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ON LAGERSTROM'S MODEL OF SLOW INCOMPRESSIBLE VISCOUS FLOW*

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Abstract. The model discussed is a nonlinear boundary value problem which contains a parameter ε that models the Reynolds number. The matched asymptotic expansions, an inner "Stokes" expansion valid near the inner boundary and an outer "Oseen" expansion valid away from it, that describe the solutions of the model problem for ε small are extended. Numerical calculations show that these matched expansions have only a small range of usefulness, with the addition of further terms generally causing a worse, rather than better, approximation at moderate values of ε. Far better results are achieved when a single expansion, the outer expansion, is used throughout. The additional terms that have been calculated then consistently give improved approximations for all ε. It is also rigorously proved that an iterative method of solution of the model equation based on the outer "Oseen" approximation, converges for all ε to a unique solution.

The results presented here for Lagerstrom's model suggest that iterative improvement of the Oseen expansion may be an effective method of approximation of viscous flows at moderate Reynolds number.

Key words. Lagerstrom's model equation, nonlinear boundary value problem, matched asymptotic expansions, numerical calculations, iterative solution, existence and uniqueness theorem

AMS(MOS) subject classifications. 34B15, 34E05, 34E10

1. Introduction. Lagerstrom's [14] model is an analytically simple ordinary differential equation that was designed to elucidate certain basic mathematical ideas introduced by Kaplun and Lagerstrom [11], [12] for the asymptotic treatment of flow past a solid at low Reynolds number. We will consider only the version of the model for slow incompressible viscous flow past an obstacle in $(n+1)$ dimensions [15]. This is the equation

$$\frac{d^2u}{dx^2}+\frac{n}{x}\frac{du}{dx}+u\frac{du}{dx}=0, \tag{1.1}$$

in the range $0<\varepsilon\leqq x<\infty$ with boundary conditions

$$u=0 \quad \text{at } x=\varepsilon, \qquad u=1 \quad \text{at } x=\infty. \tag{1.2}$$

Equation (1.1) has been scaled in such a way that the dependence of the solution $u(x, \varepsilon)$ on the positive parameter ε, the analogue of the Reynolds number R, occurs through the inner boundary condition.

In view of the origin of (1.1), much of the interest in it has been directed at developing matched asymptotic expansions to describe its solution for the case of small ε [2], [13]-[15], and with the phenomenon of *switchback* that is then encountered [16]. An inner expansion, to which the boundary condition at $x=\varepsilon$ is applied, is matched to the outer solution that is valid near $x=\infty$. In § 2, we carry these expansions and their matching one stage further than previous workers. The outer expansion is developed in a general form for all n. The inner expansion, whose form depends critically on n, is developed and matched for the important "spherical" and "cylindrical" cases of $n=2$ and $n=1$, respectively. Extensive switchback, that is, the occurrence of various powers of ln ε in addition to powers of ε, arises in the matching for the $n=2$ case. We are able to handle the switchback in a straightforward manner by

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the use of a block matching [5], [9], [25]. We then investigate, numerically, how accurate the results of the matched asymptotic expansions are. This comparison shows them to be of limited usefulness.

As is now well known [14], [15], [22], the lowest-order outer approximation in which (1.1) is linearized in u about $u=1$ is a uniformly valid approximation for all x. The nonlinear term provides corrections at higher orders. In § 3, we use our three-term outer expansion to provide an approximation for the entire region $\varepsilon \leqq x < \infty$, by applying the boundary condition at $x=\varepsilon$ directly to it. The resulting approximation is found to be good not only for the small values of ε for which the model was originally intended, but also for moderate and large values of ε.

In § 4, we analyze an iterative method of solution of (1.1) that is also based on the idea of using the outer approximation throughout the whole region. Starting with the same lowest-order approximation as in § 3, improved approximations are calculated here by using the current approximation to evaluate the nonlinear term. Although the second approximation generated by this procedure is the same as the two-term expansion of § 3, subsequent approximations differ. We prove rigorously that this procedure, called Oseen iteration by Lagerstrom and Reinelt [14], [16], converges monotonically for all $\varepsilon > 0$ and for all real $n > 0$.

Our results and their implications are discussed in § 5.

