Modeling Dependence between Loss Triangles with Hierarchical Archimedean Copulas*

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Abstract

One of the most critical problems in property/casualty insurance is to determine an appropriate reserve for incurred but unpaid losses. These provisions generally comprise most of the liabilities of a non-life insurance company. The global provisions are often determined under an assumption of independence between the lines of business. Recently, Shi and Frees (2011) proposed to put dependence between lines of business with a copula that captures dependence between two cells of two different runoff triangles. In this paper, we propose to generalize this model in two steps. First, by using an idea proposed by Barnett and Zehnwirth (1998), we will suppose a dependence between all the observations that belong to the same calendar year for each line of business. Thereafter, we will then suppose another dependence structure that links the calendar years of different lines of business. This model is done by using hierarchical Archimedean copulas. We show that the model provides more flexibility than existing models, and offers a better, more realistic and more intuitive interpretation of the dependence between the lines of business. For illustration, the model is first applied to a dataset from a major US property-casualty insurer, and then to two lines of business from a large Canadian insurer.

Keywords: runoff triangles, Copula, Hierarchical Archimedean Copula, Maximum Likelihood Estimation, Bootstrap.

*This paper is a revisited version of Abdallah et al. (2015)

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1 Introduction

Reserves are a major component of the financial statements of a financial institution. With the advent of the new regulatory standards (e.g. Solvency II in Europe and the upcoming ORSA[1] guidelines in North America), insurance companies must better understand and quantify the risks associated with their activities as a whole, not just by risk classes. Thus, it is now necessary for an insurance company to not only assess a reserve for each line of business but also to better estimate the total reserves for all its insurance products. This involves taking into account dependence between lines of business. In this context, insurance companies must be particularly able to estimate the amount of provisions for the entire portfolio. For this purpose, different reserving approaches allowing dependence between lines of business must be investigated. We will focus on the parametric approach.

Parametric reserving methods have often involved copulas to model the dependence between lines of business. For example, Brehm (2002) uses a Gaussian copula to model the joint distribution of unpaid losses, while De Jong (2012) models dependence between lines of business with a Gaussian copula correlation matrix. Shi et al. (2012) and Wüthrich et al. (2013) have also used multivariate Gaussian copula to accommodate the correlation due to accounting years within and across runoff triangles. Bootstrapping is another popular parametric approach used to forecast the predictive distribution of unpaid losses for correlated lines of business. Kirschner et al. (2008) use a synchronized bootstrap and Taylor and McGuire (2007) extend this result to a generalized linear model context.

In this paper, we propose to use a parametric approach with multivariate Archimedean copulas and hierarchical Archimedean copulas. In the same vein as Frees and Shi's model, and following an idea proposed by Barnett and Zehnwirth (1998), we propose a model that allows a dependence relation between all the observations that belong to the same calendar year for each line of business using multivariate Archimedean copulas. We use another dependence structure that links the losses of calendar years of different lines of business. We show that this complex dependence structure can be constructed using hierarchial Archimedean copulas. For illustration, the model is applied to two different datasets from a major US and a large Canadian property-casualty insurers, where we conclude that the proposed model can be considered as an interesting alternative of the model proposed by Shi and Frees (2011).

In Section 2, we review the modeling of runoff triangles, where notations are set and copulas briefly introduced. In Section 3, the model of Shi and Frees (2011) is implemented (again on their dataset from a major US property-casualty insurer), but with a different parametrization. The calendar year and hierarchical dependences are explained and applied to this data and to a new pair of runoff triangles in Section 4. In Section 5, we use a parametric bootstrap to obtain the predictive distribution of unpaid losses. Section 6 concludes the paper.

[1]ORSA: Own Risk and Solvency Assessment

2 Preliminary

2.1 Modeling and Reserves

Let us consider an insurance portfolio with ℓ lines of business ($\ell \in \{1, ..., L\}$). We define by $X_{i,j}^{(\ell)}$, the incremental payments of the i^{th} accident year ($i \in \{1, ..., I\}$), and the j^{th} development period ($j \in \{1, ..., J\}$). To take into account the volume of each line of business, we will work with standardized data which we denote by $Y_{i,j}^{(\ell)} = X_{ij}^{(\ell)}/\omega_i^{(\ell)}$, where $\omega_i^{(\ell)}$ represents the exposure variable in the i^{th} accident year for the ℓ^{th} line of business. The exposure variable can be the number of policies, the number of open claims, or the earned premiums. The latter option is the one chosen in this paper.

A regression model with two independent explanatory variables, accident year and development period, will be used. Assume that $\alpha_i^{(\ell)}$ ($i \in \{1, 2, ..., I\}$) and $\beta_j^{(\ell)}$ ($j \in \{1, 2, ..., J\}$) characterize respectively the accident year effect and the development period effect. In such a context, a systematic component for the ℓ^{th} line of business can be written as:

$$\eta_{ij}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)},\ \ell = 1, ..., L,$$

where $\zeta^{(\ell)}$ is the intercept, $I = J = n$, and for parameter identification, the constraint $\alpha_1^{(\ell)} = \beta_1^{(\ell)} = 0$ is supposed.

In our empirical illustration, we first work with the runoff triangles of cumulative paid losses exhibited in Tables 1 and 2 of Shi and Frees (2011). They correspond to paid losses of Schedule P of the National Association of Insurance Commissioners (NAIC) database. These are data of 1997 for personal auto and commercial auto lines of business, and each triangle contains losses for accident years 1988-1997 and at most ten development years.

Shi and Frees (2011) show that a lognormal and a gamma distribution provide a good fit for the Personal Auto and the Commercial Auto line data respectively. To demonstrate the reasonable model fits for the two triangles, the authors exhibit the qq-plots of marginals for personal and commercial auto lines. We work with their conclusion and then continue with the same continuous distributions for each line of business. More specifically, we consider the form $\mu_{ij}^{(1)} = \eta_{ij}^{(1)}$ for a lognormal distribution with location (log-scale) parameter $\mu_{ij}^{(1)}$ and shape parameter σ. However, for the gamma distribution, we change the parametrization and we do not use the canonical inverse link $\mu_{ij}^{(2)} = \dfrac{1}{\eta_{ij}^{(2)}\phi}$ with location (scale) parameter $\mu_{ij}^{(2)}$ and shape parameter ϕ. Such a parametrization can lead to undesirable negative values for the lower right part of the runoff triangle, especially when one uses the bootstrap technique. To assure positive means of all the cells of the runoff triangle, we use the exponential link $\mu_{ij}^{(2)} = \dfrac{\exp(\eta_{ij}^{(2)})}{\phi}$, which is always positive, even for the prediction values of the runoff triangle.

