FROM m-CLUSTERS TO m-NONCROSSING PARTITIONS VIA EXCEPTIONAL SEQUENCES

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ABSTRACT. Let W be a finite crystallographic reflection group. The generalized Catalan number of W coincides both with the number of clusters in the cluster algebra associated to W, and with the number of noncrossing partitions for W. Natural bijections between these two sets are known. For any positive integer m, both m-clusters and m-noncrossing partitions have been defined, and the cardinality of both these sets is the Fuss-Catalan number $C_m(W)$. We give a natural bijection between these two sets by first establishing a bijection between two particular sets of exceptional sequences in the bounded derived category $D^b(H)$ for any finite-dimensional hereditary algebra H.

INTRODUCTION

This paper is motivated by the following problem in combinatorics. Let W be a finite crystallographic reflection group. Associated to W is a positive integer called the generalized Catalan number, which on the one hand equals the number of clusters in the associated cluster algebra [FZ], and on the other hand equals the number of noncrossing partitions for W, see [Be]. Natural bijections between the sets of clusters and noncrossing partitions associated with W have been found in [Re, ABMW]. More generally, for any integer $m \geq 1$, there is associated with W a set of m-clusters introduced in [FR] and a set of m-noncrossing partitions defined in [Ar]. Each of these sets has cardinality the Fuss-Catalan number $C_m(W)$, see [FR, Ar]. The formula for $C_m(W)$ is as follows:

$$C_m(W) = \frac{\prod_{i=1}^{n} mh + e_i + 1}{\prod_{i=1}^{n} e_i + 1},$$

where n is the rank of W, h is its Coxeter number, and $e_1, \ldots, e_n$ are its exponents.

One of our main results is to establish a natural bijection between the m-clusters and the m-noncrossing partitions for any $m \geq 1$. We accomplish this by first solving a related, more general problem about bijections between classes of exceptional sequences in bounded derived categories of finite dimensional hereditary algebras, which is also of independent interest.

Let H be a connected hereditary artin algebra. Then H is a finite dimensional algebra over its centre, which is known to be a field k. Examples of such algebras

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are path algebras over a field of finite quivers with no oriented cycles. Let $\operatorname{mod} H$ be the category of finite dimensional left H-modules and let $\mathcal{D} = D^b(H)$ be the bounded derived category. An H-module M is called *rigid* if $\operatorname{Ext}^1(M, M) = 0$, and an indecomposable rigid H-module is called *exceptional*. The set of isomorphism classes of exceptional modules is countable, and it has interesting combinatorial structures, which have been much studied in the representation theory of algebras, and in various combinatorial applications of this theory.

We study exceptional objects and sequences in the derived category $\mathcal{D}$. With a slight modification of the definition in [KV], we say that an object T in $\mathcal{D}$ is *silting* if $\operatorname{Ext}^i(T, T) = 0$ for $i > 0$ and T is maximal with respect to this property. We say that a basic object $X = X_1 \oplus \cdots \oplus X_n$ in $\mathcal{D}$ is a $\operatorname{Hom}_{\leq 0}$-configuration if all X_i are exceptional, $\operatorname{Hom}(X_i, X_j) = 0$ for $i \neq j$, $\operatorname{Ext}^t(X, X) = 0$ for $t < 0$, and there is no subset $\{Y_1, \dots, Y_r\}$ of the indecomposable summands of X such that $\operatorname{Ext}^1(Y_i, Y_{i+1}) \neq 0$ for $1 \leq i < r$ and $\operatorname{Ext}^1(Y_r, Y_1) \neq 0$. (Here n denotes the number of isomorphism classes of simple H-modules.) It follows from the definition of $\operatorname{Hom}_{\leq 0}$-configuration that $\{X_1, \dots, X_n\}$ can be ordered into a complete exceptional sequence. For any $m \geq 1$, we say that X is an m-$\operatorname{Hom}_{\leq 0}$-configuration if the X_i lie in $\operatorname{mod} H[t]$ for $0 \leq t \leq m$.

Given the representation-theoretic interpretation of noncrossing partitions provided by [IT], it was reasonable to expect a representation-theoretic manifestation of m-noncrossing partitions. One approach to developing such a definition would have been to follow [IT] closely, and consider sequences of finitely generated exact abelian, extension closed subcategories with suitable orthogonality conditions. $\operatorname{Hom}_{\leq 0}$-configurations seemed to provide a more convenient viewpoint. When we have a Dynkin quiver, the vanishing of Hom and of Ext, which can be reduced to Hom, is easy to compute on the AR-quiver. Hence it is not hard to compute $\operatorname{Hom}_{\leq 0}$-configurations in this case.

Our main result is to obtain a natural bijection between silting objects and $\operatorname{Hom}_{\leq 0}$-configurations via a certain sequence of mutations of exceptional sequences. This induces a bijection between m-cluster tilting objects and m-$\operatorname{Hom}_{\leq 0}$-configurations, for any H. We also give a bijection between m-$\operatorname{Hom}_{\leq 0}$-configurations and m-noncrossing partitions for arbitrary H. Specializing to H being of Dynkin type, we get as an application a bijection betwen m-clusters and m-noncrossing partitions.

The paper is organized as follows. We first review preliminaries concerning exceptional sequences in module categories as well as in derived categories. In Section 2, we recall the definition of silting objects and m-cluster tilting objects, and define $\operatorname{Hom}_{\leq 0}$-configurations and m-$\operatorname{Hom}_{\leq 0}$-configurations. We also state the precise version of our main result. In the next section we give some basic results about mutations of exceptional sequences in the derived category. In Section 4 we show how to construct $\operatorname{Hom}_{\leq 0}$-configurations from silting objects. In the next two sections we finish the proof of our main result. In Section 7 we give the combinatorial interpretation of our main result, including a version for the "positive" Fuss-Catalan combinatorics. In Section 8, we discuss the relationship between our $\operatorname{Hom}_{\leq 0}$-configurations and Riedtmann's combinatorial configurations from her work on selfinjective algebras [Rie1, Rie2]. In Section 9 we show how the bijection we have constructed interacts with torsion classes in $\mathcal{D}$.

We remark that the results in Section 8 have also been obtained by Simoes [S], in the Dynkin case, with an approach which is different than ours, and and independent from it.

1. Preliminaries on exceptional sequences

As before, let H be a finite dimensional connected hereditary algebra over a field k which is the centre of H, and let $\operatorname{mod} H$ denote the category of finite dimensional left H-modules. We assume that H has n simple modules up to isomorphism. In this section we recall some basic results about exceptional sequences.

1.1. **Exceptional sequences in the module category.** A sequence of exceptional objects $\mathcal{E} = (E_1, \dots, E_r)$ in $\operatorname{mod} H$ is called an *exceptional sequence* if $\operatorname{Hom}(E_j, E_i) = 0 = \operatorname{Ext}^1(E_j, E_i)$ for $j > i$.

There are right and left mutation operations, denoted respectively μ_i and μ_i^{-1}, which take exceptional sequences to exceptional sequences. Given an exceptional sequence $\mathcal{E} = (E_1, \dots, E_r)$, right mutation replaces the subsequence (E_i, E_{i+1}) by (E_{i+1}, E_i^*), while left mutation replaces the subsequence (E_i, E_{i+1}) by $(E_{i+1}^!, E_i)$, for some exceptional objects $E_{i+1}^!$ and E_i^*.

We need the following facts about exceptional sequences in $\operatorname{mod} H$. These are proved in [C] (if the field k is algebraically closed) and in [Rin2] in general.

Proposition 1.1. *Let $\mathcal{E} = (E_1, \dots, E_r)$ in $\operatorname{mod} H$ be an exceptional sequence. Then the following hold:*

(a) $r \leq n$
(b) *if $r < n$, then there is an exceptional sequence $(E_1, \dots, E_r, E_{r+1}, \dots, E_n)$*
(c) *if $r = n-1$, then for a fixed index $j \in \{1, \dots n\}$, there is a unique indecomposable M, such that*
$$(E_1, \dots, E_{j-1}, M, E_j, \dots E_{n-1})$$
is an exceptional sequence
(d) *for any $i \in \{1, \dots, r-1\}$, we have $\mu_i^{-1}(\mu_i(\mathcal{E})) = \mathcal{E} = \mu_i(\mu_i^{-1}(\mathcal{E}))$*
(e) *the set of μ_i satisfies the braid relations, i.e. $\mu_i\mu_{i+1}\mu_i = \mu_{i+1}\mu_i\mu_{i+1}$ for $i \in \{1, \dots, r-2\}$, and $\mu_i\mu_j = \mu_j\mu_i$ for $|i-j| > 1$*
(f) *the action of the set of μ_i on the set of complete exceptional sequences is transitive.*

An exceptional sequence $\mathcal{E} = (E_1, \dots, E_r)$ is called *complete* if $r = n$.

1.2. **Exceptional sequences in derived categories.** Let $\mathcal{D} = D^b(H)$ denote the bounded derived category with translation functor [1], the shift functor. This is a triangulated category, see [H] for general properties of such categories. It is well known that since H is hereditary, the indecomposable objects of $\mathcal{D}$ are stalk complexes, i.e. they are up to isomorphism of the form $M[i]$ for some indecomposable H-module M and some integer i. If $X = M[i]$ is an indecomposable object in $\mathcal{D}$, we will write $\overline{X} = M$ for the corresponding object in $\operatorname{mod} H$.

It is well-known that the derived category $\mathcal{D}$ has almost split triangles [H], and hence an AR-translation τ, or equivalently a Serre-functor ν, where we have $\nu = \tau[1]$. We have the AR-formula $\operatorname{Hom}_{\mathcal{D}}(X, Y) \simeq D\operatorname{Hom}(Y, \tau X)$, for all objects X, Y in $\mathcal{D}$.

It is convenient to consider also exceptional sequences in the derived category $\mathcal{D}$. Let $\mathcal{E} = (E_1, E_2, \dots, E_r)$ be a sequence of indecomposable objects in $\mathcal{D}$. It is called an *exceptional sequence* in $\mathcal{D}$ if $\overline{\mathcal{E}} = (\overline{E_1}, \overline{E_2}, \dots, \overline{E_r})$ is an exceptional sequence in $\operatorname{mod} H$, and *complete* if $\overline{\mathcal{E}}$ is complete.

In Section 3 we will describe a mutation operation on exceptional sequences in $\mathcal{D}$. For this we need the following preliminary results.

