contain Schur roots from $\mathcal{W}_I$ together with some multiple of δ, and this implies that d lies in C_I.

Conversely, if d lies on C_I, then the canonical decomposition of d consists of Schur roots from $\mathcal{W}_I$, together with some non-negative multiple of δ. Lemma 7.4 implies that there exist representations of dimension vector d which lie in $^\perp Z_I$, so $Z_I \in \text{rep}(Q)_d$. $\square$

The previous two lemmas tell us that Γ_d contains enough information to determine exactly which exceptional or singular-isotropic indecomposable representations lie in $\text{rep}(Q)_d$. We will use the following lemma to show that this is enough information to reconstruct $\text{rep}(Q)_d$.

Lemma 7.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be extension-closed abelian subcategories of $\text{rep}(Q)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ have the same intersection with the exceptional representations and the indecomposables whose dimension vector is the null root. Then $\mathcal{A} = \mathcal{B}$.*

Proof. Suppose there is some object $X \in \mathcal{A}$. We want to show that $X \in \mathcal{B}$. We may assume that X is indecomposable. If X is exceptional or X is at the bottom of a homogeneous tube, then $X \in \mathcal{B}$ by assumption, so assume otherwise.

If X is in a homogeneous tube, then there is a short exact sequence

$$0 \to X \to Y \oplus Z \to X \to 0$$

where Y is the quasi-simple at the base of the tube in which X lies. It follows that $Y \in \mathcal{A}$. Since the dimension vector of Y is a null root, $Y \in \mathcal{B}$, from which it follows that $X \in \mathcal{B}$.

Now suppose that X lies in a non-homogeneous tube. Suppose that the dimension vector of X is $m\delta$ for some $m > 1$. We claim that X admits a filtration by representations each of whose dimension vectors is the null root, and each of which lies in $\mathcal{A}$. The proof is by induction on m. As in the previous case, there is a short exact sequence

$$0 \to X \to Y \oplus Z \to X \to 0$$

where Y is a singular-isotropic representation in the same tube as X. It follows that $Y \in \mathcal{A}$. Now, there is a surjection from X onto Y; let K be its kernel. Since $\mathcal{A}$ is abelian, K also lies in $\mathcal{A}$, and its dimension vector is $(m-1)\delta$. By induction, it admits a filtration as desired, and therefore so does X. Now since each of the

terms of the filtration lies in $\mathcal{A}$ and has dimension vector δ, they each also lie in $\mathcal{B}$, and therefore $X \in \mathcal{B}$.

Finally, suppose that the dimension vector of X is of the form $m\delta + \alpha$, where α is a real Schur root. Similarly to the previous situation, there is an extension of X by X which has $M(\alpha)$ as an indecomposable summand, so $M(\alpha)$ lies in $\mathcal{A}$, and so does the kernel K of the map from X to $M(\alpha)$. The dimension vector of K is $m\delta$, so by what we have already established, it lies in $\mathcal{B}$, and so does $M(\alpha)$, so X does as well. $\square$

Theorem 7.7. *Let Q be a Euclidean quiver and d_1, d_2 two dimension vectors. Then d_1 and d_2 are ss-equivalent if and only if $\Gamma_{d_1} = \Gamma_{d_2}$.*

Proof. If d_1 and d_2 are ss-equivalent, then, by definition, $\mathrm{rep}(Q)_{d_1} = \mathrm{rep}(Q)_{d_2}$, and now Lemmas 7.3 and 7.5 characterize Γ_{d_1} and Γ_{d_2} in terms of this subcategory, so they are equal.

Conversely, if $\Gamma_{d_1} = \Gamma_{d_2}$, then by Lemmas 7.3 and 7.5, we know that $\mathrm{rep}(Q)_{d_1}$ and $\mathrm{rep}(Q)_{d_2}$ agree as to their intersection with exceptional representations and singular-isotropic representations. In order to apply Lemma 7.6, we also need to check their intersections with the quasi-simples of homogeneous tubes. If Γ_{d_1} and Γ_{d_2} do not include any C_I, then neither d_i lies on H^{ss}_δ, so neither $\mathrm{rep}(Q)_{d_1}$ nor $\mathrm{rep}(Q)_{d_2}$ contains any homogeneous tubes. On the other hand, if Γ_{d_1} and Γ_{d_2} do contain some C_I, then d_1 and d_2 both lie on the regular hyperplane, and thus both $\mathrm{rep}(Q)_{d_1}$ and $\mathrm{rep}(Q)_{d_2}$ contain all the homogeneous tubes. In either case, we see that $\mathrm{rep}(Q)_{d_1}$ and $\mathrm{rep}(Q)_{d_2}$ agree as to their intersections with quasi-simples from homogeneous tubes as well, and therefore Lemma 7.6 applies to tell us that $\mathrm{rep}(Q)_{d_1} = \mathrm{rep}(Q)_{d_2}$. $\square$

8. Properties of the cd-simplicial complex and proof of Theorem 8.2

In this section, we prove Theorem 8.2, describing when two dimension vectors have canonical decompositions which include the same Schur roots. All the quivers in this section are Euclidean.

In the sequel, an additive k-category $\mathcal{C}$ will be called *representation-finite*, or of *finite type*, if it contains finitely many indecomposable objects, up to isomorphism. All the categories $\mathcal{W}_I$ are representation-finite subcategories of $\mathcal{R}$eg. The following shows that the $\mathcal{W}_I$ are the maximal such subcategories.

Lemma 8.1. *Let $\mathcal{C}$ be an abelian extension-closed subcategory of $\mathrm{rep}(Q)$ contained in $\mathcal{R}$eg. Then $\mathcal{C}$ is representation-finite if and only if $\mathcal{C}$ is contained in some $\mathcal{W}_I$.*

Proof. We only need to prove the necessity. Suppose that $\mathcal{C}$ is representation-finite but not included in any of the $\mathcal{W}_I$. If $\mathcal{C}$ contains an indecomposable non-exceptional representation X, then $\mathcal{C}$ clearly contains infinitely many indecomposable representations lying in the same component as X of the Auslander-Reiten quiver of $\mathrm{rep}(Q)$. In particular, there exists a non-homogeneous tube $\mathcal{T}$, say of rank r, for which $\mathcal{C} \cap \mathcal{T}$ is not contained in any abelian extension-closed subcategory of $\mathcal{T}$ generated by $r-1$ quasi-simple representations. Let X be an exceptional representation in $\mathcal{C} \cap \mathcal{T}$ of largest quasi-length. Then X is such that $\mathrm{Ext}^1(X, \mathcal{C} \cap \mathcal{T}) = \mathrm{Ext}^1(\mathcal{C} \cap \mathcal{T}, X) = 0$ since otherwise, this would provide a representation in $\mathcal{C} \cap \mathcal{T}$ with quasi-length larger than that of X. Suppose that the quasi-simple composition factors of X are $S, \tau S, \ldots, \tau^m S$, $0 \le m < r$. Since X is exceptional, $\tau^{m+1} S \not\cong S$ and $\tau^{m+1} S$

is not a quasi-simple composition factor of X and any $Y \in \mathcal{C} \cap \mathcal{T}$ having $\tau^{m+1}S$ as a quasi-simple composition factor is such that $\mathrm{Ext}(X, Y) \cong D\mathrm{Hom}(Y, \tau X) \neq 0$, which is impossible. Hence, $\mathcal{C} \cap \mathcal{T}$ is contained in the subcategory of $\mathcal{T}$ generated by all the quasi-simple representations except $\tau^{m+1}S$. This is a contradiction. $\square$

We say that two dimension vectors d_1, d_2 are *cd-equivalent* if the canonical decompositions of d_1, d_2 involve the same Schur roots, up to multiplicity. The following result gives an explicit description of when two dimension vectors are cd-equivalent.

