IMAGINARY CONES AND LIMIT ROOTS OF INFINITE COXETER GROUPS

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ABSTRACT. Let (W, S) be an infinite Coxeter system. To each geometric representation of W is associated a root system. While a root system lives in the positive side of the isotropic cone of its associated bilinear form, an imaginary cone lives in the negative side of the isotropic cone. Precisely on the isotropic cone, between root systems and imaginary cones, lives the set E of limit points of the directions of roots. In this article we study the close relations of the imaginary cone with the set E, which leads to new fundamental results about the structure of geometric representations of infinite Coxeter groups. In particular, we show that the W-action on E is minimal and faithful, and that E and the imaginary cone can be approximated arbitrarily well by sets of limit roots and imaginary cones of universal root subsystems of W, i.e., root systems for Coxeter groups without braid relations (the free object for Coxeter groups). Finally, we discuss open questions as well as the possible relevance of our framework in other areas such as geometric group theory.

1. INTRODUCTION

Root systems are fundamental in the theory of Coxeter groups. Finite root systems and their associated finite Coxeter groups have received a lot of attention because of their fundamental role in the theories of semisimple complex Lie algebras and Lie groups, algebraic groups, quantum groups, regular polytopes, singularities, representations of quivers etc; see for instance [Bou68, Hum90, GP00, BB05] and the references therein. This article rather focuses on infinite root systems (and their associated infinite Coxeter groups), for which many natural questions remain unexplored. Important results have been obtained on the geometry and topology of infinite Coxeter groups; see for instance [Dav08, AB08] and the references therein. In particular, we mention the strong Tits' alternative of Margulis-Noskov-Vinberg (see [NV02]), according to which any subgroup of a Coxeter group has a finite index subgroup which is either abelian or surjects onto a non-abelian free group. The approach used in this study is very often related to the Tits cone, Coxeter complex or Davis complex, which are dual objects to root systems. On the other hand, root systems are natural objects to consider and they provide tools that are not provided by their dual counterparts. Infinite crystallographic root systems and Coxeter groups have been studied because of their natural association with Lie algebras, Kac-Moody algebras and their generalizations; see for instance [Bou68,

Kac90, MP95, LN04]. Root systems of general Coxeter groups are also at the heart of fundamental work such as B. Brink and R. Howlett's proof that Coxeter groups have an automatic structure [BH93] or D. Krammer's work on the conjugacy problem for Coxeter groups [Kra09].

One of the main goals of this article is to better understand the geometric representations, and especially the associated root systems, of infinite Coxeter groups. A principal motivation for that goal is the hope it will lead to progress in the study of reflection orders of Coxeter groups and their initial sections, which play a significant role in relation to Bruhat order and Iwahori-Hecke algebras (see [BB05] for more details) and conjecturally are important for associated representation categories. Despite many important potential applications, reflection orders and their initial sections are poorly understood in general, and many of their basic properties remain conjectural. For example, Conjecture 2.5 in [Dye11] suggests that the initial sections, ordered by inclusion, form a complete lattice that may be viewed as a natural ortholattice completion of weak order. In efforts to refine and prove these conjectures for general Coxeter groups, one fundamental difficulty is that not much is known about how the roots of an infinite root system are geometrically distributed over the space, and it is our intention to begin to fill this gap. Another motivation is to study discrete subgroups of isometries in quadratic spaces; for instance modules associated to geometric representations of W are quadratic spaces and W is itself a discrete subgroup of isometries generated by reflections. The case of Lorentzian spaces is discussed in [HPR13] but the results here suggest such a study may be of considerable interest more generally.

In recent years, several studies about infinite root systems of Coxeter groups have been conducted (see for instance [BD10, Dye10, Dye11, Fu12]). One of the notions introduced is a nice generalization of the *imaginary cone*, which first appears in the context of root systems of Kac-Moody algebras (see [Kac90]), to root systems of Coxeter groups in general; see [Hée90, Fu13a, Dye12, Edg09]. While a root system lives in the positive side of the isotropic cone of its associated bilinear form, an imaginary cone lives in the negative side of the isotropic cone. Precisely on the isotropic cone, between root systems and imaginary cones, lives the set of limit points of the directions of roots, which we call *limit roots*.

In [HLR14], the second and third author, together with J.-P. Labbé, initiated a study of the set $E(\Phi)$ of limit roots of a based root system (Φ, Δ), with associated Coxeter system (W, S). In this second article we study the close relations of the set $E(\Phi)$ with the imaginary cone studied by the first author [Dye12], which leads to new fundamental results about the structure of geometric representations of infinite Coxeter groups.

To study a root system Φ in the geometric W-module V, the approach used in [HLR14] is to consider a projective version of Φ by cutting the cone $\mathrm{cone}(\Delta)$, in which the positive roots live, by an affine hyperplane V_1. We obtain this way the so-called *normalized root system* $\widehat{\Phi}$ that is the intersection of the rays spanned by the roots with V_1. By doing so, we obtain that $\widehat{\Phi}$ is contained in the polytope $\mathrm{conv}(\widehat{\Delta})$ and therefore $\widehat{\Phi}$ (when infinite) has a non-empty set of accumulation points denoted by $E(\Phi)$. The following properties of $\widehat{\Phi}$ and $E(\Phi)$ were brought to light in [HLR14]: $E(\Phi)$ is contained in the isotropic cone Q (the red curve in Figure 1) of the bilinear form associated to the geometric representation of (W, S); W acts

on $\widehat{\Phi} \sqcup E(\Phi)$ by projective transformations and has a nice geometric interpretation that can be seen on Figure 1; and $E(\Phi)$ is the closure of the set $E_2(\Phi)$ of the limit points obtained from dihedral reflection subgroups. Independently, the first author showed in [Dye12] that the closure of the imaginary cone is the convex cone $\mathrm{cone}(E(\Phi))$ spanned by the elements $E(\Phi)$ seen as vectors in V.

In §2, we recall the definition of $E(\Phi)$, of the W-action, of the imaginary cone $\mathcal{Z}(\Phi)$ and bring together, with slight improvements, the frameworks and results from [HLR14] and [Dye12]. In particular we extend in §2 the projective W-action to include the *imaginary convex set* $Z(\Phi)$ that is an affine section of $\mathcal{Z}(\Phi)$, see Figure 2. Then in §3, we prove our first fundamental fact: the W-action on $E(\Phi)$ is *minimal*, i.e., for any x in $E(\Phi)$, the orbit $W \cdot x$ of $E(\Phi)$ under W is dense in $E(\Phi)$ (Theorem 3.1). In order to do so, we study the convexity properties of $Z(\Phi)$ and give fundamental results on the set of extreme points and exposed faces of the closure $\overline{Z(\Phi)}$ of $Z(\Phi)$.

In §4 we turn our attention to two 'fractal conjectures' stated in [HLR14, §3.2] about fractal, self-similar descriptions of the set of limit roots $E(\Phi)$. We use the minimality of the W-action from §2, as well as some additional work on the case where (Φ, Δ) is weakly hyperbolic, in order to completely prove in any ranks [HLR14, Conjecture 3.9], as well as to prove in the weakly hyperbolic case the conjecture stated just above Conjecture 3.9. This will turn out to give us in §6 another fundamental result, which has a "hyperbolic discrete group taste": the accumulation set of the W-orbit of any $z \in Z$ contains $E(\Phi)$ (see Corollary 6.15). We have been made aware while preparing this article that those two fractal conjectures have been solved independently for root systems of signature $(n-1, 1)$ in [HMN14] with a different approach, see Remark 4.11.

In §5 we explore the question of the restriction of $E(\Phi)$ to a face F_I of $\mathrm{conv}(\Delta)$. In particular we prove that $E_2(\Phi)$ behaves well with the restriction to standard facial subgroups (those are exactly the standard parabolic subgroups when Δ is a basis of V). By doing so, we will be brought to give a useful interpretation of *the dominance order* and *elementary roots* in our affine normalized setting.

In §6, we study the geometry of $E(\Phi)$ and $Z(\Phi)$ in detail. We prove two other fundamental results under the assumption that the root system is irreducible and neither finite nor affine. Firstly, the W-action on $E(\Phi)$ is faithful (Theorem 6.1). Secondly, the set of limit roots $E(\Phi)$ (resp., each face of the closed imaginary cone) can be approximated arbitrarily closely (in a Hausdorff-type metric) by the sets of limits roots (resp., closed imaginary cones) of the universal root subsystems of Φ i.e., root systems for Coxeter groups without braid relations (which are the free objects for Coxeter groups, called universal Coxeter groups in [Hum90]). This second result may be viewed as asserting that Φ contains "large" universal root subsystems. As the subgroup of even length elements in a universal Coxeter group is a free group, this is a result very much in the spirit of the Tits alternative for W.

In §7 we collect open questions and discuss further avenues of research. In particular, in §7.4, we discuss the relations of our framework with hyperbolic geometry and geometric group theory. The link with Kleinian groups, which is precisely discussed in another article [HPR13], is outlined, and we also explain for instance how the convex core of W is related to the imaginary convex set and the limit roots. Considering that our framework and results apply more generally to discrete subgroups of isometries on quadratic spaces, an important question we raise
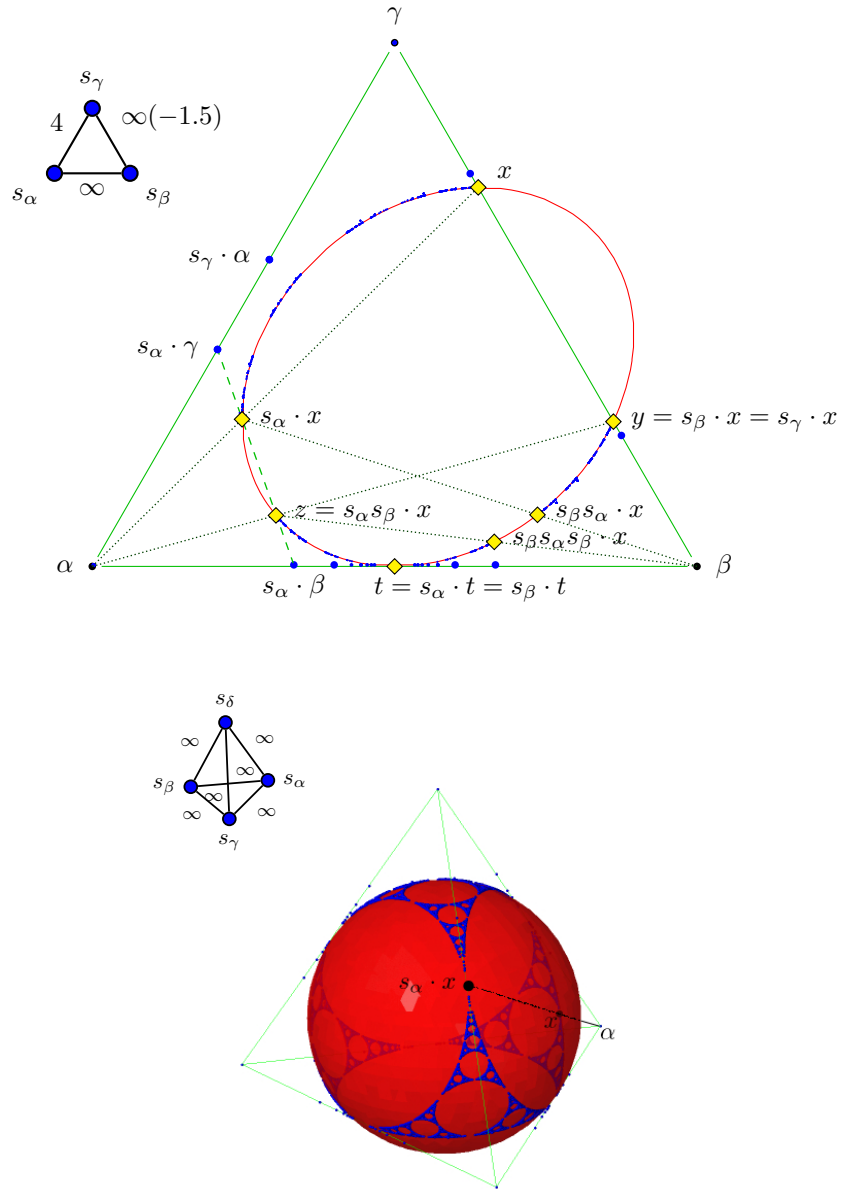
FIGURE 1. Pictures in rank 3 and 4 of the normalized isotropic cone $\widehat{Q}$ (in red), the first normalized roots (in blue dots, with depth ≤ 8) for the based root system with diagram given in the upper left of each picture. The set $E(\Phi)$ of limit roots is the limit set of the normalized roots. It is acted on by W, as explained in §2.1: for example, in the upper picture, the limit root x is sent to y by s_β, which is then sent to z by s_α.

is what part of the theory of Kleinian groups and discrete subgroups of hyperbolic isometries can be generalized to quadratic spaces.

In a final appendix, we discuss the relation of the set of limit roots defined here with a notion of limit set of a Zariski dense subgroup of the group of k-points of a connected reductive group defined over a local field k (only $k = \mathbb{R}$ here) as studied by Benoist in [Ben97].

Sections 4, 5 and 6 can be read independently of each other. In view of the length of this article, we made sure to treat each section as a small chapter by writing a short introduction and stating the main results it contains as soon as possible.

Figures. The pictures of normalized roots and the imaginary convex body were computed with the computer algebra system *Sage* [S⁺11], and drawn using the TₑX-package TikZ.

2. Imaginary cone, limit roots, and action of W

The aim of this section is to bring together, and slightly improve, the frameworks and results from [HLR14] and [Dye12]. This will lead to our first main result in §3.

Let V be a real vector space of dimension n equipped with a symmetric bilinear form (inner product) B. Let (Φ, Δ) be a *based root system* in (V, B) with associated Coxeter system (W, S), i.e., Δ is a *simple system*, W is generated by the set of simple reflections $S := \{ s_\alpha \, | \, \alpha \in \Delta \}$, where

$$s_\alpha(v) = v - 2B(\alpha, v)\alpha, \text{ for } v \in V,$$

and $\Phi := W(\Delta)$ is the associated *root system*. The set $\Phi^+ := \operatorname{cone}(\Delta) \cap \Phi$ is the set of *positive roots*. We recall[1] from [HLR14, §1] that a simple system Δ is a finite subset of V such that:

(i) Δ is positively independent: if $\sum_{\alpha \in \Delta} \lambda_\alpha \alpha = 0$ with all $\lambda_\alpha \geq 0$, then all $\lambda_\alpha = 0$;

(ii) for all $\alpha, \beta \in \Delta$, with $\alpha \neq \beta$, $B(\alpha, \beta) \in \,] - \infty, -1] \cup \{ - \cos\left(\frac{\pi}{k}\right), k \in \mathbb{Z}_{\geq 2} \}$;

(iii) for all $\alpha \in \Delta$, $B(\alpha, \alpha) = 1$.

The *rank of* (Φ, Δ) is the cardinality $|\Delta|$ of Δ. The *signature of the based root system* (Φ, Δ) is the signature of the quadratic form $q_{B,\Delta}$ associated to the restriction of B to the subspace $\operatorname{span}(\Delta)$. In the case where Δ spans V, the signature of (Φ, Δ) is the signature of $q_B = B(\cdot, \cdot)$.

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[1]**Note to the reader.** This article follows directly [HLR14]. In this spirit, we chose not to rewrite in details an introduction to based root systems. We refer the unfamiliar reader to [HLR14, §1] for a more detailed introduction to this framework, which generalizes the classical geometric representation of Coxeter groups.

Throughout the article we always assume that Δ is finite, i.e., W is a finitely generated Coxeter group. Some of the results may be extended to the general case. To lighten the notations, we will often shorten the terminology "*based root system*" and use "*root system*" instead.

2.1. **Normalized roots and limit roots.** Let V_1 be a hyperplane that is *transverse to* Φ^+, i.e., such that each ray $\mathbb{R}_{>0}\alpha$, for $\alpha \in \Delta$, intersects V_1 in one point, denoted by $\widehat{\alpha}$, see for instance [HLR14, Figure 2, Figure 3 and §5.2].

Denote by V_0 the linear hyperplane directing V_1. For any $v \in V \setminus V_0$, the line $\mathbb{R}v$ intersects the hyperplane V_1 in one point, that we denote also by $\widehat{v}$ (we also use the analog notation $\widehat{P}$ relatively to a subset P of $V \setminus V_0$). More precisely, denote by φ the linear form associated to V_0 such that the equation of V_1 is $\varphi(v) = 1$. Then we have

$$\widehat{v} = \frac{v}{\varphi(v)}, \ \forall v \in V \setminus V_0.$$

For instance, if Δ is a basis for V, we can take for V_1 the affine hyperplane spanned by Δ seen as points, so $\widehat{\Delta} = \Delta$ and $\varphi(v)$ is simply the sum of the coordinates of v in Δ (see [HLR14, §2.1 and §5.2] for more details).

Since V_1 is transverse to Φ^+, for any root $\rho \in \Phi$ we can define its associated *normalized root* $\widehat{\rho}$ in V_1. We denote by $\widehat{\Phi}$ the set of normalized roots. It is contained in the convex hull $\mathrm{conv}(\widehat{\Delta})$ of the *normalized simple roots* $\widehat{\alpha}$ in $\widehat{\Delta}$, and it can be seen as the set of representatives of the directions of the roots, i.e., the roots seen in the projective space $\mathbb{P}V$. In Figure 1, normalized roots are the blue dots, while the edges of the polytope $\mathrm{conv}(\widehat{\Delta})$ are in green. Note that since $\Phi = \Phi^+ \sqcup (-\Phi^+)$, we also have

$$\widehat{\Phi} = \widehat{\Phi^+} = V_1 \cap \bigcup_{\rho \in \Phi^+} \mathbb{R}_{>0}\rho.$$

The *set of limit roots* is the accumulation set of $\widehat{\Phi}$:

$$E(\Phi) = \mathrm{Acc}(\widehat{\Phi}).$$

In Figure 1, this is the (Apollonian gasket-like) shape to which the blue dots tend. It is well known that Φ, and therefore, $\widehat{\Phi}$, are discrete (see for instance [HLR14, Cor. 2.9]); so $E(\Phi)$ is also the complement of $\widehat{\Phi}$ in the closure of $\widehat{\Phi}$. Since the elements of $E(\Phi)$ are limit points of normalized roots, we call them for short the *limit roots* of Φ.

In [HLR14, Theorem 2.7], it was shown that $E(\Phi) \subseteq \widehat{Q} \cap \mathrm{conv}(\widehat{\Delta})$, where

$$Q := \{v \in V \mid B(v,v) = 0\}$$

is the *isotropic cone* of B, and $\widehat{Q} = Q \cap V_1$; $\widehat{Q}$ is represented in red in Figure 1. Recall also that the natural geometric action of W on V induces a W-*action on* $\widehat{\Phi} \sqcup E(\Phi)$:

$$(2.1) \qquad w \cdot x = \widehat{w(x)} = \frac{w(x)}{\varphi(w(x))} \text{ for } w \in W, x \in \widehat{\Phi} \sqcup E(\Phi),$$

where φ is, as above, the linear form such that $\ker\varphi = V_0$ is the direction of the transverse hyperplane V_1, see [HLR14, §3.1] for more details. This action has a nice geometric interpretation on $E(\Phi)$: for $\beta \in \Phi$ and $x \in E(\Phi)$ denote by $L(\widehat{\beta}, x)$ the line in V_1 passing through the points x and $\widehat{\beta}$, then either $s_\beta \cdot x = x$ if $L(\widehat{\beta}, x)$

is tangent to $\widehat{Q}$, or $s_\beta \cdot x$ is the other point of intersection of $L(\widehat{\beta}, x)$ with $\widehat{Q}$, see Figure 1.

Finally, it is interesting to notice that the signature of (Φ, Δ) is intimately linked to the shape of $E(\Phi)$. If $q_{B,\Delta}$ is:

- positive definite: (Φ, Δ) is said to be of *finite type*; in this case Φ and W are finite, and $E(\Phi)$ is empty;
- positive semi-definite (and not definite): (Φ, Δ) is said to be of *affine type*; in this case $E(\Phi)$ is finite non-empty; if in addition (Φ, Δ) is irreducible, then $\widehat{Q}$ is a singleton and $E(\Phi) = \widehat{Q}$ (see [HLR14, Cor. 2.15]);
- not positive semi-definite: (Φ, Δ) is said to be of *indefinite type*. In this case $E(\Phi)$ is infinite.

Some special cases of root systems of indefinite type are the root systems of hyperbolic type; they will be discussed in §4.1.

In the following, we will write E instead of $E(\Phi)$ if there is no possible confusion with more than one root system.

2.2. **The convex hull of limit roots and the imaginary cone.** The imaginary cone has been introduced by Kac (see [Kac90, Ch. 5]) in the context of Weyl groups of Kac-Moody Lie algebras: its name comes from the fact that it was defined as the cone pointed on 0 and spanned by the positive imaginary roots of the Weyl group. This notion has been generalized afterwards to arbitrary Coxeter groups, first by Hée [Hée90, Hée93], then by the first author [Dye12] (see also Edgar's thesis [Edg09]). The definition we use here applies to any finitely generated Coxeter group (see Remark 2.6 below).

Let (Φ, Δ) be a based root system in V with associated Coxeter group W. The *imaginary cone* $\mathcal{Z}(\Phi)$ of (Φ, Δ) is the union of the cones in the W-orbit of the cone

$$\mathcal{K}(\Phi) := \{v \in \mathrm{cone}(\Delta) \mid B(v, \alpha) \leq 0, \ \forall \alpha \in \Delta\}.$$

The imaginary cone $\mathcal{Z}(\Phi)$ is by definition stable by the action of W. Observe that for each $\alpha \in \Delta$, the reflecting hyperplane $H_\alpha = \{v \in V \mid B(v, \alpha) = 0\}$ associated to the simple reflection s_α supports a facet of $\mathcal{K}(\Phi)$. In [Dye12, Prop. 3.2.(c)], it is shown that $\mathcal{Z}(\Phi)$ is contained in the cone $\mathrm{cone}(\Delta)$ and for $x, y \in \mathcal{Z}(\Phi)$, $B(x, y) \leq 0$. In particular, letting $Q^- := \{v \in V \mid B(v, v) \leq 0\}$, we have:

$$\mathcal{Z}(\Phi) \subseteq \mathrm{cone}(\Delta) \cap Q^-.$$

The imaginary cone is intimately linked to the set of limit roots: it is proven in [Dye12] that the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}(\Phi)$ is equal to the convex cone spanned by the "limit rays of roots". These limit rays in the sense of [Dye12] are the rays spanned by limit roots in the sense of [HLR14] (see [Dye12, §5.6] for more details). We get from [Dye12, Theorem 5.4] that the set E of limit roots and the imaginary cone $\mathcal{Z}(\Phi)$ have the following relation:

$$\overline{\mathcal{Z}(\Phi)} = \mathrm{cone}(E(\Phi)).$$

We now "*normalize*" these notions. Let V_1 be an affine hyperplane transverse to Φ^+ and let

$$K(\Phi) := \widehat{\mathcal{K}(\Phi)} = \mathcal{K}(\Phi) \cap V_1 \quad \text{and} \quad Z(\Phi) := \widehat{\mathcal{Z}(\Phi)} = \mathcal{Z}(\Phi) \cap V_1.$$

In Figure 2 we draw two examples in rank 3, and an example in rank 4 is in Figure 14(b)-(c) at the end of this article. Similarly to the case of the cone $\mathcal{K}$,

we observe that for each $\alpha \in \Delta$, the trace in V_1 of the reflecting hyperplane H_α associated to the simple reflection s_α supports a facet of $K(\Phi)$. The converse is not always true: a facet of $K(\Phi)$ may be rather contained in a facet of $\mathrm{conv}(\Delta)$, see the example on the right in Figure 2. Moreover,

$$Z(\Phi) \subseteq \mathrm{conv}(\widehat{\Delta}) \cap Q^-.$$

and

(2.2) $$B(x,y) \leq 0, \quad \text{for } x, y \in \overline{Z}.$$

Definition 2.1. The closure $\overline{Z(\Phi)}$ of $Z(\Phi)$ is called the *imaginary convex body* of (Φ, Δ).

As for the set E, when the root system is unambiguous we will write simply $\mathcal{K}$, K, $\mathcal{Z}$ and Z instead of $\mathcal{K}(\Phi)$, $K(\Phi)$, $\mathcal{Z}(\Phi)$ and $Z(\Phi)$. We sometimes refer to the set Z as the *imaginary convex set*. We get this normalized version of [Dye12, Theorem 5.4].

Theorem 2.2. *The convex hull of E equals the imaginary convex body $\overline{Z}$:*

$$\mathrm{conv}(E) = \overline{Z}.$$

Using this equality, we also get the following very nice description of $\mathrm{conv}(E)$, which was mentioned in [HLR14, Remark 3.3] and proved in [Dye12, Thm. 5.1].

Theorem 2.3.

$$\mathrm{conv}(E) = V_1 \cap \bigcap_{w \in W} w(\mathrm{cone}(\Delta)).$$
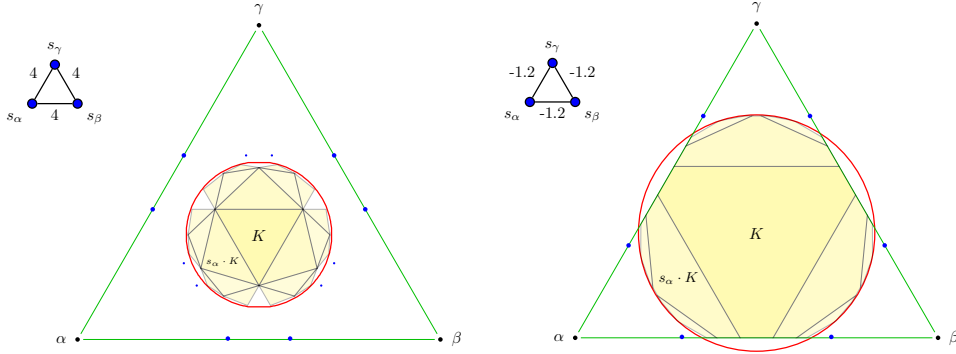
FIGURE 2. Two examples of pictures of K and of its first images by the group action (in shaded yellow), giving the first steps to construct the imaginary convex set Z. The red circle is the normalized isotropic cone $\widehat{Q}$; in black and blue are the first normalized roots. The example on the left is a group of hyperbolic type (see §4.1): K is simply a triangle and Z turns out to be the whole open disk inside $\widehat{Q}$. The example on the right is weakly hyperbolic but not hyperbolic: K is a truncated triangle and Z is strictly contained in the open disk (see §4).

Before extending the W-action on $\widehat{\Phi} \sqcup E$ to include $\overline{Z}$, we discuss the affine space $\mathrm{aff}(\overline{Z})$ spanned by the imaginary convex body $\overline{Z}$. Obviously we have

$$\mathrm{aff}(K) \subseteq \mathrm{aff}(Z) \subseteq \mathrm{aff}(\widehat{\Delta}).$$

Moreover, we have the following particular situations.

- In the case where the root system if finite, then we know that $E = \varnothing$ and therefore $\overline{Z} = \operatorname{conv}(E) = \varnothing = \operatorname{aff}(\overline{Z})$.
- If (Φ, Δ) is affine irreducible, then $E = \{x\}$ is a singleton and therefore $\{x\} = \operatorname{conv}(E) = \overline{Z} = Z = K$.
- If (Φ, Δ) is non-affine infinite dihedral, so $\Delta = \{\alpha, \beta\}$, then we obtain by direct computation that $E = \{x, y\}$ is of cardinality 2, $Z =]x, y[$ and $K \subsetneq]x, y[$ (see also [HLR14] and [Dye12, §9.10]).
- When the root system is of indefinite type and irreducible, then $\operatorname{aff}(K) = \operatorname{aff}(\widehat{\Delta})$. The essential point to prove this fact is the following result (mentioned without proof in [Dye12, §4.5] and which goes back to Vinberg [Vin71]).

Lemma 2.4. *Let (Φ, Δ) be an irreducible based root system of indefinite type. Then there exists a vector z in the topological interior, for the induced topology on $\operatorname{span}(\Delta)$, of $\operatorname{cone}(\Delta)$ such that $B(z, \alpha) < 0$, for all $\alpha \in \Delta$. Equivalently, K has non-empty interior for the induced topology on $\operatorname{aff}(\widehat{\Delta})$.*

We give a proof here for convenience.

Proof. Let $\alpha_1, \dots, \alpha_p$ be the simple roots and A be the matrix $(B(\alpha_i, \alpha_j))_{1 \le i,j \le p}$. For any $X = {}^t(x_1, \dots, x_p)$ column matrix of size n, define $v_X = \sum_i x_i \alpha_i$. If $X \in (\mathbb{R}_{>0})^p$, then v_X is in the interior of $\operatorname{cone}(\Delta)$. Moreover, $B(v_X, \alpha) < 0$ for all $\alpha \in \Delta$ if and only if $AX \in (\mathbb{R}_{<0})^p$. Thus if we prove that there is $Z \in (\mathbb{R}_{>0})^p$ such that $AZ \in (\mathbb{R}_{<0})^p$, the lemma follows by setting $z = v_Z$.

Set $M := I_p - A$. Since (Φ, Δ) is irreducible, A is indecomposable, and so is M. The matrix M has nonnegative coefficients, because A has 1's as diagonal coefficients and nonpositive coefficients elsewhere. So the Perron-Frobenius theorem implies the following two facts:

- define the spectral radius of M: $r := \max\{|\lambda|, \lambda \in \operatorname{Sp}(M)\}$. Then $r \in \operatorname{Sp}(M)$. Moreover, since the root system is of indefinite type, the signature of B has at least one -1. So Sylvester's law of inertia implies that A has at least one negative eigenvalue, so $M = I_p - A$ has an eigenvalue that is strictly greater than 1. Therefore $r > 1$.
- The eigenspace associated to the eigenvalue r is a line spanned by a vector Z of M with strictly positive coefficients.

Therefore, there is $Z \in (\mathbb{R}_{>0})^p$ such that $MZ = rZ$. Hence we obtain that $AZ = (1 - r)Z \in (\mathbb{R}_{<0})^p$, since $1 - r < 0$. $\qquad\square$

We deduce easily that K (and also E) affinely spans the same space as $\widehat{\Delta}$:

Proposition 2.5. *Suppose (Φ, Δ) is an irreducible based root system of indefinite type. Then:*
$$\operatorname{aff}(E) = \operatorname{aff}(\overline{Z}) = \operatorname{aff}(Z) = \operatorname{aff}(K) = \operatorname{aff}(\widehat{\Delta}).$$

In particular, we get $\operatorname{span}(E) = \operatorname{span}(\Delta)$. We will prove later (in Theorem 6.12) that actually any non empty open subset of E is sufficient to (affinely) span $\operatorname{aff}(\widehat{\Delta})$.

Proof. The first equality $\operatorname{aff}(E) = \operatorname{aff}(\overline{Z})$ is straightforward using $\overline{Z} = \operatorname{conv}(E)$, and the second is clear since an affine span is closed. We also have by definition $\operatorname{aff}(K) \subseteq \operatorname{aff}(Z) \subseteq \operatorname{aff}(\widehat{\Delta})$, so it will suffice to show that $\operatorname{aff}(K) = \operatorname{aff}(\widehat{\Delta})$. By

Lemma 2.4, the interior of K (for the induced topology on $\mathrm{aff}(\widehat{\Delta})$) is not empty. So K contains an open ball of $\mathrm{aff}(\widehat{\Delta})$ (of nonzero radius), and $\mathrm{aff}(K) = \mathrm{aff}(\widehat{\Delta})$. $\quad\square$

Remark 2.6. Many of the results in [Dye12] involving the Tits cone and its relationship to the imaginary cone require the assumption that B should be non-degenerate. For reasons explained in [Dye12, §12], this assumption is not necessary for results on the imaginary cone itself. Consider a based root system (Φ, Δ) in (V, B), with associated Coxeter system (W, S). If V' is any subspace of V containing Δ and B is the restriction of B to V', we may regard (Φ, Δ) as a based root system in (V', B'), which we say *arises by restriction* (of ambient vector space). The associated Coxeter systems of these two based root systems are canonically isomorphic and we identify them. The definitions show that the imaginary cones of these two based root systems are equal (as W-subsets of $\mathrm{span}(\Delta)$). We also say that (Φ, Δ), as a based root system in (V, B), is an *extension* of the based root system (Φ, Δ) in (V', B'), so the above shows that the imaginary cone is unchanged by extension or restriction. Similar facts apply to the limit roots.

Say that a based root system (Φ, Δ) in (V, B) is *spanning* if $V = \mathrm{span}(\Delta)$ and is *non-degenerate* if B is non-degenerate. Observe that any based root system has a restriction which is spanning, and also has some non-degenerate extension. The results we give in this paper are insensitive to restriction or extension, so, whenever convenient, we shall assume that a based root system under consideration is non-degenerate (so that results from [Dye12] proved for non-degenerate root systems apply) or spanning. Note however, we cannot always assume that it is simultaneously spanning and non-degenerate, as this would exclude affine Weyl groups, for instance, from consideration.

2.3. **The W-action on the imaginary convex body.** We want now to extend the W-action on $\widehat{\Phi} \sqcup E$ to include the imaginary convex body $\overline{Z}$. Recall that φ denotes the linear form such that $\ker \varphi = V_0$ is the direction of the transverse hyperplane V_1. We know from [HLR14, §3.1] that the W-action on $\widehat{\Phi} \sqcup E$ defined in Equation 2.1 is well defined on the set

$$D^+ = \bigcap_{w \in W} w(V_0^+) \cap V_1,$$

where V_0^+ is the open halfspace defined by $\varphi(x) > 0$. It is proven in [HLR14, Prop. 3.2] that $E \subseteq D^+$. Since D^+ is convex, we have necessarily that

$$\overline{Z} = \mathrm{conv}(E) \subseteq D^+.$$

So this W-action is also well-defined on $\overline{Z}$, and therefore Z. Note that W acts on $\overline{Z}$ by (restrictions of) projective transformations of V_1, and not by restrictions of affine maps. However, it does preserve convex closures. We get therefore the following result, whose illustration can be seen in Figure 2.

Proposition 2.7. *The W-action from Equation 2.1 is an action on $\widehat{\Phi} \sqcup (E \cup \overline{Z})$. More precisely:*

    (1) $Z = W \cdot K$ is stable under this W-action;
    (2) $\overline{Z}$ is stable[2] under this W-action.

---

[2]This action may be identified with the restriction to the rays in $\overline{Z}$ of the natural W-action on the set $\mathrm{Ray}(V)$ of rays of V (with origin 0), as in [Dye12].

Moreover the W-action on $\overline{Z} = \mathrm{conv}(E)$ is the restriction of projective transformations that preserve convex closures, in the sense that $\mathrm{conv}(w \cdot X) = w \cdot \mathrm{conv}(X)$ for $X \subseteq \overline{Z}$.

Proof. Almost all the statements follow from the previous discussion. We only need to show that $Z = W \cdot K$ and the statement for convex combinations.

We know that $\mathcal{Z} = W(\mathcal{K})$. Take $z \in Z = \mathcal{Z} \cap V_1$, so there is $w \in W$ and $x \in \mathcal{K} \setminus \{0\}$ such that $z = w(x)$. Since $\mathcal{K} \subseteq \mathrm{cone}(\Delta) \subseteq V_0^+$, V_1 cuts $\mathbb{R}x$ and therefore the normalized version $\widehat{x}$ of x exists. We know that $\widehat{w(x)} = \widehat{w(y)}$ for all nonzero $y \in \mathbb{R}x$. Since $\widehat{x} \in \mathbb{R}x$ we have

$$w \cdot \widehat{x} = \widehat{w \cdot \widehat{x}} = \widehat{w(x)} = \widehat{z} = z.$$

The action is clearly by restrictions of projective transformations, as already noted. We show the statement about convex closures. It is sufficient to consider the case where $X = \{x_1, x_2\}$, with $x_1, x_2 \in \overline{Z}$. Note that

$$\mathrm{conv}(X) = \{\, \lambda x_1 + (1 - \lambda)x_2 \mid 0 \leq \lambda \leq 1 \,\}.$$

Let $z = (\lambda x_1 + (1 - \lambda)x_2) \in \mathrm{conv}(X)$, with $\lambda \in \mathbb{R}$ and $0 \leq \lambda \leq 1$. Take $w \in W$, then:

$$w \cdot z = w \cdot (\lambda x_1 + (1 - \lambda)x_2) = \lambda' \, w \cdot x_1 + (1 - \lambda') \, w \cdot x_2 \,,$$

where $\lambda' = \dfrac{\lambda \, \varphi(w(x_1))}{\lambda \, \varphi(w(x_1)) + (1 - \lambda) \, \varphi(w(x_2))}$ by (2.1). Since $\varphi(w(x_i)) > 0$ for $i = 1, 2$, we have $\lambda' \in [0, 1]$ and so $w \cdot z \in \mathrm{conv}(w \cdot X)$. Therefore $w \cdot \mathrm{conv}(X) \subseteq \mathrm{conv}(w \cdot X)$. For the reverse inclusion we use this result with $w \cdot X$ instead of X and w^{-1} instead of w: $w^{-1} \cdot \mathrm{conv}(w \cdot X) \subseteq \mathrm{conv}(w^{-1} \cdot (w \cdot X)) = \mathrm{conv}(X)$. Therefore $\mathrm{conv}(w \cdot X) = w \cdot (w^{-1} \cdot \mathrm{conv}(w \cdot X)) \subseteq w \cdot \mathrm{conv}(X)$, which concludes the proof. $\square$

We end this section with this fundamental property of the convex hull of an orbit in $\overline{Z}$ shown in [Dye12].