2. Matched asymptotic expansions. It is natural to use an inner variable r, defined by the relation

$$r = \frac{x}{\varepsilon}, \tag{2.1}$$

in the neighborhood of the inner boundary. When it is used in (1.1), we obtain the equation

$$\frac{d^2u}{dr^2} + \frac{n}{r}\frac{du}{dr} = -\varepsilon u \frac{du}{dr}, \tag{2.2}$$

in which the small parameter ε appears explicitly. The inner expansion, akin to the Stokes expansion of the fluid dynamical problem, is obtained from (2.2) by neglecting the nonlinear term to lowest order and then generating iterative improvements. This expansion is not valid at large r where $u \to 1$ while $n/r \to 0$. There an outer expansion, akin to the Oseen expansion of the fluid dynamical problem, is needed. It can be generated by introducing the new and small dependent variable.

$$w = 1 - u, \tag{2.3}$$

and rewriting (1.1) as

$$\frac{d^2w}{dx^2} + \left(\frac{n}{x} + 1\right)\frac{dw}{dx} = w\frac{dw}{dx}. \tag{2.4}$$

The solution of the linearization

$$\frac{d^2w_0}{dx^2} + \left(\frac{n}{x} + 1\right)\frac{dw_0}{dx} = 0, \tag{2.5}$$

of this equation is some multiple

$$w_0 = CE_n(x), \tag{2.6}$$

of a decaying exponential integral function. Here we follow Lagerstrom and Casten [15] in using the function

$$E_n(x) = \int_x^\infty e^{-t} t^{-n}\, dt = x^{1-n} \int_1^\infty e^{-x\tau} \tau^{-n}\, d\tau = x^{1-n} \tilde{E}_n(x), \tag{2.7}$$

rather than the more standard choice, denoted here by $\tilde{E}_n(x)$, for the exponential integral function [1, Chap. 5].

The outer expansion can then be developed as a series

$$w = CE_n(x) - C^2 F_n(x) + C^3 H_n(x) + O(C^4), \tag{2.8}$$

in powers of the yet-to-be-determined multiple C of $E_n(x)$. Thus $F_n(x)$ is to be found as the solution of the equation

$$\frac{d^2 F_n}{dx^2} + \left(\frac{n}{x} + 1\right) \frac{dF_n}{dx} = x^{-n} e^{-x} \frac{d}{dx}\left[x^n e^x \frac{dF_n}{dx}\right] = -E_n \frac{dE_n}{dx}, \tag{2.9}$$

that decays to zero as $x \to \infty$ and that contains no multiple of $E_n(x)$. It can be found explicitly as [15]

$$F_n(x) = 2^{2n-1} E_{2n-1}(2x) - x^{1-n} e^{-x} E_n(x). \tag{2.10}$$

We then need the solution of the equation

$$\begin{aligned} \frac{d}{dx}\left[x^n e^x \frac{dH_n}{dx}\right] &= -x^n e^x \frac{d}{dx}[E_n(x) F_n(x)] \\ &= -(x+n-1)[E_n(x)]^2 + 2^{2n-1} E_{2n-1}(2x) \end{aligned} \tag{2.11}$$

that likewise decays to zero as $x \to \infty$ and contains no multiple of $E_n(x)$. A first integral of (2.11) can be obtained generally as

$$\begin{aligned} x^n e^x \frac{dH_n}{dx} = &-\left[\frac{x^2}{2} + (n-1)x + \frac{n(n-1)}{2}\right][E_n(x)]^2 + x^{1-n}(x+n)\, e^{-x} E_n(x) \\ &+ 2^{2n-2}(2x-n) E_{2n-1}(2x) - 3 \times 2^{2n-3} E_{2n-2}(2x). \end{aligned} \tag{2.12}$$

However, it is not also possible to obtain a general explicit expression for $H_n(x)$ in terms of elementary functions and exponential integrals. This fact is evident from the special case of $n = 1$ for which we obtain

$$\begin{aligned} H_1(x) = &\left(-1 + \frac{x}{2}\right) e^{-x} [E_1(x)]^2 + E_1(x) E_1(2x) - 2\, e^{-x} E_1(2x) \\ &- e^{-2x} E_1(x) + \frac{9}{2} E_1(3x) + \frac{3}{2} \int_x^\infty e^{-t} [E_1(t)]^2\, dt. \end{aligned} \tag{2.13}$$

The evaluation of the integral that remains here requires an infinite series. (It becomes a special case of equation (1.3.2.12) of Prudnikov, Brychkov, and Marichev [20] after an integration by parts.)