Theorem 8.2. *Let Q be a Euclidean quiver and d_1, d_2 two dimension vectors. Then d_1, d_2 are cd-equivalent if and only if they lie on the same face of $\mathcal{L}(\mathcal{J}_Q)$.*

Before we prove this theorem, we need a lemma:

Lemma 8.3. *The Schur roots appearing in a canonical decomposition are linearly independent.*

Proof. If only real Schur roots appear, then the corresponding general representation forms a partial tilting representation, which can be completed to a tilting representation, and the dimension vectors of the indecomposable summands of a tilting representations are necessarily linearly independent (since they span the Grothendieck group).

If the root δ appears, then the other roots that appear define a partial tilting representation S lying entirely inside $\mathcal{R}eg$. The summands of S can be ordered into an exceptional sequence, and completed to a full exceptional sequence by adding some modules after the summands of S. Since a full exceptional sequence cannot lie entirely inside the regular component, there is some non-regular exceptional representation M which can follow all the summands of S in an exceptional sequence. It follows that S is contained in $^{\perp}M$, and since $^{\perp}M$ is of finite representation type, so is S. Thus S is contained in some $\mathcal{W}_I$ by Lemma 8.1. Since S is partial tilting, its indecomposable summands are linearly independent, and since all the real Schur roots that appear lie on the facet F_I, the collection of roots does not become linearly dependent once δ is added, either. $\square$

Proof of Theorem 8.2. Suppose that d_1, d_2 are cd-equivalent. Any non-negative linear combination of d_1 and d_2 will also be cd-equivalent to both of them, by Proposition 3.3. If d_1, d_2 do not lie on the same face of $\mathcal{L}(\mathcal{J}_Q)$, then there is a dimension vector d_3 which lies on the straight line joining d_1, d_2 and $H \in \mathcal{J}_Q$ such that $d_3 \in H$ and at least one of d_1, d_2 does not lie on H. Without loss of generality, suppose that $d_1 \notin H$. As we have already remarked, d_3 must be cd-equivalent to d_1.

If $H = H_\alpha^{ss}$ for a real Schur root α, then there is a general representation N of dimension vector d_3 such that $N \in {}^{\perp}M(\alpha)$. But there exists a general representation N' (built from the indecomposable direct summands of N) of dimension vector d_1 that has the indecomposable summands of the same dimensions (though different multiplicities), so $N' \in {}^{\perp}M(\alpha)$, showing that $d_1 \in H_\alpha^{ss}$, a contradiction.

Similarly, if $|Q_0| > 2$ and $H = C_I$ for some $I \in R$, we deduce that a general representation N of dimension vector d_3 lies in $^{\perp}Z_I$. As argued above, there will be a general representation N' of dimension vector d_1 which lies in $^{\perp}Z_I$, showing that $d_1 \in C_I$, a contradiction. If $|Q_0| = 2$ and $H = C_\emptyset = H_\delta^{ss}$, then both d_1, d_3

are multiples of δ and it is clear that $d_1 \in H$, a contradiction. This establishes one direction.

Suppose now that d_1, d_2 lie in the same face of $\mathcal{L}(\mathcal{J}_Q)$. Suppose further that the canonical decomposition of d_1 involves distinct Schur roots $f_1, \ldots, f_r$. By Proposition 3.3, any positive linear combination of $f_1, \ldots, f_r$ will have the same canonical decomposition as d_1. Since the f_i are linearly independent, the boundary facets of this cone are spanned by any $r - 1$ of the f_i. It therefore remains to show that, for any $1 \leq i \leq r$, we have that $\{f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_r\}$ lies on some $H \in \mathcal{J}$ with $f_i \notin \mathcal{J}$.

It suffices to consider one particular value of i, so let us consider $i = r$. Suppose first that $f_1, \ldots, f_r$ consist only of real Schur roots. Therefore the direct sum of the $M(f_j)$ forms a partial tilting object, and can be ordered into an exceptional sequence. Now remove $M(f_r)$. The resulting exceptional sequence can be completed to a full exceptional sequence by adding some terms $X_r, \ldots, X_n$ to the end. If we re-insert $M(f_r)$ into the sequence at the place where it was before, the resulting sequence is too long to be exceptional, so there is some X_k with $r \leq k \leq n$ such that $M(f_r) \notin {}^\perp X_k$, while $M(f_j) \in {}^\perp X_k$ for $1 \leq j < r$, by virtue of the exceptional sequence property. This shows that $H_{X_k}^{ss}$ is a hyperplane of the desired type.

Next, suppose that $f_r = \delta$. In this case $\oplus_{1 \leq j < r} M(f_j)$ forms a regular partial tilting object, and its summands can be ordered into an exceptional sequence. Complete this to an exceptional sequence by adding some terms $X_r, \ldots, X_n$. Since a full exceptional sequence cannot consist entirely of regular objects, there is some X_k which is not regular. Now $H_{X_k}^{ss}$ contains $f_1, \ldots, f_{r-1}$, but by Lemma 4.3 it does not contain $f_r = \delta$. Thus $H_{X_k}^{ss}$ has the desired properties.

Finally, suppose that some $f_j = \delta$ for some $1 \leq j \leq r - 1$. For convenience, let $f_{r-1} = \delta$. Then $M(f_1), \ldots, M(f_{r-2})$ and $M(f_r)$ are the summands of a partial tilting object and can be ordered into an exceptional sequence which is contained in the regular component. Since they can be ordered into an exceptional sequence contained in the regular component, they generate an abelian subcategory of finite representation type, which is therefore contained in some wing $\mathcal{W}_I$. Delete $M(f_r)$ from the sequence, and then complete it to an exceptional sequence which is full inside $\mathcal{W}_I$, by adding some terms $X_{r-1}, \ldots, X_{n-2}$. Since $M(f_r)$ cannot be added back into the sequence, we see that there is some X_k such that $M(f_r)$ is not in ${}^\perp X_k$, so $f_r \notin H_{X_k}^{ss}$, while by the exceptional sequence property, $f_t \in H_{X_k}^{ss}$ for $1 \leq t \leq r - 2$, and $f_{r-1} \in H_{X_k}^{ss}$ because $f_{r-1} = \delta$ and X_k is regular. $\square$

9. Thick subcategories

This section is devoted to proving some facts concerning extension-closed abelian subcategories of $\mathrm{rep}(Q)$. For part of the section, we assume only that Q is acyclic; later, we assume that Q is Eulerian.

We begin in an even more general setting. Let $\mathcal{H}$ be a hereditary abelian k-category. A (full) subcategory $\mathcal{A}$ of $\mathcal{H}$ is *thick* if it is closed under direct summands and whenever we have a short exact sequence with two terms in $\mathcal{A}$, then the third term also lies in $\mathcal{A}$. We start with the following result which is probably well known.

Proposition 9.1. *Let $\mathcal{H}$ be a hereditary abelian k-category with a full subcategory $\mathcal{A}$. Then $\mathcal{A}$ is extension-closed abelian if and only if it is thick.*

Proof. The necessity follows trivially. Suppose that $\mathcal{A}$ is thick. We need to show that $\mathcal{A}$ has kernels and cokernels. Let $f : A \to B$ be a morphism in $\mathcal{A}$ with

kernel $u : K \to A$, cokernel $v : B \to C$ and coimage $g : A \to E$. Since $\mathcal{H}$ is hereditary and we have a monomorphism $g' : E \cong \operatorname{im}(f) \to B$, we have a surjective map $\operatorname{Ext}^1(g', K) : \operatorname{Ext}^1(B, K) \to \operatorname{Ext}^1(E, K)$. The short exact sequence $0 \to K \to A \to E \to 0$ is an element in $\operatorname{Ext}^1(E, K)$ and hence is the image of an element in $\operatorname{Ext}^1(B, K)$. We have a pullback diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{u} & A & \xrightarrow{g} & E & \longrightarrow & 0 \; . \\
& & \| & & \downarrow & & \downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

This gives rise to a short exact sequence

$$0 \to A \to A' \oplus E \to B \to 0.$$

Since $A, B \in \mathcal{A}$ and $\mathcal{A}$ is thick, we get $E \in \mathcal{A}$. Hence, $K, C \in \mathcal{A}$. $\qquad\square$

