

# BRAID GROUPS AND KLEINIAN SINGULARITIES

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ABSTRACT. We establish faithfulness of braid group actions generated by twists along an ADE configuration of 2-spherical objects in a derived category. Our major tool is the Garside structure on braid groups of type ADE. This faithfulness result provides the missing ingredient in Bridgeland's description of a space of stability conditions associated to a Kleinian singularity.

## 1. Introduction

The homological mirror symmetry program of Kontsevich [11] proposes a duality between symplectic geometry and complex geometry in the form of an equivalence between the derived Fukaya category on one side and the derived category of coherent sheaves on the other. Seidel and Thomas [14] observed that since generalized Dehn twists around Lagrangian spheres in a symplectic manifold induce autoequivalences of the derived Fukaya category, there should be corresponding autoequivalences of the derived category of coherent sheaves on a mirror dual variety. With this expectation, they developed a theory of such autoequivalences, which they named spherical twists.

In particular, they considered the case of a Kleinian singularity, constructed for example as the quotient  $\mathbb{C}^2/G$ , where  $G \subset SL_2(\mathbb{C})$  is a finite group. Any smoothing of the singularity is a symplectic manifold containing a collection of Lagrangian spheres in a configuration whose dual graph is a Dynkin diagram of type ADE. The generalized Dehn twists along these spheres are known to satisfy braid relations of type ADE in the symplectic mapping class group.

A general pattern suggests that the minimal resolution  $\pi : X \rightarrow \mathbb{C}^2/G$  should be mirror dual to the smoothing of  $\mathbb{C}^2/G$ , and so we should expect to find spherical objects in  $D^b(X)$  whose associated twists generate an action of a braid group on  $D^b(X)$  by autoequivalences. Since the exceptional divisor  $E = \pi^{-1}(0)$  consists of a tree of  $-2$ -curves whose dual graph is of type ADE, we expect that the desired spherical objects should come from the exceptional divisor. Indeed, the structure sheaves of the  $-2$ -curves are easily seen to be spherical and the associated twists are seen to satisfy braid relations in the group

of autoequivalences up to isomorphism. In type A, using results from Khovanov and Seidel [9], Seidel and Thomas were able to show that this action of the braid group is faithful.

Later, Thomas [15] and Ishii-Ueda-Uehara [6] (type A) and Bridgeland [4] (types ADE) studied the spaces of stability conditions of certain triangulated subcategories  $\mathcal{D}$  of  $D^b(X)$ . Bridgeland showed that a connected component  $\text{Stab}_0(\mathcal{D})$  of the space of stability conditions is a covering space of  $\mathfrak{h}^{reg}/W$ , the space of regular orbits of the Weyl group  $W$  corresponding to the singularity type, and that the braid group action on the derived category induces the full group of deck transformations of the cover. Moreover, he showed that faithfulness of the braid group action on the derived category implies faithfulness on  $\text{Stab}_0(\mathcal{D})$ , and since  $\pi_1(\mathfrak{h}^{reg}/W)$  is known to be the braid group, such faithfulness implies that  $\text{Stab}_0(\mathcal{D})$  is simply connected. Given the faithfulness result of [14] in type A, simply-connectedness for  $\text{Stab}_0(\mathcal{D})$  in type A follows.

Our main goal is to provide the necessary faithfulness result to complete Bridgeland's description of spaces of stability conditions associated to Kleinian singularities in all types. Specifically, we prove faithfulness for braid group actions in types ADE generated by twists along 2-spherical objects (Theorem 3.1).

Our proof makes essential use of the Garside structure on braid groups. We expect that similar methods can be used to study other braid group actions on categories appearing in algebraic geometry and representation theory.

We summarize the contents of the paper. In Section 2, we review some of the theory of braid groups to establish notation. In Section 3, we review the theory of spherical twists, and then prove our main result, Theorem 3.1: faithfulness of braid group actions generated by 2-spherical twists. In Section 4, we recall Bridgeland's work concerning spaces of stability conditions associated to Kleinian singularities, pointing out how Theorem 3.1 applies to this situation.

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## 2. Background on braid groups

We begin by establishing notation for Weyl groups, braid groups and related structures associated to a Dynkin diagram  $\Gamma$  and summarizing what we shall later need to know about Garside factorizations of braid group elements. A good general reference for this material is Kassel-Turaev, [7, Chapter 6].

The Weyl group and braid group associated to a Dynkin diagram  $\Gamma$  have various geometric, topological, and combinatorial realizations. For our purposes, however, it will be sufficient to describe these groups by presentations.

### Weyl groups and braid groups

**Definition 2.1.** *Given a Dynkin diagram  $\Gamma$  of type ADE, the associated Weyl group  $W$  has generators  $s_i$  with  $i \in \Gamma$  a node, subject to the relations  $s_i^2 = 1$ ,  $s_i s_j = s_j s_i$  if  $i, j$  are not adjacent in  $\Gamma$ , and  $s_i s_j s_i = s_j s_i s_j$  if  $i, j$  are not adjacent in  $\Gamma$ .*

Given  $w \in W$ , we call an expression  $w = s_{i_1} \cdots s_{i_k}$  *reduced* if there are no shorter expressions of  $w$  in terms of the generators. Now define a length function  $\ell : W \rightarrow \mathbb{N}$ , where  $\ell(w)$  is the length of a reduced expression. We say that a factorization  $w = uv$  is reduced if  $\ell(w) = \ell(u) + \ell(v)$ . Given a reduced factorization  $w = uv$ , we say that  $u$  is a left factor of  $w$  and  $v$  a right factor. It is known that there is a unique longest element  $w_0 \in W$  and that every element  $w \in W$  is a left factor and a right factor of  $w_0$ .

