Neutrino Mixing and Leptogenesis in $\mu-\tau$ Symmetry

E. I. Lashin[1,2,3]*, **N. Chamoun**[4,5]†, **C. Hamzaoui**[6]‡ **and S. Nasri**[7,8]§

[1] Ain Shams University, Faculty of Science, Cairo 11566, Egypt.

[2] Centre for Theoretical Physics, Zewail City of Science & Technology, Sheikh Zayed, 6 October City, 12588, Giza, Egypt.

[3] The Abdus Salam ICTP, P.O. Box 586, 34100 Trieste, Italy.

[4] Physics Department, HIAST, P.O.Box 31983, Damascus, Syria.

[5] Physikalisches Institut der Universität Bonn, Nußalle 12, D-53115 Bonn, Germany.

[6] Groupe de Physique Théorique des Particules, Département des Sciences de la Terre et de L'Atmosphère, Université du Québec à Montréal, Case Postale 8888, Succ. Centre-Ville, Montréal, Québec, Canada, H3C 3P8.

[7] Department of Physics, UAE University, P.O.Box 17551, Al-Ain, United Arab Emirates.

[8] Laboratoire de Physique Théorique, ES-SENIA University, DZ-31000 Oran, Algeria.

May 22, 2015

Abstract

We study the consequences of the Z_2-symmetry behind the μ–τ universality in neutrino mass matrix. We then implement this symmetry in the type-I seesaw mechanism and show how it can accommodate all sorts of lepton mass hierarchies and generate enough lepton asymmetry to interpret the observed baryon asymmetry in the universe. We also show how a specific form of a high-scale perturbation is kept when translated via the seesaw into the low scale domain, where it can accommodate the neutrino mixing data. We finally present a realization of the high scale perturbed texture through addition of matter and extra exact symmetries.

Keywords: Neutrino Physics; Flavor Symmetry; Matter-anti-matter

PACS numbers: 14.60.Pq; 11.30.Hv; 98.80.Cq

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1 Introduction

Flavor symmetry is commonly used in model building seeking to determine the nine free parameters characterizing the effective neutrino mass matrix M_ν, namely the three masses (m_1, m_2 and m_3), the three mixing angles (θ_{23}, θ_{12} and θ_{13}), the two Majorana-type phases (ρ and σ) and the Dirac-type phase (δ). Incorporating family symmetry at the Lagrangian level leads generally to textures of specific forms, and one may then study whether or not these specific textures can accommodate the experimental data involving the above mentioned parameters ([1] and references therein). The recent observation of a non-zero value for θ_{13} from the T2K[2], MINOS[3], and Double Chooz[4] experiments puts constraints on models based on flavor symmetry (see Table 1 where the most recent updated neutrino oscillation parameters are taken from [5]). In this regard, recent, particularly simple, choices for discrete and continuous flavor symmetry addressing the non-vanishing θ_{13} question were respectively worked out ([6] and references therein). The μ–τ symmetry [7, 8] is enjoyed by many popular mixing patterns such as tri-bimaximal mixing (TBM) [9], bimaximal mixing (BM) [10], hexagonal mixing (HM) [11] and scenarios of A_5 mixing [12], and it was largely studied in the literature [13]. Any form of the neutrino mass matrix respects a $(Z_2)^2$ symmetry [14], and we can define the μ–τ symmetry by fixing one of the two Z_2's to

*slashin@zewailcity.edu.eg, elashin@ictp.it

†nchamoun@th.physik.uni-bonn.de

‡hamzaoui.cherif@uqam.ca

§snasri@uaeu.ac.ae

express exchange between the second and third families, whereas the second Z_2 factor is to be determined later by data or, equivalently, by M_ν parameters. The whole $(Z_2)^2$ symmetry might turn out to be a subgroup of a larger discrete group imposed on the whole leptonic sector. In realizing μ–τ symmetry we have two choices namely (S_-, S_+, as explained later), and thus we have two textures corresponding to μ–τ symmetry. It is known that both of these textures lead to a vanishing θ_{13} (with S_- achieving this in a less natural way), and thus perturbations are needed to get remedy of this situation[15]. In [16] we studied the perturbed μ–τ neutrino symmetry and found the four patterns, obtained by disentangling the effects of the perturbations, to be phenomenologically viable.

Table 1: Allowed 3σ-ranges for the neutrino oscillation parameters, mixing angles and mass-square differences, taken from the global fit to neutrino oscillation data [5]. The quantities δm^2 and Δm^2 are respectively defined as $m_2^2 - m_1^2$ and $m_3^2 - \left(m_1^2 + m_2^2\right)/2$, whereas R_ν denotes the phenomenologically important quantity $\frac{\delta m^2}{|\Delta m^2|}$. Normal and Inverted Hierarchies are respectively denoted by NH and IH.

Parameter	Best fit	3σ range
$\delta m^2\left(10^{-5}\mathrm{eV}^2\right)$	7.54	$6.99 - 8.18$
$\left\|\Delta m^2\right\|\left(10^{-3}\mathrm{eV}^2\right)$ (NH)	2.43	$2.23 - 2.61$
$\left\|\Delta m^2\right\|\left(10^{-3}\mathrm{eV}^2\right)$ (IH)	2.38	$2.19 - 2.56$
R_ν (NH)	0.0310	$0.0268 - 0.0367$
R_ν (IH)	0.0317	$0.0273 - 0.0374$
θ_{12} (NH or IH)	33.71^0	$30.59^0 - 36.80^0$
θ_{13} (NH)	8.80^0	$7.62^0 - 9.89^0$
θ_{13} (IH)	8.91^0	$7.67^0 - 9.94^0$
θ_{23} (NH)	41.38^0	$37.69^0 - 52.30^0$
θ_{23} (IH)	38.07^0	$38.07^0 - 53.19^0$

In this work, we re-examine the question of exact μ–τ symmetry and implement it in a complete setup of the leptonic sector. Then, within type-I seesaw scenarios, we show the ability of exact symmetry to accommodate lepton mass hierarchies. Upon studying its effect on leptogenesis we find, in contrast to other symmetries studied in [6] and [17] that it can account for it. The reason behind this fact is that fixing just one Z_2 in μ–τ symmetry leaves one mixing angle free which can be adjusted differently in the Majorana and Dirac neutrino mass matrices (M_R and M_D), thus allowing for different diagonalizing matrices. For the mixing angles and in order to accommodate data, we introduce perturbations at the seesaw high scale and study their propagations into the low scale effective neutrino mass matrix. As in [16], we consider that the perturbed texture arising at the high scale keeps its form upon RG running which, in accordance with [20], does not affect the results in many setups. As to the origin of the perturbations, we shall not introduce explicitly symmetry breaking terms into the Lagrangian [21], but rather follow [16], and enlarge the symmetry with extra matter and then spontaneously break the symmetry by giving vacuum expectation values (vev) to the involved Higgs fields.

The plan of the paper is as follows. In Section 2, we review the standard notation for the neutrino mass matrix and the definition of the μ–τ symmetry. In Section 3 and 4, we introduce the two textures realizing the μ–τ symmetry through S_- and S_+ respectively. We then specify our analysis to the latter case (S_+), and in Section 5 we introduce the type-I seesaw scenario. We address the charged lepton sector in Subsection 5.1, whereas we study the different neutrino mass hierarchies in Subsection 5.2, and in Subsection 5.3, we study the generation of lepton asymmetry. Sections 6 and 7 examine the possible consequences for one particular possible deviation from the exact μ–τ symmetry, where we present the analytical study in the former section, while the numerical study is given in the latter section. In Section 8 we present a theoretical realization of the perturbed texture. We end by discussion and summary in Section 9.

2 Notations and preliminaries

In the Standard Model (SM) of particle interactions, there are 3 lepton families. The charged-lepton mass matrix linking left-handed (LH) to their right-handed (RH) counterparts is arbitrary, but can always be diagonalized by a bi-unitary transformation:

$$V_L^l\ M_l\ (V_R^l)^\dagger = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{1}$$

Likewise, we can diagonalize the symmetric Majorana neutrino mass matrix by just one unitary transformation:

$$V^{\nu\dagger} M_\nu\ V^{\nu *}\ = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{2}$$

with m_i (for $i = 1, 2, 3$) real and positive.

The observed neutrino mixing matrix comes from the mismatch between V^l and V^ν in that

$$V_{\text{PMNS}} \quad = \quad (V_L^l)^\dagger\ V^\nu. \tag{3}$$

If the charged lepton mass eigen states are the same as the current (gauge) eigen states, then $V_L^l = \mathbf{1}$ (the unity matrix) and the measured mixing comes only from the neutrinos $V_{\text{PMNS}} = V^\nu$. We shall assume this saying that we are working in the "flavor" basis. As we shall see, corrections due to $V_L^l \neq \mathbf{1}$ are expected to be of order of ratios of the hierarchical charged lepton masses, which are small enough to justify our assumption of working in the flavor basis. However, one can treat these corrections as small perturbations and embark on a phenomenological analysis involving them [21].

We shall adopt the parametrization of [22], related to other ones by simple relations [1], where the V_{PMNS} is given in terms of three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and three phases (δ, ρ, σ), as follows.

$$\begin{aligned}
P &= \text{diag}\left(e^{i\rho}, e^{i\sigma}, 1\right), \\
U &= R_{23}\left(\theta_{23}\right)\ R_{13}\left(\theta_{13}\right)\ \text{diag}\left(1, e^{-i\delta}, 1\right)\ R_{12}\left(\theta_{12}\right) \\
&= \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13} \\ -c_{12}\, s_{23}\, s_{13} - s_{12}\, c_{23}\, e^{-i\delta} & -s_{12}\, s_{23}\, s_{13} + c_{12}\, c_{23}\, e^{-i\delta} & s_{23}\, c_{13} \\ -c_{12}\, c_{23}\, s_{13} + s_{12}\, s_{23}\, e^{-i\delta} & -s_{12}\, c_{23}\, s_{13} - c_{12}\, s_{23}\, e^{-i\delta} & c_{23}\, c_{13} \end{pmatrix}, \\
V_{\text{PMNS}} &= U\ P = \begin{pmatrix} c_{12}\, c_{13} e^{i\rho} & s_{12}\, c_{13} e^{i\sigma} & s_{13} \\ (-c_{12}\, s_{23}\, s_{13} - s_{12}\, c_{23}\, e^{-i\delta}) e^{i\rho} & (-s_{12}\, s_{23}\, s_{13} + c_{12}\, c_{23}\, e^{-i\delta}) e^{i\sigma} & s_{23}\, c_{13} \\ (-c_{12}\, c_{23}\, s_{13} + s_{12}\, s_{23}\, e^{-i\delta}) e^{i\rho} & (-s_{12}\, c_{23}\, s_{13} - c_{12}\, s_{23}\, e^{-i\delta}) e^{i\sigma} & c_{23}\, c_{13} \end{pmatrix},
\end{aligned} \tag{4}$$

where $R_{ij}\left(\theta_{ij}\right)$ is the rotation matrix in the (i, j)-plane by angle θ_{ij}, and $s_{12} \equiv \sin\theta_{12} \ldots$. Note that in this adopted parametrization, the third column of V_{PMNS} is real.

In this parametrization, and in the flavor basis, the neutrino mass matrix elements are given by:

$$\begin{aligned}
M_{\nu\,11} &= m_1\, c_{12}^2\, c_{13}^2\, e^{2\,i\,\rho} + m_2\, s_{12}^2\, c_{13}^2\, e^{2\,i\,\sigma} + m_3\, s_{13}^2, \\
M_{\nu\,12} &= m_1\, \left(-c_{13}\, s_{13}\, c_{12}^2\, s_{23} e^{2\,i\,\rho} - c_{13}\, c_{12} s_{12}\, c_{23}\, e^{i\,(2\,\rho-\delta)}\right) \\
&\quad + m_2\, \left(-c_{13}\, s_{13}\, s_{12}^2\, s_{23} e^{2\,i\,\sigma} + c_{13}\, c_{12}\, s_{12}\, c_{23}\, e^{i\,(2\,\sigma-\delta)}\right) + m_3\, c_{13}\, s_{13}\, s_{23}, \\
M_{\nu\,13} &= m_1\, \left(-c_{13}\, s_{13}\, c_{12}^2\, c_{23}\, e^{2\,i\,\rho} + c_{13}\, c_{12}\, s_{12}\, s_{23}\, e^{i\,(2\,\rho-\delta)}\right) \\
&\quad + m_2\, \left(-c_{13}\, s_{13}\, s_{12}^2\, c_{23}\, e^{2\,i\,\sigma} - c_{13}\, c_{12}\, s_{12}\, s_{23}\, e^{i\,(2\,\sigma-\delta)}\right) + m_3\, c_{13}\, s_{13}\, c_{23}, \\
M_{\nu\,22} &= m_1\, \left(c_{12}\, s_{13}\, s_{23}\, e^{i\,\rho} + c_{23}\, s_{12}\, e^{i\,(\rho-\delta)}\right)^2 \\
&\quad + m_2\, \left(s_{12}\, s_{13}\, s_{23}\, e^{i\,\sigma} - c_{23}\, c_{12}\, e^{i\,(\sigma-\delta)}\right)^2 + m_3\, c_{13}^2\, s_{23}^2, \\
M_{\nu\,33} &= m_1\, \left(c_{12}\, s_{13}\, c_{23}\, e^{i\,\rho} - s_{23}\, s_{12}\, e^{i\,(\rho-\delta)}\right)^2
\end{aligned}$$

$$
\begin{aligned}
& +m_2\left(s_{12}\, s_{13}\, c_{23}\, e^{i\,\sigma}+s_{23}\, c_{12}\, e^{i\,(\sigma-\delta)}\right)^2+m_3\, c_{13}^2\, c_{23}^2, \\
M_{\nu\, 23} \quad = \quad & m_1\left(c_{12}^2\, c_{23}\, s_{23}\, s_{13}^2\, e^{2\,i\,\rho}+s_{13}\, c_{12}\, s_{12}\left(c_{23}^2-s_{23}^2\right) e^{i\,(2\,\rho-\delta)}-c_{23}\, s_{23}\, s_{12}^2\, e^{2\,i\,(\rho-\delta)}\right) \\
& +m_2\left(s_{12}^2\, c_{23}\, s_{23}\, s_{13}^2\, e^{2\,i\,\sigma}+s_{13}\, c_{12}\, s_{12}\left(s_{23}^2-c_{23}^2\right) e^{i\,(2\,\sigma-\delta)}-c_{23}\, s_{23}\, c_{12}^2\, e^{2\,i\,(\sigma-\delta)}\right) \\
& +m_3\, s_{23}\, c_{23}\, c_{13}^2. \qquad (5)
\end{aligned}
$$

This helps in viewing directly at the level of the mass matrix that the effect of swapping the indices 2 and 3 corresponds to the transformation $\theta_{23} \to \frac{\pi}{2} - \theta_{23}$ and $\delta \to \delta \pm \pi$. Hence, for a texture satisfying the μ-τ symmetry, one can check the correctness of any obtained formula by requesting it to be invariant under the above transformation.

As said before, any form of M_ν satisfies a Z_2^2-symmetry. This means that there are two commuting unitary Z_2-matrices (squared to unity) (S_1, S_2) which leave M_ν invariant:

$$S^T\, M_\nu\, S = M_\nu \qquad (6)$$

For a non-degenrate mass spectrum, the form of the Z_2-matrix S is given by [17]:

$$S \quad = \quad V^\nu\ \mathrm{diag}(\pm 1, \pm 1, \pm 1)\, V^{\nu\,\dagger} \qquad (7)$$

where the two S's correspond to having, in diag$(\pm 1, \pm 1, \pm 1)$, two pluses and one minus, the position of which differs in the two S's (the third Z_2-matrix, corresponding to the third position of the minus sign, is generated by multiplying the two S's and noting that the form invariance formula Eq.(6) is invariant under $S \to -S$).

In practice, however, we follow a reversed path, in that if we assume a 'real' orthogonal Z_2-matrix (and hence symmetric with eigenvalues ± 1) satisfying Eq.(6), then it commutes with M_ν, and so both matrices can be simultaneously diagonalized. Quite often, the form of S is simpler than M_ν, so one proceeds to solve the eigensystem problem for S, and find a unitary diagonalizing matrix $\tilde{U}$:

$$\tilde{U}^\dagger\, S\, \tilde{U} \quad = \quad \mathrm{Diag}\,(\pm 1, \pm 1, \pm 1) \qquad (8)$$

The conjugate matrix $\tilde{U}^*$ can 'commonly' be identified with, or related simply to, the matrix V satisfying Eq.(2)*. In this case, and in the flavor basis, the V_{PMNS} would be generally complex and equal to the one presented in Eq.(4). Determining the eigenvectors of the S matrices helps thus to determine the neutrino mixing and phase angles.

The μ–τ symmetry is defined when one of the two Z_2-matrices corresponds to switching between the 2^{nd} and the 3^{rd} families. We have, up to a global irrelevant minus sign (see again Eq.6), two choices, which would lead to two textures at the level of M_ν.