Theorem 2.8. *Let (Φ, Δ) be an irreducible based root system. Then for any $z \in \overline{Z}$, one has $\overline{\mathrm{conv}(W \cdot z)} = \mathrm{conv}(\overline{W \cdot z}) = \overline{Z}$. If $z \in Z \cup (\overline{Z} \cap \widehat{Q})$, then $\mathrm{Acc}(W \cdot z) \subseteq \widehat{Q}$. In particular, $\overline{Z}$ is the only non-empty, closed, W-invariant convex set contained in $\overline{Z}$.*

Proof. The first two assertions are equivalent, by general facts as stated in [Dye12, A11], to [Dye12, Theorem 7.5(b) and Lemma 7.4]. The third assertion follows from the first, and is a slightly weaker version of [Dye12, Theorem 7.6]. $\square$

3. The W-action on E is minimal

The aim of this section is to prove the first main result of this article: the W-action on $E(\Phi)$ is *minimal*, i.e., *every W-orbit $W \cdot x$ in $E(\Phi)$ is dense in $E(\Phi)$*.

Theorem 3.1. *Let (Φ, Δ) be an irreducible based root system.*

(a) *If $z \in \overline{Z}$, then $\overline{W \cdot z} \supseteq E$.*
(b) *If $x \in E$, then $\overline{W \cdot x} = E$, i.e., the action of W on E is minimal.*
(c) *If $\alpha \in \Phi$, then $E = \mathrm{Acc}(W \cdot \alpha) = \overline{W \cdot \alpha} \setminus W \cdot \alpha$.*

Part (b) of this theorem is a huge improvement of the only density result we had on $E(\Phi)$. In [HLR14, Theorem 4.2], it is shown that $E(\Phi)$ is the closure of the set $E_2(\Phi)$ of the limit points obtained from dihedral reflection subgroups:

$$(3.1) \qquad E_2(\Phi) := \bigcup_{\alpha,\beta \in \Phi} L(\widehat{\alpha},\widehat{\beta}) \cap \widehat{Q},$$

where $L(\widehat{\alpha},\widehat{\beta})$ denotes the line in V_1 passing through the points $\widehat{\alpha}$ and $\widehat{\beta}$. Even if Theorem 3.1 is much stronger, the density of $E_2(\Phi)$ into $E(\Phi)$ remains a very important result. Indeed, it is the main ingredient to prove Theorem 2.8, which is a main ingredient to prove this new stronger density property.

While writing this article, we have been made aware that part (b) of this theorem in the case of root systems of signature $(n-1,1)$, and with Δ linearly independent, was proven in [HMN14]. See Remark 4.11 for more details.

A remarkable consequence of Theorem 3.1 is the fact that any orbit of the W-action on $\overline{Z} = \mathrm{conv}(E)$ can get arbitrarily close to any face of $\overline{Z}$. Let C be a convex set. Recall that a *face* of C is a convex subset F of C such that whenever $tc' + (1-t)c'' \in F$ with $c', c'' \in C$ and $0 < t < 1$, one has $c' \in F$ and $c'' \in F$.

Corollary 3.2. *Let (Φ, Δ) be an irreducible based root system. Let $x \in \overline{Z}$. Then for any non-empty face F of $\overline{Z}$, for any open subset U of $\overline{Z}$ which contains F, the W-orbit of x meets U: $W \cdot x \cap U \neq \varnothing$.*

Proof. Choose a point $z \in F$. Since $\overline{Z} = \mathrm{conv}(E)$ we can express z as a convex combination $\sum_{i=1}^{p} \lambda_i z_i$ of points of E where $p > 0$, $0 < \lambda_i \leq 1$, $z_i \in E$, and $\sum_i \lambda_i = 1$. Fix any i in $\{1, \ldots, p\}$. The point z_i is in E, so for any $x \in \overline{Z}$, $z_i \in \overline{W \cdot x}$ by Theorem 3.1(a). Moreover, since F is a face and $z \in F$, one also has $z_i \in F$. Since U is an open subset of $\overline{Z}$ containing F, hence z_i, we conclude that $U \cap W \cdot x \neq \varnothing$ as required. $\qquad\square$

Remark 3.3.

(1) Suppose (Φ, Δ) is reducible, and denote its irreducible components by (Φ_i, Δ_i), for $i = 1, \ldots, p$. Then $\Delta = \bigsqcup_i \Delta_i$, $\Phi = \bigsqcup_i \Phi_i$ (where $\bigsqcup$ denotes the disjoint union), $\mathrm{span}(\Delta) = \sum_i \mathrm{span}(\Delta_i)$ (a sum of orthogonal subspaces, not necessarily direct) and $W = W_1 \times \cdots \times W_p$. We have $E(\Phi) = \bigcup_1^p E(\Phi_i)$ where the union is not necessarily disjoint (see [Dye12, Example 8.2]) but the sets $E(\Phi_i)$ are pairwise orthogonal and so any limit root in $E(\Phi_i) \cap E(\Phi_j)$, where $i \neq j$, is in the radical of the restriction of the bilinear form B to $\mathrm{span}(E(\Phi_i) \cup E(\Phi_j))$. Each subset $E(\Phi_i)$ of $E(\Phi)$ is W-invariant and the W-action on $E(\Phi_i)$ is the pullback of the natural W_i-action on this set by the projection $W \to W_i$. On the other hand, we have from [Dye12], $\mathcal{Z}(\Phi) = \mathcal{Z}(\Phi_1) + \ldots + \mathcal{Z}(\Phi_p)$ (sum of cones with pairwise orthogonal linear spans) and hence $\overline{\mathcal{Z}(\Phi)} = \overline{\mathcal{Z}(\Phi_1)} + \ldots + \overline{\mathcal{Z}(\Phi_p)}$. Consequently,

$$\overline{Z(\Phi)} = \mathrm{conv}\left(\overline{Z(\Phi_1)} \cup \ldots \cup \overline{Z(\Phi_p)}\right).$$

(2) If Δ is linearly independent, then $E(\Phi) = \bigsqcup_1^p E(\Phi_i)$, the W-action on $E(\Phi)$ is the cartesian product of the actions of each W_i on $E(\Phi_i)$ (see [HLR14, Prop. 2.14]) and

$$\overline{Z(\Phi)} = \overline{Z(\Phi_1)} * \ldots * \overline{Z(\Phi_p)},$$

where $*$ denotes the join of two (disjoint) spaces:

$$A * B := \{(1-t)a + tb \mid a \in A, b \in B, t \in [0,1]\}.$$

(3) Part (a) of the theorem implies that $\mathrm{Acc}(W \cdot z) \supseteq E$, for $z \in \overline{Z} \setminus E$ (see Corollary 6.15 for a stronger statement).

3.1. **Extreme limit roots and the proof of Theorem 3.1.**

The main ingredients for the proof of Theorem 3.1 are Theorem 2.8, as we just mentioned, along with a detailed study of convexity relations between the imaginary cone and the set of limit roots. We show in particular that the set of extreme points of $\mathrm{conv}(E(\Phi))$ is dense in $E(\Phi)$. Let us introduce this result.

Given a convex set C, recall that the *extreme points* of C are the points in C which cannot be written as a convex combination of other points of C, or, equivalently, the x in C such that $C \setminus \{x\}$ is convex. Thus, a point c in C is an extreme point of C if and only if $\{c\}$ is a face of C, i.e., c does not lie in the interior of any segment with extremities in C. If C is compact, Minkowski's theorem (finite-dimensional Krein-Milman Theorem) asserts that the set of extreme points of C is the unique inclusion-minimal subset of C with its convex hull equal to C, see [Web94] for more details.

Denote by $E_{\mathrm{ext}}(\Phi)$ (or simply E_{ext}, when there is no possible confusion) the set of extreme points of the imaginary convex body $\overline{Z} = \mathrm{conv}(E)$; so $E_{\mathrm{ext}} \subseteq E$ since $\overline{Z}$ is compact. The elements of E_{ext} are called *extreme limit roots*. By Theorem 2.8 we have $\overline{Z} = \mathrm{conv}(\overline{W \cdot z})$ for any $z \in \overline{Z}$, so by Minkowski's theorem we have $E_{\mathrm{ext}} \subseteq \overline{W \cdot z}$ for any $z \in \overline{Z}$. Thus, the statement that the closure of any orbit contains the whole of E, which is Theorem 3.1(a), is a consequence of the following theorem.

Theorem 3.4. *Let (Φ, Δ) be a based root system. Assume that either Φ is irreducible or Δ is linearly independent. Then $\overline{E_{\mathrm{ext}}(\Phi)} = E(\Phi)$.*

Remark 3.5.

(1) The equality $E_{\mathrm{ext}}(\Phi) = E(\Phi)$ holds in some cases (see Corollary 4.15). However, the based root system in [HLR14, Example 5.8] is irreducible and has linearly independent simple roots, but $\frac{\alpha+\beta+\delta+\epsilon}{4} \in E \setminus E_{\mathrm{ext}}$.

(2) The following example shows the assumption in the statement of the theorem cannot be omitted. Suppose Δ has three irreducible affine components $\{\alpha_i, \beta_i\}$ of type $\widetilde{A}_1$, for $i = 1, 2, 3$, where, setting $\delta_i := \alpha_i + \beta_i$, the space of linear relations on Δ is spanned by $\delta_1 - \delta_2 + \delta_3 = 0$. Then $E(\Phi) = \{\, \widehat{\delta_i} \mid i = 1, 2, 3 \,\}$ but $\overline{E_{\mathrm{ext}}(\Phi)} = E_{\mathrm{ext}}(\Phi) = \{\, \widehat{\delta_i} \mid i = 1, 3 \,\}$.

The proof of Theorem 3.4 needs some more detailed study of the convex geometry of E; we postpone it to §3.4 in order to present right now the proof of Theorem 3.1.

Proof of Theorem 3.1. (a) Let $z \in \overline{Z}$. As explained above, $\overline{W \cdot z}$ contains the set E_{ext} of extreme points of $\overline{Z}$ since $\mathrm{conv}(\overline{W \cdot z}) = \overline{Z} = \mathrm{conv}(E)$ by Theorem 2.8 and Theorem 2.2. Hence $\overline{W \cdot z} \supseteq \overline{E_{\mathrm{ext}}} = E$ by Theorem 3.4.

(b) For $x \in E$, the inclusion $\overline{W \cdot x} \supseteq E$ holds by (a) and the reverse inclusion holds since E is closed and W-invariant.

(c) For $\alpha \in \Phi$, since $W \cdot \alpha \subseteq \widehat{\Phi}$ and $\widehat{\Phi}$ is discrete, we get

$$\overline{W \cdot \alpha} \setminus W \cdot \alpha = \mathrm{Acc}(W \cdot \alpha) \subseteq \mathrm{Acc}(\widehat{\Phi}) = E.$$

Thus $\mathrm{Acc}(W \cdot \alpha)$ is a closed, W-stable subset of E. It is non-empty if $E \neq \varnothing$, so (b) implies that it is equal to E. □

Remark 3.6 (What it means in the imaginary cone setting from [Dye12])**.**

(1) Part (a) of Theorem 3.1 is equivalent to the assertion that for $z \in \overline{\mathcal{Z}} \setminus \{0\}$, every limit ray of positive root rays (in the space $\mathrm{Ray}(V)$ of rays of V, as defined in [Dye12, 5.2]) is contained in the closure of the set of rays in the W-orbit of the ray spanned by z; the special case ([Dye12, Theorem 7.5(a)]) in which $z \in \mathcal{Z} \setminus \{0\}$ was a key step in the proof of Theorem 2.8.

(2) Theorem 3.4 corresponds, in the setting of [Dye12], to the following statement: the set of limit rays of positive roots is equal to the closure of the set of extreme rays of the closed imaginary cone $\overline{\mathcal{Z}}$. This result amounts to a new description of the set of limit rays of roots. More precisely, Theorem 3.4 is equivalent to the assertion in [Dye12, Remark 7.9(d)] that $\overline{\mathrm{R_{ext}}} = \mathrm{R_0}$, which was unproved there. It leads to substantial strengthenings of [Dye12, Corollary 7.9(c)–(d) and Remark (2)]; for example, the inclusions in [Dye12, Corollary 7.9(c)] are actually equalities.

3.2. **Exposed faces of $\overline{Z}$.** The aim of the rest of this section is to prove Theorem 3.4. In order to do so, we need to carefully study the convexity properties that are enjoyed by the imaginary convex body $\overline{Z}$. We refer to [Web94, Chap.2] for more details on convexity theory.

A *supporting half-space* of a convex set C is a closed (affine) half-space in V which contains C and has a point of C in its boundary; the boundary (which is an affine hyperplane) of the half-space is then called a *supporting hyperplane* of C. An *exposed face* of C is defined to be a subset of C which is either the intersection of C with a supporting hyperplane of C, empty, or equal to C. Exposed faces of C are faces of C. A face or exposed face F of C is said to be proper if $F \neq C$. It is known that any proper face of C is contained in the relative boundary $\mathrm{rb}(C)$ of C.

Assume for notational convenience in this subsection that B is non-singular (see Remark 2.6). For any linear hyperplane H of V there is $x \in V \setminus \{0\}$ such that

$$x^\perp = \{v \in V \mid B(v, x) = 0\} = H.$$

Since V_1 is an affine hyperplane that does not contain 0, any affine hyperplane $\mathcal{H}$ of V_1 is the intersection of V_1 with a linear hyperplane of V: there is $x \in V$ such that $\mathcal{H} = x^\perp \cap V_1$. In particular, any (affine) half-space in V that contains the imaginary convex body $\overline{Z} = \mathrm{conv}(E)$ has a boundary of this form. The next proposition, which refines parts of [Dye12, Proposition 7.10], describes special properties of certain exposed faces of $\overline{Z}$.

Proposition 3.7. *Let $x \in \overline{Z}$ and $F := \overline{Z} \cap x^\perp$. Then:*

(a) *F is an exposed face of $\overline{Z}$.*

(b) *If U is an open subset of $\overline{Z}$ which contains F, then for some $\epsilon > 0$ one has:*

$$U \supseteq \{z \in \overline{Z} \mid B(x, z) > -\epsilon\} \supseteq F.$$

(c) *If $x \notin \widehat{Q}$ (i.e. is non-isotropic), then $x \notin F$, so F is a proper face of $\overline{Z}$.*

(d) *If $x \in \widehat{Q}$ (i.e. is isotropic), then $x \in F$, so F is a non-empty face of $\overline{Z}$.*

Proof. (a) If $x \in V^\perp$, then $F = \overline{Z}$ is an exposed face of $\overline{Z}$ (by convention). Assume now that $x \notin V^\perp$. For any $z, z' \in \overline{\mathcal{Z}}$, one has $B(z, z') \leq 0$ by Equation (2.2). This implies that $\overline{Z}$ is contained in the half-space $\{\, z \in V \mid B(x, z) \leq 0 \,\}$ of V. If $\overline{Z} \cap x^\perp = \varnothing$, then $F = \varnothing$ is an exposed face of $\overline{Z}$ by convention. Otherwise, by definition, $x^\perp$ is a supporting hyperplane of $\overline{Z}$, so $F = \overline{Z} \cap x^\perp$ is an exposed face of $\overline{Z}$.

(b) Let U be an open subset of $\overline{Z}$ containing F. Since $\overline{Z} \setminus U$ is compact, we can define $\epsilon := \min(-\{\, B(x, z) \mid z \in \overline{Z} \setminus U \,\})$. For $z \in \overline{Z}$, one has $B(x, z) \leq 0$, and $B(x, z) = 0$ if and only if $z \in F = \overline{Z} \cap x^\perp$ (which is contained in U). So $\epsilon > 0$. Then $F \subseteq \{\, z \in \overline{Z} \mid B(x, z) > -\epsilon \,\} \subseteq U$. Parts (c)–(d) follow directly from the definitions. $\square$

3.3. **Extreme limit roots and exposed limit roots.** An *exposed point* of a convex set C is a point $c \in C$ such that $\{c\}$ is an exposed face of c. In particular, any exposed point of c is an extreme point of C. The converse is not true, see for example the picture [Web94, Fig. 2.10]. Recall that if C is compact, the set of extreme points of C is the minimal subset of C whose convex hull is equal to C. Moreover, Strascewicz' theorem asserts that every extreme point of C is in the closure of the set of exposed points of C.

Let $E_{\exp}$ denote the set of exposed points of $\overline{Z} = \operatorname{conv}(E)$: we call its points *exposed limit roots*. Since $\overline{Z}$, is convex and compact, we have the following inclusions:

$$E_{\exp} \subseteq E_{\mathrm{ext}} \subseteq \overline{E_{\exp}} = \overline{E_{\mathrm{ext}}}, \quad \text{and} \quad \overline{E_{\mathrm{ext}}} \subseteq E = \overline{E} \subseteq \overline{Z} \cap \widehat{Q}.$$

Moreover,

$$(3.2) \qquad \operatorname{conv}(E_{\mathrm{ext}}) = \operatorname{conv}(\overline{E_{\exp}}) = \operatorname{conv}(E) = \overline{Z}.$$

It was already known that the W-action preserves E, Z, $\overline{Z}$ and $\widehat{Q} \cap \overline{Z}$. From Proposition 2.7, one sees that the W-action on $\overline{Z}$ sends convex sets to convex sets and preserves $E_{\exp}$, E_{ext}, and $\overline{E_{\exp}}$. It also (obviously) preserves the signs of inner products, in the sense that $B(x, y)$ and $B(w \cdot x, w \cdot y)$ have the same sign (positive, negative or zero) for all $w \in W$ and $x, y \in \overline{Z}$.

Lemma 3.8. *Let (Φ, Δ) be irreducible of indefinite type. Then:*

(a) *For any $x \in \operatorname{cone}(\Delta) \setminus \{0\}$, there exists $y \in E_{\exp}$ with $B(x, y) \neq 0$.*
(b) *If $x \in \overline{Z}$ in (a), then $B(x, y) < 0$.*

Proof. (a) Let $x \in \operatorname{cone}(\Delta) \setminus \{0\}$ be arbitrary. By Proposition 2.5, we have $\operatorname{aff}(Z) = \operatorname{aff}(\widehat{\Delta})$. By Equation (3.2), $\operatorname{aff}(E_{\exp}) = \operatorname{aff}(E_{\mathrm{ext}}) = \operatorname{aff}(\widehat{\Delta})$ as well. We claim that there is some $y \in E_{\exp}$ with $B(x, y) \neq 0$. For otherwise, the above would imply that $B(x, \widehat{\Delta}) = 0$ and so $B(x, \Delta) = 0$ also. Since $x \in \operatorname{cone}(\Delta) \setminus \{0\}$ and Φ is irreducible of indefinite type, this is impossible by Lemma 2.4 (see also [HLR14, Prop. 4.8] or [Dye12, Lemma 7.1 and §4.5]). For (b), since $y \in E_{\exp} \subseteq \overline{Z}$, if in addition $x \in \overline{Z}$, then one has $B(x, y) \leq 0$ by Equation (2.2). $\square$

Proposition 3.9. *Let $x \in \overline{Z}$ and $F := \overline{Z} \cap x^\perp$. If Φ is irreducible of indefinite type and $x \in Q$ is isotropic, then $\varnothing \neq F \subsetneq \overline{Z}$ is a proper, non-empty, exposed face of $\overline{Z}$.*

Proof. By Proposition 3.7(d), it suffices to show that $F \subsetneq \overline{Z}$. By Lemma 3.8(b), there exists $y \in E_{\exp}$ such that $B(x, y) < 0$. Thus $y \in \overline{Z} \setminus x^\perp$, and the result follows. $\square$

Remark 3.10.

(1) It is easily seen that when A and B are disjoint convex sets, the join $A * B$ of A and B is convex and the set of extreme points of $A * B$ is the disjoint union of the set of extreme points of A and of the set of extreme points of B. It follows directly from Remark 3.3 that if Δ is linearly independent, then the set $E_{\text{ext}}(\Phi)$ of extreme limit roots of Φ is the disjoint union of the sets $E_{\text{ext}}(\Phi_i)$.

(2) As the sets E_{ext} and E_{exp} can be constructed from cones, their properties do not depend on the choice of the transverse hyperplane (V_1) used to define the normalization map. If H and H' are two different affine hyperplanes, both transverse to Φ^+, denote by π_H, $\pi_{H'}$ the associated normalization maps (see [HLR14, Prop. 5.3]), such that π_H sends $\text{conv}(\pi_{H'}(\Delta))$ to $\text{conv}(\pi_H(\Delta))$ and $E(\Phi, H')$ to $E(\Phi, H)$. Then π_H also maps $E_{\text{ext}}(\Phi, H')$ to $E_{\text{ext}}(\Phi, H)$ and $E_{\text{exp}}(\Phi, H')$ to $E_{\text{exp}}(\Phi, H)$.

3.4. **A fractal property and proof of Theorem 3.4.** We want to prove that the extreme limit roots (or, equivalently, the exposed limit roots) are dense in the limit roots. Before proving Theorem 3.4, and concluding this section, we state a last theorem that concisely encapsulates certain aspects of the "fractal" (self-similar) nature of the boundary of $\overline{Z}$ (see also §4 for other fractal-like properties). Only the weaker part (b) will be needed in the proof of Theorem 3.4, but (a) will be used in §6. Of course, we write $w \cdot X$ for the set $\widehat{w(X)} = \{w \cdot x \mid x \in X\}$, where $w \in W$ and $X \subseteq V \setminus V_0$.

Theorem 3.11. *Suppose that (Φ, Δ) is irreducible of indefinite type.*

Let $x \in E$ and (α_n) be a sequence in Φ^+ such that $\widehat{\alpha_n} \to x$ as $n \to \infty$.

Let $F := \overline{Z} \cap x^\perp$ (which is a proper face of $\overline{Z}$ containing x, by Proposition 3.9). Let U be an open subset of $\overline{Z}$ containing x and P be a closed subset of $\overline{Z}$ such that $F \cap P = \varnothing$.

(a) There exists $N \in \mathbb{N}$ such that for $n \geq N$, $s_{\alpha_n} \cdot P \subseteq U$, or equivalently $s_{\alpha_n} \cdot U \supseteq P$.

(b) For any $z \in \overline{Z} \setminus F$, one has $s_{\alpha_n} \cdot z \to x$ as $n \to \infty$.

The theorem implies the following self-similarity property: given a point $x \in E$ and its associated face $F := \overline{Z} \cap x^\perp$, if we consider any open neighborhood of x inside $\overline{Z}$ and any closed subset of $\overline{Z}$ disjoint from F, we can send the latter inside the former by the action of some element of W (see Figure 3).

Proof. Recall that φ denotes the linear form such that $\widehat{v} = v / \varphi(v)$ for any $v \in V \setminus V_0$. Since $x \in E$, by definition there is a sequence (α_n) of positive roots with $\widehat{\alpha_n} = \alpha_n / \varphi(\alpha_n) \to x$ as $n \to \infty$. Since x is isotropic and $B(\alpha_n, \alpha_n) = 1$ for all n, $\varphi(\alpha_n) \to \infty$ as $n \to \infty$. Note first that the property (a) obviously implies (b) (take $P = \{z\}$). The equivalence of the conditions $s_{\alpha_n} \cdot P \subseteq U$ and $s_{\alpha_n} \cdot U \supseteq P$ in (a) is clear since $(w, z) \mapsto w \cdot z$ is a W-action on $\overline{Z}$ and s_{α_n} is an involution.

To prove the inclusion $s_{\alpha_n} \cdot P \subseteq U$, it will suffice to show that $s_{\alpha_n} \cdot z \to x$ uniformly on P. Applying Proposition 3.7(b) to the open subset $\overline{Z} \setminus P \supseteq F$ of $\overline{Z}$ shows that there is some $\epsilon > 0$ such that $P \subseteq \{z \in \overline{Z} \mid B(x, z) \leq -\epsilon\}$. Since B is bilinear, the function $B : \text{conv}(\widehat{\Delta}) \times P \to \mathbb{R}$ is uniformly continuous. By compactness of P, the function $f : \text{conv}(\widehat{\Delta}) \to \mathbb{R}$ defined by $f(y) = \sup_{z \in P} B(y, z)$ is well-defined and continuous. So $U' = f^{-1}(]-\infty, -\epsilon/2[)$ is an open neighbourhood
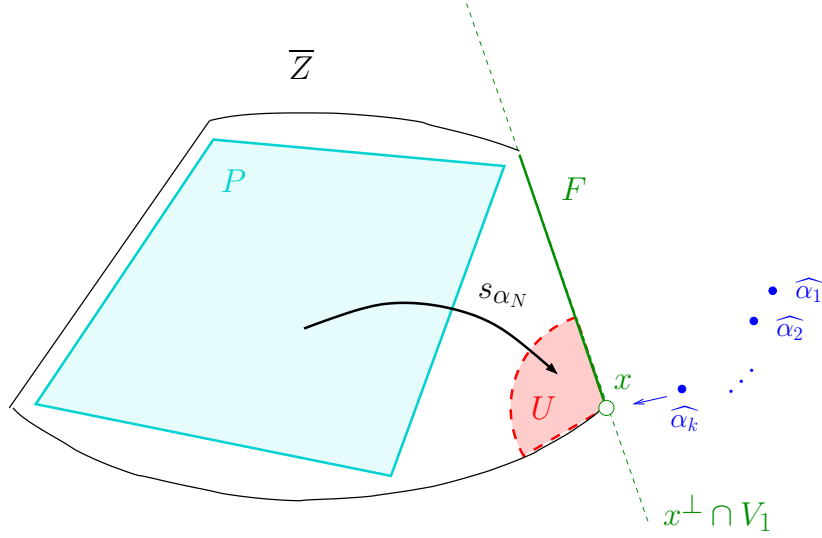
FIGURE 3. A schematic visualization of Theorem 3.11, displaying the self-similar nature of the boundary of $\overline{Z}$. Consider a limit root $x \in E$ (say $x = \lim(\widehat{\alpha_n})$), an open neighborhood U of x inside $\overline{Z}$ and a closed subset P of $\overline{Z}$ disjoint from the face F of $\overline{Z}$ containing x ($F = \overline{Z} \cap x^\perp$). Then for N large enough, s_{α_N} sends P inside U.

of x in $\mathrm{conv}(\widehat{\Delta})$ such that: $\forall z \in P, \forall y \in U', B(y,z) \le -\epsilon/2$. Moreover, for $z \in P$, one has $s_{\alpha_n}(z) = z - 2B(z, \alpha_n)\alpha_n$. Hence for n large enough so that $\widehat{\alpha}_n = \alpha_n / \varphi(\alpha_n) \in U'$, one has $\varphi(s_{\alpha_n}(z)) = 1 - 2B(z, \alpha_n)\, \varphi(\alpha_n) \ge 1 + \varphi(\alpha_n)\epsilon$. So :

(3.3) $\qquad\qquad\qquad \varphi(s_{\alpha_n}(z)) \to \infty$ as $n \to \infty$, uniformly on P.

Moreover,

$$ s_{\alpha_n} \cdot z = \frac{z}{\varphi(s_{\alpha_n}(z))} - 2\frac{B(z, \alpha_n)}{\varphi(s_{\alpha_n}(z))}\alpha_n = \frac{z}{\varphi(s_{\alpha_n}(z))} + \frac{\varphi(s_{\alpha_n}(z)) - 1}{\varphi(s_{\alpha_n}(z))}\, \widehat{\alpha}_n . $$

Since P is compact, P is bounded and therefore, using (3.3), we have $\frac{z}{\varphi(s_{\alpha_n}(z))} \to 0$ and $\frac{\varphi(s_{\alpha_n}(z)) - 1}{\varphi(s_{\alpha_n}(z))} \to 1$, uniformly on P. Since $\widehat{\alpha}_n \to x$, we conclude that $s_{\alpha_n} \cdot z$ converges to the same limit as $\widehat{\alpha}_n$, that is converges to x, uniformly on P. $\qquad\square$

We can now prove the density of E_{ext} in E.

Proof of Theorem 3.4. First we treat the case in which (Φ, Δ) is an irreducible root system. If Φ is finite, then there are no limit roots and no extreme limit roots. If Φ is affine, there is one limit root (see [HLR14, Cor. 2.15]), and the closed imaginary cone consists of the ray of this root alone. Hence the desired conclusion holds in these two cases. Suppose henceforward that (Φ, Δ) is of indefinite type. Let $x \in E$. We may choose a sequence (α_n) in Φ^+ such that $\widehat{\alpha_n} \to x$ as $n \to \infty$. By Lemma 3.8(b), there exists $y \in E_{\mathrm{exp}}$ with $B(x, y) < 0$. In particular, y does not lie in the face $\overline{Z} \cap x^\perp$, so by Theorem 3.11(b), $s_{\alpha_n} \cdot y$ converges to x as $n \to \infty$. Since E_{exp} is stable by the W-action (see Proposition 2.7), $s_{\alpha_n} \cdot y$ lies in E_{exp} for any n. Hence x is in the closure $\overline{E_{\mathrm{exp}}}$. Since exposed points are extreme, this concludes the proof in case Φ is irreducible.

It remains to deal with the case in which (Φ, Δ) is a root system such that Δ is linearly independent. Denote its irreducible components by (Φ_i, Δ_i) for $i = 1, \ldots, p$. Then $E(\Phi) = \bigsqcup_i E(\Phi_i)$, and, by Remark 3.10(1), the set of extreme limit roots satisfies $E_{\text{ext}}(\Phi) = \bigsqcup_i E_{\text{ext}}(\Phi_i)$. This reduces the proof to the case in which the root system is irreducible, which is already known.   □

4. FRACTAL PROPERTIES

We already explained in Theorem 3.11 a fractal property of E. In this section, we turn our attention to two conjectures about fractal descriptions of the set of limit roots $E(\Phi)$ that are stated in the prequel of this article, see [HLR14, §3.2] and notice Figure 4 below. We use the minimality of the W-action from Theorem 3.1(b), as well as some additional works on the case where Φ is weakly hyperbolic, to completely settle [HLR14, Conjecture 3.9] and, in the case of weakly hyperbolic Coxeter groups, to settle the conjecture stated just above Conjecture 3.9 in [HLR14, §3.2].
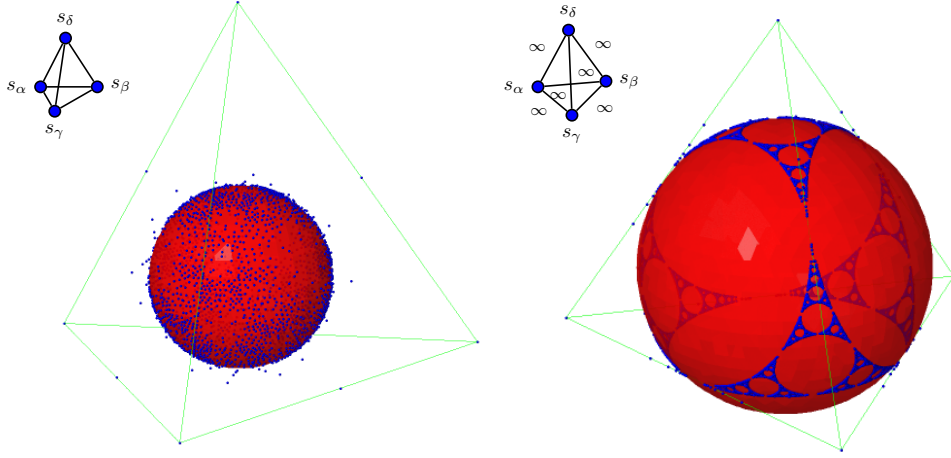

FIGURE 4. The normalized isotropic cone $\widehat{Q}$ and the first normalized roots (with depth ≤ 8) for two weakly hyperbolic root systems, whose diagrams are in in the upper left of each picture. In the left picture, the root system is of hyperbolic type and $E(\Phi)$ is the whole $\widehat{Q}$, whereas in the right picture the root system is strictly weakly hyperbolic and the roots converge toward a fractal form.

Let (Φ, Δ) be a based root system in (V, B), with associated Coxeter group (W, S). For simplicity, we assume throughout this section that $\text{span}(\Delta) = V$, and we denote by n the dimension of V. The interplay between $\widehat{Q}$, $\text{conv}(\widehat{\Delta})$ and the faces of the polytope $\text{conv}(\widehat{\Delta})$ is at the heart of the fractal properties of E.

Let us recall the existing link between the faces of $\text{conv}(\widehat{\Delta})$ and some standard parabolic subgroups of (W, S). Recall that a *standard parabolic subgroup* of W is a subgroup W_I of W generated by a subset I of S. It is well known that:

- (W_I, I) is a Coxeter system;

- (Φ_I, Δ_I) is a based root system in[3] $(V_I, B_{|V_I})$ with associated Coxeter group W_I, where:

$$\Delta_I := \{\alpha \in \Delta \mid s_\alpha \in I\}, \quad \Phi_I := W_I(\Delta_I) \quad \text{and} \quad V_I := \mathrm{span}(\Delta_I).$$

This allows us to define easily the subset $E(\Phi_I)$ of $E(\Phi)$, consisting of the accumulation points of $\widehat{\Phi_I}$ (see [HLR14, §5.4]).

We say that $I \subseteq S$ (or, $\Delta_I \subseteq \Delta$) is *facial for the based root system* (Φ, Δ) if $\mathrm{conv}(\widehat{\Delta_I})$ is a face of the polytope $\mathrm{conv}(\widehat{\Delta})$. The corresponding standard parabolic subgroup W_I is then called a *standard facial subgroup for* (Φ, Δ), and (Φ_I, Δ_I) is called a *facial root subsystem*. If Δ is a basis for V then obviously any standard parabolic subgroup is a standard facial subgroup. But if (Φ, Δ) is, for instance, a rank 4 based root system in V of dimension 3 (see for example [HLR14, Example 5.1]), then the subsets of S corresponding to the diagonals of the quadrilateral $\mathrm{conv}(\widehat{\Delta})$ are obviously not facial.

Remark 4.1. The notion of standard facial reflection subgroup which we use in this paper differs from that in [Dye12]. Their relationship may be characterized as follows: the family of standard facial reflection subgroups as defined in this paper is unchanged, as a family of subgroups of W, by extension or restriction of ambient vector space (as in Remark 2.6) and coincides with that in [Dye12] for based root systems (Φ, Δ) in (V, B) for which B is non-singular. We refer the reader to [Dye12, §2,§8] for more details and properties of standard facial subgroups.

4.1. **Facial subgroups, hyperbolicity and a self-similar dense subset of the set of limit roots.** We first completely characterize the root systems that have the same property as the one in the left picture in Figure 4 shows, i.e., such that $\widehat{Q} = E$. This settles [HLR14, Conjecture 3.9(i)].

Definition 4.2. We say that (Φ, Δ) is *weakly hyperbolic* if the signature of the bilinear form B is $(n-1, 1)$, where n is the dimension of $\mathrm{span}(\Delta)$. We say that a weakly hyperbolic based root system (Φ, Δ) is *hyperbolic* if every proper facial root subsystem of (Φ, Δ) has all its irreducible components of finite or affine type.

Remark 4.3.

(1) When $|\Delta| = 2, 3$, then any root system (Φ, Δ) of indefinite type is weakly hyperbolic. In higher ranks, there are still many families of weakly hyperbolic root system. For example, this is the case when all the inner products $B(\alpha, \beta)$ are the same (and non-zero) for any $\alpha \neq \beta \in \Delta$ (see the examples of Figure 4). In particular, any universal Coxeter group (where the labels of the Coxeter graph are all ∞) can be associated with a root system of weakly hyperbolic type. It is not true in general that all the based root systems associated to any universal Coxeter groups are of signature $(n-1, 1)$; see Figure 5 for such an example in rank 4 with signature $(2, 2)$, see also [Dye12, Example 1.4].