Thanks to $\mathcal{H}$ being hereditary, the bounded derived category of $\mathcal{H}$, written as $D^b(\mathcal{H})$, is easy to describe. Recall that a stalk complex in $D^b(\mathcal{H})$ is a complex concentrated in one degree, that is, a complex of the form $X[i]$ for an object X in $\mathcal{H}$. From [20], every object in $D^b(\mathcal{H})$ is a finite direct sum of stalk complexes. Observe also that given two objects $X, Y \in \mathcal{H}$, the condition $\operatorname{Hom}_{D^b(\mathcal{H})}(X, Y[i]) \neq 0$ implies either $i = 0$ or $i = 1$. A full subcategory $\mathcal{C}$ of $D^b(\mathcal{H})$ is *thick* if it closed under direct summands and whenever we have a map $U \to V$ in $\mathcal{C}$, then the distinguished triangle

$$W \to U \to V \to W[1]$$

lies in $\mathcal{C}$. In particular, a thick subcategory is closed under shifts and is a triangulated subcategory of $D^b(\mathcal{H})$. Given a thick subcategory $\mathcal{C}$ of $D^b(\mathcal{H})$, we denote by $t(\mathcal{C})$ the category $\mathcal{H} \cap \mathcal{C}$, that is, the complexes C in $\mathcal{C}$ for which $H^i(C) = 0$ for all $i \neq 0$. The category $t(\mathcal{C})$ is abelian and extension-closed, and hence is a thick subcategory of $\mathcal{H}$. Observe also that it is the heart of the triangulated subcategory $\mathcal{C}$ associated to the t-structure on $\mathcal{C}$ coming from the canonical t-structure on $D^b(\mathcal{H})$; see [13]. It is easily seen that $\mathcal{C}$ is triangle-equivalent to the bounded derived category of $t(\mathcal{C})$.

Given a family of objects E in $D^b(\mathcal{H})$, we write $\mathcal{D}(E)$ for the thick subcategory of $D^b(\mathcal{H})$ generated by the objects in E. We define $\mathcal{D}(E)^\perp$ (resp. $^\perp\mathcal{D}(E)$) to be the full subcategory of $D^b(\mathcal{H})$ generated by the objects Y with $\operatorname{Hom}(X, Y[i]) = 0$ (resp. $\operatorname{Hom}(Y, X[i]) = 0$) for all $X \in E$ and all $i \in \mathbb{Z}$. Observe that if $E \subseteq \mathcal{H}$, $\mathcal{C}(E) = t(\mathcal{D}(E))$ and $t(\mathcal{D}(E)^\perp) = \mathcal{C}(E)^\perp$. As on the level of abelian categories, an indecomposable object X in $D^b(\mathcal{H})$ is *exceptional* if $\operatorname{Hom}(X, X[i]) = 0$ for all nonzero integers i and $\operatorname{End}(X) \cong k$. An *exceptional sequence* $E = (X_1, X_2, \ldots, X_r)$ in $D^b(\mathcal{H})$ is a sequence of exceptional objects for which

$$\operatorname{Hom}(X_i, X_j[l]) = 0 \text{ for all } l \text{ and } 1 \leq i < j \leq r.$$

Any exceptional sequence in $\mathcal{H}$ is also an exceptional sequence in $D^b(\mathcal{H})$. Conversely, if $E = (X_1, X_2, \ldots, X_r)$ is exceptional in $D^b(\mathcal{H})$, then there exist integers $i_1, i_2, \ldots, i_r$ for which $t(E) := (X_1[i_1], \ldots, X_r[i_r])$ is an exceptional sequence in $\mathcal{H}$.

Let us introduce more terminology. Let $\mathcal{E}$ be any k-linear category. A full subcategory $\mathcal{F}$ of $\mathcal{E}$ is said to be *contravariantly finite* if for any object E in $\mathcal{E}$, there exists a morphism $f : F \to E$ with $F \in \mathcal{F}$ such that $\operatorname{Hom}(F', f)$ is surjective

for any $F' \in \mathcal{F}$. Such a morphism f is called a *right $\mathcal{F}$-approximation* of E. The following result is stated at the derived category level but is also true at the abelian category level.

Lemma 9.2. *Suppose that $\mathcal{H}$ is Hom-finite. Let $E = (X_1, X_2, \ldots, X_r)$ be an exceptional sequence in $D^b(\mathcal{H})$. Then $\mathcal{D}(E)$ is contravariantly finite in $D^b(\mathcal{H})$.*

Proof. We proceed by induction on r. If $r = 1$, then $\mathcal{D}(E)$ is the additive category containing the shifts of X_1. This is clearly contravariantly finite in $D^b(\mathcal{H})$. Suppose now that $r > 1$. We have that $E' = (X_1, \ldots, X_{r-1})$ is an exceptional sequence with $\mathcal{D}(E')$ contravariantly finite. Let M be any object in $D^b(\mathcal{H})$ with $f : C \to M$ a right $\mathcal{D}(E')$-approximation of M. Since $\mathcal{H}$ is abelian and Hom-finite, $\mathcal{H}$, and hence $D^b(\mathcal{H})$, is Krull-Schmidt. Therefore, we may assume that f is right-minimal, in the sense that if $\varphi : C \to C$ is such that $f = f\varphi$, then φ is an isomorphism. Now, we have a triangle

$$N \xrightarrow{\ g\ } C \xrightarrow{\ f\ } M \longrightarrow N[1] \ .$$

We claim that N lies in $\mathcal{D}(E')^{\perp}$. Let $u : B \to N$ be a morphism with $B \in \mathcal{D}(E')$. By the octahedral axiom, we have a commutative diagram

$$
\begin{array}{ccccccc}
B & = & B & & & & \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle gu} & & & & \\
N & \xrightarrow{\ g\ } & C & \xrightarrow{\ f\ } & M & \longrightarrow & N[1] \\
\downarrow & & \downarrow{\scriptstyle w} & & \| & & \downarrow \\
F & \longrightarrow & B' & \xrightarrow{\ v\ } & M & \longrightarrow & F[1]
\end{array}
$$

where B' is an object in $\mathcal{D}(E')$ and all rows and columns are distinguished triangles. Since f is a right $\mathcal{D}(E')$-approximation of M, v factors through f and hence, there exists $w' : B' \to C$ with $f = fw'w$. Hence, since f is right minimal, we get that w is a section, which means that $gu = 0$. Hence, u factors through $M[-1]$. But since B is in $\mathcal{D}(E')$ and $f[-1]$ is a right $\mathcal{D}(E')$-approximation of $M[-1]$, u factors through $g[-1]$ and therefore, $u = 0$. This proves that N lies in $\mathcal{D}(E')^{\perp}$.

Now, let $\mathcal{X}$ be the thick subcategory of $D^b(\mathcal{H})$ generated by X_r, which is contravariantly finite in $D^b(\mathcal{H})$ using the base case $r = 1$. Let $h : X \to N$ be a minimal right $\mathcal{X}$-approximation of N. The octahedral axiom gives a commutative diagram

$$
\begin{array}{ccccccc}
X & = & X & & & & \\
\downarrow & & \downarrow & & & & \\
N & \longrightarrow & C & \xrightarrow{\ f\ } & M & \longrightarrow & N[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
N' & \longrightarrow & F & \xrightarrow{\ w\ } & M & \longrightarrow & N'[1]
\end{array}
$$

where N' is an object in $\mathcal{D}(E')^{\perp} \cap \mathcal{X}^{\perp} = \mathcal{D}(E)^{\perp}$ and all rows and columns are distinguished triangles. It is easily seen that the morphism $w : F \to M$ is a right $\mathcal{D}(E)$-approximation of M. $\qquad\square$

Proposition 9.3. *Let $\mathcal{H}$ be a Hom-finite hereditary abelian k-category with a non-full exceptional sequence $E = (X_1, \ldots, X_r)$ in $D^b(\mathcal{H})$ and such that all indecomposable objects in $\mathcal{D}(E)^\perp$ are exceptional. Then E can be extended to an exceptional sequence $E = (X_1, \ldots, X_r, X_{r+1})$.*

Proof. We only need to prove that the category $\mathcal{D}(E)^\perp$ is nonzero. By the assumption, there is an indecomposable object M which is not in $\mathcal{D}(E)$. By Lemma 9.2, we have a triangle

$$ N \xrightarrow{g} C \xrightarrow{f} M \longrightarrow N[1] \ , $$

where f is a right $\mathcal{D}(E)$-approximation of M. Then N is nonzero in $\mathcal{D}(E)^\perp$ and has an exceptional object X_{r+1} as a direct summand. This yields an exceptional sequence $E = (X_1, \ldots, X_r, X_{r+1})$. $\qquad\square$

In the rest of this section, we specialize to $\mathcal{H} = \text{rep}(Q)$ for an acyclic quiver Q with n vertices. Given two thick subcategories $\mathcal{C}_1, \mathcal{C}_2$ of $\text{rep}(Q)$, we call the pair $(\mathcal{C}_1, \mathcal{C}_2)$ a *semi-orthogonal pair* if $\mathcal{C}_2 = \mathcal{C}_1^\perp$. In order to deal with such pair, we need the following easy observation.