3 The μ–τ symmetry manifested through S_-: ($M_{\nu\,12} = M_{\nu\,13}$ and $M_{\nu\,22} = M_{\nu\,33}$)

The Z_2-symmetry matrix is given by:

$$S_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (9)$$

The invariance of M_ν under S_- (Eq.6) forces the symmetric matrix M_ν to have a texture of the form:

$$M_\nu \quad = \quad \begin{pmatrix} A_\nu & B_\nu & B_\nu \\ B_\nu & C_\nu & D_\nu \\ B_\nu & D_\nu & C_\nu \end{pmatrix}. \qquad (10)$$

*In fact, as we shall see, starting from the general form of $\tilde{U}$ satisfying Eq.(8), one can determine (up to a diagonal phase matrix) the unitary matrix $\tilde{U}_0$ which diagonalizes simultaneously the two commuting hermitian matrices S and $M_\nu^* M_\nu$ so that $\tilde{U}_0^\dagger\ M_\nu^* M_\nu\ \tilde{U}_0 = \mathrm{Diag}\left(m_1^2, m_2^2, m_3^2\right) = D^2$. One can show then that D^2 commutes with $\tilde{U}_0^T M_\nu \tilde{U}_0$ which leads to the latter matrix being diagonal. Fixing now the phases so that the latter diagonal matrix becomes real makes $\tilde{U}_0$ play the role of V^* in Eq. (2). One then can use the freedom in rephasing the charged lepton fields to force the adopted parametrization on V_{PMNS}.

The invariance of M_ν under S_- implies that S_- commutes with both M_ν and M_ν^*, and thus also with the hermitian positive matrices $M_\nu^* M_\nu$ and $M_\nu M_\nu^*$. One can easily find the general form of the diagonalizing unitary matrix of S_- (up to an arbitrary diagonal phase matrix). The matrix S_- has normalized eigen vectors: $\left\{ v_1 = \left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)^{\mathrm{T}}, v_2 = (1, 0, 0)^{\mathrm{T}}, v_3 = \left(0, 1/\sqrt{2}, -1/\sqrt{2}\right)^{\mathrm{T}} \right\}$ corresponding respectively to the eigenvalues $(1, 1, -1)$. Since the eigenvalue 1 is two-fold degenerate, then there is still freedom for a unitary transformation defined by an angle φ and phase ξ in its eigenspace to get the new eigen vectors in the following form:

$$\begin{aligned} \overline{v}_1 &= -s_\varphi\, e^{i\,\xi}\, v_1 + c_\varphi\, v_2, \\ \overline{v}_2 &= c_\varphi\, e^{i\,\xi}\, v_1 + s_\varphi\, v_2, \end{aligned} \tag{11}$$

We have three choices as to how we order the eigenvectors forming the diagonalizing matrix U, and we chose the one which would lead to "plausible" mixing angles falling in the first quadrant. This choice for ordering the eigenvalues turns out to be $(1, -1, 1)$, as we could check that the two choices corresponding to the other two positions for the eigenvalue (-1) lead upon identification with V_{PMNS} in Eq.(4) to some mixing angles lying outside the first quadrant, and the matrix U_- which diagonalizes S_- can be cast into the form:

$$U_- = [\overline{v}_1, v_3, \overline{v}_2] = \begin{pmatrix} c_\varphi & 0 & s_\varphi \\ -s_\varphi\, e^{i\,\xi}/\sqrt{2} & 1/\sqrt{2} & c_\varphi\, e^{i\,\xi}/\sqrt{2} \\ -s_\varphi\, e^{i\,\xi}/\sqrt{2} & -1/\sqrt{2} & c_\varphi\, e^{i\,\xi}/\sqrt{2} \end{pmatrix}. \tag{12}$$

One can single out of this general form the unitary matrix which diagonalizes also the hermitian positive matrix $M_\nu^*\, M_\nu$ with different positive eigenvalues. In order to simplify the resulting formulas, the matrix $M_\nu^*\, M_\nu$ can be organized in a concise form as,

$$M_\nu^*\, M_\nu = \begin{pmatrix} a_\nu & b_\nu & b_\nu \\ b_\nu^* & c_\nu & d_\nu \\ b_\nu^* & d_\nu & c_\nu \end{pmatrix}, \tag{13}$$

where a_ν , b_ν, c_ν and d_ν are defined as follows,

$$\begin{aligned} a_\nu = |A_\nu|^2 + 2\,|B_\nu|^2\,, \qquad & b_\nu = A_\nu^*\, B_\nu + B_\nu^*\, C_\nu + B_\nu^*\, D_\nu, \\ c_\nu = |A_\nu|^2 + |B_\nu|^2 + |C_\nu|^2\,, \qquad & d_\nu = |B_\nu|^2 + C_\nu^*\, D_\nu + D_\nu^*\, C_\nu. \end{aligned} \tag{14}$$

The diagonalization of $M_\nu^*\, M_\nu$ through U_- fixes φ and ξ to be:

$$\tan(2\varphi) = \frac{2\sqrt{2}\;|b_\nu|}{c_\nu + d_\nu - a_\nu}, \qquad \xi = \mathrm{Arg}\,(b_\nu^*)\,. \tag{15}$$

Now and after having fixed φ and ξ we have,

$$U_-^\dagger\, M_\nu^*\, M_\nu U_- = U_-^T\, M_\nu\, M_\nu^*\, U_-^* = \mathrm{Diag}\left(m_1^2, m_2^2, m_3^2\right), \tag{16}$$

where

$$\begin{aligned} m_1^2 &= \frac{a_\nu + c_\nu + d_\nu}{2} + \frac{1}{2}\sqrt{(a_\nu - d_\nu - c_\nu)^2 + 8\,|b_\nu|^2}, \\ m_2^2 &= c_\nu - d_\nu, \\ m_3^2 &= \frac{a_\nu + c_\nu + d_\nu}{2} - \frac{1}{2}\sqrt{(a_\nu - d_\nu - c_\nu)^2 + 8\,|b_\nu|^2}. \end{aligned} \tag{17}$$

The above relations imply directly that $U_-^T\, M_\nu\, U_-$ commutes with $(U_-^T\, M_\nu\, U_-)^*$, and hence also with the product of these two matrices which is a diagonal matrix: $U_-^T\, M_\nu\, U_-(U_-^T\, M_\nu\, U_-)^* = U_-^T\, M_\nu\, M_\nu^*\, U_-^*$. Since we have a non-degenrate spectrum amounting to different eigenvalues of $M_\nu\, M_\nu^*$, we deduce directly that $U_-^T\, M_\nu\, U_-$ is diagonal. Actually we get:

$$U_-^T\, M_\nu\, U_- = M_\nu^{\mathrm{Diag}}, \tag{18}$$

where $M_\nu^{\rm Diag}$ is a diagonal matrix whose entries are,

$$
\begin{aligned}
M^{\rm Diag}_{\nu\,11} &= A_\nu\, c^2_\varphi - \sqrt{2}\, s_{2\varphi}\, e^{i\,\xi}\, B_\nu + \left(C_\nu + D_\nu\right)\, s^2_\varphi\, e^{2\,i\,\xi}, \\
M^{\rm Diag}_{\nu\,22} &= C_\nu - D_\nu, \\
M^{\rm Diag}_{\nu\,33} &= A_\nu\, s^2_\varphi + \sqrt{2}\, s_{2\varphi}\, e^{i\,\xi}\, B_\nu + \left(C_\nu + D_\nu\right)\, c^2_\varphi\, e^{2\,i\,\xi}.
\end{aligned} \tag{19}
$$

In order to extract the mixing and phase angles, we use the freedom of multiplying U_- by a diagonal phase matrix $Q = \text{Diag}\left(e^{-ip_1}, e^{-ip_2}, e^{-ip_3}\right)$ to ensure real positive eigenvalues for the mass matrix M_ν such that

$$
(U_-\;\; Q)^T M_\nu (U_-\;\; Q) \;\; = \;\; \text{Diag}\left(m_1, m_2, m_3\right), \tag{20}
$$

and we find that we should take

$$
p_i = \frac{1}{2}\,\text{Arg}(M^{\rm Diag}_{\nu_{ii}}), \quad i = 1, 2, 3. \tag{21}
$$

However, we get now the following form for the diagonalizing matrix $U_-\;\; Q$:

$$
U_-\;\; Q \;\; = \;\; \begin{pmatrix} c_\phi\, e^{-ip_1} & 0 & s_\phi e^{-ip_3} \\ -\frac{1}{\sqrt{2}} s_\phi e^{i(\xi - p_1)} & \frac{1}{\sqrt{2}} e^{-ip_2} & \frac{1}{\sqrt{2}} c_\phi e^{i(\xi - p_3)} \\ -\frac{1}{\sqrt{2}} s_\phi e^{i(\xi - p_1)} & -\frac{1}{\sqrt{2}} e^{-ip_2} & \frac{1}{\sqrt{2}} c_\phi e^{i(\xi - p_3)} \end{pmatrix}, \tag{22}
$$

In order to have the conjugate of this matrix in the same form as the adopted parametrization of $V_{\rm PMNS}$ in Eq.(4), where the third column is real, we can make a phase change in the charged lepton fields:

$$
e \rightarrow e^{-ip_3}\; e,\;\; \mu \rightarrow e^{i(\xi - p_3)}\; \mu,\;\; \tau \rightarrow e^{i(\xi - p_3)}\; \tau \tag{23}
$$

so that we identify now the mixing and phase angles and see that the μ–τ symmetry forces the following angles:

$$
\begin{aligned}
&\theta_{23} = \pi/4, \;\; \theta_{12} = 0, \;\; \theta_{13} = \varphi, \\
&\rho = \frac{1}{2}\,\text{Arg}\left(M^{\rm Diag}_{\nu\,11}\, M^{{\rm Diag}\,*}_{\nu\,33}\right), \;\; \sigma = \frac{1}{2}\,\text{Arg}\left(M^{\rm Diag}_{\nu\,22}\, M^{{\rm Diag}\,*}_{\nu\,33}\right), \;\; \delta = 2\,\pi - \xi.
\end{aligned} \tag{24}
$$

We can get, as phenomenology suggests, a small value for θ_{13} assuming

$$
|b_\nu| \quad \ll \quad |c_\nu + d_\nu - a_\nu|\,. \tag{25}
$$

and then the mass spectrum turns out to be:

$$
m_1^2 \approx a_\nu, \quad m_2^2 = c_\nu - d_\nu, \quad m_3^2 \approx c_\nu + d_\nu \tag{26}
$$

Inverting these relations to express the mass parameters in terms of the mass eigenvalues we get these simple direct relations,

$$
a_\nu \approx m_1^2, \quad c_\nu \approx \tfrac{m_2^2 + m_3^2}{2}, \quad d_\nu \approx \frac{m_3^2 - m_2^2}{2}. \tag{27}
$$

It is remarkable that all kinds of mass spectra can be accommodated by properly adjusting the parameters a_ν, c_ν, and d_ν according to the relations in Eq.(27). As to the mixing angles, we see that the value of θ_{23} is phenomenologically acceptable corresponding to maximal atmospheric mixing, and the parameter b_ν can be adjusted according to Eq.(25) to accommodate the small mixing angle θ_{13}. The phases are not of much concern because so far there is no serious constraint on phases. It seems that all things fit properly except the vanishing value of the mixing angle θ_{12} which is far from its experimental value $\simeq 33.7^o$.

One might argue that this symmetry pattern S_- might be viable phenomenologically if we adopt an alternative choice of ordering its eigenvalues and use the phase ambiguity to put all mixing angles in the first quadrant. We have not done this, but rather we prefer to find a phenomenologically viable symmetry leading directly to mixing angles in the first quadrant. This can be carried out in the second texture expressing the μ–τ symmetry materialized through S_+.

4 The μ–τ symmetry manifested through S_+: $(M_{\nu\,12} = -M_{\nu\,13}$ and $M_{\nu\,22} = M_{\nu\,33})$

The Z_2-symmetry matrix is given by:

$$S_+ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{28}$$

The invariance of M_ν under S_+ (Eq. 6) forces the symmetric matrix M_ν to have a texture of the form:

$$M_\nu \;=\; \begin{pmatrix} A_\nu & B_\nu & -B_\nu \\ B_\nu & C_\nu & D_\nu \\ -B_\nu & D_\nu & C_\nu \end{pmatrix} \tag{29}$$

As before, S_+ commutes with M_ν, M_ν^* and thus also with $M_\nu^* M_\nu$ and $M_\nu M_\nu^*$. The normalized eigen vectors of S_+ are: $\left\{ v_1 = \left(0, -1/\sqrt{2}, 1/\sqrt{2}\right)^{\mathrm{T}},\ v_2 = (1,0,0)^{\mathrm{T}},\ v_3 = \left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)^{\mathrm{T}} \right\}$ corresponding respectively to the eigenvalues $\{-1,-1,1\}$. We would like to find the general form (up to a diagonal phase matrix) of the unitary diagonalizing matrix of S_+. Since the eigenvalue -1 is two-fold degenerate, then there is still freedom for a unitary transformation defined by an angle φ and phase ξ in its eigenspace to get new eigen vectors in the following form:

$$\begin{aligned} \overline{v}_1 &= s_\varphi\, e^{-i\,\xi}\, v_1 + c_\varphi\, v_2, \\ \overline{v}_2 &= -c_\varphi\, e^{-i\,\xi}\, v_1 + s_\varphi\, v_2. \end{aligned} \tag{30}$$

Once again, the suitable choice of ordering the eigenvectors of S_+, which would determine the unitary matrix U_+ diagonalizing S_+ in such a way that the mixing angles fall all in the first quadrant, turns out to correspond to the eigenvalues ordering $\{-1,-1,1\}$. Hence, the matrix U_+ assumes the following form:

$$U_+ = [\overline{v}_1, \overline{v}_2, v_3] = \begin{pmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi\, e^{-i\,\xi}/\sqrt{2} & 1/\sqrt{2}\, c_\varphi\, e^{-i\,\xi} & 1/\sqrt{2} \\ s_\varphi\, e^{-i\,\xi}/\sqrt{2} & -1/\sqrt{2}\, c_\varphi\, e^{-i\,\xi} & 1/\sqrt{2} \end{pmatrix}. \tag{31}$$

The matrix $M_\nu^* M_\nu$ has the form,

$$M_\nu^*\, M_\nu \;=\; \begin{pmatrix} a_\nu & b_\nu & b_\nu \\ b_\nu^* & c_\nu & d_\nu \\ -b_\nu^* & d_\nu & c_\nu \end{pmatrix}, \tag{32}$$

where a_ν , b_ν, c_ν and d_ν are defined as follows,

$$\begin{aligned} a_\nu = |A_\nu|^2 + 2\,|B_\nu|^2, &\qquad b_\nu = A_\nu^*\, B_\nu + B_\nu^*\, C_\nu - B_\nu^*\, D_\nu, \\ c_\nu = |B_\nu|^2 + |C_\nu|^2 + |D_\nu|^2, &\qquad d_\nu = -\,|B_\nu|^2 + C_\nu^*\, D_\nu + D_\nu^*\, C_\nu. \end{aligned} \tag{33}$$

and its eigenvalues are given by:

$$\begin{aligned} m_1^2 &= \frac{a_\nu + c_\nu - d_\nu}{2} + \frac{1}{2}\sqrt{(a_\nu + d_\nu - c_\nu)^2 + 8\,|b_\nu|^2}, \\ m_2^2 &= \frac{a_\nu + c_\nu - d_\nu}{2} - \frac{1}{2}\sqrt{(a_\nu + d_\nu - c_\nu)^2 + 8\,|b_\nu|^2}, \\ m_3^2 &= c_\nu + d_\nu. \end{aligned} \tag{34}$$

The specific form of U_+ of Eq.(31) which diagonlizes also the hermitian matrix $M_\nu^* M_\nu$, which commutes with S_+, corresponds to:

$$\tan(2\varphi) = \frac{2\sqrt{2}\,|b_\nu|}{c_\nu - a_\nu - d_\nu}, \qquad \xi = \mathrm{Arg}\,(b_\nu)\,, \tag{35}$$

As in the case of U_-, one can prove that $U_+^T\, M_\nu\, U_+$, after having fixed φ and ξ according to Eq. (35), is diagonal

$$U_+^T\, M_\nu\, U_+ = M_\nu^{\mathrm{Diag}} = \mathrm{Diag}\,(M_{\nu\,11}^{\mathrm{Diag}},\ M_{\nu\,22}^{\mathrm{Diag}},\ M_{\nu\,33}^{\mathrm{Diag}}), \tag{36}$$

where

$$\begin{aligned}
M^{\text{Diag}}_{\nu\,11} &= A_\nu\, c^2_\varphi - \sqrt{2}\, s_{2\varphi}\, e^{-i\,\xi}\, B_\nu + (C_\nu - D_\nu)\; s^2_\varphi\, e^{-2\,i\,\xi}, \\
M^{\text{Diag}}_{\nu\,22} &= A_\nu\, s^2_\varphi + \sqrt{2}\, s_{2\varphi}\, e^{-i\,\xi}\, B_\nu + (C_\nu - D_\nu)\; c^2_\varphi\, e^{-2\,i\,\xi}, \\
M^{\text{Diag}}_{\nu\,33} &= C_\nu + D_\nu,
\end{aligned} \tag{37}$$

while the squared modulus of these complex eigenvalues are identified respectively with the squared mass m_1^2, m_2^2 and m_3^2 (the eigenvalues of $M_\nu^* M_\nu$ in Eq. 34).