(2) If (Φ, Δ) is weakly hyperbolic and reducible, then all but one of its irreducible components are of finite type, and the remaining one is weakly hyperbolic. Also, if (Φ, Δ) is hyperbolic, then it is irreducible. For details

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[3]Note that (Φ_I, Δ_I) can also be seen as a based root system in (V, B), since we do not require that the simple roots generate the whole space in the definition of based root system.

on these different notions of hyperbolicity, we refer to [Dye12, §9.1-2] and the references therein.

Theorem 4.4. *Assume (Φ, Δ) is irreducible of indefinite type. Then the following properties are equivalent:*

(i) (Φ, Δ) is hyperbolic;
(ii) $\widehat{Q} \subseteq \operatorname{conv}(\widehat{\Delta})$;
(iii) $E(\Phi) = \widehat{Q}$.

The proof of this theorem is postponed to §4.4, but we use the theorem now to explain the qualitative appearance of the right picture of Figure 4. The idea of Conjecture 3.9 in [HLR14] is to describe $E(\Phi)$ by acting with W only on the limit roots of parabolic root subsystems Φ_I such that $\widehat{Q} \cap \operatorname{span}(\Delta_I) \subseteq \operatorname{conv}(\widehat{\Delta_I})$. By Theorem 4.4, we know that in this case $E(\Phi_I) = \widehat{Q_I}$. This will explain why the set of limit roots in Figure 4, or in Figure 1, looks like a self-similar union of circles.

In general, say that a subset $\Delta_I \subseteq \Delta$ is generating if $\widehat{Q_I} := \widehat{Q} \cap \operatorname{span}(\Delta_I) \subseteq \operatorname{conv}(\widehat{\Delta_I})$. Denote the set of irreducible generating subsets Δ_I of Δ such that Φ_I is not finite as

$$\operatorname{Gen}(\Phi, \Delta) = \{\Delta_I \subseteq \Delta \mid (\Phi_I, \Delta_I) \text{ is irreducible and } \varnothing \neq \widehat{Q_I} \subseteq \operatorname{conv}(\Delta_I)\}$$

(note that Φ_I is infinite if and only if $\widehat{Q_I} \neq \varnothing$). Using Theorem 4.4, we have:

$$\operatorname{Gen}(\Phi, \Delta) = \{\Delta_I \subseteq \Delta \mid (\Phi_I, \Delta_I) \text{ is irreducible and of hyperbolic or affine type}\}$$

$$= \{\Delta_I \subseteq \Delta \mid (\Phi_I, \Delta_I) \text{ is irreducible and } \widehat{Q_I} = E(\Phi_I) \neq \varnothing.\}$$

This theorem settles the discussion at the end of section 2.2 in [HLR14], and proves [HLR14, Conjecture 3.9(i)]. Indeed, one easily sees that if Δ is linearly independent[4], a subset $\Delta_I \subseteq \Delta$ is generating if and only if all its components are generating and it has at most one component of infinite type; this implies by the above that if Δ_I is generating, one has $\widehat{Q_I} = E(\Phi_I)$.

The second item of the next corollary, which is a consequence of Theorem 3.1 and Theorem 4.4, settles [HLR14, Conjecture 3.9(ii)].

Corollary 4.5 ([HLR14, Conj. 3.9](ii))**.** *Let (Φ, Δ) be an irreducible root system in (V, B). Then:*

(i) $\operatorname{Gen}(\Phi, \Delta)$ is empty if and only if Φ is finite;
(ii) the set E is the topological closure of the subset $\mathcal{F}_0$ of $\widehat{Q}$ defined by:

$$\mathcal{F}_0 := W \cdot \left(\bigcup_{\Delta_I \in \operatorname{Gen}(\Phi, \Delta)} \widehat{Q_I} \right).$$

In the example of right-hand side of Figure 4, the set $\mathcal{F}_0$ is the self-similar fractal constituted by the circles appearing on the facets (which are the $\widehat{Q_I}$'s) and all their W-orbits, the first of them are the smaller visible circles; thus $\mathcal{F}_0$ is an infinite (countable) union of circles, similar to an Apollonian gasket drawn on a sphere; see also the example of Figure 14(a), where the only generating subsets of Δ are

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[4]In [HLR14], it is assumed that Δ is a basis of V; in this case any parabolic based root subsystem is facial.
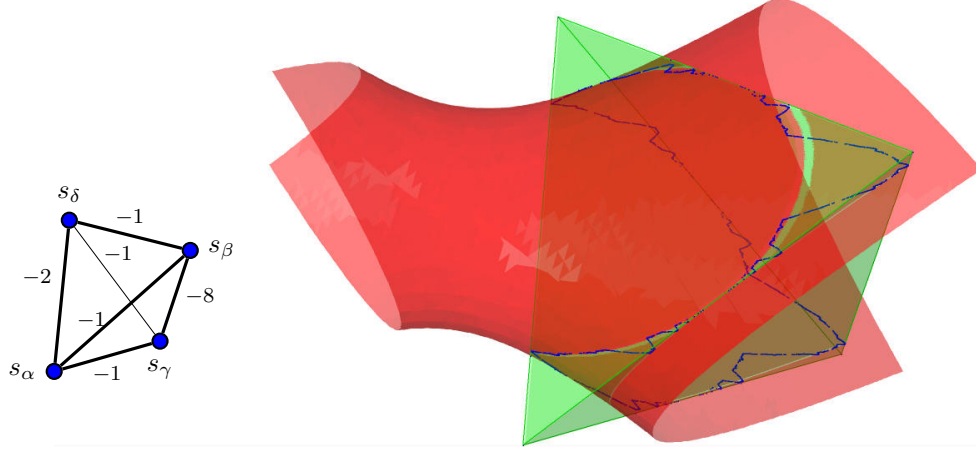
FIGURE 5. A based root system which is not weakly hyperbolic, but whose associated Coxeter group is the universal Coxeter group of rank 4. The inner products between the simple roots are indicated in the diagram on the left. The signature of the bilinear form is $(2, 2)$, and $\widehat{Q}$ is a hyperboloid of one sheet. The normalized roots are drawn until depth 8. Note that the geometry of the limit shape E looks very different from the case of weakly hyperbolic root systems.

$\{\alpha, \beta, \gamma\}$ and $\{\alpha, \gamma, \delta\}$, which produce, respectively, the single limit root of the bottom face and the ellipse of limit roots of the left face of the tetrahedron.

Remark 4.6.

(1) Define a based root system (Φ, Δ) to be *compact hyperbolic* if every proper facial root subsystem of (Φ, Δ) has all its irreducible components of finite type. The notion of hyperbolic (resp., compact hyperbolic) Coxeter group defined in [Hum90, §6.8] corresponds in our setting to a Coxeter group with a hyperbolic (resp., compact hyperbolic) based root system such that the simple system is a basis of V. Using the theorem and its corollary, it is easy to deduce the following addition: an irreducible root system of indefinite type (Φ, Δ) is compact hyperbolic if and only if $\widehat{Q} \subseteq \mathrm{relint}(\mathrm{conv}(\widehat{\Delta}))$ where $\mathrm{relint}(X)$ denotes relative interior of X, see for instance [Dye12, Appendix A] where it is denoted by "ri(X)".

(2) By looking carefully at the proof below, we note that the corollary still holds if we replace $\mathrm{Gen}(\Phi, \Delta)$ by the set $\mathrm{Gen}'(\Phi, \Delta)$ of $\Delta_I \subseteq \Delta$ such that (Φ_I, Δ_I) is irreducible of affine type or compact hyperbolic type.

Proof of Corollary 4.5. (i) Clearly $\mathrm{Gen}(\Phi, \Delta)$ is empty when Φ is finite. Suppose now that Φ is irreducible infinite. We prove that $\mathrm{Gen}(\Phi, \Delta) \neq \varnothing$ by induction on the rank of the root system. When (Φ, Δ) has rank 2, $\Delta \in \mathrm{Gen}(\Phi, \Delta)$. Suppose now that the property is true until some rank $n-1 \geq 2$, and take (Φ, Δ) irreducible infinite of rank n. If (Φ, Δ) has a proper facial root subsystem Φ_I which is infinite, we can conclude by applying the induction hypothesis on one infinite irreducible component of Φ_I. Suppose now that every proper facial root subsystem is finite.

Then the positive index in the signature of B is at least $n-1$, so (Φ, Δ) is (irreducible) of finite, affine, or weakly hyperbolic type. It is not finite by hypothesis, and if it is affine, then Δ is in $\mathrm{Gen}(\Phi, \Delta)$. If (Φ, Δ) is weakly hyperbolic, since every proper facial root subsystem is finite, (Φ, Δ) is hyperbolic by definition (it is even compact hyperbolic), so Δ is in $\mathrm{Gen}(\Phi, \Delta)$.

(ii) If $\Delta_I \in \mathrm{Gen}(\Phi, \Delta)$, then $\widehat{Q_I} = E(\Phi_I) \subseteq E(\Phi)$: indeed, the affine case is clear (see [HLR14, Cor. 2.15]), and the hyperbolic case is one implication of Theorem 4.4. Moreover $\widehat{Q_I} \neq \varnothing$, so by (i), $\mathcal{F}_0$ is not empty and contained in E. It is also stable by W, and from Corollary 3.1(b), the orbit of any point of E is dense in E, so $\mathcal{F}_0$ is dense in E. $\qquad\square$

4.2. **Fractal description of E using parts of the isotropic cone $\widehat{Q}$.** In this subsection, we study the conjecture mentioned in [HLR14, §3.2] before Conjecture 3.9. We start by constructing a natural subset $\mathcal{F}$ of $\widehat{Q}$ by removing the parts of $\widehat{Q}$ that cannot belong to E for straightforward reasons[5]. We conjecture below that $\mathcal{F}$ is actually equal to E, and prove this conjecture when the root system is weakly hyperbolic.

The construction of the set $\mathcal{F}$ was roughly described in [HLR14, §3.2]. Here we need a more precise definition, taking care of the fact that the W-action is not defined everywhere on V_1. Let (Φ, Δ) be an irreducible root system. We denote by D the part of V_1 on which W acts (see [HLR14, §3.1]):

$$D = V_1 \cap \bigcap_{w \in W} w(V \setminus V_0).$$

Is clear that the domain D^+ considered in §2.3 is contained in D.

Definition 4.7. Denote by $\widehat{Q}_{\mathrm{act}} := \widehat{Q} \cap D$ the part of $\widehat{Q}$ where W acts. Then

$$\mathcal{F} := \widehat{Q}_{\mathrm{act}} \setminus \bigcup_{w \in W} w \cdot (\widehat{Q}_{\mathrm{act}} \setminus \widehat{Q}_{\mathrm{act}} \cap \mathrm{conv}(\widehat{\Delta})).$$

The idea of the construction of $\mathcal{F}$ is actually mostly naive. We know the limit roots are in $\widehat{Q}_{\mathrm{act}} = \widehat{Q} \cap D$, but there cannot be any limit root outside $\mathrm{conv}(\widehat{\Delta})$. Since E is W-stable, there cannot be any limit root also in the orbits of the parts of $\widehat{Q}$ that are outside $\mathrm{conv}(\widehat{\Delta})$. We remove all these parts from $\widehat{Q}$ to construct $\mathcal{F}$. In the example pictured in Figure 4(right), $\mathcal{F}$ is the complement, in the red sphere, of the union of the open spherical caps associated to the circles used to define $\mathcal{F}_0$. Similarly, in Figure 14(a), $\mathcal{F}$ is the complement of the union of the open ellipsoidal caps which are the images of the one cut by the left face.

We give in Proposition 4.19 other characterizations of $\mathcal{F}$, in particular $\mathcal{F} = \mathrm{conv}(E) \cap \widehat{Q}$.

Remark 4.8. As explained in [HLR14, §3.2], E is contained in $\mathcal{F}$. Indeed, it is contained in Q ([HLR14, Theorem 2.7]), contained in $\mathrm{conv}(\widehat{\Delta})$ (clear), contained in D and stable by W ([HLR14, Prop. 3.1]).

Question 4.9. For any irreducible root system, is E equal to $\mathcal{F}$?

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[5]We use the letter $\mathcal{F}$ for the sake of consistency with [HLR14], and because $\mathcal{F}$ can be thought of as a fractal set.

We are able to answer this question for any weakly hyperbolic root system, using the specific geometry of $\widehat{Q}$ in this case. We do not know if this result extends to more general root systems.

Theorem 4.10. *Let (Φ, Δ) be an irreducible root system. Assume that it is* weakly hyperbolic. *Then:*

(i) $\mathcal{F} = E = \widehat{Q} \cap \operatorname{conv}(E)$;

(ii) *E is the unique non-empty, closed subset of $\widehat{Q}_{\mathrm{act}} \cap \operatorname{conv}(\widehat{\Delta})$, which is stable by W.*

Remark 4.11.

(1) In [HMN14], Higashitani, Mineyama and Nakashima prove both conjectures of [HLR14, §3.2] in the case of rank $(n-1, 1)$ root systems with Δ linearly independent. They also obtain, as a by-product of their work, Theorem 3.1(b) under these assumptions. Their proof is different from ours and based on a careful analysis of the limit roots by looking at $\widehat{Q}$ as a metric space $(\widehat{Q}, d_B)$. It would be interesting to see which part of their work could be generalized to based root systems of arbitrary ranks that are not necessarily weakly hyperbolic.

(2) In the particular case where the root system is *hyperbolic*, the theorem is implied by Theorem 4.4, and $E = \mathcal{F} = \widehat{Q}$ is homeomorphic to an $(n-2)$-sphere (see Remark 4.20).

The proof of Theorem 4.10 is postponed to §4.5. The theorem implies that $\operatorname{Acc}(W \cdot z) \subseteq E$ for $z \in Z$ if W is weakly hyperbolic; we show equality in Corollary 6.15.

Remark 4.12. Suppose that the rank of Φ is 3 or 4 and Φ is weakly hyperbolic. In this case, we can always describe E using Theorem 4.10. The Coxeter group W acts on E as a group generated by hyperbolic reflections, so can be seen as a Fuchsian or Kleinian group. Using this point of view, the set E is no other than the limit set of a Kleinian group, which explains the shape of Apollonian gasket obtained in [HLR14, Fig. 9]. These relations are explored in the general context of Lorentzian spaces in the article [HPR13], and are outlined in §7.4 at the end of this article.

4.3. **The normalized isotropic cone for weakly hyperbolic groups.** Before proving Theorem 4.4 and Theorem 4.10, we first describe what $\widehat{Q}$ looks like when the root system (Φ, Δ) is weakly hyperbolic. Recall that we assume (without loss of generality) that $\operatorname{span}(\Delta) = V$ and denote by n the dimension of V, which is smaller than or equal to the rank $|\Delta|$ of (Φ, Δ). Recall that a hyperplane of V is said to be *transverse to* Φ^+ if for any $\rho \in \Phi^+$, H intersects the ray $\mathbb{R}_{>0}\rho$ in one point. We prove below that we can find a hyperplane H of V, transverse to Φ^+, and such that $Q \cap H$ is an $(n-2)$-dimensional sphere. This will be useful in the following subsections.

Proposition 4.13. *Let (Φ, Δ) be an irreducible based root system in (V, B) of dimension n. Suppose that (Φ, Δ) is weakly hyperbolic. Then there exists a basis $(e_1, \ldots, e_n)$ for V such that:*

(i) *the restriction of B to the hyperplane $H_0 := \operatorname{span}(e_1, \ldots, e_{n-1})$ is positive definite;*

(ii) *the matrix of B in this basis is $\operatorname{Diag}(1, \ldots, 1, -1)$;*

(iii) Q intersects the affine hyperplane $H := e_n + H_0$ in an $(n-2)$-dimensional sphere; if $x_1, \ldots, x_n$ are the coordinates in the basis,

$$Q \cap H = \{(x_1, \ldots, x_{n-1}, 1) \mid x_1^2 + \cdots + x_{n-1}^2 = 1\} \; ;$$

(iv) the vector e_n satisfies $B(e_n, \alpha) < 0$, for all $\alpha \in \Delta$;

(v) H is transverse to Φ^+.

Remark 4.14. In the case (Φ, Δ) is weakly hyperbolic but not irreducible, then only one of its irreducible components (say (Φ', Δ')) is infinite, and (Φ', Δ') is weakly hyperbolic (see Remark 4.2). So Q lives in $\mathrm{span}(\Delta')$ and Proposition 4.13 implies that there is a transverse hyperplane H such that $Q \cap H$ is a sphere of dimension $\dim(\mathrm{span}(\Delta')) - 2$.

Note that items (i)-(iii) are straightforward. Indeed, as the signature of B is $(n-1, 1)$, there exists a basis such that the equation of Q is $x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = 0$, and Q is a conical surface on $\mathbb{S}^{n-2}$. The fact that H is transverse to Φ^+ (item (v)) is not direct and will follow from (iv) and Lemma 2.4.

Proof. Let (Φ, Δ) be an irreducible root system of weakly hyperbolic type. Take z as in Lemma 2.4. Since $B(z, \alpha) < 0$ for all $\alpha \in \Delta$, and z is in[6] $\mathrm{cone}(\Delta) \setminus \{0\}$, we have $B(z, z) < 0$. Denote $e_n := z/\sqrt{-B(z, z)}$, so that $B(e_n, e_n) = -1$, and e_n satisfies item (iv).

Since the signature of B is $(n-1, 1)$, we can complete $\{e_n\}$ in a basis $(e_1, \ldots, e_n)$ such that the matrix of B in this basis is $\mathrm{Diag}(1, \ldots, 1, -1)$. Let H_0 be the hyperplane spanned by $e_1, \ldots, e_{n-1}$. The restriction of B to H_0 is positive definite, so $(H_0, B_{|H_0})$ is a Euclidean plane with $(e_1, \ldots, e_{n-1})$ as an orthonormal basis. Note that $B(e_n, v) = 0$ for any $v \in H_0$. Consider $H := e_n + H_0$ the affine hyperplane directed by H_0 and passing through the point e_n.

This proves items (i) and (ii). Item (iii) follows since the equation of Q in the chosen basis is $x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = 0$, so:

$$Q \cap H = \{(x_1, \ldots, x_n) \mid x_n = 1 \text{ and } x_1^2 + \cdots + x_{n-1}^2 = 1\}.$$

We are left to proving item (v), that is, H is transverse to Φ^+: we have to show that $\mathbb{R}_{>0}\alpha \cap H$ is nonempty for all $\alpha \in \Delta$. Since $0 \notin H$, the line $\mathbb{R}\alpha$ cannot be contained in H, and therefore neither can $\mathbb{R}_{>0}\alpha$. Assume by contradiction that $\mathbb{R}_{>0}\alpha \cap H = \varnothing$ for some $\alpha \in \Delta$. We have two cases:

- either $\mathbb{R}_{>0}\alpha$ is in H_0 and so is the line $\mathbb{R}\alpha$. Therefore $\alpha \in H_0$. So $B(\alpha, e_n) = 0$ contradicting $B(\alpha, e_n) < 0$;
- or $\mathbb{R}_{<0}\alpha \cap H$ is a point $\lambda\alpha = e_n + u$, with $\lambda < 0$ and $u \in H_0$. Since $B(e_n, u) = 0$ we have $-1 = B(e_n, e_n) = B(\lambda\alpha - u, e_n) = \lambda B(\alpha, e_n)$. But $\lambda < 0$ and $B(\alpha, e_n) < 0$, so we get a contradiction.

Thus, overall, $\mathbb{R}_{>0}\alpha \cap H$ must be a point. $\qquad\square$

Proposition 4.13 will be used in the two following subsections. We can already deduce an interesting consequence on the set E_{ext} of extreme points of $\overline{Z} = \mathrm{conv}(E)$: as they live on a sphere, no limit root can be written as a convex combination of other limit roots.

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[6]In this proof we do not need the stronger statement that z is in the topological interior of $\mathrm{cone}(\Delta)$.

Corollary 4.15. *Let (Φ, Δ) be an irreducible based root system. If (Φ, Δ) is weakly hyperbolic, then $E_{\mathrm{ext}}(\Phi) = E(\Phi)$.*

After Theorem 3.4, it is always true that $\overline{E_{\mathrm{ext}}} = E$. The equality $E_{\mathrm{ext}} = E$ is not always valid (see Remark 3.5(3)). We do not know how to characterize root systems such that $E_{\mathrm{ext}} = E$.

Proof. From Proposition 4.13, we can choose the transverse hyperplane V_1 such that $\widehat{Q} = Q \cap V_1$ is a sphere. This does not change the properties of E and E_{ext}, as explained in Remark 1. Suppose there exists $x \in E \setminus E_{\mathrm{ext}}$. Then x is a linear combination with positive coefficients of points $x_1, \ldots, x_p$ in E (with $p > 1$). Since $E \subseteq \widehat{Q}$, $x_1, \ldots, x_p$ lie on the sphere $\widehat{Q}$. So x can not be in $\widehat{Q}$ (since every point in $\widehat{Q}$ is an extreme point of the ball $\mathrm{conv}(\widehat{Q})$), which contradicts the inclusion $E \subseteq \widehat{Q}$. □

4.4. **Proof of Theorem 4.4.** To prove Theorem 4.4 we will use the following (technical but elementary) lemma, which answers the question: when can $\widehat{Q}$ be bounded? Note that the letters Q and B below are specific to the lemma and its proof, and more general than in the framework of the article.

Lemma 4.16. *Let B be a symmetric bilinear form on a n-dimensional vector space V (over a field of characteristic 0). Define the associated quadric*

$$Q := \{v \in V \mid B(v, v) = 0\}.$$

Let H be an affine (nonlinear) hyperplane in V. If $Q \cap H$ is bounded, then we have one of the following:

- *B is positive (definite or not);*
- *B is negative (definite or not);*
- *B has signature $(n - 1, 1)$ or $(1, n - 1)$.*

More precisely, if B does not satisfy any of these conditions, then $Q \cap H$ contains an affine line.

Remark 4.17.

(1) The converse is obviously false: for example when the signature of B is $(n - 1, 1)$, Q is a cone on a sphere, so the boundedness of $Q \cap H$ depends on the choice of the hyperplane H.

(2) Of course Q is unchanged if we replace B by $(-B)$, so if we assume the signature of B to be (p, q) with $p \geq q$, the lemma is equivalent to

$$Q \cap H \text{ bounded} \implies q = 0 \text{ or } (p, q) = (n - 1, 1).$$

Proof. Denote $\mathrm{sgn}\, B = (p, q)$, and set $r = n - (p + q)$. Let us choose an adapted basis for V such that $X = (x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r) \in Q \cap H$ if and only if:

$$(4.1) \quad \begin{cases} x_1^2 + \ldots + x_p^2 - y_1^2 - \ldots - y_q^2 = 0 & (Q) \\ a_1 x_1 + \ldots + a_p x_p + b_1 y_1 + \ldots + b_q y_q + c_1 z_1 + \ldots + c_r z_r = 1 & (H) \end{cases}$$

where $(a_1, \ldots, b_1, \ldots, c_1, \ldots) \neq (0, \ldots, 0)$. Suppose that $Q \cap H$ is bounded. For simplicity we assume that $p \geq q$ (see Remark 4.17(2)), and we want to prove that $q = 0$ or $(p, q) = (n - 1, 1)$. Supposing this is not the case, we will reach a contradiction by constructing an affine line contained in $Q \cap H$.

(1) Suppose $r = 0$ and $q \geq 2$. By reordering the coordinates if needed, we can assume that $(a_1, b_1) \neq (0, 0)$. For any $s, t \in \mathbb{R}$, define $X(s, t) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$

where $x_1 = s$, $y_1 = \epsilon s$ (for some $\epsilon = \pm 1$), $x_2 = t$, $y_2 = t$, and $x_i = 0$, $y_j = 0$ for all $i, j \geq 3$. Then we clearly have $X(s,t) \in Q$, and $X(s,t) \in H$ if and only if

$$(4.2) \qquad (a_1 + b_1\epsilon)s + (a_2 + b_2)t = 1.$$

Since $(a_1, b_1) \neq (0,0)$ we can choose ϵ such that $a_1 + b_1\epsilon \neq 0$. Thus for any t there is a unique solution $s(t)$ in Equation 4.2. Hence $(X(s(t), t))_{t \in \mathbb{R}}$ is an affine line contained in $Q \cap H$.

(2) Suppose now $r \geq 1$ and $q \geq 1$. By reordering the coordinates, we can suppose that $(a_1, b_1, c_1) \neq (0,0,0)$. For any $s, t \in \mathbb{R}$, define $Y(s,t) = (x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r)$ where $x_1 = s$, $y_1 = \epsilon s$ (for some $\epsilon = \pm 1$), $z_1 = t$, and $x_i = 0$, $y_j = 0$, $z_k = 0$ for all $i, j, k \geq 2$. Then we clearly have $Y(s,t) \in Q$, and $Y(s,t) \in H$ if and only if

$$(4.3) \qquad (a_1 + b_1\epsilon)s + c_1 t = 1.$$

Since $(a_1, b_1, c_1) \neq (0,0,0)$ we can choose ϵ such that $(a_1 + b_1\epsilon, c_1) \neq (0,0)$. Thus Equation 4.3 has an affine line (L) of solutions for (s,t). Hence we obtain an affine line $(Y(s,t))_{(s,t) \in L}$ living in $Q \cap H$.

$\square$

By definition of finite, affine, and weakly hyperbolic type, Lemma 4.16 automatically implies the following property, which is the key point in the proof of Theorem 4.4.

Proposition 4.18. *Let (Φ, Δ) be an irreducible root system, and Q be the isotropic cone of its associated bilinear form. Let H be an affine nonlinear hyperplane. If the intersection $Q \cap H$ is bounded, then (Φ, Δ) is of finite, affine or weakly hyperbolic type. If (Φ, Δ) is of another type, then $Q \cap H$ contains an affine line.*

Proof of Theorem 4.4. $(iii) \implies (ii)$ is straightforward, since $E \subseteq \mathrm{conv}(\widehat{\Delta})$.

$(ii) \implies (i)$: $\widehat{Q} \subseteq \mathrm{conv}(\widehat{\Delta})$, so in particular $\widehat{Q} = Q \cap V_1$ is bounded. By Proposition 4.18, the root system (Φ, Δ) (which is assumed to be of indefinite type) is necessarily weakly hyperbolic. Let us choose the transverse hyperplane V_1 as in Proposition 4.13 such that $\widehat{Q} = Q \cap V_1$ is an $(n-2)$-dimensional sphere. Let I be a facial subset of Δ, and consider the facial root subsystem (Φ_I, I). Its normalized isotropic cone $\widehat{Q_I}$ is $\widehat{Q} \cap \mathrm{span}(I)$, so it is either (1) empty, or (2) a singleton, or (3) an $(|I| - 2)$-dimensional sphere of positive radius. Since $\widehat{Q} \subseteq \mathrm{conv}(\widehat{\Delta})$, $\widehat{Q}$ cannot cross nontrivially the faces of $\mathrm{conv}(\widehat{\Delta})$, and case (3) cannot arise. So any component of (Φ_I, I) is of finite or affine type. Hence, (Φ, Δ) is hyperbolic.

$(i) \implies (iii)$: Suppose (Φ, Δ) is hyperbolic. Choose the transverse hyperplane V_1 as in Proposition 4.13 such that $\widehat{Q} = Q \cap V_1$ is an $(n-2)$-dimensional sphere. From the specific study of the imaginary cone $\mathcal{Z}$ for hyperbolic groups in [Dye12], we have that $\overline{\mathcal{Z}} \cap V_1$ is equal to the ball $\mathrm{conv}(\widehat{Q})$ (this corresponds to the statement $\overline{\mathcal{Z}} = \overline{\mathcal{L}}$ in [Dye12, Prop. 9.4(c)]). From Theorem 2.2, we know that $\overline{\mathcal{Z}} \cap V_1 = \overline{Z} = \mathrm{conv}(E)$, so we get $\mathrm{conv}(E) = \mathrm{conv}(\widehat{Q})$, i.e., E contains the set of extreme points of $\widehat{Q}$. But since $\widehat{Q}$ is a sphere, any point in $\widehat{Q}$ is an extreme point of the ball $\mathrm{conv}(\widehat{Q})$. Hence $E \supseteq \widehat{Q}$, and $E = \widehat{Q}$.
$\square$

4.5. **Proof of Theorem 4.10.** We start by giving several equivalent descriptions for the set $\mathcal{F}$, arising from the characterization of the closed imaginary cone in Theorem 2.3.

Proposition 4.19. *Let $\mathcal{F}$ be defined as above. We have:*

(i) $\mathcal{F} = \displaystyle\bigcap_{w \in W} w \cdot (\widehat{Q}_{\text{act}} \cap \text{conv}(\widehat{\Delta}))$;

(ii) $\mathcal{F} = Q \cap \overline{\mathcal{Z}} \cap V_1$;

(iii) $\mathcal{F} = \widehat{Q} \cap \text{conv}(E)$;

(iv) $\mathcal{F}$ *is the maximal closed subset of* $\widehat{Q}_{\text{act}} \cap \text{conv}(\widehat{\Delta})$, *which is stable by* W.

Remark 4.20. In the case where $\widehat{Q} \subseteq \text{conv}(\widehat{\Delta})$ (i.e., finite, affine, or hyperbolic type, according to Theorem 4.4), we have $\mathcal{F} = \widehat{Q}_{\text{act}}$. But by the same theorem, $E = \widehat{Q}$, so $\widehat{Q} \subseteq D$, $\widehat{Q}_{\text{act}} = \widehat{Q}$ and $\mathcal{F} = \widehat{Q} = E$.

Proof. Recall that we denote the W-action inside V_1 (defined in D) as "$w \cdot v$", whereas the geometric action of W on V is denoted by "$w(v)$".

(i) is clear since $\widehat{Q}_{\text{act}}$ is stable by the W-action. For (ii), note that since $\widehat{Q}_{\text{act}}$ is stable,

$$\mathcal{F} = \bigcap_{w \in W} w \cdot (\widehat{Q}_{\text{act}} \cap \text{conv}(\widehat{\Delta})) = \widehat{Q}_{\text{act}} \cap \bigcap_{w \in W} w \cdot (\text{conv}(\widehat{\Delta}) \cap D) \ .$$

Moreover $\bigcap_{w \in W} w(\text{cone}(\widehat{\Delta})) \cap V_1 \subseteq D$, so

$$\mathcal{F} = \widehat{Q}_{\text{act}} \cap \bigcap_{w \in W} w(\text{cone}(\widehat{\Delta})) \cap V_1$$

$$= \widehat{Q} \cap \bigcap_{w \in W} w(\text{cone}(\widehat{\Delta})) \cap V_1$$

$$= \widehat{Q} \cap \bigcap_{w \in W} w(\text{cone}(\Delta)) \cap V_1$$

Now, from [Dye12, Thm. 5.1] (see also [Dye12, Def. 3.1]), we have

$$\bigcap_{w \in W} w(\text{cone}(\Delta)) = \overline{\mathcal{Z}} \ ,$$

so (ii) is proved. Since $\overline{\mathcal{Z}} \cap V_1 = \overline{Z} = \text{conv}(E)$ (see Theorem 2.2), the equality (iii) follows (see also Theorem 2.3). Finally, the characterization (iv) is clear from equality (i). $\qquad\square$

Proof of Theorem 4.10. Let us prove the equality (i), which is equivalent to $E = \mathcal{F}$ from Proposition 4.19(iii). The inclusion $E \subseteq \widehat{Q} \cap \text{conv}(E)$ is always true. From Proposition 4.13, we can choose the transverse hyperplane V_1 such that $\widehat{Q}$ is a sphere. Let x be a point in $\text{conv}(E) \cap \widehat{Q}$, and suppose $x \notin E$. Then x is a convex combination of some points in E, which are points in the sphere $\widehat{Q}$. So x cannot lie on the sphere, i.e., $x \notin \widehat{Q}$ (same argument as in the proof of Corollary 4.15), which is contradictory. Thus $\widehat{Q} \cap \text{conv}(E) \subseteq E$.

(ii) Let G be a non-empty, closed subset of $\widehat{Q}_{\text{act}} \cap \text{conv}(\widehat{\Delta})$, which is stable by W. Then $G \subseteq \mathcal{F}$ by Proposition 4.19(iv). So $G \subseteq E$. As it is non-empty, closed and W-stable, Corollary 3.1(b) implies that $G = E$. $\qquad\square$

5. ON FACIAL RESTRICTIONS OF SUBSETS OF E AND THE DOMINANCE ORDER

Let (Φ, Δ) be a based root system, with associated Coxeter system (W, S). Take $I \subseteq S$ a facial subset, i.e., $F_I := \mathrm{conv}(\widehat{\Delta_I})$ is a face of the polytope $\mathrm{conv}(\widehat{\Delta})$. Recall that (Φ_I, Δ_I) is facial root subsystem, as recalled in the introduction of §4, and the set $E(\Phi_I)$ is the set of limit roots which are accumulation points of $\widehat{\Phi_I}$. A natural question to ask is whether it is possible to describe $E(\Phi_I)$ from $E(\Phi)$. Clearly $E(\Phi_I)$ is contained in $E(\Phi) \cap F_I$; however, the equality is not true in general, and a counterexample was provided in [HLR14, Ex. 5.8]. It is interesting to note that the imaginary convex set behaves well with facial restriction: $K(\Phi_I) = K(\Phi) \cap F_I$ and $Z(\Phi_I) = Z(\Phi) \cap F_I$, see [Dye12, Lemma 3.4].

In this section we explore the question of the restriction of some subsets of $E(\Phi)$ to a face F_I of $\mathrm{conv}(\widehat{\Delta})$. By doing so, we will be brought to interpret *the dominance order* and *elementary roots* in our geometrical setting.

A natural, (countable and dense) subset of $E(\Phi)$ that we start to consider for facial restriction is the set of *dihedral limit roots* already considered in Equation (3.1):

$$E_2(\Phi) = \bigcup_{\alpha, \beta \in \Phi} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q},$$

where $L(\widehat{\alpha}, \widehat{\beta})$ denotes again the line containing $\widehat{\alpha}$ and $\widehat{\beta}$. One of the main results of this section is that the set of dihedral limit roots respects the facial structure.

Theorem 5.1. *For all $I \subseteq S$ facial, $E_2(\Phi_I) = E_2(\Phi) \cap F_I$.*

The proof will be given in §5.6. The question of characterizing subsets of E that verify the same facial restriction equality is open and is discussed a little bit more in §7.2. Still, we are able to give more examples of such subsets in this section. They are all subsets of $E_2(\Phi)$ built from the notion of the *dominance order*, *elementary roots* and *the root poset*. We also provide, along the way, useful geometric interpretations of the *"normalized version"* of these important combinatorial tools using the geometry of the normalized isotropic cone $\widehat{Q}$.

5.1. **Dominance order, elementary roots, elementary limit roots.** We collect here some definitions used throughout this section.

Definition 5.2. Let (Φ, Δ) be a based root system.
- The *dominance order* is a partial order on Φ defined by:

$$\alpha \preceq \beta \quad \text{if and only if} \quad \forall w \in W, \quad w(\beta) \in \Phi^- \implies w(\alpha) \in \Phi^-$$

  (we say that β *dominates* α).
- A positive root β is called *elementary*[7] when β dominates no other positive roots than itself:

$$\forall \alpha \in \Phi^+, \ \alpha \preceq \beta \implies \alpha = \beta.$$

- We denote by $\Sigma(\Phi)$ (or Σ when Φ is clear) the set of elementary roots.

For instance, the simple roots are elementary: $\Delta \subseteq \Sigma$, but there can be other elementary roots than the simple roots. For example, in Figure 1, the elementary roots are α, β, γ, $s_\alpha \cdot \gamma$ and $s_\gamma \cdot \alpha$; in Figure 6 they are the points in purple.