With both parametrizations, the estimated total reserve is $\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n}\widehat{y}_{ij}^{(\ell)}\omega_i^{(\ell)}$, where $\widehat{y}_{ij}^{(\ell)}$ is the projected unpaid loss ratio, and $\omega_i^{(\ell)}$ represents the net premiums earned in the corresponding accident year i. For the lognormal distribution, we have $\widehat{y}_{ij}^{(1)} = \exp^{\widehat{\mu}_{ij}^{(1)} + (\widehat{\gamma}^{(1)})^2/2}$, and for the gamma distribution, $\widehat{y}_{ij}^{(2)} = \widehat{\mu}_{ij}^{(2)}\widehat{\gamma}^{(2)}$, where $\widehat{\mu}_{ij}^{(\ell)}$ and $\widehat{\gamma}^{(\ell)}$ are respectively the scale (location) and the shape parameters. Also, $\widehat{\gamma}^{(1)} = \widehat{\sigma}$ and $\widehat{\gamma}^{(2)} = \widehat{\phi}$.

2.2 Copulas

Copulas are a useful and flexible tool to model a dependence relation between runoff triangles of different lines of business. They allow a separate interpretation of the relationship (linear and non-linear) between linked random variables and their marginals. See Joe (1997) further details. We briefly recall below definitions and results that will be used later.

A multivariate copula $C(u_1, u_2, ..., u_n)$ is an application from $[0,1]^n$ to $[0,1]$, that has the same properties as a joint cumulative distribution. In other words, a copula is a function that links a multidimensional distribution to its one-dimensional margins. Let F be a n-dimensional cumulative joint function with margins $F^{(1)}$,$F^{(2)}$,...,$F^{(n)}$. Then, if the margins are all continuous, the joint distribution of n random variables $(Y^{(1)}, Y^{(2)}, ..., Y^{(n)})$, can be represented by a unique copula function:

$$F(y^{(1)}, y^{(2)}, ..., y^{(n)}) = C(F^{(1)}, F^{(2)}, ..., F^{(n)}; \theta),$$

where $F^{(i)}$, with $i \in \{1, 2, ..., n\}$, are the respective distribution functions of $Y^{(i)}$, and θ is the dependence parameter, also called the association parameter.

In this paper, we choose to use the Archimedean family of copulas, given its several interesting properties. This family of copulas offers a wide choice of copulas for which many have a closed form expression in a multivariate setting. This last property will prove to be useful in what follows. Finally, Archimedean copulas can be constructed easily with a simple generator. Formally, we can define multivariate Archimedean copulas as

$$C(u_1, u_2, ..., u_n) = \phi^{-1}(\phi(u_1) + ... + \phi(u_n)) \ , \tag{1}$$

where the function ϕ is called the generator of the copula. From (1), one can derive the expression for the multivariate density function of an Archimedean copula. According to McNeil and Nešlehová (2009), an Archimedean copula C admits a density c if and only if $\phi^{(n-1)}$ exists and is absolutely continuous on $(0, \infty)$. In such a case, c is given by

$$c(u_1, u_2, ..., u_n) = \phi^{(n)}(\phi^{-1}(u_1) + ... + \phi^{-1}(u_n)) \prod_{i=1}^{n} (\phi^{-1})'(u_i),$$

where functions $\phi^{(n)}$ and ϕ^{-1} correspond to the n^{th} derivative of the generator function of the copula and the inverse generator respectively. Hofert et al. (2012) derive closed form expressions for the multivariate density function of a few Archimedean copulas, notably the Clayton and the Gumbel copula used in this paper.

3 Pairwise dependence

Dividing a portfolio into homogeneous sub-portfolios and deriving the total reserve by summing the reserve for each segment implicitly assumes independence between risks. It is generally admitted that common social or economic factors may affect several lines of business simultaneously. Allowing a possible dependence relation between the runoff triangles of different lines of business of a portfolio provides a better representation of the portfolio's behavior as a whole and hence permits to take better advantage of diversification. It is also helpful to risk managers in determining the risk capital for an insurance portfolio.

Shi and Frees (2011) propose a model that incorporates a dependence structure between two runoff triangles in a pairwise manner. More precisely, the dependence between two lines of business is based on an identical association between cells of a given accident year and development period, coming from different lines of business. This means that two paid loss ratios $Y_{i,j}^{(1)}$ and $Y_{i,j}^{(2)}$ are correlated for a given couple (i, j). This form of dependence goes back to Braun (2004). Throughout the paper, we refer to Frees and Shi's model as the pairwise dependence model (PWD).

3.1 Modeling

The PWD model associates two elements of the same accident year and development period, $(Y_{i,j}^{(1)}, Y_{i,j}^{(2)})$ with a bivariate copula. Mathematically, and following Sklar's theorem, the joint distribution of normalized incremental payments $(Y_{i,j}^{(1)}, Y_{i,j}^{(2)})$ will be represented by the unique copula function:

$$F_{ij}(y_{ij}^{(1)}, y_{ij}^{(2)}) = \Pr(Y_{ij}^{(1)} \leq y_{ij}^{(1)}, Y_{ij}^{(2)} \leq y_{ij}^{(2)}) = C(F_{ij}^{(1)}, F_{ij}^{(2)}; \theta) \ , \tag{2}$$

where $C(., \theta)$ denotes the copula function with parameter θ, that captures the dependence between two runoff triangles. Also, this model has the flexibility of choosing a different cumulative density function for each line of business. The log-likelihood expression can be easily derived from equation (2):

$$L = \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{ij}^{(1)}) + \log(f_{ij}^{(2)}) + \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log \ c(F_{ij}^{(1)}, F_{ij}^{(2)}; \theta) \ , \tag{3}$$

where $c(.)$ denotes the probability density function corresponding to the copula distribution function $C(.)$, $f_{ij}^{(\ell)}$ denotes the density of marginal distribution $F_{ij}^{(\ell)}$, for $\ell = 1, 2$. These marginals are noted as:

$$F_{ij}^{(\ell)} = \mathrm{Prob}(Y_{ij}^{(\ell)} \leq y_{ij}^{(\ell)}) = F^{(\ell)}(y_{ij}^{(\ell)}; \eta_{ij}^{(\ell)}, \gamma^{(\ell)}),$$

for $i = 1, ..., I$, $j = 1, ...J$ and $\ell = 1, ..., L$. Shi and Frees (2011) choose the Gaussian and the Frank copula to model dependence, as well as the product copula that supposes independence between the cells. Their model selection is based on a likelihood-based goodness-of-fit measure, namely Akaike's Information Criterion (AIC). We will also use this criterion to select our models.