Lemma 1.2. *Let (E, F) be an exceptional sequence in $\mathcal{D}$. Then $\operatorname{Ext}^i(E, F) = \operatorname{Hom}_{\mathcal{D}}(E, F[i])$ is nonzero for at most one integer i, and $\operatorname{Ext}^i(F, E) = \operatorname{Hom}_{\mathcal{D}}(F, E[i]) = 0$ for all $i \in \mathbb{Z}$.*

Proof. We provide the short proof from [BRT2] for the convenience of the reader.

It suffices to check the statements for $(\overline{E}, \overline{F})$. By results from [C, Rin2], we can consider $(\overline{E}, \overline{F})$ as an exceptional sequence in a hereditary module category, say $\operatorname{mod} H'$, with H' of rank 2, and such that $\operatorname{mod} H'$ has a full and exact embedding into $\operatorname{mod} H$. For a hereditary algebra H' of rank 2, the only exceptional indecomposable modules are preprojective or preinjective. Hence, a case analysis of the possible exceptional sequences in $\operatorname{mod} H'$ for such algebras, gives the first statement. The second statement is immediate from the definition of exceptional sequence. $\square$

There is a general notion of exceptional sequences in triangulated categories, see [Bond, GK]. Note that in our setting, this definition is equivalent to our definition. This follows from combining the fact that indecomposables in $\mathcal{D}$ are stalk complexes with the second part of Lemma 1.2.

2. Silting objects and $\operatorname{Hom}_{\leq 0}$-configurations

In this section we recall some basic properties of silting objects, and introduce the notion of $\operatorname{Hom}_{\leq 0}$-configurations.

2.1. **Silting objects.** A basic object Y in $\mathcal{D}$ is called a *partial silting object* if $\operatorname{Ext}^i_{\mathcal{D}}(Y, Y) = 0$ for $i \geq 1$, and *silting* if it is maximal with respect to this property. Note that this differs slightly from the original definition in [KV]. It is known (see [BRT2]) that a partial silting object Y is silting if and only if it has n indecomposable direct summands. If a silting object Y is in $\operatorname{mod} H$, it is called a *tilting module*.

The following connection with exceptional sequences is a special case of [AST, Theorem 2.3]. We include the sketch of a proof for convenience.

Lemma 2.1. *If Y is partial silting in $\mathcal{D}$, there is a way to order its indecomposable direct summands to obtain an exceptional sequence $(Y_1, \dots, Y_r)$ in $\mathcal{D}$.*

Proof. Assume that $A[u]$ and $B[v]$ are indecomposable direct summands of Y where A, B are H-modules. If $v > u$, then $\operatorname{Ext}^i_{\mathcal{D}}(B, A) = 0$ for all i, so we put $A[u]$ before $B[v]$ in the exceptional sequence. For a fixed degree d, assume there are $t \geq 1$ direct summands of Y of degree d. The direct sum of these t summands is the shift of a rigid module in $\operatorname{mod} H$. By [HR], there are no oriented cycles in the quiver of the endomorphism ring of a rigid module. Hence, there is an ordering on these t summands, say $A_1, \ldots, A_t$, such that $\operatorname{Hom}(A_j, A_k) = 0$ for $j > k$. □

An exceptional sequence is called *silting* if it is induced by a silting object as in Lemma 2.1.

An object M in $\mathcal{D}$ is called a *generator* if $\operatorname{Hom}_{\mathcal{D}}(M, X[i]) = 0$ for all i only if $X = 0$. For a hereditary algebra, the indecomposable projectives can be ordered to form an exceptional sequence. Hence, by transitivity of the action of mutation on exceptional sequences (Proposition 1.1) (e)), the direct sum of the objects in an exceptional sequence is a generator. Thus we obtain the following consequence of Lemma 2.1.

Lemma 2.2. *Any silting object in $\mathcal{D}$ is a generator.*

For a positive integer m, the m-cluster category is the orbit category $\mathcal{C}_m = \mathcal{D}/\tau^{-1}[m]$, see [BMRRT, K, T, Z, W]. It is canonically triangulated by [K]. An object T in $\mathcal{C}_m$ is called *maximal rigid* if $\operatorname{Ext}^i_{\mathcal{C}_m}(T, T) = 0$ for $i = 1, \ldots, m$, and T is maximal with respect to this property. An object T in $\mathcal{C}_m$ is called *m-cluster tilting* if for any object X, we have that X is in $\operatorname{add} T$ if and only if $\operatorname{Ext}^i_{\mathcal{C}_m}(T, X) = 0$ for $i = 1, \ldots, m$. It is known by [W, ZZ] that T is maximal rigid if and only if it is an m-cluster tilting object. Let $\mathcal{D}^{(\geq 1)+}_{\leq m}$ denote the full subcategory of $\mathcal{D}$ additively generated by: the injectives in $\operatorname{mod} H$, together with $\operatorname{mod} H[i]$ for $1 \leq i \leq m$. Every object in $\mathcal{C}_m$ is induced by an object contained in $\mathcal{D}^{(\geq 1)+}_{\leq m}$.

In [BRT2] it is shown that silting objects contained in $\mathcal{D}^{(\geq 1)+}_{\leq m}$ are in 1-1 correspondence with m-cluster tilting objects. We consider this an identification, and from now on we will refer to silting objects contained in $\mathcal{D}^{(\geq 1)+}_{\leq m}$ as m-cluster tilting objects.

2.2. $\operatorname{Hom}_{\leq 0}$**-configurations and** m**-**$\operatorname{Hom}_{\leq 0}$**-configurations.** In this subsection we introduce new types of objects in $\mathcal{D}$. They are related to the combinatorial configurations investigated in [Rie1, Rie2], and will turn out to be closely related to noncrossing partitions.

A basic object X in $\mathcal{D}$ is a $\operatorname{Hom}_{\leq 0}$*-configuration* if

- (H1) X is the direct sum of n exceptional indecomposable summands $X_1, \ldots, X_n$, where n is the number of simple modules of H.
- (H2) $\operatorname{Hom}(X_i, X_j) = 0$ for $i \neq j$.
- (H3) $\operatorname{Ext}^t(X, X) = 0$ for $t < 0$.
- (H4) there is no subset $\{Y_1, \ldots, Y_r\}$ of the indecomposable summands of X such that $\operatorname{Ext}^1(Y_i, Y_{i+1}) \neq 0$ and $\operatorname{Ext}^1(Y_r, Y_1) \neq 0$.

Lemma 2.3. *The indecomposable summands of a* $\operatorname{Hom}_{\leq 0}$*-configuration can be ordered into a complete exceptional sequence.*

Proof. Let X be a $\operatorname{Hom}_{\leq 0}$-configuration, and let $X = \bigoplus_i A_i[i]$, with $A_i \in \operatorname{mod} H$, and where all but finitely many of the A_i are zero. Each A_i is the direct sum of finitely many indecomposables in $\operatorname{mod} H$ with no morphisms between them, so (H4) suffices to conclude that they can be ordered into an exceptional sequence.

Now concatenate the sequences in *decreasing* order with respect to i. This implies that if E, F are indecomposable summands of X lying in $\operatorname{mod} H[e]$ and $\operatorname{mod} H[f]$ respectively with $e < f$, then F will precede E in the sequence. We must therefore show that $\operatorname{Ext}^j(E, F) = 0$ for all j. This is true for $j \leq 0$ by (H2) and (H3), and for $j > 0$ because $e < f$. □

Note that it is also the case that, for X in $\mathcal{D}$, if the summands of X can be ordered into a complete exceptional sequence, then (H4) is necessarily satisfied.

If X is a $\mathrm{Hom}_{\leq 0}$-configuration, we refer to an exceptional sequence on the indecomposable summands of X as a $\mathrm{Hom}_{\leq 0}$-configuration exceptional sequence. A $\mathrm{Hom}_{\leq 0}$-configuration will be called an m-$\mathrm{Hom}_{\leq 0}$-*configuration* if it is contained in the full subcategory $\mathcal{D}^{\geq 0}_{\leq m}$, whose indecomposables are in $\operatorname{mod} H[i]$ for $0 \leq i \leq m$.

2.3. **Main results.** One aim of this paper is to use mutation of exceptional sequences to establish the following result.

Theorem 2.4. *There are bijections between*

(a) *exceptional sequences which are silting and exceptional sequences which are* $\mathrm{Hom}_{\leq 0}$*-configurations.*
(b) *silting objects and* $\mathrm{Hom}_{\leq 0}$*-configurations.*
(c) *m-cluster tilting objects and m-$\mathrm{Hom}_{\leq 0}$-configurations (for any $m \geq 1$).*

We prove (a) in Section 4 and (b) in Section 5, while (c) is proved in Section 6. In Section 7 we apply (c) in finite type to obtain a bijection between m-noncrossing partitions in the sense of [Ar] and m-clusters in the sense of [FR].

3. Mutations in the derived category

In this section we give some basic results on mutations of exceptional sequences in the bounded derived category. This is the main tool used in the proof of Theorem 2.4. We also compare mutations in $\mathcal{D}$ with mutations in $\operatorname{mod} H$. The results in this section can also be found in e.g. [Bond, GK]. We include proofs, for completeness and for the convenience of the reader.

We start with the following observation.

Lemma 3.1. *Let $(E_1, \dots, E_n)$ be a complete exceptional sequence in $\mathcal{D}$. For any complete exceptional sequence $(E_2, \dots, E_n, X)$, we have $X = \nu^{-1} E_1[k]$ for some k.*

Proof. Since the sequence $(E_1, \dots, E_n)$ is exceptional, we know that $\operatorname{Ext}^i_{\mathcal{D}}(E_j, E_1) = 0$ for $j > 1$ and all i. By Serre duality for $\mathcal{D}$, this implies that $\operatorname{Ext}^i_{\mathcal{D}}(\nu^{-1}E_1, E_j)=0$ for all i and all $j > 1$. From this it follows that $(\overline{E_2}, \dots, \overline{E_n}, \nu^{-1}\overline{E_1})$ is exceptional in $\operatorname{mod} H$. The claim now follows from Proposition 1.1 (c). □

We now describe mutation of exceptional sequences in $\mathcal{D}$, and show that it is compatible with mutation in the module category.

For an object Y in $\mathcal{D}$, we write $\operatorname{th}(Y)$ for the thick additive full subcategory of $\mathcal{D}$ generated by Y. Note that if Y is exceptional, the objects of $\operatorname{th}(Y)$ are direct sums of objects of the form $Y[i]$.

Define an operation $\hat{\mu}_i$ on exceptional sequences in $\mathcal{D}$ by replacing the pair (E_i, E_{i+1}) by the pair (E_{i+1}, E_i^*), where E_i^* is defined by taking $E_i \to Z$ to be the minimal left $\mathrm{th}(E_{i+1})$-approximation of E_i, and completing to a triangle:

$$E_i^* \to E_i \to Z \to E_i^*[1]$$

Note that Z is of the form $E_{i+1}^r[p]$, by Lemma 1.2 (in other words, Z is concentrated in one degree).