**Definition 2.2.** *The braid group  $B$  is generated by  $\sigma_i, i \in \Gamma$ , subject to the braid relations  $\sigma_i \sigma_j = \sigma_j \sigma_i$  if  $i, j$  are not adjacent in  $\Gamma$  and  $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$  if  $i, j$  are adjacent in  $\Gamma$ .*

*The braid monoid  $B^+$  is given by the same presentation, but now in the category of monoids.*

It is known that the natural monoid homomorphism  $B^+ \rightarrow B$  sending  $\sigma_i$  to  $\sigma_i$  is an injection and identifies  $B$  with the group of fractions of  $B^+$ . That is to say, any monoid homomorphism  $\rho^+ : B^+ \rightarrow G$  to a group  $G$  extends uniquely to a group homomorphism  $\rho : B \rightarrow G$ . Note that the image of  $B^+ \rightarrow B$  is just the submonoid of  $B$  generated by the  $\sigma_i$ .

**Remark 2.1.** It is well-known that the braid group  $B$  of type  $\Gamma$  may be realized as the fundamental group of the space of regular orbits  $\mathfrak{h}^{reg}/W$  for the Weyl group  $W$  acting on the the Cartan algebra  $\mathfrak{h}$  of type  $\Gamma$ .

Since the relations among the generators  $\sigma_i$  of  $B$  are also satisfied by the generators  $s_i$  of the associated Weyl group  $W$ , we have a natural surjection  $\pi : B \rightarrow W$  sending  $\sigma_i$  to  $s_i$ . There is, moreover, a set-theoretic section  $\varphi : W \rightarrow B$  of  $\pi$ , which sends  $s_i$  to  $\sigma_i$  and sends an element  $w \in W$  to the product of  $\sigma_i$  corresponding to a reduced expression in terms of the  $s_i$ . This prescription is well-defined since any two reduced expressions of an element  $w \in W$  can be related by braid relations, which also hold in  $B$ . In particular, let  $\Delta \in B$  be the image of the longest word  $w_0$  under  $\varphi$ . Note that the image of the section  $\varphi(W)$  is by construction contained in the braid monoid  $B^+$ . For brevity, we shall often denote  $\varphi(w)$  by  $\tilde{w}$ . Note that  $\widetilde{uv} = \tilde{u}\tilde{v}$  if  $\ell(uv) = \ell(u) + \ell(v)$ .

On the braid monoid  $B^+$ , we have a length function  $\ell : B^+ \rightarrow \mathbb{N}$ , again defined as the length of a shortest expression of an element in terms of the generators. Note that  $\alpha \in \varphi(W)$  if and only if  $\ell(\alpha) = \ell(\pi(\alpha))$ . In particular, the length of  $w \in W$  is the same as the length of  $\tilde{w} \in B^+$ .

A factorization  $\alpha = \beta\gamma$  in  $B^+$  is said to be reduced if  $\ell(\alpha) = \ell(\beta) + \ell(\gamma)$ . Given a reduced factorization  $\alpha = \beta\gamma$  in  $B^+$ , we say that  $\beta$  is a left factor of  $\alpha$  and  $\gamma$  a right factor. Note that the image  $\varphi(W)$  can be described as the set of left factors or the set of right factors of  $\Delta$ . In particular,  $\varphi(W)$  is closed under taking left or right factors.

## Garside factorization

We now want to describe a normal form, the *Garside factorization*, for elements of the braid group  $B$ . For a more thorough discussion and further references, see [7, Chapter 6].

The Garside factorization for an element  $\alpha \in B^+$  is a reduced expression  $\alpha = \alpha_k \cdots \alpha_1$ , where  $\alpha_i = \tilde{w}_i \in \varphi(W)$  are canonically defined elements of the braid monoid. Let  $\alpha \in B^+$ . It can be shown that  $\alpha$  has a unique longest right factor lying in  $\varphi(W)$  and that all other right factors are right factors of the longest one. By definition, we take this longest right factor to be  $\alpha_1$ . Writing  $\alpha = \alpha'\alpha_1$ , the succeeding factors are defined by applying the same procedure recursively to  $\alpha'$ .

The following standard result says that the property of being a Garside factorization can be checked locally.

**Lemma 2.1.** *For  $\alpha_i \in B^+$ ,  $1 \leq i \leq k$ ,  $(\alpha_k, \dots, \alpha_1)$  is the Garside factorization for  $\alpha_k \cdots \alpha_1$  if and only if the Garside factorization for  $\alpha_i \alpha_{i-1}$  is  $(\alpha_i, \alpha_{i-1})$ .*

The previous lemma combines well with the following, which gives a more explicit procedure for checking that  $(\alpha, \beta)$  is a Garside factorization of  $\alpha\beta$ .

**Lemma 2.2.**  *$(\tilde{u}, \tilde{v})$  is the Garside factorization of  $\tilde{u}\tilde{v}$  if and only if for any  $s_i$  which can appear as the rightmost factor of  $u$ ,  $s_i$  can also appear as the leftmost factor of  $v$ .*

*Proof.* Suppose first that  $(\tilde{u}, \tilde{v})$  is the Garside factorization of  $\tilde{u}\tilde{v}$ . Consider any  $s_i$  which can appear as a rightmost factor of  $u$ . If  $\ell(s_iv) = \ell(v) + 1$ , then  $(\tilde{u}s_i, \tilde{v})$  would be a factorization of  $\tilde{u}\tilde{v}$  with longer right factor, contradicting our assumption. Thus  $\ell(s_iv) < \ell(v)$ , which implies that  $v$  can be written with  $s_i$  as its leftmost factor.