3.2 Empirical Illustration

We provide in Tables 1 and 2, the fit statistics and the reserves for the PWD model. Note that even if the results are close to those obtained in Shi and Frees (2011), we do not obtain the same estimates because we have changed the link function of the mean of the gamma distribution to avoid inconsistencies, as explained in Section 2.2.

		Copula			
Fit Statistics	**Independence**	**Frank**		**Gaussian**	
Dependence parameter	.	-2.7978	(1.0243)	-0.3655	(0.1190)
Log-Likelihood	346.6	350.3		350.5	
AIC	-613.2	-618.5		-618.9	

Table 1: Fit Statistics of PWD model with Shi and Frees (2011) database

On the other hand, even if we have chosen a different parametrization, we obtain the same conclusion as their and find that the copula that leads to the smallest AIC is the Gaussian copula. This model generates a reserve of almost 7 million dollars. Interestingly, the dependence parameter obtained for the pairwise model with the Gaussian and the Frank copula is negative, meaning that the model supposes that the two lines of business are negatively correlated.

4 Calendar Year and Hierarchical Dependence

We propose here to further investigate the model of Shi and Frees (2011) to better capture the interactions within and between the runoff triangles of different lines of business. For that purpose, we first propose to consider a dependence construction for the different elements of a diagonal of a given runoff triangle to take into account a calendar year effect. Second, we add another level of dependence to capture the dependence between the lines of business.

4.1 Calendar Year Effect

We propose in this section a model that allows a dependence relation within paid claims belonging to a diagonal of a runoff triangle. This reflects a calendar year (CY) effect, more precisely the changes or inflections on paid claims in a calendar year due to jurisprudence

		Copula	
Reserves estimation	**Independence**	**Frank**	**Gaussian**
Personal	6,464,090	6,511,363	6,423,180
Commercial	490,657	487,904	495,989
Total	6,954,747	6,999,267	6,919,169

Table 2: Reserves estimation with the PWD model with Shi and Frees (2011) database

modifications or inflationary trends for example. A CY effect can also highlight the impact of strategic decisions made in a calendar year such as an incentive to increase payments in a particular calendar year for all lines of business.

This dependence structure assumes that all cells from the same diagonal are correlated, which implies that the number of cells in the dependence structure is different for each diagonal. Indeed, the number of cells in the dependence structure varies from 1 to t for the t^{th} diagonal, with $t \in \{1, ..., n\}$, and $t = i + j - 1$. Evidently, the first cell at the top left of the runoff triangle is not linked to any other cell within the triangle.

Such a calendar year effect has already been analyzed before, for example by Barnett and Zehnwirth (1998) who added a covariate to capture the calendar year effect. The systematic component of such a model can be written as:

$$\eta_{ij}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)} + \Upsilon_t^{(\ell)},\ \ell = 1, ..., L\ , \tag{4}$$

where $\zeta^{(\ell)}$ is the intercept, $\alpha_i^{(\ell)}$ ($i \in \{1, 2, ..., I\}$) and $\beta_j^{(\ell)}$ ($j \in \{1, 2, ..., J\}$) characterize respectively the accident year effect and the development period effect, while $\Upsilon_t^{(\ell)}$ ($t = i + j - 1$) captures the calendar year effect.

De Jong (2006) modeled the growth rates in cumulative payments in a calendar year, and Wüthrich (2010) examined the accounting year effect for a single line of business. Wüthrich and Salzmann (2012) used a multivariate Bayes Chain-Ladder model that allows the modeling of dependence along accounting years within runoff triangles. The authors showed that they are able to derive closed form solutions for the posterior distribution, the claims reserves and the corresponding prediction uncertainty. Kuang et al. (2008) have also considered a canonical parametrization with three factors for a single line of business. Each factor represents time scale, in such way the inflation is taken into account. Also, they added an assumption ensuring that the forecasts do not depend on these arbitrary linear trends. They extended this assumption later by combining the canonical parametrization with a non-stationary time series forecasting model in Kuang et al. (2011).

In our proposed model, instead of adding an explanatory variable for the calendar year effect, the dependence relation between the paid claims of a diagonal will be based on a multivariate Archimedean copula. More specifically, the same Archimedean copula with an identical dependence parameter is assumed for each diagonal of a runoff triangle. Hence, all random variables of the same calendar year $t = i + j - 1$ and ℓ^{th} line of business are included in the vector $\boldsymbol{Y}_{\ell t} = \{Y_{\ell ij} : i + j - 1 = t\}$. The log-likelihood function of this model can be written as:

$$L \ = \ \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{ij}) + \sum_{t=2}^{n} \log\ c\left(F_{t-j+1,j}, ..., F_{1,t}; \theta\right)_{j=1,...,t}\ , \tag{5}$$

where f denotes the density of marginal distribution F, and $c(.)$ the probability density function corresponding to the copula distribution function $C(.)$.

The main advantage of the copula approach instead of adding a calendar year covariate in the mean specification, lies in the fact that the copula approach allows a more general structure of dependence between the observations of a given calendar year and allows more flexibility. Also, the use of covariates would lead to a great number of parameters to explain the calendar year effect instead of only one (dependence copula parameter). For example, for two lines of business, we would have 20 parameters instead of 2 (see equation (4)). This might lead to over-parametrization. Furthermore, the parameter describing a given calendar year effect, would not have any predictive power, as we cannot use it to compute the lower triangle.

4.2 Line of Business Dependence

A natural extension to the model behind (5) is to introduce a dependence structure between lines of business based on copulas, more precisely here with the Gaussian copula and hierarchical Archimedean copulas.

Another way to add dependence between lines of business is by modifying equation (4) and use the same calendar year covariate for the two lines of business, i.e. $\Upsilon_t = \Upsilon_t^{(1)} = \Upsilon_t^{(2)}$. The correlation induced by common calendar year effects would then be introduced through the mean specification. Also, as done in Shi et al. (2012), in addition to the common calendar year covariate, a pair-wise correlation between the two runoff triangles can be added. This approach has the disadvantage however of adding a new parameter for each diagonal (Υ_t).

4.2.1 Multivariate Gaussian Copula

We first propose to use the Gaussian copula to capture the dependence within and between runoff triangles. The Gaussian copula which arises from the multivariate normal distribution is the most widely known copula of the elliptical family of copulas. Such a copula allows great flexibility to model dependences simply by modifying its correlation matrix.