Similarly, we define $\hat{\mu}_i^{-1}$ of $(E_1, \dots, E_r)$ by taking $Z \to E_{i+1}$ to be the minimal right $\mathrm{th}(E_i)$ approximation of E_{i+1}, completing to a triangle

$$Z \to E_{i+1} \to E_{i+1}^! \to Z[1]$$

and replacing the pair (E_i, E_{i+1}) with $(E_{i+1}^!, E_i)$.

We recall the following well-known properties of exceptional objects in $\operatorname{mod} H$. The proofs of these are contained in [Bong], [HR] and [RS2], see also [Hu].

Lemma 3.2. *Let E, F be exceptional modules, and assume* $\operatorname{Hom}(F, E) = 0 = \operatorname{Ext}^1(F, E)$.

(a) *Let $f \colon E \to F^r$ be a minimal left* $\operatorname{add} F$*-approximation. If* $\operatorname{Hom}(E, F) \neq 0$*, then f is either an epimorphism or a monomorphism.*

(b) *If* $\operatorname{Ext}^1(E, F) \neq 0$*, there is an extension*

$$0 \to F^r \to U \to E \to 0$$

with the property that $\operatorname{Hom}(F^r, F') \to \operatorname{Ext}^1(E, F')$ *is a surjection for any* $F' \in \operatorname{add} F$.

Lemma 3.3. (a) *The operations $\hat{\mu}_i$ and $\hat{\mu}_i^{-1}$ are mutual inverses.*

(b) *Let $\mathcal{E}$ be an exceptional sequence in $\mathcal{D}$, then $\mu_i(\overline{\mathcal{E}}) = \overline{\hat{\mu}_i(\mathcal{E})}$ and $\mu_i^{-1}(\overline{\mathcal{E}}) = \overline{\hat{\mu}_i^{-1}(\mathcal{E})}$.*

(c) *If $(E_1, \dots, E_n)$ is a complete exceptional sequence in $\mathcal{D}$, then*

$$\mu_{n-1} \dots \mu_1(E_1, \dots, E_n) = (E_2, \dots, E_n, \nu^{-1} E_1).$$

(d) *The operators $\hat{\mu}_i$ and $\hat{\mu}_j$ satisfy the braid relations.*

(e) *Let (A, B, C) be an exceptional sequence in $\mathcal{D}$. Let $\hat{\mu}_1\hat{\mu}_2(A, B, C) = (C, A^*, B^*)$. Then* $\operatorname{Ext}^\bullet(A, B) \simeq \operatorname{Ext}^\bullet(A^*, B^*)$.

Proof. (a) Let

$$E_{i+1}^r[p-1] \to E_i^* \to E_i \to E_{i+1}^r[p]$$

be the approximation triangle defining E_i^*. Since $\operatorname{Hom}(E_{i+1}, E_i[j]) = 0$ for all j by Lemma 1.2, it is clear that the map $E_{i+1}^r[p-1] \to E_i^*$ is a right $\mathrm{th}(E_{i+1})$-approximation of E_i^*. The assertion follows from this and the dual argument.

For (b), let us recall how right mutation μ_i is defined in $\operatorname{mod} H$. For (E, F) an exceptional pair in $\operatorname{mod} H$ we have that $\mu_1(E, F) = (F, E^*)$. Let $f : E \to F^r$ be the minimal left $\operatorname{add} F$-approximation. Then the module E^* is defined as follows:

$$E^* = \begin{cases} E & \text{if } \mathrm{Hom}(E,F) = 0 = \mathrm{Ext}^1(E,F) \\ \ker f & \text{if } \mathrm{Hom}(E,F) \neq 0 \text{ and } f \text{ is an epimorphism} \\ \operatorname{coker} f & \text{if } \mathrm{Hom}(E,F) \neq 0 \text{ and } f \text{ is a monomorphism} \\ U & \text{if } \mathrm{Ext}^1(E,F) \neq 0 \text{ with } U \text{ as in (3.2 (b))} \end{cases}$$

Note that at most one of $\mathrm{Hom}(E,F)$ and $\mathrm{Ext}^1(E,F)$ is non-zero, by Lemma 1.2. It is now straightforward to check that in all cases we have $\mu_i(\overline{\mathcal{E}}) = \overline{\hat{\mu}_i(\mathcal{E})}$. This proves part (b) of Lemma 3.3.

For (c) consider the approximation triangles

$$X_j \overset{f_{j-1}}{\to} X_{j-1} \to E_j^{r_j}[k_j] \to X_j[1]$$

where we let X_i be the object in the i-th place of $\mu_{i-1} \ldots \mu_1(E_1, \ldots, E_n)$; that is to say, the object obtained by $i-1$ successive mutations of E_1. We want to show that $\mathrm{Hom}(X_n, X_1) \neq 0$. We have a sequence of morphisms

$$X_n \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

We claim that the composition of the morphisms is nonzero. Without loss of generality we can replace $E_j^{r_j}[k_j]$ with $E_j^{r_j}$, and assume that all approximations are non-zero. We first consider the composition $f_1 f_2$.

Apply the octahedral axiom:

$$\begin{tikzcd}
& X_1 \arrow[r, equal] & X_1 & \\
E_3^{r_3}[-1] \arrow[r] \arrow[d, equal] & X_3 \arrow[r, "f_2"] \arrow[u] & X_2 \arrow[r] \arrow[u, "f_1"] & E_3^{r_3} \arrow[d, equal] \\
E_3^{r_3}[-1] \arrow[r] & Y \arrow[r] \arrow[u] & E_2^{r_2}[-1] \arrow[r] \arrow[u] & E_3^{r_3} \\
& X_1[-1] \arrow[r, equal] \arrow[u] & X_1[-1] \arrow[u] &
\end{tikzcd}$$

If the composition $f_1 f_2$ is zero then the left column splits, and $Y \simeq X_3 \oplus X_1[-1]$. By Lemma 1.2, since (X_1, E_3) is exceptional, then $\mathrm{Hom}(E_3, X_1) = 0$.

Thus we have a pair of triangles and a commutative diagram, where the second vertical arrow is projection onto the second summand

$$\begin{tikzcd}
E_3^{r_3}[-1] \arrow[r] \arrow[d] & X_3 \oplus X_1[-1] \arrow[r] \arrow[d] & E_2^{r_2}[-1] \arrow[r] \arrow[d, dotted] & E_3^{r_3} \arrow[d] \\
0 \arrow[r] & X_1[-1] \arrow[r] & X_1[-1] \arrow[r] & 0
\end{tikzcd}$$

which implies the existence of the dotted arrow. This forces the right column in the previous diagram to split, which is a contradiction. Hence $f_1 f_2 \neq 0$.

The same argument can be iterated, taking the left column from the previous diagram and using it as the right column for another octahedron. One then uses the fact that $\mathrm{Hom}(E_4, X_1) = 0$ in a similar way to the above, and obtains $(f_1 f_2) f_3 \neq 0$. By further iterations one obtains $f_1 f_2 \dots f_{n-1} \neq 0$.

By Lemma 3.1, we know that $X_n = \nu^{-1} X_1[j]$ for some j, so the fact that there is a nonzero morphism from X_n to X_1 implies that $X_n = \nu^{-1} X_1$ as desired. This completes the proof of (c).

The nontrivial case of (d) is to show

$$(\hat{\mu}_1 \hat{\mu}_2 \hat{\mu}_1)(X, Y, Z) = (\hat{\mu}_2 \hat{\mu}_1 \hat{\mu}_2)(X, Y, Z)$$

(and similarly for left mutation). The first terms of the sequences on the left and right hand sides are both Z, and the second terms agree by definition. The third terms agree by part (c), after passing to the derived category of the rank 3 abelian category containing $\bar{X}, \bar{Y}, \bar{Z}$.

For (e) consider the exchange triangles

$$B^* \to B \to C^u \to B^*[1]$$

and

$$A^* \to A \to C^v \to A^*[1]$$

Applying $\mathrm{Hom}(A^*, \ )$ to the first and $\mathrm{Hom}(\ , B)$ to the second triangle one obtains the long exact sequences

$$\mathrm{Ext}^{i-1}(A^*, C^u) \to \mathrm{Ext}^i(A^*, B^*) \to \mathrm{Ext}^i(A^*, B) \to \mathrm{Ext}^i(A^*, C^u)$$

and

$$\mathrm{Ext}^i(C^v, B) \to \mathrm{Ext}^i(A, B) \to \mathrm{Ext}^i(A^*, B) \to \mathrm{Ext}^{i+1}(C^v, B)$$

The first and last term of both sequences vanish. Hence we obtain the isomorphisms $\mathrm{Ext}^i(A, B) \simeq \mathrm{Ext}^i(A^*, B) \simeq \mathrm{Ext}^i(A^*, B^*)$ for each i. □

From now on, we shall omit the carets from $\hat{\mu}_i$, $\hat{\mu}_i^{-1}$.

4. From silting objects to $\mathrm{Hom}_{\leq 0}$-configurations via exceptional sequences

In this section we consider exceptional sequences induced by silting objects and by $\mathrm{Hom}_{\leq 0}$-configurations in $\mathcal{D}$. Recall that an exceptional sequence $\mathcal{Y} = (Y_1, Y_2, \dots, Y_n)$ is called silting if $Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n$ is a silting object. Note that different exceptional sequences can in this way give rise to the same silting object, and recall that by Lemma 2.1 any silting object can be obtained from an exceptional sequence in this way.

Recall also that any $\mathrm{Hom}_{\leq 0}$-configuration gives rise to a (not necessarily unique) exceptional sequence, and that such exceptional sequences are called $\mathrm{Hom}_{\leq 0}$-configuration exceptional sequences.

We will prove part (a) of Theorem 2.4: that silting exceptional sequences are in 1–1 correspondence with $\mathrm{Hom}_{\leq 0}$-configuration exceptional sequences. This will be proved by considering the following product of mutations:

$$\mu^{(n)}_{\mathrm{rev}} = \mu_{n-1}(\mu_{n-2}\mu_{n-1})\dots(\mu_1\dots\mu_{n-1}). \tag{1}$$

where we sometimes omit the superscript (n) from $\mu^{(n)}_{\mathrm{rev}}$. The same sequence of mutations has been considered in [Bond] in a related context.