Conversely, assume that any  $s_i$  appearing as a right factor of  $u$  also appears as a left factor of  $v$ . If  $\tilde{u}\tilde{v}$  is not already the Garside factorization, then let  $\tilde{v}'$  be the rightmost Garside factor. Then  $v'$  must have  $v$  as a right factor and there is some reduced expression  $v' = ws_iv$ , so  $\ell(s_iv) > \ell(v)$ . Then we can write  $\tilde{u}\tilde{v} = \alpha\sigma_i\tilde{v}$ . Thus  $\alpha\sigma_i = \tilde{u}$ . It follows that  $\pi(\alpha)s_i$  is a reduced factorization of  $u$ , so  $u$  has a factorization  $u = xs_i$ . Since  $u$  has  $s_i$  as a right factor, but  $\ell(s_iv) > \ell(v)$ ,  $v$  does not have  $s_i$  as a left factor, contrary to assumption.  $\square$

More generally, for  $\alpha \in B$ , the Garside factorization is of the form  $(\alpha_k, \dots, \alpha_1)$  where  $\alpha_i \in \varphi(W) \cup \{\Delta^{-1}\}$ . More precisely, there is some  $j$  such that:

- $\alpha_1 = \dots = \alpha_j = \Delta$  or  $\alpha_1 = \dots = \alpha_j = \Delta^{-1}$ .
- For  $i > j$ ,  $\alpha_i \in \varphi(W) \setminus \{\Delta\}$ .
- $(\alpha_k, \dots, \alpha_{j+1})$  is the Garside factorization of  $\alpha_k \dots \alpha_{j+1}$  in  $B^+$ .

Further, any factorization satisfying these properties is the Garside factorization of some element of  $B$ .

The following lemma will be useful in establishing faithfulness of braid group actions.

**Lemma 2.3.** *A group homomorphism  $\rho : B \rightarrow G$  is injective if and only if the induced monoid homomorphism  $\rho^+ : B^+ \hookrightarrow B \rightarrow G$  is injective.*

*Proof.* Injectivity of  $\rho$  clearly implies injectivity of  $\rho^+$ . Conversely, suppose  $\rho^+$  is injective and let  $\rho(\alpha) = 1$  for  $\alpha \in B$ . Using the Garside factorization, write  $\alpha = \beta\Delta^m$  for  $\beta \in B^+$  and  $m \in \mathbb{Z}$ . If  $m \geq 0$ , then  $\alpha \in B^+$  and so  $\rho(\alpha) = \rho^+(\alpha) = 1$  implies  $\alpha = 1$ . If  $m < 0$ , then  $\rho(\alpha) = 1$  gives  $\rho^+(\beta) = \rho^+(\Delta^{-m})$ , so injectivity of  $\rho^+$  implies  $\beta = \Delta^{-m}$  and hence  $\alpha = 1$ .  $\square$

### 3. Spherical twists and braid group actions

We begin by reviewing some aspects of the theory of spherical objects (introduced in [14] and refined in [13] and [1]). Since it is sufficient for our applications and the statement of our main result, we restrict to the two dimensional case to simplify our exposition.

Throughout this paper, any triangulated category  $\mathcal{D}$  is assumed to be linear over a field  $k$  and to come with a fixed enhancement. For instance, we may take  $\mathcal{D}$  to be the homotopy category of a stable  $\infty$ -category in the sense of [12], an algebraic triangulated category in the sense of [8], or a pre-triangulated differential-graded category in the sense of [2].

Since  $\mathcal{D}$  comes with a fixed enhancement, we have functorial cones, derived Hom-complexes  $\mathrm{RHom}(X, Y)$  for any two objects  $X, Y \in \mathcal{D}$ , and the adjoint pair of functors  $? \otimes X : D(k) \rightarrow \mathcal{D}$  and  $\mathrm{RHom}(X, ?) : \mathcal{D} \rightarrow D(k)$ . If  $X$  is such that  $\mathrm{RHom}(M, X)$  has total finite dimensional cohomology for all  $M \in \mathcal{D}$ , then the functor  $? \otimes X : D(k) \rightarrow \mathcal{D}$  also has a left adjoint  $\mathrm{RHom}(?, X)^\vee : \mathcal{D} \rightarrow D^b(k)$ , where  $\vee$  denotes the dualization  $\mathrm{RHom}(?, k) : D^b(k) \rightarrow D^b(k)$ . (Here  $D(k)$  is the unbounded derived category of vector spaces and  $D^b(k)$  is the bounded derived category of finite dimensional  $k$ -vector spaces.)

In order to simplify notation, we often write  $[X, Y]$  for  $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ ,  $[X, Y]_d$  or  $\mathrm{Ext}^d(X, Y)$  for  $\mathrm{Hom}(X, Y[d])$ , and  $[X, Y]_*$  for  $\bigoplus_d \mathrm{Hom}(X, Y[d])$ .

For brevity, let us temporarily denote  $\mathrm{RHom}(X, ?)$  by  $R$ ,  $? \otimes X$  by  $X$ , and  $\mathrm{RHom}(?, X)^\vee$  by  $L$ . Then we have the following units and counits:

$$(1) \quad \mathrm{Id}_k \rightarrow RX$$

$$(2) \quad XR \rightarrow \mathrm{Id}_{\mathcal{D}}$$

$$(3) \quad \mathrm{Id}_{\mathcal{D}} \rightarrow XL$$

$$(4) \quad LX \rightarrow \mathrm{Id}_k$$

Now define an exact endofunctor  $\Phi_X$  of  $D(k)$  as the cone of the unit in (1), so that we have a triangle of functors  $\mathrm{Id}_k \rightarrow RX \rightarrow \Phi_X$ . Applying  $R$  on the left of (3), we get a morphism  $R \rightarrow RXL$ , and applying  $L$  to the right of  $RX \rightarrow \Phi_X$ , we get a morphism  $RXL \rightarrow \Phi_X L$ . Composing these two morphisms gives a morphism

$$(5) \quad R \rightarrow \Phi_X L.$$

**Definition 3.1.** Let  $S \in \mathcal{D}$  and suppose that  $\mathrm{RHom}(?, S)^\vee$  always has total finite dimensional cohomology so that it gives a left adjoint to the functor  $? \otimes S$ . Then  $S$  is 2-spherical if

- a) the cone  $\Phi_S$  of  $\mathrm{Id}_k \rightarrow \mathrm{RHom}(S, ? \otimes S)$  is isomorphic to the shift functor  $[2]$
- b) the natural morphism  $\mathrm{RHom}(S, ?) \rightarrow \mathrm{RHom}(?, S[2])^\vee$  from (5) is an isomorphism.