Again, as was the case for the S_- pattern, we use the freedom of multiplying U_+ by a diagonal phase matrix Q in order that

$$(U_+\, Q)^T M_\nu (U_+\, Q) \quad = \quad \text{Diag}\,(m_1,\; m_2,\; m_3)\,. \tag{38}$$

Moreover, we re-phase the charged lepton fields to make the conjugate of $(U_+\, Q)$ in the same form as the adopted parametrization for V_{PMNS} in Eq.(4), so that to identify the mixing and phase angles. We find that the μ–τ symmetry realized through S_+ entails the followings:

$$\begin{aligned}
&\theta_{23} = \pi/4, \;\; \theta_{12} = \varphi, \;\; \theta_{13} = 0, \\
&\rho = \frac{1}{2}\,\text{Arg}\,(M^{\text{Diag}}_{\nu\,11})\,, \;\; \sigma = \frac{1}{2}\,\text{Arg}\,(M^{\text{Diag}}_{\nu\,22})\,, \;\; \delta = \frac{1}{2}\,\text{Arg}\,(M^{\text{Diag}}_{\nu\,33}) - \xi.
\end{aligned} \tag{39}$$

These predictions are phenomenologically "almost" viable (the non-vanishing value of θ_{13} will be attributed to small deviations from the exact symmetry), and furthermore do not require a special adjustment for the parameters $a_\nu, b_\nu, c_\nu, d_\nu$ which can be of the same order, in contrast to Eq.(25), and still accommodate the experimental value of $\theta_{12} \simeq 33.7^o$.

The various neutrino mass hierarchies can also be produced as can be seen from Eq.(34) and Eq.(35) where the three masses and the angle φ are given in terms of four parameters $a_\nu, |b_\nu|\,, c_\nu$, and d_ν. Therefore, one can solve the four given equations to get $a_\nu, |b_\nu|\,, c_\nu$, and d_ν in terms of the masses and the angle φ.

5 The seesaw mechanism and the S_+ realized μ–τ symmetry

We impose now the $\mu - \tau$-symmetry, defined by the matrix $S = S_+$, at the Lagrangian level within a model for the Leptons sector. Then, we shall invoke the type-I see-saw mechanism to address the origin of the effective neutrino mass matrix, with consequences on leptogenesis. The procedure has already been done in [17] for other Z_2-symmetries.

5.1 The charged lepton sector

We start with the part of the SM Lagrangian responsible for giving masses to the charged leptons:

$$\mathcal{L}_1 = Y_{ij}\, \overline{L}_i\, \phi\, \ell^c_j, \tag{40}$$

where the SM Higgs field ϕ and the right handed (RH) leptons ℓ^c_j are assumed to be singlet under S, whereas the left handed (LH) leptons transform in the fundamental representation of S:

$$L_i \quad \longrightarrow \quad S_{ij} L_j. \tag{41}$$

Invariance under S implies:

$$S^T Y \quad = \quad Y, \tag{42}$$

and this forces the Yukawa couplings to have the form:

$$Y \quad = \quad \left(\begin{array}{ccc} 0 & 0 & 0 \\ a & b & c \\ a & b & c \end{array} \right), \tag{43}$$

which leads, when the Higgs field acquires a vev v, to a charged lepton squared mass matrix of the form:

$$M_l M_l^\dagger \quad = \quad v^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \left(|a|^2 + |b|^2 + |c|^2\right). \tag{44}$$

As the eigenvectors of $M_l M_l^\dagger$ are $\left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)^{\mathrm{T}}$ with eigenvalue $2v^2 \, \left(|a|^2 + |b|^2 + |c|^2\right)$ and $\left(0, 1/\sqrt{2}, -1/\sqrt{2}\right)^{\mathrm{T}}$ and $\left(\ 1, 0, 0\right)^{\mathrm{T}}$ with a degenerate eigenvalue 0, then the charged lepton mass hierarchy can not be accommodated. Moreover, the nontrivial diagonalizing matrix, illustrated by non-canonical eigenvectors, means we are no longer in the flavor basis. To remedy this, we introduce SM-singlet scalar fields Δ_k coupled to the lepton LH doublets through the dimension-5 operator:

$$\mathcal{L}_2 = \frac{f_{ikr}}{\Lambda} \, \overline{L}_i \, \phi \, \Delta_k \, \ell_r^c. \tag{45}$$

This way of adding extra SM-singlets is preferred, for suppressing flavor–changing neutral currents, than to have additional Higgs fields . Also, we assume the Δ_k's transform under S as:

$$\Delta_i \longrightarrow S_{ij} \, \Delta_j. \tag{46}$$

Invariance under S implies,

$$S^T f_r S = f_r, \quad \text{where} \quad (f_r)_{ij} = f_{ijr}, \tag{47}$$

and thus we have the following form

$$f_r = \begin{pmatrix} A^r & B^r & -B^r \\ E^r & C^r & D^r \\ -E^r & D^r & C^r \end{pmatrix}, \tag{48}$$

when the fields Δ_k and the neutral component of the Higgs field ϕ° take vevs ($\langle \Delta_k \rangle = \delta_k$, $v = \langle \phi^\circ \rangle$) we get a charged lepton mass matrix:

$$(M_l)_{ir} \quad = \quad \frac{v f_{ikr}}{\Lambda} \delta_k, \tag{49}$$

if $\delta_1, \delta_2 \ll \delta_3$ then

$$(M_l)_{ir} \simeq \frac{v f_{i3r}}{\Lambda} \delta_3 \quad \simeq \quad \begin{pmatrix} -B^1 & -B^2 & -B^3 \\ D^1 & D^2 & D^3 \\ C^1 & C^2 & C^3 \end{pmatrix}, \tag{50}$$

with $f_{13j} = -B^j$, $f_{23j} = D^j$, $f_{33j} = C^j$ for $j = 1, 2, 3$. In Ref. [17], a charged lepton matrix of exactly the same form was shown to represent the lepton mass matrix in the flavor basis with the right charged lepton mass hierarchies, assuming just the ratios of the magnitudes of the vectors comparable to the lepton mass ratios.

5.2 Neutrino mass hierarchies

The effective light LH neutrino mass matrix is generated through the seesaw mechanism formula

$$M_\nu \quad = \quad M_\nu^D M_R^{-1} M_\nu^{D\mathrm{T}}, \tag{51}$$

where the Dirac neutrino mass matrix M_ν^D comes from the Yukawa term

$$g_{ij} \, \overline{L}_i \, i\tau_2 \, \Phi^* \nu_{Rj}, \tag{52}$$

upon the Higgs field acquiring a vev, whereas the symmetric Majorana neutrino mass matrix M_R comes from a term (C is the charge conjugation matrix)

$$\frac{1}{2} \, \nu_{Ri}^T \, C^{-1} \, (M_R)_{ij} \, \nu_{Rj}. \tag{53}$$

We assume the RH neutrino to transform under S as:

$$\nu_{Rj} \longrightarrow S_{jr}\nu_{Rr}, \tag{54}$$

and thus the S-invariance leads to

$$S^T\, g\, S = g \quad , \quad S^T\, M_R\, S = M_R. \tag{55}$$

This forces the following textures:

$$M_\nu^D = v \begin{pmatrix} A_D & B_D & -B_D \\ E_D & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix} \quad , \quad M_R = \Lambda_R \begin{pmatrix} A_R & B_R & -B_R \\ B_R & C_R & D_R \\ -B_R & D_R & C_R \end{pmatrix}, \tag{56}$$

where the explicitly appearing scales Λ_R and v characterize respectively the heavy RH Majorana neutrino masses and the electro-weak scale. Later, for numerical estimates, we shall take Λ_R and v to be respectively around 10^{14} GeV and 175 GeV, so the scale characterizing the effective light neutrino $\frac{v^2}{\Lambda_R}$ would be around 0.3 eV. Throughout the work, where no risk of confusion, these scales will not be written explicitly in the formulae in order to simplify the notations. The resulting effective matrix M_ν will have the form of Eq.(29) with

$$\begin{aligned}
A_\nu &= [(C_R^2 - D_R^2)\, A_D^2 - 4\, B_R\, (C_R + D_R)\, A_D\, B_D + 2\, A_R\, (C_R + D_R)\, B_D^2]/\mathrm{det}M_R, \\
B_\nu &= -(C_R + D_R)\, \{(D_D - C_D)\, B_D\, A_R + (D_R - C_R)\, E_D\, A_D + [A_D\, (C_D - D_D) + 2\, B_D\, E_D]\, B_R\}/\mathrm{det}M_R, \\
C_\nu &= \{(A_R\, C_R - B_R^2)\, D_D^2 + [-2\, (A_R\, D_R + B_R^2)\, C_D + 2\, B_R\, (C_R + D_R)\, E_D]\, D_D \\
&\quad +(A_R\, C_R - B_R^2)\, C_D^2 - 2\, B_R\, (C_R + D_R)\, E_D\, C_D + E_D^2\, (C_R^2 - D_R^2)\}/\mathrm{det}M_R, \\
D_\nu &= \{-(A_R\, D_R + B_R^2)\, D_D^2 + [-2\, (-A_R\, C_R + B_R^2)\, C_D - 2\, B_R\, (C_R + D_R)\, E_D]\, D_D \\
&\quad -(A_R\, D_R + B_R^2)\, C_D^2 + 2\, B_R\, (C_R + D_R)\, E_D\, C_D - E_D^2\, (C_R^2 - D_R^2)\}/\mathrm{det}M_R, \\
\mathrm{det}M_R &= (C_R + D_R)\, [A_R\, (C_R - D_R) - 2\, B_R^2].
\end{aligned} \tag{57}$$

Concerning the mass spectrum of the light neutrinos, it can be related to that of the RH neutrinos through the following equation connecting the product of the square eigenmasses of M_ν to those of M_D and M_R:

$$\det\left(M_\nu^*\, M_\nu\right) = \det\left(M_\nu^{D\dagger}\, M_\nu^D\right)^2\, \det\left(M_R^*\, M_R\right)^{-1}. \tag{58}$$

As was the case for the effective neutrino squared mass matrix, we choose to write:

$$M_\nu^{D\dagger}\, M_\nu^D = \begin{pmatrix} a_D & b_D & -b_D \\ b_D^* & c_D & d_D \\ -b_D^* & d_D & c_D \end{pmatrix}, \qquad M_R^*\, M_R = \begin{pmatrix} a_R & b_R & b_R \\ b_R^* & c_R & d_R \\ -b_R^* & d_R & c_R \end{pmatrix}, \tag{59}$$

with

$$\begin{aligned}
a_D &= |A_D|^2 + 2\,|E_D|^2, & a_R &= |A_R|^2 + 2\,|B_R|^2, \\
b_D &= A_D^*\, B_D + E_D^*\, C_D - E_D^*\, D_D, & b_R &= A_R^*\, B_R + B_R^*\, C_R - B_R^*\, D_R, \\
c_D &= |B_D|^2 + |C_D|^2 + |D_D|^2, & c_R &= |B_R|^2 + |C_R|^2 + |D_R|^2, \\
d_D &= -|B_D|^2 + C_D^*\, D_D + D_D^*\, C_D, & d_R &= -|B_R|^2 + C_R^*\, D_R + D_R^*\, C_R.
\end{aligned} \tag{60}$$

so that one can write concisely the mass spectrum of $M_\nu^*\, M_\nu$, $M_R^*\, M_R$ and $M_\nu^{D\dagger}\, M_\nu^D$ as:

$$\left\{ c_{\nu,R,D} + d_{\nu,R,D},\ \frac{a_{\nu,R,D} + c_{\nu,R,D} - d_{\nu,R,D}}{2} \pm \frac{1}{2}\sqrt{\left(a_{\nu,R,D} + d_{\nu,R,D} - c_{\nu,R,D}\right)^2 + 8\, |b_{\nu,R,D}|^2} \right\}. \tag{61}$$

The mass spectrum and its hierarchy type are determined by the eigenvaules presented in Eq.(61). One of the simple realizations which can be inferred from Eq.(58) is to adjust the spectrum of $M_R^*\, M_R$ so that to follow the same kind of hierarchy as $M_\nu^*\, M_\nu$. However, this does not necessarily imply that $M_\nu^{D\dagger}\, M_\nu^D$ will behave similarly. Also, this does not exhaust all possible realizations producing the desired hierarchy and what is stated is just a mere simple possibility.

5.3 Leptogenesis

In this kind of models, the unitary matrix diagonalizing M_R is not necessarily diagonalizing M_ν^D. In fact, the Majorana and Dirac neutrino mass matrices have different forms dictated by the S-symmetry and the angle φ in Eq.(35) depends on the corresponding mass parameters. This point is critical in generating lepton asymmetry, in contrast to other symmetries [17] where no freedom was left for the mixing angles leading to the same form on M_R and M_ν^D with identical diagonalizing matrices. This is important when computing the CP asymmetry induced by the lightest RH neutrinos, say N_1, since it involves explicitly the unitary matrix diagonalizing M_R:

$$\varepsilon_1 \quad = \quad \frac{1}{8\,\pi\,v^2}\,\frac{1}{\left(\tilde{M}_\nu^{D\dagger}\,\tilde{M}_\nu^D\right)_{11}}\sum_{j=2,3}\mathrm{Im}\left\{\left[\left(\tilde{M}_\nu^{D\dagger}\,\tilde{M}_\nu^D\right)_{1j}\right]^2\right\}F\left(\frac{m_{Rj}^2}{m_{R1}^2}\right). \tag{62}$$

where $F(x)$ is the function containing the one loop vertex and self-energy corrections [23], and which, for a hierarchical heavy neutrinos mass spectrum far from almost degenerate, is given by

$$F(x) = \sqrt{x}\left[\frac{1}{1-x} + 1 - (1+x)\ln\left(1+\frac{1}{x}\right)\right] \tag{63}$$

Assuming that there is a strong hierarchy among RH neutrino masses with $m_{R1} << m_{R2} << m_{R3}$, the CP asymmetry can be approximated as

$$\varepsilon_1 \simeq -6\times 10^{-2}\,\frac{\mathrm{Im}\left\{\left[\left(\tilde{M}_\nu^{D\dagger}\,\tilde{M}_\nu^D\right)_{12}\right]^2\right\}}{v^2\left(\tilde{M}_\nu^{D\dagger}\,\tilde{M}_\nu^D\right)_{11}}\,\frac{m_{R1}}{m_{R2}}. \tag{64}$$

The matrix $\tilde{M}_\nu^D$ is the Dirac neutrino mass matrix in the basis where the RH neutrinos are mass eigenstates:

$$\tilde{M}_\nu^D = M_\nu^D\,V_R\,F_0 \tag{65}$$

Here V_R is the unitary matrix, defined up to a phase diagonal matrix, that diagonalizes the symmetric matrix M_R, and F_0 is a phase diagonal matrix chosen such that the eigenvalues of M_R are real and positive.