Using that the μ_i satisfy the braid relations, μ_{rev} can be expressed in various ways, in particular as

$$\mu_{\mathrm{rev}} = \mu_1(\mu_2\mu_1)(\mu_3\mu_2\mu_1)\dots(\mu_{n-1}\dots\mu_1). \tag{2}$$

We say that a mutation μ_i of an exceptional sequence $Y = (Y_1,\dots,Y_n)$ is *negative* if the left approximation is of the form:

$$Y^r_{i+1}[j] \to Y^*_i \to Y_i \to Y^r_{i+1}[j+1]$$

where j is negative and *non-negative* if $j \geq 0$.

Similarly, we say that μ_i^{-1} is *negative* if j is negative in the approximation

$$Y^r_{i-1}[j] \to Y_i \to Y^*_i \to Y^r_{i-1}[j+1]$$

and *non-negative* if $j \geq 0$.

It is immediate from the definitions that if μ_i is negative, then μ_i^{-1} applied to $\mu_i(Y)$ will also be negative.

Lemma 4.1. *Assume that the exceptional sequence $(Y_1,\dots,Y_n)$ is silting. Consider the process of applying μ_{rev} to it in the order given by (1). Then each mutation will be negative.*

Proof. Note that $\mu_1\dots\mu_{n-1}(Y_1,\dots,Y_n) = (Y_n, Y^*_1,\dots,Y^*_{n-1})$. Since Y is a silting object, each of the mutations $Y_i \to Y^r_n[j]$ is negative. The claim can now be proved by induction, after using Lemma 3.3 (e), which guarantees that $(Y^*_1,\dots,Y^*_{n-1})$ form a silting object in the subcategory of $\mathcal{D}$ which they generate. □

Lemma 4.2. *If the exceptional sequence $\mathcal{Y}$ is silting, then the exceptional sequence $\mu_{\mathrm{rev}}(\mathcal{Y})$ is a $\mathrm{Hom}_{\leq 0}$-configuration.*

Proof. The proof is by induction. First consider the case $n = 2$. Let (E, F) be an exceptional sequence, and apply $\mathrm{Hom}(F, \ )$ to the approximation triangle

$$E^* \to E \to F^r \to E^*[1]$$

It follows that (F, E^*) is a $\mathrm{Hom}_{\leq 0}$-configuration.

Now, let $n > 2$. We use the presentation of μ_{rev} defined by (1). After applying $\mu_1\dots\mu_{n-1}$, we obtain the exceptional sequence $(Y_n, Y^*_1,\dots,Y^*_{n-1})$. Then $\mathrm{Hom}(Y_n, Y^*_i[j]) = 0$ for $i < n$, $j \leq 0$. By Lemma 3.3 (e), we know that $(Y^*_1,\dots,Y^*_{n-1})$ is silting. By induction, applying $\mu^{(n-1)}_{\mathrm{rev}}$ to this silting exceptional sequence will yield a $\mathrm{Hom}_{\leq 0}$-configuration. We know that the mutations which are used are negative, that is to say, of the form

$$E_{i-1}[j] \to E_i^* \to E_i \to E_{i-1}[j+1]$$

with $j < 0$. It follows that if we know that $\mathrm{Hom}(Y_n, E_i[t])$ and $\mathrm{Hom}(Y_n, E_{i-1}[t])$ vanish for $t \leq 0$, then also $\mathrm{Hom}(Y_n, E_i^*[t])$ and $\mathrm{Hom}(Y_n, E_{i-1}^*[t])$ vanish for $t \leq 0$. This shows that the mutations $\mu_{\mathrm{rev}}^{(n-1)}$ which we apply to reverse $(Y_1^*, \ldots, Y_{n-1}^*)$ preserve the property that $\mathrm{Ext}^t(Y_n, \ ) = 0$ for $t \leq 0$, and thus we are done. □

Lemma 4.3. *Let $(Y_1, \ldots, Y_n)$ be a $\mathrm{Hom}_{\leq 0}$-configuration exceptional sequence. Consider the process of applying μ_{rev} in the order given by (1). Each mutation will be non-negative.*

Proof. The mutations which move Y_n are all non-negative since we begin with a $\mathrm{Hom}_{\leq 0}$-configuration. The result holds by induction, as in the proof of Lemma 4.1. □

Lemma 4.4. *If $\mathcal{Y}$ is a $\mathrm{Hom}_{\leq 0}$-configuration exceptional sequence, then the exceptional sequence $\mu_{\mathrm{rev}}(\mathcal{Y})$ is silting.*

Proof. The proof is by induction, and the statement is easily verified in the case $n = 2$. Assume $n > 2$. We prove that $\mu_{\mathrm{rev}}(\mathcal{Y})$ is silting using the order (1). We apply $\mu_1\mu_2 \ldots \mu_{n-1}$ to obtain the exceptional sequence $(Y_n, Y_1^*, \ldots, Y_{n-1}^*)$, and hence $\mathrm{Ext}^j(Y_n, Y_i^*) = 0$ for $j \geq 1$ and $1 \leq i \leq n-1$. The sequence $(Y_1^*, \ldots, Y_{n-1}^*)$ is a $\mathrm{Hom}_{\leq 0}$-configuration, by Lemma 3.3, and hence applying $\mu_{\mathrm{rev}}^{(n-1)}$ to this it will give a silting object by induction.

We then have to check that the mutations $\mu_{\mathrm{rev}}^{(n-1)}$ used in reversing the Y_i^*'s preserve the property of $\mathrm{Ext}^j(Y_n, \ )$ vanishing for $j > 1$.

By Lemma 4.3, the approximations are of the form

$$E_{i-1}[j] \to E_i^* \to E_i \to E_{i-1}[j+1]$$

with $j \geq 0$. The desired result is immediate. □

Proposition 4.5. *Let $\mathcal{Y}$ be an exceptional sequence. Then $\mu_{\mathrm{rev}}(\mu_{\mathrm{rev}}(\mathcal{Y})) = \nu^{-1}(\mathcal{Y})$.*

Proof. We know that the effect of $(\mu_{n-1} \ldots \mu_1)$ is to remove the left end term Y_1 from the exceptional sequence and replace it with $\nu^{-1}(Y_1)$ at the right end. Thus, the effect of $(\mu_{n-1} \ldots \mu_1)^n$ is to apply ν^{-1} to every element of the exceptional sequence, maintaining the same order.

Consider the operation

$$(\mu_{n-1} \ldots \mu_1)(\mu_{n-1} \ldots \mu_1) \ldots (\mu_{n-1} \ldots \mu_1) \tag{3}$$

with n repetitions of the product $(\mu_{n-1} \ldots \mu_1)$. This operation can be written as the composition of the following two operations

$$\mu' = (\mu_{n-1} \ldots \mu_1)(\mu_{n-1} \ldots \mu_2) \ldots (\mu_{n-1}\mu_{n-2})(\mu_{n-1})$$

and

$$\mu'' = (\mu_1)(\mu_2\mu_1) \ldots (\mu_{n-1} \ldots \mu_1).$$

This can be done using only commutation relations, by taking the expression (3) and moving to the left the rightmost generator in the second parenthesis, the two rightmost in the third parenthesis, etc. (counting from the left). Both μ' and μ'' are expressions for μ_{rev}.

□

Summarizing, we have proved part (a) of Theorem 2.4, i.e. we have the following.

Theorem 4.6. *The operation μ_{rev} gives a bijection between silting exceptional sequences and* $\text{Hom}_{\leq 0}$*-configuration exceptional sequences.*

Proof. This is a direct consequence of Lemmas 4.2 and 4.4 and Proposition 4.5, since obviously ν gives a bijection on the set of all exceptional sequences. □

5. The bijection between silting objects and $\text{Hom}_{\leq 0}$-configurations

We have given a bijection from exceptional sequences coming from silting objects to exceptional sequences coming from $\text{Hom}_{\leq 0}$-configurations. We would like to show that this also determines a bijection from silting objects to $\text{Hom}_{\leq 0}$-configurations. This is not immediate from Theorem 4.6, because there can be more than one way to order a silting object or a $\text{Hom}_{\leq 0}$-configuration into an exceptional sequence.

We proceed as follows. Suppose we have a silting object T, and consider some exceptional sequence $\mathcal{E} = (E_1, \ldots, E_n)$ obtained from it. Consider the braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, where the action of B_n on exceptional sequences is defined by having σ_i act like μ_i.

Let $R_{\mathcal{E}} = \{(i,j) \mid \text{Ext}^\bullet(E_i, E_j) = 0\}$. Clearly, if we know $R_{\mathcal{E}}$, we know exactly which reorderings of $\mathcal{E}$ will be exceptional sequences. Let $\text{Stab}_{\mathcal{E}} = \{\sigma \in B_n \mid \sigma\mathcal{E} = \mathcal{E}\}$ be the stabilizer.

Lemma 5.1. *$(i,j) \in R_{\mathcal{E}}$ if and only if $\mu_{j-1}^{-1} \ldots \mu_{i+1}^{-1} (\mu_i)^2 \mu_{i+1} \ldots \mu_{j-1} \in \text{Stab}_{\mathcal{E}}$.*

Proof. The effect of $\mu_{i+1} \ldots \mu_{j-1}$ is to move E_j to the left so it is adjacent on the right to E_i. (This also modifies the elements it passes over.) We claim that μ_i^2 does not change E if and only if $\text{Ext}^\bullet(E_i, E_j) = 0$. In case $\text{Ext}^\bullet(E_i, E_j) = 0$, the remaining mutations $\mu_{j-1}^{-1} \ldots \mu_{i+1}^{-1}$ undo the effect of the first mutations $\mu_{i+1} \ldots \mu_{j-1}$, so the result is the identity.

If $\text{Ext}^\bullet(E_i, E_j) \neq 0$, then the i-th element will be modified, and hence the composition $\mu_{j-1}^{-1} \ldots \mu_{i+1}^{-1} (\mu_i)^2 \mu_{i+1} \ldots \mu_{j-1}$ is not in $\text{Stab}_{\mathcal{E}}$. □

Denote by σ_{rev} the element of B_n corresponding to μ_{rev}.

From a basic lemma about group actions, we have that $\text{Stab}_{\mu_{\text{rev}}(E)} = \sigma_{\text{rev}} \, \text{Stab}_E \, \sigma_{\text{rev}}^{-1}$. To determine $\text{Stab}_{\mu_{\text{rev}}}(\mathcal{E})$, we need the following lemma. (See [Br] for a different proof.)