**Remark 3.1.** Applying the first condition in the above definition to the object  $k$ , we see that  $\mathrm{Ext}^*(S, S)$  is one-dimensional in degrees 0 and 2 and zero elsewhere, hence isomorphic to the cohomology of the 2-sphere (hence the name ‘spherical object’). The second condition is a Calabi-Yau condition and says that the shift functor  $[2]$  restricted to  $S$  realizes a kind of Serre duality.

Note that condition b) implies that composition of morphisms gives a perfect pairing

$$(6) \quad [S, X] \otimes [X, S]_2 \rightarrow [S, S]_2 \simeq k.$$

**Example 3.1.** Let  $X$  be a smooth quasi-projective surface,  $C \subset X$  a  $-2$ -curve. Then any twisted structure sheaf  $\mathcal{O}_C(d)$  is 2-spherical in  $D^b(X)$ , the bounded derived category of coherent sheaves on  $X$ .

We shall be particularly interested in special configurations of spherical objects.

**Definition 3.2.** Let  $\Gamma$  be a Dynkin diagram of type ADE. We say that a collection of 2-spherical objects  $S_i$ ,  $i \in \Gamma$ , is a  $\Gamma$ -configuration if for  $i \neq j$ , the space  $[S_i, S_j]_*$  is one-dimensional and concentrated in degree 1 when  $i$  and  $j$  are adjacent in  $\Gamma$  and is zero otherwise.

**Example 3.2.** Consider a finite subgroup  $G \subset SL_2(\mathbb{C})$  and the quotient  $\mathbb{C}^2/G$ , a singular surface with a unique singular point 0, commonly called a Kleinian singularity after Felix Klein who first determined the ring of invariants  $\mathbb{C}[x, y]^G$  in his classic book *The Icosahedron* [10]. The minimal resolution  $\pi : X \rightarrow \mathbb{C}^2/G$  was later studied by du Val [5], who showed that the exceptional divisor  $E = \pi^{-1}(0)$  consists of a tree of  $-2$ -curves whose dual graph  $\Gamma$  is a Coxeter-Dynkin diagram  $\Gamma$  of type ADE. The geometry of the minimal resolution  $\pi : X \rightarrow \mathbb{C}^2/G$  thus provides a beautiful bijection between (conjugacy classes of) finite subgroups  $G \subset SL_2(\mathbb{C})$  and Dynkin diagrams of type ADE.

Letting  $E_i$ ,  $i \in \Gamma$  be the irreducible components of  $\pi^{-1}(0) = E$ , a standard computation shows that the collection  $S_i = \mathcal{O}_{E_i}$  is a  $\Gamma$ -configuration of 2-spherical objects. (The same is of course true for the



collection  $S_i = \mathcal{O}_{E_i}(-1)[1]$ , which, despite its appearance, is sometimes more convenient to work with.)

Returning to a general  $\Gamma$ -configuration, we see that since the objects  $S_i$  are 2-spherical, we have perfect pairings as in (6):

$$[S_i, S_j]_1 \otimes [S_j, S_i]_1 \rightarrow [S_i, S_i]_2 \simeq k.$$

As a consequence, we have the following lemma that will be useful in the proof of Proposition 3.1 below.

**Lemma 3.1.** *If  $i$  and  $j$  are adjacent in  $\Gamma$ , then any morphism  $S_i \rightarrow S_i[2]$  factors as  $S_i \rightarrow S_j[1] \rightarrow S_i[2]$ .*

We now recall some results of Seidel-Thomas [14] concerning spherical twists in enhanced triangulated categories, stated for 2-spherical objects for simplicity, and then give the main result of this paper, Theorem 3.1: braid group actions of types ADE, generated by 2-spherical twists, are faithful in types ADE.

Given an object  $S \in \mathcal{D}$ , we have a functor  $? \otimes S : D(k) \rightarrow \mathcal{D}$  and its right adjoint  $\mathrm{RHom}(S, ?) : \mathcal{D} \rightarrow D(k)$ . We therefore can define a ‘twist’ functor  $t_S : \mathcal{D} \rightarrow \mathcal{D}$  as the cone of the counit  $\mathrm{RHom}(S, ?) \otimes S \rightarrow \mathrm{Id}_{\mathcal{D}}$ . When  $S$  is a spherical object, Seidel and Thomas [14] showed that  $t_S$  is an (auto)equivalence.

We collect some standard facts about spherical twists in the following lemma.

**Lemma 3.2.**

- a) *If  $S$  is 2-spherical,  $t_S(S) \simeq S[-1]$*
- b) *If  $F$  is an autoequivalence of  $\mathcal{D}$ , then  $F \circ t_S \simeq t_{F(S)} \circ F$ .*
- c) *If  $S_i$  and  $S_j$  are not adjacent in a  $\Gamma$ -configuration of spherical objects, then  $t_i S_j \simeq S_j$ .*
- d) *If  $S_i$  and  $S_j$  are adjacent in a  $\Gamma$ -configuration of spherical objects, then  $t_i t_j S_i \simeq S_j$ .*

From Lemma 3.2 c) and d), it can be shown that if  $S_i, i \in \Gamma$  form a  $\Gamma$ -configuration, then the associated twist functors  $t_i := t_{S_i}$  satisfy the braid relations of type  $\Gamma$ , up to isomorphism, and so there is a homomorphism

$$\rho : B \rightarrow \mathrm{Aut}(\mathcal{D})$$

from the braid group  $B$  of type  $\Gamma$  to the group  $\mathrm{Aut}(\mathcal{D})$  of isomorphism classes of autoequivalences of  $\mathcal{D}$ . We denote the functor  $\rho(\alpha)$  by  $t_\alpha$ .

