

Expansion of k -Schur functions for maximal k -rectangles within the affine nilCoxeter algebra

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Abstract

We give several explicit combinatorial formulas for the expansion of k -Schur functions indexed by maximal rectangles in terms of the standard basis of the affine nilCoxeter algebra. Using our result, we also show a commutation relation of k -Schur functions corresponding to rectangles with the generators of the affine nilCoxeter algebra.

1 Introduction and Prerequisites

k -Schur functions ($s_\lambda^{(k)}$, indexed by k -bounded partitions) were first introduced by Lapointe, Lascoux and Morse in [10], in an attempt to better understand Macdonald polynomials. Since that time it developed that k -Schur functions play an important role in many other areas of mathematics (see [6, 7, 9, 8, 11, 12, 13]). As a result, one fundamental open problem in the theory of k -Schur functions is the understanding of their structure constants, the k -Littlewood Richardson coefficients $c_{\lambda,\mu}^{\nu,(k)}$ defined by $s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda,\mu}^{\nu,(k)} s_\nu^{(k)}$.

Recent work of Thomas Lam [5] has applied the work of Fomin and Green [3] by viewing the ring $\Lambda_{(k)}$ of symmetric functions spanned by the k -Schur

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functions as a subalgebra of the affine nilCoxeter algebra. In particular, we recall in Section 4 that understanding k -Littlewood Richardson coefficients $c_{\lambda,\mu}^{\nu,(k)}$ is equivalent to understanding the explicit expansion of $s_{\lambda}^{(k)}$ in the standard basis of the affine nilCoxeter algebra (the basis formed by the words in the generators).

With this in mind, this article is about expanding the k -Schur function $s_{\lambda}^{(k)}$ in a specific case, namely when λ is a maximal rectangle, i.e. a rectangular partition with maximal hook length k . In fact, we give four different formulas giving four different points of view. Two of them are central to our proof, but the other two are of interest in their own right. It is not clear yet which point of view will generalize to other shapes and we choose to present all four formulas as a starting point of this investigation. However, a reader chiefly interested in getting to the proof of the main theorem could skip Sections 2.3, 2.4, 3.2 and 3.3.

In Section 2 we state our four combinatorial definitions of elements of the affine nilCoxeter algebra. In Section 3 we prove that all four definitions are equivalent. In Section 4 we prove that these formulas give an explicit expansion for a k -Schur function indexed by a maximal rectangle in the standard basis of the affine nilCoxeter algebra. Section 5 contains a short application of our formulas: we give a commutation relation of the k -Schur function indexed by a maximal rectangle with the generators of the affine nilCoxeter algebra.

1.1 k -bounded partitions, $(k + 1)$ -cores and the affine symmetric group

Throughout the paper, we work with $k \geq 1$ a fixed integer.

For a Young diagram of a partition λ , we associate to each box (i, j) (row i , column j) of the diagram a *content* defined by $c_{(i,j)} = (j - i) \bmod (k + 1)$. We will let $\mathcal{P}^{(k)}$ denote the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ whose first part λ_1 is at most k .

A p -core is a partition which has no removable rim hooks of length p . Lapointe and Morse [12, Theorem 7] showed that the set $\mathcal{P}^{(k)}$ bijects with the set of $(k + 1)$ -cores. Following their notation, we let $\mathfrak{c}(\lambda)$ denote the $(k + 1)$ -core corresponding to the partition λ , and $\mathfrak{p}(\mu)$ denote the k -bounded partition corresponding to the $(k + 1)$ -core μ . We will also use $\mathcal{C}^{(k+1)}$ to represent the set of all $(k + 1)$ -cores.

The affine symmetric group W is generated by reflections s_i for $i \in \{0, 1, \dots, k\}$, subject to the relations:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i \in \{0, 1, \dots, k\} \\ s_i s_j &= s_j s_i \text{ if } i - j \neq \pm 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in \{0, 1, \dots, k\} \end{aligned}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

An element $w \in W$ has a length, denoted $len(w)$, defined to be the minimal m for which $w = s_{i_1} \cdots s_{i_m}$ for some i_1, \dots, i_m .

W has an action on $\mathcal{C}^{(k+1)}$. Specifically, if λ is a $(k+1)$ -core then $s_i \lambda$ is λ union all addable positions of content i , if λ has such an addable position, $s_i \lambda$ is λ minus all removable boxes of content i from λ if λ has such a removable box (a $(k+1)$ -core cannot have both a removable box and an addable position of the same content), and $s_i \lambda = \lambda$ otherwise.

We let W_0 denote the parabolic subgroup obtained from W by removing the generator s_0 . This is naturally isomorphic to the symmetric group S_{k+1} . W^0 will denote the set of minimal length coset representatives of W/W_0 . W^0 is naturally identified with $\mathcal{C}^{(k+1)}$ in the following way. To a core $\lambda \in \mathcal{C}^{(k+1)}$, we associate the unique element $w \in W^0$ for which $w\emptyset = \lambda$. For a k -bounded partition μ , we let w_μ denote the element of W^0 which satisfies $w_\mu \emptyset = \mathbf{c}(\mu)$. More details on this can be found in [2].

Fix an orthonormal basis $\{\epsilon_1, \dots, \epsilon_{k+1}\}$ of \mathbb{R}^{k+1} . There is a left action of W on $V := \mathbb{R}^{k+1} / (\sum_{i=1}^{k+1} \epsilon_i)$ which occurs from viewing W as the affine type A Weyl group (see for example [4]):

$$\begin{aligned} s_i \diamond (a_1, \dots, a_{k+1}) &= (a_1, \dots, a_{i+1}, a_i, \dots, a_{k+1}) \text{ for } i \neq 0 \\ s_0 \diamond (a_1, \dots, a_{k+1}) &= (a_{k+1} + 1, a_2, \dots, a_k, a_1 - 1) \end{aligned} \tag{1}$$

We let

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \dots, \alpha_k = \epsilon_k - \epsilon_{k+1}, \alpha_0 = \epsilon_{k+1} - \epsilon_1.$$

The \diamond action comes from viewing s_i as a reflection in V across the hyperplane through the origin perpendicular to α_i for $i \neq 0$, and across the hyperplane $\{(a_1, \dots, a_{k+1}) \in V : a_1 - a_{k+1} = 1\}$ for s_0 . The collection of $\{\alpha_1, \dots, \alpha_k\}$ are called the *simple roots* for the finite root system of type A .

We let \langle, \rangle denote the inner product on V defined by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{else} \end{cases}$$

The \mathbb{Z} -span of $\{\epsilon_1, \epsilon_2, \dots, \epsilon_{k+1}\}$ contained in the k -dimensional space V are called *weights*.

Let $\Lambda_+ = \{a \in V : a_1 \geq a_2 \geq \dots \geq a_{k+1}\}$ denote the *dominant chamber*. A weight η that is in Λ_+ will be called a dominant weight.

An element of $W_0 \diamond \alpha_1$ is called a *root*. A root α is positive if $\langle \alpha, v \rangle \geq 0$ for all vectors v in the dominant chamber.

Let $A_\emptyset := \{(a_1, \dots, a_{k+1}) \in V : a_1 \geq a_2 \geq \dots \geq a_{k+1} \geq a_1 - 1\}$. This is called the *fundamental alcove*. The action of W on V is faithful on A_\emptyset ($w \diamond A_\emptyset \neq v \diamond A_\emptyset$ for $w \neq v$). For $w \in W$, we define $A_w := w^{-1} \diamond A_\emptyset$. The A_w are called *alcoves*. The union of all alcoves is V and two alcoves overlap on at most an affine hyperplane.

There is another way of calculating the location of A_w given a reduced word of the element $w = s_{i_1} s_{i_2} \dots s_{i_r}$ that we picture as an *alcove walk*.

To a weight $\eta = (\eta_1, \dots, \eta_{k+1})$, we associate a label $L(\eta) = (\sum_{i=1}^{k+1} \eta_i) \bmod (k+1)$. Every alcove contains exactly one weight of every label, corresponding to the vertices of the alcove. Figure 1 shows what this picture looks like in the case of $k = 2$.

The following proposition can be found as Lemma 6.1 in [16].

Proposition 1.1. *Suppose A_w has vertices v_0, v_1, \dots, v_k with $L(v_j) = j$. Then $A_{s_{i_w}}$ is the alcove which has vertices $\{v_j : j \neq i\}$ and the vertex obtained by reflecting v_i across the affine hyperplane spanned by $\{v_j : j \neq i\}$.*

An alcove A_w is contained in Λ_+ if and only if $w \in W^0$ (see [4]). From the correspondence between $(k+1)$ -cores and W^0 , we obtain a correspondence between $(k+1)$ -cores and alcoves of Λ_+ .

Given a word $w = s_{i_1} s_{i_2} \dots s_{i_r}$, the location of A_w is calculated by a path starting at A_\emptyset followed by the alcove $A_{s_{i_r}}$, then

$$A_{s_{i_{r-1}} s_{i_r}}, A_{s_{i_{r-2}} s_{i_{r-1}} s_{i_r}}, \dots, A_{s_{i_1} s_{i_2} \dots s_{i_{r-1}} s_{i_r}}.$$

Each of these alcoves is adjacent due to the previous proposition and the word for w determines a path which travels from the fundamental alcove to A_w traversing a single hyperplane for each simple reflection in the word. See Figure 3 for an example of this.

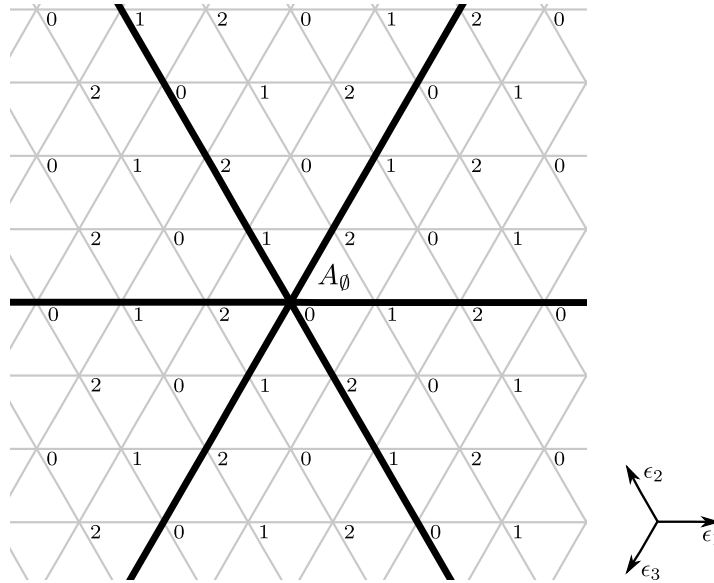


Figure 1: With $k = 2$, a portion of the vector space V divided into alcoves. We have labeled the fundamental alcove A_\emptyset and each of the other alcoves corresponds to an element A_w for $w \in W$.