Lemma 5.2. *In B_n, we have $\sigma_{\text{rev}} \sigma_i \sigma_{\text{rev}}^{-1} = \sigma_{n-i}$.*

Proof. Let S_n be the symmetric group generated by the simple reflections $s_1, \ldots, s_{n-1}$ and let w_0 be the longest element in S_n. This is the permutation which takes i to $n+1-i$ for all i. For any i, we can write $w_0 = (w_0 s_i w_0^{-1})(w_0 s_i)$. Note that $w_0 s_i w_0^{-1} = s_{n-i}$.

For any $w \in S_n$, write σ_w for the element of the braid group B_n obtained by taking any reduced word for w and replacing each occurrence of s_i by σ_i for all i.

This produces a well-defined element of B_n because any two reduced words for w are related by braid relations, which also hold in B_n.

Fix i, and write $u = w_0 s_i$. We now have that $\sigma_{\mathrm{rev}} = \sigma_{w_0} = \sigma_{n-i}\sigma_u$. So $\sigma_{\mathrm{rev}}\sigma_i\sigma_{\mathrm{rev}}^{-1} = \sigma_{n-i}\sigma_u\sigma_i\sigma_{\mathrm{rev}}^{-1} = \sigma_{n-i}\sigma_{\mathrm{rev}}\sigma_{\mathrm{rev}}^{-1} = \sigma_{n-i}$. □

It follows that $(n-j, n-i) \in R_{\mu_{\mathrm{rev}}(\mathcal{E})}$ if and only if $(i,j) \in R_{\mathcal{E}}$. Hence we have proved the following, which is part (b) of our main theorem.

Theorem 5.3. *The operation μ_{rev} induces a bijection between silting objects and $\mathrm{Hom}_{\leq 0}$-configurations.*

6. Specializing to m-cluster tilting objects and m-$\mathrm{Hom}_{\leq 0}$-configurations

In this section we prove part (c) of our main theorem. We need to recall the following notions. A full subcategory $\mathcal{T}$ of $\mathcal{D}$ is called *suspended* if it satisfies the following:

(S1) If $A \to B \to C \to A[1]$ is a triangle in $\mathcal{D}$ and A, C are in $\mathcal{T}$, then B is in $\mathcal{T}$.
(S2) If A is in $\mathcal{T}$, then $A[1]$ is in $\mathcal{T}$.

A suspended subcategory $\mathcal{U}$ is called a *torsion class* in [BR] (or *aisle* in [KV]) if the inclusion functor $\mathcal{U} \to \mathcal{D}$ has a right adjoint. For a subcategory $\mathcal{U}$ of $\mathcal{D}$, we let $\mathcal{U}^{\perp} = \{X \in \mathcal{D} \mid \mathrm{Hom}(\mathcal{U}, X) = 0\}$. For a torsion class $\mathcal{T}$, let $\mathcal{F} = \mathcal{T}^{\perp}$ be the corresponding *torsion-free* class. Recall that a torsion class in $\mathcal{D}$ is called *splitting* if every indecomposable object in $\mathcal{D}$ is either torsion or torsion-free; in other words, any indecomposable object which is not in the torsion class, does not admit any morphisms from any object of the torsion class.

We prove the following easy lemmas:

Lemma 6.1. *If $\mathcal{E}$ is an exceptional sequence contained in a splitting torsion-free class $\mathcal{F}$, then $\mu_{\mathrm{rev}}(\mathcal{E})$ is also contained in $\mathcal{F}$.*

Proof. This follows from the fact that each object in $\mu_{\mathrm{rev}}(\mathcal{E})$ has a sequence of non-zero morphisms to an object in $\mathcal{E}$. □

Lemma 6.2. *If $\mathcal{E}$ is an exceptional sequence contained in a splitting torsion class $\mathcal{T}$, then $\mu_{\mathrm{rev}}(\mathcal{E})$ is contained in $\nu^{-1}(\mathcal{T})$.*

Proof. This follows from the fact that, applying μ_{rev} to $\mu_{\mathrm{rev}}(\mathcal{E})$, we obtain $\nu^{-1}(\mathcal{E})$ by Proposition 4.5, which implies that there is a sequence of non-zero morphisms to every element in $\mu_{\mathrm{rev}}(\mathcal{E})$ from an element in $\nu^{-1}(\mathcal{E})$. □

By combining the above lemmas, we obtain that μ_{rev}, applied to an exceptional sequence in $\mathcal{D}^{(\geq 1)+}_{\leq m}$, yields a sequence with elements in $\mathcal{D}^{\geq 0}_{\leq m}$. Recall that an m-$\mathrm{Hom}_{\leq}$-configuration is a $\mathrm{Hom}_{\leq}$-configuration contained in $\mathcal{D}^{\geq 0}_{\leq m}$, and that an m-cluster tilting object is a silting object contained in $\mathcal{D}^{(\geq 1)+}_{\leq m}$. Hence, in particular we have the following.

Proposition 6.3. *Let $\mathcal{E}$ be an exceptional sequence which is an m-cluster tilting object. Then $\mu_{\mathrm{rev}}(\mathcal{E})$ is an m-$\mathrm{Hom}_{\leq 0}$-configuration.*

We aim to show that the converse also holds. For this we need the following lemmas.

Lemma 6.4. *Let $\mathcal{E}$ be an exceptional sequence contained in a torsion class $\mathcal{T}$. Then if μ_i is non-negative, $\mu_i(\mathcal{E})$ is also contained in $\mathcal{T}$.*

Proof. When we apply a non-negative μ_i, we have the approximation sequence:

$$E_{i-1}^r[j] \to E_i^* \to E_i \to E_{i-1}^r[j+1]$$

with $j \geq 0$. The left and right terms of this triangle are in $\mathcal{T}$, so the middle term is also. □

Corollary 6.5. *If a* $\mathrm{Hom}_{\leq 0}$*-configuration exceptional sequence $\mathcal{E}$ is contained in a torsion class $\mathcal{T}$, so is $\mu_{\mathrm{rev}}(\mathcal{E})$.*

Proof. By Lemma 4.3, we can calculate $\mu_{\mathrm{rev}}(\mathcal{E})$ using only non-negative mutations. The claim now follows from Lemma 6.4. □

Lemma 6.6. *If $\mathcal{E}$ is an exceptional sequence contained in a torsion-free class $\mathcal{F}$, and μ_i^{-1} is a negative mutation, then $\mu_i^{-1}(\mathcal{E})$ is also contained in $\mathcal{F}$.*

Proof. The proof is similar to that of Lemma 6.4. The approximation is of the form

$$E_{i-1} \to E_{i-1}^* \to E_i^r[j] \to E_{i-1}[1]$$

and $j \leq 0$. Again, the left and right terms of the triangle are in $\mathcal{F}$, hence the middle is also. □

Corollary 6.7. *If $\mathcal{E}$ is a* $\mathrm{Hom}_{\leq 0}$*-configuration exceptional sequence which is contained in a torsion-free class $\mathcal{F}$, then $\mu_{\mathrm{rev}}^{-1}(\mathcal{E})$ is contained in $\mathcal{F}$.*

Proof. We know that μ_{rev}^{-1} can be expressed as a product of negative mutations by Lemma 4.1. □

Proposition 6.8. *Let $\mathcal{E}$ be an m-*$\mathrm{Hom}_{\leq}$*-configuration exceptional sequence. Then $\mu_{\mathrm{rev}}^{-1}(\mathcal{E})$ is an m-cluster tilting object.*

Proof. By Proposition 4.5 we have that $\mu_{\mathrm{rev}}(\mathcal{E}) = \nu^{-1}\mu_{\mathrm{rev}}^{-1}(\mathcal{E})$ and hence $\mu_{\mathrm{rev}}^{-1}(\mathcal{E}) = \nu\mu_{\mathrm{rev}}(\mathcal{E})$. By Corollary 6.5 we have that $\mu_{\mathrm{rev}}(\mathcal{E})$ is contained in $\mathcal{D}^{\geq 0}$. Hence $\mu_{\mathrm{rev}}^{-1}(\mathcal{E})$ is contained in $\nu(\mathcal{D}^{\geq 0})$. We also know that $\mu_{\mathrm{rev}}^{-1}(\mathcal{E})$ is contained in $\mathcal{D}_{\leq m}$ by Corollary 6.7. This completes the proof. □

Summarizing, we obtain part (c) of Theorem 2.4.

Theorem 6.9. *The product of mutations μ_{rev} defines a bijection between m-cluster-tilting objects and m-*$\mathrm{Hom}_{\leq 0}$*-configurations.*

Proof. This is a direct consequence of Propositions 6.8 and 6.3, using the already established bijections from Theorems 4.6 and 5.3. □

7. A combinatorial interpretation: m-noncrossing partitions

In this section, we give our desired combinatorial interpretation of part (c) of Theorem 2.4. The main task of this section is to construct, for an arbitrary connected hereditary artin algebra H, a bijection between m-$\mathrm{Hom}_{\leq 0}$-configurations and m-noncrossing partitions in the sense of [Ar] for the reflection group W corresponding to H.

The set of m-clusters is only defined in the case that H is of finite type; in this case, they are known to be in bijection with the m-cluster tilting objects [T, Z]. Thus, once we have accomplished the main task of this section, we will have obtained a bijection between m-clusters and m-noncrossing partitions for H of finite type (or equivalently, for W any finite crystallographic reflection group). A description of the resulting bijection, in purely Coxeter-theoretic terms, has already been presented, without proof, in [BRT1].

7.1. **Weyl groups and noncrossing partitions.** We define the Weyl group W associated to H following [Rin2]. Let k be the centre of H (which is a field since we have assumed that H is connected). Number the simple objects of H in such a way that $(S_1, \dots, S_n)$ is an exceptional sequence. The Grothendieck group of H, denoted $K_0(H)$, is a free abelian group generated by the classes $[S_i]$.

For $i < j$, define

$$\Delta_{ij} = -\dim_{\mathrm{End}(S_i)}(\mathrm{Ext}^1(S_i, S_j))$$
$$\Delta_{ji} = -\dim_{\mathrm{End}(S_j)}(\mathrm{Ext}^1(S_i, S_j))$$

Write d_i for the k-dimension of $\mathrm{End}(S_i)$. Note that $d_i\Delta_{ij} = d_j\Delta_{ji}$. Now define a symmetric, bilinear form on $K_0(H)$ by $([S_i], [S_j]) = d_i\Delta_{ij}$ for $i \neq j$, and $([S_i], [S_i]) = 2d_i$.

For x in $K_0(H)$, with $(x, x) \neq 0$, define t_x, the reflection along x, by:

$$t_x(v) = v - \frac{2(v, x)}{(x, x)}x$$

We now have the following lemma:

Lemma 7.1. (a) *For A, B modules, we have* $([A], [B]) = \dim_k \mathrm{Hom}(A, B) + \dim_k \mathrm{Hom}(B, A) - \dim_k \mathrm{Ext}^1(A, B) - \dim_k \mathrm{Ext}^1(B, A)$.