In type A, Seidel and Thomas, based on work of Khovanov and Seidel [9], showed that the homomorphism  $\rho$  is injective. We will generalize this to types ADE. Our strategy is to show that a braid group element



$\alpha$  is completely determined by the action of the corresponding twist  $t_\alpha$  on  $\mathcal{S} = \bigoplus_i S_i$ . To do this, we will probe  $t_\alpha \mathcal{S}$  by considering the Hom spaces  $[S_i, t_\alpha \mathcal{S}]_k$ . The following proposition provides the necessary information.

**Proposition 3.1.** *Let  $1 \neq \alpha \in B^+$  have Garside factorization  $\alpha = \tilde{w}_k \cdots \tilde{w}_1$ . Then*

- a)  $[\mathcal{S}, t_\alpha \mathcal{S}]_d = 0$  for  $d > k + 2$
- b)  $[S_i, t_\alpha \mathcal{S}]_{k+2} \neq 0$  if and only if  $s_i$  is a left factor of  $w_k$  (in particular  $[\mathcal{S}, t_\alpha \mathcal{S}]_{k+2} \neq 0$ ).
- c) The maximal  $p$  such that  $[\mathcal{S}, t_\alpha \mathcal{S}]_p \neq 0$  is precisely  $p = k + 2$ .

In words, we can determine the number  $k$  of Garside factors of  $\alpha$  from the maximal degree  $p$  of a map from  $\mathcal{S}$  to  $t_\alpha \mathcal{S}$  and we can determine whether or not  $s_i$  is a left factor of the final Garside factor of  $\alpha$  by studying maps from  $S_i$  to  $t_\alpha \mathcal{S}$ .

Before proving Proposition 3.1, let us see how it implies our main result.

**Theorem 3.1.** *The homomorphism  $\rho : B \rightarrow \text{Aut}(\mathcal{D})$  is injective.*

*Proof.* By Lemma 2.3, it is enough to show injectivity on  $B^+$ . To do this, we show that  $\alpha \in B^+$  can be recovered from the functor  $t_\alpha$ . Thus if two functors  $t_\alpha$  and  $t_\beta$  for  $\alpha, \beta \in B^+$  are isomorphic, then we must have  $\alpha = \beta$ .

To recover  $\alpha$  from  $t_\alpha$ , we study the mapping space  $[\mathcal{S}, t_\alpha \mathcal{S}]_*$ . By Proposition 3.1 c), we know that the number of Garside factors of  $\alpha$  is  $k = p - 2$  where  $p$  is the maximal degree of a non-zero map from  $\mathcal{S}$  to  $t_\alpha \mathcal{S}$ . Now let  $\alpha = \tilde{w}_k \cdots \tilde{w}_1$  be the Garside factorization of  $\alpha$  and let  $\beta = \tilde{w}_{k-1} \cdots \tilde{w}_1$

First, we will determine a reduced decomposition of  $w_k$  and hence  $w_k$  itself. Since  $[\mathcal{S}, t_\alpha \mathcal{S}]_{k+2} \neq 0$ , there must be some  $S_i$  such that  $[S_i, t_\alpha \mathcal{S}]_{k+2} \neq 0$ , and by Proposition 3.1 b),  $s_i$  must then be a left factor of  $w_k$ , so write a reduced expression  $w_k = s_i u$ . Then  $t_i^{-1} t_\alpha = t_{\tilde{u}\beta}$ . Now consider  $[\mathcal{S}, t_{\tilde{u}\beta} \mathcal{S}]_{k+2}$ . If it is zero, then  $\tilde{u}\beta$  has  $k - 1$  Garside factors, so  $u = 1$  and we have determined that  $w_k = s_i$ . Otherwise we repeat the above argument to find a left factor  $s_j$  of  $u$ . Proceeding in this way, we eventually find a reduced decomposition  $w_k = s_i s_j \cdots$ .

Once we have determined  $w_k$ , we repeat the whole process on  $t_{w_k}^{-1} t_\alpha = t_\beta$  to determine  $w_{k-1}$ , and so on, until we have determined in order all of the Garside factors of  $\alpha$  and hence  $\alpha$  itself.  $\square$

We shall need the following lemma in the proof of Proposition 3.1.

**Lemma 3.3.** *Let  $Y \in \mathcal{D}$ .*

- a) *If  $l$  is maximal such that  $[S_i, Y]_l \neq 0$ , then  $m = l + 1$  is maximal such that  $[S_i, t_i Y]_m \neq 0$ .*
- b) *Let  $p$  be maximal such that  $[\mathcal{S}, Y]_p \neq 0$ . If  $q$  is maximal such that  $[\mathcal{S}, t_i Y]_q \neq 0$ , then  $p \leq q \leq p + 1$ . Further,  $q = p + 1$  if and only if  $[S_i, Y]_p \neq 0$ .*

In words, part a) says that twisting an object  $Y$  by  $t_i$  increases by one the maximal degree of a map from  $S_i$ . Part b) says that twisting by  $t_i$  cannot decrease the maximal degree of a map from  $\mathcal{S}$ , it increases the maximal degree if and only if there is a map of degree  $p$  from  $S_i$  to  $Y$ , and, if so, it increases the maximal degree by one.