1.2 The affine nilCoxeter algebra and k -Schur functions

The affine nilCoxeter algebra \mathbb{A} is the algebra generated by u_i for $i \in \{0, 1, \dots, k\}$, subject to the relations (see for instance [5]):

$$\begin{aligned} u_i^2 &= 0 \text{ for } i \in \{0, 1, \dots, k\} \\ u_i u_j &= u_j u_i \text{ if } i - j \neq \pm 1 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \text{ for } i \in \{0, 1, \dots, k\} \end{aligned}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

If $s_{i_1} \dots s_{i_m}$ is a reduced word for an element $w \in W$, we let $\mathbf{u}(w) = u_{i_1} \dots u_{i_m}$. Then $U := \{\mathbf{u}(w) : w \in W\}$ is a basis of \mathbb{A} , which we will call the *standard basis of \mathbb{A}* . For a k -bounded partition $\lambda \in \mathcal{P}^{(k)}$, $u_\lambda := \mathbf{u}(w_\lambda) \in U$. We let $U^0 = \{\mathbf{u}(w) : w \in W^0\}$. We note that $w = xy$ with $\text{len}(w) = \text{len}(x) + \text{len}(y)$ if and only if $\mathbf{u}(w) = \mathbf{u}(x)\mathbf{u}(y)$.

The affine nilCoxeter algebra has an action on the free abelian group with basis the $k + 1$ -cores. Let $\nu \in \mathcal{C}^{(k+1)}$ and then define $u_i \nu$ to be the $k + 1$ core

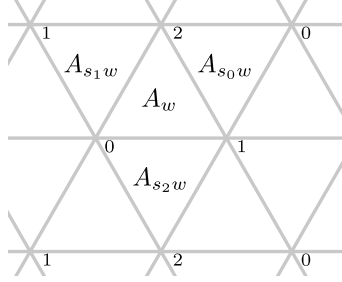


Figure 2: With $k = 2$, a single alcove A_w is adjacent to each of $A_{s_0 w}$, $A_{s_1 w}$ and $A_{s_2 w}$.

formed by adding all addable boxes of content i if ν has at least one such addable box, and $u_i \nu$ is 0 otherwise.

Example 1.2. Let $k + 1 = 5$ and let ν be the 5-core $(6, 4, 3, 1)$.

$$\nu = \begin{array}{|c|c|c|c|c|c|} \hline & 2 & & & & \\ \hline 3 & 4 & 0 & & & \\ \hline 4 & 0 & 1 & 2 & & \\ \hline 0 & 1 & 2 & 3 & 4 & 0 \\ \hline \end{array} \quad u_1 \nu = \begin{array}{|c|c|c|c|c|c|} \hline 1 & & & & & \\ \hline 2 & & & & & \\ \hline 3 & 4 & 0 & 1 & & \\ \hline 4 & 0 & 1 & 2 & & \\ \hline 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ \hline \end{array} \quad u_3 \nu = \begin{array}{|c|c|c|c|c|c|} \hline 2 & 3 & & & & \\ \hline 3 & 4 & 0 & & & \\ \hline 4 & 0 & 1 & 2 & 3 & \\ \hline 0 & 1 & 2 & 3 & 4 & 0 \\ \hline \end{array}$$

Then $u_1 \nu = (7, 4, 4, 1, 1)$, $u_3 \nu = (6, 5, 3, 2)$ and $u_i \nu = 0$ for $i \in \{0, 2, 4\}$.

Within the affine nilCoxeter algebra, Lam [5] found elements \mathbf{h}_i for $1 \leq i \leq k$ which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions h_1, \dots, h_k , or equivalently to a commutative polynomial ring in k variables.

Definition 1.3. An element $u = u_{i_1} u_{i_2} \cdots u_{i_m} \in U$ is said to be cyclically decreasing if each of i_1, \dots, i_m are distinct, and whenever $j = i_s$ and $j+1 = i_t$ then $t < s$ (here $j+1$ is taken modulo $k+1$). To a strict subset $D \subset \{0, 1, \dots, k\}$, we let u_D denote the unique element of U which is cyclically decreasing and is a product of the generators u_m for $m \in D$.

Lam then defines elements $\mathbf{h}_i := \sum_{|D|=i} u_D \in \mathbb{A}$ for $i \in \{0, 1, \dots, k\}$.

Theorem 1.4 (Lam [5] Proposition 8 and Corollary 14). *The \mathbf{h}_i commute with one another (i.e. $\mathbf{h}_i \mathbf{h}_j = \mathbf{h}_j \mathbf{h}_i$) and \mathbf{h}_i for $i \in \{1, 2, \dots, k\}$ generate*

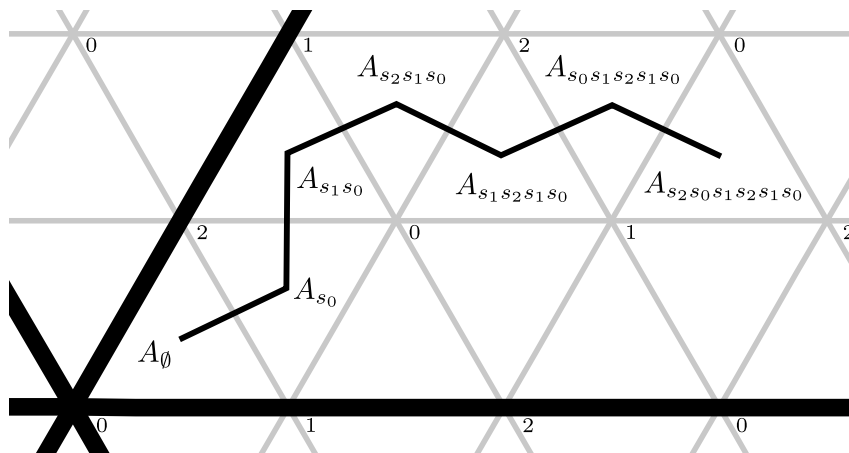


Figure 3: With $k = 2$, an example of a walk represented by a reduced word $w = s_2 s_0 s_1 s_2 s_1 s_0$. The terminal alcove corresponds to the 3-core $s_2 s_0 s_1 s_2 s_1 s_0 \emptyset = (4, 2, 2, 1, 1)$.

a subalgebra isomorphic to the ring generated by the complete homogeneous symmetric functions h_i for $i \in \{1, 2, \dots, k\}$. The isomorphism identifies \mathbf{h}_i and h_i .

One can then define the k -Schur functions.

Definition 1.5. Let $\lambda \in \mathcal{P}^{(k)}$. Then we define $s_\lambda^{(k)}$ to be the unique elements of the subring generated by the \mathbf{h}_i which satisfy the following rule:

$$\mathbf{h}_i s_\lambda^{(k)} = \sum_{\mu} s_\mu^{(k)}; \quad s_\emptyset^{(k)} = 1.$$

where $u_\mu = y u_\lambda$ and y is a cyclically decreasing word of length i .

Remark 1.6. This defining formula for k -Schur functions is called the k -Pieri rule. It is an analogue of the classical Pieri rule for Schur functions, and was first used as a definition of the k -Schur functions in the ring of symmetric functions by Lapointe and Morse in [12].

It is conjecturally equivalent to earlier definitions of the k -Schur functions found in [10] and [11]. The form of the k -Pieri rule we have stated here in terms of the action of the nil-Coxeter algebra follows Lam [5] and is equivalent to the k -Pieri rule of Lapointe and Morse by Proposition 27 of [5].

Remark 1.7. We note that by the definition of \mathbf{h}_i and the action of \mathbb{A} that $\mathbf{h}_i \mathbf{c}(\lambda) = \sum_{\mu} \mathbf{c}(\mu)$, where the sum is indexed by the same conditions as in Definition 1.5.

The following is the start of a running example which demonstrates the formulas presented in this article.

Example 1.8. If the largest hook of λ is sufficiently small, the k -Pieri rule and the usual Pieri rule are the same. In particular, we have that $s_{\lambda}^{(k)} = s_{\lambda}$ whenever $\lambda = \mathbf{c}(\lambda)$.

Let $k = 4$. For example we have the 4-Schur function indexed by $(2, 2, 2)$ has an expansion in terms of the homogeneous symmetric functions given by the Jacobi-Trudi formula.

$$s_{(222)}^{(4)} = \mathbf{h}_{(222)} - 2\mathbf{h}_{(321)} + \mathbf{h}_{(33)} + \mathbf{h}_{(411)} - \mathbf{h}_{(42)}.$$

By definition,

$$\begin{aligned} \mathbf{h}_1 &= u_0 + u_1 + u_2 + u_3 + u_4, \\ \mathbf{h}_2 &= u_1u_0 + u_2u_1 + u_3u_2 + u_4u_3 + u_0u_4 + u_0u_2 + u_0u_3 + u_1u_3 + u_1u_4 + u_2u_4, \\ \mathbf{h}_3 &= u_2u_1u_0 + u_3u_2u_1 + u_4u_3u_2 + u_0u_4u_3 + u_1u_0u_4 + u_1u_0u_3 + u_0u_4u_2 \\ &\quad + u_0u_3u_2 + u_4u_3u_1 + u_4u_2u_1, \\ \mathbf{h}_4 &= u_4u_3u_2u_1 + u_0u_4u_3u_2 + u_1u_0u_4u_3 + u_2u_1u_0u_4 + u_3u_2u_1u_0. \end{aligned}$$

By substituting each of the \mathbf{h}_i in the expansion of $s_{(2,2,2)}^{(4)}$, we see with a bit of cancellation that

$$\begin{aligned} s_{(2,2,2)}^{(4)} &= u_4u_3u_0u_4u_1u_0 + u_2u_4u_3u_0u_4u_1 + u_3u_2u_4u_3u_0u_4 + u_1u_2u_4u_3u_0u_1 \\ &\quad + u_1u_3u_2u_4u_3u_0 + u_0u_1u_2u_4u_0u_1 + u_2u_1u_3u_2u_4u_3 + u_0u_1u_3u_2u_4u_0 \\ &\quad + u_0u_2u_1u_3u_2u_4 + u_1u_0u_2u_1u_3u_2. \end{aligned}$$

2 Four formulas for k -Schur functions indexed by a maximal rectangle

The first explicit formulas for expansions of k -Schur functions in the standard basis of the affine nilCoxeter algebra come from Lam [5], where he gives the formula stated above for \mathbf{h}_i , since $\mathbf{h}_i = s_{(i)}^{(k)}$.