(b) *If (A, B) is an exceptional sequence in $\mathcal{D}$, and (B, A^*) is the result of mutating it, then:*

$$[A^*] = t_{[B]}[A].$$

Proof. (a) See [Rin1, p. 279].

(b) Let (A, B) be an exceptional sequence in $\mathcal{D}$. Consider the triangle $A^* \to A \to B^r[p]$, where $f : A \to B^r[p]$ is a minimal left $\mathrm{th}(B)$-approximation. Note that we know that $\mathrm{Hom}(A, B[j]) = 0$ for all but at most one j. Without loss of generality, assume $p = 0$.

Assume that $f : A \to B^r$ is non-zero. We have that $K_0(H) \simeq K_0(\mathcal{D})$, and the above triangle gives $[A^*] = [A] - r[B]$. Since B is exceptional, and hence $\mathrm{End}(B)$

is a division ring, it follows directly that $f = (f_1, \dots, f_r)$ where $\{f_1, \dots, f_r\}$ is an $\mathrm{End}(B)$-basis for $\mathrm{Hom}(A, B)$. Hence, $r = \dim_{\mathrm{End}(B)} \mathrm{Hom}(A, B)$. Since

$$t_{[B]}[A] = [A] - \frac{2([A],[B])}{([B],[B])}[B],$$

it suffices to show that $r = 2([A],[B])/([B],[B])$. We have that $([A],[B]) = \dim_k \mathrm{Hom}(A, B)$ by extending (a) to $\mathcal{D}$. (We use here that $\mathrm{Ext}^i(B, A) = 0$ since (A, B) is an exceptional sequence, and $\mathrm{Ext}^1(A, B) = 0$ since $\mathrm{Hom}(A, B) \neq 0$.) By (a), we similarly get that $([B],[B]) = 2\dim_k(\mathrm{Hom}(B, B))$. Hence we have:

$$\frac{2([A],[B])}{([B],[B])} = \frac{2\dim_k \mathrm{Hom}(A,B)}{2\dim_k \mathrm{Hom}(B,B)} = \dim_{\mathrm{End}(B)} \mathrm{Hom}(A, B) = r,$$

and we are done in this case.

If $f = 0$, so that $r = 0$, then $[A^*] = [A]$, and $([A],[B]) = 0$, so $t_{[B]}([A]) = [A]$, as desired. □

We now define $s_i = t_{[S_i]}$ for $1 \leq i \leq n$, and let W be the group generated by the set $\{s_1, \dots, s_n\}$; it is a Weyl group. By definition, we say that an element of W is a *reflection* if it is the conjugate of some s_i. We denote the set of all reflections in W by T.

It follows directly from Lemma 7.1 that if (E_i, E_{i+1}) and (E_{i+1}, E_i^*) are related by mutation, then

$$t_{[E_i]}t_{[E_{i+1}]} = t_{[E_{i+1}]}t_{[E_i^*]}. \tag{4}$$

Since the reflections corresponding to the simple objects are all in W, it follows from (4) that $t_{[E]}$ is in W for each exceptional module E. It also follows from (4) that the product of the reflections corresponding to any exceptional sequence is the Coxeter element c (see [IT]).

Conversely, we have the following result from [IS]:

Theorem 7.2. *If $c = t_1 \dots t_n$ in W, with $t_i \in T$, then each t_i must be of the form $t_{[E_i]}$ for some exceptional module E_i, where $(E_1, \dots, E_n)$ forms an exceptional sequence.*

Now we give the (purely Coxeter-theoretic) definition of an m-noncrossing partition. First of all, define a function $\ell_T : W \to \mathbb{N}$, where $\ell_T(w)$ is the length of the shortest expression for w as a product of reflections. (Note that this is not the classical length function on W, which is the minimum length of an expression for w as a product of *simple* reflections.) We note that $\ell_T(c) = n$.

We say that $(u_1, \dots, u_r)$, an r-tuple of elements of W, is a *T-reduced expression* for $u_1 \dots u_r$ if $\ell_T(u_1) + \dots + \ell_T(u_r) = \ell_T(u_1 \dots u_r)$. We can now follow Armstrong [Ar] in defining the *m-noncrossing partitions* for W to consist of the set of T-reduced expressions for c with $m + 1$ terms.

Now we define the bijection. Let $(u_1, \dots, u_{m+1})$ be a T-reduced expression for c. By Theorem 7.2, pick an exceptional sequence $E_1, \dots, E_n$ such that the first $\ell_T(u_1)$ terms correspond to some factorization of u_1 into reflections, and similarly for the next $\ell_T(u_2)$ terms, and so on. For each i with $1 \leq i \leq m + 1$, we then have an exceptional sequence $\mathcal{E}_i$. Write $\mathcal{C}_i$ for the minimal abelian subcategory containing $\mathcal{E}_i$. Let F_i be the sum of the simples of $\mathcal{C}_i$. Set $\phi(u_1,\dots,u_{m+1}) = \bigoplus F_i[m+1-i]$.

Theorem 7.3. *The above map ϕ from T-reduced expressions of c to objects in $\mathcal{D}$ is a bijection from m-noncrossing partitions to m-$\mathrm{Hom}_{\leq 0}$-configurations.*

Proof. First, we show that if $(u_1,\dots,u_{m+1})$ is a T-reduced expression for c, then $\bigoplus F_i[m+1-i] = \phi(u_1,\dots,u_{m+1})$ is an m-$\mathrm{Hom}_{\leq 0}$-configuration. By definition, $\bigoplus F_i[m+1-i]$ is contained in $\mathcal{D}^{\leq m}_{\geq 0}$. We check the four conditions in the definition of a $\mathrm{Hom}_{\leq 0}$-configuration. (H1) is immediate. It is possible to transform each sequence $\mathcal{E}_i$ into (an ordering of) the summands of F_i by mutations, thanks to the transitivity of the action of mutations within $\mathcal{C}_i$. (H4) follows, and the form of this exceptional sequence guarantees (H2) and (H3).

Next, we show that any m-$\mathrm{Hom}_{\leq 0}$-configuration arises in this way. Take X to be an m-$\mathrm{Hom}_{\leq 0}$-configuration and order it into an exceptional sequence in such a way that the objects in $\operatorname{mod} H[m]$ come first, then those in $\operatorname{mod} H[m-1]$, etc. This was shown to be possible in the proof of Lemma 2.3.

Now, for $1 \leq i \leq m+1$, define $\mathcal{C}_i$ to be the subcategory of $\operatorname{mod} H$ consisting of modules admitting a filtration by modules corresponding to the summands in X of degree $m+1-i$. This is the minimal abelian subcategory of $\operatorname{mod} H$ containing these summands of X, and the summands of X are obviously the simple objects in this subcategory. We can therefore define u_i by taking the product of these summands of X, ordered as in the exceptional sequence, and we obtain a T-reduced expression for c. □

7.2. **Combinatorics of positive Fuss-Catalan numbers.** When H is of finite type, corresponding to a finite crystallographic group W, there is a variant of the Fuss-Catalan number called the *positive Fuss-Catalan number*, denoted $C^+_m(W)$. By definition, $C^+_m(W) = |C_{-m-1}(W)|$.

It is known that the number of m-cluster tilting objects contained in $\mathcal{D}^{\geq 1}_{\leq m}$ is $C^+_m(W)$, see [FR].

Write $\mathcal{D}^{(\geq 0)-}_{\leq m}$ for the full subcategory of $\mathcal{D}^{\geq 0}_{\leq m}$ additively generated by the indecomposable objects of $\mathcal{D}^{\geq 0}_{\leq m}$ other than the summands of H. The following is an immediate corollary of Theorem 2.4.

Corollary 7.4. (a) *There is a bijection between silting objects contained in $\mathcal{D}^{\geq 1}_{\leq m}$ and m-$\mathrm{Hom}_{\leq 0}$-configurations contained in $\mathcal{D}^{(\geq 0)-}_{\leq m}$ given by μ_{rev}.*

(b) *If H is of Dynkin type with corresponding crystallographic reflection group W, then the number of m-$\mathrm{Hom}_{\leq 0}$-configurations contained in $\mathcal{D}^{(\geq 0)-}_{\leq m}$ is $C^+_m(W)$.*

It is possible to give a Coxeter-theoretic description of the subset of m-noncrossing partitions which correspond, under the bijection of Theorem 7.3, to the m-$\mathrm{Hom}_{\leq 0}$-configurations contained in $\mathcal{D}^{(\geq 0)-}_{\leq m}$. See [BRT1] for more details.

8. The link between $\mathrm{Hom}_{\leq 0}$-configurations and Riedtmann's combinatorial configurations

In this section we show that our $\mathrm{Hom}_{\leq 0}$-configurations contained in $\mathcal{D}_{\leq 1}^{(\geq 0)-}$ are related to the combinatorial configurations introduced by Riedtmann in connection with her work on selfinjective algebras of finite representation type. Note that an alternative and independent approach to this, dealing with the Dynkin case, is given by Simoes [S]. She also gives a bijection from combinatorial configurations to a subset of the 1-noncrossing partitions, and thus to the positive clusters (in the sense of the previous section).

8.1. **Complements of tilting modules and cluster-tilting objects.** In this subsection we recall some basic facts about complements of tilting modules in $\mathrm{mod}\, H$ and cluster tilting objects in the associated cluster category. For more on complements of tilting modules, see [HU, RS1, U, CHU]; for more on complements in cluster categories, see [BMRRT].

Suppose that $T = \bigoplus_{i=1}^{n} T_i$ is a tilting object in $\mathrm{mod}\, H$. Write $\overline{T}$ for $\bigoplus_{j\neq i} T_j$.

We say that an indecomposable object X in $\mathrm{mod}\, H$ is a complement to $\overline{T}$ if $X \oplus \overline{T}$ is tilting. If $\overline{T}$ is not sincere, then T_i is its only complement; otherwise, it has exactly two complements up to isomorphism, T_i and one other one, T_i'. We say that $T_i' \oplus \overline{T}$ is the result of mutating T at T_i.