The generated baryon asymmetry can be written as

$$Y_B := \frac{n_B - n_{\bar{B}}}{s} \simeq 1.3\times 10^{-3}\times\varepsilon_1\times\mathcal{W}(\tilde{m}, m_{R1}), \qquad \tilde{m} = \frac{\left(\tilde{M}_\nu^{D\dagger}\,\tilde{M}_\nu^D\right)_{11}}{m_{R1}} \tag{66}$$

where $n_B, n_{\bar{B}}$ and s are the number densities of baryons, anti-baryons, and entropy, respectively, and $\mathcal{W}$ is a dilution factor which accounts for the wash-out of the total lepton asymmetry due to the $\Delta L = 1$ inverse decays and the lepton violating 2-2 scattering processes, and its value can be determined by solving the Boltzmann equation. However, analytical expressions for $\mathcal{W}$ have been obtained for the cases where ($\tilde{m} > 1\ eV$) and ($1\ eV > \tilde{m} > 10^{-3}\ eV$), known as the strong and the weak wash-out regimes respectively[24]. For instance, in the strong wash out regime (SW), $\mathcal{W}$ is approximated as

$$\mathcal{W}^{(SW)} \simeq \left(\frac{10^{-3}\ eV}{2\tilde{m}}\right)^{1.2} \tag{67}$$

In our case where the S-symmetry imposes a particular form on the symmetric M_R (Eq. 56), we can take V_R as being the rotation matrix U_+ of Eq.(31) corresponding to

$$\theta_{R\,23} = \pi/4,\ \theta_{R\,12} = \varphi_R = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{2}\ |b_R|}{c_R - a_R - d_R}\right),\ \theta_{R\,13} = 0,\ \xi_R = \mathrm{Arg}\left(b_R\right). \tag{68}$$

As to the diagonal phase matrix, $F_0 = \text{Diag}\left(e^{-i\alpha_1},\, e^{-i\alpha_2},\, e^{-i\alpha_3}\right)$, it can be chosen according to Eq.(37) to be

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}\text{Arg}\left[A_R\, c^2_{\varphi_R} - \sqrt{2}\, s_{2\varphi_R}\, e^{-i\,\xi_R}\, B_R + (C_R - D_R)\; s^2_{\varphi_R}\, e^{-2\,i\,\xi_R}\right], \\
\alpha_2 &= \frac{1}{2}\text{Arg}\left[A_R\, s^2_{\varphi_R} + \sqrt{2}\, s_{2\varphi_R}\, e^{-i\,\xi_R}\, B_R + (C_R - D_R)\; c^2_{\varphi_R}\, e^{-2\,i\,\xi_R}\right], \\
\alpha_3 &= \frac{1}{2}\text{Arg}\,(c_R + d_R)\,. \qquad (69)
\end{aligned}$$

We assume here that the resulting mass spectrum of M_R via the diagonalizing matrix $V_R F_0$ is in increasing order, otherwise one needs to apply a suitable permutation on the columns of the latter matrix in order to get this. Note here that had the matrix V_R diagonalized M^D_ν, which would have meant that $N = V_R^\dagger\, M^D_\nu\, V_R$ is diagonal, then we would have reached a diagonal $\tilde{M}^{D\dagger}_\nu \tilde{M}^D_\nu$ equaling a product of diagonal matrices, and no leptogenesis:

$$\tilde{M}^{D\dagger}_\nu\, \tilde{M}^D_\nu \;=\; F_0^\dagger \left(V_R^\dagger M^{D\dagger}_\nu V_R\right)\left(V_R^\dagger M^D_\nu V_R\right) F_0 = F_0^\dagger N^\dagger N F_0 \qquad (70)$$

In contrast, we get in our case:

$$\begin{aligned}
\left(\tilde{M}^{D\dagger}_\nu \tilde{M}^D_\nu\right)_{12} &= e^{i(\alpha_1-\alpha_2)}\,\Big[-\sqrt{2}\, e^{i\,\xi_R}\,(A_D\, B^*_D + E_D\, C^*_D - E_D\, D^*_D)\, s^2_{\varphi_R} \\
&\quad + \sqrt{2}\, e^{-i\,\xi_R}\,(A^*_D\, B_D - E^*_D\, D_D + E^*_D\, C_D) \\
&\quad + s_{\varphi_R}\, c_{\varphi_R}\left(-2\,|B_D|^2 - |C_D|^2 - |D_D|^2 + 2\,|E_D|^2 + |A_D|^2 + C^*_D\, D_D + D^*_D\, C_D\right)\Big] \\
\left(\tilde{M}^{D\dagger}_\nu \tilde{M}^D_\nu\right)_{13} &= 0 \\
\left(\tilde{M}^{D\dagger}_\nu \tilde{M}^D_\nu\right)_{11} &= c^2_{\varphi_R}\,\left(|A_D|^2 + 2\,|E_D|^2\right) \\
&\quad + s^2_{\varphi_R}\,\left(2\,|B_D|^2 + |C_D|^2 + |D_D|^2 - C^*_D\, D_D - C_D\, D^*_D\right) \\
&\quad - \sqrt{2}\, s_{\varphi_R}\, c_{\varphi_R}\left(A_D\, B^*_D\, e^{i\,\xi_R} - E_D\, D^*_D\, e^{i\,\xi_R} + E_D\, C^*_D\, e^{i\,\xi_R} + \text{h.c}\right). \qquad (71)
\end{aligned}$$

We see that $\left(\tilde{M}^{D\dagger}_\nu \tilde{M}^D\right)_{12}$ is complex in general, and the question is asked whether or not one can tune it to produce the correct CP asymmetry. Clearly, the phase of $\left(\tilde{M}^{D\dagger}_\nu \tilde{M}^D_\nu\right)_{12}$ would be the triggering factor in producing the baryon asymmetry. More explicitly,

$$\text{Im}[\left(M^{\dagger D}_\nu M^D_\nu\right)_{12}]^2 \propto \sin\left[2\,(\phi + \alpha_1 - \alpha_2)\right], \qquad (72)$$

where ϕ is the phase of the entry $\left(V_R^\dagger\, M^D_\nu\, V_R\right)_{12}$.

Considering that $m_{R1} < 10^{14}\; GeV$ and the Yukawa neutrino couplings to be not too small compared to the one which makes the see-saw mechanism more natural, which corresponds to $\tilde{m} > 10^{-3}\; eV$, and hence the baryon asymmetry can be expressed as

$$Y_B \simeq 1.1\times 10^{-9}\left(\frac{r_{12}}{0.1}\right)\left(\frac{m_{R1}}{10^{13}\; GeV}\right)\left(\frac{10^{-3}\text{eV}}{\tilde{m}}\right)^{0.2}\left[\frac{|(M^{D\dagger}_\nu M^D_\nu)_{12}|}{(M^{D\dagger}_\nu M^D_\nu)_{11}}\right]^2 \sin\left[2\,(\phi + \alpha_1 - \alpha_2)\right] \qquad (73)$$

with $r_{12} = m_{R1}/m_{R2}$ which parametrizes how strong is the hierarchy of the RH neutrinos mass spectrum. If the matrix elements $\left(M^{D\dagger}_\nu M^D_\nu\right)_{11}$ and $\left(M^{D\dagger}_\nu M^D_\nu\right)_{12}$ are of the same order, then, for $\tilde{m}$ of the order of $\frac{v^2}{\Lambda_R} \simeq 0.3\; eV$, we have

$$Y_B \simeq 0.35\times 10^{-9}\left(\frac{r_{12}}{0.1}\right)\left(\frac{m_{R1}}{10^{13}\; GeV}\right)\sin\left[2\,(\phi + \alpha_1 - \alpha_2)\right] \qquad (74)$$

So, for hierarchical heavy RH neutrino mass spectrum and with $m_{R1} > 10^{13}\; GeV$ one can adjust the value of Majorana phase difference $(\alpha_1 - \alpha_2)$ to obtain Y_B equals to the observed value[25].

The above estimate for the baryon asymmetry assumed $|(M^{D\dagger}_\nu M^D_\nu)_{12}|/(M^{D\dagger}_\nu M^D_\nu)_{11} \sim 1$, and it is not generic by any mean. However, from the equation (73) it is clear that one can easily obtain a value of Y_B, that is in agreement with the observation, corresponding to many other possible choices for the values of the matrix elements of $(M^{D\dagger}_\nu M^D_\nu)$, and the mass of the lightest RH neutrino [17].

6 A possible deviation from the μ–τ symmetry through S_+ and its consequences

We saw that exact μ–τ-symmetry implied a vanishing value for the mixing angle θ_{13}. Recent oscillation data pointing to a small but non-vanishing value for this angle suggest then a deviation on the exact symmetry texture in order to account for the observed mixing. We showed in [16] how "minimal" perturbed textures disentangling the effects of the perturbations can account for phenomenology. We shall consider now, within the scheme of type-I seesaw, a specific perturbed texture imposed on Dirac neutrino mass matrix M_ν^D, and parameterized by only one small parameter α, and show how it can resurface on the effective neutrino mass matrix M_ν, which is known to be phenomenologically viable. We compute then the "perturbed" eigenmasses and mixing angles to first order in α, whereas we address in the next section the question of finding numerically a viable pattern for M_ν^D and M^R leading to M_ν consistent with the phenomenology. Thus, we assume a perturbed M_ν^D of the form

$$M_\nu^D = \begin{pmatrix} A_D & B_D\,(1+\alpha) & -B_D \\ E_D & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix} \tag{75}$$

The small parameter α affects only one condition defining the exact S-symmetry texture, and can be expressed as:

$$\alpha = -\frac{\left(M_\nu^D\right)_{12} + \left(M_\nu^D\right)_{13}}{(M_\nu^D)_{13}}. \tag{76}$$

Applying the seesaw formula of Eq.(51) with M_R given by Eq.(56) we get then:

$$\begin{aligned}
M_\nu\,(1,1) &= M_\nu^0\,(1,1) + \alpha^2 \frac{B_D^2\left(C_R A_R - B_R^2\right)}{\det M_R} + \alpha\frac{2B_D\,(C_R + D_R)\,(A_R B_D - B_R A_D)}{\det M_R} \\
M_\nu\,(1,2) &= M_\nu^0\,(1,2) + \alpha\frac{B_D\left[A_R\,(C_R C_D - D_R D_D) - B_R^2\,(D_D + C_D) - E_D B_R\,(D_R + C_R)\right]}{\det M_R} \\
M_\nu\,(1,3) &= M_\nu^0\,(1,3) + \alpha\frac{B_D\left[A_R\,(C_R D_D - D_R C_D) - B_R^2\,(D_D + C_D) + E_D B_R\,(D_R + C_R)\right]}{\det M_R} \\
M_\nu\,(2,2) &= M_\nu^0\,(2,2) = M_\nu^0\,(3,3) = M_\nu\,(3,3) \\
M_\nu\,(2,3) &= M_\nu^0\,(2,3)
\end{aligned} \tag{77}$$

where M_ν^0 is the 'unperturbed' effective neutrino mass matrix (corresponding to $\alpha = 0$) and thus can be diagonalized by U_+^0 of Eq.(31) corresponding to the following angles,

$$\theta_{23}^0 = \pi/4,\ \theta_{12}^0 = \varphi^0 = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{2}\left|b_\nu^0\right|}{c_\nu^0 - a_\nu^0 - d_\nu^0}\right),\ \theta_{13}^0 = 0,\ \text{and}\ \xi^0 = \mathrm{Arg}\left(b_\nu^0\right), \tag{78}$$

Here, the superscript 0 denotes quantities corresponding to the unperturbed effective neutrino mass matrix M_ν^0.

The mass matrix M_ν can be organized in the following form,

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu\,(1+\chi) & -B_\nu \\ B_\nu\,(1+\chi) & C_\nu & D_\nu \\ -B_\nu & D_\nu & C_\nu \end{pmatrix} \tag{79}$$

where the perturbation parameter χ is given by:

$$\chi = -\frac{(M_\nu)_{12} + (M_\nu)_{13}}{(M_\nu)_{13}}. \tag{80}$$

The two parameters χ and α are generally complex and of the same order provided we do not have unnatural cancelations between the mass parameters of M_ν^D and M_R. Nevertheless and without loss of generality, α can be made positive and real. Furthermore, as will be explained later in our numerical investigation, α can be adjusted to have the same value as $|\chi|$.

In order to compute the new eigenmasses and mixing angles of M_ν, we write it in the following form working only to first order in α:

$$M_\nu = M_\nu^0 + M_\alpha, \tag{81}$$

where the matrix M_α is given as,

$$M_\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & 0 & 0 \\ \alpha_{13} & 0 & 0 \end{pmatrix}, \tag{82}$$

and the non-vanishing entries of M_α are found to be,

$$\begin{aligned} \alpha_{11} &= \frac{2\alpha B_D \left(C_R + D_R\right)\left(A_R B_D - B_R A_D\right)}{\det M_R}, \\ \alpha_{12} &= \frac{\alpha B_D \left[A_R \left(C_R C_D - D_R D_D\right) - B_R^2 \left(D_D + C_D\right) - E_D B_R \left(D_R + C_R\right)\right]}{\det M_R}, \\ \alpha_{13} &= \frac{\alpha B_D \left[A_R \left(C_R D_D - D_R C_D\right) - B_R^2 \left(D_D + C_D\right) + E_D B_R \left(D_R + C_R\right)\right]}{\det M_R}. \end{aligned} \tag{83}$$

Note here that $M_\nu\,(1,1)$ gets distorted by terms of order α and α^2. However, this will not "perturb" the relations defining μ–τ symmetry, which are expressed only through $M_\nu\,(1,2)\,, M_\nu\,(1,3)\,, M_\nu\,(2,2)$ and $M_\nu\,(3,3)$.

We seek now a unitary matrix Q diagonalizing $M_\nu^*\,M_\nu$, and we write it in the form:

$$Q \;=\; U_+^0\left(1 + I_\epsilon\right), \quad I_\epsilon = \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1^* & 0 & \epsilon_3 \\ -\epsilon_2^* & -\epsilon_3^* & 0 \end{pmatrix}, \tag{84}$$

where I_ϵ is an antihermitian matrix due to the unitarity of Q. Imposing the diagonalization condition on $M_\nu^*\,M_\nu$, and knowing that U_+^0 diagonalizes $M_\nu^{0*}\,M_\nu^0$, we have:

$$\begin{aligned} Q^\dagger M_\nu^*\,M_\nu Q &= \mathrm{Diag}\left(|M_{\nu\,11}^{\mathrm{Diag}}|^2, |M_{\nu\,22}^{\mathrm{Diag}}|^2, |M_{\nu\,33}^{\mathrm{Diag}}|^2\right), \\ U_+^{0\dagger}\,M_\nu^{0*}\,M_\nu^0\,U_+^0 &= \mathrm{Diag}\left(\left|M_{\nu\,11}^{0\,\mathrm{Diag}}\right|^2, \left|M_{\nu\,22}^{0\,\mathrm{Diag}}\right|^2, \left|M_{\nu\,33}^{0\,\mathrm{Diag}}\right|^2\right). \end{aligned} \tag{85}$$

Keeping only terms up to first order in α, which is consistent with aiming to compute I_ϵ up to this order in α and thus with dropping higher orders of I_ϵ, we get the condition:

$$i, j \in \{1,2,3\}, i \neq j, \left(Q^\dagger\,M_\nu^*\,M_\nu\,Q\right)_{ij} = 0 \Longrightarrow \left[I_\epsilon\,,\,M_\nu^{0\mathrm{Diag}*}\,M_\nu^{0\mathrm{Diag}}\right]_{ij} = \left[U_+^{0\dagger}\left(M_\nu^{0*}\,M_\alpha + M_\alpha^*\,M_\nu^0\right)U_+^0\right]_{ij}. \tag{86}$$

One can solve analytically for ϵ_1, ϵ_2, ϵ_3 to get:

$$\begin{aligned} \epsilon_1 &= \frac{1}{\left|M_{\nu\,22}^{0\mathrm{Diag}}\right|^2 - \left|M_{\nu\,11}^{0\mathrm{Diag}}\right|^2}\left\{\frac{1}{\sqrt{2}}\,e^{-i\,\xi^0}\;\left[\left(\alpha_{13}^* - \alpha_{12}^*\right)\,\left(D_\nu^0 - C_\nu^0\right) - A_\nu^{0*}\,\left(\alpha_{13} - \alpha_{12}\right) + 2\,\alpha_{11}^*\,B_\nu^0\right]\,c_\varphi^2 + \right. \\ &\qquad \left. 2\,\mathrm{Re}\left(\alpha_{11}^*\,A_\nu^0\right)\,s_\varphi\,c_\varphi - \frac{1}{\sqrt{2}}\,e^{i\,\xi^0}\;\left[\left(\alpha_{13} - \alpha_{12}\right)\,\left(D_\nu^{0*} - C_\nu^{0*}\right) - A_\nu^0\,\left(\alpha_{13}^* - \alpha_{12}^*\right) + 2\,\alpha_{11}\,B_\nu^{0*}\right]\,s_\varphi^2\;\right\}, \\ \epsilon_2 &= \frac{1}{\left|M_{\nu\,33}^{0\mathrm{Diag}}\right|^2 - \left|M_{\nu\,11}^{0\mathrm{Diag}}\right|^2}\left\{\frac{1}{\sqrt{2}}\,\left[\left(\alpha_{13} + \alpha_{12}\right)\,A_\nu^{0*} + \left(C_\nu^0 + D_\nu^0\right)\,\left(\alpha_{13}^* + \alpha_{12}^*\right)\right]\,c_\varphi - \,e^{i\,\xi^0}\,B_\nu^{0*}\,\left(\alpha_{12} + \alpha_{13}\right)\,s_\varphi\right\}, \\ \epsilon_3 &= \frac{1}{\left|M_{\nu\,33}^{0\mathrm{Diag}}\right|^2 - \left|M_{\nu\,22}^{0\mathrm{Diag}}\right|^2}\left\{\frac{1}{\sqrt{2}}\,\left[\left(\alpha_{13} + \alpha_{12}\right)\,A_\nu^{0*} + \left(C_\nu^0 + D_\nu^0\right)\,\left(\alpha_{13}^* + \alpha_{12}^*\right)\right]\,s_\varphi + \,e^{-i\,\xi^0}\,B_\nu^{0*}\,\left(\alpha_{12} + \alpha_{13}\right)\,c_\varphi\right\}, \end{aligned} \tag{87}$$

and the resulting diagonal matrix $M_\nu^{\mathrm{Diag}} = Q^T M_\nu Q$ is such that

$$\begin{aligned} M_{\nu\,11}^{\mathrm{Diag}} &= M_{\nu\,11}^{0\,\mathrm{Diag}} + c_{\varphi^0}^2\alpha_{11} - \sqrt{2}\,s_{\varphi^0}\,c_{\varphi^0}\,\left(\alpha_{12} - \alpha_{13}\right)\,e^{-i\,\xi^0}, \\ M_{\nu\,22}^{\mathrm{Diag}} &= M_{\nu\,22}^{0\,\mathrm{Diag}} + s_{\varphi^0}^2\alpha_{11} + \sqrt{2}\,s_{\varphi^0}\,c_{\varphi^0}\,\left(\alpha_{12} - \alpha_{13}\right)\,e^{-i\,\xi^0}, \\ M_{\nu\,33}^{\mathrm{Diag}} &= M_{\nu\,33}^{0\,\mathrm{Diag}}. \end{aligned} \tag{88}$$

where the diagonalized mass matrix entries $M_{\nu\,11}^{0\,\mathrm{Diag}}$, $M_{\nu\,22}^{0\,\mathrm{Diag}}$ and $M_{\nu\,33}^{0\,\mathrm{Diag}}$ can be inferred from those in Eq.(37) to be,

$$\begin{aligned} M_{\nu\,11}^{0\,\mathrm{Diag}} &= A_\nu^0\,c^2_{\varphi^0} - \sqrt{2}\,s_{2\varphi^0}\,e^{-i\,\xi^0}\,B_\nu^0 + \left(C_\nu^0 - D_\nu^0\right)\,s^2_{\varphi^0}\,e^{-2\,i\,\xi^0}, \\ M_{\nu\,22}^{0\,\mathrm{Diag}} &= A_\nu^0\,s^2_{\varphi^0} + \sqrt{2}\,s_{2\varphi^0}\,e^{-i\,\xi^0}\,B_\nu^0 + \left(C_\nu^0 - D_\nu^0\right)\,c^2_{\varphi^0}\,e^{-2\,i\,\xi^0}, \\ M_{\nu\,33}^{0\,\mathrm{Diag}} &= C_\nu^0 + D_\nu^0. \end{aligned} \tag{89}$$