When λ is a hook shape and $\lambda_1 + \text{len}(\lambda) \leq k$, the k -Schur function $s_\lambda^{(k)}$ has an explicit formula in the standard basis of the affine nilCoxeter algebra [1].

An explicit formula in general would give an explicit solution to the k -Littlewood Richardson problem of computing the coefficients $c_{\lambda,\mu}^{\nu,(k)}$ from the expansion $s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda,\mu}^{\nu,(k)} s_\nu^{(k)}$ (this was first realized by Thomas Lam in [5], see Remark 4.4 for more details). The primary contribution of this paper is to give a combinatorial formula for a k -Schur function $s_R^{(k)}$ where R is a maximal rectangle (a rectangular partition with maximal hook length exactly k). It is worth noting that there is only a finite amount of work to determine all of the k -Littlewood Richardson coefficients. This is because maximal rectangles factor out of k -Schur functions (see Theorem 4.1) and there are only finitely many partitions (in fact, $k!$) which do not contain a maximal rectangle, so one is left with the problem of understanding $s_\lambda^{(k)} s_\mu^{(k)}$ when λ and μ contain no maximal rectangle.

In fact, we give four combinatorial formulas for $s_R^{(k)}$. In this section, we explicitly state the four formulas.

We fix two integers $c, r \geq 1$ such that $r + c = k + 1$. Let R be the maximal rectangle $\underbrace{(c, c, \dots, c)}_r = (c^r)$ (a rectangle with c columns and r rows).

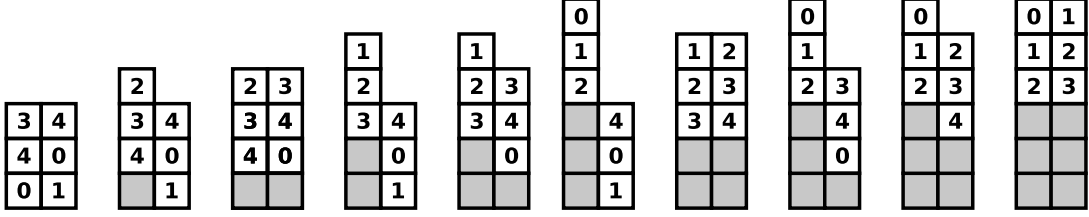
2.1 Def. 1: Partitions contained in the rectangle

For a skew shape ν/μ , let $V_{\nu/\mu} \in W$ be the reading word in the products of the generators s_i of the contents (mod $k + 1$) of the rows of ν/μ starting in the top row and reading from right to left.

For a maximal rectangle $R = (c^r)$, we let $\mathfrak{R} := \{\nu : \nu \subset R\}$ and, in particular, we are interested in the terms $V_{(R,\nu)/\nu}$ for $\nu \in \mathfrak{R}$. It is well known that there are $\binom{k+1}{c} = \binom{k+1}{r}$ such terms.

Definition 2.1. Define $X_R = \sum_{\nu \in \mathfrak{R}} \mathbf{u}(V_{(R,\nu)/\nu})$.

Example 2.2. Let $k = 4$ and $R = (2, 2, 2) = (2^3)$. There are ten partitions contained in \mathfrak{R} . They are $\emptyset, (1), (2), (1, 1), (2, 1), (1, 1, 1), (2, 2), (2, 1, 1), (2, 2, 1), (2, 2, 2)$. These ten partitions are represented by the diagrams



The element $V_{(R, \nu)/\nu}$ is a reading of the cells in white by placing the cells of the partition ν on top of the rectangle R and then reading the rows of the resulting shape $(R, \nu)/\nu$. This defines the following words,

$$\begin{aligned}
V_{(2^3)} &= s_4 s_3 s_0 s_4 s_1 s_0 & V_{(2^3, 1)/(1)} &= s_2 s_4 s_3 s_0 s_4 s_1 \\
V_{(2^4)/(2)} &= s_3 s_2 s_4 s_3 s_0 s_4 & V_{(2^3, 1, 1)/(1, 1)} &= s_1 s_2 s_4 s_3 s_0 s_1 \\
V_{(2^4, 1)/(2, 1)} &= s_1 s_3 s_2 s_4 s_3 s_0 & V_{(2^3, 1^3)/(1^3)} &= s_0 s_1 s_2 s_4 s_0 s_1 \\
V_{(2^5)/(2, 2)} &= s_2 s_1 s_3 s_2 s_4 s_3 & V_{(2^4, 1, 1)/(2, 1, 1)} &= s_0 s_1 s_3 s_2 s_4 s_0 \\
V_{(2^5, 1)/(2, 2, 1)} &= s_0 s_2 s_1 s_3 s_2 s_4 & V_{(2^6)/(2^3)} &= s_1 s_0 s_2 s_1 s_3 s_2.
\end{aligned}$$

Therefore, $X_{(2, 2, 2)} = u_4 u_3 u_0 u_4 u_1 u_0 + u_2 u_4 u_3 u_0 u_4 u_1 + u_3 u_2 u_4 u_3 u_0 u_4 + u_1 u_2 u_4 u_3 u_0 u_1 + u_1 u_3 u_2 u_4 u_3 u_0 + u_0 u_1 u_2 u_4 u_0 u_1 + u_2 u_1 u_3 u_2 u_4 u_3 + u_0 u_1 u_3 u_2 u_4 u_0 + u_0 u_2 u_1 u_3 u_2 u_4 + u_1 u_0 u_2 u_1 u_3 u_2$. This is equal to $s_{(2, 2, 2)}^{(4)}$ from Example 1.8.

2.2 Def. 2: Pseudo-translations in the alcove picture

Let

$$\Gamma = \{(a_1, \dots, a_{k+1}) \in V : a_i \in \{0, 1\}, \sum a_i = c\} \quad (2)$$

and let

$$\Lambda_c = \underbrace{(1, 1, \dots, 1)}_c, \underbrace{(0, 0, \dots, 0)}_{k+1-c} \in \Gamma.$$

Definition 2.3. Let η be a weight. We say $y \in W$ is a pseudo-translation of A_w in direction η if $A_{yw} = A_w + \eta$.

Remark 2.4. If $\alpha = (a_1, \dots, a_{k+1})$ satisfies $\sum_i a_i = 0$ and $a_i \in \mathbb{Z}$ then α is an element of the *root lattice*. The affine Weyl group is the semi direct product of the finite Weyl group and translations by the root lattice (see for instance [2]). For α an element of the root lattice we use t_α to denote a translation by $-\alpha$ (i.e. $t_\alpha \diamond v = v - \alpha$ for all $v \in V$). It is also the case that t_α is a pseudo-translation of A_\emptyset in direction α since $A_{t_\alpha} = t_\alpha^{-1} \diamond A_\emptyset = A_\emptyset + \alpha$. However, it is not the case that t_α acts as a pseudo-translation on all alcoves in the same direction. In other words, t_α is a pseudo-translation of A_w in

direction β (i.e. $A_{t_\alpha w} = A_w + \beta$) for some weight β depending on w and α . This is stated precisely in Lemma 3.3.

Remark 2.5. Throughout the paper we will determine if y is a pseudo-translation of A_w in direction η by taking the centroid G_w of A_w (the average of the vertices of A_w) and checking whether $w^{-1}y^{-1}w \diamond G_w = G_w + \eta$. Since η is a weight, $A_w + \eta$ is an alcove, so it suffices to check what happens to the centroid. Letting G_\emptyset be the centroid of A_\emptyset , we can see the equivalence of the two notions because the centroid G_w of A_w is $w^{-1} \diamond G_\emptyset$ and the centroid of A_{yw} is $w^{-1}y^{-1} \diamond G_\emptyset = w^{-1}y^{-1}ww^{-1} \diamond G_\emptyset = w^{-1}y^{-1}w \diamond G_w$.

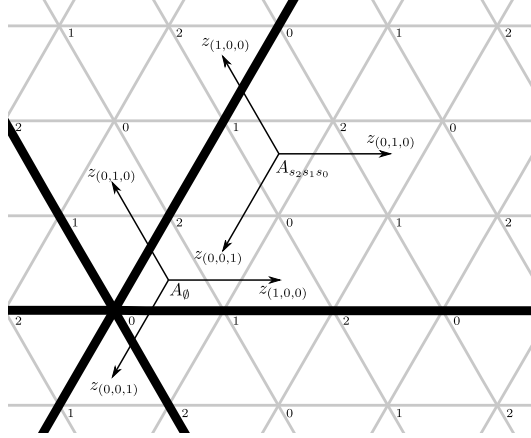
The vertices of the fundamental alcove are 0 and $\Lambda_i = (\underbrace{1, 1, \dots, 1}_i, \underbrace{0, 0, \dots, 0}_{k+1-i})$

for $i \in \{0, 1, \dots, k\}$. The centroid of the fundamental alcove is the average of the vertices and has coordinates

$$G_\emptyset = \frac{1}{k+1} \sum_i \Lambda_i = \left(\frac{k}{k+1}, \frac{k-1}{k+1}, \dots, \frac{1}{k+1}, 0 \right). \quad (3)$$

For a weight γ (in particular for $\gamma \in \Gamma$), we let z_γ denote the pseudo-translation of the fundamental alcove A_\emptyset in direction γ . Note that when z_γ acts on other alcoves it also acts as a pseudo-translation, but perhaps not in the same direction as when it acts on the fundamental alcove (see Lemma 3.3). Note that t_α is equal to z_α for α an element of the root lattice.