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[7]These roots are also called *humble* or *small* in the literature. We adopt here the terminology of [BH93]. See [BB05, Notes, p.130] for more detail.
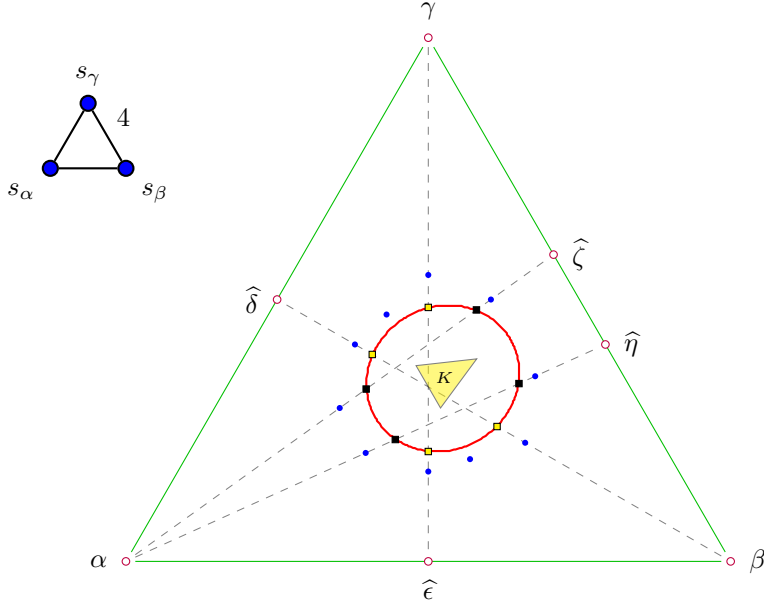
FIGURE 6. Example of elementary roots and elementary limit roots for the rank 3 root system with diagram in the upper left corner. As in Figure 1, $\widehat{Q}$ is in red. In blue and purple are the first normalized roots (with depth ≤ 4): the elementary (normalized) roots are in purple (small circles), while the non-elementary ones are in blue (full dots) (see §5.2 for an interpretation). The elementary limit roots are the small squares on $\widehat{Q}$ (yellow and black). In yellow are the ones in $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi)$, constructed from fundamental covers of dominance (see §5.4); for example, $-\delta \mathrel{\dot{\prec}_{\mathrm{f}}} \beta$ since $\beta + \delta \in \mathcal{K}$. The polytope K is in shaded yellow, and illustrates Remark 5.15.

We give in §5.3 other characterizations of the dominance order, including a geometric interpretation. The dominance order and elementary roots are a fundamental tool allowing one to build a finite state automaton for the language of words in Coxeter groups, as shown by B. Brink and R. Howlett [BH93]. The key point in their construction is the property that Σ is a finite set (see [BH93], or also [BB05, §4.7]). Translating this fact into our framework, we construct the set of *elementary root limits*:

(5.1) $$E_{\mathrm{elem}}(\Phi) := \bigcup_{\alpha,\beta \in \Sigma(\Phi)} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q},$$

which is finite since Σ is finite and $|L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q}| \leq 2$ for all $\alpha, \beta \in \Phi^+$. Note that by definition, one has: $E_{\mathrm{elem}}(\Phi) \subseteq E_2(\Phi) \subseteq E(\Phi)$. By Theorem 3.1(b) we get immediately the following result.

Proposition 5.3. *The union of the W-orbits of points in the finite set $E_{\mathrm{elem}}(\Phi)$ is dense in E:*
$$E(\Phi) = \overline{W \cdot E_{\mathrm{elem}}(\Phi)}.$$

Remark 5.4. In an early version of this research, before we had the idea of Theorem 3.1(b), $E_{\mathrm{elem}}(\Phi)$ allowed us to prove the existence of a dense subset of E that

is a finite union of W-orbits of points of E. We think the techniques in this direct elementary proof of Proposition 5.3 are still of interest and it will be presented in §5.7.

An example is shown in Figure 6. We will explain the picture in more detail once we have described a way to compute algorithmically the elementary roots. In order to do so, we need an alternative characterization of Σ, using another partial order on Φ^+, which we introduce now (see [BB05, §4.6] for details).

5.2. Geometric construction of elementary roots and the root poset. We
consider Φ^+ with the partial order defined as the transitive closure of the following relation: let $\alpha \in \Delta$ and $\beta \in \Phi^+ \setminus \{\alpha\}$, we write $\beta \leq s_\alpha(\beta)$ if $B(\alpha, \beta) \leq 0$ and $\beta \geq s_\alpha(\beta)$ otherwise. The poset $(\Phi^+, \leq)$ is called the *root poset on* Φ^+. This poset is an interesting tool to build roots algorithmically by means of the depth. Recall that the depth of $\beta \in \Phi^+$ is:

$$\mathrm{dp}(\beta) = 1 + \min\{k \,|\, \beta = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}(\alpha_{k+1}), \text{ for } \alpha_1, \ldots, \alpha_k, \alpha_{k+1} \in \Delta\}.$$

Note that for $\alpha \in \Delta$ and $\beta \in \Phi^+ \setminus \{\alpha\}$, $\beta \leq s_\alpha(\beta)$ precisely when the depth increases (see [BB05, §4.6]). The root poset is ranked by the depth, i.e., the length of a chain from β to γ in Φ^+ such that $\beta \leq \gamma$ is $\mathrm{dp}(\gamma) - \mathrm{dp}(\beta)$.

The root poset is intimately linked with elementary roots: given $\alpha \in \Delta$ and $\beta \in \Phi^+$, the edge $\beta \,\text{—}\, s_\alpha(\beta)$ in the root poset of Φ^+ is called *a short edge* if $|B(\alpha, \beta)| < 1$; otherwise it is called *a long edge*. With this terminology, one has the following characterization of elementary roots.

Proposition 5.5 (see [BB05, §4.6]). *Let $\beta \in \Phi^+$, then $\beta \in \Sigma$ (i.e., β is an elementary root) if and only if there is a chain from a simple root to β constituted only in short edges.*

This statement gives a useful way to construct elementary roots. First of all, recall from [HLR14] that the dihedral reflection subgroup generated by s_α and s_β (with $\alpha, \beta \in \Phi$) is finite if and only if $|B(\alpha, \beta)| < 1$, if and only if the line $L(\widehat{\alpha}, \widehat{\beta})$ does not intersect $\widehat{Q}$. We obtain the following geometric characterization of short and long edges: let $\alpha \in \Delta$ and $\beta \in \Phi^+$, then:

- $\beta \,\text{—}\, s_\alpha(\beta)$ is a *short edge* if and only if $L(\widehat{\alpha}, \widehat{\beta}) \cap Q = \varnothing$;
- $\beta \,\text{—}\, s_\alpha(\beta)$ is a *long edge* if and only if $L(\widehat{\alpha}, \widehat{\beta}) \cap Q \neq \varnothing$.

We build Σ by induction:

Initial step: Take $\Sigma_1 := \Delta$;
Inductive step: Assume Σ_k is built. Draw the lines between the roots in Σ_k and the simple roots and select those that do not cut $\widehat{Q}$:

$$\Sigma_{k+1} := \Sigma_k \cup \{s_\alpha(\beta) \,|\, \alpha \in \Delta, \ \beta \in \Sigma_k, \ L(\alpha, \widehat{\beta}) \cap Q = \varnothing\}.$$

In Figure 1, we only have to consider the line $L(\alpha, \gamma)$ to build Σ_2.
Final step: By Proposition 5.5, Σ_k is composed of elementary roots for any k; moreover, $\Sigma_k \setminus \Sigma_{k-1}$ is constituted of roots of depth k. So $\bigcup_k \Sigma_k = \Sigma$. Since Σ is finite, there exists $N \geq 1$ such that $\Sigma = \Sigma_N = \Sigma_{N+1}$. In the example of Figure 1, $\Sigma = \Sigma_2$.

5.3. **Geometric interpretation of dominance order.** The ingredients here are not new, and already appear in [BH93]. However, we provide here a geometric interpretation of the dominance order inside $\mathrm{conv}(\widehat{\Delta})$, using the normalized isotropic cone $\widehat{Q}$ and the particular geometry of infinite dihedral reflection subgroups. The idea is the following: given $\rho, \gamma \in \Phi^+$, $\rho \preceq \gamma$ if and only if when looking at $\widehat{Q}$, $\widehat{\rho}$ is able to see $\widehat{\gamma}$ (see Figure 8). As far as we know, such a description never appeared in the literature, we feel that to get a geometric intuition of the properties of the dominance order could be very useful in future works (for example, many of the properties proved in [Fu13a] have a natural interpretation in this setting).

We need first to recall basic properties of the dominance order, from the seminal work in [BH93].

Proposition 5.6 (see [BH93, §2]). *Let $\alpha, \beta \in \Phi$.*
  (i) *There is a dominance relation between α and β of and only if $B(\alpha, \beta) \geq 1$.*
  (ii) *Let $w \in W$, then $w(\alpha) \preceq w(\beta)$ if and only if $\alpha \prec \beta$.*
  (iii) *Assume that $\alpha \preceq \beta$. Then:*
      (a) *if $\alpha \in \Phi^+$, then $\beta \in \Phi^+$;*
      (b) *if $\beta \in \Phi^-$, then $\alpha \in \Phi^-$;*
      (c) *$-\beta \preceq -\alpha$.*
  (iv) *If $\alpha, \beta \in \Phi^+$, then $\alpha \preceq \beta$ if and only if $B(\alpha, \beta) \geq 1$ and $\mathrm{dp}(\alpha) \leq \mathrm{dp}(\beta)$.*
  (v) *If $\alpha \in \Phi^-$ and $\beta \in \Phi^+$, then $\alpha \preceq \beta$ if and only if $B(\alpha, \beta) \geq 1$.*
  (vi) *Let (Φ', Δ') be a root subsystem of (Φ, Δ) (i.e. it is a based root system such that $\Delta' \subseteq \Phi^+$ and therefore $\Phi' \subseteq \Phi$). Denote by $\preceq'$ the dominance order of the root system (Φ', Δ'). Then:*

$$\forall \rho, \gamma \in \Phi', \rho \preceq \gamma \Leftrightarrow \rho \preceq' \gamma.$$

Proof. The proof of *(i)-(v)* can be found in [BH93, §2]. For *(vi)*, Let (Φ', Δ') be a root subsystem of (Φ, Δ), and denote by W' the associated reflection subgroup of W. Let $\rho \neq \gamma \in \Phi'$.

First suppose that $\rho \preceq \gamma$. For $w \in W'$, if $w(\gamma) \in \Phi'^- \subseteq \Phi^-$, then $w(\rho) \in \Phi^-$ by hypothesis. But as W' preserves Φ', we also have $w(\rho) \in \Phi'$, so $w(\rho) \in \Phi^- \cap \Phi' = \Phi'^-$. Hence $\rho \prec' \gamma$.

Conversely, suppose now that $\rho \preceq' \gamma$. Then, by *(i)* applied to Φ', we have $B(\rho, \gamma) \geq 1$. So by the same property *(i)*, we have either $\rho \preceq \gamma$ or $\gamma \preceq \rho$. If we suppose $\gamma \preceq \rho$, we get by the first implication that $\gamma \preceq' \rho$ as well, so $\gamma = \rho$, which is a contradiction. $\square$

The above proposition is a key to give a geometric interpretation of the dominance order: for a dominance relation to be possible between ρ and γ in Φ, the line $L(\widehat{\rho}, \widehat{\gamma})$ must intersect Q. However, this does not imply that there will be a dominance relation between ρ and γ, since $B(\rho, \gamma)$ could be negative. We aim to get a criterion on the line $L(\widehat{\rho}, \widehat{\gamma})$, which represents the dihedral reflection subgroup $W' := \{s_\rho, s_\gamma\}$, for a root on this line to dominate another.

Let $\alpha, \beta \in \Phi^+$ such that $B(\alpha, \beta) \leq -1$ (so $L(\widehat{\alpha}, \widehat{\beta})$ intersects Q). Set:

$$\Delta' = \{\alpha, \beta\}, \ W' := \{s_\alpha, s_\beta\} \text{ and } \Phi' := W'(\Delta').$$

We know that in this case (Φ', Δ') is a root system with associated infinite dihedral reflection subgroup W' (see [HLR14, Proposition 1.5(2)]). Moreover, by [HLR14, Proposition 3.5], we know that $L(\widehat{\alpha}, \widehat{\beta}) \cap Q = \{x, s_\alpha \cdot x\}$ for some x. Choose notation

so $x \in \mathrm{conv}(\{\alpha, s_\alpha \cdot x\})$. We recall here some basic facts on W', see for instance [HLR14, Example 3.8] or [Fu13a, Prop. 3.7].

We refer to Figure 7 for a pictorial representation of the following description. Let us give to the line $L(\widehat{\alpha}, \widehat{\beta})$ the total order inherited from $\mathbb{R}$ such that

$$\widehat{\alpha} <_{\mathbb{R}} x \leq_{\mathbb{R}} s_\alpha \cdot x <_{\mathbb{R}} \widehat{\beta}.$$

So

$$[\widehat{\alpha}, \widehat{\beta}] = [\widehat{\alpha}, x[\sqcup[x, s_\alpha \cdot x]\sqcup]s_\alpha \cdot x, \widehat{\beta}],$$

and $\widehat{\Phi'}$ is ordered as follows:

(5.2)

$$\widehat{\alpha} <_{\mathbb{R}} s_\alpha \cdot \widehat{\beta} <_{\mathbb{R}} s_\alpha s_\beta \cdot \widehat{\alpha} <_{\mathbb{R}} \cdots <_{\mathbb{R}} (s_\alpha s_\beta)^n \cdot \widehat{\alpha} <_{\mathbb{R}} (s_\alpha s_\beta)^n s_\alpha \cdot \widehat{\beta} <_{\mathbb{R}} \cdots <_{\mathbb{R}} x$$

$$\text{and} \quad s_\alpha \cdot x <_{\mathbb{R}} \cdots <_{\mathbb{R}} s_\beta (s_\alpha s_\beta)^n \cdot \widehat{\alpha} <_{\mathbb{R}} (s_\beta s_\alpha)^n \cdot \widehat{\beta} <_{\mathbb{R}} \cdots <_{\mathbb{R}} s_\beta \cdot \widehat{\alpha} <_{\mathbb{R}} \widehat{\beta}.$$

Moreover, note that:

(a) the function B is positive on $[\widehat{\alpha}, x[\times[\widehat{\alpha}, x[$ and $]s_\alpha \cdot x, \widehat{\beta}]\times]s_\alpha \cdot x, \widehat{\beta}]$; and negative on $[\widehat{\alpha}, x[\times]s_\alpha \cdot x, \widehat{\beta}]$.

(b) the depth function dp' on Φ' is increasing on $[\widehat{\alpha}, x[\cap\widehat{\Phi}$ and decreasing on $]s_\alpha \cdot x, \widehat{\beta}] \cap \widehat{\Phi}$ (we obviously mean here that $\mathrm{dp}'(\widehat{\nu}) := \mathrm{dp}'(\nu)$ for any $\nu \in \widehat{\Phi'}$).

Now, let $\rho \neq \gamma \in \Phi'^+$. Using Proposition 5.6 *(iv)* and *(vi)*, we have $\rho \preceq \gamma$ if and only if $\mathrm{dp}'(\rho) \leq \mathrm{dp}'(\gamma)$ and $B(\rho, \gamma) \geq 1$. Note that we always have $|B(\rho, \gamma)| \geq 1$ (since Φ' is infinite), so $B(\rho, \gamma) \geq 1$ if and only if $B(\rho, \gamma) \geq 0$. Then, by (a) above we obtain:

(5.3) $\rho \preceq \gamma$ if and only if $\mathrm{dp}'(\rho) \leq \mathrm{dp}'(\gamma)$ and $\widehat{\rho}, \widehat{\gamma} \in [\widehat{\alpha}, x[$ or $\widehat{\rho}, \widehat{\gamma} \in]s_\alpha \cdot x, \widehat{\beta}]$.

Now by (b) we get:

(5.4) $\rho \preceq \gamma$ if and only if $\widehat{\gamma} \in [\widehat{\rho}, x[$ or $\widehat{\gamma} \in]s_\alpha \cdot x, \widehat{\rho}]$.

So we deduce that the dominance order on the positive roots corresponding to normalized roots in $[\widehat{\alpha}, x[\cap\widehat{\Phi}$ is precisely the order $<_{\mathbb{R}}$ on this interval:

(5.5) $\alpha \prec s_\alpha(\beta) \prec s_\alpha s_\beta(\alpha) \prec \cdots \prec (s_\alpha s_\beta)^n(\alpha) \prec (s_\alpha s_\beta)^n s_\alpha(\beta) \prec \ldots,$

while it is the reverse of $<_{\mathbb{R}}$ for the positive roots corresponding to $]s_\alpha \cdot x, \widehat{\beta}]$:

(5.6) $\cdots \succ s_\beta(s_\alpha s_\beta)^n(\alpha) \succ (s_\beta s_\alpha)^n(\beta) \succ \cdots \succ s_\beta s_\alpha(\beta) \succ s_\beta(\alpha)\cdot \succ \beta.$
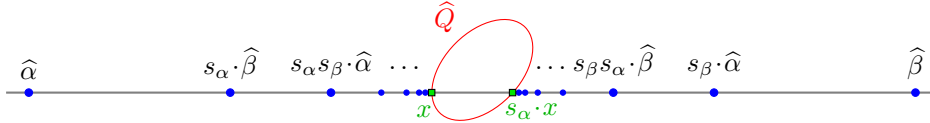
FIGURE 7. Schematic visualization of the dominance order restricted to the root subsystem with simple roots $\{\alpha, \beta\}$ (where $\alpha, \beta \in \Phi^+$ are such that $B(\alpha, \beta) < -1$, and $x, s_\alpha \cdot x$ are the two limit roots of this dihedral root subsystem). There are two chains of dominance, given in Equations (5.5) and (5.6).

From this discussion, we obtain an interpretation of the relation of dominance within our framework (and therefore that can be easily seen in our affine pictures, see Figure 8 and Figure 7). We say that a point $x \in \widehat{Q}$ *is visible from* $v \in V_1$ if

$[x, v] \cap Q = \{x\}$ ([HLR14, Definition 3.7]). More generally, if $u, v \in V_1$ are two points, we say that u *is visible from v looking at $\widehat{Q}$* if the line $L(u, v)$ cuts $\widehat{Q}$ in x such that x is visible from v and $u \in [v, x]$.

Proposition 5.7. *Let $\rho \neq \gamma \in \Phi^+$. Then $\rho \prec \gamma$ if and only if there is $x \in L(\widehat{\rho}, \widehat{\gamma}) \cap Q$ that is visible from $\widehat{\rho}$ and such that $\widehat{\gamma} \in [\widehat{\rho}, x]$. In other words, $\rho \prec \gamma$ if and only if $\widehat{\gamma}$ is visible from $\widehat{\rho}$ looking at $\widehat{Q}$.*

In particular, there is a dominance relation between ρ and γ if and only if $L(\widehat{\rho}, \widehat{\gamma}) \cap Q \neq \varnothing$ and $[\widehat{\rho}, \widehat{\gamma}] \cap Q = \varnothing$.
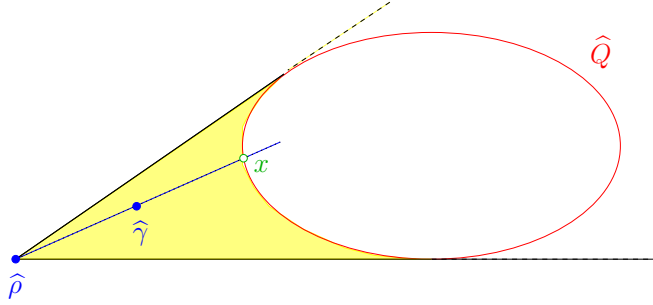

FIGURE 8. Geometric interpretation of the dominance order: $\rho \prec \gamma$ if and only if $\widehat{\rho}$ can see $\widehat{\gamma}$ when looking at $\widehat{Q}$ (see Proposition 5.7).

Example 5.8. On the top picture of Figure 1, we can see that $\alpha \prec s_\alpha(\beta)$, since t is visible from both α and $s_\alpha \cdot \beta$ and $s_\alpha \cdot \beta \in [\alpha, t]$. However, there is no dominance relation between β and $s_\alpha(\beta)$, since $t \in [s_\alpha \cdot \beta, \beta] \cap Q$. By considering the line $L(s_\alpha \cdot \gamma, s_\alpha \cdot \beta)$, we also see that there is no dominance relation between $s_\alpha(\gamma)$ and $s_\alpha(\beta)$ for a similar reason.

Proof. If $\rho \prec \gamma$ then $B(\rho, \gamma) \geq 1$, by Proposition 5.6 (i) and so $L(\widehat{\rho}, \widehat{\gamma})$ intersects Q. So, for both directions of the equivalence we have to show $L(\widehat{\rho}, \widehat{\gamma}) \cap Q \neq \varnothing$. Therefore there is $\Delta' := \{\alpha, \beta\} \subseteq \Phi^+$ that is a simple system for the infinite dihedral reflection subgroup $W' := \{s_\rho, s_\gamma\}$; see for instance [HLR14, Proposition 1.5(2)]. Set $\Phi' := W'(\Delta')$. We are therefore in the situation of the discussion above. In particular Properties (5.3) and (5.4) hold, which completes the proof. □

5.4. **Fundamental dominance and cover of dominance.** Here we construct fundamental dominances, which is a useful tool when considering W-orbits since with the action of W they generate all the dominances.

Recall that the imaginary cone of (Φ, Δ) was constructed as the union of the sets in the W-orbit of the set $\mathcal{K}(\Phi)$ (see Definition 2.1):

$$\mathcal{K}(\Phi) := \{v \in \mathrm{cone}(\Delta) \,|\, B(v, \alpha) \leq 0, \ \forall \alpha \in \Delta\}.$$

Definition 5.9 (see [Fu13a, Dye12]). A dominance $\alpha \prec \beta$ of distinct roots is called a *fundamental dominance*, denoted by $\alpha \prec_{\mathrm{f}} \beta$, if $\beta - \alpha \in \mathcal{K}(\Phi)$, $\beta \in \Phi^+$ and $\alpha \in \Phi^-$.

The term *fundamental dominance* is strongly suggested by the following result, which is a direct consequence of [Fu13a, Theorem 4.13]:

Theorem 5.10 (X. Fu [Fu13a])**.** *Let α, β be two roots such that $\alpha \preceq \beta$ with $\alpha \neq \beta$, then there is $w \in W$ such that $w(\alpha) \prec_{\mathrm{f}} w(\beta)$.*

Note that given the definition of the imaginary cone $\mathcal{Z}$, Theorem 5.10 implies that if $\alpha \preceq \beta$, then $\beta - \alpha \in \mathcal{Z}$. It turns out that the converse of this property is true, which gives another geometric characterization of the dominance order.

Theorem 5.11 (see [Fu13a, Cor. 4.15])**.** *Let $\alpha, \beta \in \Phi$. Then $\alpha \preceq \beta$ if and only if $\beta - \alpha \in \mathcal{Z}$.*

Using Theorem 5.10, we can construct easily another subset of E_2, whose orbit is exactly E_2. Define the set of fundamental limit roots as the set of limit roots obtained by the lines associated to couples of fundamental dominance:

$$E_{\mathrm{f}}(\Phi) := \bigcup_{\alpha \prec_{\mathrm{f}} \beta} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q}.$$

Then we have the following property.

Proposition 5.12. *The orbit $W \cdot E_{\mathrm{f}}(\Phi)$ of $E_{\mathrm{f}}(\Phi)$ is equal to $E_2(\Phi)$.*

Note that this implies (without having to use the minimality of the W-action from Corollary 3.1) that the orbit of $E_{\mathrm{f}}(\Phi)$ is dense in $E(\Phi)$, by [HLR14, Thm. 4.2].

Proof. Recall that for all $w \in W$, $x \in L(\widehat{\alpha}, \widehat{\beta})$ if and only if $w \cdot x \in L(w \cdot \widehat{\alpha}, w \cdot \widehat{\beta})$ ([HLR14, Proposition 3.5 (ii)]). So by Theorem 5.10 we have

$$W \cdot E_{\mathrm{f}}(\Phi) = \bigcup_{\substack{\alpha \prec_{\mathrm{f}} \beta \\ w \in W}} L(w \cdot \widehat{\alpha}, w \cdot \widehat{\beta}) \cap \widehat{Q} = \bigcup_{\gamma \prec \rho} \big(L(\widehat{\gamma}, \widehat{\rho}) \cap \widehat{Q} \big).$$

On the other hand, $E_2(\Phi) = \bigcup (L(\widehat{\gamma}, \widehat{\rho}) \cap \widehat{Q})$ where the union is over all $\gamma, \rho \in \Phi$ with $\gamma \neq \pm\rho$. But for any such roots, and any $\epsilon, \eta \in \{\pm 1\}$, one has $L(\widehat{\epsilon\gamma}, \widehat{\eta\rho}) = L(\widehat{\gamma}, \widehat{\rho})$ and $L(\widehat{\gamma}, \widehat{\rho}) \cap \widehat{Q} \neq \varnothing$ if and only if $|B(\gamma, \rho)| \geq 1$. The equality $W \cdot E_{\mathrm{f}}(\Phi) = E_2(\Phi)$ follows on noting that $|B(\gamma, \rho)| \geq 1$ if and only if $\epsilon\gamma \preceq \eta\rho$ for some $\epsilon, \eta \in \{\pm 1\}$, by Proposition 5.6(i). $\qquad\square$

Definition 5.13.
 - A relation of dominance is a *cover of dominance*, denoted by $\alpha \dot{\prec} \beta$, if there are no roots in between α and β, i.e., if $\alpha \neq \beta$ and

$$\forall \gamma \in \Phi, \ \alpha \preceq \gamma \preceq \beta \Longrightarrow \alpha = \gamma \text{ or } \gamma = \beta.$$

 - A cover of dominance that is a fundamental dominance is called a *fundamental cover of dominance* and is denoted by $\alpha \dot{\prec}_{\mathrm{f}} \beta$.

Note that, as dominance order is preserved by the action of W, so are the covers of dominance. The following result will provide a relation between limit roots coming from fundamental covers of dominance, and elementary limit roots.

Proposition 5.14 (X. Fu [Fu13a, Corollary 4.17])**.** *If $\alpha \dot{\prec}_{\mathrm{f}} \beta$, then $-\alpha, \beta \in \Sigma$. In particular, there is a finite number of fundamental covers of dominance.*

Remark 5.15. In Figure 6, we describe the pairs $(\widehat{\rho}, \widehat{\sigma})$ such that $-\rho \dot{\prec}_{\mathrm{f}} \sigma$, and draw the polytope $K = \mathcal{K} \cap V_1$. Note that for any of these pairs, we have $\sigma + \rho \in \mathcal{K}$ and $\rho, \sigma \in \Phi^+$, so there is $\lambda \in\,]0, 1[$ such that $\lambda\widehat{\sigma} + (1 - \lambda)\widehat{\rho} \in K$. Thus a necessary condition for $(\widehat{\rho}, \widehat{\sigma})$ to be such a pair is that the open interval $\widehat{\rho}, \widehat{\sigma}$ cuts K. From

this we see already on the example of Figure 6 that the only candidates for elements of $E_{\mathrm{f}}^{\mathrm{cov}}$ are the points in yellow, coming from the pairs $(\widehat{\beta}, \widehat{\delta})$ and $(\widehat{\gamma}, \widehat{\epsilon})$.

More precisely, one has:

$$\widehat{\sigma + \rho} = \frac{\varphi(\sigma)}{\varphi(\sigma) + \varphi(\rho)}\widehat{\sigma} + \frac{\varphi(\rho)}{\varphi(\sigma) + \varphi(\rho)}\widehat{\rho},$$

where φ is the linear form such that the hyperplane transverse to Φ^+ is $V_1 = \varphi^{-1}(1)$. A definition of fundamental dominance involving K instead of $\mathcal{K}$ is therefore: $\rho \prec_{\mathrm{f}} \sigma$ if and only if the barycenter of the system $(\widehat{\rho}, \varphi(\rho))$ and $(\widehat{\sigma}, \varphi(\sigma))$ is in K. In the example (where the transverse hyperplane V_1 is assumed to be the one passing through Δ), one has $\delta = \alpha + \gamma$, so the barycenter to consider is the one of $(\widehat{\beta}, 1)$, $(\widehat{\delta}, 2)$, which is just in K.

We consider:

- the finite set $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi)$ of limit points obtained by the lines corresponding to fundamental covers of dominance:

$$E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) := \bigcup_{\alpha \prec_{\mathrm{f}} \beta} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q} \subseteq E_{\mathrm{f}}(\Phi);$$

- the set $E^{\mathrm{cov}}(\Phi)$ of limits points obtained by the lines corresponding to covers of dominance:

$$E^{\mathrm{cov}}(\Phi) := \bigcup_{\alpha \prec \beta} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q} \subseteq E_2(\Phi).$$

Recall that $E_{\mathrm{elem}}(\Phi)$ denotes the set of elementary limits of roots, introduced in §5.1:

$$E_{\mathrm{elem}}(\Phi) := \bigcup_{\alpha, \beta \in \Sigma} L(\widehat{\alpha}, \widehat{\beta}) \cap \widehat{Q}.$$

The following proposition follows easily from the definitions and Proposition 5.14.

Proposition 5.16. $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) \subseteq E_{\mathrm{elem}}(\Phi)$ *and* $E^{\mathrm{cov}}(\Phi) = W \cdot E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) \subseteq E_2(\Phi)$.

Figure 6 demonstrates an example where the inclusion $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) \subseteq E_{\mathrm{elem}}(\Phi)$ is strict.

Remark 5.17. It is easy to see that $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) \subseteq E_{\mathrm{f}}(\Phi) \cap E^{\mathrm{cov}}(\Phi)$. However, we do not know if the equality holds.

5.5. **Restriction of the dominance order to facial root subsystems.** Before discussing the restriction of the dominance order to facial root subsystems, let us briefly recall some basic properties of cosets of standard parabolic subgroups. For a fixed subset $I \subseteq S$, the set

$$W^I = \{u \in W \mid \forall \alpha \in \Delta_I,\, u(\alpha) \in \Phi^+\}$$

is a set of minimal length coset representatives for W/W_I. It is well known and a useful fact ([Hum90, §5.12]) that for each element $w \in W$, there is a unique decomposition $w = w^I w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Moreover:

$$w^I(\Phi_I^+) \subseteq \Phi^+ \setminus \Phi_I \text{ and } w_I(\Phi_I) = \Phi_I.$$

We call the pair (w^I, w_I) the parabolic components of w along I.

Recall that $I \subseteq S$ is said to be facial for the root system (Φ, Δ) if $\mathrm{conv}(\widehat{\Delta_I})$ is a face of $\mathrm{conv}(\widehat{\Delta})$ (see §4.1). By [Dye12, Proposition 2.4(b)] that $\Phi_I = \Phi \cap V_I$ where $V_I = \mathrm{span}(I)$.

Denote again $F_I = \mathrm{conv}(\widehat{\Delta_I})$ the face of $\mathrm{conv}(\widehat{\Delta})$ corresponding to I, so that $\widehat{\Phi_I} = \widehat{\Phi} \cap F_I$. We list below some other useful properties related to the restriction of the dominance order to W_I, for I facial.

Proposition 5.18. *Let $I \subseteq S$ be facial.*

(i) *Let $\alpha, \beta \in \Phi^+$ such that $\alpha \preceq \beta$. If $\beta \in \Phi_I$, then $\alpha \in \Phi_I$.*

(ii) *If $\alpha, \beta \in \Phi_I$, then:*

    (a) *$\alpha \prec \beta$ in Φ_I if and only if $\alpha \prec \beta$ in Φ.*

    (b) *$\alpha \dot{\prec} \beta$ is a cover of dominance in Φ_I if and only if $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ.*

    (c) *$\alpha \prec_{\mathrm{f}} \beta$ in Φ if and only if $\alpha \prec_{\mathrm{f}} \beta$ in Φ_I.*

(iii) *Denote by Σ the set of elementary roots and by $\Sigma_I = \Sigma(\Phi_I)$ the set of elementary roots of the root subsystem (Φ_I, Δ_I). Then $\Sigma_I = \Sigma \cap \Phi_I$, so $\Sigma_I = \Sigma \cap V_I$ and $\widehat{\Sigma_I} = \widehat{\Sigma} \cap F_I$.*

(iv) *Let $\alpha \in \Sigma$ and $w \in W^I$ such that $w^{-1}(\alpha) \in \Phi_I$, then $w^{-1}(\alpha) \in \Sigma_I$.*

Remark 5.19. Properties (i), (ii)(a)-(b), and $\Sigma_I = \Sigma \cap \Phi_I$ in (iii) remain in fact valid even if I is not facial, because they can be written as combinatorial properties of the group W itself so they do not depend on the choice of a root system.

Proof. (i) Since $\alpha \preceq \beta$, then by Proposition 5.7 there is $x \in L(\widehat{\alpha}, \widehat{\beta}) \cap Q$ that is visible from $\widehat{\alpha}$ and such that $\widehat{\beta} \in [\widehat{\alpha}, x]$. Assume that $\alpha \notin \Phi_I$, i.e., $\alpha \notin V_I$. We know, since I is facial, that $V_I \cap V_1$ is the affine subspace of V supporting the face F_I of the polytope $\mathrm{conv}(\widehat{\Delta})$. Let H be the subspace of V_1 spanned by α and $V_I \cap V_1$. So $\widehat{\beta}, \widehat{\alpha}, x \in H$. Since $\widehat{\beta} \in V_I \cap [\widehat{\alpha}, x]$, we necessarily have that $\widehat{\alpha}$ and x are separated by the affine hyperplane $V_I \cap V_1$ in H. Since F_I is a face of $\mathrm{conv}(\widehat{\Delta}) \cap H$, that means that either $\widehat{\alpha}$ or x is outside $\mathrm{conv}(\widehat{\Delta})$, contradicting the inclusion $\widehat{\Phi} \sqcup E(\Phi) \subseteq \mathrm{conv}(\Delta)$.

(ii) (a) is a particular case of Proposition 5.6*(vi)*.

(ii) (b) Let $\alpha, \beta \in \Phi_I$. We just have to show that if $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ_I then $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ. Let $\gamma \in \Phi$ such that $\alpha \preceq \gamma \preceq \beta$. First, suppose that $\gamma \in \Phi^+$. Then $\beta \in \Phi_I^+$, and $\gamma \in \Phi_I$ by (i). Since $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ_I, we get $\alpha = \gamma$ or $\beta = \gamma$, which proves that $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ. Suppose now that $\gamma \in \Phi^-$. Then $\alpha \in \Phi^-$, so $-\gamma \preceq -\alpha$ by Proposition 5.6. Then the same line of reasoning as above, with $-\alpha$ in the role of β and $-\gamma$ in the role of γ, shows that $\alpha = \gamma$ or $\beta = \gamma$, which proves again that $\alpha \dot{\prec} \beta$ is a cover of dominance in Φ.

(ii) (c) We will first show that $\mathcal{K}(\Phi_I) = \mathcal{K}(\Phi) \cap V_I$. Recall that

$$\mathcal{K}(\Phi_I) := \{v \in \mathrm{cone}(\Delta_I) \,|\, \forall \alpha \in \Delta_I, \ B(v, \alpha) \leq 0\}$$

and

$$\mathcal{K}(\Phi) := \{v \in \mathrm{cone}(\Delta) \,|\, \forall \alpha \in \Delta, \ B(v, \alpha) \leq 0\}.$$

So, since $\mathrm{cone}(\Delta) \cap V_I = \mathrm{cone}(\Delta_I)$, it is clear that $\mathcal{K}(\Phi) \cap V_I \subseteq \mathcal{K}(\Phi_I)$. Now let $v \in \mathcal{K}(\Phi_I)$ and let $\alpha \in \Delta$. If $\alpha \in \Delta_I$ we know by definition that $B(v, \alpha) \leq 0$. If $\alpha \in \Delta \setminus \Delta_I$, then $B(\gamma, \alpha) \leq 0$ for all $\gamma \in \Delta_I$, by definition of a root system.

Since $v \in \mathrm{cone}(\Delta_I)$, we can write $v = \sum_{\gamma \in \Delta_I} v_\gamma \gamma$ with $v_\gamma \geq 0$, so by linearity $B(v, \alpha) \leq 0$. Hence $v \in \mathcal{K}(\Phi) \cap V_I$.

Now let $\alpha, \beta \in \Phi_I$ such that $\alpha \prec_{\mathrm{f}} \beta$ in Φ. So $\beta - \alpha \in \mathcal{K}(\Phi)$, $\beta \in \Phi^+$ and $\alpha \in \Phi^-$. Therefore $\beta - \alpha \in \mathcal{K}(\Phi) \cap V_I = \mathcal{K}(\Phi_I)$, $\beta \in \Phi_I^+$ and $\alpha \in \Phi_I^-$, so $\alpha \prec_{\mathrm{f}} \beta$ in Φ_I. The converse implication is trivial.