Thus one can obtain the squared masses up to order α as,

$$\begin{aligned} m_1^2 &= \left|M_{\nu\,11}^{0\,\mathrm{Diag}}\right|^2 - \sqrt{2}\,\mathrm{Re}\left\{e^{-i\,\xi^0}\,\left[(\alpha^*_{13}-\alpha^*_{12})\,\left(D_\nu^0 - C_\nu^0\right) - A_\nu^{0*}\,(\alpha_{13}-\alpha_{12}) + 2\,\alpha^*_{11}\,B_\nu^0\right]\,s_\varphi\,c_\varphi\right\} + \\ &\quad 2\,\mathrm{Re}\left[A_\nu^0\,\alpha^*_{11}\,c^2_\varphi + (\alpha^*_{12}-\alpha^*_{13})\,B_\nu^0\right], \\ m_2^2 &= \left|M_{\nu\,22}^{0\,\mathrm{Diag}}\right|^2 + \sqrt{2}\,\mathrm{Re}\left\{e^{-i\,\xi^0}\,\left[(\alpha^*_{13}-\alpha^*_{12})\,\left(D_\nu^0 - C_\nu^0\right) - A_\nu^{0*}\,(\alpha_{13}-\alpha_{12}) + 2\,\alpha^*_{11}\,B_\nu^0\right]\,s_\varphi\,c_\varphi\right\} + \\ &\quad 2\,\mathrm{Re}\left[A_\nu^0\,\alpha^*_{11}\,c^2_\varphi + (\alpha^*_{12}-\alpha^*_{13})\,B_\nu^0\right], \\ m_3^2 &= \left|M_{\nu\,33}^{0\,\mathrm{Diag}}\right|^2. \end{aligned} \tag{90}$$

In order to extract the mixing and phase angles corresponding to $Q = U_+^0\,(1 + I_\epsilon)$, the matrix Q should be multiplied by a suitable diagonal phase matrix to ensure that the eigenvalues of M_ν are real and positive. Moreover, as mentioned before, the charged lepton fields should be properly re-phased in order that one can match the adopted parameterization in Eq.(4). Thus, identifying Q, after having been multiplied by the diagonal phase matrix and made to have a third column of real values, with the V_{PMNS} one can get the "perturbed" mixing angles,

$$t_{12} \approx t_{\varphi^0}\left|1 + \frac{\epsilon_1}{t_{\varphi^0}} + \epsilon_1^*\,t_{\varphi^0}\right|, \quad t_{13} \approx \left|\epsilon_2\,c_{\varphi^0} + \epsilon_3\,s_{\varphi^0}\right|, \quad t_{23} \approx \left|1 - 2\,\epsilon_2\,s_{\varphi^0}\,e^{-i\,\xi^0} + 2\,\epsilon_3\,c_{\varphi^0}\,e^{-i\,\xi^0}\right|, \tag{91}$$

and the "perturbed" phases

$$\begin{aligned} \delta &\approx 2\,\pi - \xi^0 - \mathrm{Arg}\left(\epsilon_1^*\,c_{\varphi^0}\,e^{-i\,\xi^0} + \epsilon_2^*\right), \\ \rho &\approx \pi - \mathrm{Arg}\left[\left(c_{\varphi^0} - \epsilon_1^*\,s_{\varphi^0}\right)\left(\epsilon_2^*\,c_{\varphi^0} + \epsilon_3^*\,s_{\varphi^0}\right)\right] - \frac{1}{2}\mathrm{Arg}\left(M_{\nu\,33}^{\mathrm{Diag}}\,M_{\nu\,11}^{\mathrm{Diag}*}\right), \\ \sigma &\approx \pi - \mathrm{Arg}\left[\left(s_{\varphi^0} + \epsilon_1\,c_{\varphi^0}\right)\left(\epsilon_2^*\,c_{\varphi^0} + \epsilon_3^*\,s_{\varphi^0}\right)\right] - \frac{1}{2}\mathrm{Arg}\left(M_{\nu\,33}^{\mathrm{Diag}}\,M_{\nu\,22}^{\mathrm{Diag}*}\right). \end{aligned} \tag{92}$$

7 Numerical investigation for the deviation from the S_+-realized μ–τ symmetry

The numerical investigation turns out to be quite subtle due to the huge number of involved parameters which describe the relevant mass matrices and the possible deviation. Therefore, we start by studying numerically the perturbed mass matrix texture at the level of the effective light neutrino mass matrix, then, working backward, we reconstruct the Dirac and Majorana neutrino mass matrices together with the parameter α. For our numerical purpose, it is convenient to recast the effective neutrino light mass matrix, by using Eqs.(2-5), into the form,

$$M_{\nu\,ab} = \sum_{j=1}^{3} U_{aj}\,U_{bj}\,\lambda_j, \tag{93}$$

where λ_1, λ_2 and λ_3 are defined as,

$$\lambda_1 = m_1\,e^{2i\rho}, \lambda_2 = m_2\,e^{2i\sigma}, \lambda_3 = m_3. \tag{94}$$

Then the texture characterized by the deviation χ, where χ is a complex parameter equal to $|\chi|\;e^{i\theta}$, can be written as

$$M_{\nu\,12} + M_{\nu\,13}\,(1+\chi) = 0 \quad \Rightarrow \quad \sum_{j=1}^{3}\left[U_{1j}\,U_{2j} + (U_{1j}\,U_{3j})\,(1+\chi)\right]\,\lambda_j = 0,$$

$$\begin{aligned} & \Rightarrow \quad A_1\,\lambda_1 + A_2\,\lambda_2 + A_3\,\lambda_3 = 0, \\ M_{\nu\,22} - M_{\nu\,33} = 0 & \Rightarrow \quad \sum_{j=1}^{3} \left(U_{2j}\,U_{2j} - U_{3j}\,U_{3j}\right)\;\lambda_j = 0, \\ & \Rightarrow \quad B_1\,\lambda_1 + B_2\,\lambda_2 + B_3\,\lambda_3 = 0, \end{aligned} \tag{95}$$

where

$$A_j = U_{1j}\,U_{2j} + U_{1j}\,U_{3j}\,(1+\chi)\,, \quad \text{and} \quad B_j = U_{2j}^2 - U_{3j}^2, \qquad \text{(no sum over } j). \tag{96}$$

Then the coefficients A and B can be written explicitly in terms of mixing angles and Dirac phase as,

$$\begin{aligned} A_1 &= -c_{\theta_{12}}\,c_{\theta_{13}}\left(c_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} - s_{\theta_{12}}s_{\theta_{23}}e^{-i\,\delta}\right)(1+\chi) - c_{\theta_{12}}c_{\theta_{13}}\left(c_{\theta_{12}}s_{\theta_{23}}s_{\theta_{13}} + s_{\theta_{12}}c_{\theta_{23}}e^{-i\,\delta}\right), \\ A_2 &= -s_{\theta_{12}}\,c_{\theta_{13}}\left(s_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} + c_{\theta_{12}}s_{\theta_{23}}e^{-i\,\delta}\right)(1+\chi) - s_{\theta_{12}}c_{\theta_{13}}\left(s_{\theta_{12}}s_{\theta_{23}}s_{\theta_{13}} - c_{\theta_{12}}c_{\theta_{23}}e^{-i\,\delta}\right), \\ A_3 &= s_{\theta_{13}}c_{\theta_{23}}c_{\theta_{13}}\,(1+\chi) + s_{\theta_{13}}s_{\theta_{23}}c_{\theta_{13}}, \\ B_1 &= \left(-c_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} + s_{\theta_{12}}s_{\theta_{23}}e^{-i\,\delta}\right)^2 - \left(c_{\theta_{12}}s_{\theta_{23}}s_{\theta_{13}} + s_{\theta_{12}}c_{\theta_{23}}e^{-i\,\delta}\right)^2, \\ B_2 &= \left(s_{\theta_{12}}c_{\theta_{23}}s_{\theta_{13}} + c_{\theta_{12}}s_{\theta_{23}}e^{-i\,\delta}\right)^2 - \left(s_{\theta_{12}}s_{\theta_{23}}s_{\theta_{13}} - c_{\theta_{12}}c_{\theta_{23}}e^{-i\,\delta}\right)^2, \\ B_3 &= c_{\theta_{23}}^2 c_{\theta_{13}}^2 - s_{\theta_{23}}^2 c_{\theta_{13}}^2. \end{aligned} \tag{97}$$

Assuming $\lambda_3 \neq 0$, Eqs.(95) can be solved to yield λ's ratios as,

$$\begin{aligned} \frac{\lambda_1}{\lambda_3} &= \frac{A_3\;B_2 - A_2\;B_3}{A_2\;B_1 - A_1\;B_2}, \\ \frac{\lambda_2}{\lambda_3} &= \frac{A_1\;B_3 - A_3\;B_1}{A_2\;B_1 - A_1\;B_2}, \end{aligned} \tag{98}$$

From the λ's ratios, one can get exact results for the mass ratios $m_{13} \equiv \frac{m_1}{m_3}$ and $m_{23} \equiv \frac{m_2}{m_3}$ as well as for the phases ρ and σ in terms of the mixing angles, remaining Dirac phase δ and the parameter χ. In addition, one can compute the expressions for many phenomenologically relevant quantities such as:

$$\begin{aligned} R_\nu \equiv \frac{\delta m^2}{|\Delta m^2|}, &\qquad \Sigma = \sum_{i=1}^{3} m_i. \\ \langle m \rangle_e = \sqrt{\sum_{i=1}^{3}\left(|V_{ei}|^2 m_i^2\right)}, &\qquad \langle m \rangle_{ee} = \left|m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2\right| \;=\; |M_{\nu 11}|\,. \end{aligned} \tag{99}$$

Here, R_ν characterizes the hierarchy of the solar and atmospheric mass square differences, while the effective electron-neutrino mass $\langle m \rangle_e$ and the effective Majorana mass term $\langle m \rangle_{ee}$ are sensitive to the absolute neutrino mass scales and can be respectively constrained from reactor nuclear experiments on beta-decay kinematics and neutrinoless double-beta decay. As to the mass 'sum' parameter Σ, its upper bound can be constrained from cosmological observations. As regards the values of the non oscillation parameters $\langle m \rangle_e$, $\langle m \rangle_{ee}$ and Σ, we adopt the less conservative 2-σ range, as reported in [18] for $\langle m \rangle_e$ and Σ, and in [19] for $\langle m \rangle_{ee}$.

$$\begin{aligned} \langle m \rangle_e &< 1.8 \text{ eV}, \\ \Sigma &< 1.19 \text{ eV}, \\ \langle m \rangle_{ee} &< 0.34 - 0.78 \text{ eV}. \end{aligned} \tag{100}$$

The exact expressions turn out to be cumbersome to be presented, but for the sake of illustration, we state the relevant expressions up to leading order in $s_{\theta_{13}}$ as

$$\begin{aligned} m_{13} \approx 1 + \frac{2\,s_\delta s_\theta\,|\chi|\,s_{\theta_{13}}}{t_{\theta_{12}} T}, &\qquad m_{23} \approx 1 - \frac{2\,t_{\theta_{12}} s_\delta s_\theta\,|\chi|\,s_{\theta_{13}}}{T}, \\ \rho \approx \delta + \frac{s_\delta\;s_{\theta_{13}}\;\left(s_{\theta_{23}}c_{\theta_{23}}\,|\chi|^2 + |\chi|\;c_\theta\;(-c_{2\theta_{23}} + s_{2\theta_{23}}) - c_{2\theta_{23}}\right)}{t_{\theta_{12}}\,T}, &\qquad R_\nu \approx -\frac{8\,s_\delta\;s_\theta\,|\chi|\;s_{\theta_{13}}}{s_{2\theta_{12}}\,T}, \end{aligned} \tag{101}$$

$$\sigma \approx \delta - \frac{s_\delta\, t_{\theta_{12}}\, s_{\theta_{13}}\, \left(s_{\theta_{23}} c_{\theta_{23}}\, |\chi|^2 + |\chi|\; c_\theta\; (-c_{2\theta_{23}} + s_{2\theta_{23}}) - c_{2\theta_{23}}\right)}{T}, \qquad m^2_{23} - m^2_{13} \approx -\frac{8\, s_\delta\;\, s_\theta\, |\chi|\;\, s_{\theta_{13}}}{s_{2\theta_{12}}\, T},$$

$$\langle m\rangle_e \approx m_3 \left[1 + \frac{4\, s_\theta\, s_\delta\;\, |\chi|\;\, s_{\theta_{13}}}{t_{2\theta_{12}}\, T}\right], \qquad \langle m\rangle_{ee} \approx m_3 \left[1 + \frac{4\, s_\theta\, s_\delta\;\, |\chi|\;\, s_{\theta_{13}}}{t_{2\theta_{12}}\, T_1}\right].$$

where T is defined as,

$$T = |\chi|^2\;\, s^2_{\theta_{23}} + 2\;|\chi|\;\, c_\theta\, s_{\theta_{23}}\;\, (s_{\theta_{23}} - c_{\theta_{23}}) + 1 - s_{2\theta_{23}}. \tag{102}$$

Our expansion in terms of $s_{\theta_{13}}$ is justified since $s_{\theta_{13}}$ is typically small for phenomenological acceptable values where the best fit for $s_{\theta_{13}} \approx 0.15$. This kind of expansion in terms of $s_{\theta_{13}}$, in the case of partial μ–τ symmetry, has many subtle properties which were fully discussed in [16] and no need to repeat them here.

For the numerical generation of M_ν consistent with those relations in Eq.(95), we vary θ_{12}, θ_{13} and δm^2 within their allowed ranges at the 3–σ level precision reported in Table (1), while θ_{23} is varied in the range $\left[43^0, 47^0\right]$ in order to keep it not far away from the value predicted upon imposing exact μ–τ symmetry. The Dirac phase δ and the phase θ are varied in their full ranges, while the parameter $|\chi|$ characterizing the small deviation from the exact μ–τ symmetry is consistently kept small satisfying $|\chi| \leq 0.3$. Scanning randomly the 7-dim free parameter space (reading "random" values of $\theta_{12}, \theta_{23}, \theta_{13}, \delta, \delta m^2, \theta, |\chi|$ in their prescribed ranges), then determining the A, B's coefficients (Eq. 97) and producing the mass ratios and Majorana phases as determined by Eqs.(98) allow us, after computing the quantities of Eq.(99), to confront the theoretical predictions of the texture versus the experimental constraints in Table (1), and whence to figure out the admissible 7-dim parameter space region. Knowing the masses and the angles in the admissible region allows us to reconstruct the whole neutrino mass matrix M_ν which, as should be stressed, is based on numerical calculations using the exact formulas in Eqs.(98–99).