Example 2.6. Let $k = 2$ and $R = (1, 1)$. Then $\Gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. The element $z_{(1,0,0)} = s_2s_0$ because $(s_2s_0)^{-1} \diamond A_\emptyset = A_\emptyset + (1, 0, 0)$. Similarly, $z_{(0,1,0)} = s_0s_1$ and $z_{(0,0,1)} = s_1s_2$. In the figure below the arrows represent the action of z_γ when they act on two alcoves A_\emptyset and $A_{s_2s_1s_0}$. Notice that the sum over all the pseudo-translations z_γ for $\gamma \in \Gamma$ has the same effect on all alcoves. The element $z_{(1,0,0)}$ by definition is a pseudo-translation of A_\emptyset in the direction of $(1, 0, 0)$, but $z_{(1,0,0)}$ is also a pseudo-translation of $A_{s_2s_1s_0}$ in the direction of $(0, 1, 0)$ (see Lemma 3.3).



Definition 2.7. Define $Y_R = \sum_{\gamma \in \Gamma} \mathbf{u}(z_\gamma)$.

Remark 2.8. Stated in these terms, the formula Y_R and results in Section 4 can be viewed as a strengthening of Lam's formula in Proposition 4.5 of [7]. His formula is valid for roots and translations while our results in Section 4 show that analogous statements hold for weights and pseudo-translations.

Example 2.9. Continuing Example 2.2 from above, we show here that $z_{(0,0,0,1,1)} = s_2 s_1 s_3 s_2 s_4 s_3$. By Remark 2.5, it is enough to show that

$$(s_2 s_1 s_3 s_2 s_4 s_3)^{-1} \diamond G_\emptyset = G_\emptyset + (0, 0, 0, 1, 1).$$

$$\begin{aligned} (s_2 s_1 s_3 s_2 s_4 s_3)^{-1} \diamond \left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0 \right) &= (s_3 s_4 s_2 s_3 s_1 s_2) \diamond \left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0 \right) \\ &= (s_3 s_4 s_2 s_3 s_1) \diamond \left(\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, 0 \right) = (s_3 s_4 s_2 s_3) \diamond \left(\frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, 0 \right) \\ &= (s_3 s_4 s_2) \diamond \left(\frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5}, 0 \right) = (s_3 s_4) \diamond \left(\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, \frac{3}{5}, 0 \right) = (s_3) \diamond \left(\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, 0, \frac{3}{5} \right) \\ &= \left(\frac{2}{5}, \frac{1}{5}, 0, \frac{4}{5}, \frac{3}{5} \right) = \left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0 \right) + \left(\frac{-2}{5}, \frac{-2}{5}, \frac{-2}{5}, \frac{3}{5}, \frac{3}{5} \right) \\ &\equiv \left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0 \right) + (0, 0, 0, 1, 1) \text{ in } V. \end{aligned}$$

Therefore $s_2 s_1 s_3 s_2 s_4 s_3 = z_{(0,0,0,1,1)}$.

Similarly,

$$\begin{aligned}
s_4s_3s_0s_4s_1s_0 &= z_{(1,1,0,0,0)}, & s_0s_1s_3s_2s_4s_0 &= z_{(1,0,1,0,0)}, & s_0s_1s_2s_4s_0s_1 &= z_{(0,1,1,0,0)}, \\
s_1s_3s_2s_4s_3s_0 &= z_{(1,0,0,1,0)}, & s_1s_2s_4s_3s_0s_1 &= z_{(0,1,0,1,0)}, & s_1s_0s_2s_1s_3s_2 &= z_{(0,0,1,1,0)}, \\
s_3s_2s_4s_3s_0s_4 &= z_{(1,0,0,0,1)}, & s_2s_4s_3s_0s_4s_1 &= z_{(0,1,0,0,1)}, & s_0s_2s_1s_3s_2s_4 &= z_{(0,0,1,0,1)}.
\end{aligned}$$

We compare this calculation with Example 1.8 and find that $s_{(2,2,2)}^{(4)} = X_{(2,2,2)} = Y_{(2,2,2)}$.

2.3 Def. 3: Choosing columns

For A a subset of $\{0, 1, 2, \dots, k\}$ of size c , and $r + c = k + 1$, we set

$$\tilde{v}_A = u_A u_{A+1} u_{A+2} \cdots u_{A+r-1}$$

where $A + d = \{i + d : i \in A\}$.

We can represent this graphically by imagining a cylinder with r rows of $k + 1$ cells as appears in the diagram below. The top row of this cylinder has boxes which are labeled with $\{0, 1, 2, \dots, k\}$ and each subsequent row has the labels increased by 1 just below.

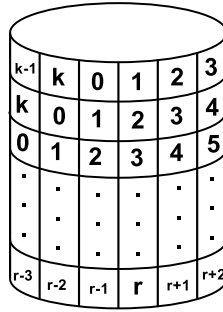


Figure 4: A cylinder divided into r rows of $k + 1$ cells.

From this cylinder we will choose c columns in all possible ways. For each choice of columns we read the rows starting with the top row. A reading of the row will begin at one of the non-chosen columns and proceed clockwise with the top of the cylinder as the point of reference (or from right to left as the face of cylinder is laid flat). The entries in selected columns are part of the word, the others are not.

Definition 2.10. Define $Z_R = \sum_{A \in \binom{[k+1]}{c}} \tilde{v}_A$.

Example 2.11. Fix $k = 4$ and again let $R = (2, 2, 2)$. For each subset of size 2 from $\{0, 1, 2, 3, 4\}$ we fix a 3×5 rectangle that we imagine lies on a cylinder.

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1

The reading words for these pictures are (respectively) given by the following elements.

$$\begin{aligned}
 \tilde{v}_{\{0,1\}} &= u_1 u_0 u_2 u_1 u_3 u_2 & \tilde{v}_{\{0,2\}} &= u_2 u_0 u_3 u_1 u_4 u_2 & \tilde{v}_{\{0,3\}} &= u_3 u_0 u_4 u_1 u_0 u_2 \\
 \tilde{v}_{\{0,4\}} &= u_0 u_4 u_1 u_0 u_2 u_1 & \tilde{v}_{\{1,2\}} &= u_2 u_1 u_3 u_2 u_4 u_3 \\
 \tilde{v}_{\{1,3\}} &= u_3 u_1 u_4 u_2 u_0 u_3 & \tilde{v}_{\{1,4\}} &= u_4 u_1 u_0 u_2 u_1 u_3 & \tilde{v}_{\{2,3\}} &= u_3 u_2 u_4 u_3 u_0 u_4 \\
 \tilde{v}_{\{2,4\}} &= u_4 u_2 u_0 u_3 u_1 u_4 & \tilde{v}_{\{3,4\}} &= u_4 u_3 u_0 u_4 u_1 u_0
 \end{aligned}$$

This calculation can again be compared with Example 1.8 and we observe that $s_{(2,2,2)}^{(4)} = X_{(2,2,2)} = Y_{(2,2,2)} = Z_{(2,2,2)}$.

2.4 Def. 4: Windows

The affine symmetric group W can also be thought of (see [2]) as the group of bijections w on \mathbb{Z} for which

1. $w(i + k + 1) = w(i) + k + 1$ for all $i \in \mathbb{Z}$;
2. $\sum_{i=1}^{k+1} w(i) = \binom{k+2}{2}$.

The bijections corresponding to the generators s_i are given by

$$s_i(j) = \begin{cases} j + 1 & \text{if } j \equiv i; \\ j - 1 & \text{if } j \equiv i + 1; \\ j & \text{otherwise.} \end{cases}$$

To describe the bijection, it is enough to understand what w does to $\{1, \dots, k+1\}$. So we identify $\mathbf{win}(w) = [w(1), w(2), \dots, w(k+1)]$. This notation is analogous to one line notation in the symmetric group and is called *window notation*.

For a c element subset B of $\{1, \dots, k+1\}$, we let j_B denote the element of W with window

$$\mathbf{win}(j_B) = [1-r\delta_{1 \in B} + c\delta_{1 \notin B}, 2-r\delta_{2 \in B} + c\delta_{2 \notin B}, \dots, k+1-r\delta_{(k+1) \in B} + c\delta_{(k+1) \notin B}].$$

Definition 2.12. Define $W_R = \sum_{B \in \binom{[k+1]}{c}} \mathbf{u}(j_B)$.

Example 2.13. Continuing our comparable running example with $k = 4$, $r = 3$ and $c = 2$,

$$\binom{[5]}{2} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$

The windows for these elements are given by

$$\begin{aligned} \mathbf{win}(j_{\{1,2\}}) &= [1-3, 2-3, 3+2, 4+2, 5+2] = [-2, -1, 5, 6, 7] \\ \mathbf{win}(j_{\{1,3\}}) &= [1-3, 2+2, 3-3, 4+2, 5+2] = [-2, 4, 0, 6, 7] \\ \mathbf{win}(j_{\{1,4\}}) &= [1-3, 2+2, 3+2, 4-4, 5+2] = [-2, 4, 5, 1, 7] \\ \mathbf{win}(j_{\{1,5\}}) &= [1-3, 2+2, 3+2, 4+2, 5-3] = [-2, 4, 5, 6, 2] \\ \mathbf{win}(j_{\{2,3\}}) &= [1+2, 2-3, 3-3, 4+2, 5+2] = [3, -1, 0, 6, 7] \\ \mathbf{win}(j_{\{2,4\}}) &= [1+2, 2-3, 3+2, 4-3, 5+2] = [3, -1, 5, 1, 7] \\ \mathbf{win}(j_{\{2,5\}}) &= [1+2, 2-3, 3+2, 4+2, 5-3] = [3, -1, 5, 6, 2] \\ \mathbf{win}(j_{\{3,4\}}) &= [1+2, 2+2, 3-3, 4-3, 5+2] = [3, 4, 0, 1, 7] \\ \mathbf{win}(j_{\{3,5\}}) &= [1+2, 2+2, 3-3, 4+2, 5-3] = [3, 4, 0, 6, 2] \\ \mathbf{win}(j_{\{4,5\}}) &= [1+2, 2+2, 3+2, 4-3, 5-3] = [3, 4, 5, 1, 2]. \end{aligned}$$