Lemma 9.4. *Let $\mathcal{C}$ be a finitely generated abelian extension-closed subcategory of $\text{rep}(Q)$. Then $^\perp(\mathcal{C}^\perp) = \mathcal{C}$.*

Proof. The category $\mathcal{C}$ is generated by an exceptional sequence $(X_1, X_2, \ldots, X_r)$ that can be completed to a full exceptional sequence $(X_1, \ldots, X_r, \ldots, X_n)$. Then $\mathcal{C}^\perp$ is generated by $X_{r+1}, \ldots, X_n$. It is clear that $\mathcal{C} \subseteq \cap_{i=r+1}^n {}^\perp X_i$ and that $\cap_{i=r+1}^n {}^\perp X_i$ is equivalent to the category of representations of a quiver with $n - (n - r) = r$ vertices. If equality does not hold, then $(X_1, \ldots, X_r)$ is not full in $\cap_{i=r+1}^n {}^\perp X_i$, which is a contradiction. $\qquad\square$

Given a full exceptional sequence $E = (X_1, X_2, \ldots, X_n)$ in $\text{rep}(Q)$ and an integer $0 \le s \le n$, there is a semi-orthogonal pair

$$ \mathcal{P}(E, s) := (\mathcal{C}(E_{\le s}), \mathcal{C}(E_{>s})), $$

where $E_{\le s} = (X_1, \ldots, X_s)$ and $E_{>s} = (X_{s+1}, \ldots, X_n)$. We set $E_{\le 0} = \emptyset$ and $E_{>n} = \emptyset$. Indeed, it is clear that $\mathcal{C}(E_{>s}) \subseteq \mathcal{C}(E_{\le s})^\perp$. Moreover, $E_{>s}$ is an exceptional sequence in $\mathcal{C}(E_{\le s})^\perp$, which is equivalent to $\text{rep}(Q')$ for some acyclic quiver Q' with $n - s$ vertices. Since any exceptional sequence in $\text{rep}(Q')$ can be completed to a full exceptional sequence, see [6], and E is full, we see that $\mathcal{C}(E_{>s}) = \mathcal{C}(E_{\le s})^\perp$. An exceptional pair of the form $\mathcal{P}(E, s)$ where $E = (X_1, X_2, \ldots, X_n)$ is a full exceptional sequence and $1 \le s \le n$ will be called an *exceptional semi-orthogonal pair*. The following is an easy observation.

Lemma 9.5. *Let $(\mathcal{C}_1, \mathcal{C}_1^\perp)$, $(\mathcal{C}_2, \mathcal{C}_2^\perp)$ be exceptional semi-orthogonal pairs in $\text{rep}(Q)$ with $\mathcal{C}_1^\perp = \mathcal{C}_2^\perp$. Then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. By Lemma 9.4, $\mathcal{C}_1 = {}^\perp(\mathcal{C}_1^\perp) = {}^\perp(\mathcal{C}_2^\perp) = \mathcal{C}_2$. $\qquad\square$

Lemma 9.6. *Let Q be any acyclic quiver and let $\mathcal{C}$ be any thick subcategory of $\text{rep}(Q)$ which is representation-finite. Then $\mathcal{C}$ is equivalent to $\text{rep}(Q')$ for some (possibly disconnected) Dynkin quiver Q'.*

Proof. Let $\{C_1, \ldots, C_r\}$ be a complete set of indecomposable objects in $\mathcal{C}$. Since $\mathcal{C}$ is thick and representation-finite, every C_i is exceptional. Moreover, define $C_i \preceq C_j$ if and only if there exists a sequence of nonzero morphisms $C_i \to C_{i_1} \to \cdots \to C_{i_m} \to C_j$ where each $i_j \in \{1, 2, \ldots, r\}$. By the structure of $\mathrm{rep}(Q)$ and by the assumption on the C_i, we see that this is indeed a partial order on the set $\{C_1, \ldots, C_r\}$. Note that, since $\mathcal{C}$ is abelian, the Ext-projective objects are exactly the projective objects in $\mathcal{C}$. We claim that $\mathcal{C}$ has enough projective objects. Otherwise, let C_t be a least element, with respect to $\preceq$, not generated by a projective object. Since C_t is not projective, it has a non split epimorphism $M \to C_t$ where M is a direct sum $C_{j_1} \oplus \cdots \oplus C_{j_m}$ with each $C_{j_i} \prec C_t$. Since each C_{j_i} is generated by a projective object, so is C_t, a contradiction. Now since $\mathcal{C}$ has enough projectives, it has a minimal projective generator P and $\mathcal{C}$ is equivalent to $\mathrm{mod}(\mathrm{End}(P))$. Since $\mathcal{C}$ is clearly hereditary, $\mathrm{End}(P)$ is a finite-dimensional hereditary k-algebra, hence $\mathrm{End}(P)$ is equivalent to $\mathrm{rep}(Q')$ for a (possibly disconnected) Dynkin quiver Q'. $\square$

The preceding lemma yields the following: For an acyclic quiver Q, any thick subcategory of $\mathrm{rep}(Q)$ which is representation-finite is generated by an exceptional sequence.

A *connecting component* of $D^b(\mathrm{rep}(Q))$ is a connected component of the Auslander-Reiten quiver of $D^b(\mathrm{rep}(Q))$ containing projective representations. It is unique if and only if Q is connected. The following result strengthens a result appearing in [10].

Proposition 9.7. *Any thick subcategory generated by a finite set of objects in connecting components of $D^b(\mathrm{rep}(Q))$ is generated by an exceptional sequence E where all the terms can be chosen to be in connecting components.*

Proof. It is sufficient to consider the case where Q is connected and non-Dynkin. In [10], it is proven that if $\mathcal{C}$ is a thick subcategory of $\mathrm{rep}(Q)$ which is generated by a finite set of preprojective representations, then $\mathcal{C}$ is generated by an exceptional sequence where each term can be chosen to be preprojective. Let $\mathcal{D}$ be a thick subcategory of $D^b(\mathrm{rep}(Q))$ generated by the exceptional objects $X_1, \ldots, X_r$ where all the X_i lie in the connecting component. Let τ denote the Auslander-Reiten translate in $D^b(\mathrm{rep}(Q))$. The connecting component contains only the preprojective representations and the inverse shifts of the preinjective representations. There exists a positive integer t for which all $\tau^{-t}X_i$ are preprojective indecomposable representations. Hence $\mathcal{C}(\tau^{-t}X_1, \ldots, \tau^{-t}X_r)$ is a thick subcategory of $\mathrm{rep}(Q)$ generated by preprojective representations. Therefore, by the previous observation,

$$\mathcal{C}(\tau^{-t}X_1, \ldots, \tau^{-t}X_r) = \mathcal{C}(E)$$

for an exceptional sequence $E = (Y_1, \ldots, Y_m)$ where all Y_i are preprojective. Clearly, we also have

$$\mathcal{D}(\tau^{-t}X_1, \ldots, \tau^{-t}X_r) = \mathcal{D}(E)$$

where here, E is seen as an exceptional sequence in $D^b(\mathrm{rep}(Q))$. From this, we see that

$$\mathcal{D}(X_1, \ldots, X_r) = \mathcal{D}(E'),$$

where E' is the exceptional sequence $E' = (\tau^t Y_1, \ldots, \tau^t Y_m)$. $\square$

Using the fact that the thick subcategories $\mathcal{D}$ of $D^b(\mathrm{rep}(Q))$ correspond to the thick subcategories $t(\mathcal{D})$ of $\mathrm{rep}(Q)$, we get the following result.