Let us suppose, for a given calendar year t, the following set of observations $\mathbf{u}_t = \left(u_{t-j+1,j}^{(1)}, ..., u_{1,t}^{(1)}, u_{t-j+1,j}^{(2)}, ..., u_{1,t}^{(2)}\right)_{j=1,...,t}$, with multivariate Gaussian copula density:

$$c\left(\mathbf{u}_t\right) = \left|\mathbf{\Sigma}_t\right|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{\xi}_t^T\left(\mathbf{\Sigma}_t^{-1} - I\right)\boldsymbol{\xi}_t\right) ,$$

where $\boldsymbol{\xi}_t = \left(\Phi^{-1}\left(u_{t-j+1,j}^{(1)}\right), ..., \Phi^{-1}\left(u_{1,t}^{(1)}\right), \Phi^{-1}\left(u_{t-j+1,j}^{(2)}\right), ..., \Phi^{-1}\left(u_{1,t}^{(2)}\right)\right)_{j=1,...,t}^T$. The correlation matrix $\mathbf{\Sigma}_t$ for the calendar year t can be represented as a block matrix as follows, given the assumptions of the model:

$$\mathbf{\Sigma}_t = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{21} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{12} \end{pmatrix}. \tag{6}$$

In (6), the matrices $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{12}$ are correlation matrices with unit main diagonal and off-diagonal parameters $\theta_{1,1}$ and $\theta_{1,2}$ corresponding to the calendar year dependence for the first and second line of business respectively. $\mathbf{\Sigma}_{21}$ is a matrix filled with parameter $\theta_{2,1}$ representing the dependence between the two lines of business.

Numerical results obtained with the Gaussian copula are presented in the empirical illustration of section 4.3.

4.2.2 Hierarchical Archimedean Copulas

Hierarchical Archimedean copulas permit to have different levels of dependence within our framework. We use them here to add another level of dependence to the one proposed in section 4.1. With this second level of dependence, we capture the dependence between two different runoff triangles in a pairwise manner between corresponding diagonals, instead of between cells. Pairing diagonals instead of cells with a copula has the advantage of being applicable even in a case of missing data in one of the runoff triangles.

The hierarchical approach allows us to visualize the multi-level dependence. Indeed, this dependence structure is illustrated in Figure 1, where a dependence structure between cells of the same calendar year is supposed as well as a dependence structure between the two lines of business. In the next section, we will also be interested in comparing the hierarchical copula approach with the multivariate Gaussian copula approach, as the latter is often considered as a benchmark model.

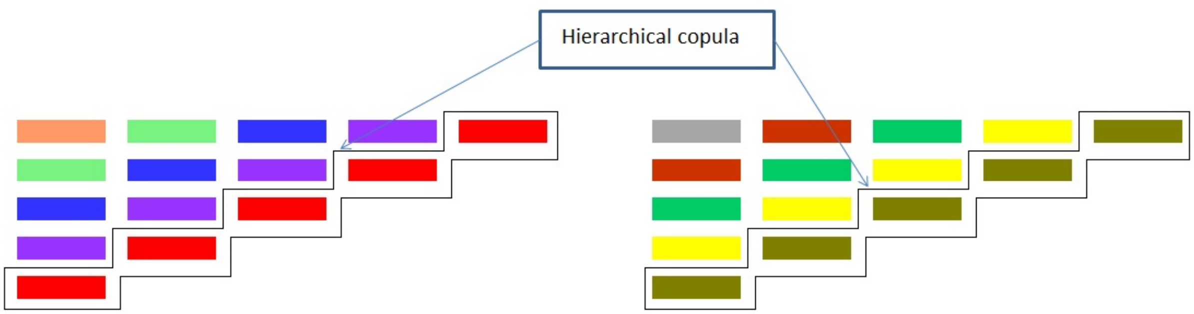

Figure 1: Dependence implied by hierarchical dependence

The CY effect has not been often studied with more than one line of business. Two recent examples are De Jong (2012), where the calendar year effect was introduced through the correlation matrix and Shi et al. (2012), who used random effects to accommodate the correlation due to accounting year effects within and across runoff triangles. In Shi et al. (2012), they work with a Bayesian perspective, using a multivariate lognormal distribution, along with a multivariate Gaussian correlation matrix. The predictive distributions of outstanding payments are generated through Monte Carlo simulations. The calendar year effect is taken into account through an explanatory variable. A discussion of this paper is suggested in Wüthrich (2012), and where it is also explained that for the method it does not really matter whether we consider incremental or cumulative claims, as long as we have a multivariate Gaussian structure. Also, still with a Bayesian framework, Wüthrich et al. (2013) used a multivariate lognormal Chain-Ladder model and derived predictors and confidence bounds in closed form. Their analytical solutions are such that they allow for any correlation structure. Their models allow a dependence between and within runoff triangles, and for any correlation structure. It has also been shown in this paper that the pair-wise dependence form is a rather weak one compared to calendar year dependence. More recently, Shi (2014) captures the dependencies introduced by various sources, including the common calendar year effects via the family of elliptical copulas, and use a parametric bootstrapping to quantify the associated reserving variability.

In this paper, to model the complex dependence structure between two runoff triangles, we introduce models based on hierarchical Archimedean copulas. The idea is to use Archimedean copulas at each level, from the lowest (calendar years) to the highest (lines of business). Hierarchical Archimedean copulas have first been mentioned in the literature by Joe (1997), and appeared in more details in Savu and Trede (2010). More recently, Okhrin et al. (2013) provided a method to estimate multivariate distributions defined through hierarchical Archimedean copulas.

The main advantage of using Archimedean and hierarchical Archimedean copulas is that they can be explicitly defined in terms of a one-dimensional function called the generator of the Archimedean copula. Elliptical copulas, used in Shi (2014), do not possess this nice property; they do not have a closed form. Archimedean copulas are also flexible and allow to model many kinds of dependencies, while Elliptical copulas, have equal lower and upper tail dependence coefficients. In high dimensions, Archimedean copulas are restricted given the exchangeability of the components. This assumption is relaxed with hierarchical Archimedean copulas.