The resulting mass patterns are found to be classifiable into three categories:

- Normal hierarchy: characterized by $m_1 < m_2 < m_3$ and is denoted by **N** satisfying numerically the bound:
$$\frac{m_1}{m_3} < \frac{m_2}{m_3} < 0.7 \tag{103}$$
- Inverted hierarchy: characterized by $m_3 < m_1 < m_2$ and is denoted by **I** satisfying the bound:
$$\frac{m_2}{m_3} > \frac{m_1}{m_3} > 1.3 \tag{104}$$
- Degenerate hierarchy (meaning quasi- degeneracy): characterized by $m_1 \approx m_2 \approx m_3$ and is denoted by **D**. The corresponding numeric bound is taken to be:
$$0.7 < \frac{m_1}{m_3} < \frac{m_2}{m_3} < 1.3 \tag{105}$$

Moreover, we studied for each pattern the possibility of having a singular (non-invertible) mass matrix characterized by one of the masses (m_1, and m_3) being equal to zero (the data prohibits the simultaneous vanishing of two masses and thus m_2 can not vanish). It turns out that the violation of exact μ–τ symmetry does not allow for the singular neutrino mass matrix. The reason behind this is rather simple and can be clarified through examining the mass ratio expressions $\frac{m_2}{m_3}$ and $\frac{m_2}{m_1}$ which respectively characterize the cases $m_1 = 0$ and $m_3 = 0$. The mass ratio expressions can be evaluated in terms of A's or B's coefficients defined in Eq.(97) and can also be related to R_ν leading to the following results, for the case $m_1 = 0$:

$$\frac{m_2}{m_3} \;=\; \left\{ \begin{array}{lcl} \left|\frac{A_3}{A_2}\right| & \approx & \sqrt{\frac{|\chi|^2\, c^2_{\theta_{23}} + 2\,|\chi|\, c_\theta\, c_{\theta_{23}}\left(s_{\theta_{23}} + c_{\theta_{23}}\right) + 1 + s_{2\theta_{23}}}{|\chi|^2\, s^2_{\theta_{23}} + 2\,|\chi|\, c_\theta\, c_{\theta_{23}}\left(s_{\theta_{23}} - c_{\theta_{23}}\right) + 1 - s_{2\theta_{23}}}}\; \frac{s_{\theta_{13}}}{s_{\theta_{12}}\; c_{\theta_{12}}} + O(s^2_{\theta_{13}}), \\ & \approx & \sqrt{\frac{1 + s_{2\theta_{23}}}{1 - s_{2\theta_{23}}}}\; \frac{s_{\theta_{13}}}{s_{\theta_{12}}\; c_{\theta_{12}}} + O(s_{\theta_{13}}\, |\chi|), \\ \left|\frac{B_3}{B_2}\right| & \approx & \frac{1}{c^2_{\theta_{12}}} \left(1 + 2\, t_{\theta_{12}}\, t_{2\theta_{23}}\, c_\delta\, s_{\theta_{13}}\right) + O(s^2_{\theta_{13}}), \end{array} \right\} \approx \sqrt{R_\nu}, \tag{106}$$

and for the case $m_3 = 0$:

$$\frac{m_2}{m_1} = \left\{ \begin{array}{lcl} \left|\frac{A_1}{A_2}\right| & \approx & 1 - \frac{|\chi|^2 s_{\theta_{23}} c_{\theta_{23}} c_\delta + |\chi| \left[c_\delta c_\theta \left(s_{2\theta_{23}} - c_{2\theta_{23}}\right) + s_\theta s_\delta\right] - c_\delta c_{2\theta_{23}}}{|\chi|^2 s^2_{\theta_{23}} + 2|\chi| c_\theta s_{\theta_{23}} \left(s_{\theta_{23}} - c_{\theta_{23}}\right) + 1 - s_{2\theta_{23}}} \frac{s_{\theta_{13}}}{s_{\theta_{12}} c_{\theta_{12}}} + O(s^2_{\theta_{13}}), \\ & \approx & 1 + \frac{c_\delta c_{2\theta_{23}} s_{\theta_{13}}}{s_{\theta_{12}} c_{\theta_{12}} \left(1 - s_{2\theta_{23}}\right)} + O(s_{\theta_{13}} |\chi|), \\ \left|\frac{B_1}{B_2}\right| & \approx & t^2_{\theta_{12}} \left(1 + \frac{2\, t_{2\theta_{23}} c_\delta s_{\theta_{13}}}{s_{\theta_{12}} c_{\theta_{12}}}\right) + O(s^2_{\theta_{13}}), \end{array} \right\} \approx 1 + \frac{R_\nu}{2}. \quad (107)$$

The mass ratio $\frac{m_2}{m_3}$ for the case $m_1 = 0$ should be approximately equal to $\sqrt{R_\nu}$, which means that it should be much less than one. The expression obtained from the A's, although it starts from $O(s_{\theta_{13}})$, can not be tuned to a small value compatible with $\sqrt{R_\nu}$ for any admissible values for the mixing angles. The mixing angle θ_{13} plays the decisive role in this failure for not being small enough as Table (1) shows. Thus no need to examine the second expression derived from the B's, and we conclude the impossibility of having $m_1 = 0$ with an approximate $\mu - \tau$-symmetry. Regarding the case $m_3 = 0$, the mass ratio $\frac{m_2}{m_1}$ should be approximately equal to $\left(1 + \frac{R_\nu}{2}\right)$ and accordingly would be slightly greater than one. Each one of the two available expressions providing the mass ratio can be separately tuned to fit the desired value within the admissible ranges for the mixing angles and the Dirac phase δ. However, the compatibility of the two expressions purports the condition, $\frac{c_{2\theta_{23}}}{2 s_{2\theta_{23}} \left(1 - s_{2\theta_{23}}\right)} \approx R_\nu$, which can not be met for any admissible choice for θ_{23}. Our numerical study confirms this conclusion where all the phenomenologically acceptable ranges for mixing angles and Dirac phase are scanned, but no solutions could be found satisfying the mass constraint expressed in Eqs. (106–107). Obviously, our conclusions remain the same when we consider the exact μ–τ symmetry corresponding to $\chi = 0$.

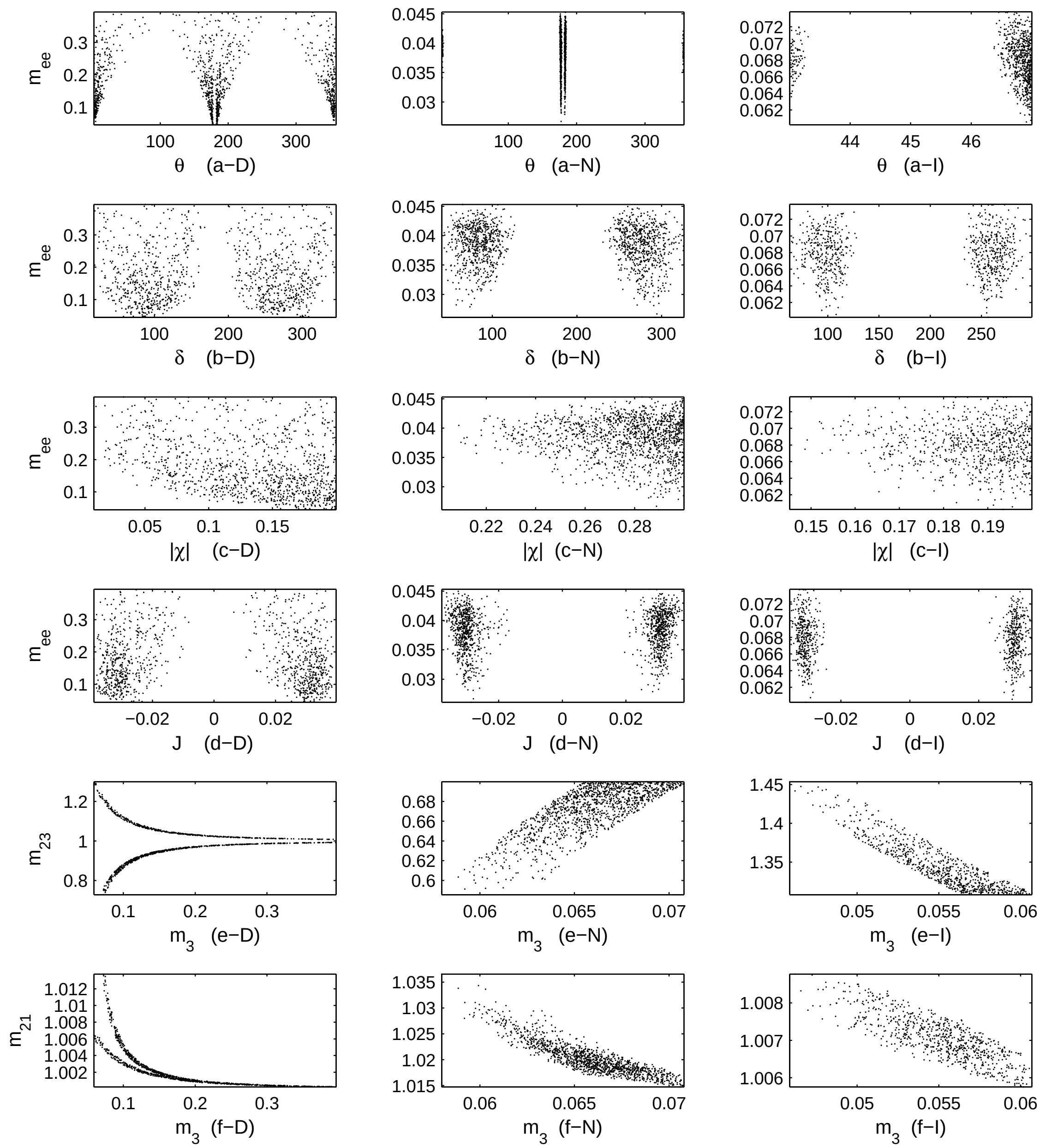

Figure 1: The correlations of $\langle m\rangle_{ee}$ against θ, δ, $|\chi|$ and J are depicted in the first four rows, wheras the last two rows are reserved for the correlations of mass ratios m_{23} and m_{21} against m_3.

The pattern: $M_{\nu\,12} + M_{\nu\,13}\,(1+\chi) = 0$, and $M_{\nu\,22} - M_{\nu\,33} = 0$													
θ_{12}	θ_{23}	θ_{13}	m_1	m_2	m_3	ρ	σ	δ	$\langle m \rangle_e$	$\langle m \rangle_{ee}$	J	$\lvert\chi\rvert$	θ
Degenerate Hierarchy													
30.98 − 36.2	[43, 44.9] ⋃ [45.1 − 47]	7.67 − 9.94	0.0521 − 0.3955	00.0529 − 0.3955	0.0590 − 0.3960	[0.003 − 14.12] ⋃ [166.3 − 179.89]	[0.55 − 31] ⋃ [154.4 − 179.45]	[18.3 − 168.1] ⋃ [197.7 − 345.61]	0.0528 − 0.3954	0.0452 − 0.3941	[−0.0390 − −0.0082] ⋃ [0.0064 − 0.0397]	0.01 − 0.2	[2.19 − 83.3] ⋃ [119 − 178.9] ⋃ [181.5 − 248.3] ⋃ [282.3 − 358.82]
Normal Hierarchy													
30.98 − 36.11	[43, 43.78] ⋃ [46.61 − 47]	7.66 − 9.87	0.0329 − 0.0487	0.0329 − 0.0487	0.0580 − 0.0708	[0.003 − 19.47] ⋃ [160.2 − 179.72]	[11.04 − 35.59] ⋃ [144.9 − 169.34]	[40.44 − 126.3] ⋃ [230.7 − 326.22]	0.0339 − 0.0495	0.0261 − 0.0453	[−0.0379 − −0.0168] ⋃ [0.0196 − 0.0382]	0.2 − 0.3	[2.8 − 4.6] ⋃ [175.6 − 178.6] ⋃ [181.8 − 185] ⋃ [355.6 − 358.9]
Inverted Hierarchy													
30.99 − 36.08	[43, 43.31] ⋃ [46.38 − 47]	7.66 − 9.37	0.0660 − 0.0790	0.0666 − 0.0795	0.0459 − 0.0607	[3.35 − 10.76] ⋃ [169.8 − 176.88]	[12.68 − 22.76] ⋃ [157.8 − 168.64]	[62.73 − 127.9] ⋃ [233.2 − 299.14]	0.0659 − 0.0788	0.0602 − 0.0738	[−0.0349 − −0.0243] ⋃ [0.0232 − 0.0354]	0.15 − 0.2	[3.4 − 4.33] ⋃ [175.5 − 177.6] ⋃ [182.4 − 184.7] ⋃ [355.5 − 356.91]

Table 2: Various predictions of allowed ranges for one pattern violating the exact μ–τ symmetry. All the angles (masses) are evaluated in degrees (eV).

Degenerate Hierarchy																		
A_ν	B_ν	C_ν	D_ν	A_R	B_R	C_R	D_R	A_D	B_D	C_D	D_D	E_D	χ	α	θ_{12}	φ	θ_{23}	θ_{13}
0.8187 + 0.0085 i	−0.0278 − 0.0300 i	0.4165 − 0.4094 i	0.3890 + 0.4097 i	0.8188 + 0.0086 i	−0.0297 − 0.0313 i	0.4165 − 0.4094 i	0.3890 + 0.4097 i	0.8187 + 0.0086 i	−0.0337 − 0.0232 i	0.4165 − 0.4093 i	0.3890 + 0.4096 i	−0.0238 − 0.0380 i	0.1089 − 0.0243 i	0.1116	32.63	34.33	44.49	9.44
0.8045 − 0.0260 i	−0.0229 + 0.0331 i	0.5365 + 0.3771 i	0.2557 − 0.3780 i	0.8046 − 0.0259 i	−0.0248 + 0.0366 i	0.5365 + 0.3771 i	0.2557 − 0.3780 i	0.8046 − 0.0259 i	−0.0185 + 0.0358 i	0.5365 + 0.3771 i	0.2557 − 0.3780 i	−0.0293 + 0.0339 i	0.1960 − 0.0257i	0.1977	35.81	34.53	44.33	9.64
0.5440 + 0.0119 i	−0.0351 − 0.0074 i	0.0152 − 0.1167 i	0.5077 + 0.1169 i	0.5441 + 0.0118 i	−0.0376 − 0.0087 i	0.0152 − 0.1167 i	0.5077 + 0.1169 i	0.5440 + 0.0118 i	−0.0320 − 0.0162 i	0.0152 − 0.1166 i	0.5076 + 0.1169 i	−0.0407 + 0.0002 i	0.1558 + 0.0417 i	0.1613	32.50	34.60	44.55	8.43
δ_ν	δ_ν^0	$\rho^{\text{exa.}}$	ρ^{per}	σ^{exa}	$\sigma^{\text{per.}}$	m_1^0	m_2^0	m_3^0	m_1	m_2	m_3	m_{R3}	m_{R2}	m_{R1}	m_{D1}	m_{D2}	m_{D3}	
42.36	42.76	1.69	2.31	176.94	178.04	0.2511	0.2517	0.2466	0.2515	0.2517	0.2465	8.22	8.21	8.06	144.22	143.23	140.96	
142.75	142.72	0.68	1.23	175.93	176.79	0.2469	0.2475	0.2426	0.2473	0.2475	0.2424	8.0813	8.0735	7.9223	141.88	140.75	138.64	
260.66	259.97	178.79	177.89	5.18	3.63	0.1671	0.1679	0.1601	0.1676	0.1678	0.1599	5.48	5.47	5.23	96.37	95.15	91.50	
Normal Hierarchy																		
A_ν	B_ν	C_ν	D_ν	A_R	B_R	C_R	D_R	A_D	B_D	C_D	D_D	E_D	χ	α	θ_{12}	φ	θ_{23}	θ_{13}
0.1287 − 0.0021 i	0.0538 + 0.0038 i	0.0485 − 0.0115 i	0.1758 + 0.0115 i	0.1297 − 0.0016 i	0.0611 + 0.0040 i	0.0485 − 0.0115 i	0.1758 + 0.0115 i	0.1294 − 0.0016 i	0.0540 + 0.0001 i	0.0485 − 0.0115 i	0.1758 + 0.0115 i	0.0609 + 0.0078 i	0.2700 − 0.0192 i	0.2707	35.75	33.03	46.94	7.86
0.1333 − 0.0148 i	0.0480 + 0.0104 i	0.0544 − 0.0355 i	0.1689 + 0.0353 i	0.1344 − 0.0142 i	0.0553 + 0.0115 i	0.0544 − 0.0355 i	0.1689 + 0.0353 i	0.1341 − 0.0143 i	0.0486 + 0.0070 i	0.0544 − 0.0355 i	0.1689 + 0.0353 i	0.0546 + 0.0150 i	0.2985 − 0.0213 i	0.2992	35.44	32.62	46.87	8.08
0.1325 + 0.0127 i	0.0488 − 0.0093 i	0.0537 + 0.0318 i	0.1716 − 0.0316 i	0.1337 + 0.0122 i	0.0562 − 0.0103 i	0.0537 + 0.0318 i	0.1716 − 0.0316 i	0.1334 + 0.0122 i	0.0494 − 0.0058 i	0.0538 + 0.0317 i	0.1715 − 0.0316 i	0.0555 − 0.0140 i	0.2978 + 0.0221 i	0.2986	36.08	33.02	46.84	7.93
δ_ν	δ_ν^0	$\rho^{\text{exa.}}$	ρ^{per}	σ^{exa}	$\sigma^{\text{per.}}$	m_1^0	m_2^0	m_3^0	m_1	m_2	m_3	m_{R2}	m_{R1}	m_{R3}	m_{D1}	m_{D2}	m_{D3}	
97.63	99.65	167.90	177.15	24.17	78.78	0.0457	0.0461	0.0687	0.0471	0.0479	0.0691	1.57	1.55	2.24	27.71	25.82	39.24	
82.97	84.76	166.26	175.94	19.23	99.86	0.0461	0.0465	0.0684	0.0474	0.0482	0.0688	1.58	1.55	2.23	27.91	26.02	39.08	
275.52	273.35	13.75	4.18	160.58	88.91	0.0460	0.0464	0.0690	0.0473	0.0481	0.0694	1.58	1.55	2.25	27.86	25.94	39.43	
Inverted Hierarchy																		
A_ν	B_ν	C_ν	D_ν	A_R	B_R	C_R	D_R	A_D	B_D	C_D	D_D	E_D	χ	α	θ_{12}	φ	θ_{23}	θ_{13}
0.2322 + 0.0012 i	−0.0613 − 0.0085 i	−0.0165 − 0.0282 i	0.2113 + 0.0283 i	0.2329 + 0.0016 i	−0.0674 − 0.0090 i	−0.0165 − 0.0282 i	0.2113 + 0.0283 i	0.2326 + 0.0016 i	−0.0617 − 0.0046 i	−0.0164 − 0.0282 i	0.2113 + 0.0283 i	−0.0669 − 0.0131 i	0.1960 − 0.0129 i	0.1964	33.63	23.15	43.17	8.03
0.2158 − 0.0033 i	−0.0600 − 0.0021 i	−0.0194 − 0.0058 i	0.1987 + 0.0058 i	0.2165 − 0.0030 i	−0.0658 − 0.0019 i	−0.0194 − 0.0058 i	0.1987 + 0.0058 i	0.2162 − 0.0029 i	−0.0600 + 0.0023 i	−0.0194 − 0.0058 i	0.1987 + 0.0058 i	−0.0657 − 0.0064 i	0.1909 − 0.0142 i	0.1914	32.66	24.02	43.18	7.69
0.2219 − 0.0043 i	−0.0603 − 0.0002 i	−0.0200 + 0.0003 i	0.2044 − 0.0004 i	0.2226 − 0.0040 i	−0.0663 + 0.0001 i	−0.0200 + 0.0003 i	0.2044 − 0.0004 i	0.2223 − 0.0039 i	−0.0602 + 0.0040 i	−0.0199 + 0.0003 i	0.2043 − 0.0004 i	−0.0664 − 0.0041 i	0.1990 − 0.0140 i	0.1995	35.68	24.00	43.16	7.93
δ_ν	δ_ν^0	$\rho^{\text{exa.}}$	ρ^{per}	σ^{exa}	$\sigma^{\text{per.}}$	m_1^0	m_2^0	m_3^0	m_1	m_2	m_3	m_{R3}	m_{R2}	m_{R1}	m_{D1}	m_{D2}	m_{D3}	
73.18	62.18	7.52	5.98	162.91	158.33	0.0758	0.0769	0.0597	0.0773	0.0778	0.0593	2.54	2.52	1.95	44.75	43.14	34.10	
76.62	67.45	6.86	5.94	160.97	157.89	0.0708	0.0719	0.0549	0.0723	0.0729	0.0546	2.38	2.35	1.79	41.88	40.33	31.38	
81.34	69.41	7.58	5.69	163.23	158.12	0.0726	0.0737	0.0565	0.0741	0.0746	0.0561	2.44	2.41	1.84	42.91	41.31	32.27	

Table 3: Numerically generated relevant parameters for M_ν, M_R and M_ν^D. Light neutrino masses are evaluated in units of eV, Dirac neutrino masses in units of GeV, and Majorana masses in units of 10^{13} Gev. The angles are evaluated in degrees.