(iii) Let $\alpha \in \Sigma_I$. Then $\alpha \in \Phi_I^+$. Consider $\gamma \in \Phi^+$ such that $\gamma \prec \alpha$. By (i) we know that $\gamma \in \Phi_I^+$. Since α is elementary in Φ_I, we obtain $\gamma = \alpha$. So α is also elementary in Φ, i.e., $\alpha \in \Sigma$. Hence $\Sigma_I \subseteq \Sigma \cap \Phi_I$. The reverse inclusion is straightforward.

(iv) Let $\alpha \in \Sigma$ and $w \in W^I$ such that $w^{-1}(\alpha) \in \Phi_I$. We know that $w(\Phi_I^+) \subseteq \Phi^+$, so $w(\Phi_I^-) \subseteq \Phi^-$. Since $\alpha \in \Sigma \subseteq \Phi^+$, necessarily $w^{-1}(\alpha) \notin \Phi_I^-$, i.e., $w^{-1}(\alpha) \in \Phi_I^+$. Now consider $\gamma \in \Phi_I^+$ such that $\gamma \preceq w^{-1}(\alpha)$. By Proposition 5.6 (ii), we get $w(\gamma) \prec \alpha$. Since $\gamma \in \Phi_I^+$ and $w \in W^I$, we have $w(\gamma) \in \Phi^+$. But α is elementary, so we obtain $w(\gamma) = \alpha$ and therefore $w^{-1}(\alpha) = \gamma$. Hence $w^{-1}(\alpha) \in \Sigma_I$. $\qquad\square$

5.6. **Facial restriction to subsets of $E_2(\Phi)$ related to dominance.** In [HLR14, Example 5.8], it is shown that, in general, the restriction of $E(\Phi)$ to the face F_I is not equal to $E(\Phi_I)$. It turns out that however, this property of good facial restrictions holds for all the subsets of E that we have defined in this section. Recall that, given a based root system (Φ, Δ), we have constructed in §5.1 and §5.4 the set $E_{\mathrm{f}}(\Phi)$, its W-orbit $E_2(\Phi)$, the set $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi)$, its W-orbit $E^{\mathrm{cov}}(\Phi)$, and the set $E_{\mathrm{elem}}(\Phi)$. The following theorem states that all these six "functorial" subsets of E restrict well to facial root subsystems. Theorem 5.1 is the first item below.

Theorem 5.20. *Let (Φ, Δ) be a based root system with associated Coxeter group (W, S). Let $I \subseteq S$ be facial, and $F_I = \mathrm{conv}(\widehat{\Delta_I})$ denote the associated face of $\mathrm{conv}(\widehat{\Delta})$. Then:*

(i) $E_2(\Phi_I) = E_2(\Phi) \cap F_I$;
(ii) $E_{\mathrm{f}}(\Phi_I) = E_{\mathrm{f}}(\Phi) \cap F_I$;
(iii) $E^{\mathrm{cov}}(\Phi_I) = E^{\mathrm{cov}}(\Phi) \cap F_I$ and $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi_I) = E_{\mathrm{f}}^{\mathrm{cov}}(\Phi) \cap F_I$;
(iv) $E_{\mathrm{elem}}(\Phi_I) = E_{\mathrm{elem}}(\Phi) \cap F_I$ and $W_I \cdot E_{\mathrm{elem}}(\Phi_I) = (W \cdot E_{\mathrm{elem}}(\Phi)) \cap F_I$.

Example 5.21. In Figure 1, take $I = \{\beta, \gamma\}$. Then $E_{\mathrm{elem}}(\Phi_I) = \{x, y\} = W_I \cdot E_{\mathrm{elem}}(\Phi_I)$.

Before getting into the proof of Theorem 5.20, we first need the following key lemma.

Lemma 5.22. *Let $x \in E_2(\Phi)$ and $\alpha, \beta \in \Phi$ be distinct such that $x \in L(\widehat{\alpha}, \widehat{\beta})$. Let $I \subseteq S$ be facial, and denote by F_I the face $\mathrm{conv}(\widehat{\Delta_I})$.*

(i) We have $x \in F_I$ if and only if $\widehat{\alpha}, \widehat{\beta} \in F_I$, if and only if $\alpha, \beta \in \Phi_I$.
(ii) Assume $\alpha, \beta \in \Phi^+$ and let $y \in E_2(\Phi_I)$ such that $x = w \cdot y$ for some $w \in W^I$. Then $w^{-1}(\alpha), w^{-1}(\beta) \in \Phi_I^+$ and $y \in L(w^{-1} \cdot \widehat{\alpha}, w^{-1} \cdot \widehat{\beta})$.

Proof. (i) First, note that $\widehat{\alpha}, \widehat{\beta} \in F_I$ if and only if $L(\widehat{\alpha}, \widehat{\beta}) \subseteq V_I$, since F_I is a face of $\mathrm{conv}(\widehat{\Delta})$; that is, if and only if $\alpha, \beta \in \Phi_I = \Phi \cap V_I$. Then, note that any line $L(\widehat{\alpha}, \widehat{\beta})$ contains two normalized roots $\widehat{\alpha_0}, \widehat{\beta_0}$ such that α_0, β_0 form a simple system for the dihedral reflection subgroup W' generated by s_α and s_β. So, $\alpha, \beta \in \Phi_I$ if and only if $\alpha_0, \beta_0 \in \Phi_I$. Therefore, we assume without loss of generality that α, β is a simple system for W'. In particular, x lies in the interior of the segment $[\widehat{\alpha}, \widehat{\beta}]$.

We just have to show that if $x \in E_2(\Phi_I)$, then $\widehat{\alpha}, \widehat{\beta} \in F_I$ (the remaining implication is trivial). If $\widehat{\alpha} \in F_I$, then $L(\widehat{\alpha}, \widehat{\beta}) = L(\widehat{\alpha}, x) \subseteq V_I$. Therefore $\widehat{\alpha}, \widehat{\beta} \in F_I$. The symmetric case $\widehat{\beta} \in F_I$ is handled the same way.

Suppose now that neither $\widehat{\alpha}$ nor $\widehat{\beta}$ is in V_I. We know that $V_I \cap V_1$ is an affine subspace in V_1 supporting the face F_I of $\mathrm{conv}(\widehat{\Delta})$. Let H be the affine subspace of V_1 spanned by $\widehat{\alpha}$ and $V_I \cap V_1$. Since $x \in V_I \cap [\widehat{\alpha}, \widehat{\beta}]$, necessarily $\widehat{\alpha}$ and $\widehat{\beta}$ are separated by $V_I \cap V_1$ in H. Since F_I is a face of $\mathrm{conv}(\widehat{\Delta}) \cap H$, that means that either $\widehat{\alpha}$ or $\widehat{\beta}$ is outside $\mathrm{conv}(\widehat{\Delta})$, contradicting the inclusion $\widehat{\Phi} \subseteq \mathrm{conv}(\widehat{\Delta})$. (Remark: this last argument is almost the same as the one used to prove Proposition 5.18 (i).)

(ii) Set $\alpha' = w^{-1}(\alpha)$ and $\beta' = w^{-1}(\beta)$. Then by [HLR14, Proposition 3.6 (ii)], we have that $z = w^{-1} \cdot x \in L(\widehat{\alpha'}, \widehat{\beta'})$. So by (i) we have that $\alpha', \beta' \in \Phi_I$, since $y \in E_2(\Phi_I) \subseteq F_I$. Finally $\alpha', \beta' \in \Phi_I^+$ because $\alpha, \beta \in \Phi^+$ and $w \in W^I$ (same argument as the beginning of the proof of 5.18(iv)). $\qquad \square$

Now we are ready to prove Theorem 5.20, namely, that the six subsets of E defined in the previous subsections all have the property of good facial restriction.

Proof of Theorem 5.20. (i) The inclusion $\supseteq$ is a direct consequence of Lemma 5.22(i) and $\subseteq$ is clear.

(ii) The inclusion $\subseteq$ follows from follows Lemma 5.22(i) on recalling from the proof of 5.18(ii)(c) that $\mathcal{K}(\Phi_I) = \mathcal{K} \cap V_I$, while $\supseteq$ is proved using Proposition 5.18(ii) and Lemma 5.22(i) as follows. Let $x \in E_{\mathrm{f}}(\Phi) \cap F_I$, so there is $\alpha, \beta \in \Phi$ such that $x \in L(\widehat{\alpha}, \widehat{\beta})$ and $\alpha \prec_{\mathrm{f}} \beta$ in Φ. Since $x \in F_I$, $\alpha, \beta \in \Phi_I$ and therefore $\alpha \prec_{\mathrm{f}} \beta$ in Φ_I. So $x \in E_{\mathrm{f}}(\Phi_I)$.

(iii) is proved similarly as (ii), using Proposition 5.18(ii) and Lemma 5.22(i).

(iv) From Proposition 5.18(iii) we know that $\Sigma_I = \Sigma \cap \Phi_I$, so:

$$E_{\mathrm{elem}}(\Phi_I) \subseteq E_{\mathrm{elem}}(\Phi) \cap F_I \quad \text{and} \quad W_I \cdot E_{\mathrm{elem}}(\Phi_I) \subseteq (W \cdot E_{\mathrm{elem}}(\Phi)) \cap F_I.$$

Now let $x \in E_{\mathrm{elem}}(\Phi) \cap F_I$. So there is $\alpha, \beta \in \Sigma$ such that $x \cap L(\widehat{\alpha}, \widehat{\beta})$. By Lemma 5.22(i) we know that $\alpha, \beta \in \Phi_I$, since $x \in E_2(\Phi) \cap F_I$. So $\alpha, \beta \in \Sigma_I$ and therefore $x \in E_{\mathrm{elem}}(\Phi_I)$.

Let $z \in (W \cdot E_{\mathrm{elem}}(\Phi)) \cap F_I$. So there is $w \in W$ and $x \in E_{\mathrm{elem}}(\Phi)$ such that $w \cdot z = x$. Write $w = w^I w_I$ with $v = w^I \in W^I$ and $w_I \in W_I$ and set $y = w_I \cdot z$. Since $w_I(\Phi_I) = \Phi_I$, we have $y \in (W \cdot E_{\mathrm{elem}}(\Phi)) \cap F_I \subseteq E_2(\Phi) \cap F_I = E_2(\Phi_I)$.

Let $\alpha, \beta \in \Sigma$ such that $x \in L(\widehat{\alpha}, \widehat{\beta})$. Since $x = v \cdot y$ with $v \in W^I$, $y \in E_2(\Phi_I)$ and $x \in E_2(\Phi)$, we have by Lemma 5.22(ii) that $v^{-1}(\alpha), v^{-1}(\beta) \in \Phi_I^+$ and $y \in L(v^{-1} \cdot \widehat{\alpha}, v^{-1} \cdot \widehat{\beta})$. So by Proposition 5.18(iv) we have that $v^{-1}(\alpha), v^{-1}(\beta) \in \Sigma_I$, since $\alpha, \beta \in \Sigma$. Therefore $y \in E_{\mathrm{elem}}(\Phi_I)$, hence $z = w_I^{-1} \cdot y \in W_I \cdot E_{\mathrm{elem}}(\Phi_I)$. $\quad \square$

5.7. **Direct proof of the density of the fundamental limit roots.** Proposition 5.3 is a consequence of the following statement.

Proposition 5.23. *The set $E^{\mathrm{cov}}(\Phi)$ is dense in $E(\Phi)$. Consequently, both $E_{\mathrm{f}}^{\mathrm{cov}}(\Phi)$ and $E_{\mathrm{elem}}(\Phi)$ provide examples of finite subsets, the union the orbits of which is dense in E.*

Even if it is a straightforward consequence of Theorem 3.1(b), we will give a direct (geometric) proof of Proposition 5.23, without using the minimality of the action, i.e., avoiding the reliance on the machinery of the imaginary cone developed in [Dye12] and used in §2. This direct proof is elementary and relies only on a

careful study of the geometry in the case of a root system of rank 3, and on the density of E_2 proved in [HLR14, Theorem 4.2]. It illustrates techniques that may be useful in the study of open questions involving the relationship of dominance order and limit roots.

Assume for now that (Φ, Δ) is a rank 3, irreducible root system in (V, B). It is well known that the signature of B is then $(3, 0)$, $(2, 0)$ or $(2, 1)$, see for instance §2.1. In the case where (Φ, Δ) is weakly hyperbolic (signature $(2, 1)$), we show the following property. This will allow us to pass from dihedral limit roots coming from dominance to dihedral limit roots coming from covers of dominance.

Proposition 5.24. *Let (Φ, Δ) be a based root system of rank 3, of weakly hyperbolic type. Let $\alpha, \beta, \gamma \in \Phi^+$ such that $B(\alpha, \beta) \leq -1$ and $\alpha \prec \gamma$ (with $\alpha \neq \gamma$). Let $x \in \widehat{Q} \cap L(\widehat{\alpha}, \widehat{\beta})$ and $y \in \widehat{Q} \cap L(\widehat{\alpha}, \widehat{\gamma})$ that are visible from $\widehat{\alpha}$. Set $w := s_\alpha s_\beta$. Then we have:*

(i) the sequence $w^n \cdot \widehat{\alpha}$ converges to x when n tends to infinity. Moreover, $w^{n+1} \cdot \widehat{\alpha} \in \,]w^n \cdot \widehat{\alpha}, x[$ for all $n \in \mathbb{N}$.

(ii) The sequence $y_n := w^n \cdot y$ converges to x when n tends to infinity. Moreover, $y_n \in \widehat{Q} \cap L(w^n \cdot \widehat{\alpha}, w^n \cdot \widehat{\gamma})$ is visible from $w^n \cdot \widehat{\alpha}$ for all $n \in \mathbb{N}$.

This property and its proof are illustrated in Figure 9.

Proof. Set $W' := \langle s_\alpha, s_\beta \rangle$, so $w \in W'$, and set $L := L(\widehat{\alpha}, \widehat{\beta})$. Since $B(\alpha, \beta) \leq -1$, $\Delta' := \{\alpha, \beta\}$ is a simple system for the root system $\Phi' := W'(\Delta')$. (i) By Equation (5.2) in §5.3, we know that x is visible from $\widehat{\alpha}$ on the line L and that $w^{n+1} \cdot \widehat{\alpha} \in \,]w^n \cdot \widehat{\alpha}, x]$. Therefore, the sequence $(w^n \cdot \widehat{\alpha})_{n \in \mathbb{N}}$ is increasing (in the line L ordered from $\widehat{\alpha}$ to $\widehat{\beta}$) and entirely contained in $[\widehat{\alpha}, x]$. So $(w^n \cdot \widehat{\alpha})_{n \in \mathbb{N}}$ has a limit ℓ in $[\widehat{\alpha}, x]$. This limit is in E, so also in Q by [HLR14, Theorem 2.7]. Since $Q \cap [\widehat{\alpha}, x] = \{x\}$, we obtain $\ell = x$.

(ii) By [HLR14, §2.3], we can assume without loss of generality that V_1 is the transverse plane of Proposition 4.13, so $\widehat{Q}$ is a circle in the plane V_1. Since $y \in L(\widehat{\alpha}, \widehat{\gamma})$ is visible from $\widehat{\alpha}$, we obtain by [HLR14, Proposition 3.6 (ii) and Proposition 3.8 (iii)] that $y_n = w^n \cdot y \in L(w^n \cdot \widehat{\alpha}, w^n \cdot \widehat{\gamma})$ is visible from $w^n \cdot \widehat{\alpha}$ for all $n \in \mathbb{N}$.

In order to finish to prove (ii), we need to cover some basic facts from classical Euclidean geometry. Any point p outside the closed disk bounded by $\widehat{Q}$ has two tangent lines to $\widehat{Q}$ passing through p; these tangent lines define two tangent points $t(p), t'(p)$. Let z be a point of the circle, visible from p. The arc $\mathcal{C}_z(p)$ of the circle $\widehat{Q}$ containing z and bounded by $t(p)$ and $t'(p)$ is precisely the set of elements in the circle that are visible from p, see Figure 9. It is an easy exercise to show the following two statements:

- for any $q \in [p, z]$, z is visible from q, and $\mathcal{C}_z(q) \subseteq \mathcal{C}_z(p)$;
- let $(p_n)_{n \in \mathbb{N}}$ be a sequence of points outside the closed disk, such that z is visible from p_n for any n; if (p_n) converges to z, then the length of the arc $\mathcal{C}_z(p_n)$ tends to 0, and so $\bigcap_{n \in \mathbb{N}} \mathcal{C}_z(p_n) = \{z\}$.

We apply these facts to our situation. For all $n \in \mathbb{N}$, we have:

- $y_n \in \mathcal{C}_x(w^n \cdot \widehat{\alpha})$, since y_n is visible from $w^n \cdot \widehat{\alpha}$;
- x is visible from $w^n \cdot \widehat{\alpha}$ and $\mathcal{C}_x(w^{n+1} \cdot \widehat{\alpha}) \subseteq \mathcal{C}_x(w^n \cdot \widehat{\alpha})$, by (1) (see Figure 9).
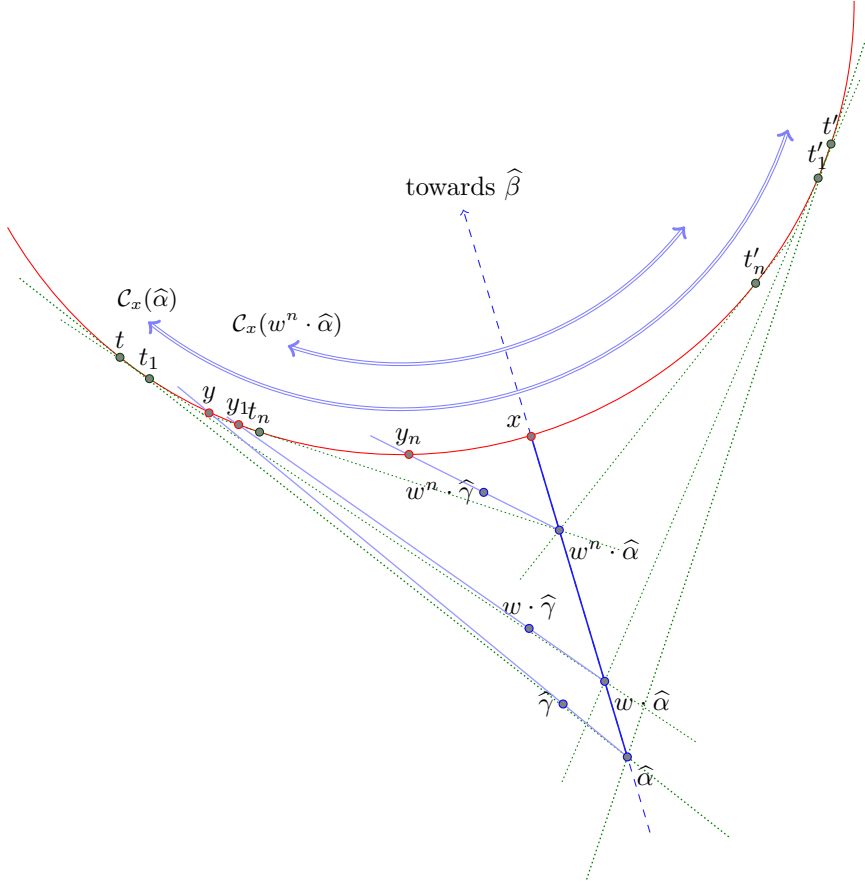
FIGURE 9. Illustration of the proof of Proposition 5.24: in red is a part of the circle $\widehat{Q}$ on which lives the arc $\mathcal{C}_x(\widehat{\alpha})$ of all the points on $\widehat{Q}$ visible from $\widehat{\alpha}$. Here we adopt the notation $t := t(\widehat{\alpha})$ and $t' := t'(\widehat{\alpha})$ for the intersection points of the two tangents to the circle $\widehat{Q}$ passing through $\widehat{\alpha}$; similarly we consider $t_1 := t(w \cdot \widehat{\alpha})$ and $t_n := t(w^n \cdot \widehat{\alpha}_n)$. Note that in the case where $B(\alpha, \beta) = -1$, the line $L(\widehat{\alpha}, \widehat{\beta})$ is tangent to $\widehat{Q}$ and x is equal to t or t'.

Since $w^n \cdot \widehat{\alpha}$ converges to x, we have therefore that y_n converges to a limit that lives in the set

$$\bigcap_{n \in \mathbb{N}} C_x(w^n \cdot \widehat{\alpha}) = \{x\}.$$

$\square$

Proof of Theorem 5.23. Since $E_2(\Phi)$ is dense in $E(\Phi)$ by [HLR14, Thm. 4.2], it is enough to show that any $x \in E_2(\Phi)$ is the limit of a sequence in $E^{\mathrm{cov}}(\Phi)$.

Let $x \in E_2(\Phi)$ and let $\alpha, \beta \in \Phi^+$ such that $x \in L(\widehat{\alpha}, \widehat{\beta})$. We can choose α, β such that $B(\alpha, \beta) \leq -1$ and x is visible from $\widehat{\alpha}$.

We have to prove that there is a sequence $(y_n)_{n \in \mathbb{N}} \subseteq E^{\mathrm{cov}}(\Phi)$ converging to x. Let $\gamma \in \Phi^+$ such that $\alpha \mathrel{\dot{\prec}} \gamma$; for instance, any $\gamma \in \Phi^+$ with $\alpha \prec \gamma \prec s_{\alpha(\beta)}$, $\gamma \neq \alpha$

and $l(s_\gamma)$ is minimal amongst γ with these properties will do. Since $\alpha \prec \gamma$, there is $y \in L(\widehat{\alpha}, \widehat{\gamma})$ such that y is visible from $\widehat{\alpha}$ (Proposition 5.7). We have two cases:

(i) If $\widehat{\gamma} \in L(\widehat{\alpha}, \widehat{\beta})$, then $y \in L(\widehat{\alpha}, \widehat{\gamma}) = L(\widehat{\alpha}, \widehat{\beta})$ and therefore $y = x \in E^{\mathrm{cov}}(\Phi)$ is the unique point in $L(\widehat{\alpha}, \widehat{\beta})$ that is visible from $\widehat{\alpha}$.

(ii) Assume that $\widehat{\gamma} \notin L(\widehat{\alpha}, \widehat{\beta})$, so $x \neq y$. Set $V' := \mathrm{span}\{\alpha, \beta, \gamma\}$, $W' := \langle s_\alpha, s_\beta, s_\gamma \rangle$ and $\Phi' := W'(\{s_\alpha, s_\beta, s_\gamma\})$. Using [Dye90], there is a simple system such that (Φ', Δ') is a root system of rank 3 in (V', B) with associated Coxeter group W'. Recall that $B(\alpha, \beta) \leq -1$, so $w := s_\alpha s_\beta$ has infinite order. Since $\alpha \dot{\prec} \gamma$, $\alpha \neq \gamma$ and $B(\alpha, \gamma) \geq 1$. Therefore $s_\alpha s_\gamma$ also has infinite order. Thus W' must be irreducible, of rank 3, infinite and cannot be affine (since $y \neq x$). So the signature of the restriction of B to V' is $(2, 1)$. Since $\alpha \dot{\prec} \gamma$ in W, then $\alpha \prec \gamma$ in W'. Since x and y are visible from $\widehat{\alpha}$, Proposition 5.24 applies to our situation: there is a sequence $y_n := w^n \cdot y$ in $E(\Phi') \subseteq E(\Phi)$ that converges to x; moreover, $y_n \in \widehat{Q} \cap L(w^n \cdot \widehat{\alpha}, w^n \cdot \widehat{\gamma})$ for all $n \in \mathbb{N}$. Since cover of dominance is preserved under the action of W, we have $w^n(\alpha) \dot{\prec} w^n(\gamma)$ and therefore $y_n \in E^{\mathrm{cov}}(\Phi)$ for all $n \in \mathbb{N}$.

$\square$

6. Faithfulness of the action on limit roots and universal Coxeter groups

Let (Φ, Δ) be a based root system in (V, B) with associated Coxeter system (W, S). A question that naturally arose in [HLR14, Remark 3.4] is: is the W-action on the set E of limit roots faithful?

Obviously, if Φ is finite, it is not the case since E is empty. If Φ is of affine type or is indefinite dihedral, then E is finite whereas W is infinite, so the action cannot be faithful either. If Φ is not irreducible, and we write the decomposition in irreducible parts $\Phi = \bigsqcup_{i=1}^p \Phi_i$, $W = W_1 \times \cdots \times W_p$, then one sees from Remark 3.3 that the W-action on $E(\Phi)$ is faithful if and only if the W_i action on each $E(\Phi_i)$ is faithful. The answer to the question is therefore given by the following result, the proof of which is one of the aims of this section.

Theorem 6.1. *Assume that (Φ, Δ) is irreducible of indefinite type, of rank ≥ 3. Then the W-action on E is faithful.*

We actually prove a stronger property: for any open set U with $U \cap E \neq \varnothing$, if some $w \in W$ fixes $U \cap E$ pointwise, then $w = 1$ (Theorem 6.12(c)).

The main ingredient of the proof of Theorem 6.1 is the following existence property. Given a based root system (Φ, Δ) (irreducible, indefinite) with Coxeter group W, one can find a root subsystem (Φ', Δ') of (Φ, Δ) such that $\mathrm{span}(\Delta') = \mathrm{span}(\Delta)$ and for all $\alpha, \beta \in \Delta'$ ($\alpha \neq \beta$), $B(\alpha, \beta) < -1$ (Proposition 6.4). This implies the existence of non-dihedral universal reflection subgroups of W (see §6.1 for the definitions).

Refinement of the proof leads to a positive answer to another important question, raised by the first author in [Dye12, Question 9.8]: can one approximate, with arbitrary precision, the set of limit roots (resp., imaginary convex body) of Φ with the sets of limit roots (resp., imaginary convex body) of its root subsystems associated to reflection subgroups which are universal Coxeter groups. Actually, we extend the result in two directions. First, we establish a more general result on

similar approximations of arbitrary faces of the imaginary convex body. Second, we also consider approximations by finite subsets of the W-orbit of a point in the imaginary convex body, and their convex closures. In order to be able to state precisely these results, we first collect in §6.1 some definitions and known properties of reflection subgroups and universal Coxeter groups. In §6.2 we recall the definition of the Hausdorff metric on compact sets, and state the approximation theorem (Theorem 6.2).

We then continue to the core of this section, starting the steps of the proofs of Theorems 6.1 and 6.2. In §6.3 we state Proposition 6.4 mentioned above, concerning the existence of reflection subgroups of W that are universal Coxeter groups. In §6.4 we give some notations and facts related to the way lines in V_1 intersect $\widehat{Q}$; they will be helpful in shortening the proofs in the following subsections. In §6.6 we check the faithfulness of the W-action on E (Theorem 6.1), by proving a stronger result (Theorem 6.12); the main component of the proof of which is the existence of universal Coxeter subgroups from Proposition 6.4. Among the consequences of this stronger theorem are also the facts that E has no isolated points, and that E is not contained in any countable union of proper affine subspaces of aff(E). In §6.8-6.9 we state and prove two other direct consequences of Theorem 6.12: $\overline{Z} = \text{conv}(E)$ has uncountably many extreme points, and E contains a subset homeomorphic to the Cantor set (in particular, E has the cardinality of $\mathbb{R}$). In the last part, §6.10, the proof of the approximation theorem 6.2 is completed, using Theorem 6.12 and the tools introduced in §6.4. Although some of the steps of the proof are a bit technical, they are always constructed from a geometric intuition which is explicitly given.

From §6.6 on in this section, we assume unless otherwise stated that (Φ, Δ) is an irreducible based root system of indefinite type and rank at least three.

6.1. **Reflection subgroups and universal Coxeter groups.** Let (W, S) be a Coxeter group. The set T of reflections of (W, S) is the conjugacy closure of S. A *reflection subgroup* W' of (W, S) is a subgroup of W generated by the reflections it contains, i.e., $W' = \langle W' \cap T \rangle$. It is easy to see that $T = \{s_\beta \mid \beta \in \Phi^+\}$, so reflection subgroups are subgroups of W that naturally associated with subsets of Φ^+. Let us discuss this relation a bit more (see [Dye90, §3.3], or also [BD10], for more details).

Let (Φ, Δ) be a based root system associated to (W, S). For $A \subseteq \Phi^+$, we consider the reflection subgroup W' generated by the reflections associated to A: $W' = \langle s_\beta, \ \beta \in A \rangle$. Set:

$$\Phi' = \Phi_{W'} := \{\alpha \in \Phi \mid s_\alpha \in W'\} \ \text{ and } \ \Phi_{W'}^+ := \Phi_{W'} \cap \Phi^+.$$

To find a set of canonical generators for W', we will first build a simple system for $\Phi_{W'}$. Let

$$\Delta' = \Delta_{W'} := \left\{ \alpha \in \Phi_{W'}^+ \mid s_\alpha(\Phi_{W'}^+ \setminus \{\alpha\}) = \Phi_{W'}^+ \setminus \{\alpha\} \right\}.$$

Set $S' = S_{W'} = \{s_\alpha \mid \alpha \in \Delta_{W'}\}$, then $(\Phi_{W'}, \Delta_{W'})$ is a based root system in (V, B) with positive roots $\Phi_{W'}^+$ and associated Coxeter system (W', S') (see [BD10, Lemma 3.5]). Any based root system arising this way will be called below a *root subsystem* of (Φ, Δ). In particular, facial root subsystems defined in §4 are examples of root subsystems.

Note that, even when Φ is the standard root system of (W, S) and S is finite, $\Delta_{W'}$ may be linearly dependent, and one may have $B(\alpha, \beta) < -1$ for some $\alpha, \beta \in \Delta_{W'}$. This is the main reason for having considered from the beginning of this article

(and already in [HLR14] and [Dye12]) a larger class of root systems than the usual one (see for instance [HLR14, §5.1] for some examples). Actually $\Delta_{W'}$ may even be infinite. When it is finite, the reflection subgroup (resp., the root subsystem) is said to be of finite rank, or finitely generated.

The following fact (from [Dye90, Theorem 4.4]) is fundamental: given a subset $\Gamma \subseteq \Phi^+$, its associated reflections $R = \{s_\gamma \mid \gamma \in \Gamma\}$ and reflection subgroup $W' = \langle R \rangle$, one has $R = S_{W'}$, i.e., $\Gamma = \Delta_{W'}$, if and only if

for all distinct $\alpha, \beta \in \Gamma$, $B(\alpha, \beta) \in\ ] - \infty, -1] \cup \{ - \cos\left(\dfrac{\pi}{k}\right), k \in \mathbb{Z}_{\geq 2}\}$.

This is equivalent to saying that Γ satisfies the axiom (ii) of a simple system seen in the introduction of §2. Geometrically, this means that $\widehat{\Gamma}$ is the set of extreme points of $\operatorname{conv}(\widehat{\Phi'})$.

A Coxeter group with no non-trivial braid relations, canonically isomorphic to the free product of cyclic groups of order two generated by its simple reflections, is called below a *universal Coxeter group*. It is the free object for Coxeter groups. Given the characterization above, a reflection subgroup W' of W is universal if and only if $B(\alpha, \beta) \leq -1$ for all distinct $\alpha, \beta \in \Delta_{W'}$. It is easily seen that any reflection subgroup of a universal Coxeter group is universal. If all the $B(\alpha, \beta)$ are equal, then a simple computation of eigenvalues shows that the root system is weakly hyperbolic. Otherwise, it is not always the case; see for instance the example of Figure 5 and also [Dye12, Example 1.4]. We shall say that a based root system (Φ, Δ) is *generic universal* if $B(\alpha, \beta) < -1$ for all distinct $\alpha, \beta \in \Delta$.

6.2. Approximation of E and $\overline{Z}$ using reflection subgroups.

Let $\mathbf{X}$ denote the set of non-empty compact subsets of V. There is a natural distance on $\mathbf{X}$, called the *Hausdorff metric*, that may be defined as follows (see for instance [Web94, §2.7] for details of the definition and proofs of the few basic properties needed here). Fix a norm $\|\cdot\|$ on V (inducing the standard topology on V). For $K \in \mathbf{X}$ and $\epsilon \in \mathbb{R}_{\geq 0}$, define the ϵ-neighbourhood of K (see [Web94, Fig. 2.12]):

$$K_\epsilon := \{\, v \in V \mid \|v - a\| \leq \epsilon \text{ for some } a \in K \,\}.$$

The Hausdorff metric $\mathrm{d} \colon \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ on $\mathbf{X}$ is defined by

$$\mathrm{d}(K, L) := \inf(\{\, \epsilon \in \mathbb{R}_{\geq 0} \mid K \subseteq L_\epsilon \text{ and } L \subseteq K_\epsilon \,\}), \text{ for } K, L \in \mathbf{X}.$$

It is well known that d is a metric on $\mathbf{X}$, that the resulting topology on $\mathbf{X}$ is independent of the choice of norm on V and that

$$(6.1) \qquad \mathrm{d}(\operatorname{conv}(K), \operatorname{conv}(L)) \leq \mathrm{d}(K, L), \text{ for } K, L \in \mathbf{X}.$$

Another simple property used later in this section is that for $K, L \in \mathbf{X}$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}_{\geq 0}$ with $L \subseteq K_{\epsilon_2}$, one has $L_{\epsilon_1} \subseteq K_{\epsilon_1 + \epsilon_2}$. Finally, we shall use the fact that

$$(6.2) \qquad \mathrm{d}(\bigcup_{i=1}^{n} K_i, \bigcup_{i=1}^{n} L_i) \leq \max(\{\, \mathrm{d}(K_i, L_i) \mid i = 1, \ldots, n \,\})$$

if $K_i, L_i \in \mathbf{X}$ for $i = 1, \ldots, n$, where $n > 0$.

The second main result of this section is the theorem below. In the special case in which $F = \overline{Z}$, parts of (a) were raised as a question in [Dye12, Question 9.8] and were previously established in the case of hyperbolic W by Tom Edgar in [Edg09].

Theorem 6.2. *Assume that (Φ, Δ) is an irreducible based root system of indefinite type. Abbreviate $Z := Z(\Phi)$ and $E := E(\Phi)$. Fix a face F of $\overline{Z}$ (e.g. $F = \overline{Z}$) and any $\epsilon > 0$.*

(a) *There is a finite rank based root subsystem (Φ', Δ') with associated reflection subgroup W' of W such that, writing $E' = E(\Phi')$ and $Z' = Z(\Phi')$:*

  (i) (Φ', Δ') *is generic universal, so W' is a universal Coxeter group;*
  (ii) $\operatorname{span}(\Delta') = \operatorname{span}(\Delta)$;
  (iii) $\operatorname{d}(E', F \cap E) < \epsilon$, $\operatorname{d}(\widehat{\Phi'} \cup E', F \cap E) < \epsilon$ *and* $\operatorname{d}(\widehat{\Delta'}, F \cap E) < \epsilon$;
  (iv) $\operatorname{d}(\overline{Z'}, F) < \epsilon$ *and* $\operatorname{d}\left(\operatorname{conv}(\widehat{\Delta'}), F\right) < \epsilon$.

  Moreover, given any non-empty set of W-orbits on Φ, one may chose Δ' so it contains roots from those W-orbits and no others.

(b) *For any $z \in \overline{Z}$, there exists a finite subset $G \subseteq W \cdot z$ with $\operatorname{span}(G) = \operatorname{span}(\Delta)$, $\operatorname{d}(G, F \cap E) < \epsilon$ and $\operatorname{d}(\operatorname{conv}(G), F) < \epsilon$.*

If Φ has rank at least three, one may in addition, for any $m \in \mathbb{N}$, choose W' and G above so W' has rank at least m and G has cardinality at least m.
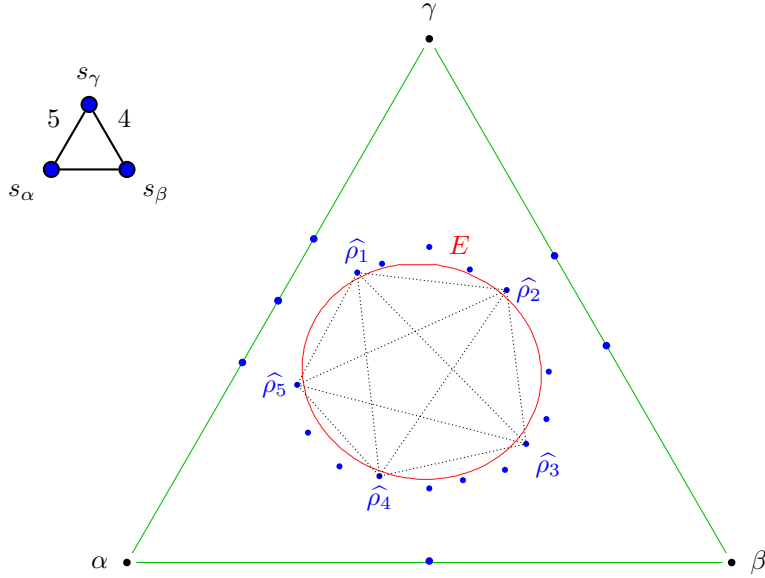

FIGURE 10. The geometric intuition behind Theorem 6.2 when $F = \overline{Z}$, on a simple example (see discussion below the theorem): for any $\epsilon > 0$, one choose a subset of positive roots $\Delta' = \{\rho_1, \ldots, \rho_k\}$, such that $\widehat{\Delta'}$ is close enough to E and numerous enough in order to verify $B(\rho_i, \rho_j) < -1$ for all $i \neq j$, and $\operatorname{d}\left(\operatorname{conv}(\widehat{\Delta'}), \operatorname{conv}(E(\Phi))\right) < \epsilon$.