At the lowest level, and for the calendar year t, we have $2 \times t$ standard uniformly distributed random variables $U^{(1)}_{t-j+1,j}, ..., U^{(1)}_{1,t}, U^{(2)}_{t-j+1,j}, ..., U^{(2)}_{1,t}$ where j designates the development period ($j = 1, ..., t$).
The joint distribution function is evaluated at $\mathbf{u} = (u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}) \in [0,1]^{2t}$. Let there be H hierarchy levels indexed by h. For example, the set of elements $\mathbf{u}$ is located at level $h = 0$. At each level $h = 0, ..., H$ we have n_h distinct objects with index $k = 1, ..., n_h$.

At level $h = 1$, the $u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}$ are grouped into n_1 ordinary multivariate Archimedean copulas $C_{1,k}$, $k = 1, ..., n_1$ (in our case with two lines of business, we have $n_1 = 2$), of the form

$$C_{1,k}(\mathbf{u}_{1,k}) = \phi^{-1}_{1,k}\left(\sum_{\mathbf{u}_{1,k}} \phi_{1,k}(\mathbf{u}_{1,k})\right) ,$$

where $\phi_{1,k}$ denotes the generator of the copula $C_{1,k}$. Let $\mathbf{u}_{1,k}$ denote the set of elements of $u^{(k)}_{t-j+1,j}, ..., u^{(k)}_{1,t}$ belonging to the copula $C_{1,k}$ for $k = 1, ..., n_1$, which represents the elements of a given calendar year for a single line of business ℓ. At this level only, k corresponds to ℓ .In our model, we have three levels, i.e $H = 2$. At the highest level, we have a single object ($n_2 = 1$), which is the hierarchical Archimedean copula $C_{2,1}$, that aggregates the multivariate Archimedean copulas of the previous level, and can be represented as

$$C_{2,k}(\mathbf{C}_{2,k}) = \phi^{-1}_{2,k}\left(\sum_{\mathbf{C}_{2,k}} \phi_{2,k}(\mathbf{C}_{2,k})\right) ,$$

where $\phi_{2,k}$ denotes the generator of the copula $C_{2,k}$ and $\mathbf{C}_{2,k}$ represents the set of all copulas from level $h = 1$ entering copula $C_{2,k}$ for $k = 1, ..., n_2$.

Obviously, there are numerous conditions to be satisfied for the existence of a hierarchical Archimedean copula. The number of copulas must decrease at each level, i.e. $n_h < n_{h-1}$, as well as the degree of dependence, i.e. $\theta_{h+1,k'} < \theta_{h,k}$ for all $h = 0, ..., H$ and $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ such that $C_{h,k} \in C_{h+1,k'}$ where $\theta_{h,k}$ is the parameter belonging to the generator $\phi_{h,k}$. This means that for runoff triangles, elements of a same line of business can have a higher degree of dependence than elements of different lines of business. Mathematically, the conditions that have to be verified by a hierarchical Archimedean copula are summarized as follows:

1. All inverse generator functions $\phi^{-1}_{h,k}$ are completely monotone.

2. The composite $\phi_{h+1,k'} \circ \phi^{-1}_{h,k}$ are convex functions for all $h = 0, ..., H$ and $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ such that $C_{h,k} \in C_{h+1,k'}$.

In our application, we will limit the number of levels to three, and the number of lines of business to two. This means that we will have at the highest level ($h = 2$), one (hierarchical) bivariate Archimedean copula between lines of business, and for $h = 1$, two (ordinary) multivariate Archimedean copula within a runoff triangle.

As an illustration, let us consider a dependence structure between two runoff triangles for the second calendar year. The resulting hierarchical Archimedean copula has the following analytical form

$$
\begin{aligned}
C_{2,1}(\mathbf{u}) &= C_{2,1}(u^{(1)}_{2,1}, u^{(1)}_{1,2}, u^{(2)}_{2,1}, u^{(2)}_{1,2}) \\
&= C_{2,1}(C_{1,1}(u^{(1)}_{2,1}, u^{(1)}_{1,2}), C_{1,2}(u^{(2)}_{2,1}, u^{(2)}_{1,2})) \\
&= \phi^{-1}_{2,1}\left(\phi_{2,1} \circ \phi^{-1}_{1,1}[\phi_{1,1}(u^{(1)}_{2,1}) + \phi_{1,1}(u^{(1)}_{1,2})] + \phi_{2,1} \circ \phi^{-1}_{1,2}[\phi_{1,2}(u^{(2)}_{2,1}) + \phi_{1,2}(u^{(2)}_{1,2})]\right).
\end{aligned}
$$

This hierarchical Archimedean copula will be applied to each calendar year, with the dataset described in Section 3.2. The calendar year t takes values from 1 to 10 because the runoff triangles both have 10 diagonals, i.e. $I = J = 10$. The resulting hierarchical Archimedean copula for our model has the following general analytical form:

$$
\begin{aligned}
C_{2,1}(\mathbf{u}) &= C_{2,1}(u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}) \\
&= C_{2,1}(C_{1,1}(u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}), C_{1,2}(u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t})) \\
&= \phi^{-1}_{2,1}\left(\phi_{2,1} \circ \phi^{-1}_{1,1}[\phi_{1,1}(u^{(1)}_{t-j+1,j}) + ... + \phi_{1,1}(u^{(1)}_{1,t})] + \phi_{2,1} \circ \phi^{-1}_{1,2}[\phi_{1,2}(u^{(2)}_{t-j+1,j}) + ... + \phi_{1,2}(u^{(2)}_{1,t})]\right).
\end{aligned} \tag{7}
$$

Finally, the log-likelihood function of the hierarchical model can be written as follows:

$$L=\sum_{\ell=1}^{2}\sum_{i=1}^{I}\sum_{j=1}^{I-i+1}\log(f_{ij}^{(\ell)})+\sum_{t=2}^{n}\log\left(c_{2,1}\left(F_{t,1}^{(1)},F_{t-1,1}^{(1)},...,F_{1,t}^{(1)},F_{t,1}^{(2)},F_{t-1,1}^{(2)},...,F_{1,t}^{(2)}\right)\right),\qquad(8)$$

where $c_{2,1}$ denotes the density of a hierarchical Archimedean copula, which is obtained by differentiating the copula using the chain rule. More precisely, for a given diagonal t, we have the following expression:

$$c_{2,1}\left(\mathbf{u}\right)=\frac{\partial^{2t}C_{2,1}\left(\mathbf{u}\right)}{\partial u_{t-j+1,j}^{(1)}\cdots\partial u_{1,t}^{(1)}\partial u_{t-j+1,j}^{(2)}\cdots\partial u_{1,t}^{(2)}}.\qquad(9)$$

As we have 10 diagonals, we need to derive up to 20 times. We show an example of a 4-variables case, corresponding to the second diagonal in Appendix A. However, the density is computationally intensive in high dimensions when the number of observations in the diagonal increases, and a closed form expression for the maximum likelihood estimators is no longer available.