Corollary 9.8. *Let Q be a connected acyclic quiver. Any thick subcategory of* $\mathrm{rep}(Q)$ *generated by non-regular representations is generated by an exceptional sequence whose terms can be chosen to be non-regular.*

For the rest of this section, we specialize to the Euclidean case. Let Q be a Euclidean quiver and let $E = (X_1, X_2, \ldots, X_n)$ be a full exceptional sequence in $\mathrm{rep}(Q)$. Let s be an integer with $1 \leq s \leq n$. Clearly, one of the X_i is not regular since E is full. Then, by Lemma 4.3, one of $\mathcal{C}(E_{\leq s})$, $\mathcal{C}(E_{>s})$ only contains indecomposable representations that are exceptional. Since each of $\mathcal{C}(E_{\leq s})$, $\mathcal{C}(E_{>s})$ is equivalent to the category of representations of some acyclic quiver, we get that one of $\mathcal{C}(E_{\leq s})$, $\mathcal{C}(E_{>s})$ is representation-finite.

Theorem 9.9. *Let Q be a Euclidean quiver. Any thick subcategory of* $\mathrm{rep}(Q)$ *is either of the form $\mathcal{C}(E)$ for an exceptional sequence E in* $\mathrm{rep}(Q)$ *or is entirely contained in $\mathcal{R}eg$.*

Proof. Let $\mathcal{T}$ be a thick subcategory of $\mathrm{rep}(Q)$ which contains at least one preprojective or preinjective indecomposable object X. By Proposition 9.1, $\mathcal{T}$ is a Hom-finite hereditary abelian category containing X. Clearly, X is an exceptional representation, hence providing an exceptional sequence $E' = (X)$ in $\mathcal{T}$. From Lemma 4.3, $\mathcal{D}(E')^{\perp}$ in $D^b(\mathcal{T})$ only contains exceptional objects. By Lemma 9.3, we see that E' can be completed to a full exceptional sequence E in $D^b(\mathcal{T})$, proving that $\mathcal{T}$ is also generated by an exceptional sequence. $\square$

The following lemma is well known; see for example [16]. We will need it shortly.

Lemma 9.10. *Let R be a Dynkin quiver. Any thick subcategory of* $\mathrm{rep}(R)$ *is equivalent to $V^{\perp}$ for a rigid representation $V \in \mathrm{rep}(R)$. Moreover, $V^{\perp}$ is equivalent to* $\mathrm{rep}(R')$, *where R' is a (possibly disconnected) Dynkin quiver.*

In order to understand thick subcategories of $\mathrm{rep}(Q)$ which are contained in $\mathcal{R}eg$, we need to classify the thick subcategories of a single tube. Let $\mathcal{T}$ be a tube of rank r, which is identified with the additive subcategory of $\mathrm{rep}(Q)$ that it generates. Let J be a subset of the quasi-simples of $\mathcal{T}$. Write $\mathcal{E}_J$ for $X^{\perp} \cap \mathcal{T}$, where X is the direct sum of the simples not in J.

We say that $(\mathcal{E}_J, \mathcal{F})$ is a *regular orthogonal pair* if $\mathcal{F} \subseteq {}^{\perp}\mathcal{E}_J \cap \mathcal{E}_J^{\perp}$, $\mathcal{F}$ is thick and contains only exceptional indecomposables. In this case, let $\mathcal{S}_{J,\mathcal{F}}$ be the additive hull of $\mathcal{E}_J$ and $\mathcal{F}$, which is clearly a thick subcategory of $\mathcal{T}$.

Proposition 9.11. *Let $\mathcal{T}$ be a tube in* $\mathrm{rep}(Q)$. *Any thick subcategory of $\mathcal{T}$ can be written as $\mathcal{S}_{J,\mathcal{F}}$ for a unique subset J of the quasi-simples, and subcategory $\mathcal{F}$ such that $(\mathcal{E}_J, \mathcal{F})$ is a regular orthogonal pair.*

Proof. Let $\mathcal{C}$ be a thick subcategory of $\mathcal{T}$. Let J be the set of quasi-socles of the singular-isotropic representations in $\mathcal{C}$, if any. We claim that $\mathcal{E}_J$ is contained in $\mathcal{C}$, and that if we set $\mathcal{F}$ to be the additive hull of $\mathrm{ind}\,\mathcal{C} \setminus \mathrm{ind}\,\mathcal{E}_J$, then $(\mathcal{E}_J, \mathcal{F})$ is a regular orthogonal pair, so $\mathcal{C} = \mathcal{S}_{J,\mathcal{F}}$.

First, we establish that $\mathcal{E}_J$ is contained in $\mathcal{C}$. If $J = \emptyset$, then $\mathcal{E}_J = 0$, so this is obvious. Assume otherwise. Let X be the direct sum of the quasi-simples not in J. Then $\mathcal{E}_J = \mathcal{T} \cap X^{\perp}$. Suppose that $\mathcal{T}$ has rank r_1. Since the summands of X form an exceptional sequence, from Proposition 4.2, $X^{\perp}$ is equivalent to the category of representations of a quiver Q' with $|Q_0| - r_1 + |J|$ vertices. It is clear that Q' is a possibly disconnected Euclidean quiver. Since $\mathcal{T} \cap X^{\perp}$ contains exactly

$|J|$ singular-isotropic representations, we see that the ranks of the tubes of $\mathrm{rep}(Q')$ will be the same as the ones for $\mathrm{rep}(Q)$, but one rank will decrease by $r_1 - |J|$. If the r_i denote the ranks of the non-homogeneous tubes for $\mathrm{rep}(Q)$, the well know formula $\sum(r_i - 1) = n - 2$ gives $|J| + \sum_{i \neq 1} r_i = n - 2 - (r_1 - |J|)$ which then tells us that Q' is connected. It follows that $\mathcal{E}_J$ is equivalent to some tube $\mathcal{T}'$ of $\mathrm{rep}(Q')$. Since the singular-isotropic representations in $\mathcal{T}'$ generate all of $\mathcal{T}'$ as a thick subcategory, it follows that the smallest thick subcategory containing the singular-isotropic representations in $\mathcal{C}$ is $\mathcal{E}_J$. Thus, $\mathcal{E}_J$ is contained in $\mathcal{C}$.

Suppose that $J \neq \emptyset$. Let $\mathcal{Q}$ denote the set of objects of $\mathcal{E}_J$ which correspond to the quasi-simples of $\mathcal{T}'$. The objects in $\mathcal{E}_J$ consist of representations which have filtrations by objects from $\mathcal{Q}$.