Lemma 8.1. *Exactly one of the following three possibilities occurs:*

(a) T_i **has no replacement.** *This occurs if and only if $\overline{T}$ is not sincere.*

(b) T_i **admits a monomorphism to a module in** $\mathrm{add}\,\overline{T}$**.** *In this case, let $T_i \to B$ be the minimal left $\mathrm{add}\,\overline{T}$-approximation to T_i. Then there is a short exact sequence:*

$$0 \to T_i \to B \to T_i' \to 0.$$

(c) T_i **admits a epimorphism from a module in** $\mathrm{add}\,\overline{T}$**.** *In this case, let $B \to T_i$ be the minimal right $\mathrm{add}\,\overline{T}$-approximation to T_i. Then there is a short exact sequence*

$$0 \to T_i' \to B \to T_i \to 0.$$

We also think of $\mathrm{mod}\, H$ as embedded inside the cluster category associated to H. A tilting object in $\mathrm{mod}\, H$ is thereby identified with a (1-)cluster tilting object in the cluster category. In the cluster category, there is always exactly one way to replace T_i by some other indecomposable object while preserving the property of being a cluster tilting object. If there is a replacement for T_i in $\mathrm{mod}\, H$, that replacement is also a replacement in the cluster category; otherwise, the replacement for T_i is of the form $P[1]$, where P is indecomposable projective.

8.2. **Torsion classes arising from partitions of exceptional sequences.** This subsection is mainly devoted to the proof of Lemma 8.3, which says that if a complete exceptional sequence in $\mathrm{mod}\, H$ is divided into two parts, $(E_1, \ldots, E_r)$ and $(E_{r+1}, \ldots, E_n)$, for some $0 < r < n$, and the objects from the second part are used to generate a torsion class, then the corresponding torsion-free class is generated (in a suitable sense) by the objects from the first part of the exceptional sequence.

Let T be a tilting module, $\mathcal{T} = \operatorname{Fac} T$ the torsion class generated by T, and $\mathcal{F} = \operatorname{Sub} \tau T$ the corresponding torsion-free class.

Some summand U of T (typically not indecomposable) is minimal among modules such that $\operatorname{Fac} U = \mathcal{T}$. We refer to U as the minimal generator of $\mathcal{T}$. Similarly, there is a minimal cogenerator of $\mathcal{F}$.

We have the following lemma, based on an idea from [IT].

Lemma 8.2. *Let T_i be an indecomposable summand of T. Then T_i is a summand of the minimal generator of $\mathcal{T}$ if and only if τT_i is not a summand of the minimal cogenerator of $\mathcal{F}$. (By convention, if $\tau T_i = 0$, then we do not consider it a summand of the minimal cogenerator of $\mathcal{F}$.)*

Proof. If T_i is projective, then it must be a summand of the minimal generator for $\mathcal{T}$, and then τT_i is zero, so (by convention) it is not a summand of the minimal generator for $\mathcal{F}$. We may therefore assume that T_i is not projective.

For the rest of the proof, we embed $\operatorname{mod} H$ into the corresponding cluster category. Note that τ is an autoequivalence on the cluster category.

Let T_i' be the result of mutating T at T_i in the cluster category. Since τ is an autoequivalence, the effect of mutating τT at τT_i is to replace τT_i by $\tau T_i'$. Write $\overline{T}$ for $\bigoplus_{j \neq i} T_j$.

Suppose now that T_i is a summand of the minimal generator for $\mathcal{T}$. Then there is no epimorphism from $\operatorname{add} \overline{T}$ to T_i, so either there is a short exact sequence in the module category

$$0 \to T_i \to B \to T_i' \to 0,$$

where B is in $\operatorname{add} \overline{T}$, or else T_i' is a shifted projective.

In the former case, applying τ to the above sequence shows that τT_i is not a summand of the minimal cogenerator of $\mathcal{F}$, since τT_i injects into $\tau B \in \operatorname{add} \tau \overline{T}$. In the latter case, $\tau T_i'$ is injective, so the exchange sequence in $\operatorname{mod} H$ again has the same form (τT_i is on the left, and therefore injects into an object of $\operatorname{add} \tau \overline{T}$, so is not a summand of the minimal cogenerator of $\mathcal{F}$).

Next suppose that T_i is not a summand of the minimal generator for $\mathcal{T}$. So there is an epimorphism from some B in $\operatorname{add} \overline{T}$ to T_i, and thus we have a short exact sequence in $\operatorname{mod} H$ of the form

$$0 \to T_i' \to B \to T_i \to 0.$$

Therefore either τ applied to the above sequence in $\operatorname{mod} H$ is still a short exact sequence, or else T_i' is projective, and hence $\tau T_i'$ is a shifted projective. In the first case, the exchange sequence for τT_i has τT_i on the right; in particular, τT_i does not admit a monomorphism to any B' in $\operatorname{add} \tau \overline{T}$. Thus τT_i is a summand of the minimal cogenerator of $\mathcal{F}$. In the second case, τT_i has no complement in $\operatorname{mod} H$, so $\tau \overline{T}$ is not sincere and thus τT_i is again a summand of the minimal cogenerator for $\mathcal{F}$. □

A subcategory of $\operatorname{mod} H$ is called *exact abelian* if it is abelian with respect to the exact structure inherited from $\operatorname{mod} H$. If $(E_1, \ldots, E_r)$ is an exceptional sequence in

$\operatorname{mod} H$, it naturally determines an exact abelian and extension-closed subcategory of $\operatorname{mod} H$, the smallest such subcategory of $\operatorname{mod} H$ containing $E_1, \dots, E_r$. This subcategory is a module category for a hereditary algebra H' with r simples [Rin2]. If $(E_1, \dots, E_n)$ is a complete exceptional sequence, then the minimal exact abelian and extension-closed subcategory of $\operatorname{mod} H$ containing $E_1, \dots, E_r$ can also be described as the full subcategory of $\operatorname{mod} H$ consisting of all Z such that $\operatorname{Hom}(E_i, Z) = 0 = \operatorname{Ext}^1(E_i, Z) = 0$ for all $r+1 \leq i \leq n$.

Lemma 8.3. *Let $(E_1, \dots, E_n)$ be a complete exceptional sequence in $\operatorname{mod} H$. Let $\mathcal{B}$ be the exact abelian extension-closed subcategory generated by $E_1, \dots, E_r$, with $0 < r < n$, and let $\mathcal{C}$ be the exact abelian extension-closed subcategory generated by $E_{r+1}, \dots, E_n$. Let $\mathcal{T} = \operatorname{Fac} \mathcal{C}$, and $\mathcal{G} = \operatorname{Sub} \mathcal{B}$. Then $(\mathcal{T}, \mathcal{G})$ forms a torsion pair.*

Proof. Since $\mathcal{C}$ is closed under extensions, it is straightforward to see that $\mathcal{T}$ is also closed under extensions, and hence that it is a torsion class. Let $\mathcal{F}$ be the torsion-free class corresponding to $\mathcal{T}$. Clearly $\mathcal{G}$ is a full subcategory of $\mathcal{F}$. Suppose first that $\mathcal{T}$ is generated by a tilting object $T = \bigoplus T_i$, so we can apply Lemma 8.2. Let P be the minimal generator of $\mathcal{T}$. This consists of the direct sum of the indecomposable Ext-projectives of $\mathcal{C}$. (Note that $\mathcal{C}$ is again a module category.) Let T_i be a summand of T which is not a summand of the minimal generator of $\mathcal{T}$. Since τT_i is in $\mathcal{F}$, we know that $\operatorname{Hom}(P, \tau T_i) = 0$. Let P_j be an indecomposable summand of P. We want to show that $\operatorname{Hom}(T_i, P_j) = 0$. Morphisms between indecomposable summands of a tilting object are epimorphisms or monomorphisms [HR]. Since P_j is by assumption a summand of the minimal generator of $\mathcal{T}$, it cannot admit an epimorphism from T_i. Since T_i admits an epimorphism from $\overline{T}$, it cannot also admit a monomorphism into P_j (by Lemma 8.1). Therefore, $\operatorname{Hom}(T_i, P) = 0$, and hence $\operatorname{Ext}^1(P, \tau T_i) \simeq D\underline{\operatorname{Hom}}(T_i, P) = 0$. Using the remarks before the statement of the lemma, we conclude that τT_i lies in $\mathcal{B}$. By Lemma 8.2 we conclude that all the indecomposable summands of the minimal cogenerator of $\mathcal{F}$ lie in $\mathcal{B}$, and therefore in $\mathcal{G}$. So $\mathcal{F} = \mathcal{G}$, as desired.

Suppose now that $\mathcal{T}$ is not generated by a tilting module. It is still generated by the direct sum of the indecomposable non-isomorphic Ext-projectives of $\mathcal{C}$, which we denote by T. Let $I_1, \dots, I_s$ be the indecomposable injectives such that $\operatorname{Hom}(T, I_i) = 0$. These are objects of $\mathcal{B}$. Suitably ordered, $(I_1, \dots, I_s)$ form an exceptional sequence in $\mathcal{B}$; we can therefore extend this sequence to a complete exceptional sequence in $\mathcal{B}$, which we denote by $(I_1, \dots, I_s, F_1, \dots, F_{r-s})$. Note that this sequence can be further extended to a complete exceptional sequence in $\operatorname{mod} H$ by appending $(E_{r+1}, \dots, E_n)$.

Consider the category $\mathcal{M}$ with objects $\{M \mid \operatorname{Hom}(M, I_i) = 0 \text{ for } 1 \leq i \leq s\}$. This is a module category for some hereditary algebra H' with $n - s$ simples. $\mathcal{T}$ is a torsion class for $\operatorname{mod} H'$, and T is tilting in $\operatorname{mod} H'$. We can therefore apply the previous case to conclude that the torsion-free class in $\operatorname{mod} H'$ associated to $\mathcal{T}$ is cogenerated by $\mathcal{C}'$, the smallest exact abelian extension-closed subcategory of $\operatorname{mod} H'$ containing $F_1, \dots, F_{r-s}$. Now if Z is any object in $\operatorname{mod} H$, we want to show that there is an exact sequence

$$0 \to K \to Z \to Z/K \to 0$$

with Z/K in $\mathcal{G}$ and K in $\mathcal{T}$. If we can do this, then that shows that $\mathcal{G}$ is "big enough", that is to say, it coincides with $\mathcal{F}$.

To do this, let N be the maximal quotient of Z which is a subobject of add $\oplus_{i=1}^{s} I_i$, and let the kernel be Z'. So Z' admits no non-zero morphisms to $\oplus_{i=1}^{s} I_i$; in other words, Z' is in $\operatorname{mod} H'$. So Z' has a maximal torsion submodule K, and Z'/K is in the torsion-free class associated to $\mathcal{T}$ in $\operatorname{mod} H'$. It follows that Z/K is in $\mathcal{G}$, and we are done. □

8.3. **Riedtmann's combinatorial configurations.** Define the autoequivalence $F = [-2]\tau^{-1}$ of $\mathcal{D}$.