We can check that

$$s_2 s_1 s_3 s_2 s_4 s_3(1) = 3, s_2 s_1 s_3 s_2 s_4 s_3(2) = 4, s_2 s_1 s_3 s_2 s_4 s_3(3) = 5,$$

$$s_2 s_1 s_3 s_2 s_4 s_3(4) = 1, s_2 s_1 s_3 s_2 s_4 s_3(5) = 2.$$

Therefore $\mathbf{win}(j_{\{4,5\}}) = \mathbf{win}(s_2 s_1 s_3 s_2 s_4 s_3) = [3, 4, 5, 1, 2]$. Similarly:

$$j_{\{1,2\}} = s_4 s_3 s_0 s_4 s_1 s_0, j_{\{1,3\}} = s_0 s_1 s_3 s_2 s_4 s_0, j_{\{2,3\}} = s_0 s_1 s_2 s_4 s_0 s_1,$$

$$\begin{aligned} \dot{j}_{\{1,4\}} &= s_1 s_3 s_2 s_4 s_3 s_0, & \dot{j}_{\{2,4\}} &= s_1 s_2 s_4 s_3 s_0 s_1, & \dot{j}_{\{3,4\}} &= s_1 s_0 s_2 s_1 s_3 s_2, \\ \dot{j}_{\{1,5\}} &= s_3 s_2 s_4 s_3 s_0 s_4, & \dot{j}_{\{2,5\}} &= s_2 s_4 s_3 s_0 s_4 s_1, & \dot{j}_{\{3,5\}} &= s_0 s_2 s_1 s_3 s_2 s_4. \end{aligned}$$

Again comparing this calculation with Example 1.8 we see that $s_{(2,2,2)}^{(4)} = X_{(2,2,2)} = Y_{(2,2,2)} = Z_{(2,2,2)} = W_{(2,2,2)}$.

3 Equivalence of definitions

3.1 Equivalence of def. 1 and def. 2

We start by defining a new action of W on V :

$$\begin{aligned} s_i \star (a_1, \dots, a_{k+1}) &= (a_1, \dots, a_{i+1}, a_i, \dots, a_{k+1}) \text{ for } i \neq 0 \\ s_0 \star (a_1, \dots, a_{k+1}) &= (a_{k+1}, a_2, \dots, a_k, a_1) \end{aligned}$$

This action is connected with the \diamond action in the following way.

Lemma 3.1. *Let $w \in W$. Then $w \diamond (a + b) = w \diamond a + w \star b$.*

Remark 3.2. From the definition of \star and that of Γ (equation (2)), we see that $w \star \Gamma = \Gamma$ for $w \in W$.

We have already defined $z_\gamma \in W$ to be the element of the Weyl group which acts on the fundamental alcove as a pseudo-translation by $\gamma \in \Gamma$. The next lemma shows that z_γ is a pseudo-translation on all alcoves, but in different directions.

Lemma 3.3. *For $w \in W$ and a weight γ , z_γ is a pseudo-translation of A_w by $w^{-1} \star \gamma$.*

Proof. Let G_w be the centroid of A_w . Then $w^{-1} \diamond G_\emptyset = G_w$. We compute:
 $(w^{-1} z_\gamma^{-1} w) \diamond G_w = (w^{-1} z_\gamma^{-1} w) w^{-1} \diamond G_\emptyset = w^{-1} z_\gamma^{-1} \diamond G_\emptyset = w^{-1} \diamond (G_\emptyset + \gamma) = w^{-1} \diamond G_\emptyset + w^{-1} \star \gamma = G_w + w^{-1} \star \gamma$. By Remark 2.5, the statement follows. \square

Definition 3.4. For $w \in W$, $w = s_{i_1} \cdots s_{i_k}$, we let $w^{(m)} = s_{i_1+m} \cdots s_{i_k+m}$.

Lemma 3.5. *For $x, y \in W$, if $A_x = A_y + \eta$ for some weight η , then for any word $w \in W$, $A_{w^{(m)}x} = A_{wy} + \eta$ where $m = L(\eta)$.*

Proof. It is enough to show this when $w = s_i$ is a single generator (if w is of longer length, then repeated application of the single generator case yields the result). The vertex with label i will change when we pass from A_y to $A_{s_i y}$; all others are fixed. Suppose that $\beta \in A_y$ and $s_i \diamond \beta \in A_{s_i y}$ are the vertices of A_y and $A_{s_i y}$ (respectively) of label i . It is enough to show that $s_{i+L(\eta)} \diamond (\beta + \eta) = (s_i \diamond \beta) + \eta$.

However $L(\beta + \eta) = i + L(\eta)$. Therefore the vertices β when acted on by s_i and $\beta + \eta$ when acted on by $s_{i+L(\eta)}$ are reflected across parallel affine hyperplanes, which implies that $s_{i+L(\eta)} \diamond (\beta + \eta) = (s_i \diamond \beta) + \eta$. \square

Definition 3.6. Let W_c denote the subgroup of W without generator s_c , let $W_{c,0}$ denote the subgroup without s_0 and s_c , and let W_c^0 denote the minimal length coset representatives of $W_c/W_{c,0}$.

Lemma 3.7. For $\nu \in \mathfrak{R}$, $V_{(R,\nu)/\nu}$ is a pseudo-translation of A_{w_ν} by Λ_c .

Proof. We first show this for $\nu = \emptyset$. V_R is the longest word in W_c^0 . We see that, for $w \in W^0$, we have $w \in W_c^0$ if and only if A_w has Λ_c as a vertex (this is because the fundamental alcove has Λ_c as a vertex, and no s_c appearing in w implies that the vertex with label c in A_w is always Λ_c). The longest word in W_c^0 corresponds to the alcove furthest away from the fundamental alcove inside the dominant chamber and having Λ_c as a vertex. This is the alcove $A_\emptyset + \Lambda_c$.

Now let $\nu \in \mathfrak{R}$. It is easy to see that $V_{R/\nu} w_\nu = V_R$ and $V_{(R,\nu)/\nu} = w_\nu^{(c)} V_{R/\nu}$. Also $A_{V_R} = A_{V_{R/\nu} w_\nu} = A_\emptyset + \Lambda_c$ by the first paragraph. Applying Lemma 3.5 we then get that $A_{V_{(R,\nu)/\nu} w_\nu} = A_{w_\nu^{(c)} V_{R/\nu} w_\nu} = A_{w_\nu} + \Lambda_c$, since $L(\Lambda_c) = c$. \square

Lemma 3.8. $\nu \in \mathfrak{R}$ if and only if $w_\nu \in W_c^0$.

Proof. It is enough to notice that an $r \times c$ rectangle will contain no cell of content c , so a partition ν is contained in R if and only if w_ν has no s_c generator. \square

Theorem 3.9. For $R = (c^r)$ with $r + c = k + 1$, $X_R = Y_R$.

Proof. It is enough to find a bijection $\psi : \mathfrak{R} \rightarrow \Gamma$ such that $V_{(R,\nu)/\nu} = z_{\psi(\nu)}$.

Define $\psi(\nu) := w_\nu \star \Lambda_c$. The map $\star \Lambda_c : W_c \rightarrow \Gamma$, sending w to $w \star \Lambda_c$, is onto. The fixed point set of this map, $Fix(\star \Lambda_c) = \{w \in W_c : w \star \Lambda_c = \Lambda_c\}$, is exactly $W_{c,0}$. Therefore $\star \Lambda_c : W_c^0 \rightarrow \Gamma$ is a bijection. By Lemma 3.8 ψ is also a bijection.

By Lemma 3.7, $A_{V_{(R,\nu)}/\nu} w_\nu = A_{w_\nu} + \Lambda_c$. Applying $w_\nu \diamond$ to this equation yields

$$A_{V_{(R,\nu)}/\nu} = w_\nu \diamond A_{V_{(R,\nu)}/\nu} w_\nu = w_\nu \diamond (A_{w_\nu} + \Lambda_c) = A_\emptyset + w_\nu \star \Lambda_c = A_\emptyset + \psi(\nu).$$

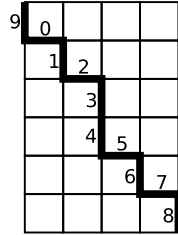
This shows that $V_{(R,\nu)}/\nu$ is the pseudo-translation of A_\emptyset by $\psi(\nu)$. Hence $V_{(R,\nu)}/\nu = z_{\psi(\nu)}$. \square

3.2 Equivalence of def. 1 and def. 3

For a partition $\nu \subseteq (c^r)$, there is a lattice path of length $k + 1$ consisting of the edges in the rectangle (c^r) which trace the path just above ν . Label the edges of this path by equivalence classes of integers modulo $k + 1$, by starting from $-2r + 1 \pmod{k + 1}$ in the upper left hand corner and increasing by 1 with each step down or to the right. Let $\phi(\nu)$ be the set of labels on the horizontal edges.

Alternatively, set $\phi(\nu) = \bigcup_{j=0}^r \{i : \nu_{r-j} - 2r + 1 + j > i \geq \nu_{r-j+1} - 2r + 1 + j\}$ with all entries taken $\pmod{k + 1}$ where we set $\nu_0 = c$ and $\nu_d = 0$ for $d > \text{len}(\nu)$. Note that these entries are also the contents of the cells in the tops of the columns of $(R, \nu)/\nu$. If we define an element of \mathbb{A} by reading the label i as indexing the element u_i , and we read in our standard order, the result is a word for $u_{\phi(\nu)}$ since each of the numbers 0 through k appear exactly once along this path and it is easy to check that if i and $i + 1$ appear in $\phi(\nu)$, then $i + 1$ appears before i .

Example 3.10. Let $r = 6$ and $c = 4$ so that $k + 1 = 6 + 4 = 10$. Let $\nu = (4, 3, 2, 2, 1)$. The path corresponding to ν is labelled as in the diagram below, starting at $-2 \cdot r + 1 = -11 \equiv 9 \pmod{10}$.



In this case we find $\phi((4, 3, 2, 2, 1)) = \{-10, -8, -5, -3\} \equiv \{0, 2, 5, 7\} \pmod{10}$.

Theorem 3.11. For $R = (c^r)$ with $r + c = k + 1$ and ν a partition which is contained in R ,

$$\mathbf{u}(V_{(R,\nu)/\nu}) = \tilde{v}_{\phi(\nu)} .$$

As a consequence, $X_R = Z_R$.