A numerically efficient way to evaluate the log-density is presented in ?, where an implementation of the hierarchical Clayton and Gumbel copulas is provided using the R package `copula`; see Hofert and Mächler (2011).

The simpler form of hierarchical dependence is to suppose a product copula between the two runoff triangles, meaning independence between lines of business. In this situation, the log-likelihood of the model is simply $L=L^{(1)}+L^{(2)}$, where $L^{(\ell)}$, $\ell=1,2$ is simply the log-likelihood obtained by (5). Of course, it is very easy to extend this model to more than two lines of business.

4.3 Empirical Illustration

Hierarchical models based on different copulas have been applied to the runoff triangles used in Section 3.2. For this model, the CY dependence has been modeled with four different copulas (product, Gumbel, Clayton and Gaussian). In our empirical study, we first use a model that supposes independence between lines of business, i.e. a product copula between runoff triangles. We call this model ICYD, for independence calendar year dependence. Fit statistics as long as dependence parameters of this model are shown in Table 3, while the estimated reserves are presented in Table 4. In terms of AIC, we observe that all Archimedean copula models offer a better fit than the multivariate Gaussian copula. Note that a CY dependence with a product copula within and between the two lines of business is simply a cell-by-cell modeling. The empirical results of this simple model have already been given in Section 3.2, for the PWD model with a product copula.

The two available copulas in the R package `copula`, which are Clayton and Gumbel, have been considered in a hierarchical model to investigate dependence between the two lines of business. The same copula is used for each level, meaning for example that if a Gumbel copula is chosen within a runoff triangle, then it is also used between the business lines. This is due to the convexity condition on hierarchical Archimedean copulas. We call this model HCYD, for hierarchical calendar year dependence. When we apply this model

	Copulа - Estimates and Standard Errors					
	Gaussian		**Clayton**		**Gumbel**	
$\theta_{1,1}$	0.6091	(0.1366)	2.2695	(0.4463)	2.7267	(0.6762)
$\theta_{1,2}$	0.7634	(0.0983)	2.9759	(0.5743)	2.7103	(0.6045)
Log-Lik.	391.5		403.9		404.3	
AIC	-699.0		-723.9		-724.6	

Table 3: Fit Statistics of ICYD model with Shi and Frees (2011) database

	Copula		
Reserves estimation	**Gaussian**	**Clayton**	**Gumbel**
Personal	6 175 574	6 425 748	6 965 466
Commercial	751 725	550 179	593 945
Total	6 927 299	6 975 927	7 559 412

Table 4: Reserves estimation of ICYD model with Shi and Frees (2011) database

to the dataset used in Section 3.2, the hierarchical model do not improve the independent calendar year model for the three copulas (Gaussian, Clayton and Gumbel). The values of the dependence parameters $\theta_{2,1}$ are not statistically significant, meaning that the two lines of business are uncorrelated.

To better emphasize the features of the hierarchical model, we work with two other runoff triangles that were recently used in Côté et al. (2015), which come from a Canadian property-casualty insurer. The two lines of business comprise personal and commercial auto insurance. The first triangle contains paid losses of the Accident Benefits (AB) coverage from Ontario, while the second one constitutes paid losses from Bodily Injuries (BI) coverage from the Western region. The Accident Benefits (AB) coverage provides compensation, regardless of fault, if driver, passengers, or pedestrians suffer injury or death in an automobile collision. On the other hand, the Bodily Injury (BI) coverage provides compensation to the insured if he is injured or killed through the fault of a motorist who has no insurance, or by an unidentified vehicle.

Côté et al. (2015) demonstrate that a gamma distribution provides a good fit for the two lines of business. We work with their conclusion and then continue with the same continuous distribution for each line of business. The cumulative paid losses and earned premiums for the two lines of business are displayed in Appendix B.

We first apply the PWD model to these two lines of business, the estimation parameters and the reserves estimation are shown in Table 5. Whereas, the fit statistics and the reserves obtained for the independent and hierarchical calendar year models are shown in Table 6. To compare the degree of dependence between different copulas, we also provide the two non-linear association measures Spearman's rho ρ_S and Kandall's tau τ_K for the two copulas, see Table 6. We notice that the Clayton copula captures a smaller dependence than the Gumbel copula, whose association measures are slightly higher. Indeed, the Clayton family is characterized by a lower tail dependence. Also, the hierarchical calendar year model offers a better fit than the independent calendar year model as shown by the values of the log-likelihood function. This finding leads to a statistically significant dependence between the two lines of business ($\theta_{2,1}$), captured through the calendar year effects. This is also confirmed by looking at the value of the AIC, which points to the Gumbel hierarchical copula model as the one which better adjusts the data.

Fit Statistics	Independence	Copula Frank	Gaussian
Dependence parameter	.	-0.6649 (0.9430)	0.0149 (0.1362)
Log-Likelihood	423.7	424.0	423.8
AIC	-767.4	-766.0	-765.6
Total Reserve	96 954	96 994	96 949

Table 5: Fit Statistics and Reserves of PWD model with Côté et al. (2015) database

When we incorporate a calendar year correlation within the lines of business (level 1), the residual dependence becomes positive. Intuitively, this can be explained by the trends and common effects that are detected with the introduction of the proposed dependence structure but not with the Chain-Ladder coefficients. In a given calendar year, exogenous common factors such as inflation, interest rates, jurisprudence or strategic decisions such as the acceleration of the payments for the entire portfolio can have simultaneous impacts on all lines of business of a given sector, such as the two lines of business considered in the present paper. These effects may as well result in trends in the development period parameters.

It is interesting to note that, unlike the slightly negative pairwise association obtained by the PWD model in Table 5 and also displayed for these two lines of business in Table 4 of Côté et al. (2015), hierarchical models generate positive dependence between loss triangles with the same dataset.