Now think of the filtration by quasi-simples of the objects from $\mathcal{Q}$. Each Q in $\mathcal{Q}$ has a filtration $\mathcal{K}_Q$ by a consecutive sequence of the quasi-simples; these consecutive sequences are disjoint and their union is the set of all the quasi-simples of $\mathcal{T}$. Suppose we have an indecomposable object $X \in \mathcal{F}$. Consider its filtration by quasi-simples, which also gives rise to a consecutive sequence of quasi-simples. Since $X \notin \mathcal{E}_J$, this sequence of quasi-simples is not the concatenation of subsequences corresponding to elements of $\mathcal{Q}$: it either begins, or ends, or both, out of step with the subdivision of quasi-simples of $\mathcal{T}$ into the sets $\mathcal{K}_Q$. We would like to show that the quasi-simples in the filtration of X all lie inside $\mathcal{K}_Q$ for some Q, and do not include either the quasi-socle or the quasi-top of that Q. Suppose that this is not the case. Then there is some $Q \in \mathcal{Q}$ such that X admits a non-epimorphism to Q, or a non-monomorphism from Q. Suppose we are in the first case. (The second is dual.) Since $\mathcal{C}$ is extension-closed, $\mathcal{C}$ contains the image A of X in Q, so that there is a short exact sequence $0 \to A \to Z \to B \to 0$ in $\mathcal{C}$, where Z is singular-isotropic and lies in $\mathcal{E}_J$. But then, there is another short exact sequence $0 \to B \to Z' \to A \to 0$ where Z' is singular-isotropic. Since $\mathcal{C}$ is extension-closed, $Z' \in \mathcal{C}$ and since Z' is singular-isotropic, we must have $Z' \in \mathcal{E}_J$. This means that there exists $Q' \in \mathcal{Q}$ which lies on the co-ray where Z', X and A lies. This contradicts that $\mathcal{K}_Q$ and $\mathcal{K}_{Q'}$ have to be disjoint. Therefore, we know that, for any indecomposable X in $\mathcal{F}$, there is some $Q \in \mathcal{Q}$ such that X admits a filtration by the quasi-simples in the filtration of Q, excluding its quasi-socle and quasi-top. This implies, in particular, that $X \subseteq {}^{\perp}\mathcal{E}_J \cap \mathcal{E}_J^{\perp}$. It also shows that X is necessarily exceptional.

Now we consider the case that $J = \emptyset$. The situation which we must rule out is that $\mathcal{C}$ contains some non-rigid indecomposables, but no singular-isotropic representations. That this is impossible is a by-product of the proof of Lemma 7.6. $\square$

Now we are able to describe the semi-stable subcategories of $\mathrm{rep}(Q)$ which lie in $\mathcal{R}\mathrm{eg}$.

Proposition 9.12. *A thick subcategory of $\mathcal{R}\mathrm{eg}$ is semi-stable if and only if it contains all the homogeneous tubes and its intersection with each non-homogeneous tube $\mathcal{T}_i$ is of the form $\mathcal{S}_{J_i, \mathcal{F}_i}$ where each J_i is non-empty.*

Proof. By Theorem 5.4, a semi-stable subcategory of $\mathcal{R}\mathrm{eg}$ can also be written as $\mathcal{A} \cap \mathcal{R}\mathrm{eg}$ for $\mathcal{A}$ some finitely generated abelian category, equivalent to the representations of a (possibly disconnected) Euclidean quiver. Since $\mathcal{A}$ is finitely generated, it can be written as $V^{\perp}$ for some rigid object V, and since $\mathcal{A}$ is equivalent to a possibly disconnected Euclidean quiver, V must be regular.

Since V is regular, $V^\perp$ contains all the homogeneous tubes. Now consider V_i, the maximal direct summand of V lying in $\mathcal{T}_i$. Since V_i is rigid, V_i is contained in some wing, and thus $V_i^\perp$ contains some singular-isotropic representation. It follows that $V^\perp \cap \mathcal{T}_i$ is of the form $\mathcal{S}_{J_i, \mathcal{F}_i}$ where J_i is non-empty.

Conversely, suppose we have a thick subcategory as described in the statement of the proposition. We want to show that it is of the form $V^\perp \cap \mathcal{R}\mathrm{eg}$ where V is rigid. Clearly, it suffices to consider the case of one tube $\mathcal{T}$, and a subcatgory $\mathcal{S}_{J,\mathcal{F}}$, with $J \neq \emptyset$. We want to show that there is some rigid representation $V \in \mathcal{T}$ such that $V^\perp \cap \mathcal{T} = \mathcal{S}_{J,\mathcal{F}}$.

Let $\mathcal{Q}$ be the collection of the (relative) quasi-simples of $\mathcal{E}_J$, as before. For each $Q \in \mathcal{Q}$, let R_Q be the target of the irreducible epimorphism from Q (or 0 if there is no such epimorphism). Then $\bigoplus R_Q$ rigid, and $(\bigoplus R_Q)^\perp$ consists of the additive hull of $\mathcal{E}_J$ together with each of the wings W_Q. Since each of the W_Q is Dynkin of type $\mathbb{A}$, Lemma 9.10 tells us that adding further summands to $\bigoplus R_Q$, it is possible to find a rigid object V such that $V^\perp \cap \mathcal{T} = \mathcal{S}_{J,\mathcal{F}}$. $\square$

Using the classification of thick subcategories inside the regular representations, the previous proposition can be restated as follows:

Proposition 9.13. *The semi-stable subcategories in (ii) of Theorem 5.4 can also be described as those abelian extension-closed subcategories of the regular part of* $\mathrm{rep}(Q)$, *which contain infinitely many indecomposable objects from each tube.*

We end this section will the following, which will be used in the next section.

Proposition 9.14. *Let Q be a Euclidean quiver and $\mathcal{C}$ be a thick subcategory of* $\mathrm{rep}(Q)$. *Then one of $\mathcal{C}$, $\mathcal{C}^\perp$ is not contained in $\mathcal{R}\mathrm{eg}$ if and only if both $\mathcal{C}$ and $\mathcal{C}^\perp$ are generated by exceptional sequences. In this case, $(\mathcal{C}, \mathcal{C}^\perp)$ is an exceptional semi-orthogonal pair.*

Proof. If $\mathcal{C}$ is not contained in $\mathcal{R}\mathrm{eg}$, then it follows from Theorem 9.9 that $\mathcal{C}$ is generated by an exceptional sequence. Since this exceptional sequence can be completed to a full exceptional sequence, see [6], $\mathcal{C}^\perp$ is also generated by an exceptional sequence. If $\mathcal{C}^\perp$ is not contained in $\mathcal{R}\mathrm{eg}$, then $\mathcal{C}^\perp$ is generated by an exceptional sequence by Theorem 9.9. By Lemma 9.4, we have $^\perp(\mathcal{C}^\perp) = \mathcal{C}$ and hence $\mathcal{C}$ is generated by an exceptional sequence. Suppose now that both $\mathcal{C}, \mathcal{C}^\perp$ are contained in $\mathcal{R}\mathrm{eg}$. If $\mathcal{C}$ is generated by an exceptional sequence, then all the representations in homogeneous tubes are contained in $\mathcal{C}^\perp$ and hence, $\mathcal{C}^\perp$ cannot be generated by an exceptional sequence. If $\mathcal{C}^\perp$ is generated by an exceptional sequence, then all the representations in homogeneous tubes are contained in $^\perp(\mathcal{C}^\perp) = \mathcal{C}$ and $\mathcal{C}$ cannot be generated by an exceptional sequence. $\square$

10. Intersection of semi-stable subcategories

In this section, we start with Q any connected acyclic quiver and later specialize to the Euclidean case. We first look at some situations where the intersection of semi-stable subcategories is again semi-stable, and we end the section by describing, in the Euclidean case, how to construct the subcategories of $\mathrm{rep}(Q)$ arising as an intersection of semi-stable subcategories.

A prehomogeneous dimension vector d such that all the indecomposable summands of $M(d)$ are non-regular will be called *strongly prehomogeneous*. The following result says that when d_1, d_2 are strongly prehomogeneous, the intersection $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2}$ remains semi-stable.

Proposition 10.1. *Let d_1, d_2 be two strongly prehomogeneous dimension vectors. Then $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2} = \mathrm{rep}(Q)_{d_3}$, where d_3 is prehomogeneous.*

Proof. By assumption, $\mathrm{rep}(Q)_{d_i} = M(d_i)^{\perp}$, $i = 1, 2$, where the $M(d_i)$ are rigid representations whose indecomposable direct summands are non-regular. Now, $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2} = \mathcal{C}(M(d_1), M(d_2))^{\perp}$. However, from Corollary 9.8,

$$\mathcal{C}(M(d_1), M(d_2)) = \mathcal{C}(E),$$

where E is an exceptional sequence whose terms are non-regular. This gives a rigid representation V with $\mathcal{C}(M(d_1), M(d_2)) = \mathcal{C}(V)$. Then, $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2} = V^{\perp}$. Since V is rigid, the orbit of V is open in $\mathrm{rep}(Q, d_V)$. Hence, the canonical decomposition of V is given by the dimension vectors of its indecomposable direct summands. In particular, d_3 is prehomogeneous. $\square$

Here is a simple examples that illustrate some results of Section 9 and Proposition 10.1.