Proof. Let $\tilde{\nu} = (R, \nu)$. For $0 \leq a \leq r$, consider $\tilde{\nu}^{(a)} = (c^{r-a}, \nu)$ (so that by definition, $V_{(R,\nu)/\nu} = V_{\tilde{\nu}^{(0)}/\nu}$). We will show that

$$\mathbf{u}(V_{\tilde{\nu}^{(a)}/\nu}) = u_{\phi(\nu)+a} \mathbf{u}(V_{\tilde{\nu}^{(a+1)}/\nu})$$

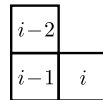
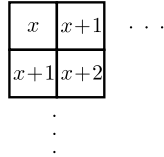
for $0 \leq a < r-1$, and hence $\mathbf{u}(V_{(R,\nu)/\nu}) = \mathbf{u}(V_{\tilde{\nu}/\nu}) = u_{\phi(\nu)} u_{\phi(\nu)+1} \cdots u_{\phi(\nu)+r-1} = \tilde{v}_{\nu}$.

We note that since $\phi(\nu)$ is the set of contents of the highest cells of each of the columns of $\tilde{\nu} = \tilde{\nu}^{(0)}$, then $\phi(\nu) + a$ are the contents of the highest cells in each of the columns of $\tilde{\nu}^{(a)}$.

Since the word $V_{\tilde{\nu}^{(r-1)}/\nu}$ is a cyclically decreasing reading of the elements of $\phi(\nu) + r - 1$, we have that $\mathbf{u}(V_{\tilde{\nu}^{(r-1)}/\nu}) = u_{\phi(\nu)+r-1}$.

For $0 \leq a < r - 1$, consider each of the cells $p \in \tilde{\nu}^{(a)}/\tilde{\nu}^{(a+1)}$. Say that p has content $i \pmod{k+1}$. We need to show that if p is not in the highest row of $\tilde{\nu}^{(a)}$, then u_i commutes with all u_j which correspond to cells (of content j) in $\tilde{\nu}^{(a+1)}/\nu$ and lie to the left of u_i in $\mathbf{u}(V_{\tilde{\nu}^{(a)}/\nu})$. The u_j that lie to the left of the letter corresponding to p in $\mathbf{u}(V_{\tilde{\nu}^{(a)}/\nu})$ are those that correspond to cells in $\tilde{\nu}^{(a+1)}/\nu$ which are strictly to the left and strictly above p .

Consider a cell in $\tilde{\nu}^{(a+1)}/\nu$ that is strictly above and strictly to the left of p and let j be the content of this cell. Let x be the content of the cell in the first column of $\tilde{\nu}^{(a)}/\tilde{\nu}^{(a+1)}$. We see this on the clipped picture of the cells of $\tilde{\nu}^{(a)}$ where the content increases (mod $(k+1)$) as we move down and to the right between the cell with content x and p with content i .



Therefore j is in the interval $\{x + 1, x + 2, \dots, i - 2\} \pmod{k + 1}$. Note that the Manhattan metric distance from the cell p to the cell at the top of the first column of $\tilde{\nu}^{(a)}$ is less than $k + 1$. It follows that j differs from i by at least 2 $\pmod{k + 1}$ and hence u_j commutes with u_i .

We conclude that since each of the letters u_i commutes with the letters that are read before u_i in $\mathbf{u}(V_{\tilde{\nu}^{(a)}/\nu})$ and correspond to cells in $\tilde{\nu}^{(a+1)}/\nu$ that it is possible to factor to the left the terms in $u_{\phi(\nu)+a}$ and hence $\mathbf{u}(V_{\tilde{\nu}^{(a)}/\nu}) = u_{\phi(\nu)+a}\mathbf{u}(V_{\tilde{\nu}^{(a+1)}/\nu})$. \square

3.3 Equivalence of def. 2 and def. 4

The following lemma is stated in Section 8.3 of [2].

Lemma 3.12. *Suppose $\mathbf{win}(w) = [a_1, \dots, a_{k+1}]$. Then*

$$\mathbf{win}(ws_i) = [a_1, \dots, a_{i+1}, a_i, \dots, a_{k+1}] \text{ for } i \neq 0;$$

$$\mathbf{win}(ws_0) = [a_{k+1} - (k + 1), a_2, \dots, a_k, a_1 + (k + 1)] \text{ for } i = 0.$$

The relationship between windows and centroids is made precise in Lemma 3.16.

Definition 3.13. If $w = s_{i_1}s_{i_2}\dots s_{i_m}$, then let $\hat{w} = s_{-i_1}s_{-i_2}\dots s_{-i_m}$.

Definition 3.14. For a sequence $\mathbf{a} = (a_1, a_2, \dots, a_k, a_{k+1})$, let $rev(\mathbf{a}) = (a_{k+1}, a_k, \dots, a_2, a_1)$.

Remark 3.15. There is a map between weights and windows. For $v \in V$, there is a unique vector $y \in \mathbb{R}^{k+1}$ in the equivalence class for v which also satisfies $\sum_i y_i = \binom{k+2}{2}$. We associate to the vector v the window $[v] := [y_1, y_2, \dots, y_{k+1}]$.

Lemma 3.16. *For $w \in W$, $\mathbf{win}(w) = [(k + 1)rev(G_{\hat{w}})]$.*

Proof. The proof is by induction on the length of w . If $len(w) = 0$, then $w = 1$ and $[(k + 1)rev(G_{\emptyset})] = [(0, 1, \dots, k)] = [1, 2, \dots, k + 1]$, which is the window of the identity element.

Now assume that the statement is true for w and we compute the window corresponding to ws_i ,

$$\begin{aligned}
[(k+1)rev(G_{\widehat{ws_i}})] &= [(k+1)rev(\widehat{ws_i}^{-1} \diamond G_\emptyset)] \\
&= [(k+1)rev(s_{-i}\widehat{w}^{-1} \diamond G_\emptyset)] \\
&= [s_i \diamond (k+1)rev(G_{\widehat{w}}) + \delta_{i,0}(-k, 0, \dots, 0, +k)] \\
&= [s_i \diamond (w(1), \dots, w(k+1)) + \delta_{i,0}(-k, 0, \dots, 0, +k)] \\
&= \mathbf{win}(ws_i)
\end{aligned}$$

by induction and Lemma 3.12. □

Lemma 3.17. *There is a bijection between*

$$\Gamma = \{\gamma \in V : \sum_i \gamma_i = c, \gamma_i \in \{0, 1\}\}$$

and

$$\Gamma^t := \{\gamma \in V : \sum_i \gamma_i = r, \gamma_i \in \{0, 1\}\}$$

$\tau : \Gamma \rightarrow \Gamma^t$ such that if $\gamma \in \Gamma$, then $\widehat{z}_\gamma = z_{\tau(\gamma)}$.

Proof. If ν is a partition, let ν^t denote the transpose of ν . Let \mathfrak{R}^t denote the set of partitions $\{\nu^t : \nu \in \mathfrak{R}\}$.

By Theorem 3.9, there is a bijection ψ from \mathfrak{R} to Γ which satisfies $V_{(R,\nu)/\nu} = z_{\psi(\nu)}$. There is also an analogous bijection ψ^t from \mathfrak{R}^t to Γ^t which satisfies that $V_{(R^t,\nu^t)/\nu^t} = z_{\psi^t(\nu^t)}$ for $\nu \in \mathfrak{R}$.

Now, let $\gamma \in \Gamma$. Let $\nu = \psi^{-1}(\gamma)$. We have that $z_\gamma = V_{(R,\nu)/\nu}$. Thinking of ν as a $(k+1)$ core, we can act on it by z_γ . We obtain $z_\gamma \nu = (R + \nu, \nu)$, where we write $R + \nu$ for the partition $(c + \nu_1, \dots, c + \nu_r)$.

It follows that $\widehat{z}_\gamma \nu^t = (R^t + \nu^t, \nu^t)$, and thus (reversing the previous argument) that $\widehat{z}_\gamma = V_{(R^t,\nu^t)/\nu^t}$. Thus $\widehat{z}_\gamma = z_{\psi^t(\psi^{-1}(\gamma)^t)}$. We may therefore take $\tau(\gamma) = \psi^t(\psi^{-1}(\gamma)^t)$; it is clear that this is a bijection. □

Theorem 3.18. *For $R = (c^r)$ with $r + c = k + 1$, $Y_R = W_R$.*

Proof. Let $\gamma \in \Gamma$. Let $B = \{i : rev(\tau(\gamma))_i = 1\}$. We will show that $\mathbf{win}(z_\gamma) = \mathbf{win}(j_B)$.

We know that

$$\mathbf{win}(z_\gamma) = [(k+1)rev(G_{\widehat{z}_\gamma})]$$

by Lemma 3.16. By Lemma 3.17, $\widehat{z}_\gamma = z_{\tau(\gamma)}$, so

$$\mathbf{win}(z_\gamma) = [(k+1)\text{rev}(G_\emptyset + \tau(\gamma))] = [(k+1)\text{rev}(G_\emptyset) + (k+1)\text{rev}(\tau(\gamma))].$$

By direct computation, $(k+1)\text{rev}(G_\emptyset) + (1, 1, \dots, 1) = (1, 2, \dots, k+1)$, which sums to $\binom{k+2}{2}$, and $[v] = [v + (1, 1, \dots, 1)]$ by the definition of $[v]$. The term $(k+1)\text{rev}(\tau(\gamma))$ has $k+1$ in the positions indexed by B and 0 elsewhere. It follows that $(k+1)\text{rev}(\tau(\gamma)) - r(1, 1, \dots, 1)$ has c in the positions indexed by B and $-r$ in the other positions, and that this vector sums to zero. Therefore, $\mathbf{win}(j_B) = \mathbf{win}(z_\gamma)$, and we are done. \square

4 Proof of Main Theorem

Maximal rectangles were studied by Lapointe and Morse in [13, Theorem 40] where they showed the following property:

Theorem 4.1. *For a k -bounded partition λ , if R is a maximal rectangle then $s_\lambda^{(k)} s_R^{(k)} = s_{\lambda \cup R}^{(k)}$, where $\lambda \cup R$ is the partition obtained by combining and sorting the parts of R and λ .*