We observe that the positive parameter $\theta_{2,1}$ is statistically significant for the Clayton and Gumbel copulas. This results highlights the fact that the choice of the dependence structure

	ICYD model Clayton	Gumbel	HCYD model Clayton	Gumbel
$\theta_{1,1}$	0.0294 (0.0708)	1.0829 (0.1292)	0.0495 (0.0608)	1.0692 (0.0515)
$\theta_{1,2}$	0.2384 (0.1881)	1.1548 (0.1315)	0.2034 (0.2259)	1.0692 (0.0496)
$\theta_{2,1}$	.	.	0.0495	1.0692
ρ_S	.	.	0.0362	0.0948
τ_K	.	.	0.0241	0.0648
LogLik	424.4	425.8	426.3	427.7
AIC	-764.8	-767.6	-766.6	-769.4
Total Reserve	84 172	81 650	96 496	83 202

Table 6: Parameter and Reserves estimation of ICYD and HCYD models with Côté et al. (2015) database

can lead to different conclusions for the dependence analysis. This was also well illustrated in Figure 4 of Shi et al. (2012).

Finally, a hierarchical copula model requires a higher degree of dependence for variates linked at a lower level than those linked at a higher level. In our context, this means that the degree of dependence within lines of business should be greater than between lines of business, as illustrated in Figure 1. One can observe in Table 6 that this condition is respected with a dependence parameter $\theta_{2,1}$ lower than the dependence parameters $\theta_{1,1}$ and $\theta_{1,2}$. In this sense, this condition could also be seen as a restriction for the dependence parameter between the two lines of business. In fact, we observe that the parameters $\theta_{2,1}$ are on the boundary of their domain in Table 6. This actually could constitute a limitation of the hierarchical model.

5 Predictive distribution

In practice, actuaries are interested in knowing the uncertainty of the reserve. A parametric technique, the bootstrap, not only provides such information but most importantly lets one determine the entire predictive distribution, rarely obtained for non-Bayesian models. The predictive distribution notably allows assessment of risk capital for an insurance portfolio. Bootstrapping is also ideal from a practical point of view, because it avoids the complex theoretical calculations and can easily be implemented. Moreover, it tackles the potential model overfitting, typically encountered in loss reserving problems, due to the small sample size.

The bootstrap technique is increasingly popular in loss reserving, and allows a wide range of applications. It was first introduced in a loss reserving context with a distribution-free approach by Lowe (1994). For a multivariate loss reserving analysis, Kirschner et al. (2008) used a synchronized parametric bootstrap to model dependence between correlated lines of business, and Taylor and McGuire (2007) extended this result to a generalized linear model context. Shi and Frees (2011) and more recently Shi (2014) have also performed a parametric bootstrap to incorporate the uncertainty in parameter estimates, while modeling dependence between loss triangles using copulas.

5.1 Parametric Bootstrap

The parametric bootstrap allows us to obtain the whole distribution of the reserves. We follow the same bootstrap algorithm of Taylor and McGuire (2007), and summarized in Shi and Frees (2011).

5.1.1 Copula simulation

The first step of the parametric bootstrap is to generate pseudo-responses of normalized incremental paid losses $y_{ij,r}^{*(\ell)}$, for i, j such that $i + j \leq I$ and $\ell = 1, 2$. We know that $y_{ij,r}^{*(\ell)} = F^{(-1)(\ell)}(u_{ij}^{(\ell)}, \widehat{\mu}_{ij}^{(\ell)}, \widehat{\gamma}^{(\ell)})$, with $\widehat{\mu}_{ij}^{(\ell)}$ and $\widehat{\gamma}^{(\ell)}$ already estimated. Therefore, a technique to generate the realizations of the copula $u_{ij}^{(\ell)}$, with $\ell = 1, 2$ should be used.

Model	Copula reserve	Bootstrap reserve	Bias	Std Error
Gumbel hierarchical model	83 202	81 574	1.95%	8 555

Table 7: Bootstrap bias for the Gumbel HCYD model with Côté et al. (2015) database

Given that the Gumbel copula generates the best fit for many models in this paper, we have decided to focus on this copula for the bootstrap. Below, the bootstrap study is performed with the datasets of Côté et al. (2015).

To generate a multivariate Gumbel copula, we follow the method based on the inversion of the Laplace transform, an idea that can be traced back to Marshall and Olkin (1988).

The above cited algorithm allows us to generate the set of realizations $\mathbf{u}_{1,1}^{(1)}$ and $\mathbf{u}_{1,2}^{(2)}$ for the first level of hierarchy (CY level at $h = 1$) from the ordinary multivariate Archimedean copulas $C_{1,1}$ and $C_{1,2}$, for a given calendar year t and development period j ($j = 1, ..., t$), with $\mathbf{u}_{1,1}^{(1)} = (u_{t-j+1,j}^{(1)}, ..., u_{1,t}^{(1)})$ and $\mathbf{u}_{1,2}^{(2)} = (u_{t-j+1,j}^{(2)}, ..., u_{1,t}^{(2)})$. To generate realizations with a Gumbel copula at the highest level of the hierarchy (line of business level at $h = 2$), we used the sampling algorithm of Nested Archimedean copulas from the R package `copula`.

Consequently, we have obtained the set of realizations $\mathbf{u}_{2,1}^{(1)}$ and $\mathbf{u}_{2,1}^{(2)}$ for the second level of hierarchy (business line level at $h = 2$) from the hierarchical Archimedean copula $C_{2,1}$.

5.1.2 Bias and MLE

The maximum likelihood estimation technique is known to be asymptotically unbiased. In practice, we work with a finite number of observations, particularly with runoff triangles. Indeed, in our empirical illustrations, only 55 observations have been used in each triangle. Consequently, regardless of the number of simulations, our estimation is done each time on limited datasets of 55 observations.

The impact of the bias on the estimation has been analyzed. Recently, the lognormal MLE bias has been studied in Johnson et al. (2011), along with the gamma and Weibull distributions. Consequently, inter alia, a bias is necessarily observed in the bootstrapping procedure. In our empirical illustration, the bootstrap bias obtained for the hierarchical model is exhibited in Table 7.

5.2 Reserve indications

We show a histogram of the reserve distribution of the hierarchical model in Figure 2, which is important and useful for actuaries when they want to select a reserve at a desired level of conservatism.

In Table 7, we exhibit the bootstrap results for the Gumbel hierarchical model which mainly refers to the mean reserve and the prediction uncertainty. The latter may substantially be increased by the introduction of the accounting year dependence. In contrast, PWD models can under-estimate the variability because they implicitly assume an independence between accident years. This was also stated in Wüthrich et al. (2013), where it has been shown that the CY modeling is more performant than the PWD modeling. It is worth mentioning that to compute the mean squared error of prediction, the process uncertainty must be added to this prediction error (see England and Verrall (2002)).