Example 10.2. Let Q be the following quiver
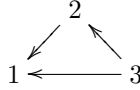
There is only one non-homogeneous tube of rank two with two quasi-simple representations M_1, M_2 where $d_{M_1} = (1, 0, 1)$ and $d_{M_2} = (0, 1, 0)$. Observe that $\mathcal{C}(M_1, M_2)$ is not generated by an exceptional sequence. This shows that the assumption that M_1, M_2 are non-regular in Corollary 9.8 is essential.

For $i = 1, 2, 3$, denote by S_i the simple representation at i, by P_i the projective representation at i and by I_i the injective representation at i. Observe that d_{S_1}, d_{S_3} are strongly prehomogeneous. However, $\mathrm{rep}(Q)_{d_{S_1}} \cap \mathrm{rep}(Q)_{d_{S_3}} = \mathrm{rep}(Q)_{d_3}$ where d_3 is prehomogeneous but not strongly prehomogeneous. Any such d_3 is cd-equivalent to one of $(1, 0, 2), (2, 0, 1)$.

Let $d_1 = (1, 0, 0)$ and $d_2 = (2, 1, 1)$. Since d_1 is the dimension vector of P_1, we have

$$\mathrm{rep}(Q)_{d_1} = P_1^{\perp} = \mathrm{add}(S_2, S_3, I_2) = \mathcal{C}(I_3, I_2).$$

Similarly,

$$\mathrm{rep}(Q)_{d_2} = P_3^{\perp} = \mathrm{add}(S_1, S_2, P_2) = \mathcal{C}(P_1, P_2).$$

Therefore, each of $\mathrm{rep}(Q)_{d_1}, \mathrm{rep}(Q)_{d_2}$ is generated by a representation whose indecomposable direct summands are non-regular. We see that the intersection $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2}$ is $\mathcal{C}(P_1, P_3)^{\perp} = \mathrm{add}(S_2)$, where S_2 is regular.

From now on, we suppose that Q is a Euclidean quiver. We give a complete description of the possible intersections of semi-stable subcategories of $\mathrm{rep}(Q)$, and in particular, a description of those intersections that are not semi-stable. Let us start with some notations and reminders.

Given two semi-orthogonal pairs $(\mathcal{A}, \mathcal{A}^{\perp})$ and $(\mathcal{B}, \mathcal{B}^{\perp})$, we define

$$(\mathcal{A}, \mathcal{A}^{\perp}) * (\mathcal{B}, \mathcal{B}^{\perp}) = (\mathcal{C}(\mathcal{A}, \mathcal{B}), \mathcal{A}^{\perp} \cap \mathcal{B}^{\perp}),$$

where we recall that $\mathcal{C}(\mathcal{A}, \mathcal{B})$ denotes the smallest thick subcategory of $\mathrm{rep}(Q)$ containing $\mathcal{A}$ and $\mathcal{B}$. Observe that $\mathcal{A}^{\perp} \cap \mathcal{B}^{\perp} = \mathcal{C}(\mathcal{A}, \mathcal{B})^{\perp}$, hence defining a new semi-orthogonal pair $(\mathcal{C}(\mathcal{A}, \mathcal{B}), \mathcal{A}^{\perp} \cap \mathcal{B}^{\perp})$.

Recall that for each facet F_I of H_δ^{ss}, we have the representation-finite thick subcategory $\mathcal{W}_I$ generated by the exceptional representations whose dimension vectors lie in F_I. Given a dimension vector d, recall that $M(\hat{d})$ is a rigid representation of dimension vector $\hat{d}$, where $\hat{d}$ is the sum of the real Schur roots appearing in the canonical decomposition of d. We start with the following.

Proposition 10.3. *Let Q be a Euclidean quiver and d_1, d_2 be dimension vectors. Then $(\mathcal{C}(M(\hat{d}_1)), \mathcal{C}(M(\hat{d}_1))^{\perp}) * (\mathcal{C}(M(\hat{d}_2)), \mathcal{C}(M(\hat{d}_2))^{\perp})$ is exceptional if and only if one of the following occurs:*

(a) *At least one of d_1, d_2 does not lie on H_δ^{ss},*
(b) *There exists $I \in R$ with $d_1, d_2 \in C_I$.*

Proof. If at least one of d_1, d_2 is not in H_δ^{ss}, then $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))$ is not contained in $\mathcal{R}eg$. We apply Proposition 9.14 in this case. If both d_1, d_2 lie in the same cone C_I, then $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))$ is clearly representation-finite by Lemma 8.1. By Lemma 9.6, $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))$ is generated by an exceptional sequence and so is $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))^{\perp}$. Suppose now that d_1, d_2 both lie in H_δ^{ss} but in different cones. If $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))$ is representation-finite, then it lies in some $\mathcal{W}_I$ by Lemma 8.1, and hence both d_1, d_2 lie in C_I, a contradiction. Since $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))^{\perp}$ contains all the representations in homogeneous tubes, it is not representation-finite. Therefore, $(\mathcal{C}(M(\hat{d}_1)), \mathcal{C}(M(\hat{d}_1))^{\perp}) * (\mathcal{C}(M(\hat{d}_2)), \mathcal{C}(M(\hat{d}_2))^{\perp})$ cannot be exceptional since neither $\mathcal{C}(\mathcal{C}(M(\hat{d}_1)), \mathcal{C}(M(\hat{d}_2)))$ nor $\mathcal{C}(\mathcal{C}(M(\hat{d}_1)), \mathcal{C}(M(\hat{d}_2)))^{\perp}$ is representation-finite. $\square$

From what we just proved, if $(\mathcal{C}(M(\hat{d}_1)), \mathcal{C}(M(\hat{d}_1))^{\perp}) * (\mathcal{C}(M(\hat{d}_2)), \mathcal{C}(M(\hat{d}_2))^{\perp})$ is not exceptional, then both $\mathcal{C}(M(\hat{d}_1), M(\hat{d}_2))$ and $\mathcal{C}(M(\hat{d}_1))^{\perp} \cap \mathcal{C}(M(\hat{d}_2))^{\perp}$ are contained in $\mathcal{R}eg$ and are not representation-finite. The following proposition gives a first partial answer on how to compute the intersection of two semi-stable subcategories in the Euclidean case.

Proposition 10.4. *Let Q be a Euclidean quiver and d_1, d_2 be dimension vectors satisfying the equivalent conditions of Proposition 10.3. Then $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2} = \mathrm{rep}(Q)_{d_3}$ for some dimension vector d_3.*

(a) *If at least one of d_1, d_2 is not in H_δ^{ss}, then d_3 is not in H_δ^{ss}. In particular, $\mathrm{rep}(Q)_{d_3}$ is representation-finite.*
(b) *If both d_1, d_2 lie in some cone C_I but not both on F_I, then d_3 can be chosen to be in C_I. In this case, $\mathrm{rep}(Q)_{d_3} = V^{\perp} \cap \mathcal{R}eg$ where V is rigid and $\mathcal{C}(V)$ is representation-finite.*
(c) *If both d_1, d_2 lie on some facet F_I, then $\mathrm{rep}(Q)_{d_3} = V^{\perp}$ where V is rigid and $\mathcal{C}(V)$ is representation-finite.*

Proof. The main statement follows from Propositions 10.3 and 4.4. For part (a), suppose that d_1 does not lie in H_δ^{ss}. Then $\mathrm{rep}(Q)_{d_1}$ is representation-finite and so is $\mathrm{rep}(Q)_{d_1} \cap \mathrm{rep}(Q)_{d_2} = \mathrm{rep}(Q)_{d_3}$. If d_3 lies in H_δ^{ss}, then all the homogeneous tubes

are contained in $\text{rep}(Q)_{d_3}$, contradicting that $\text{rep}(Q)_{d_3}$ is representation-finite. For part (c), $d_1 = \hat{d}_1$ and $d_2 = \hat{d}_2$ and we have $M(d_1), M(d_2) \in \mathcal{W}_I$, and hence, $\mathcal{C}(M(d_1), M(d_2)) \subseteq \mathcal{W}_I$ is representation-finite. Therefore, $\mathcal{C}(M(d_1), M(d_2))$ is generated by some rigid representation V and $\text{rep}(Q)_{d_3} = V^\perp$. Part (b) follows from part (c). $\qquad\square$

Now, we concentrate on the remaining case, that is, to describe the intersection $\text{rep}(Q)_{d_1} \cap \text{rep}(Q)_{d_2}$ when both d_1 and d_2 lie in H_δ^{ss}, and not in the same cone C_I. We can clearly assume that Q has more than 2 vertices.