Note that $B(\alpha, \beta) < -1$ if and only if the segment $[\widehat{\alpha}, \widehat{\beta}]$ intersects $\widehat{Q}$ in two distinct points (see for instance [HLR14, Figure 3]). In particular, in this case they provide two limit points. In Figure 10 we give a schematic representation of Theorem 6.2 in the case where the face F is $\overline{Z}$ (to which we temporarily restrict attention for simplicity). The ellipse represents the normalized isotropic cone $\widehat{Q}$,

which in general contains E, and in the example of the picture is exactly E. The idea is to choose a sufficiently numerous subset Δ' of roots such that $\widehat{\Delta'}$ is close enough to $\widehat{Q}$ that the line joining any two of the roots cuts $\widehat{Q}$ in two points. It is intuitively clear that one could do this if, say, $\widehat{Q}$ really is (as in the diagram) the boundary of some strictly convex body with the roots outside it (this is essentially the hyperbolic case). If one can do this, one has (i) automatically since the segment between distinct normalized roots in $\widehat{\Delta'}$ cuts $\widehat{Q}$ in two points. The main subtlety is that the strict convexity need not hold: for example, $\widehat{Q}$ may contain affine subspaces of positive dimension (recall Proposition 4.18) and choosing roots close to two limit roots in such an affine subspace does not guarantee that the segment joining the roots cuts $\widehat{Q}$ (since the subspace is flat). In fact, even a (non-strictly) convex body need not exist with properties as above (recall from [Dye12, Example 1.4] that there are only very weak restrictions on the signature of B on span(Δ)). However, the part of $\widehat{Q}$ near limit roots behaves enough like such a body for the proof to go through (see for example Figure 5). Since E is the set of limit points of $\widehat{\Phi}$, it is intuitively reasonable that, given some technical device to get around the subtleties, one should be able to choose Δ' large enough so that $\widehat{\Delta'}$ is arbitrarily close to E and Δ' has the same span as Δ, giving (ii) and the last part of (iii) (with $F = \overline{Z}$). The other parts of (iii) hold since E', $\widehat{\Phi}' \cup E'$ and $\widehat{\Delta'}$ are automatically close for generic universal Φ' for which the set of limit roots of rank two standard parabolic subsystems is sufficiently close to $\widehat{\Delta'}$ (see Lemma 6.11) and the position of the latter limit roots can be controlled. Then part (iv) follows from (iii) by inequality (6.1). These intuitive geometric arguments will be made rigorous in §6.3 and §6.10.

Remark 6.3. We could state an equivalent version of Theorem 6.2 in the context of closed but possibly non-convex cones, by replacing each subset of V_1 in the statement of the theorem by the union of rays through its points and defining a distance on the set of closed non-empty, nonzero (possibly non-convex) cones included in cone(Δ) by $\mathrm{d}'(C, C') := \mathrm{d}(C \cap V_1, C' \cap V_1)$. It is easily seen that the resulting topology is independent of choice of V_1.

The rest of this section is now devoted to the proofs of Theorem 6.1 and Theorem 6.2.

6.3. **Existence of reflection subgroups that are universal Coxeter groups.**
The following proposition proves the existence of reflection subgroups that are universal Coxeter groups; it corresponds to parts (i)-(ii) of Theorem 6.2.

Proposition 6.4. *Assume* (Φ, Δ) *is irreducible of indefinite type and rank two or greater. Then there exists a generic, universal based root subsystem* (Λ, Ψ) *of* (Φ, Δ) *such that* span(Ψ) = span(Δ). *In particular* W_Ψ *is a universal Coxeter group.*

Geometrically, this means that one can always find normalized roots $\{\widehat{\rho_1}, \ldots, \widehat{\rho_k}\}$ such that for $i \neq j$, $\widehat{\rho_i}$ and $\widehat{\rho_j}$ are "on both sides" of $\widehat{Q}$, i.e., the segment $[\widehat{\rho_i}, \widehat{\rho_j}]$ intersects $\widehat{Q}$ in two distinct points (see Figure 10).

Remark 6.5. The proposition implies that the possible signatures of the restrictions of B to span(Δ), for irreducible Φ of indefinite type and rank at least three, coincide with those of generic universal root systems of the same rank. These are described in [Dye12, Example 1.4]; in fact, the proposition provides a conceptual explanation for the observation at the end of that example.

Before proving this proposition, we need a technical lemma. A subset Ψ of Φ is said to be *indecomposable* if there is no partition $\Psi = \Psi_1 \sqcup \Psi_2$ of Ψ into non-empty, disjoint, pairwise orthogonal subsets Ψ_1, Ψ_2. If Ψ is finite, this is equivalent to indecomposability of the Gram matrix $(B(\alpha, \beta))_{\alpha, \beta \in \Psi}$ in the usual sense, and, if Ψ is a simple system, it corresponds to irreducibility of W_Ψ.

Lemma 6.6. *Let Ψ be an indecomposable finite subset of Φ^+ such that $|\Psi| \geq 3$, $B(\alpha, \beta) \leq 0$ for all distinct $\alpha, \beta \in \Psi$ and $B(\alpha, \beta) \leq -1$ for some $\alpha, \beta \in \Psi$. Let $N \in \mathbb{R}_{>1}$. Then there exists $\Psi' \subseteq \Phi^+$ with the following properties:*

(i) $W_{\Psi'} \subseteq W_\Psi$, and $\mathrm{span}(\Psi') = \mathrm{span}(\Psi)$.

(ii) There is a bijection $\Psi \xrightarrow{\cong} \Psi'$ which maps each element of Ψ to an element of Ψ' in the same W_Ψ-orbit.

(iii) If $\alpha, \beta \in \Psi'$ are distinct, then $B(\alpha, \beta) < -N$.

In particular, there is a generic universal based root subsystem (Λ, Ψ') of (Φ, Δ) with simple system Ψ'.

Note that the result does not hold if $|\Psi| = 2$: take for example a dihedral reflection subgroup $\langle s_\alpha, s_\beta \rangle$ with $B(\alpha, \beta) = -1$.
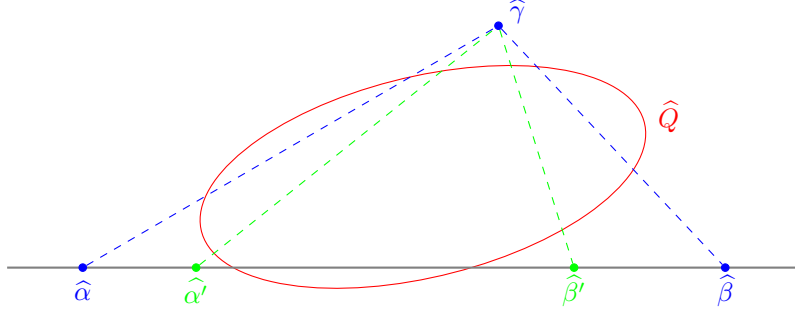

FIGURE 11. Condensation of a pair of roots $\{\alpha, \beta\}$: replacement of such a pair by a new pair $\{\alpha', \beta'\}$ which is closer to Q. This is the main idea of the proof of Lemma 6.6.

The geometric idea of the proof is to replace progressively any pair (α, β) such that $B(\alpha, \beta) \leq -1$ by a pair (α', β') such that $(\widehat{\alpha'}, \widehat{\beta'})$ is on the on the same line as $(\widehat{\alpha}, \widehat{\beta})$ but closer to $\widehat{Q}$, and with α', β' conjugate to α, β respectively in $\langle s_\alpha, s_\beta \rangle$ (we call this a *condensation* at (α, β)): this will not increase the other inner products $B(\gamma, \alpha)$, $B(\gamma, \beta)$ (see Figure 11) and may decrease some of them. Properties (i)-(ii) are clearly preserved by this replacement; it turns out one can iterate this process in order to obtain property (iii).

Proof. Fix $N \in \mathbb{R}_{>1}$. We will first describe a relation $\to$ on *admissible k-tuples* built up using condensation, and use this relation to show the existence of Ψ'.

For $k \in \mathbb{N}_{\geq 3}$, call a k-tuple $a = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive roots $\alpha_i \in \Phi^+$ an *admissible k-tuple* if $B(\alpha_i, \alpha_j) \leq 0$ for all $1 \leq i < j \leq k$, $B(\alpha_i, \alpha_j) \leq -1$ for some $1 \leq i < j \leq k$ and $\Gamma_a := \{\alpha_1, \ldots, \alpha_k\}$ is indecomposable. In particular, this implies that the α_i are pairwise distinct. Let A_k be the set of all admissible k-tuples. For instance, by the assumptions on Ψ, there is an admissible k-tuple

$a = (\alpha_1, \ldots, \alpha_k) \in A_k$ such that $B(\alpha_1, \alpha_2) \leq -1$ and $\Psi = \Gamma_a = \{\alpha_1, \ldots, \alpha_j\}$ is indecomposable for $j = 1, \ldots, k$.

Define the *incompleteness index* $I(a)$ of an admissible k-tuple $a \in A_k$ to be $I(a) := \{\, (j,i) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j \leq k, B(\alpha_i, \alpha_j) \geq -N \,\}$. This index can be seen as the margin of error for Ψ to be a desired Ψ'.

Condensation on k-tuples. Consider an admissible k-tuple $a \in A_k$ as above. For any distinct $i, j \in \mathbb{N}$ with $1 \leq i, j \leq k$ and $B(\alpha_i, \alpha_j) \leq -1$, define another admissible k-tuple $b = (\beta_1, \ldots, \beta_k) \in A_k$, which we call a *condensation* of a (at $\{\alpha_i, \alpha_j\}$), as follows. First, we set $\beta_l = \alpha_l$ for all l with $1 \leq l \leq k$ and $l \neq i, j$. We shall set

$$\beta_i := (s_{\alpha_i} s_{\alpha_j})^n(\alpha_i), \qquad \beta_j := (s_{\alpha_j} s_{\alpha_i})^n(\alpha_j)$$

for any $n \in \mathbb{N}_{>0}$ chosen sufficiently large to satisfy the conditions described below.

Write $B(\alpha_i, \alpha_j) = -\cosh \lambda$ where $\lambda \in \mathbb{R}_{\geq 0}$. Define p_n for $n \in \mathbb{Z}$ by

$$p_n = \begin{cases} n, & \text{if } \lambda = 0; \\ \frac{\sinh(n\lambda)}{\sinh \lambda}, & \text{if } \lambda > 0. \end{cases}$$

It is well known and easily checked[8] that

$$\beta_i = p_{2n+1}\alpha_i + p_{2n}\alpha_j, \qquad \beta_j = p_{2n}\alpha_i + p_{2n+1}\alpha_j$$

and $B(\beta_i, \beta_j) = -\cosh((4n+1)\lambda)$. For $\gamma \in \Gamma_a \setminus (\{\alpha_i, \alpha_j\} \cup \{\alpha_i, \alpha_j\}^\perp)$ one has $B(\gamma, \alpha_i) \leq 0$ and $B(\gamma, \alpha_j) \leq 0$ with at least one of the inequalities being strict. Since $p_m \to +\infty$ as $m \to +\infty$, we may (and do) choose $n \in \mathbb{N}_{>0}$ so that

$$(6.3) \quad B(\beta_i, \gamma) < -N, \quad B(\beta_j, \gamma) < -N \text{ for all } \gamma \in \Gamma_\alpha \setminus (\{\alpha_i, \alpha_j\} \cup \{\alpha_i, \alpha_j\}^\perp).$$

This completes the definition of the k-tuple b. We show now that $b \in A_k$. Using that $p_{2n}, p_{2n+1} \geq 1$ and $-\cosh((4n+1)\lambda) \leq -\cosh \lambda$, it follows that

$$(6.4) \qquad\qquad B(\beta_l, \beta_m) \leq B(\alpha_l, \alpha_m) \text{ for all } 1 \leq l, m \leq k.$$

This implies that $b \in A_k$ with $I(b) \subseteq I(a)$.

$(\star)$: Write $a \xrightarrow{i,j} b$ to indicate that $b \in A_k$ is a condensation of $a \in A_k$ at $\{\alpha_i, \alpha_j\}$. This implies that (6.4) holds, that $W_{\Gamma_b} \subseteq W_{\Gamma_a}$, that $\mathrm{span}(\Gamma_b) = \mathrm{span}(\Gamma_a)$ and that $\alpha_l \mapsto \beta_l$ defines a bijection $\Gamma_a \to \Gamma_b$ with the property that β_l is in the W_{Γ_a}-orbit of α_l for $l = 1, \ldots, k$. Moreover, if $J \subseteq \{1, \ldots, k\}$ with $\{\, \alpha_i \mid i \in J \,\}$ indecomposable, then $\{\, \beta_i \mid i \in J \,\}$ is also indecomposable.

Write $a \xrightarrow{*,*} b$ to indicate that $b \in A_k$ is a condensation of $a \in A_k$ for some $\{\alpha_i, \alpha_j\}$ and let $\to$ be the transitive closure of this relation.

Back to the proof. Now take $k = |\Psi| \geq 3$. By the assumptions on Ψ, we already noted at the beginning of the proof that there is an admissible k-tuple $a = (\alpha_1, \ldots, \alpha_k) \in A_k$ such that $B(\alpha_1, \alpha_2) \leq -1$ and $\Psi = \Gamma_a = \{\alpha_1, \ldots, \alpha_j\}$ is indecomposable for $j = 1, \ldots, k$. Consider the set $A_k(a)$ of admissible k-tuples $b \in A_k$ such that $a \to b$. Fix $b \in A_k(a)$ with $I(b)$ minimal under inclusion. It will suffice to show that $I(b) = \varnothing$; for then, setting $\Psi' = \Gamma_b = \{\beta_1, \ldots, \beta_k\}$, the above remarks $(\star)$ imply that (i) holds and that (ii) is satisfied by the bijection $\alpha_i \mapsto \beta_i \colon \Psi \to \Psi'$, and (iii) holds since $I(b) = \varnothing$.

Suppose to the contrary that $I(b) \neq \varnothing$. Then there exists a minimum element $(j, i) \in I(b)$ in the lexicographic total order on $I(b) \subseteq \mathbb{N} \times \mathbb{N}$ induced by the

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[8]See for example [How96, p.3].

standard total order of $\mathbb{N}$. First assume $j \leq 2$, so $(j, i) = (2, 1)$. Since $B(\beta_1, \beta_2) \leq B(\alpha_1, \alpha_2) \leq -1$, we have $b \xrightarrow{1,2} g$ for some $g = (\gamma_1, \ldots, \gamma_k) \in A_k(a)$. Since $\{\alpha_l \mid l = 1, 2, 3\}$ is indecomposable, so is $\{\beta_l \mid l = 1, 2, 3\}$ i.e. $B(\beta_3, \beta_l) < 0$ for some $l \in \{1, 2\}$. From (6.3), we get $B(\gamma_3, \gamma_l) < -N < -1$ for $l = 1, 2$. Now we may define $d = (\delta_1, \ldots, \delta_k) \in A_k(a)$ such that $g \xrightarrow{1,3} d$. Since $B(\gamma_1, \gamma_2) \leq B(\alpha_1, \alpha_2) \leq -1$, we have $B(\delta_1, \delta_2) < -N$ by (6.3). In particular, $d \in A_k(a)$, as desired, with $I(d) \subseteq I(b) \setminus \{(2, 1)\} \subsetneq I(\beta)$, contrary to minimality of $I(b)$.

Hence $j \geq 3$ and $1 \leq i < j$. Since $\Psi' = \{\alpha_1, \ldots, \alpha_j\}$ is indecomposable, there is $l \in \mathbb{N}$ so $1 \leq l < j$ and $B(\alpha_j, \alpha_l) < 0$. Since $j \geq 3$, we may choose $m \in \mathbb{N}$ so $1 \leq m < j$, $m \neq l$ and $i \in \{l, m\}$. Since $l, m < j$, the minimality of $(j, i) \in I(b)$ in lexicographic order implies $B(\beta_l, \beta_m) < -N < -1$. Hence $b \xrightarrow{l,m} g$ for some $g \in A_k(a)$. Since $B(\beta_j, \beta_i) \leq B(\alpha_j, \alpha_i) < 0$ with $i \in \{l, m\}$, it follows by (6.3) that $B(\gamma_i, \gamma_j) < -N$. Therefore, $\gamma \in A'$ with $I(g) \subseteq I(b) \setminus \{(j, i)\} \subsetneq I(b)$ contrary to minimality of $I(b)$. This contradiction completes the proof of Lemma 6.6. $\square$

Proof of Proposition 6.4. If W is hyperbolic, this is proved in [Edg09]. We sketch an argument for that case since it provides the starting point for the proof in general. Since W is hyperbolic, from §4 we get that $\widehat{Q}$ is the boundary of an ellipsoid inside $\mathrm{conv}(\widehat{\Delta})$ and $E_{\mathrm{ext}} = \overline{E_{\mathrm{ext}}} = \widehat{Q}$ (see Corollary 4.15 and Theorem 4.4). Choose a subset $\{e_1, \ldots, e_n\}$ of $\widehat{Q}$ which is affinely independent and which has affine span equal to $\mathrm{aff}(\widehat{\Delta})$. By the Cauchy-Schwarz inequality, one has $B(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_i + \frac{1}{2}e_j) < 0$ for $i \neq j$. Choose positive roots ρ_i such that $\widehat{\rho_i}$ is sufficiently close to e_i that $\widehat{\rho}_1, \ldots, \widehat{\rho}_n$ are affinely independent and $B(\frac{1}{2}\widehat{\rho}_i + \frac{1}{2}\widehat{\rho}_j, \frac{1}{2}\widehat{\rho}_i + \frac{1}{2}\widehat{\rho}_j) < 0$ for $i \neq j$. This implies that $\{\rho_1, \ldots, \rho_n\}$ spans $\mathrm{span}(\Delta)$, and a quick calculation shows that for $i \neq j$, $B(\rho_i, \rho_j) < -1$, as required in this case.

Next we prove the result in the case Δ is linearly independent, by induction on $|\Delta|$. If Φ is of rank two, the required conditions are satisfied by taking $\Psi = \Delta$. Assume henceforward Φ is of rank three or greater and is not hyperbolic. We claim that Φ contains a proper irreducible standard parabolic subsystem of indefinite type. If $B(\alpha, \beta) < -1$ for some $\alpha, \beta \in \Delta$ then $\Phi_{\langle s_\alpha, s_\beta \rangle}$ is such a subsystem, so we may assume $B(\alpha, \beta) \geq -1$ for all $\alpha, \beta \in \Delta$. Then the root system Φ is the standard root system associated to W in [Hum90, Chapter 5]. Since W is not hyperbolic, by [Hum90, §6.8], there is a maximal proper standard parabolic subsystem of Φ which is not of "positive type" i.e. which has a component of indefinite type. This proves the claim.

By the claim and irreducibility of Φ, we may choose $\alpha \in \Delta$ such that $\Delta \setminus \{\alpha\}$ is irreducible and contains the simple roots of an irreducible standard parabolic subsystem of Φ of indefinite type. Hence $\Delta \setminus \{\alpha\}$ is itself the set of simple roots of an irreducible standard parabolic subsystem Ψ of Φ of indefinite type. By induction, there are roots $\alpha_1, \ldots, \alpha_{n-1} \in \Psi$ such that $\mathrm{span}(\alpha_1, \ldots, \alpha_{n-1}) = \mathrm{span}(\Delta \setminus \{\alpha\})$ and $B(\alpha_i, \alpha_j) \leq -1$ for all i, j with $1 \leq i < j \leq n - 1$. Since $\alpha_i \in \mathrm{cone}(\Delta \setminus \{\alpha\})$, one has $B(\alpha, \alpha_i) \leq 0$ for all $i = 1, \ldots, n - 1$. The inequality must be strict for some i by irreducibility of Δ. Hence $\Psi := \{\alpha_1, \ldots, \alpha_{n-1}, \alpha\}$ satisfies the conditions of Lemma 6.6. Then any subset Ψ' of Φ^+ satisfying the conditions in the statement of Lemma 6.6 meets the requirements on Ψ' here. This completes the proof in the special case in which Δ is linearly independent. Finally, the case in which Δ is not linearly independent immediately reduces to that in which Δ is linearly dependent by use of [Dye12, §1.4]. $\square$

6.4. **Intersection of lines with the isotropic cone.** The proofs of Lemmas 6.14 and 6.21) require a slight extension of a computation in [HLR14, §4.2] that describes the intersection points of a line cutting Q. We give the details here for ease of reference. In these results, (Φ, Δ) can be an arbitrary based root system.

Let $u, v \in V$. If u, v are distinct, the line $L(u, v)$ passing through u, v consists of all points $(1 - t)u + tv$ for $t \in \mathbb{R}$, whereas if $u = v$, $(1 - t)u + tv = u$ for all $t \in \mathbb{R}$. In any case, for $t \in \mathbb{R}$, the point $(1 - t)u + tv \in Q \cap L(u, v)$ if and only if

(6.5) $\ B(u + t(v - u), u + t(v - u)) = t^2 B(v - u, v - u) + 2t B(u, v - u) + B(u, u) = 0.$

The above equation, regarded as an equation for $t \in \mathbb{R}$, has exactly two distinct solutions if and only if

(6.6) $\qquad B(v - u, v - u) \neq 0 \text{ and } B(u, v)^2 - B(u, u)B(v, v) > 0.$

In that case, the solutions are

$$t = \frac{B(u - v, u) \pm \sqrt{B(v, u)^2 - B(u, u)B(v, v)}}{B(v - u, v - u)}$$

which we shall denote as $t = t_{\min}(u, v)$, $t = t_{\max}(u, v)$ where $t_{\min}(u, v) < t_{\max}(u, v)$. So

$$L(u, v) \cap Q = \{(1 - t_{\min}(u, v))u + t_{\min}(u, v)v, (1 - t_{\max}(u, v))u + t_{\max}(u, v)v\}.$$

Observe that by the symmetry between u and v, one has

(6.7) $\qquad\qquad\qquad t_{\max}(u, v) + t_{\min}(v, u) = 1.$

So by setting

$$u_Q(u, v) := (1 - t_{\min}(u, v))u + t_{\min}(u, v)v$$

one obtains $u_Q(v, u) = (1 - t_{\max}(u, v))u + t_{\max}(u, v)v$. Therefore for pairs (u, v) verifying (6.6) we have

$$L(u, v) \cap Q = \{u_Q(u, v), u_Q(v, u)\},$$

and $u_Q(u, v) < u_Q(v, u)$ for the order induced by the oriented line $\overrightarrow{(uv)}$. In other words, $u_Q(u, v)$ is the point of intersection of $L(u, v)$ with Q that is seen from u when looking at Q; whereas $u_Q(v, u)$ is the point of intersection of $L(u, v)$ with Q that is seen from v when looking at Q. For example, in Figure 7, we have $x = u_Q(\widehat{\alpha}, \widehat{\beta}) = x$ and $y = u_Q(\widehat{\beta}, \widehat{\alpha})$.

Let U_Q be the open set of all pairs (u, v) such that $L(u, v) \cap Q$ consists of two distinct points, i.e., such that (u, v) satisfies (6.6). The functions $t_{\min}, t_{\max} : U_Q \to \mathbb{R}$ are continuous. Therefore the function $u_Q : U_Q \to V$ given by $(u, v) \mapsto u_Q(u, v)$ is also continuous.

The following lemma will be needed in the proof of Lemma 6.21; it is however natural to state and prove it here.

Lemma 6.7. *Suppose that $u, v \in Q$ and $B(u, v) < 0$. Then there exist an open neighbourhood Ω_u of u and an open neighbourhood Ω_v of v with the following properties:*

(i) $\Omega_u \times \Omega_v \subseteq U_Q$.

(ii) *If $x \in \Omega_u$ and $y \in \Omega_v$ with $B(x, x) > 0$ and $B(y, y) > 0$, then $0 < t_{\min}(x, y) < t_{\max}(x, y) < 1$. That is, x, $u_Q(x, y)$, $u_Q(y, x)$, y are distinct and aligned in this order for the order induced by the oriented line $\overrightarrow{(xy)}$.*
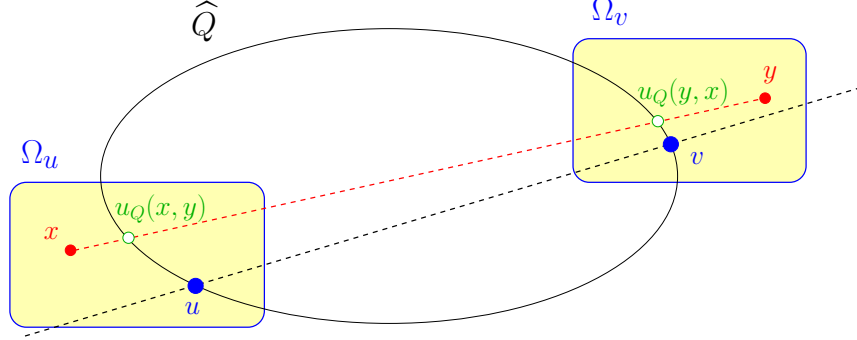
Figure 12 illustrates this lemma.

FIGURE 12. Illustration of Lemma 6.7: for $u, v \in Q$, there are neighborhoods Ω_u and Ω_v of u and v such that for any x, y in the "outer side" of Q, if $x \in \Omega_u$ and $y \in \Omega_v$, then $x, u_Q(x, y), u_Q(y, x), y$ are distinct and aligned in this order.

Proof. From the definitions, $(u, v) \in U_Q$, and U_Q is open. Therefore there exist open neighbourhoods Ω_u of u and Ω_v of v in V such that $\Omega_u \times \Omega_v \subseteq U_Q$ and $B(x, y) < 0$ for $(x, y) \in \Omega_u \times \Omega_v$. Observe that from Equation (6.5) one has, for all $(x, y) \in U_Q$:

$$(6.8) \qquad t_{\max}(x, y) + t_{\min}(x, y) = \frac{2B(x, x - y)}{B(y - x, y - x)}$$

$$(6.9) \qquad t_{\max}(x, y) t_{\min}(x, y) = \frac{B(x, x)}{B(y - x, y - x)}.$$

If $(x, y) \in \Omega_u \times \Omega_v$ are such that $B(x, x) > 0$ and $B(y, y) > 0$, then $B(y - x, y - x) > 0$ (since $B(x, y) < 0$) and $t_{\min}(x, y)$ and $t_{\max}(x, y)$ have positive sum and positive product, by (6.8) and (6.9). Hence $t_{\min}(x, y)$ and $t_{\max}(x, y)$ are both positive. Similarly, $t_{\min}(y, x)$ and $t_{\max}(y, x)$ are also positive. Using (6.7), we get that they all lie between 0 and 1 exclusive. This implies that $0 < t_{\min}(x, y) < t_{\max}(x, y) < 1$. The rest of (ii) follows directly. $\qquad \square$

Although it is not needed in the proof of the main results of this section, we note also the following consequence of Lemma 6.7.

Proposition 6.8.

(a) *Let $(\rho_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ be sequences of positive roots. Suppose that for each $n \in \mathbb{N}$, $\rho_n \prec \tau_n$. Let $u \in E$ (resp., $v \in E$) be a limit point of $(\widehat{\rho}_n)_{n \in \mathbb{N}}$ (resp., $(\widehat{\tau}_n)_{n \in \mathbb{N}}$). Then $B(u, v) = 0$.*

(b) *Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of positive roots which is strictly increasing in dominance order: for each $n \in \mathbb{N}$, $\rho_n \prec \rho_{n+1}$. Then the limit points of $(\widehat{\rho}_n)$ are pairwise orthogonal (and isotropic).*

Proof. For the proof of (a), one may assume, by passing to subsequences of (ρ_n), (τ_n) if necessary, that $\widehat{\rho}_n \to u$ and $\widehat{\tau}_n \to v$ as $n \to \infty$. One has $u, v \in E$ by definition of E. Hence $u, v \in Q$ since $E \subseteq \widehat{Q}$. If $u = v$ then $B(u, v) = 0$ so assume $u \neq v$. From (2.2), it follows that $B(u, v) \leq 0$. Suppose for a contradiction that $B(u, v) < 0$, and choose Ω_u and Ω_v as in Lemma 6.7. Choose m sufficiently

large that $\widehat{\rho}_n \in \Omega_u$ and $\widehat{\tau}_n \in \Omega_v$ for all $n \geq m$. Then Proposition 5.7 says that $[\widehat{\rho}_m, \widehat{\tau}_m] \cap Q = \varnothing$ while Lemma 6.7(ii) says that $[\widehat{\rho}_m, \widehat{\tau}_m] \cap Q$ consists of two points. This contradiction completes the proof of (a). Part (b) follows by taking $\tau_n := \rho_{n+1}$ in (a). $\qquad\square$

6.5. **Decomposition of** $\mathrm{conv}(\widehat{\Delta})$ **in the case of generic universal based root systems.** The function u_Q gives, in the case of a generic universal based root system, a very nice decomposition of the polytope $\mathrm{conv}(\widehat{\Delta})$. For instance, in the right-hand side picture in Figure 2, it looks like $\mathrm{conv}(\widehat{\Delta})$ is the union of Z with the union of the triangle $\mathrm{conv}(\{\gamma, u_Q(\gamma, \beta), u_Q(\gamma, \alpha)\})$, of the triangle $\mathrm{conv}(\{\alpha, u_Q(\alpha, \beta), u_Q(\alpha, \gamma)\})$ and of the triangle $\mathrm{conv}(\{\beta, u_Q(\beta, \gamma), u_Q(\beta, \alpha)\})$. The same kind of geometric intuition holds in the case of generic universal root systems that are weakly hyperbolic, since in this case the transverse hyperplane can be chosen for $\widehat{Q}$ to be a sphere (Proposition 4.13). However in the general case it is not that obvious, see for instance Figure 5. Still, this phenomenon is true in general and was observed first in the framework of the imaginary cone in [Dye12, §9.12].

Suppose that (Φ, Δ) is a generic universal based root system. Then for any distinct $\alpha, \beta \in \Delta$, we have $(\widehat{\alpha}, \widehat{\beta}) \in U_Q$, so $u_Q(\widehat{\alpha}, \widehat{\beta}) \in \mathrm{conv}(\{\widehat{\alpha}, \widehat{\beta}\})$ is defined. It is remarked after the statement of Theorem 6.2 that, quite generally, for distinct $\alpha, \beta \in \Phi^+$ with $B(\alpha, \beta) < -1$, the line joining $\widehat{\alpha}$ to $\widehat{\beta}$ cuts $\widehat{Q}$ in the two limit roots $u_Q(\widehat{\alpha}, \widehat{\beta}) \neq u_Q(\widehat{\beta}, \widehat{\alpha})$, which are in the open line segment joining $\widehat{\alpha}$ and $\widehat{\beta}$. For $\alpha \in \Delta$, let

$$D_\alpha := \mathrm{conv}\left(\{\widehat{\alpha}\} \cup \{u_Q(\widehat{\alpha}, \widehat{\beta}) \mid \beta \in \Delta \setminus \{\alpha\}\}\right) \subseteq \mathrm{conv}(\widehat{\Delta}).$$

It is clear that D_α is a polytope that spans the same affine space, and therefore is of the same dimension as $\mathrm{conv}(\widehat{\Delta})$.

Proposition 6.9. *Suppose that (Φ, Δ) is a generic universal based root system. The polytope $\mathrm{conv}(\widehat{\Delta})$ is the union of the imaginary convex set Z together with the polytopes D_α, $\alpha \in \Delta$:*

$$\mathrm{conv}(\widehat{\Delta}) = Z \cup \bigcup_{\alpha \in \Delta} D_\alpha.$$

Moreover $Z \cap Q = \varnothing$ and $D_\alpha \cap D_\beta = \varnothing$ for distinct $\alpha, \beta \in \Delta$.

This proposition is a direct reformulation in affine terms of (parts of) [Dye12, Lemma 9.11 and Lemma 9.12]. We give a proof here for convenience.

Proof. Assume that Δ is linearly independent. The general case is dealt with by choosing a lift up as in [HLR14, §5.3], see the details in [Dye12, Proof of Lemma 9.12 (d)]. Let P be the polytope $P := \mathrm{conv}\{u_Q(\widehat{\alpha}, \widehat{\beta}) \mid \alpha \neq \beta \in \Delta\}$. Since Δ is linearly independent and the root system is universal, the set $\{u_Q(\widehat{\alpha}, \widehat{\beta}) \mid \beta \in \Delta \setminus \{\alpha\}\}$ is affinely independent of cardinality $|\Delta| - 1$ for any $\alpha \in \Delta$. Therefore, for any $\alpha \in \Delta$, the polytope $\mathrm{conv}(\{u_Q(\widehat{\alpha}, \widehat{\beta}) \mid \beta \in \Delta \setminus \{\alpha\}\})$ is a facet of both P and D_α; the other facets of P and D_α being contained in the facets of $\mathrm{conv}(\widehat{\Delta})$. In particular, it is not difficult to see that

$$\mathrm{conv}(\widehat{\Delta}) = P \cup \bigcup_{\alpha \in \Delta} D_\alpha.$$

Since $u_Q(\widehat{\alpha}, \widehat{\beta})$ is a limit root, we have that $P \subseteq \mathrm{conv}(E) = \overline{Z}$, by Theorem 2.2. Therefore $\mathrm{relint}(P) \subseteq \mathrm{relint}(\overline{Z}) = \mathrm{relint}(Z) \subseteq Z$, where $\mathrm{relint}(X)$ denotes the relative interior of X (see for instance [Dye12, Appendix A and Eq. A.2.2]).

To show that $\mathrm{conv}(\widehat{\Delta}) = Z \cup \bigcup_{\alpha \in \Delta} D_\alpha$, we show by induction on the dimension of $\mathrm{conv}(\widehat{\Delta})$ that $P \subseteq Z \cup \bigcup_{\alpha \in \Delta} D_\alpha$.

If the dimension $\mathrm{conv}(\widehat{\Delta})$ is 2 then $\Delta = \{\alpha, \beta\}$ and $\mathrm{conv}(\widehat{\Delta}) = [\widehat{\alpha}, \widehat{\beta}]$. Moreover

$$D_\alpha = [\widehat{\alpha}, u_Q(\widehat{\alpha}, \widehat{\beta})], \ D_\beta = [\widehat{\beta}, u_Q(\widehat{\beta}, \widehat{\alpha})] \text{ and } P = [u_Q(\widehat{\alpha}, \widehat{\beta}), u_Q(\widehat{\beta}, \widehat{\alpha})].$$

But $Z =]u_Q(\widehat{\alpha}, \widehat{\beta}), u_Q(\widehat{\beta}, \widehat{\alpha})[$ which concludes the dimension 2 case. Assume now that the dimension of $\mathrm{conv}(\widehat{\Delta})$ is strictly greater than 2. Let $z \in P$, we may assume $z \in P \setminus \mathrm{relint}(P)$, since $\mathrm{relint}(P) \subseteq Z$. So z is in a facet P' of P. We know from the discussion above that either P' is also a facet of D_α, for $\alpha \in \Delta$, and $z \in D_\alpha$; or P' is contained in a facet $\mathrm{conv}(\widehat{\Delta} \setminus \{\widehat{\gamma}\})$, $\gamma \in \Delta$, of $\mathrm{conv}(\widehat{\Delta})$. In this last case, it is easy to see that, for any $\alpha \in \Delta' = \Delta \setminus \{\gamma\}$, the set $D_\alpha \cap P'$ is the equivalent object to D_α for the based root subsystem associated to Δ'. Moreover, $Z \cap \mathrm{conv}(\widehat{\Delta'})$ is the imaginary convex set for for the root subsystem associated to Δ', as noted at the beginning of §5 (see also [Dye12, Lemma 3.4]) and

$$P' = \mathrm{conv}\{u_Q(\widehat{\alpha}, \widehat{\beta}) \,|\, \alpha \neq \beta \in \Delta'\}.$$

So by induction we have that

$$z \in P' \subseteq \big(Z \cap \mathrm{conv}(\widehat{\Delta'})\big) \bigcup_{\alpha \in \Delta'} (D_\alpha \cap \mathrm{conv}(\widehat{\Delta'})) \subseteq Z \cup \bigcup_{\alpha \in \Delta} D_\alpha.$$

This concludes the first part of the proof.