Lemma 4.2. *For $\lambda \in \mathcal{P}^{(k)}$, $s_\lambda^{(k)}(\emptyset) = \mathbf{c}(\lambda)$.*

Proof. By Remark 1.7,

$$\mathbf{h}_i \mathbf{c}(\lambda) = \sum_{\mu} \mathbf{c}(\mu),$$

where the sum is over the same set of objects as described in Definition 1.5. Therefore $s_\lambda^{(k)} \mathbf{c}(\mu) = \sum_{\nu} c_{\lambda, \mu}^{\nu, (k)} \mathbf{c}(\nu)$. In particular, $s_\lambda^{(k)}(\emptyset) = \mathbf{c}(\lambda)$, since $c_{\lambda, \emptyset}^{\nu, (k)} = \delta_{\lambda, \nu}$. \square

Lemma 4.3. *Let $\mu, \nu \in \mathcal{P}^{(k)}$. There is at most one $w \in W$ such that $w\mathbf{c}(\mu) = \mathbf{c}(\nu)$.*

Proof. If $w\mathbf{c}(\mu) = w'\mathbf{c}(\mu) = \mathbf{c}(\nu)$, then $ww_\mu = w_\nu = w'w_\mu$, so $w = w'$. \square

Remark 4.4. [5, Proposition 42] An explicit expression for $s_\lambda^{(k)}$ in terms of the standard basis of the nil-Coxeter algebra would give a combinatorial method of obtaining the k -Littlewood-Richardson coefficients. If $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}(w)$ then Lemma 4.2 implies that the coefficient of $s_\nu^{(k)}$ in the product $s_\lambda^{(k)} s_\mu^{(k)}$ would be c_w if there exists a w such that $w\mathbf{c}(\mu) = \mathbf{c}(\nu)$ and zero otherwise. By Lemma 4.3, there can be at most one such w .

Lemma 4.5. (see for instance [17]) *Minimal length expressions of $w \in W$ correspond to alcove walks which do not cross the same affine hyperplane twice.*

Proposition 4.6. *If $w \in W^0$ such that $A_{xw} = A_w + \eta$ for $x \in W$ and for some weight η , then $\eta \in \Lambda_+$ if and only if $\mathbf{u}(x)\mathbf{u}(w)\emptyset \neq 0$.*

Proof. Let \mathcal{J} be the set of reflection hyperplanes which separate A_w from A_\emptyset , and let \mathcal{K} be the set of reflection hyperplanes which separate A_w and A_{xw} . The set \mathcal{J} consists of the hyperplanes crossed by a reduced alcove walk from A_\emptyset to A_w , while \mathcal{K} consists of the hyperplanes crossed by a reduced alcove walk from w to xw .

Write $\underline{x}, \underline{w}$ for reduced expressions for x and w . Then \underline{xw} corresponds to an alcove walk from A_\emptyset to A_{xw} passing through A_w . Therefore $\mathbf{u}(x)\mathbf{u}(w) \neq 0$ if and only if this alcove walk is reduced if and only if $\mathcal{J} \cap \mathcal{K} = \emptyset$ by Lemma 4.5.

For a positive root α , we let \mathcal{H}_α denote the set of reflection hyperplanes perpendicular to α . If we write $H_{\alpha,i} = \{x : \langle \alpha, x \rangle = i\}$ then $\mathcal{H}_\alpha = \{H_{\alpha,i} : i \in \mathbb{Z}\}$.

Since w is in the dominant chamber,

$$\mathcal{J} \cap \mathcal{H}_\alpha = \{H_{\alpha,i} : \langle \alpha, G_\emptyset \rangle < i < \langle \alpha, G_w \rangle\}.$$

Similarly, $\mathcal{K} \cap \mathcal{H}_\alpha$ consists of those $H_{\alpha,i}$ with i between $\langle \alpha, G_w \rangle$ and $\langle \alpha, G_w + \eta \rangle$.

If η is dominant, then $\langle \alpha, \eta \rangle \geq 0$ for all positive roots α , and thus $\mathcal{J} \cap \mathcal{H}_\alpha$ and $\mathcal{K} \cap \mathcal{H}_\alpha$ are disjoint for all α , so \mathcal{J} and \mathcal{K} are disjoint, and thus $\mathbf{u}(x)\mathbf{u}(w) = \mathbf{u}(xw)$, and $xw \in W^0$, so $\mathbf{u}(xw)\emptyset \neq 0$.

Contrariwise, if η is not dominant, then there exists a positive root α , which we may take to be simple, such that $\langle \alpha, \eta \rangle < 0$. If $\langle \alpha, G_w \rangle > 1$, then it follows that $\mathcal{K} \cap \mathcal{H}_\alpha$ and $\mathcal{J} \cap \mathcal{H}_\alpha$ are not disjoint, so $\mathbf{u}(x)\mathbf{u}(w) = 0$. If, on the other hand, $0 < \langle \alpha, G_w \rangle < 1$, then A_{xw} is not dominant, so $\mathbf{u}(xw)\emptyset = 0$. \square

Lemma 4.7. *Let $w \in W^0$, and let $\alpha \in \Lambda_+$ be an element of the root lattice. Then $\mathbf{u}(w)\mathbf{u}(t_\alpha)\emptyset \neq 0$.*

Proof. $\mathbf{u}(t_\alpha)\emptyset \neq 0$ since $\alpha \in \Lambda_+$. Any reduced word for w determines an alcove walk from A_{t_α} to A_{wt_α} . This alcove walk is the translate by α of the

corresponding reduced walk from A_\emptyset to A_w ; it therefore crosses no affine hyperplane more than once. Further, as in the proof of the previous proposition, we see that, for any positive root β , we have

$$\langle \beta, G_\emptyset \rangle \leq \langle \beta, G_{t_\alpha} \rangle \leq \langle \beta, G_{wt_\alpha} \rangle,$$

from which it follows that no hyperplane perpendicular to β is crossed by both a reduced walk from G_\emptyset to G_{t_α} and a reduced walk from G_{t_α} to G_{wt_α} . By Lemma 4.5, $\mathbf{u}(w)\mathbf{u}(t_\alpha)\emptyset = \mathbf{u}(wt_\alpha)\emptyset \neq 0$. \square

Lemma 4.8. *For every $w \in W$, there is a $\lambda \in \mathcal{C}^{(k+1)}$ such that $\mathbf{u}(w)\lambda \neq 0$.*

Proof. Let $\alpha = 2(\Lambda_1 + \Lambda_2 + \cdots + \Lambda_k)$. Since $\Lambda_i = \sum_{j=1}^i \epsilon_j$ we have that $\alpha = k\epsilon_1 + (k-2)\epsilon_2 + \cdots + (-k)\epsilon_{k+1}$ as an element of V so we conclude that α is both a dominant weight and an element of the root lattice. Let $t_\alpha \in W$ denote the translation corresponding to α (i.e. $t_\alpha v = v - \alpha$ for all $v \in V$). Let $w = xy$ for $x \in W^0$ and $y \in W_0$.

The alcove $A_{y^{-1}t_\alpha} = t_\alpha^{-1} \diamond A_{y^{-1}} = A_{y^{-1}} + \alpha$. Since $y \in W_0$, $A_{y^{-1}}$ contains the vertex 0, so $A_{y^{-1}t_\alpha}$ has α as a vertex. The dominant weight α is in the interior of the dominant chamber (i.e. α is not on the wall of the dominant chamber), so the alcove $A_{y^{-1}t_\alpha}$ is contained in the dominant chamber.

Since $A_{y^{-1}t_\alpha}$ is in the dominant chamber, it is associated to a $(k+1)$ -core $\lambda := y^{-1}t_\alpha\emptyset$. We claim that $\mathbf{u}(w)\lambda = \mathbf{u}(x)\mathbf{u}(y)\lambda \neq 0$.

We claim that $\text{len}(s_i t_\alpha) < \text{len}(t_\alpha)$ for all $i \neq 0$. The fundamental weights Λ_i form a dual basis with the simple roots α_j (i.e. $\langle \Lambda_i, \alpha_j \rangle = \delta_{i,j}$). Since A_{t_α} is a translate of A_\emptyset , the wall which separates A_{t_α} and $A_{s_i t_\alpha}$ is a translate of the wall which separates A_\emptyset and A_{s_i} . The latter is the wall $\{\nu : \langle \nu, \alpha_i \rangle = 0\}$, so the former is the wall $\{\nu + \alpha : \langle \nu, \alpha_i \rangle = 0\} = \{\nu : \langle \nu, \alpha_i \rangle = 2\}$ since $\langle \alpha, \alpha_i \rangle = 2$. The fundamental alcove satisfies $0 \leq \langle A_\emptyset, \alpha_i \rangle \leq 1$, so the wall in question separates A_\emptyset and A_{t_α} and hence $A_{s_i t_\alpha}$ is on the same side of the hyperplane $\{\nu : \langle \nu, \alpha_i \rangle = 2\}$ as A_\emptyset , so $\text{len}(s_i t_\alpha) < \text{len}(t_\alpha)$.

Let w_0 be the longest element of W_0 . Since w_0 is characterized as the element for which $\text{len}(s_i w_0) < \text{len}(w_0)$ for each $1 \leq i \leq k$, we have that $t_\alpha = w_0 \sigma$ for some $\sigma \in W^0$ with $\text{len}(t_\alpha) = \text{len}(w_0) + \text{len}(\sigma)$. Since $\text{len}(y^{-1} w_0 \sigma) = \text{len}(w_0 \sigma) - \text{len}(y^{-1})$ we have that $\text{len}(t_\alpha) = \text{len}(w_0 \sigma) = \text{len}(y) + \text{len}(y^{-1} w_0 \sigma)$, so we know that $\mathbf{u}(y)\mathbf{u}(y^{-1} t_\alpha) = \mathbf{u}(t_\alpha)$.

By Lemma 4.7, $\mathbf{u}(x)\mathbf{u}(t_\alpha)\emptyset \neq 0$ since $x \in W^0$ and $\alpha \in \Lambda_+$. \square

Corollary 4.9. *For all $\lambda \in \mathcal{P}^{(k)}$, $s_R^{(k)}\mathbf{c}(\lambda) = \mathbf{c}(\lambda \cup R)$. Moreover, this uniquely determines $s_R^{(k)}$: it is the unique element of \mathbb{A} with this property.*

Proof. From Theorem 4.1 we know that $s_R^{(k)} s_\lambda^{(k)} = s_{\lambda \cup R}^{(k)}$. Then $s_R^{(k)} \mathbf{c}(\lambda) = s_R^{(k)} s_\lambda^{(k)} \emptyset = s_{\lambda \cup R}^{(k)} \emptyset = \mathbf{c}(\lambda \cup R)$.