A collection $\mathcal{I}$ of indecomposable objects in $\mathcal{D}$ is called a (Riedtmann) combinatorial configuration if it satisfies the following two properties:

- For X and Y non-isomorphic objects in $\mathcal{I}$, we have $\operatorname{Hom}(X, Y) = 0$.
- For any nonzero Z in $\mathcal{D}$, there is some $X \in \mathcal{I}$ such that $\operatorname{Hom}(X, Z) \neq 0$.

Note that Riedtmann only considers combinatorial configurations for path algebras of type ADE, but the above definition does not require that restriction.

A combinatorial configuration is called *periodic* if it satisfies the additional property that (in our notation) for any $X \in \mathcal{I}$, we have $F^i(X) \in \mathcal{I}$ for all i. Riedtmann showed that if H is a path algebra of type A or D, then any combinatorial configuration is periodic [Rie1, Rie2].

Theorem 8.4. *If T is a $\operatorname{Hom}_{\leq 0}$-configuration contained in $\mathcal{D}_{\leq 1}^{(\geq 0)-}$, then the set of indecomposable summands of $F^i(T)$ for all i is a periodic combinatorial configuration in the sense of Riedtmann.*

Proof. To verify the Hom-vanishing condition in the definition of a combinatorial configuration, it suffices to verify, for any non-isomorphic indecomposable summands A, B of T, that $\operatorname{Hom}(A, F^i(B)) = 0$. It is clear that $\operatorname{Hom}(A, F^i(B))$ is zero unless $i = 0$ or $i = -1$. If $i = 0$, the vanishing follows directly from the definition of a $\operatorname{Hom}_{\leq 0}$-configuration. For $i = -1$, observe that $\operatorname{Hom}(A, F^{-1}B) \simeq D \operatorname{Ext}^{-1}(B, A) = 0$.

Let $\hat{T} = \bigoplus_i F^i(T)$. Now we consider the property that for each X in $\mathcal{D}$, we have that $\operatorname{Hom}(\hat{T}, X) \neq 0$. We may assume that X is indecomposable. We can clearly assume that $X \in \mathcal{D}_{\leq 1}^{(\geq 0)-}$.

Let $E_1[1], \ldots, E_r[1]$ be the indecomposable summands of T in degree 1, and let $E_{r+1}, \ldots, E_n$ be the indecomposable summands of T in degree 0, ordered so that $(E_1, \ldots, E_n)$ forms an exceptional sequence in $\operatorname{mod} H$.

Assume first that X is in degree 0. Let $\mathcal{B}$ be the smallest exact abelian extension-closed subcategory containing $E_1, \ldots, E_r$. This is the category of objects of $\operatorname{mod} H$ filtered by $\{E_1, \ldots, E_r\}$. Similarly, let $\mathcal{C}$ be the smallest exact abelian extension-closed subcategory containing $E_{r+1}, \ldots, E_n$.

Let $\mathcal{T} = \operatorname{Fac} \mathcal{C}$, and $\mathcal{F} = \operatorname{Sub} \mathcal{B}$. By Lemma 8.3, $(\mathcal{T}, \mathcal{F})$ is a torsion pair.

If X has non-zero torsion, then we have shown that X admits a non-zero morphism from some object in $\mathcal{T}$, and therefore from some object of $\mathcal{C}$, so X admits a non-zero morphism from some E_i with $r+1 \leq i \leq n$. Since this E_i is a summand of $\hat{T}$, we are done with this case.

Now suppose that X has no torsion, which is to say, it is torsion-free. X therefore admits a monomorphism into some object of $\mathcal{B}$, and thus a non-zero morphism to some E_i with $1 \leq i \leq r$. Hence there is a non-zero morphism from $\nu^{-1}(E_i)$ to X. But $\nu^{-1}(E_i) = F(E_i[1])$, which is a summand of $\hat{T}$, and we are done.

Now consider the case that X lies in degree 1. Let $Z = (\bigoplus_{i=1}^{r} E_i[1]) \oplus (\bigoplus_{i=r+1}^{n} F^{-1}E_i)$. We claim that Z is a $\mathrm{Hom}_{\leq 0}$-configuration contained in $\mathcal{D}^{(\geq 1)}_{(\leq 2)-}$ (by which we mean $\mathcal{D}^{(\geq 1)}_{(\leq 2)}$ with $DH[2]$ removed). (H1) is clear. For (H2), the nontrivial requirement is to show $\mathrm{Hom}(E_i[1], F^{-1}E_j) = 0$ with $i \leq r$ and $j > r$. Now $\mathrm{Hom}(E_i[1], F^{-1}E_j) \simeq D\,\mathrm{Ext}^{-1}(E_j, E_i[1]) = 0$. For (H3), the nontrivial requirement is to show that $\mathrm{Ext}^{-1}(E_i[1], F^{-1}E_j) = 0$ for $i \leq r$ and $j > r$, and we see that $\mathrm{Ext}^{-1}(E_i[1], F^{-1}E_j) \simeq \mathrm{Hom}(E_j, E_i[1]) = 0$. For (H4), observe that, by Lemma 3.3(c), $(\mu_1 \ldots \mu_{n-1})^{n-r}$ transforms $(E_1[1], \ldots, E_r[1], E_{r+1}, \ldots, E_n)$ to $(\nu E_{r+1}, \ldots, \nu E_n, E_1[1], \ldots, E_r[1])$. Up to some shifts of degrees, the terms in this exceptional sequence coincide with the summands of Z, which implies (H4).

Now apply the argument from the case that X is in degree zero to $X[-1]$ and the $\mathrm{Hom}_{\leq 0}$-configuration $Z[-1]$. □

Theorem 8.5. *If $\mathcal{I}$ is a periodic combinatorial configuration, and H is of finite type, then the objects of $\mathcal{I}$ lying inside $\mathcal{D}^{(\geq 0)-}_{\leq 1}$ form a* $\mathrm{Hom}_{\leq 0}$*-configuration.*

Proof. Let X be the direct sum of the objects of $\mathcal{I}$ lying inside $\mathcal{D}^{(\geq 0)-}_{\leq 1}$. We show first of all that X has at least n non-isomorphic indecomposable summands. By the definition of combinatorial configuration, any object in $\mathrm{mod}\, H[1]$ admits a non-zero morphism from some object in $\mathcal{I}$. By degree considerations, such an object must be a summand of X. Thus X generates $\mathcal{D}$, and therefore contains at least n non-isomorphic indecomposable summands.

It follows from the definition of combinatorial configuration that $\mathrm{Hom}(X, X)$ has as basis the identity maps on the indecomposable summands of X. If A, B are two non-isomorphic indecomposable summands of X, we have that $\mathrm{Ext}^{-1}(A, B) \simeq D\,\mathrm{Hom}(F(B), A)$ is zero, since $F(B) \in \mathcal{I}$. Further, $\mathrm{Ext}^{t}(A, B) = 0$ for $t < -1$ because X is contained in $\mathcal{D}^{(\geq 0)-}_{\leq 1}$. Since H is of Dynkin type, the summands of X are exceptional and also (H4) holds. It follows that the summands of X can be ordered into an exceptional sequence, which means that there are at most n of them, so there are exactly n, and X is a $\mathrm{Hom}_{\leq 0}$-configuration. □

Note that silting objects in $\mathcal{D}^{\geq 1}_{\leq 1}$ naturally correspond to tilting H-modules. Combining Theorems 8.4 and 8.5 with Corollary 7.4, we obtain the following corollary.

Corollary 8.6. *Assume that the hereditary algebra H is of Dynkin type. Then there is a natural bijection between the tilting H-modules and the periodic combinatorial configurations.*

A bijection between the tilting H-modules and the periodic combinatorial configurations was constructed in type ADE in [BLR].

9. Torsion classes in the derived category

Both silting objects and torsion classes play an important role in this paper. Here we point out that there is a close relationship between these concepts.

For an object M in $\mathcal{D}$ we can define (as in [KV]) the subcategory

$$A(M) = \{X \in \mathcal{D} \mid \mathrm{Ext}^i(M, X) = 0 \text{ for } i \geq 1\}.$$

In this section we prove that $A(M)$ is preserved under application of μ_{rev}.

Lemma 9.1. *If M is silting, $A(M)$ is a torsion class.*

Proof. By [AST, Cor. 3.2] (see [KV] in the Dynkin case) the smallest suspended subcategory $U(M)$ containing M is a torsion class. We claim that $A(M) = U(M)$. Since $A(M)$ is clearly suspended, we need only to show $A(M) \subset U(M)$. Assume X is in $A(M)$. Since $U(M)$ is a torsion class, there is (see [AST, BR]) a triangle

$$U \to X \to Z \to U[1]$$

in $\mathcal{D}$ with U in $U(M)$ and with Z in $U(M)^{\perp}$. Since $U[1]$ is in $U(M) \subseteq A(M)$, and $A(M)$ is suspended, we also have that Z is in $A(M)$.

By Lemma 2.2 we have that M is a generator. Since Z is in $U(M)^{\perp}$ and $M[i]$ is in $U(M)$ for $i \geq 0$, we have that $\mathrm{Hom}_{\mathcal{D}}(M, Z[i]) = 0$ for $i \leq 0$. On the other hand, since Z is in $A(M)$ we have by definition that $\mathrm{Hom}_{\mathcal{D}}(M, Z[i]) = 0$ for $i > 0$. Hence $Z = 0$, and $X \simeq U$ is in $U(M)$. □

The following can be found in [AST].

Proposition 9.2. *If A is a torsion class which is $A(Y)$ for some silting object Y, then Y can be recovered as the* Ext*-projectives of A.*

From this we obtain the following direct consequence.

Corollary 9.3. *The map $Y \mapsto A(Y)$ is an injection from silting objects to torsion classes.*

The following shows that the torsion class associated to an exceptional sequence is not affected by negative mutations.

Proposition 9.4. *If μ_i is a negative mutation for Y, then $A(Y) = A(\mu_i(Y))$.*

Proof. Consider an approximation triangle

$$Y_{i+1}^r[j] \to Y_i^* \to Y_i \to Y_{i+1}^r[j+1]$$

with j negative. The result follows from the long exact sequence obtained by applying $\mathrm{Hom}(\ , X)$ to this triangle. □

Hence the correspondence described in Section 4 preserves the torsion classes.

Corollary 9.5. *If $Y = (Y_1, \ldots, Y_n)$ is silting, then $A(\mu_{\mathrm{rev}}(Y)) = A(Y)$.*

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