*Proof.* a) Since  $t_i S_i \simeq S_i[-1]$ , twisting a non-zero morphism  $S_i \rightarrow Y[l]$  produces a non-zero morphism  $S_i \rightarrow t_i Y[l + 1]$ . It must be of maximal degree, since if there were a non-zero morphism  $S_i \rightarrow t_i Y[m]$  with  $m > l + 1$ , then twisting by  $t_i^{-1}$  and translating would give a non-zero map  $S_i \rightarrow Y[m - 1]$  of degree greater than  $l$ , contrary to assumption.

b) Consider the triangle

$$[S_i, Y]_* \otimes S_i \xrightarrow{f} Y \xrightarrow{g} t_i Y$$

Let  $p$  be maximal such that  $[\mathcal{S}, Y]_p \neq 0$  and  $q$  be maximal such that  $[\mathcal{S}, t_i Y]_q \neq 0$ . For  $j \in \Gamma$ , let  $l_j$  be the maximum degree of a map from  $S_j$  to  $Y$  and note that  $l_j \leq p$  for all  $j$ . By part a), the maximum degree of a map from  $S_i$  to  $t_i Y$  is  $l_i + 1$ .

Now consider some  $j \neq i$ . The maximum degree of a map from  $S_j$  to  $[S_i, Y]_* \otimes S_i[1]$  is  $l_i$ . Since  $t_i Y$  lies in a triangle between  $Y$  and  $[S_i, Y]_* \otimes S_i[1]$ , the maximum degree of a map from  $S_j$  to  $t_i Y$  is no greater than  $\max(l_i, l_j) \leq p + 1$ .

This shows that  $q \leq p + 1$ , and that  $q = p + 1$  if and only if the maximum degree of a map from  $S_i$  to  $Y$  is  $p$ .

To show that  $q \geq p$ , observe first that if  $l_i \geq p - 1$ , we are done, so assume that  $l_i < p - 1$ . Now, by assumption, for some  $j \neq i$ , there must be a nonzero map in  $[S_j, Y]_p$ . Since  $[S_i, Y]_{p-1} = 0$ , we know by the condition of a  $\Gamma$ -configuration that  $[S_j, [S_i, Y]_* \otimes S_i]_p = 0$ , so the nonzero map in  $[S_j, Y]_p$  must compose with  $g$  to give a non-zero map in  $[S_j, t_i Y]_p$ . This shows that  $q \geq p$ .  $\square$

We now prove Proposition 3.1.

*Proof of Proposition 3.1.* We prove a) and b) together by simultaneous induction on  $k$  and  $\ell(w_k)$ . Statement c) follows immediately from a) and b).

The base case of the induction is when  $k = 1$  and  $w_1 = s_i$  for some  $i \in \Gamma$ . In this case, the statements follow from a straight-forward calculation or directly from Lemma 3.3.

Now we suppose that we know a) and b) for any  $1 \neq \beta \in B^+$  with fewer Garside factors or with shorter final Garside factor than  $\alpha$ , and we prove a) and b) for  $\alpha$ .

a) Suppose first that  $w_k = s_i$  for some  $i$ . In this case, we know by the induction hypothesis that  $[\mathcal{S}, t_{\tilde{w}_{k-1}\dots\tilde{w}_1}\mathcal{S}]_d = 0$  for  $d > k + 1$ , so it follows from Lemma 3.3 that  $[\mathcal{S}, t_\alpha\mathcal{S}]_d = 0$  for  $d > k + 2$ .

Now suppose that  $\ell(w_k) > 1$ , and fix a reduced decomposition  $w_k = s_i u$ . Let  $\beta = \tilde{w}_{k-1}\dots\tilde{w}_1$ . By the induction hypothesis, we know that  $[\mathcal{S}, t_\beta\mathcal{S}]_d = 0$  for  $d > k + 2$ . Since  $s_i u$  is reduced, we know that  $u$  cannot be written with  $s_i$  as a leftmost factor. Therefore, by b), we know that  $[S_i, t_\beta\mathcal{S}]_{k+2} = 0$ . By Lemma 3.3, it follows that  $[\mathcal{S}, t_i t_\beta\mathcal{S}]_d = 0$  for  $d > k + 2$ , as desired.

b) First we prove that if  $s_i$  is a left factor of  $w_k$ , then  $[S_i, t_\alpha\mathcal{S}]_{k+2} \neq 0$ . For brevity, write  $\beta = \tilde{w}_{k-1}\dots\tilde{w}_1$ .

Consider first the case that  $\ell(w_k) = 1$ . Since we have already disposed of the case that  $k = 1$  and  $\ell(w_1) = 1$ , we may assume that  $k > 1$ . By Lemma 2.2,  $w_{k-1}$  must have  $s_i$  as a left factor, so by induction on  $k$ , there is a non-zero map  $S_i \rightarrow t_\beta\mathcal{S}[k + 1]$ . Again by Lemma 3.3, part a), twisting with  $t_i = t_{\tilde{w}_k}$  then produces a non-zero map  $S_i \rightarrow t_\alpha\mathcal{S}[k + 2]$ , as needed.

Now assume that  $\ell(w_k) > 1$  and suppose then that we have a reduced expression  $w_k = s_i s_j u$ . This leads us to consider various cases.

**Case 1** Suppose  $s_i$  and  $s_j$  commute (so  $[S_i, S_j]_* = 0$ ) and thus we may write  $w_k = s_j s_i u$ . By induction on the length of  $w_k$ , we have a non-zero map  $S_i \rightarrow t_i t_{\tilde{u}} t_\beta\mathcal{S}[k + 2]$ . Since  $t_j S_i \simeq S_i$ , twisting with  $t_j$  produces a non-zero map  $S_i \rightarrow t_\alpha\mathcal{S}[k + 2]$ , as desired.

**Case 2** Suppose  $s_i$  and  $s_j$  do not commute and we have a reduced decomposition  $w_k = s_i s_j s_i v = s_j s_i s_j v$ . By induction on the length of  $w_k$ , we have a non-zero map  $S_j \rightarrow t_j t_{\tilde{v}} t_\beta\mathcal{S}[k + 2]$ . Applying  $t_j t_i$  gives a non-zero map  $S_i \rightarrow t_\alpha\mathcal{S}[k + 2]$  (since  $t_j t_i S_j \simeq S_i$  by Lemma 3.2).