Note that to obtain the lower triangle in the Bootstrap procedure, we can either calculate the projected mean for each cell of the lower triangle, as shown in this paper (projected mean approach), or generating (by simulation) each cell of the lower triangle starting from the new estimates obtained for each bootstrap sample. The second approach (the simulation based approach) offers a wider range of possible reserves, and will consequently have a larger standard error. This second approach can be particularly interesting from a capital risk standpoint where extreme loss events have to be considered. Both bootstrap approaches (projected mean approach and simulation based approach) are relevant information for property-casualty insurers.

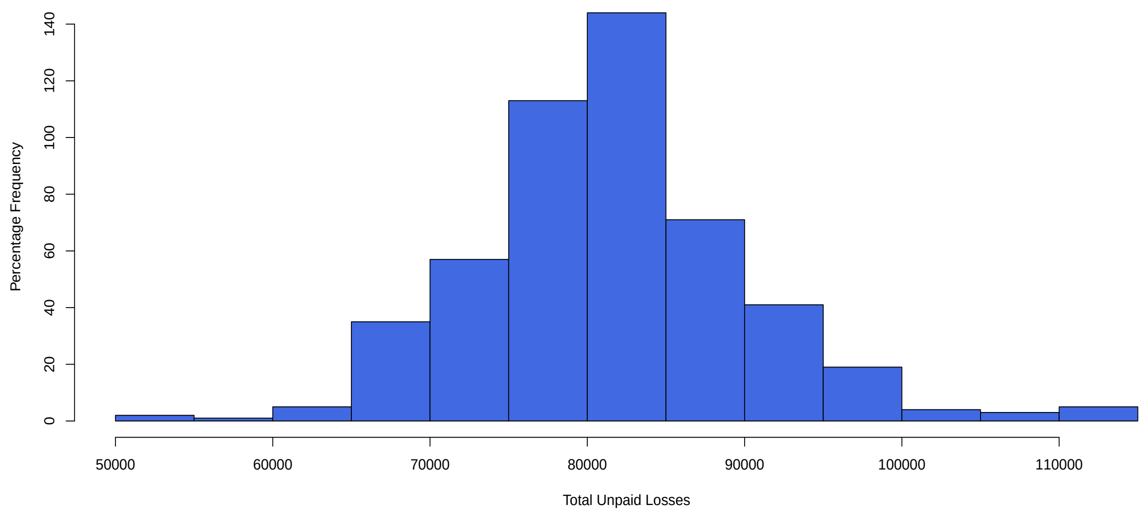

Figure 2: Predictive distribution of total unpaid losses - Complete hierarchical model

6 Conclusion

In this paper, we have studied different approaches to model the dependence between loss triangles using multivariate copulas. If losses in different lines of business are correlated, aggregate reserves must reflect this dependence. To allow a complex dependence relation, we propose the use of new models using hierarchial Archimedean copulas. To illustrate the model, an empirical illustration was performed using the same data as the one used by Shi and Frees (2011). Based on the AIC, we show that the ICYD models provide a better fit than PWD models. Furthermore, to show the interest of HCYD models and better highlight their properties, the empirical illustration has also been performed on two other runoff triangles from a major Canadian insurance company, which also allows us to expose the proposed model to a wider range of situations. A hierarchical calendar year dependence seemed to be relevant because the hierarchical Gumbel copula model was one of the best to adjust the data.

With the proposed models, we can derive analytically the value of the reserve. However, to obtain the distribution of the reserve and to estimate the parameters, numerical evaluation is necessary. Indeed, estimation and sampling are implemented in the `R` package `copula`. Also, the total reserve estimate in the presence of dependence relies heavily on the choice of the dependence structure and the selected copula. This is a limitation of the joint estimation of the marginal and dependence parameters. This undesirable effect will be addressed in a future work within a two-stage inference strategy; see Côté et al. (2015) for more details.

These new models that use hierarchical copula theory constitute a new way to model the dependence structures of runoff triangles. Those models are promising tools to better take into account dependencies within and between business lines. Indeed, this approach can easily be generalized to more than two lines of business because hierarchical Archimedean copulas are flexible and allow more refined possible dependence constructions. Because of their flexibility, hierarchical copula models should also be considered in other areas of actuarial science.

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A Nested Archimedean Copula Density

To lighten the notation, let $C_\theta^{(i,j)}(u,v) = \dfrac{\partial^{i+j} C_\theta(u,v)}{\partial u^i \partial v^j}$ for $i, j \in \{0, 1, 2\}$.

Following equation (9), the 4-dimensional density of the hierarchical Archimedean copula $C_{2,1}$ for the second diagonal ($t = 2$) will be written as follows:

$$
\begin{aligned}
c_{2,1}\left(u_1,u_2,u_3,u_4\right) &= \frac{\partial^4}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} C_{2,1}\left(C_{1,1}\left(u_1,u_2\right), C_{1,2}\left(u_3,u_4\right)\right) \\
&= \frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} C_{2,1}^{(0,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(0,1)}(u_3,u_4) \\
&= \frac{\partial^2}{\partial u_1 \partial u_2} [C_{2,1}^{(0,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(0,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(1,1)}(u_3,u_4)] \\
&= \frac{\partial}{\partial u_1} [C_{2,1}^{(1,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4)] \\
&= C_{2,1}^{(2,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,0)}(u_1,u_2) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(2,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,0)}(u_1,u_2) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4)
\end{aligned}
$$

B Data

Table 8: Cumulative paid losses for Ontario AB.

Accident	Development Lag (in months)									
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	3043	5656	7505	8593	9403	10380	10450	10812	10856	10860	116491
2004	2070	4662	6690	8253	9286	9724	9942	10086	10121		111467
2005	2001	4825	7344	8918	9824	10274	10934	11155			107241
2006	1833	4953	7737	9524	10986	11267	11579				105687
2007	2217	5570	7898	8885	9424	10402					105923
2008	2076	5681	8577	10237	12934						111487
2009	2025	6225	9027	10945							113268
2010	2024	5888	8196								121606
2011	1311	3780									110610
2012	912										104304

Table 9: Cumulative paid losses for West BI.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	2279	8683	15136	21603	27650	30428	32004	32592	33009	34140	76620
2004	2139	7077	13159	16435	20416	22598	24171	25034	25714		65691
2005	1420	4888	8762	12184	14482	15633	17089	17710			55453
2006	1510	5027	10763	15799	19269	22504	24807				54006
2007	1693	5175	8216	12263	16918	20792					55425
2008	2097	7509	10810	15673	19791						59100
2009	2094	5174	8062	12389							54438
2010	1487	4789	7448								53483
2011	1868	6196									52978
2012	2080										57879