Regarding the non-singular pattern, one can deduce some restrictions concerning mixing angles and phase just by considering the approximate expression for R_ν as given in Eq.(102). The parameter R_ν must be positive, non-vanishing ($R_\nu \approx 0.03$) and its valued at the $3-\sigma$ level is reported in Table (1). This clearly requires non-vanishing values for $s_{\theta_{13}}$, s_δ, s_θ and $|\chi|$. The nonvanishing of $s_{\theta_{13}}$ implies $\theta_{13} \neq 0$ which is phenomenologically favorable, while the nonvanishing of s_δ and s_θ excludes 0, π and 2π for both δ and θ. The reported allowed range for θ and δ in Table (2) confirms these exclusions. The nonvanishing of $|\chi|$ is naturally expected otherwise there would not be a deviation from exact $\mu-\tau$ symmetry. These conclusions remain valid if one used the exact expression for R_ν instead of the first order expression. Explicit computations of R_ν using its exact expression reveal that θ_{23} cannot be exactly equal to $\frac{\pi}{4}$, otherwise R_ν would be zero, but nevertheless θ_{23} can possibly stay very close to $\frac{\pi}{4}$, and this again is confirmed by the reported allowed values for θ_{23} in Table (2).

For the sake of illustration, we show correlations involving $\langle m\rangle_{ee}$ against θ, δ, $|\chi|$ and J where J is the Jarlskog rephasing invariant quantity which is given by $J = s_{\theta_{12}}\, c_{\theta_{12}}\, s_{\theta_{23}}\, c_{\theta_{23}}\, s_{\theta_{13}}\, c^2_{\theta_{13}} \sin\delta$ [27]. The quantity $\langle m\rangle_{ee}$ is extremely important as a measure of neutrinoless double beta decay and provides a clear signature for the true nature of neutrino. The non-vanishing value for $\langle m\rangle_{ee}$, if experimentally confirmed, will definitely establish the nature of neutrino as being Majorana particle. But so far, no convincing experimental evidence of the decay exists. Other important correlations are also displayed for those involving the mass ratios m_{12} and m_{23} against m_3 which could reveal the hierarchy strength.

In Fig. 1, the plots (a) and (b) clearly reveal the allowed band regions for both θ and δ which are quite distinct in the case of normal and inverted hierarchy, and in addition they show also the excluded region around 0 and π. This behavior can be mainly attributed to the constraint imposed by the parameter R_ν. As to the plots (c), they do not point out any clear correlation between $\langle m\rangle_{ee}$ and $|\chi|$, but remarkably one can realize that in case of inverted and normal hierarchy the parameter $|\chi|$ generally tends to be larger than what is required to be in the quasi degenerate case. Regarding the correlation of $\langle m\rangle_{ee}$ against J (plots (d)), it is, as expected, another manifestation of the correlation $\langle m\rangle_{ee}$ against δ, since in our investigation the size of J is only controlled by δ while it is apparently insensitive to the other mixing angles. The values of $\langle m\rangle_{ee}$ can not attain the zero-limit in all types of hierarchy, which is evident from the graphs or explicitly from the corresponding covered ranges in Table (2). There are some characteristic features for the possible hierarchies as can be observed from the plots (e) and (f), and which turn out to be crucial in deriving a simple formula for $\langle m\rangle_{ee}$. First, the masses m_1 and m_2 are approximately equal, as is clear in Fig. 1 (plots: f); second, the hierarchy is mild in both normal and inverted cases, as is evident from Fig. 1 (plots: e-N, e-I). The simple approximate formula for $\langle m\rangle_{ee}$, capturing the essential observed features for all kinds of hierarchies, can be deduced, assuming $m_1 \approx m_2$, from Eq. (99) to be in the form:

$$\langle m\rangle_{ee} \approx m_1\, c^2_{\theta_{13}} \sqrt{\left[1 - s^2_{2\theta_{12}} \sin^2(\rho-\sigma)\right]}. \tag{108}$$

The formula clearly points out that the $\langle m\rangle_{ee}$ scale is of the order of the scale of $m_1(\approx m_2)$ as is confirmed from the corresponding covered ranges stated in Table 2.

The numerical generation for possible M_R and M_ν^D for a given numerically generated M_ν proceeds through the following routine (Again, this does not exhaust all possible M_ν^D, M^R leading to the given M_ν). The first step consists in assuming that M_R is "proportional" to M_ν but obeying exact μ–τ symmetry. Thus the entries of M_R can be assumed to be:

$$\begin{aligned} A_R &= \Lambda_R\, M_{\nu\,11}/v^2 = A_\nu, \quad & B_R &= \Lambda_R\, \left(M_{\nu\,11} - M_{\nu\,13}\right)/(2v^2) \approx B_\nu, \\ C_R &= \Lambda_R\, M_{\nu\,22}/v^2 = C_\nu, & D_R &= \Lambda_R\, M_{\nu\,23}/v^2 = D_\nu, \end{aligned} \tag{109}$$

As said before, we took v the electroweak scale characterizing the Dirac neutrino to be 175 GeV (around the top quark mass), whereas Λ_R the high energy scale characterizing the heavy RH Majorana neutrino is taken to be around 10^{14} GeV, so the scale characterizing the effective light neutrino v^2/Λ_R would be around 0.3 eV in agreement with data. In the second step, we assume the equality of α and $|\chi|$. Consequently, the system of five equations given by the seesaw formula (Eq. 51) applied to the symmetric matrix M_ν with ($M_{\nu 22} = M_{\nu 33}$) can then be solved for the five unknowns residing in the Dirac mass matrix having the form described in Eq.(75). We have solved this non-linear system of equations by iteration starting with the initial guess ($A_D = A_R, B_D = B_R, C_D = C_R$ and $E_D = B_R$).

Having all parameters $A_R, \cdots, D_R$, $A_D, \cdots, E_D$ and α enables us to numerically produce the neutrino relevant quantities. In Table (3), we report for each possible type of hierarchy three representative points containing all the parameters describing M_ν, M_R and M_ν^D. In addition, the same table also contains the values of the mixing angles, the phase angles and the masses of the light neutrinos, computed on one hand according to the exact formulae and on the other hand according to the perturbative formulae, and the two ways of computing showed good agreement. We did the perturbative calculations starting from (M_R, M_ν^D, α), deduced in turn from M_ν and the corresponding χ, by computing M_α (Eqs. 83 and 82) and M_ν^0 (Eq. 81) and then deducing the ϵ's (Eq. 87), followed by plugging them into the perturbative formulae for the mixing angles (Eq. 91), the phases (Eq. 92) and the masses (Eq. 90).

Furthermore, the eigen masses for M_R and unperturbed M_ν^D are as well reported in Table (3). We note here that we get an almost degenerate RH neutrino mass spectrum. Actually, we get for the degenerate- and inverted-hierarchy examples a mild hierarchy in the RH eigenmasses ($m_{R1} \leq m_{R2} \simeq m_{R3}$), and so one would expect a scenario where a considerable part of the CP asymmetry is due to the decay of the lightest RH neutrino N_1. In order to estimate the baryon asymmetry in these examples one can follow the analysis of subsection 5.3 but with caution considering that we assumed there a strong hierarchy in the RH neutrino eigen masses leading often to N_1-dominated scenario. On the other hand, we obtain for the normal-hierarchy examples a mild hierarchy where the two lightest RH neutrinos are the almost degenerate ones ($m_{R1} \simeq m_{R2} \leq m_{R3}$), and so we would expect a scenario where the CP asymmetry is due to the decay of, at least, both N_1 and N_2. Here, one should go beyond the hierarchical limit assumed in subsection 5.3 to estimate the baryon asymmetry. In [28, 29], analytical formulae for the baryon asymmetry, corresponding to the case $m_{R1} \simeq m_{R2} \ll m_{R3}$, were obtained, and in [30] other approximate expressions, which were shown [31] to agree well with the former ones, were derived. Although the extrapolation from the almost-degenerate two RH neutrinos case to the case of three RH neutrinos of approximately similar masses may plausibly be smooth regarding the fit to the Boltzmann equations, however we did not carry out the estimation of the baryon asymmetry in Table 3 in any of the numerical examples we had, as the precise calculations go beyond the scope of the paper and the formulated expressions are approximate, so one needs a more refined analysis in order to draw conclusions. Nonetheless, we have checked our assumption that the ϵ's (Eqs. 87) are far smaller than 1 in accordance with them being as perturbative factors.

8 Realization of perturbed texture

As we saw, perturbed textures are needed in order to account for phenomenology. We have two ways to seek models for achieving these perturbations. The first method consists of introducing a term in the Lagrangian which breaks explicitly the symmetry [21], and then of expressing the new perturbed texture in terms of this breaking term. The second method is to keep assuming the exact symmetry, but then we break it spontaneously by introducing new matter and enlarging the symmetry. We follow here the second approach in order to find a realization of the forms given in Eq.(75) for M_D and in Eq.(56) for M_R, while assuring that we work in the flavor basis. However, for the sake of minimum added matter, we shall not force the most general forms of M_R and M_D, but rather be content with special forms of them leading to an effective mass matrix M_ν of the desired perturbed texture (Eq. 79). In [16] a realization was given for a perturbed texture corresponding to the S_--symmetry, whereas here we treat the more phenomenologically motivated S_+-symmetry (we shall drop henceforth the +suffix). We present two ways, not meant by whatsoever to be restrictive but rather should be looked at as proof of existence tools, to get the three required conditions of a "perturbed" M_D, non-perturbed M_R and diagonal $M_l M_l^\dagger$. Both ways add new matter, but whereas the first approach adds just a $(Z_2)^2$ factor to the $S-$symmetry while requiring some Yukawa couplings to vanish, the second approach enlarges the symmetry larger to $S \times Z_8$ but without need to equate Yukawa couplings to zero by hand. Some "form invariance" relations are in order:

$$\left\{ \left(M = M^{\mathrm{T}} \right) \wedge \left[S^{\mathrm{T}} \cdot M \cdot S = M \right] \right\} \quad \Leftrightarrow \quad \left[M = \begin{pmatrix} A & B & -B \\ B & C & D \\ -B & D & C \end{pmatrix} \right], \tag{110}$$

$$\left\{\left(M=M^{\mathrm{T}}\right)\wedge\left[S^{\mathrm{T}}\cdot M\cdot S=-M\right]\right\} \;\Leftrightarrow\; \left[M=\begin{pmatrix}0 & B & B\\ B & C & 0\\ B & 0 & -C\end{pmatrix}\right], \tag{111}$$

$$\left[S^{\mathrm{T}}\cdot M\cdot S=M\right] \;\Leftrightarrow\; \left[M=\begin{pmatrix}A & B & -B\\ E & C & D\\ -E & D & C\end{pmatrix}\right], \tag{112}$$

$$\left[S^{\mathrm{T}}\cdot M\cdot S=-M\right] \;\Leftrightarrow\; \left[M=\begin{pmatrix}0 & B & B\\ E & C & D\\ E & -D & -C\end{pmatrix}\right], \tag{113}$$

We denote $L^{\mathrm{T}}=(L_1,L_2,L_3)$ with L_i's,$(i=1,2,3)$ are the components of the i^{th}-family LH lepton doublets (we shall adopt this notation of 'vectors' in flavor space even for other fields, like l^c the RH charged lepton singlets, ν_R the RH neutrinos, ...).

8.1 $S\times Z_2\times Z_2'$-flavor symmetry

- **Matter content and symmetry transformations**

  We have three SM-like Higgs doublets (ϕ_i, $i=1,2,3$) which would give mass to the charged leptons and another three Higgs doublets (ϕ_i', $i=1,2,3$) for the Dirac neutrino mass matrix. All the fields are invariant under Z_2' except the fields ϕ' and ν_R which are multiplied by -1, so that we assure that neither ϕ can contribute to M_D, nor ϕ' to M_l. The fields transformatios are as follows.

$$\nu_R \xrightarrow{Z_2} \mathrm{diag}\,(1,-1,-1)\,\nu_R,\;\; \phi' \xrightarrow{Z_2} \mathrm{diag}\,(1,-1,-1)\,\phi', \tag{114}$$
$$L \xrightarrow{Z_2} \mathrm{diag}\,(1,-1,-1)\,L,\;\; l^c \xrightarrow{Z_2} \mathrm{diag}\,(1,1,-1)\,l^c,\;\; \phi \xrightarrow{Z_2} \mathrm{diag}\,(1,-1,-1)\,\phi, \tag{115}$$

$$\nu_R \xrightarrow{S} S\nu_R,\;\; \phi' \xrightarrow{S} \mathrm{diag}\,(1,1,-1)\,\phi', \tag{116}$$
$$L \xrightarrow{S} SL,\;\; l^c \xrightarrow{S} l^c,\;\; \phi \xrightarrow{S} S\phi, \tag{117}$$

- **Charged lepton mass matrix-flavor basis**

  The Lagrangian responsible for M_l is given by:

$$\mathcal{L}_2 \;=\; f^j_{ik}\overline{L}_i\phi_k l^c_j \tag{118}$$

  The transformations under S and Z_2, with the "form invariance" relations Eqs. (110–113), lead to:

$$f^{(1)}=\begin{pmatrix}A^1 & 0 & 0\\ 0 & C^1 & D^1\\ 0 & D^1 & C^1\end{pmatrix}, f^{(2)}=\begin{pmatrix}A^2 & 0 & 0\\ 0 & C^2 & D^2\\ 0 & D^2 & C^2\end{pmatrix}, f^{(3)}=\begin{pmatrix}0 & B^3 & -B^3\\ E^3 & 0 & 0\\ -E^3 & 0 & 0\end{pmatrix} \tag{119}$$

  where f^j_{ik} is the $(i,k)^{\mathrm{th}}$-entry of the matrix $f^{(j)}$. Assuming $(v_3\gg v_1,v_2)$ we get:

$$M_l=v_3\begin{pmatrix}0 & 0 & -B^3\\ D^1 & D^2 & 0\\ C^1 & C^2 & 0\end{pmatrix} \;\Rightarrow\; M_l\,M_l^{\dagger}=v_3^2\begin{pmatrix}|\mathbf{B}|^2 & 0 & 0\\ 0 & |\mathbf{D}|^2 & \mathbf{D}\cdot\mathbf{C}\\ 0 & \mathbf{C}\cdot\mathbf{D} & |\mathbf{C}|^2\end{pmatrix}, \tag{120}$$

  where $\mathbf{B}=\left(0,0,-B^3\right)^T$, $\mathbf{D}=\left(D^1,D^2,0\right)^T$ and $\mathbf{C}=\left(C^1,C^2,0\right)^T$, and where the dot product is defined as $\mathbf{D}\cdot\mathbf{C}=\sum_{i=1}^{i=3}D^iC^{i*}$. Under the reasonable assumption that the magnitudes of the Yaukawa couplings come in ratios proportional to the lepton mass ratios as $|B|:|C|:|D|\sim m_e:m_\mu:\mu_\tau$, one can show, as was done in [16], that the diagonalization of the charged lepton mass matrix can be achieved by infinitesimally rotating the LH charged lepton fields, which justifies working in the flavor basis to a good approximation.