Now let us prove the "Moreover" part of the proposition. If $z \in \widehat{Q} \cap Z$ then there is $w \in W$ such that $w \cdot z \in K \cap \widehat{Q}$. Since $w \cdot z \in K \subseteq \mathrm{conv}(\widehat{\Delta}) \subseteq \mathrm{cone}(\Delta)$, we can write $w \cdot z = \sum_{\alpha \in \Delta} a_\alpha \alpha$ with $a_\alpha \geq 0$. Let $\Delta' = \{\alpha \in \Delta \,|\, a_\alpha > 0\}$. This set is non-empty, since $w \cdot z \neq 0$, and $|\Delta'| \geq 2$ since $B(w \cdot z, w \cdot z) = 0$ and $B(\alpha, \alpha) > 0$ for any root α. Since $w \cdot z \in K \cap Q$, we have

$$0 = B(w \cdot z, w \cdot z) = \sum_{\alpha \in \Delta'} a_\alpha B(\alpha, w \cdot z) \leq 0.$$

This forces $B(\alpha, w \cdot z) = 0$, for all $\alpha \in \Delta'$, since $a_\alpha \neq 0$. Hence $w \cdot z$ is in the radical of B restricted to $\mathrm{span}(\Delta')$. Since $w \cdot z = \sum_{\alpha \in \Delta'} a_\alpha \alpha$ with all $a_\alpha \neq 0$ and Δ' is connected, this implies Δ' is the simple system of an irreducible affine standard parabolic subsystem of Φ (see for instance [Dye12, §4.5]). But since (Φ, Δ) is a *generic* universal based root system, there are no such subsystems, a contradiction which completes the proof that $Q \cap Z = \varnothing$. The last parts of the proposition hold since for $\alpha \neq \beta$ in Δ we have $B(\widehat{\alpha}, \widehat{\alpha}) > 0$, $B(\widehat{\alpha}, u_Q(\widehat{\alpha}, \widehat{\beta})) > 0$ and $B(\widehat{\beta}, u_Q(\widehat{\alpha}, \widehat{\beta})) < 0$, whereas for $\gamma \neq \alpha$ in Δ we have $B(\widehat{\gamma}, \alpha) < 0$ and $B(\widehat{\gamma}, u_Q(\widehat{\alpha}, \widehat{\beta})) < 0$. $\square$

Remark 6.10.

(1) This decomposition still holds in the case of universal based root systems that are not necessarily generic, see for an example the right-hand side picture in Figure 4. But in this case, Z may meet Q and the D_α's may not be disjoint. For instance Z meets Q in limit points arising from facial affine dihedral based root subsystems, i.e., when $B(\alpha, \beta) = -1$.

(2) Let $Q^+ = \{v \in V \mid B(v,v) \geq 0\}$. If the based root system is generic universal, we have necessarily $Q^+ \cap \operatorname{conv}(\widehat{\Delta}) \subseteq \bigcup_{\alpha \in \Delta} D_\alpha$, since $Z \subseteq Q^-$ and $Z \cap Q = \varnothing$, and the union is disjoint.

As a direct consequence of the last statement in the remark above is the following lemma that will be required for the proof of Theorem 6.2.

Lemma 6.11. *Suppose that (Φ, Δ) is a generic universal based root system. Then $\widehat{\Phi}^+ \cup (\widehat{Q} \cap \operatorname{conv}(\widehat{\Delta})) \subseteq \bigcup_{\alpha \in \Delta} D_\alpha$, and the union is disjoint.*

6.6. **Faithfulness of the W-action.** For the remainder of Section §6, we assume unless otherwise stated that (Φ, Δ) is an irreducible based root system of indefinite type and rank at least three. We aim to prove next that the W-action on E is faithful. Here is roughly the idea of the proof. Assume $w \in W$ is such that for all $x \in E$, $w \cdot x = x$. By the definition of the W-action, this means that any x in E is an eigenvector for w with positive eigenvalue (for the linear action of w in V). Hence, E is contained in the union of the eigenspaces of w. It would then be easy to conclude provided we prove the following fact:
(6.10)
E is not contained in any finite union of proper linear subspaces of $\operatorname{span}(\Delta)$.

It is clear that E cannot be contained in *one* proper linear subspace of $\operatorname{span}(\Delta)$, since we know that $\operatorname{span}(E) = \operatorname{span}(\Delta)$ (see Proposition 2.5). To prove (6.10), we need to show that even after removing a proper subspace, there are still enough points in E to span $\operatorname{span}(\Delta)$, and we must be able to repeat this process indefinitely. It would be sufficient to prove that for any x in E, there exists an open neighbourhood U of x in V such that $U \cap E$ is enough to span $\operatorname{span}(E)$. This is what is stated in the theorem below, together with the logical consequences.

Theorem 6.12. *Let U be an open subset of V with $U \cap E(\Phi) \neq \varnothing$. Then*
   (a) $\operatorname{aff}(U \cap E(\Phi)) = \operatorname{aff}(E(\Phi))$.
   (b) $U \cap E(\Phi)$ *is not contained in the union of any countable collection of proper affine subspaces of* $\operatorname{aff}(E(\Phi))$.
   (c) If for all $x \in U \cap E(\Phi)$, *we have* $w \cdot x = x$, *then* $w = 1$.

Remark 6.13. Part (c) is a stronger version of the faithfulness of the W-action on E, i.e., implies Theorem 6.1. Together with part (b), it has several consequences on W-orbits in $\overline{Z}$ and on the cardinality of E and E_{ext}, statements that we postpone to §6.7–6.9. From (a) we already have that E is perfect (i.e. contains no isolated points), so in particular it is infinite.

In the proof of Theorem 6.12, part (b) will be naturally deduced from (a) and will imply quite easily part (c) (as explained before the theorem). The difficult part is to prove (a). The idea is to exhibit enough points in $U \cap E$ to affinely generate $\operatorname{aff}(E)$. This will be a consequence of the following fact.

Lemma 6.14. *There exists a finite subset P of E which is not contained in the union of any two proper affine subspaces of $\operatorname{aff}(E)$.*

We give below a proof for Lemma 6.14, and deduce afterwards the proof of Theorem 6.12.

Proof. Note that for any roots $\alpha, \beta \in \Phi^+$ with $B(\alpha, \beta) < -1$, one has $(\widehat{\alpha}, \widehat{\beta}) \in U_Q$ and so $u_Q(\widehat{\alpha}, \widehat{\beta})$ and $u_Q(\widehat{\beta}, \widehat{\alpha})$ are defined and distinct. Using Proposition 6.4, one

can find $\alpha_1, \ldots, \alpha_n \in \Phi^+$ with $B(\alpha_i, \alpha_j) < -1$ if $1 \le i < j \le n$, and which form a basis for $\mathrm{span}(\Delta)$ (by taking a subset of Ψ if necessary). The set $A = \{\, \widehat{\alpha}_i \mid i = 1, \ldots, n \,\}$ is affinely independent and $\mathrm{aff}(A) = \mathrm{aff}(\widehat{\Delta})$. Set $\alpha_{i,j} = u_Q(\widehat{\alpha}_i, \widehat{\alpha}_j) \in E$. Then

(6.11) $$P := \{\, \alpha_{i,j} \mid 1 \le i \le n, 1 \le j \le n, i \ne j \,\}$$

is a set of $n(n-1)$ points in the simplex $A' := \mathrm{conv}(A)$ with A as vertex set. Note that A does not contain any vertex of A' and contains exactly two points on each of the $\binom{n}{2}$ edges of A'. In particular, $\mathrm{aff}(\{\alpha_{i,j}, \alpha_{j,i}\}) = \mathrm{aff}(\{\widehat{\alpha_i}, \widehat{\alpha_j}\})$ for $i \ne j$.

Clearly, $\mathrm{aff}(P) = \mathrm{aff}(A) = \mathrm{aff}(\widehat{\Delta})$. Denote $V' := \mathrm{aff}(\widehat{\Delta})$. It suffices to show that if H is any affine hyperplane in V', then $\mathrm{aff}(P \setminus H) = \mathrm{aff}(P)$. This follows by a simple general argument as follows. One has $\widehat{\alpha}_i \notin H$ for some i, say for $i = 1$ by reindexing. Then for each $j = 2, \ldots, n$, H cannot contain both $\alpha_{1,j}$ and $\alpha_{j,1}$ since their affine span contains $\widehat{\alpha}_1$. We now consider two cases as follows. Assume first that for some k with $2 \le k \le n$, $P \setminus H$ contains both $\alpha_{1,k}$ and $\alpha_{k,1}$. Then $\mathrm{aff}(P \setminus H)$ contains $\widehat{\alpha}_1$. For each $j = 2, \ldots, n$, $P \setminus H$ contains either $\alpha_{1,j}$ or $\alpha_{j,1}$, so $\mathrm{aff}(P \setminus H)$ contains $\widehat{\alpha}_j$. Thus, $\mathrm{aff}(P \setminus H)$ contains A and so is equal to V' as required in this case. In the other case, for each k with $2 \le k \le n$, H contains exactly one of $\alpha_{1,k}$ and $\alpha_{k,1}$. Then H strictly separates $\widehat{\alpha}_1$ from the other vertices $\widehat{\alpha}_2, \ldots, \widehat{\alpha}_n$ of A', and so $\alpha_{j,k} \in P \setminus H$ for all distinct j, k in $\{2, \ldots, n\}$. Obviously $\mathrm{aff}(P \setminus H) \supseteq \mathrm{aff}(\{\widehat{\alpha}_2, \ldots, \widehat{\alpha}_n\})$. Since also either $\alpha_{1,2}$ or $\alpha_{2,1}$ is in $P \setminus H$, we therefore get $\widehat{\alpha}_1 \in \mathrm{aff}(P \setminus H)$ as well and so $\mathrm{aff}(P \setminus H) = \mathrm{aff}(A)$ as required in this case too. $\qquad\square$

Proof of Theorem 6.12. (a) Choose $x \in U \cap E$. By Proposition 3.9, $x^\perp \cap \overline{Z}$ is a proper face of $\overline{Z} = \mathrm{conv}(E)$ and in particular, $x^\perp \cap \mathrm{aff}(E)$ is a proper affine subspace of $\mathrm{aff}(E)$. By Lemma 6.14, there is a finite subset P of $E \setminus x^\perp$ with $\mathrm{aff}(P) = \mathrm{aff}(E)$. By Theorem 3.11, there exists $w \in W$ such that $w \cdot P \subseteq E \cap U$. Since the geometric action of W on V is by invertible linear maps, we have $\dim(\mathrm{span}(w(P))) = \dim(\mathrm{span}(P)) = \dim(\mathrm{span}(E))$. But $\mathrm{span}(w(P)) = \mathrm{span}(w \cdot P) \subseteq \mathrm{span}(E)$, so we get $\mathrm{span}(w \cdot P) = \mathrm{span}(E)$. Hence $\mathrm{aff}(E) = \mathrm{aff}(w \cdot P) \subseteq \mathrm{aff}(E \cap U)$, which completes the proof of (a).

(b) It will suffice to show that if $(H_n)_{n \in \mathbb{N}}$ is a family of proper affine subspaces of $\mathrm{aff}(E)$, then there is a point $x \in (U \cap E) \setminus (\cup_{n \in \mathbb{N}} H_n)$. By (a), there is a point $x_1 \in (U \cap E) \setminus H_1$. Choose an open neighbourhood U_1 of x_1 in V with compact closure $\overline{U_1} \subseteq U \setminus H_1$. Since $x_1 \in E \cap U_1$, there exists by (a) a point $x_2 \in (U_1 \cap E) \setminus H_2$. Choose an open neighbourhood U_2 of x_2 with compact closure $\overline{U_2} \subseteq U_1 \setminus H_2$. Continuing to use (a) in this way, choose for each $n \in \mathbb{N}_{\ge 3}$ a point $x_n \in (E \cap U_{n-1}) \setminus H_n$ and an open neighbourhood U_n of x_n in V with compact closure $\overline{U_n} \subseteq U_{n-1} \setminus H_n$. Since E is compact, the sequence $(x_n)_{n \in \mathbb{N}}$ in E has a limit point $x \in E$. Let $n \in \mathbb{N}_{\ge 1}$. Since $x_m \in U_m \subseteq U_n$ for all $m \ge n$, it follows that $x \in \overline{U_n}$. Since $\overline{U_n} \cap H_n = \varnothing$, $x \notin H_n$. Since $x \in \overline{U_1} \subseteq U$, this completes the proof of (b).

(c) Assume that $w \in W$ fixes $E \cap U$ pointwise. Then for each $x \in E \cap U$, $wx = \lambda_x x$ for some $\lambda_x \in \mathbb{R}_{>0}$. For each $\lambda \in \mathbb{R}$, let V_λ be the λ-eigenspace of w on V. The above shows that $E \cap U \subseteq \cup_{\lambda \in \mathbb{R}}(V_\lambda \cap \mathrm{aff}(E))$. Only finitely many affine subspaces $V_\lambda \cap \mathrm{aff}(E)$ of $\mathrm{aff}(E)$ are non-empty, so by (b) we must have $\mathrm{aff}(E) \subseteq V_\lambda$ for some $\lambda \in \mathbb{R}$. Hence $\Phi \subseteq \mathrm{span}(E) \subseteq V_\lambda$. It follows that w is an homothety on

V. Since W is a reflection group, we have $\det(w) = \pm 1$, so $w = \pm 1$. But it is well known that for an infinite Coxeter group W, -1 cannot be an element of W (for example by using the characterization of the length function in terms of roots, see [Hum90, §5.6]). Therefore $w = 1$. $\square$

6.7. **Accumulation points.** The fact that E is perfect enables us to prove the following results on accumulation points of W-orbits on $\overline{Z}$.

Corollary 6.15.

(a) *If $z \in E$, then $\mathrm{Acc}(W \cdot z) = E$.*
(b) *If $z \in \overline{Z}$ then $\mathrm{Acc}(W \cdot z) \supseteq E$.*
(c) *If (Φ, Δ) is of hyperbolic (resp., weakly hyperbolic) type, then $\mathrm{Acc}(W \cdot z) = E$ for all $z \in \overline{Z}$ (resp., for all $z \in Z$).*

Remark 6.16.

(a) If (Φ, Δ) is irreducible of affine type, (a)–(c) all fail. If it is hyperbolic dihedral, then for $z \in \overline{Z}$, one has $\mathrm{Acc}(W \cdot z) \supseteq E$ if and only if $\mathrm{Acc}(W \cdot z) = E$.
(b) We do not know if $E = \mathrm{Acc}(W \cdot z)$ for arbitrary irreducible, non-dihedral Φ of indefinite type and all $z \in \overline{Z}$ (or even just all $z \in Z$).

Proof. Let $z \in E$. Then $\overline{W \cdot z} = E$ by Theorem 3.1. Hence $E \backslash W \cdot z = \overline{W \cdot z} \backslash W \cdot z \subseteq \mathrm{Acc}(W \cdot z) \subseteq \overline{W \cdot z} = E$. The equality $\mathrm{Acc}(W \cdot z) = E$ therefore holds since E has no isolated points, as observed in Remark 6.13. This proves (a). By (a), it suffices to prove (b) for $z \in \overline{Z} \setminus E$. Then $W \cdot z \cap E = \varnothing$, so

$$\mathrm{Acc}(W \cdot z) \supseteq \overline{W \cdot z} \setminus W \cdot z \supseteq E$$

by Theorem 3.1. This proves (b). To prove (c), assume Φ is hyperbolic (resp., weakly hyperbolic). By (b), it suffices to prove that if $z \in \overline{Z}$ (resp., $z \in Z$), then $\mathrm{Acc}(W \cdot z) \subseteq E$. Note that if (Φ, Δ) is hyperbolic, then $\overline{Z} \subseteq Z \cup (\overline{Z} \cap \widehat{Q})$. Hence in both the hyperbolic and weakly hyperbolic cases, Theorem 2.8 gives the inclusion $\mathrm{Acc}(W \cdot z) \subseteq \widehat{Q}$. So, using Theorem 4.10, we have $\mathrm{Acc}(W \cdot z) \subseteq \widehat{Q} \cap \overline{Z} = E$. This gives the required equalities. $\square$

6.8. **Cardinality of E_{ext}.** The fact that E is not contained in any countable union of affine proper subspaces of $\mathrm{aff}(E)$ (implied by Theorem 6.12(b)) has the following easy consequence.

Corollary 6.17.

(a) *The imaginary convex body $\overline{Z(\Phi)}$ and the closed imaginary cone $\overline{\mathcal{Z}(\Phi)}$ both have uncountably many faces;*
(b) *$E_{\mathrm{ext}}(\Phi)$ is uncountable.*

Remark 6.18. Part (a) is a consequence of (b) (since extreme points are particular faces), but will be the first step in the proof of (b). In particular $E(\Phi)$ is strictly bigger than its countable (dense) subset E_2. In §6.9 we prove a stronger property, namely, any open neighbourhood of a point in E is uncountable, by constructing a Cantor set inside such a neighbourhood (Corollary 6.19).

Proof. (a) The set $\overline{Z} = \mathrm{conv}(E_{\mathrm{ext}})$ is a convex body living in the affine space $A := \mathrm{aff}(Z)$. We call relative interior $\mathrm{relint}(\overline{Z})$ of $\overline{Z}$ the interior of $\overline{Z}$ for the induced topology on A, and relative boundary of $\overline{Z}$ the set $\mathrm{rb}(\overline{Z}) = \overline{Z} \setminus \mathrm{relint}(\overline{Z})$. It is well known that $\mathrm{rb}(\overline{Z})$ is equal to the union of the proper faces of $\overline{Z}$ (see for example

[Web94, Thm. 2.4.12]). In particular, E_{ext} is contained in $\mathrm{rb}(\overline{Z})$. Moreover, $\mathrm{rb}(\overline{Z})$ is closed, and $E = \overline{E_{\mathrm{ext}}}$ from Theorem 3.4. Thus $E \subseteq \mathrm{rb}(\overline{Z}) = \bigcup_F F \subseteq \bigcup_F \mathrm{aff}(F)$, where F runs over the proper faces of $\overline{Z}$. Using Theorem 6.12(b), this implies that $\overline{Z}$ has uncountably many faces, as well as $\overline{\mathcal{Z}} = \mathrm{cone}(\overline{Z})$.

(b) We first prove that any proper face can be constructed from a finite number of extreme points. Let F be a face of $\overline{Z}$. Thus $F = \overline{Z} \cap \mathrm{aff}(F)$. Denote by X the set of extreme points of F. We have $F = \mathrm{conv}(X)$, so $\mathrm{aff}(X) = \mathrm{aff}(F)$. Since $\mathrm{aff}(F)$ is finite-dimensional, one can choose in X a finite number of points $x_1, \ldots, x_p$ such that $\mathrm{aff}(\{x_1, \ldots, x_p\}) = \mathrm{aff}(F)$. Note that since F is a face of $\overline{Z}$, extreme points of F are also extreme points of $\overline{Z}$, so the points x_i are in E_{ext}. Thus we can associate to any face F of $\overline{Z}$ a finite subset $\{x_1, \ldots, x_p\}$ of E_{ext} such that $F = \overline{Z} \cap \mathrm{aff}(\{x_1, \ldots, x_p\})$. This construction is clearly injective.

Now suppose by contradiction that E_{ext} is countable. Then there are also countably many finite subsets of E_{ext}. From the injection constructed above, this would imply that the set of faces of $\overline{Z}$ is countable, contradicting (a). $\square$

6.9. **A Cantor space inside E.** We know from Corollary 6.17 that E is uncountable. We prove below (as another corollary of Theorem 6.12) that any open neighbourhood of a point in E is also uncountable. In order to do so, we construct, for any open subset U of V such that $U \cap E \neq \varnothing$, a Cantor space living inside $U \cap E$. Recall that a *Cantor space* is a topological space that is homeomorphic to the classical (ternary) Cantor set, or, equivalently, to a product $\prod_{n \in \mathbb{N}}\{0,1\}$ of countably infinitely many copies of a discrete two-point space. A space is a Cantor space if and only if it is non-empty, compact, metrizable, totally disconnected (i.e., it has no non-trivial connected subsets), and perfect, see [Wil70, §30]. A Cantor space has the cardinality of $\mathbb{R}$.

Corollary 6.19. *Let U be an open subset of V with $U \cap E \neq \varnothing$. Then $U \cap E$ contains a subset homeomorphic to the Cantor set. Consequently, $U \cap E$ has the cardinality of $\mathbb{R}$.*

Proof. We give a proof using well-known facts of general topology. Recall that a topological space is said to be *topologically complete* if its topology is induced by a complete metric. Any closed or open subspace of a topologically complete space is topologically complete [Mun75, §7.2, Exercise 6]. It is known that any non-empty, perfect, topologically complete space contains a Cantor space as a subspace [KS06, Proposition 3.2.8]. Now we have already seen that E is perfect (see Remark 6.13). Let U be open in V with $E \cap U \neq \varnothing$. Since E is perfect and U is open, $E \cap U$ is perfect. As V is topologically complete and E is closed (in fact, compact) in V, $E \cap U$ is topologically complete. Hence $E \cap U$ contains a Cantor subspace. $\square$

Remark 6.20. For root systems of universal Coxeter groups with no affine dihedral subgroups, it is known that E itself is a Cantor set in some cases and it is conjectural that it has a Cantor set as quotient space, with topological balls (of unknown dimensions, possibly all 0), as the fibers of the quotient map more generally (see [Dye12, §9]). For general irreducible root systems, Corollary 4.5 gives a description of E as the closure of the union of countably many (not necessarily pairwise disjoint) topological spheres and points (see Figure 2). In particular, E is not necessarily a Cantor set.

6.10. **Proof of Theorem 6.2.** The following technical statement contains the main content of the proof of Theorem 6.2.

Lemma 6.21. *Let $\alpha_1, \ldots, \alpha_n \in \Phi$, $x_1, \ldots, x_n \in E$, and for any $i = 1, \ldots, n$, let U_i be an open neighbourhood of x_i in V. Then*
- (a) *There exist some $y_i \in E \cap U_i$, for $i = 1, \ldots, n$, such that $B(y_i, y_j) < 0$ for all $i \neq j$.*
- (b) *For $i = 1, \ldots, n$, there exists some $\rho_i \in W\alpha_i \cap \Phi^+$ with the following properties:*
  - (i) *$\widehat{\rho}_i \in U_i$ for all i.*
  - (ii) *$(\widehat{\rho}_i, \widehat{\rho}_j) \in U_Q$ and $u_Q(\widehat{\rho}_i, \widehat{\rho}_j) \in U_i$ for all $i \neq j$.*
  - (iii) *$B(\rho_i, \rho_j) < -1$ for $i \neq j$.*
- (c) *If $\mathrm{aff}(\{x_1, \ldots, x_n\}) = \mathrm{aff}(E)$, one may further require $\mathrm{aff}(\{y_1, \ldots, y_n\}) = \mathrm{aff}(E)$ in (b) and $\mathrm{aff}(\{\widehat{\rho}_1, \ldots, \widehat{\rho}_n\}) = \mathrm{aff}(E)$ in (c).*

The lemma is straightforward for $n = 1$, and the case $n = 2$ can be done using the construction of Lemma 6.7; the technical part is to be able to approximate all the x_i's at the same time, preserving the properties we want. We illustrate this construction in Figure 13.
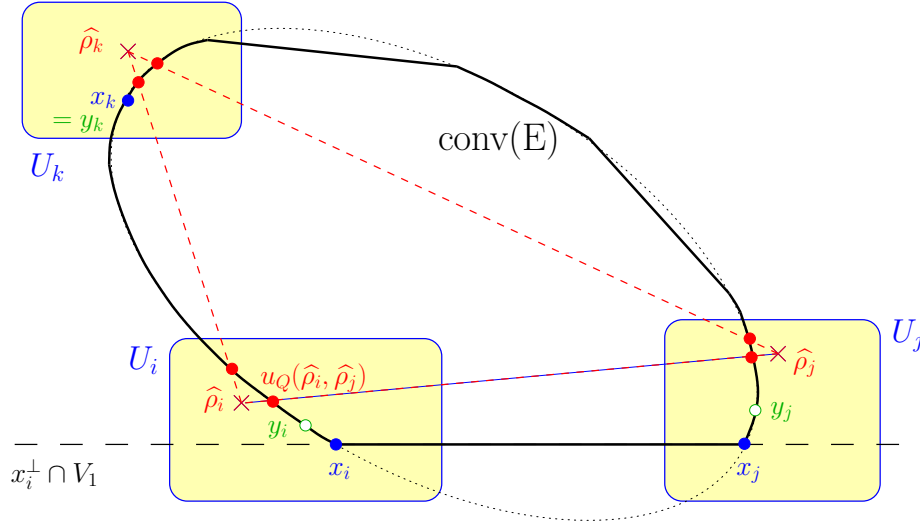

FIGURE 13. Illustration of Lemma 6.21: $\widehat{Q}$ is drawn in black dotted line, and the boundary of $\overline{Z} = \mathrm{conv}(E)$ in full line. Given three limit roots x_i, x_j, x_k and their respective neighborhoods U_i, U_j, U_k, some of the constructions given by Lemma 6.21 are depicted. Here we can choose $y_k = x_k$, but, since $B(x_i, x_j) = 0$, we have to find other $y_i, y_j \in E$ such that $y_i \in U_i$, $y_j \in U_j$ and $B(y_i, y_j) < 0$. The normalized roots $\widehat{\rho}_i$, $\widehat{\rho}_j$ and $\widehat{\rho}_k$ illustrate item (b) of Lemma 6.21.

Proof. The assertions are easily checked if Φ is dihedral, so we assume Φ has rank at least three.

(a) We proceed by induction on n. For $n = 1$, it holds trivially. Assume that $n > 1$ and choose by induction $y_1, \ldots, y_{n-1}$ with $y_i \in U_i$ and $B(y_i, y_j) < -1$ for all $1 \leq i < j \leq n - 1$. We show by induction on m that for $m = 0, 1, \ldots, n - 1$ there is

a point $z_m \in U_n \cap E$ such that $B(z_m, y_i) < 0$ for $i = 1, \ldots, m$. For $m = 0$, one may take $z_0 = x_n$. Assume that $1 \le m \le n - 1$ and z_{m-1} exists with $z_{m-1} \in U_n \cap E$ and $B(z_{m-1}, y_i) < 0$ for $i = 1, \ldots, m - 1$. There is an open neighbourhood $\Omega \subseteq U_n$ of z_{m-1} in V such that $B(v, y_i) < 0$ for all $v \in \Omega$ and all $i = 1, \ldots, m - 1$. By Theorem 6.12(a), $\mathrm{span}(E \cap \Omega) = \mathrm{span}(E)$. Since $y_m \not\perp E$ by Lemma 3.8(a), there is some $z_m \in E \cap \Omega$ with $B(z_m, y_m) \ne 0$. By Equation (2.2), one has $B(z_m, y_m) < 0$. Since $z_m \in \Omega$, $B(z_m, y_i) < 0$ for $i = 1, \ldots, m - 1$. Hence z_m has the required properties, and the induction on m is complete. In particular, z_{n-1} is defined. We set $y_n := z_{n-1} \in E \cap U_n$. Then $y_i \in E \cap U_i$ for $i = 1, \ldots, n$ and $B(y_i, y_j) < 0$ for $1 \le i < j \le n$ as required. This completes the induction on n and hence the proof of (a).

(b) For each $i = 1, \ldots, n$, choose by Theorem 3.1(c) a sequence $(\rho_{i,k})_{k \in \mathbb{N}}$ in $W\alpha_i \cap \Phi^+$ such that $\widehat{\rho_{i,k}} \to y_i$ as $k \to \infty$. It follows immediately from the definitions that for all distinct i, j in $\{1, \ldots, n\}$, one has $(y_i, y_j) \in U_Q$. By Lemma 6.7, by passing to subsequences of the sequences $(\rho_{i,k})_{k \in \mathbb{N}}$ if necessary, we may assume without loss of generality that

  (1) $\widehat{\rho_{i,k}} \in U_i$ for all $i = 1, \ldots, n$ and all $k \in \mathbb{N}$
  (2) $(\widehat{\rho_{i,k}}, \widehat{\rho_{j,\ell}}) \in U_Q$ for all distinct i, j and all $k, \ell \in \mathbb{N}$.
  (3) $u_Q(\widehat{\rho_{i,k}}, \widehat{\rho_{j,\ell}})$ (resp., $u_Q(\widehat{\rho_{j,\ell}}, \widehat{\rho_{i,k}})$) is in the open interval with endpoints $\widehat{\rho_{i,k}}$ and $u_Q(\widehat{\rho_{j,\ell}}, \widehat{\rho_{i,k}})$) (resp., $u_Q(\widehat{\rho_{i,k}}, \widehat{\rho_{j,\ell}})$ and $\widehat{\rho_{j,\ell}}$) for all distinct i, j and all $k, \ell \in \mathbb{N}$.

Condition (3) implies that $B(\rho_{i,k}, \rho_{j,\ell}) < -1$ since the closed interval with the normalized roots $\widehat{\rho_{i,k}}$ and $\widehat{\rho_{j,\ell}}$ as endpoints cuts the isotropic cone in two (distinct) points. As $k \to \infty$, $u_Q(\widehat{\rho_{i,k}}, \widehat{\rho_{j,k}}) \to u_Q(y_i, y_j) = y_i$ for distinct i, j, by continuity of u_Q. Hence we may choose a sufficiently large integer N such that $u_Q(\widehat{\rho_{i,N}}, \widehat{\rho_{j,N}}) \in U_i$ for all distinct $i, j = 1, \ldots, n$. Setting $\rho_i := \rho_{i,N}$ for $i = 1, \ldots, n$, the conditions (i)–(iii) above hold as required. This proves (b).

(c) Choose open neighbourhoods $U_i' \subseteq U_i$ of x_i such that for any $v_i \in U_i' \cap \mathrm{aff}(E)$ for $i = 1, \ldots, n$, one still has $\mathrm{aff}(\{v_1, \ldots, v_n\}) = \mathrm{aff}(E)$. Then one can apply (a)–(b) with U_i replaced by U_i'. $\qquad\square$

Proof of Theorem 6.2. We assume for simplicity that Φ has rank at least three, leaving the indefinite dihedral case to the reader. Fix $\epsilon > 0$ and $m \in \mathbb{N}$. Set $\epsilon' = \epsilon/5$. First we prove (a). Note that F is closed in $\overline{Z}$, hence compact and convex. Therefore F is the convex hull of its extreme points, which are extreme points of $\overline{Z}$ and hence are contained in $E \cap F$. Thus, $F = \mathrm{conv}(E \cap F)$. Since $E \cap F$ is compact, there is a finite subset Y of $E \cap F$ such that $F \cap E \subseteq Y_{\epsilon'}$. Since $Y \subseteq F \cap E \subseteq (F \cap E)_{\epsilon'}$, it follows that $\mathrm{d}(F \cap E, Y) \le \epsilon'$. Since E has no isolated points and the affine span of any non-empty open subset of E coincides with $\mathrm{aff}(E)$, one may choose a subset $X = \{x_1, \ldots, x_n\}$ of E, where $|X| = n \ge \max(m, 2)$, $\mathrm{aff}(X) = \mathrm{aff}(E)$ and $\mathrm{d}(X, Y) < \epsilon'$. Hence $\mathrm{d}(F \cap E, X) \le 2\epsilon'$. Set U_i to be the open ball with center x_i and radius ϵ'. Choose $\rho_i \in \Phi^+$ as in Lemma 6.21(b)–(c) (for any choice of the $\alpha_i \in \Phi$). Define the reflection subgroup $W' := \langle s_{\rho_i} \mid i = 1, \ldots, n \rangle$, and denote by (Φ', Δ') the associated based root system. To simplify notation, write $E'' := \widehat{\Phi'} \cup E'$ for the closure of $\widehat{\Phi'}$. By Lemma 6.21(b)–(c), $\Delta' = \{\rho_1, \ldots, \rho_n\}$ and (a)(i)–(ii) hold. For $i = 1, \ldots, n$, let $D_i' = \{\rho_i\} \cup \{u_Q(\widehat{\rho_i}, \widehat{\rho_j}) \mid j \in \{1, \ldots, n\}, j \ne i\}$ and $D_i := \mathrm{conv}(D_i')$. Also set $D := \bigcup_{j=1}^{n} D_j$. One has $D_i' \subseteq U_i$ by Lemma 6.21(a),(b). Hence $D_i \subseteq U_i$ since U_i is convex. Note that $\widehat{\Delta'} \cap D_i$, $E'' \cap D_i$ and $E' \cap D_i$ are all non-empty. In fact,

the first is equal to $\{(\widehat{\rho}_i)\}$ (using Lemma 6.11), the second contains the first and the last contains $u_Q(\widehat{\rho}_i, \widehat{\rho}_j)$ for all $1 \le j \le n$ with $j \ne i$ (recall $n \ge 2$). This implies $\mathrm{d}(\widehat{\Delta'} \cap D_i, \{x_i\}) \le \epsilon'$, $\mathrm{d}(E'' \cap D_i, \{x_i\}) \le \epsilon'$ and $\mathrm{d}(E' \cap D_i, \{x_i\}) \le \epsilon'$. Note that, $\widehat{\Delta'} \subseteq D$ (trivially) and, by Lemma 6.11, $\widehat{\Phi'} \subseteq D$ and $E' \subseteq \widehat{Q} \cap \mathrm{conv}(\widehat{\Delta'}) \subseteq D$. Hence $E'' = \widehat{\Phi'} \cup E' \subseteq D$ also. It follows from (6.2) that $\mathrm{d}(\widehat{\Delta'}, X) \le \epsilon'$, $\mathrm{d}(E'', X) \le \epsilon'$ and $\mathrm{d}(E', X) \le \epsilon'$. Together with $\mathrm{d}(X, F \cap E) \le 2\epsilon'$ as already established, the triangle inequality now implies (a)(iii) (with ϵ replaced by $4\epsilon' < \epsilon$). Then (a)(iv) follows using (6.1) and $\mathrm{conv}(F \cap E) = F$.

The final claim in (a) (with statement beginning by "Moreover") is proved by taking X sufficiently large (which is possible since E is infinite, see Remark 6.13) and choosing the $\alpha_i \in \Phi$ above so $\{\alpha_1, \ldots, \alpha_n\}$ contains at least one root from each of the specified W-orbits, and no roots from the other W-orbits.

Now we prove (b). Choose X as in the proof of (a). By Corollary 6.15(b), one may choose a finite subset G of $W \cdot Z$, with $|G| \ge m$ and $\mathrm{aff}(G) = \mathrm{aff}(X)$, such that $\mathrm{d}(X, G) < \epsilon'$. Then $\mathrm{d}(G, F \cap E) < 3\epsilon' < \epsilon$ and, using (6.1) again, $\mathrm{d}(\mathrm{conv}(G), F) < \epsilon$. $\qquad\square$

7. Open problems

7.1. **Geometric characteristics of $E(\Phi)$ and $\overline{Z(\Phi)}$.** We already formulated an important open question in §4, about whether the equality $E = \widehat{Q} \cap \mathrm{conv}(E)$ (valid for the weakly hyperbolic case by Theorem 4.10) is true in general irreducible systems. This would provide a nice "fractal" description of E, see Proposition 4.19.

In view of Remark 6.20, a natural problem would be to understand the root systems Φ, Φ' such that $E(\Phi)$ and $E(\Phi')$ are homeomorphic. A more general question is the following: to which extent does the set of limit roots $E(\Phi)$ characterize the root system Φ? It would be interesting to characterize the root systems for which E is connected, or locally connected, or totally disconnected.

Some questions asked in [Dye12, §9.7], on the geometry of the imaginary convex body, also remain unanswered. For example, we do not know whether the equality $E_{\mathrm{exp}} = E_{\mathrm{ext}}$ holds (see §3.3).

We present in this section other avenues of research and open problems that should be investigated. The questions raised above and below are generally of greatest interest for irreducible Φ, even if we do not explicitly make that assumption.