**Case 3** Suppose  $s_i$  and  $s_j$  do not commute and there is no reduced decomposition  $w_k = s_i s_j s_i v$ , so  $s_i$  is *not* a left factor of  $u$ . (Note that this includes the case  $u = 1$ ).

We need to show that there is a non-zero map  $S_i[-k-2] \rightarrow t_i t_j t_{\tilde{u}} t_\beta\mathcal{S}$ . Applying  $t_i^{-1}$ , we see that this is equivalent to having a non-zero map  $S_i \rightarrow t_j t_{\tilde{u}} t_\beta\mathcal{S}[k + 1]$  (since  $t_i^{-1} S_i \simeq S_i[1]$ ).

Now consider the triangle

$$(7) \quad [S_j, t_{\tilde{u}}t_\beta\mathcal{S}]_* \otimes S_j \rightarrow t_{\tilde{u}}t_\beta\mathcal{S} \rightarrow t_jt_{\tilde{u}}t_\beta\mathcal{S}$$

Note  $[S_j, t_{\tilde{u}}t_\beta\mathcal{S}]_{k+2} = 0$ , or else  $s_j$  would be a left factor of  $u$  (by induction on the length of  $w_k$ ) and then the expression for  $w_k$  would not be reduced, or  $u = 1$ , and then we may apply induction on the number of factors.

Now we claim that  $[S_j, t_{\tilde{u}}t_\beta\mathcal{S}]_{k+1} \neq 0$ . Suppose otherwise. By induction on length of  $w_k$ , there is a non-zero map  $S_j[-k-2] \rightarrow t_jt_{\tilde{u}}t_\beta\mathcal{S}$ . But in the triangle (7), the objects to the left and right of  $t_jt_{\tilde{u}}t_\beta\mathcal{S}$  admit no map from  $S_j[-k-2]$ , a contradiction. Thus  $[S_j, t_{\tilde{u}}t_\beta\mathcal{S}]_{k+1} \neq 0$ .

Then since  $i$  and  $j$  are neighbors, we get a map  $\varphi : S_i[-k-2] \rightarrow [S_j, t_{\tilde{u}}t_\beta\mathcal{S}]_{k+1} \otimes S_j[-k-1]$ . Considering again the triangle (7), we see that the composition to  $t_{\tilde{u}}t_\beta\mathcal{S}$  vanishes, so  $\varphi$  must factor through  $t_jt_{\tilde{u}}t_\beta\mathcal{S}[-1]$ , giving a non-zero map  $S_i[-k-2] \rightarrow t_jt_{\tilde{u}}t_\beta\mathcal{S}[-1]$ . We twist this with  $t_i$  and shift to get a non-zero map  $S_i[-k-2] \rightarrow t_\alpha\mathcal{S}$ , as desired.

We now establish the opposite implication by proving the contrapositive: if  $s_i$  is not a left factor of  $w_k$ , then  $[S_i, t_\alpha\mathcal{S}]_{k+2} = 0$ .

Fix a reduced factorization  $w_k = s_ju$ . Write  $\beta = \tilde{u}\tilde{w}_{k-1} \cdots \tilde{w}_1$ . Consider the triangle:

$$(8) \quad [S_j, t_\beta\mathcal{S}]_* \otimes S_j \rightarrow t_\beta\mathcal{S} \rightarrow t_\alpha\mathcal{S}$$

Suppose that  $\ell(w_k) = 1$ , so that  $w_k = s_j$  and  $u = 1$ . By induction on the number of Garside factors, we know that  $[S_i, t_\beta\mathcal{S}]_{k+2} = 0$ , and  $[S_j, t_\beta\mathcal{S}]_{k+2} = 0$ , and that  $[S_i, [S_j, t_\beta\mathcal{S}]_* \otimes S_j]_{k+3} = 0$ , so (8) implies  $[S_i, t_\alpha\mathcal{S}]_{k+2} = 0$ , as desired.

Now let  $\ell(w_k) > 1$ . We consider two cases.

**Case 1** Suppose  $s_i$  and  $s_j$  commute. Then  $u$  does not admit  $s_i$  as a left factor, since we know that  $s_ju$  does not admit  $s_i$  as a left factor. The induction hypothesis applied to  $\beta$  implies that  $[S_i, t_\beta\mathcal{S}]_{k+2} = 0$ . Apply  $t_j$  to both  $S_i$  and  $t_\beta\mathcal{S}$ ; since  $t_jS_i \simeq S_i$ , we obtain that  $[S_i, t_\alpha\mathcal{S}]_{k+2} = 0$ , as desired.

**Case 2** Suppose  $s_i$  and  $s_j$  do not commute. Applying  $\text{Hom}(S_i, ?)$  to (8) we get the following portion of a long exact sequence:

$$(9) \quad [S_j, t_\beta\mathcal{S}]_{k+1} \otimes [S_i, S_j]_1 \rightarrow [S_i, t_\beta\mathcal{S}]_{k+2} \rightarrow [S_i, t_\alpha\mathcal{S}]_{k+2} \rightarrow [S_j, t_\beta\mathcal{S}]_{k+2} \otimes [S_i, S_j]_1$$

Since  $s_j$  is not a left factor of  $u$ , we see by the induction hypothesis applied to  $t_\beta\mathcal{S}$  that the final term is zero. This implies that the map  $[S_i, t_\beta\mathcal{S}]_{k+2} \rightarrow [S_i, t_\alpha\mathcal{S}]_{k+2}$  is surjective, so any morphism in  $[S_i, t_\alpha\mathcal{S}]_{k+2}$  factors through  $t_\beta\mathcal{S}[k+2]$ . If  $u$  does not admit  $s_i$  as a left factor, then we are done, since, by the induction hypothesis,  $[S_i, t_\beta\mathcal{S}]_{k+2} = 0$ . So suppose otherwise, and write  $w_k = s_j s_i v$ . Note that  $v$  does not admit  $s_j$  as a left factor, since  $s_j s_i s_j = s_i s_j s_i$ , which would imply that  $w_k$  admits  $s_i$  as a left factor. Let  $\gamma = \tilde{v} \tilde{w}_{k-1} \cdots \tilde{w}_1$ . We now have:

$$[S_i, t_\gamma\mathcal{S}]_* \otimes S_i \rightarrow t_\gamma\mathcal{S} \rightarrow t_\beta\mathcal{S}$$