Lemma 10.5. *If d_1 and d_2 both lie in H_δ^{ss}, and do not both lie in the same cone C_I, then $\text{rep}(Q)_{d_1} \cap \text{rep}(Q)_{d_2}$ is contained in $\mathcal{R}eg$.*

Proof. We can clearly assume, from Proposition 5.2, that both d_1, d_2 lie on the boundary of H_δ^{ss}. Let $\mathcal{C} = \text{rep}(Q)_{d_1} \cap \text{rep}(Q)_{d_2}$. Let $f = d_1 + d_2$. We observe that $X = M(d_1) \oplus M(d_2)$ is a representation with dimension vector f, such that $\text{Hom}(X, Y) = 0$ and $\text{Ext}^1(X, Y) = 0$ for any Y in $\mathcal{C}$. Thus $\mathcal{C}$ is contained in $\text{rep}(Q)_f$. Since d_1 and d_2 do not both lie in the same cone C_I, we know that f lies in the interior of H_δ^{ss}. Proposition 5.2 tells us that $\text{rep}(Q)_f$ is contained in $\mathcal{R}eg$, proving the result. $\qquad\square$

Since we now know that, in the case we are presently studying, $\text{rep}(Q)_{d_1} \cap \text{rep}(Q)_{d_2}$ is contained in $\mathcal{R}eg$, it suffices to restrict our attention to the regular parts of $\text{rep}(Q)_{d_1}$ and $\text{rep}(Q)_{d_2}$. It therefore suffices to assume that neither d_i lies on the boundary of H_δ^{ss}. If d_1 does lie on the boundary, replace it by $d_1' = d_1 + \delta$, and observe that $\text{rep}(Q)_{d_1'} = \mathcal{R}eg \cap \text{rep}(Q)_{d_1}$.

By Proposition 9.13, we know exactly what kind of subcategories can arise as semi-stable subcategories of $\mathcal{R}eg$: namely, they are the subcategories with the property that their intersection with each tube is of the form $\mathcal{S}_{J,\mathcal{F}}$ where J is non-empty. Observe that the condition $J \neq \emptyset$ applied to a homogeneous tube $\mathcal{T}$ just means $\mathcal{S}_{J,\mathcal{F}} = \mathcal{T}$ (and $\mathcal{F} = 0$).

Given a non-homogeneous tube $\mathcal{T}$ of $\text{rep}(Q)$, denote by $\mathcal{O}_\mathcal{T}$ the set of indecomposable representations in $\mathcal{T}$ which admit an irreducible monomorphism to a singular-isotropic representation. The set $\mathcal{O}_\mathcal{T}$ hence form a τ-orbit of $\mathcal{T}$, which lie just below the τ-orbit of singular-isotropic representations.

Lemma 10.6. *Let $\mathcal{T}$ be a non-homogeneous tube, and $\mathcal{S}_{J_i, \mathcal{F}_i}$ be thick subcategories of $\mathcal{T}$ with each J_i non-empty. Let $\mathcal{C} = \bigcap_i \mathcal{S}_{J_i, \mathcal{F}_i}$. Then*
(a) *The category $\mathcal{C}$ is given by some $\mathcal{S}_{K, \mathcal{G}}$, where $\mathcal{G}$ does not contain any indecomposables from $\mathcal{O}_\mathcal{T}$.*
(b) *Conversely, any thick subcategory $\mathcal{S}_{K, \mathcal{G}}$ of $\mathcal{T}$ such that $\mathcal{G}$ does not contain any indecomposables from $\mathcal{O}_\mathcal{T}$ can be written as the intersection $\mathcal{S}_{K_1, \mathcal{G}_1} \cap \mathcal{S}_{K_2, \mathcal{G}_2}$ where both K_1, K_2 are non-empty.*

Proof. Clearly, $K = \bigcap_i J_i$. If $K \neq \emptyset$, let V be a singular-isotropic representation contained in $\mathcal{C}$. Since $\mathcal{G} \subseteq {}^\perp V \cap V^\perp$, it follows that $\mathcal{G} \cap \mathcal{O}_\mathcal{T} = \emptyset$, as desired.

Suppose now that $K = \emptyset$. Since $\bigcap_i \mathcal{E}_{J_i} = 0$, any indecomposable of $\bigcap_i \mathcal{S}_{J_i, \mathcal{F}_i}$ must be contained in some $\mathcal{F}_i$. But by the previous argument, $\mathcal{F}_i \cap \mathcal{O}_\mathcal{T} = \emptyset$.

Now we prove the converse direction. If $\mathcal{S}_{K, \mathcal{G}}$ were itself semi-stable there would be nothing to prove, so we may assume that it is not, so $K = \emptyset$. Since $\mathcal{G}$ is a thick subcategory whose indecomposable objects are all exceptional and does

not contain objects in $\mathcal{O}_{\mathcal{T}}$, there is a singular-isotropic representation Z such that $\mathcal{G} \subseteq {}^{\perp}Z \cap Z^{\perp}$. Let J be the set consisting of the quasi-socle of Z, and let J^c be all the other quasi-simples. Then $\mathcal{S}_{K,\mathcal{G}} = \mathcal{S}_{J,\mathcal{G}} \cap \mathcal{S}_{J^c,0}$. $\qquad\square$

We have now essentially proved our final main theorem.

Theorem 10.7. *Let Q be a Euclidean quiver. There are finitely many subcategories of $\mathrm{rep}(Q)$ which arise as an intersection of semi-stable subcategories, and which are not themselves semi-stable subcategories for any stability condition. Moreover, such a subcategory is characterized by the following.*

(a) *It is contained entirely in the regular part of Q.*

(b) *It contains all the homogeneous tubes.*

(c) *Its intersection with each non-homogeneous tube $\mathcal{T}_i$ can be written as $\mathcal{S}_{J_i,\mathcal{F}_i}$ where $(\mathcal{E}_{J_i}, \mathcal{F}_i)$ is a regular orthogonal pair in $\mathcal{T}_i$ with $\mathcal{F}_i \cap \mathcal{O}_{\mathcal{T}_i} = \emptyset$. Moreover, at least one of the J_i is empty.*

(d) *It can be expressed as the intersection of two semi-stable subcategories.*

Proof. We have shown that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two semi-stable subcategories, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is not itself semi-stable, then $\mathcal{C}_1 \cap \mathcal{C}_2$ is contained in the regular part $\mathcal{R}\mathrm{eg}$, and contains the homogeneous tubes. Since it is thick, its intersection with each non-homogeneous tube can be written as $\mathcal{S}_{J_i,\mathcal{F}_i}$, and since, by assumption, it is not semi-stable, some J_i must be empty. Lemma 10.6 says further that $\mathcal{F}_i \cap \mathcal{O}_{\mathcal{T}_i} = \emptyset$. The same lemma also shows that any such subcategory can be written as the intersection of two semi-stable subcategories. $\qquad\square$

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