7.2. **Facial structure for subsets of E.** In the same way as many combinatorial properties of a Coxeter group behave well through restriction to parabolic subgroups, the geometric properties of a based root system usually behave as expected through restriction to facial root subsystems. Given a based root system (Φ, Δ) with Coxeter group (W, S), recall that $I \subseteq S$ is said to be facial if $\mathrm{conv}(\widehat{\Delta}_I)$ is a face of $\mathrm{conv}(\widehat{\Delta})$. The root subsystem (Φ_I, Δ_I) is then said to be facial (when Δ is a basis for V, this construction corresponds to standard parabolic subgroups of W; see §4.1 for details). For I facial, denote by F_I the face $\mathrm{conv}(\widehat{\Delta}_I)$. The root system respects the facial structure: for I facial, we have $\Phi_I = \Phi \cap \mathrm{span}(\Delta_I)$, so $\widehat{\Phi}_I = \widehat{\Phi} \cap F_I$ (hence $\widehat{\Phi}_I \cap \widehat{\Phi}_J = \widehat{\Phi}_{I \cap J}$ for I, J facial).

Consider a mapping E_* which associates to any based root system (Φ, Δ) a subset $E_*(\Phi)$ of $E(\Phi)$. We say that E_* is a *functorial subset of E* if for a based root system (Φ, Δ) and a facial root subsystem (Φ_I, Δ_I), $E_*(\Phi_I)$ is contained in $E_*(\Phi)$.

Obviously E itself is a functorial subset of E. In addition, all the subsets constructed in §5 are also functorial: E_f (as well as its $W(\Phi)$-orbit E_2), E_f^{cov} (also its orbit E^{cov}), and E_{elem} (and its orbit). Let us say that a functorial subset E_* of E respects the facial structure if in addition,

$$(7.1) \qquad\qquad E_*(\Phi_I) = E_*(\Phi) \cap F_I, \quad \text{for any } I \text{ facial.}$$

All the six subsets mentioned above have this property, by Theorem 5.20. However, as it was already noted, E does not satisfy (7.1) (see [HLR14, Ex. 5.8]). We ask the general question about how to characterize the functorial subsets of E which respect the facial structure. A first direction to follow would be to explore what happens in the case of the W-orbit $W \cdot x$ of a point x.

7.3. Facial restriction for $E(\Phi)$.
As mentioned above, E does not respect the facial structure as in (7.1). We would still like to understand the relation, for I facial, between $E(\Phi_I)$ and $E(\Phi) \cap F_I$. Let us describe an approach towards understanding this. The counterexample in [HLR14, Ex. 5.8] can be generalized in the following way. Suppose (Φ, Δ) is irreducible, and I is facial such that (Φ_I, Δ_I) is not irreducible. Write $\Phi_I = \Phi_1 \sqcup \Phi_2$, with Φ_1 B-orthogonal to Φ_2 (this corresponds to taking two subsets of S which are not connected in the Coxeter diagram of W). Then we have $E(\Phi_I) = E(\Phi_1) \sqcup E(\Phi_2)$. Calculations suggest the possibility that for any $x \in E(\Phi_1)$, $y \in E(\Phi_2)$, the segment joining x and y is contained in $E(\Phi)$. This would create many counterexamples to the facial restriction formula (7.1) for E, provided $E(\Phi_1)$ and $E(\Phi_2)$ are non-empty. We do not know whether this property is the only obstruction to the facial formula, i.e., whether in this setting $E(\Phi) \cap V_I$ is exactly the join of all the $E(\Phi_{I,k})$ where the $\Phi_{I,k}$ are the irreducible components of Φ_I. If so, this would imply in particular that when Φ_I is irreducible, $E(\Phi_I) = E(\Phi) \cap F_I$.

7.4. Relation with hyperbolic geometry and geometric group theory.
The relations of our setting with relevant topics in hyperbolic geometry or geometric group theory are mainly unexplored, but look fertile. For instance, consider (Φ, Δ) a based root system of rank 3 or 4, and of indefinite type which is weakly hyperbolic. The Coxeter group W acts on E as a group generated by hyperbolic reflections, so can be seen as a Fuchsian or Kleinian group, which explains the shape of Apollonian gasket obtained in the figures (see Remark 4.12). Some of our results are generalizations of known theorems in Kleinian group theory, such as the minimality of the action.

In [HPR13], which was written after a first version of this paper was circulated, the authors explore some of the relations between hyperbolic geometry and our setting. If Φ is weakly hyperbolic, it means that Φ is a root system in the Lorentzian space (V, B), which contains models for the hyperbolic space $\mathbb{H}^n$, where $n + 1 = \dim(V)$. In particular, each root is Lorentzian-normal to a hyperbolic hyperplane, so W turns out to be a discrete subgroup of isometries of $\mathbb{H}^n$ generated by reflections. Moreover, the set of limit roots $E(\Phi)$ is precisely the limit set $\Lambda(W)$ of W seen as a Kleinian group.

Starting from this point, a dictionary between our terminology and the terminology commonly used in hyperbolic geometry can be developed. As an example, we may interpret the *convex core* associated to W as follows, see for instance [Rat06, p.637] for the definition of convex core.

From Proposition 4.13 and the remark that follows, we know that, in the weakly hyperbolic type, the transverse hyperplane can be chosen so that $\widehat{Q}$ is a sphere. Therefore $\mathrm{conv}(\widehat{Q})$ is a W-invariant ball, and its interior $\mathcal{B}^n$ is a W-invariant open ball of dimension n. Recall from §2.2 and Proposition 2.7, that the imaginary convex set $Z(\Phi) = W \cdot K$, i.e., the projective version of the imaginary cone, is the W-orbit of the fundamental convex polytope K, and that the closure of $Z(\Phi)$ is $\overline{Z(\Phi)} = \mathrm{conv}(E(\Phi))$, which is contained in the ball $\mathrm{conv}(\widehat{Q})$. So the *convex core of* $\mathcal{B}^n/W$ is by definition

$$C(\mathcal{B}^n/W) = \big(\mathrm{conv}(E(\Phi)) \cap \mathcal{B}^n\big)/W.$$

We also point out that W is of finite covolume if and only if the fundamental polyhedron for W in $\mathbb{H}^n$ is contained in the conical hull of the simple roots, see [HPR13, §3.5.2]. Our results and framework presented here are valid for all discrete reflection groups generated by reflections in the isometry group of $\mathbb{H}^n$, so in particular for all discrete reflection groups of infinite covolume.

Actually, our framework (limit roots E, imaginary convex body Z, fundamental convex polytope K) and many of the results are valid for any Coxeter group geometrically represented as a subgroup of an orthogonal group $O_B(V)$, where B is a (not necessarily non-degenerate) symmetric bilinear form; in this sense our work could be relevant for the community studying infinite covolume actions of discrete groups in more generality. An interesting approach would be then to try to generalize in our framework other classical properties of limit sets of Kleinian groups relative to the dynamics of the W-action.

7.5. **Dynamics of the projective action of W.** Another natural question concerns the dynamics of the projective action of W on all directions of the vector space V, not only on the roots and the imaginary cone. After a first version of this paper was circulated, H. Chen and J.-P. Labbé gave some answers to this question for W associated to a weakly hyperbolic root system. It turns out that in this case, $E(\Phi)$ is also equal to the set of limit directions arising from the projective action on the *weights* of the root system [CL15], but some directions outside $E(\Phi)$ can occur in limit sets of orbits of another direction [CL14].

7.6. **Ergodic theory for the W-action on E.** It is a classical question in ergodic theory of discrete groups, given a limit set of a group, whether there exists a (unique) invariant measure on this set, and how to construct it (see for example [Nic89, Ch. 3]). Thus a natural problem in our framework is the search for W-invariant measures on E. When W acts on a hyperbolic space (in the context of generalizations of Kleinian groups), these are well-known questions (see [Sul81]).

When the root system is of indefinite type, and not hyperbolic, E can be qualified as "fractal" (see Theorem 3.11, Corollaries 4.5 and 6.19 for some fractal properties). Thus a natural question is to compute the Hausdorff dimension of E. When the root system corresponds to the universal Coxeter group of rank 4, E is the usual Apollonian gasket inscribed in a sphere (see [HLR14, Fig. 9]), and its Hausdorff dimension is about 1.3057 (see [McM98]).

7.7. **Construction of converging sequences, combinatorics and dominance order.** Many questions on the precise way in which the normalized roots converge to E have been left open. For example, the rate of convergence (as a function of the depth of roots) is unknown. Also it would be interesting to describe explicitly for

which sequences of roots the associated normalized roots converge. More precisely, given a sequence of positive roots $(\rho_1 \leq \rho_2 \leq \dots)$ (increasing in the root poset, see §5.1), when does the sequence $(\widehat{\rho_n})_{n \in \mathbb{N}}$ converge? This comes down to study the possible limit points of a sequence $(s_k s_{k-1} \dots s_1(\alpha_0))_{k \in \mathbb{N}}$, where $\alpha_0 \in \Delta$ and $(\dots s_k \dots s_1)$ is a (left-)infinite reduced word of W. The case where the word is periodic is of special interest. When the period is 2, it will provide limit roots in $E_2(\Phi)$. In general, this question requires the precise study of the asymptotics of sequences of the form $(w^n(\alpha))_{n \in \mathbb{N}}$ for $w \in W$, $\alpha \in \Delta$.

Other related questions are as follows. Consider a sequence $(\rho_n)_{n \in \mathbb{N}}$ of positive roots, with $\rho_1 \prec \rho_2 \prec \dots$ i.e. strictly increasing in the dominance order (see §5.2). We do not know if $(\widehat{\rho_n})_n$ has a unique limit root. However, it follows from Proposition 6.8(b) that any two limit roots of a fixed such sequence are orthogonal (compare Proposition 3.7). It can be shown that in general, not every limit root in E is a limit root from a dominance increasing sequence $(\rho_n)_n$, but it is a limit root of some sequence $(\widehat{\tau_n})$ where (τ_n) is related to some dominance increasing (ρ_n) as in Proposition 6.8(a).

This suggests a way to associate subsets of E to ends of dominance order. The result [Dye12, Proposition 7.10(c)] also suggests an approach to attaching isotropic faces of the imaginary cone to ends of weak order. These ideas have been worked out most fully for generic universal root systems (see [Dye12, 9.9–9.16]) but basic questions remain open even in that specially simple case. Clarifying these ideas and their relationships in general would contribute to a better understanding of the relation between the combinatorics of the root system and the distribution of the normalized roots.

As shown in [Dye12], the Coxeter system (W, S), specified by its Coxeter graph with vertex set Δ, together with the set of facial subsets of Δ, suffice to determine the face lattice of the imaginary cone (as lattice with W-action). One might speculate that this information together with the additional data given by the set of "affine edges" $\{\, \{\alpha, \beta\} \mid \alpha, \beta \in \Delta, \ B(\alpha, \beta) = -1 \,\}$ may determine the face lattice of the closed imaginary cone combinatorially. It is not incompatible with what is currently known that the set of limit roots may admit a combinatorial description which determines it as a set (or perhaps even up to homeomorphism) in terms of this data.

7.8. **Generalization to other frameworks.** The concept of root system has many different incarnations, depending on the framework: Coxeter groups, semi-simple Lie algebras, Kac-Moody Lie algebras, extended affine Lie algebras, reductive algebraic groups...; see the many references in the introduction of [LN11], where Loos-Neher developed a general framework in order to clarify all these structures (see also Hée [Hée91] and the recent work of Fu [Fu13b]). In most of these contexts, the limit roots can still be defined, and in some cases, the isotropic cone as well. We expect that a part of the results in [HLR14] and in the present work generalize well to these other settings.

For example, some classes of based root systems appear naturally in the context of quiver representation, where the positive roots can be interpreted as dimension vectors for the indecomposable representations (see [DW05]). The question of an interpretation of the limit roots in this setting is intriguing.
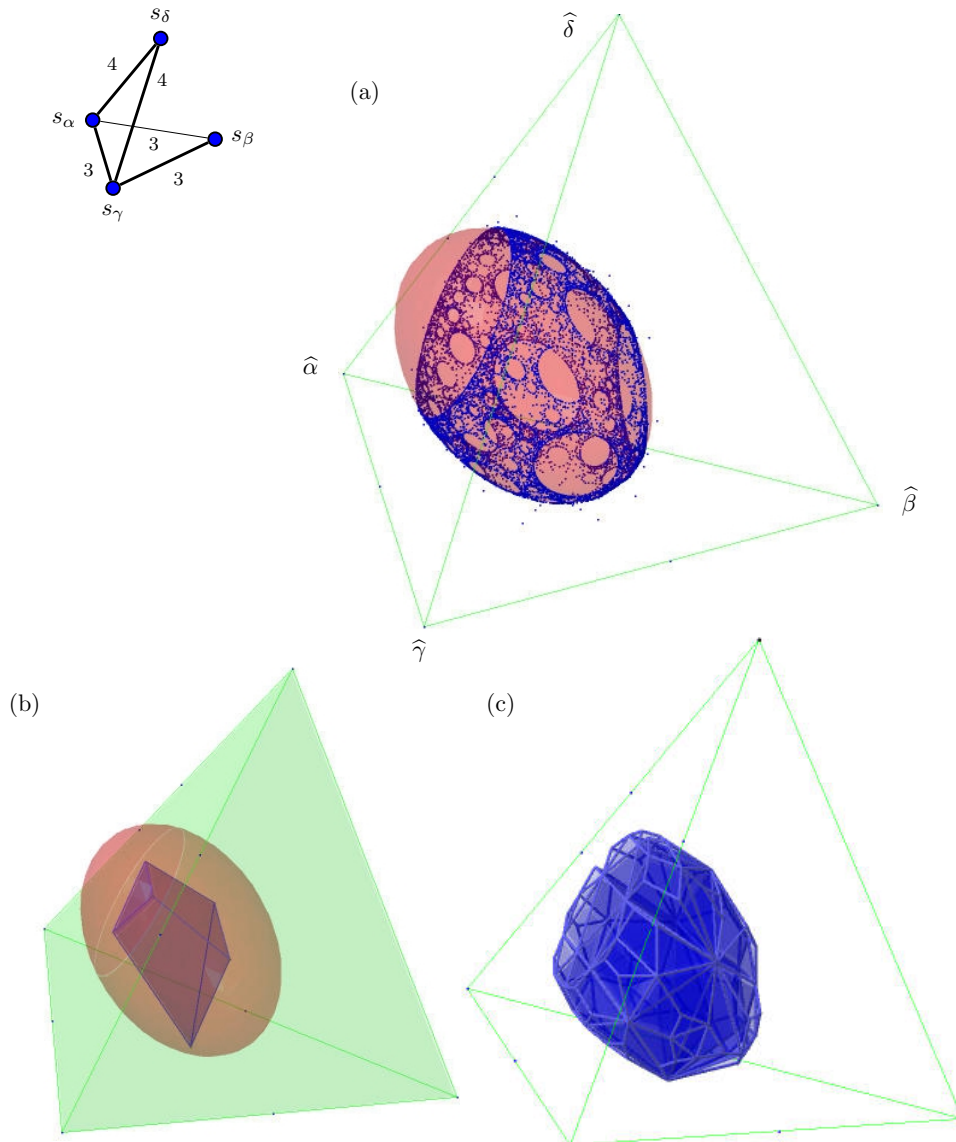
FIGURE 14. An example of weakly hyperbolic root system (with diagram on the top left corner). (a) Normalized roots (blue dots, drawn until depth 11), which quickly tend to an Apollinian gasket-like shape living on $\widehat{Q}$ (in red), as explained in §4; the sets $\mathcal{F}_0$ and $\mathcal{F}$ described in §4.1-4.2 appear clearly. (b) The polytope K defined in §2.2; note how it is truncated by the left face of $\mathrm{conv}(\widehat{\Delta})$ and it touches the bottom face exactly on the limit root of the affine root subsystem generated by $\{\alpha, \beta, \gamma\}$. (c) The first steps of construction of the imaginary convex set Z, defined in §2.2. We draw all the polytopes $w \cdot K$ for w of Coxeter length ≤ 3 ($\widehat{Q}$ is not drawn, to lighten the picture).

Appendix A. Relation of limit roots to Benoist's limit sets

Let Φ be a based root system, associated to a Coxeter group W. We assume here that Φ is irreducible, of indefinite type and of rank at least three, and that Φ spans V linearly. When Φ is non-degenerate (i.e., the associated bilinear form is non-degenerate), we explain in this appendix how the set of limit roots $E(\Phi)$ can be identified with one of the *projected limit sets* of Benoist [Ben97], which are limit sets associated to a Zariski dense subgroup of a connected reductive algebraic group. Benoist's framework is described in §A.1. Constructing the identification involves generalizing first a result by Benoist-De la Harpe [BdlH04] on the Zariski closure of a Coxeter group (§A.2). In §A.5, we prove the identification of E with Benoist's limit set (Theorem A.3) and we obtain this way a new characterization of the set of limit roots in the non-degenerate case (Corollary A.4).

These results do not extend directly to the case where Φ is degenerate, because the natural ambient algebraic group is not reductive (§A.3). However, in this case $E(\Phi)$ will project onto some $E(\Psi)$ with Ψ non-degenerate, as explained in §A.6.

A.1.   This subsection describes, somewhat informally and imprecisely, a part of the results of [Ben97], referring to [Bor91] and [Mar91, Chapter 1] for the necessary background. Let Γ be a Zariski dense subsemigroup of the group of k-points $G(k)$ of a connected, reductive algebraic group G defined over a local field k. Benoist attaches to Γ certain (equivalent) notions of "limit set" for Γ in G. We discuss the realization of the limit set as a subset Λ of a suitable flag variety Y. Our applications involve only the special case in which $k = \mathbb{R}$, G is semisimple and Γ is a group, and we assume this henceforward for simplicity. Below, the set of k-points $X(k)$ of a complex algebraic variety X defined over k is always considered as an analytic k-variety (in particular, it is taken to have the standard Hausdorff topology induced from that of k).

The standard parabolic k-subgroups of G may be naturally indexed as P_θ for subsets θ of the set Π of restricted simple roots, so that $P_\theta \supseteq P_{\theta'}$ if $\theta \subseteq \theta'$. Attached to P_θ, one has a "flag variety" $Y_\theta := G(k)/P_\theta(k)$ which is a compact analytic k-manifold on which a maximal compact subgroup K of $G(k)$ acts transitively. There is a K-invariant probability measure μ_θ on Y_θ. Define Λ_θ to be the set of all points x in Y_θ such that there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in Γ such that the sequence $\gamma_n^*(\mu_\theta)$ of pullback measures converges to a Dirac mass concentrated at x.

The following facts are from [Ben97, §3] (see especially the first paragraph of §3 and 3.5–3.6). The set Λ_θ is a closed, Γ-invariant subset of Y_θ. Let us denote by $Y := Y_\Pi$ the flag variety associated to the minimal parabolic k-subgroup, and $\Lambda := \Lambda_\Pi$ the set of associated limit points in Y for Γ. Because of our assumption $k = \mathbb{R}$, one has $\Lambda \neq \varnothing$ and any non-empty Γ-invariant closed subset of Y contains Λ. For $\theta \subseteq \Pi$, Λ_θ is the image of Λ under the natural projection $Y \to Y_\theta$.

The results of the preceding paragraph imply that for all $\theta \subseteq \Pi$, Λ_θ is a non-empty, closed, Γ-invariant subset of Y_θ, and that any non-empty Γ-invariant closed subset of Y_θ contains Λ_θ. These properties uniquely characterize Λ_θ and may be taken as definitions for our purposes below. They imply in particular that the Γ-action on each Λ_θ is minimal. We call Λ the limit set of Γ. By a *projected limit set*, we mean a set Λ_θ for some $\theta \subseteq \Pi$.

A.2.   For non-degenerate, spanning based root systems (associated to a Coxeter group W), the set of limit roots will be identified below with a suitable projected

limit set. First we need to understand what is the right algebraic group to consider. The Zariski closure of W in its standard reflection representation is described by Benoist-De la Harpe in [BdlH04]. We extend below their result to the more general class of reflection representations considered in our paper. Although this result is used here only for non-degenerate forms, we state it in natural generality corresponding to that in op. cit.

Fix a based root system (Φ, Δ) in (V, B), together with its Coxeter group W. We assume from now on that Φ is irreducible of indefinite type and of rank at least three, and that Φ spans V linearly. Let $O(V) = O(V, B)$ denote the orthogonal group of (V, B); that is,

$$O(V) := \{g \in \mathrm{GL}(V) \mid B(gv, gv') = B(v, v') \text{ for all } v, v' \in V\}.$$

Let $V^{\perp} = \{v \in V \mid B(v, v') = 0 \text{ for all } v' \in V\}$ denote the radical of (V, B), and define the following subgroup of $O(V)$:

$$H(B) := \{g \in O(V) \mid g(v) = v \text{ for all } v \in V^{\perp}\}.$$

Let $(V_{\mathbb{C}}, B_{\mathbb{C}})$ denote the quadratic space arising as the complexification of (V, B) (i.e., $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $B_{\mathbb{C}}$ is the symmetric bilinear form on $V_{\mathbb{C}}$ arising by extension of scalars to $\mathbb{C}$ from B on V). Similarly as above, we define the orthogonal group $O(V_{\mathbb{C}}) = O(V_{\mathbb{C}}, B_{\mathbb{C}})$ and its subgroup $H(B_{\mathbb{C}}) = H(V_{\mathbb{C}}, B_{\mathbb{C}})$. Regarding the natural map $\mathrm{GL}(V, \mathbb{R}) \to \mathrm{GL}(V_{\mathbb{C}}, \mathbb{C})$ as an inclusion, we regard $H(B)$ as a subgroup of $H(B_{\mathbb{C}})$. Note that $H(B_{\mathbb{C}})$ is a linear algebraic group, since it is closed in the Zariski topology of $\mathrm{GL}(V_{\mathbb{C}}, \mathbb{C})$. More precisely, we view $H := H(B_{\mathbb{C}})$ as a complex linear algebraic group defined over k with $H(k) := H(B)$ as its (Zariski dense) group of k-points.

The main result of [BdlH04] extends to this setting as follows:

Proposition A.1. *The Zariski closure of W is H (i.e., W is Zariski dense in $H(k)$).*

Proof. In case Δ is linearly independent the argument is the same, mutatis mutandis, as that in [BdlH04]. The general case can be reduced to that case as follows. Choose a subset Δ' of Δ which is inclusion maximal subject to the requirements that Δ' is linearly independent and the corresponding standard parabolic subgroup $W_{\Delta'} = \langle s_{\alpha} \mid \alpha \in \Delta' \rangle$ is irreducible. We claim that Δ' spans V. Otherwise, there is some $\alpha \in \Delta \setminus \mathrm{span}(\Delta')$. By irreducibility of W, one may suppose without loss of generality that $B(\alpha, \beta) \neq 0$ for some $\beta \in \Delta \cap \mathrm{span}(\Delta')$. This implies $B(\alpha, \gamma) \neq 0$ for some $\gamma \in \Delta'$. Then $\Delta'' := \Delta' \cup \{\alpha\}$ is linearly independent and $W_{\Delta''}$ is irreducible, contrary to maximality of Δ'. Note $W_{\Delta'}$ is of rank at least three and is of indefinite type, since its type is determined by the signature of B. By the case of linearly independent simple roots, $W_{\Delta'}$ is Zariski dense in $H(k)$ and hence so is $W \supseteq W_{\Delta'}$. $\square$

A.3. To apply Benoist's results, the ambient algebraic group should be connected and reductive. Since H is not connected, we will first need to replace H with H^0, its connected component of the identity, and W with $W \cap H^0(k)$. We therefore need the following simple fact.

Proposition A.2. *The algebraic group H^0 is reductive if and only if $V^{\perp} = \{0\}$.*

Proof. Choose a complementary subspace U to $V^\perp$ in V and let B_U be the restriction of B to a symmetric bilinear form on U. Let r denote the dimension of $V^\perp$ and m that of U. Let A denote the $m \times m$ matrix (with respect to some basis) of B_U on U. Then (with respect to a basis obtained by extending that basis by a basis of $V^\perp$) the matrix of B on V is a diagonal block matrix $\mathrm{diag}(A, 0_r)$, where 0_r is the $r \times r$ zero matrix. Then $H(k)$ (resp., H) identifies with the group of all real (resp., complex) block matrices of the form

$$
\text{(A.1)} \qquad\qquad \begin{bmatrix} X & 0 \\ Y & \mathrm{Id}_r \end{bmatrix}
$$

where Y, of size $r \times m$, is arbitrary and X satisfies $X^t A X = A$. The subgroup of such (complex) matrices with $Y = 0$ identifies with the complex semisimple algebraic group $(O(U)_\mathbb{C}, (B_U)_\mathbb{C}) \cong O(m, \mathbb{C})$. On the other hand, the set of complex matrices (A.1) with $X = \mathrm{Id}_m$ is a unipotent normal (abelian) subgroup of H; it is the unipotent radical $R_u H^0$. It follows that H^0 is reductive if and only if $V^\perp = \{0\}$, in which case H^0 is semisimple. $\square$

A.4. The following notation will prove convenient below. For any finite-dimensional real vector space U', let $\mathbb{P}(U')$ denote the projective space with points the real lines in U', in the usual (Hausdorff) topology. For $X \subseteq U'$, let $[X] \subseteq \mathbb{P}(U')$ denote the set of lines spanned by non-zero points of X.

A.5. We assume in this subsection that (Φ, Δ) is (spanning and) non-degenerate, i.e. $V^\perp = \{0\}$. Let us denote as usual by Q the isotropic cone of B, $Q := \{ v \in V \mid B(v,v) = 0 \}$. In the following we explain how to identify the set of isotropic lines $[Q]$ with some partial flag variety Y_θ, for some $\theta \subseteq \Pi$ as in §A.1.

The assumed non-degeneracy of B implies that $H = O(V_\mathbb{C}, B_\mathbb{C})$, so the connected component $G := H^0 = SO(V_\mathbb{C}, B_\mathbb{C})$ of the identity of H is a semisimple complex algebraic k-group. Let $n = \dim V$, so that $G \cong SO(n, \mathbb{C})$. Its group $G(k)$ of k-points identifies with $SO(V, B) \cong SO(p, q)$ where (p, q) is the signature of (V, B). Let $r := \min(p, q)$ be the Witt index of (V, B). Since (Φ, Δ) is of indefinite type and rank at least three, we have $r \geq 1$ and $p + q \geq 3$. Fix a choice of maximal k-split torus in G and a set Π of simple relative roots for the corresponding relative root system for G, which is of type B_r if $p \neq q$ and D_r (interpreted as $A_1 \times A_1$, A_3 for $r = 2, 3$) if $p = q$ (see [Bor91, 23.4]). The standard minimal parabolic k-subgroup P_Π identifies (see loc. cit.) with the stabilizer in G of a standard maximal flag, defined over k, of totally isotropic subspaces of $(V_\mathbb{C}, B_\mathbb{C})$. The standard parabolic k-subgroups P_θ, where $\theta \subseteq \Pi$, of G are precisely the subgroups of G which contain P_Π. The standard k-parabolic subgroups all have interpretations similar to that of P_Π, as stabilizers of standard isotropic flags in $V_\mathbb{C}$ defined over k, but there are complications in type D because there are two G-orbits of maximal isotropic $\mathbb{C}$-subspaces of $V_\mathbb{C}$.

For purposes here, it suffices to note that there is a standard parabolic k-subgroup of G, which we write as P_θ for some $\theta \subseteq \Pi$, given by the stabilizer of the isotropic line in that standard maximal isotropic flag. (We do not need the explicit description of θ as a subset of Π, but it may easily be determined for each type of root system). The corresponding (partial) flag variety $Y_\theta = G(k)/P_\theta(k)$ identifies with the $G(k)$-orbit in $\mathbb{P}(V)$ of the corresponding (real) line. Now all isotropic lines in (V, B) are in the same $G(k) = SO(V, B)$ orbit, since (by Witt's theorem) they are in the same

orbit for $O(V, B)$, and any one of them is stabilized by the reflection in some non-isotropic vector orthogonal to that line (such a line always exists since $p + q \geq 3$). Hence Y_θ naturally identifies (as homogeneous spaces for $G(k)$) with the set $[Q]$ of all isotropic lines in $[V] = \mathbb{P}(V)$. The above identification $[Q] = Y_\theta$ can be made as analytic manifolds, but it suffices here to make it as topological spaces (which is straightforward since both are compact Hausdorff spaces).

Let $\Gamma := W \cap G(k) = \{w \in W \mid l(w) \text{ is even}\}$ be the "rotation subgroup" of W, regarded as Zariski dense subgroup of $G(k)$. Denote the projected limit set for Γ in Y_θ as Λ_θ, as in §A.1.

Theorem A.3. *Assume (V, B) is non-degenerate and Δ spans V, and make the identification $[Q] = Y_\theta$, with the specific θ defined in the previous paragraphs. Then $\Lambda_\theta = [E(\Phi)]$, i.e., the projected limit set $\Lambda_\theta \subseteq Y_\theta$ for Γ as a Zariski dense subgroup of $G(k)$ identifies with the set of limit roots $E(\Phi)$, as subsets of $[Q]$. In particular, Λ_θ is W-stable.*

Proof. Note Γ is a normal subgroup of W, which acts on $[Q]$ by restriction of the natural $O(V, B)$-action given by $g[\mathbb{R}\alpha] = [\mathbb{R}g\alpha]$ for any $g \in O(V, B)$ and non-zero $\alpha \in Q$. For any $w \in W$, $w\Lambda_\theta$ is a minimal non-empty closed Γ-invariant subset of $[Q]$ (since $\Gamma = w\Gamma w^{-1}$) and therefore coincides with Λ_θ (which is the unique minimal such subset). Hence Λ_θ is stable under the W-action on $[Q]$.

Since $[E(\Phi)]$ is a non-empty, closed, Γ-invariant subset of $[Q]$, one has $\Lambda_\theta \subseteq [E(\Phi)]$, by minimality of Λ_θ amongst sets with those properties. But then Λ_θ is a non-empty, closed W-invariant subset of $[E(\Phi)]$, and the minimality of the W-action (Theorem 3.1) on $[E(\Phi)]$ forces equality in the inclusion. □

Corollary A.4. *If (V, B) is degenerate and Δ spans V, any non-empty, closed W-invariant subset of $[Q]$ contains $[E(\Phi)]$.*

Proof. This follows since the previous theorem implies it holds with $[E(\Phi)]$ replaced by Λ_θ and W by its subgroup Γ. □

Remark A.5.
  (1) Although [Ben97] and (our extension of) [BdlH04] easily imply as above the existence of a unique non-empty closed W-invariant subset of $[Q]$ on which the W-action is minimal, we do not know how to prove that set identifies with $[E(\Phi)]$ except as above, i.e., by use of our Theorem 3.1. In particular, we do not have a way to relate the two notions of limit sets (Λ_θ and $E(\Phi)$) directly from their definitions, without using their characterizations via minimality.
  (2) We do not know how to prove Corollary A.4 without use of [Ben97]. The related result we have (Theorem 4.10) assumes that the root system is weakly hyperbolic and states only that any non-empty, closed W-invariant subset of $[Q]$, that is *also contained in* $[\text{conv}(\Delta)]$, is equal to $[E(\Phi)]$.
  (3) For Φ non-degenerate, Theorem A.3 provides an interpretation of $[E(\Phi)]$ as a projected limit set, and [Ben97] then yields many additional facts about $E(\Phi)$ which seem likely to have significant applications (see for example Remark A.6(3)).
  (4) It is an interesting question whether other projected limit sets $\Lambda_{\theta'}$ for Γ, and especially the limit set Λ itself, can be given interpretations similar to those in the theorem in terms of the root system. The corresponding flag

varieties $Y_{\theta'}$ involve flags containing higher dimensional totally isotropic spaces, and such isotropic subspaces already appear naturally in the study of limit roots (see for instance Proposition 6.8).

A.6. We now consider the situation for possibly degenerate root bases. Let us explain a classical way (after Krammer) to obtain from a degenerate root system a non-degenerate one, with the same Coxeter group. Let $\pi\colon V \to V/V^{\perp}$ be the natural map. The restriction of π to U identifies U isomorphically with $V/V^{\perp}$ as real vector space, and we further identify (U, B_U) with the quotient of (V, B) by its radical $V^{\perp}$. Since (Φ, Δ) is of indefinite type, one has $\mathrm{cone}(\Delta) \cap V^{\perp} = \{0\}$, and there is a (non-degenerate, spanning) based root system (Ψ, Σ) for (U, B_U) where $\Psi = \pi(\Phi)$ and $\Sigma = \pi(\Delta)$ (see [Kra09, 6.1]). The Coxeter system attached to (Ψ, Σ) identifies canonically with W, with its natural action on the quotient space $U = V/V^{\perp}$.

Let V_1 be the fixed affine subspace of V transverse to Φ and V_0 be its translate through 0, as in §2.1. The map $v \mapsto [\{v\}]$ for $v \in V_1$ identifies V_1 homeomorphically with $[V_1] = \mathbb{P}(V) \setminus \mathbb{P}(V_0)$, an open subset of $\mathbb{P}(V)$. The rule $[\{v\}] \mapsto [\{\pi(v)\}]$ for v in $\mathrm{cone}(\Delta) \cap V_1$ defines a continuous surjective map $[\mathrm{cone}(\Delta)] \to [\mathrm{cone}(\Sigma)]$ of compact Hausdorff spaces (hence it is a closed, proper quotient map). Since v is isotropic if and only if $\pi(v)$ is isotropic, it easily follows that this map restricts to a surjective continuous (closed, proper, quotient) map $[E(\Phi)] \to [E(\Psi)]$ (where $E(\Phi)$ and $E(\Psi)$ are the sets of limit roots for Φ and Ψ respectively) and that this latter map is in addition W-equivariant.

By Theorem 3.1(b), the W-actions on $[E(\Phi)]$ and $[E(\Psi)]$ are minimal. One easily sees that minimality on $[E(\Phi)]$ directly implies that on $[E(\Psi)]$, but we do not know any direct argument for the converse implication. Therefore, results on the limit sets for non-degenerate root systems do not easily extend to degenerate ones.

A.7. We give a simple example to show that the above map $[E(\Phi)] \to [E(\Psi)]$ is not bijective in general. Let (Φ, Δ) be the standard based root system attached to the following Coxeter graph, in which vertices are labeled by the corresponding simple roots:
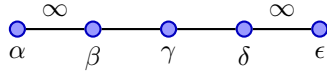
Denote the associated quadratic space as (V, B). One easily checks that $\alpha + \beta - \delta - \epsilon$ is in $V^{\perp}$. One has $\widehat{\alpha + \beta} = \lim_{n \to \infty} \widehat{(s_\alpha s_\beta)^n} \alpha \in E(\Phi)$ and similarly $\widehat{\delta + \epsilon} \in E(\Phi)$. Hence the above map $[E(\Phi)] \to [E(\Psi)]$ sends the distinct elements $[\{\alpha + \beta\}]$ and $[\{\delta + \epsilon\}]$ of $[E(\Phi)]$ to the same element of $[E(\Psi)]$.

Note also that although (Φ, Δ) is a standard based root system, we are not able to deduce the minimality of the W-action on $E(\Phi)$ from Benoist's results.

Remark A.6.

(1) We do not know if, for degenerate, spanning (Φ, Δ), with $V^{\perp}$ defined as the radical of (V, B), any closed non-empty W-invariant subset of $[Q] \setminus [V^{\perp}]$ contains $[E(\Phi)]$ (though the corresponding statement with $[Q] \setminus [V^{\perp}]$ replaced by $[Q]$ obviously fails in general).

(2) In the case Φ is degenerate, §A.6 gives a W-equivariant surjection from $E(\Phi)$ to some $E(\Psi)$ with Ψ non-degenerate. This construction may allow

one to transfer some of the properties known in the non-degenerate case to the degenerate case, but not all. Remark (3) below illustrates both this point and Remark A.5(3).

(3) We sketch another proof of faithfulness of the W-action on $E(\Phi)$ in the setting of Theorem 6.12 (Φ indefinite of rank at least 3, and irreducible) as follows. From §A.6 one always has a surjective W-equivariant map from $E(\Phi)$ to some $E(\Psi)$ where Ψ is non-degenerate. So it is sufficient to prove the faithfulness property in the non-degenerate case.

Thus, assume now that Φ is non-degenerate. Using the notations and result of Theorem A.3, $[Q]$ identifies to Y_θ and $E(\Phi)$ to Λ_θ. Using the projection $Y \to Y_\theta$, the Zariski density of Λ in Y (see [Ben97, Lemma 3.6]) implies that of Λ_θ in Y_θ. Therefore, if $w \in W$ acts as the identity on $[E(\Phi)]$, it acts as the identity on $[Q]$. This implies w fixes each isotropic line in V. Since Φ is irreducible and non-degenerate, this readily implies that w acts as the identity on V and hence $w = 1$ by faithfulness of the W-action on Φ.

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