- **Majorana neutrino mass matrix**
The mass term is directly present in the Lagrangian

$$\mathcal{L}_R \quad = \quad M_{R\,ij}\,\nu_{Ri}\,\nu_{Rj}\ . \tag{121}$$

The invariance under Z_2' is trivially satisfied while the one under $S \times Z_2$ is more involved. The symmetry S constrains M_R to satisfy

$$S^T\,M_R\,S = M_R, \tag{122}$$

whereas the restrictions due to Z_2 are imprinted in the bilinear of $\nu_{Ri}\,\nu_{Rj}$ determining their transformations under Z_2 as:

$$\nu_{Ri}\,\nu_{Rj} \overset{Z_2}{\sim} B \quad = \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \tag{123}$$

which means:

$$\nu_{Ri}\,\nu_{Rj} \xrightarrow{Z_2} Z_2(\nu_{Ri}\,\nu_{Rj}) \quad = \quad B_{ij}\nu_{Ri}\,\nu_{Rj}(\text{no sum}) \tag{124}$$

Thus the symmetry through Eqs.(110,122,123) entails that M_R would assume the following form,

$$M_R \quad = \quad \begin{pmatrix} A_R & 0 & 0 \\ 0 & C_R & D_R \\ 0 & D_R & C_R \end{pmatrix}. \tag{125}$$

which is of the general form (Eq. 56) with $B_R = 0$.

- **Dirac neutrino mass matrix**

The Lagrangian responsible for the neutrino mass matrix is

$$\mathcal{L}_D \quad = \quad g^k_{ij}\overline{L}_i\tilde{\phi}'_k\nu_{Rj}\ ,\ \text{where } \tilde{\phi}' = i\sigma_2\phi'^* \tag{126}$$

Because of the fields transformations under S and Z_2 we get:

$$S^{\mathrm{T}}g^{(k=1,2)}S = g^{(k=1,2)} \quad , \quad S^{\mathrm{T}}g^{(k=3)}S = -g^{(k=3)}, \overline{L}_i\nu_{Rj} \overset{Z_2}{\sim} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \tag{127}$$

where $g^{(k)}$ is the matrix whose $(i,j)^{th}$-entry is the Yukawa coupling g^k_{ij}. Then, the "form invariance" relations (Eqs.110–113) lead to:

$$g^{(1)} = \begin{pmatrix} \mathcal{A}^1 & 0 & 0 \\ 0 & \mathcal{C}^1 & \mathcal{D}^1 \\ 0 & \mathcal{D}^1 & \mathcal{C}^1 \end{pmatrix}, g^{(2)} = \begin{pmatrix} 0 & \mathcal{B}^2 & -\mathcal{B}^2 \\ \mathcal{E}^2 & 0 & 0 \\ -\mathcal{E}^2 & 0 & 0 \end{pmatrix}, g^{(3)} = \begin{pmatrix} 0 & \mathcal{B}^3 & \mathcal{B}^3 \\ \mathcal{E}^3 & 0 & 0 \\ \mathcal{E}^3 & 0 & 0 \end{pmatrix} \tag{128}$$

Upon acquiring vevs (v'_i, $i = 1,2,3$) for the Higgs fields (ϕ'_i), we get for Dirac neutrino mass matrix the form:

$$M_D \quad = \quad \begin{pmatrix} v'_1\,\mathcal{A}^1 & v'_2\,\mathcal{B}^2 + v'_3\,\mathcal{B}^3 & -v'_2\,\mathcal{B}^2 + v'_3\,\mathcal{B}^3 \\ v'_2\,\mathcal{E}^2 + v'_3\,\mathcal{E}^3 & v'_1\,\mathcal{C}^1 & v'_1\,\mathcal{D}^1 \\ -v'_2\,\mathcal{E}^2 + v'_3\,\mathcal{E}^3 & v'_1\,\mathcal{D}^1 & v'_1\,\mathcal{C}^1 \end{pmatrix}, \tag{129}$$

which can be put into the form,

$$M_D \quad = \quad \begin{pmatrix} A_D & B_D\,(1+\alpha) & -B_D \\ E_D\,(1+\beta) & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix}. \tag{130}$$

with

$$\alpha = \frac{2v_3'\mathcal{B}^3}{v_2'\mathcal{B}^2 - v_3'\mathcal{B}^3} \quad , \quad \beta = \frac{2v_3'\mathcal{E}^3}{v_2'\mathcal{E}^2 - v_3'\mathcal{E}^3}. \tag{131}$$

If the vevs satisfy $v_3' \ll v_2'$ and the Yukawa couplings are of the same order, then we get perturbative parameters $\alpha, \beta \ll 1$.

The deformations appearing in the Dirac mass matrix as described in Eqs.(129–131) would resurface in the effective light neutrino mass matrix M_ν through the seesaw formula (Eq.51) with M_R given in Eq.(125). The resulting deformations in M_ν can be described by two parameters:

$$\chi \equiv -\frac{M_\nu\,(1,2) + M_\nu\,(1,3)}{M_\nu\,(1,3)} \quad , \quad \xi \equiv \frac{M_\nu\,(2,2) - M_\nu\,(3,3)}{M_\nu\,(3,3)}. \tag{132}$$

One can repeat now the analysis of the last subsection in order to compute χ, ξ in terms of α, β and other mass parameters to get:

$$\begin{aligned} \chi &= -\frac{\alpha\, A_R B_D\,(C_R - D_R)\,(C_D + D_D) + \beta A_D E_D\left(C_R^2 - D_R^2\right)}{\alpha A_R B_D\,(C_R D_D - D_R C_D) + B_D A_R\,(D_R + C_R)\,(D_D - C_D) - E_D A_D\,(C_R^2 - D_R^2)} \\ \xi &= \frac{\beta\,(\beta - 2)\, E_D^2\left(C_R^2 - D_R^2\right)}{A_R\left[C_R\,(D_D^2 + C_D^2) - 2C_D D_D D_R\right] + E_D^2\,(C_R^2 - D_R^2)}, \end{aligned} \tag{133}$$

We note here that we do not get in general the desired pattern (Eq. 79) corresponding to disentanglement of the perturbations ($\xi = 0$). However, for specific choices of Yukawa couplings, for e.g. $\mathcal{E}^3 = 0$ leading to $\beta = 0$ and hence $\xi = 0$, we get this form, in which case M_D is of the form (Eq.75) and χ of Eq.(133) would also be given by Eq.(80) with $B_R = 0$.

8.2 $S \times Z_8$-flavor symmetry

- **Matter content and symmetry transformations**

  In addition to the left doublets ($L_i,\ i = 1,2,3$), the RH charged singlets ($l_j^c,\ j = 1,2,3$), the RH neutrinos ($\nu_{Rj},\ j = 1,2,3$) and the SM-Higgs three doublets ($\phi_i,\ i = 1,2,3$) responsible for the charged lepton masses, we have now four Higgs doublets ($\phi_j',\ j = 1,2,3,4$) giving rise when acquiring a vev to Dirac neutrino mass matrix, and also two Higgs singlet scalars ($\Delta_k,\ k = 1,2$) related to Majorana neutrino mass matrix. We denote the octic root of the unity by $\omega = e^{\frac{i\pi}{4}}$. The fields transform as follows.

$$L \xrightarrow{S} SL,\ \ l^c \xrightarrow{S} l^c,\ \ \phi \xrightarrow{S} S\phi, \tag{134}$$

$$\nu_R \xrightarrow{S} S\nu_R,\ \ \phi' \xrightarrow{S} \text{diag}\,(1,1,1,-1)\,\phi',\ \ \Delta \xrightarrow{S} \Delta \tag{135}$$

$$L \xrightarrow{Z_8} \text{diag}\,(1,-1,-1)\,L,\ \ l^c \xrightarrow{Z_8} \text{diag}\,(1,1,-1)\,l^c,\ \ \phi \xrightarrow{Z_8} \text{diag}\,(1,-1,-1)\,\phi, \tag{136}$$

$$\nu_R \xrightarrow{Z_8} \text{diag}\left(\omega,\omega^3,\omega^3\right)\nu_R,\ \ \phi' \xrightarrow{Z_8} \text{diag}\left(\omega,\omega^3,\omega^7,\omega^3\right)\phi',\ \ \Delta \xrightarrow{Z_8} \text{diag}\left(\omega^6,\omega^2\right)\Delta \tag{137}$$

  Note here that we have the following transformation rule for $\tilde{\phi}' \equiv i\sigma_2\phi'^*$:

$$\tilde{\phi}' \xrightarrow{S} \text{diag}(1,1,1,-1)\tilde{\phi}' \quad , \quad \tilde{\phi}' \xrightarrow{Z_8} \text{diag}(\omega^7,\omega^5,\omega,\omega^5)\tilde{\phi}' \tag{138}$$

- **Charged lepton mass matrix-flavor basis**

  The symmetry restriction in constructing the charged lepton mass Lagrangian as given by Eq.(118) is similar to what is obtained in the case of ($S \times Z_2 \times Z_2'$). The similarity orginates from the fact that the charges assigned to the fields (L, l^c, ϕ) corresponding to the factor Z_2 (of $S \times Z_2 \times Z_2'$) and that of Z_8 (of $S \times Z_8$) are the same. Thus we end up, assuming again a hierarchy in the Higgs ϕ's fields vevs ($v_3 \gg v_2, v_1$), with a charged lepton mass matrix adjustable to be approximately in the flavor basis. Note also here that the symmetry forbids the term $\overline{L}_i\phi_k' l_j^c$ since we have:

$$\overline{L}_i l_j^c \overset{Z_8}{\approx} \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \quad \overset{\text{Eq.137}}{\Longrightarrow} \quad \nexists i,j,k : \overline{L}_i\phi_k' l_j^c = Z_8(\overline{L}_i\phi_k' l_j^c) \tag{139}$$

- **Dirac neutrino mass matrix**

  The Lagrangian responsible for the Dirac neutrino mass matrix is given by Eq. (126). By means of fields transformations we have:

$$S^{\mathrm{T}} g^{(k=1,2,3)} S = g^{(k=1,2,3)} \quad , \quad S^{\mathrm{T}} g^{(k=4)} S = -g^{(k=4)}, \overline{L}_i \nu_{Rj} \overset{Z_8}{\sim} \begin{pmatrix} \omega & \omega^3 & \omega^3 \\ \omega^5 & \omega^7 & \omega^7 \\ \omega^5 & \omega^7 & \omega^7 \end{pmatrix} \tag{140}$$

  where $g^{(k)}$ is the matrix whose $(i,j)^{th}$-entry is the Yukawa coupling g^k_{ij}. Then, the "form invariance" relations impose the following forms:

$$g^{(1)} = \begin{pmatrix} \mathcal{A}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, g^{(2)} = \begin{pmatrix} 0 & \mathcal{B}^2 & -\mathcal{B}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, g^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{C}^3 & \mathcal{D}^3 \\ 0 & \mathcal{D}^3 & \mathcal{C}^3 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} 0 & \mathcal{B}^4 & \mathcal{B}^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{141}$$

  When the Higgs fields (ϕ'_i) get vevs (v'_i, $i = 1, 2, 3, 4$), we obtain:

$$M_D = \Sigma_{k=1}^{k=4} v'_k g^{(k)} = \begin{pmatrix} v'_1 \mathcal{A}^1 & v'_2 \mathcal{B}^2 + v'_4 \mathcal{B}^4 & -v'_2 \mathcal{B}^2 + v'_4 \mathcal{B}^4 \\ 0 & v'_3 \mathcal{C}^3 & v'_3 \mathcal{D}^3 \\ 0 & v'_3 \mathcal{D}^3 & v'_3 \mathcal{C}^3 \end{pmatrix}, \tag{142}$$

  which is of the form of Eq.(75) with $E_D = 0$:

$$M_D = \begin{pmatrix} A_D & B_D (1+\alpha) & -B_D \\ 0 & C_D & D_D \\ 0 & D_D & C_D \end{pmatrix}, \tag{143}$$

  where

$$\alpha = \frac{2 v'_4 \mathcal{B}^4}{v'_2 \mathcal{B}^2 - v'_4 \mathcal{B}^4} \tag{144}$$

  If the vevs satisfy $v'_4 \ll v'_2$ and the Yukawa couplings are of the same order then we get a perturbative parameter $\alpha \ll 1$.

- **Majorana neutrino mass matrix**

  The mass term is generated from the Lagrangian

$$\mathcal{L}_R = h^k_{ij} \, \Delta_k \, \nu_{Ri} \, \nu_{Rj} \tag{145}$$

  Under Z_8 we have the bilinear:

$$\nu_{Ri} \, \nu_{Rj} \overset{Z_8}{\sim} \begin{pmatrix} \omega^2 & \omega^4 & \omega^4 \\ \omega^4 & \omega^6 & \omega^6 \\ \omega^4 & \omega^6 & \omega^6 \end{pmatrix} \overset{\text{Eq.137}}{\Longrightarrow}$$

$$\mathcal{L}_R = h^1_{11} \, \Delta_1 \, \nu_{R1} \, \nu_{R1} + h^2_{22} \, \Delta_2 \, \nu_{R2} \, \nu_{R2} + h^2_{23} \, \Delta_2 \, \nu_{R2} \, \nu_{R3} + h^2_{32} \, \Delta_2 \, \nu_{R3} \, \nu_{R2} + h^2_{33} \, \Delta_2 \, \nu_{R3} \, \nu_{R3} \tag{146}$$

  If we call $h^{(k)}$ the matrix whose $(i,j)^{th}$-entry is the coupling h^k_{ij} then we have (the cross sign denote a non-vanishing entry):

$$h^{(1)} = \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \tag{147}$$

Then the "form invariance" relations lead to:

$$S^{\mathrm{T}} h^{(k)} S = h^{(k)}, \overset{\text{Eqs.}110,147}{\Longrightarrow}$$

$$h^{(1)} = \begin{pmatrix} a_R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_R & d_R \\ 0 & d_R & c_R \end{pmatrix}, \tag{148}$$

Thus when the Higgs singlets Δ acquire vevs $\left(\Delta_1^0, \Delta_2^0\right)$ we get the following form for M_R,

$$M_R = \begin{pmatrix} \Delta_1^0 \, a_R & 0 & 0 \\ 0 & \Delta_2^0 \, c_R & \Delta_2^0 \, d_R \\ 0 & \Delta_2^0 \, d_R & \Delta_2^0 \, c_R \end{pmatrix}. \tag{149}$$

which of the form of Eq.(56) with $B_R = 0$. The analysis of the last subsection shows then that the deformation α in M_D resurfaces as a 'sole' perturbation χ in M_ν which would get the desired form of Eq.(79) with χ given by Eq.(80) after putting $B_R = E_D = 0$:

$$\chi = \frac{\alpha \, (d_R - c_R) \, (C_D + D_D)}{(D_D - C_D) \, (c_R + d_R) + \alpha \, (c_R D_D - d_R C_D)}. \tag{150}$$

Before ending this section, we would like to mention that having multiple Higgs doublets in our constructions might display flavor-changing neutral currents. However, the effects are calculable and in principle one can adjust the Yukawa couplings so that to suppress processes like $\mu \to e\gamma$ [26]. Moreover, the constructions are carried out at the seesaw high scale, but the RG running effects are expected to be small when multiple Higgs doublets are present, and so we expect the predictions of the symmetry will still be valid at low scale.

9 Discussion and summary

We studied the properties of the Z_2 symmetry behind the $\mu - \tau$ neutrino universality. We singled out the texture (S_+) which imposes naturally a maximal atmospheric mixing $\theta_{23} = \pi/4$ and vanishing θ_{13}. The remaining mixing angle θ_{12} remains free, and the other Z_2 necessary to characterize the neutrino mass matrix can be used to fix it at its experimentally measured value ($\sim 33^0$). We showed how the S_+-texture accommodates all the neutrino mass hierarchies. Later, we implemented the S_+-symmetry in the whole lepton sector, and showed how it can accommodate the charged lepton mass hierarchies with small mixing angles of order of the 'acute' charged lepton mass hierarchies. We computed, within type-I seesaw, the CP asymmetry generated by the symmetry and found that the phases of the RH Majorana fields may be adjusted to produce enough baryon asymmetry. The fact that the μ–τ symmetry does not determine fully the mixing angles, but leaves θ_{12} as a free parameter able to take different values in M_R and M_D is crucial for obtaining leptogenesis within type-I seesaw scenarios. We found also that "complex-valued" perturbations on Dirac neutrino mass matrix can account for the correct neutrino mixing angles.

We carried out a complete numerical study to find phenomenologically acceptable M_ν respecting the approximate S_+, and we generated possible corresponding M_R and M_ν^D. Crucially, we found in our numerical scanning that no "real-valued" neutrino mass matrices can account for the experimental constraints, and so one has to take complex matrices from the outset. The perturbation at the level of M_ν should also be complex in order to account for phenomenology. .

Finally, we presented a theoretical realization of the perturbed Dirac mass matrix, where the symmetry is broken spontaneously and the perturbation parameter originates from ratios of different Higgs fields vevs.

Acknowledgements

Part of the work was done within the short visits program of ICTP.
N.C. thanks Graham Ross for discussions, and acknowledges funding provided by the Alexander von Humboldt Foundation.

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