Since  $v$  does not admit  $s_i$  as a left factor, by the induction hypothesis we know that  $[S_i, t_\gamma\mathcal{S}]_{k+2} = 0$ , and thus any non-zero map  $\varphi : S_i \rightarrow t_\beta\mathcal{S}[k+2]$  composes to give a non-zero map  $S_i \rightarrow [S_i, t_\gamma\mathcal{S}]_* \otimes S_i[k+3]$ . Since  $[S_i, t_\gamma\mathcal{S}]_d = 0$  for  $d > k+1$  by induction, such a map must land in the summand  $[S_i, t_\gamma\mathcal{S}]_{k+1} \otimes S_i[2]$  of  $[S_i, t_\gamma\mathcal{S}]_* \otimes S_i[k+3]$ . By Lemma 3.1,  $\varphi$  must therefore factor through  $S_j[1]$ , so we have the arrows in the diagram below other than the dotted arrow.

$$\begin{array}{ccccc} t_\beta\mathcal{S}[k+2] & \longrightarrow & [S_i, t_\gamma]_* \otimes S_i[k+3] & \longrightarrow & t_\gamma\mathcal{S}[k+3] \\ \uparrow & & \swarrow \cdots & & \uparrow \\ S_i & \longrightarrow & S_j[1] & & \end{array}$$

By the induction hypothesis (on length if  $v \neq 1$  and on the number of Garside factors if  $v = 1$ ), we know that  $\text{Hom}(S_j[1], t_\gamma\mathcal{S}[k+3]) = 0$ . It follows that the dotted arrow can be filled in, making the upper triangle commutative.

We claim that the lower triangle must also be commutative. If it were not, the difference between the two maps from  $S_i$  to  $t_\beta\mathcal{S}[k+2]$  would induce a non-zero map to  $t_\gamma\mathcal{S}[k+2]$ , but no such map exists.

Now consider (9) again. We have seen that any  $\varphi : S_i \rightarrow t_\beta\mathcal{S}[k+2]$  factors through  $S_j[1]$ , so the map  $[S_j, t_\beta\mathcal{S}]_{k+1} \otimes [S_i, S_j]_1 \rightarrow [S_i, t_\beta\mathcal{S}]_{k+2}$  must be surjective. But we have already argued that the map from  $[S_i, t_\beta\mathcal{S}]_{k+2}$  to  $[S_i, t_\alpha\mathcal{S}]_{k+2}$  is surjective, so by exactness of (9) we have  $[S_i, t_\alpha\mathcal{S}]_{k+2} = 0$ , as desired.  $\square$

#### 4. Application to spaces of stability conditions

We briefly recall the notion of stability condition on a triangulated category as introduced by Bridgeland in [3], review the results of Bridgeland in [4], and point out how our faithfulness result Theorem 3.1 answers a question left open in [4].

A stability condition on a triangulated category  $\mathcal{D}$  consists of a bounded  $t$ -structure on  $\mathcal{D}$  together with a group homomorphism  $\mathcal{Z} : K_0(\mathcal{D}) \rightarrow \mathbb{C}$ , known as the ‘central charge’ such that each object in the heart of the  $t$ -structure has a Harder-Narasimhan filtration with respect to the slope  $Im(\mathcal{Z})/Re(\mathcal{Z})$  of the central charge. The set of stability conditions satisfying some technical hypotheses can be given the structure of a complex manifold  $Stab(\mathcal{D})$ .

Now consider the minimal resolution  $\pi : X \rightarrow \mathbb{C}^2/G$  of the Kleinian singularity, as described in Example 3.2. Inside the bounded derived category of coherent sheaves  $D^b(X)$ , let  $\mathcal{D}$  be the thick subcategory generated by the  $\Gamma$ -configuration of 2-spherical objects  $S_i = \mathcal{O}_{E_i}(-1)[1]$ . Bridgeland [4] has shown that there is a connected component  $Stab_0(\mathcal{D})$  of  $Stab(\mathcal{D})$ , stable under the action of the braid group induced by the spherical twists  $t_i$ , together with a covering map  $p : Stab_0(\mathcal{D}) \rightarrow \mathfrak{h}^{reg}/W$ . As noted in Remark 2.1, it is well-known that the fundamental group  $\pi_1(\mathfrak{h}^{reg}/W)$  is the braid group  $B$  of the corresponding type. Bridgeland further shows that the image of  $\rho : B \rightarrow Aut(\mathcal{D})$  gives the full group of deck transformations for the covering  $p$ , and therefore that if  $\rho$  is injective,  $p$  is in fact a universal cover. When Bridgeland was writing [4], the injectivity of  $\rho$  was known in type  $A$  due to Seidel-Thomas [14]. From Theorem 3.1, such injectivity is now known, and thus  $p$  is a universal cover, in all types ADE.

**Remark 4.1.** In [4], Bridgeland also considers an affine analogue of the category  $\mathcal{D}$ , with an extra generator  $S_0 = \mathcal{O}_E$ , the structure sheaf of the exceptional divisor  $\pi^{-1}(0) = E$ , and establishes analogous results for its space of stability conditions.

It is therefore an interesting problem to determine if the affine braid group action on this category (with extra generator given by the spherical twist along  $S_0$ ) is faithful. This has been established by Ishii-Ueda-Uehara [6] in type  $A$ . Somewhat more generally, one can ask if the extended affine braid group action generated by the finite type braid group and the Picard group of  $X$  is faithful.

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