For the second statement, let $s_R^{(k)} = \sum_{w \in W} c_w \mathbf{u}(w)$ and let $x = \sum_{w \in W} d_w \mathbf{u}(w)$ be another element of \mathbb{A} satisfying $x \mathbf{c}(\lambda) = \mathbf{c}(\lambda \cup R)$ for all $\lambda \in \mathcal{P}^{(k)}$. Then $(s_R^{(k)} - x) \mathbf{c}(\lambda) = \sum_{w \in W} (c_w - d_w) \mathbf{u}(w) \mathbf{c}(\lambda)$. So $c_w = d_w$ for all w by Lemma 4.8. \square

The following lemma follows from the definition of $V_{(R,\lambda)/\lambda}$ and the action of \mathbb{A} on $\mathcal{C}^{(k+1)}$.

Lemma 4.10. *If $\nu \in \mathfrak{R}$, then $\nu = \mathbf{c}(\nu)$ and $\mathbf{u}(V_{(R,\nu)/\nu})\nu = (R + \nu, \nu) = \mathbf{c}(R \cup \nu)$.*

Lemma 4.11. *For any $\lambda \in \mathcal{C}^{(k+1)}$, $X_R \lambda$ consists of exactly one term.*

Proof. Since $X_R = Y_R$ we will instead prove this for Y_R .

An arbitrary term in Y_R looks like $\mathbf{u}(z_\gamma)$, so we will compute $\mathbf{u}(z_\gamma)\lambda = \mathbf{u}(z_\gamma)\mathbf{u}(w_\lambda)\emptyset$.

In order to use Proposition 4.6, we calculate the difference between the alcoves $A_{z_\gamma w_\lambda}$ and A_{w_λ} .

$$A_{z_\gamma w_\lambda} = (z_\gamma w_\lambda)^{-1} \diamond A_\emptyset = w_\lambda^{-1} \diamond A_{z_\gamma} = w_\lambda^{-1} \diamond (A_\emptyset + \gamma) = A_{w_\lambda} + w_\lambda^{-1} \star \gamma.$$

By Proposition 4.6, $\mathbf{u}(z_\gamma)\mathbf{u}(w_\lambda)\emptyset \neq 0$ if and only if $w_\lambda^{-1} \star \gamma \in \Lambda_+$. Since $\Gamma \cap \Lambda_+ = \{\Lambda_c\}$, there is a unique term in $Y_R \lambda$ which is nonzero; specifically it is the term $\mathbf{u}(z_{w_\lambda^{-1} \star \Lambda_c})\lambda$. \square

Theorem 4.12. *For a maximal rectangle $R = (c^r)$ with $c + r = k + 1$, $s_R^{(k)} = X_R$.*

Proof. We will show that X_R has the defining property of $s_R^{(k)}$ as outlined in Corollary 4.9. Let $s_R^{(k)} = \sum_w c_w \mathbf{u}(w)$ and let $X_R = \sum_w d_w \mathbf{u}(w)$. Lemma 4.10, implies that X_R and $s_R^{(k)}$ act the same on $\mathbf{c}(\nu)$ when $\nu \in \mathfrak{R}$:

$$s_R^{(k)} \mathbf{c}(\nu) = X_R \mathbf{c}(\nu) = \mathbf{c}(\nu \cup R).$$

This implies that $c_w = d_w$ for all w in the support of X_R by Lemma 4.10, since every element $\mathbf{u}(w)$ of the support of X_R has a $\nu \in \mathfrak{R}$ for which $\mathbf{u}(w)\mathbf{c}(\nu) \neq 0$. Therefore the support of X_R is contained in the support of $s_R^{(k)}$.

Now suppose $\nu \notin \mathfrak{R}$. By Lemma 4.11, $X_R \mathbf{c}(\nu)$ consists of a single term, let's say μ . But then:

$$s_R^{(k)} \mathbf{c}(\nu) = \mathbf{c}(\nu \cup R) \text{ and } X_R \mathbf{c}(\nu) = \mu,$$

for some core μ . We know the support of X_R is contained in the support of $s_R^{(k)}$ and that they agree on the support of X_R . Therefore, some term $\mathbf{u}(w)$ of X_R and $s_R^{(k)}$ has $\mathbf{u}(w) \mathbf{c}(\nu) = \mu$. There cannot be a $v \neq w$ for which $\mathbf{u}(v) \mathbf{c}(\nu) = \mu$ by Lemma 4.3, so μ must appear in the expansion of $s_R^{(k)} \mathbf{c}(\nu)$. Therefore $\mu = \mathbf{c}(\nu \cup R)$. \square

5 An application

Finally, we use our formula for $s_R^{(k)}$ to prove a surprising observation about $s_R^{(k)}$ which motivated this article.

Theorem 5.1. *Let $R = (c^r)$ be a maximal rectangle with $c + r = k + 1$, then $s_R^{(k)} u_i = u_{i+c} s_R^{(k)}$.*

Proof. Let $\gamma \in \Gamma$. We start by computing $(s_{i+c} z_\gamma s_i)^{-1} \diamond A_\emptyset$. We let $\Lambda_0 = 0$. Then A_\emptyset is the alcove with vertices $\{\Lambda_j : j \in \{0, 1, \dots, k\}\}$. $s_i \diamond A_\emptyset$ is the alcove with vertices $\{\Lambda_j : j \neq i\}$ and $\Lambda_i - \alpha_i$. By Lemma 3.3, z_γ is a pseudo-translation of A_{s_i} by $s_i \star \gamma$. The vertices of $(z_\gamma s_i)^{-1} \diamond A_\emptyset$ are thus $\{\Lambda_j + s_i \star \gamma : j \neq i\}$ and $\Lambda_i - \alpha_i + s_i \star \gamma$. This shares all but one vertex with the alcove $A_\emptyset + s_i \star \gamma$. By Proposition 1.1, $(s_{i+c} z_\gamma s_i)^{-1} \diamond A_\emptyset$ shares all the vertices of the alcove $(z_\gamma s_i)^{-1} \diamond A_\emptyset$ except for the vertex with label $c + i$, which is $\Lambda_i - \alpha_i + s_i \star \gamma$ (since $s_i \star \gamma$ has label c while $\Lambda_i - \alpha_i$ has label i). Thus it follows that $(s_{i+c} z_\gamma s_i)^{-1} \diamond A_\emptyset = A_\emptyset + s_i \star \gamma$. But then $A_{s_{i+c} z_\gamma s_i} = (s_{i+c} z_\gamma s_i)^{-1} \diamond A_\emptyset = A_\emptyset + s_i \star \gamma = A_{z_{s_i \star \gamma}}$, so $s_{i+c} z_\gamma s_i = z_{s_i \star \gamma}$. Therefore $z_\gamma s_i = s_{i+c} z_{s_i \star \gamma}$.

Replace each s_j by u_j in the above expression. We still have equality because the two expressions have the same length, so either both are reduced or neither is. Summing over all γ shows that $s_R^{(k)} u_i = u_{i+c} s_R^{(k)}$. \square

Remark 5.2. Let $\widetilde{s}_R^{(k)}$ denote the element of $\mathbb{C}W$ which has the same expression as $s_R^{(k)}$, but in the s_i generators instead of the u_i generators. Then we have actually proved in Theorem 5.1 that $\widetilde{s}_R^{(k)} s_i = s_{i+c} \widetilde{s}_R^{(k)}$.

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References

- [1] J. Bandlow, A. Schilling, and M. Zabrocki, The Murnaghan-Nakayama rule for k -Schur functions, *Journal of Combinatorial Theory, Series A*, 118(5) (2011) 1588-1607.
- [2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.
- [3] S. Fomin and C. Greene, Noncommutative Schur functions and their applications, *Discrete Mathematics* **193** (1998), 179–200. Reprinted in the *Discrete Math Anniversary Volume* **306** (2006), 1080–1096.
- [4] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, no. 29, 1992 .
- [5] T. Lam, Affine Stanley symmetric functions, *Amer. J. Math.* **128** (2006), no. 6, 1553–1586.
- [6] T. Lam, Stanley symmetric functions and Peterson algebras, [arXiv:1007.2871v1](https://arxiv.org/abs/1007.2871v1) (2010).
- [7] T. Lam, Schubert polynomials for the affine Grassmannian, *J. Amer. Math. Soc.*, 21 (2008), 259-281.
- [8] T. Lam, L. Lapointe, J. Morse, and M. Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, *Memoirs of the AMS*, Volume 208, Number 977, November 2010.
- [9] T. Lam and M. Shimozono, Quantum cohomology of G/P and homology of affine Grassmannian, *Acta. Math.* 204 (2010), 49–90.
- [10] L. Lapointe, A. Lascoux, and J. Morse, Tableau atoms and a new Macdonald positivity conjecture, *Duke Math. J.* **116** (2003), no. 1, 103–146.

- [11] L. Lapointe and J. Morse, Schur function analogs for a filtration of the symmetric function space, *J. Combin. Theory Ser. A*, (2003), no. 101, 191-224.
- [12] L. Lapointe and J. Morse, Tableaux on $k + 1$ -cores, reduced words for affine permutations, and k -Schur expansions, *J. Combin. Theory Ser. A* **112** (2005), no. 1, 44–81.
- [13] L. Lapointe and J. Morse, A k -tableau characterization of k -Schur functions, *Adv. Math.* **213** (2007), no. 1, 183–204.
- [14] W. A. Stein et al., *Sage Mathematics Software (Version 4.3.3)*, The Sage Development Team, 2010, <http://www.sagemath.org>.
- [15] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, <http://combinat.sagemath.org>, 2008.
- [16] J.Y. Shi, Alcoves corresponding to an affine Weyl group. *J. London Math. Soc.* (2) 35 (1987), no. 1, 42–55.
- [17] D. J. Waugh, Upper bounds in affine Weyl groups under the weak order, *Order*, 16 no. 1, (1999), pp. 77–87.