For the case of $n = 2$, (2.12) can first be reduced using the recurrence formula

$$mE_{m+1}(x) = x^{-m} e^{-x} - E_m(x), \tag{2.14}$$

to

$$x^2 e^x \frac{dH_2}{dx} = -\left(1 + x + \frac{x^2}{2}\right)[E_1(x)]^2 + (1+x)\, e^{-x} E_1(x) + 2(1+2x) E_1(2x) - \frac{5}{2} e^{-2x}. \tag{2.15}$$

From this, the integral

$$H_2(x)=\left(\frac{1}{x}+\frac{7}{2}\right)e^{-x}[E_1(x)]^2-2E_1(x)E_1(2x)-\frac{2\,e^{-x}}{x}E_1(2x)$$
$$-\frac{3}{x}e^{-2x}E_1(x)+\frac{45}{2}E_2(3x)-3\int_x^{\infty}e^{-t}[E_1(t)]^2\,dt, \tag{2.16}$$

can be obtained, in which the same integral as that in (2.13) remains.

The matching to be carried out requires the calculation of the inner limit of the outer expansion. A basic result needed for this is [1, eq. 5.1.12].

$$E_n(x)=\frac{(-1)^{n-1}}{(n-1)!}[-\ln x+\psi(n)]-\sum_{\substack{m=0\\ m\neq n-1}}^{\infty}\frac{(-1)^m x^{m+1-n}}{(m-n+1)m!}. \tag{2.17}$$

The function $\psi(n)$ is the digamma function with values

$$\psi(1)=-\gamma,\quad \psi(n)=-\gamma+\sum_{m=1}^{n-1}\frac{1}{m}\quad \text{for } n>1, \tag{2.18}$$

where γ is Euler's constant. The first term on the right-hand side of (2.17) can be regarded as replacing the $m=(n-1)$ term that is omitted from the summation. Its relative significance, and hence the relative prominence of logarithmic terms, decreases with increasing n.

2.1. The case $n=2$. As is evident from (2.2), the form of the simpler inner expansion depends even more critically on the value of n. Here too, the smaller n is, the sooner logarithmic terms appear. For $n=2$, a straightforward computation of the solution that vanishes at $r=1$ yields the expansion

$$u=A\left(1-\frac{1}{r}\right)-\varepsilon A^2\left(1+\frac{1}{r}\right)\ln r+\varepsilon^2A^3\left[\frac{r}{2}-\frac{1}{2r}-\frac{\ln r}{r}-\frac{(\ln r)^2}{r}\right]$$
$$+\varepsilon^3A^4\left\{\frac{-r^2}{12}-\frac{r\ln r}{2}+\frac{3r}{4}-\frac{(\ln r)^2}{2}+\frac{3\ln r}{2}\right.$$
$$\left.-\frac{1}{r}\left[(\ln r)^3+\frac{5(\ln r)^2}{2}+5\ln r+\frac{2}{3}\right]\right\}+O(\varepsilon^4A^5). \tag{2.19}$$

The parameter A, like C in expansion (2.8), is a yet-to-be-determined multiple of the solution of the linearized form of (2.2).

The matching for the $n=2$ case will now be carried out using Crighton and Leppington's [5] modified version of Van Dyke's asymptotic matching principle [25]. Terms are matched in blocks according to the power of ε that they contain with no distinction being made for any additional factors of $\ln\varepsilon$ (Fraenkel [9]). The basic structure of the expansions becomes apparent when the first term of the inner expansion (2.19) is matched to the two-term outer expansion $u=1-CE_2(x)$. Equating the two-term outer expansion of the one-term inner expansion to the one-term inner expansion of the two-term outer expansion, we obtain the equation

$$A\left(1-\frac{1}{r}\right)=1-\frac{C}{x}. \tag{2.20}$$

To lowest order, therefore, we have $A=1$ and $C=\varepsilon$. Hence both series (2.8) and (2.19) start as simple power series in ε. Logarithmic switchback terms appear at the next stage. They are accounted for automatically in expansions

$$A(\varepsilon)=1+\varepsilon A_1(\varepsilon)+\varepsilon^2A_2(\varepsilon)+\varepsilon^3A_3(\varepsilon)+O(\varepsilon^4), \tag{2.21}$$

$$C(\varepsilon)=\varepsilon+\varepsilon^2C_2(\varepsilon)+\varepsilon^3C_3(\varepsilon)+O(\varepsilon^4), \tag{2.22}$$

in which the coefficients $A_i(\varepsilon)$ and $C_i(\varepsilon)$ are allowed to include logarithmic terms. With the expansions noted, we can evaluate the four term (i.e., through $O(\varepsilon^3)$) outer expansion of the four term inner expansion (2.19) by setting $r=x/\varepsilon$ and working through $O(\varepsilon^3)$. The result, when re-expressed in terms of the inner variable r, is

$$\begin{aligned} u=&(1+\varepsilon A_1+\varepsilon^2A_2+\varepsilon^3A_3)-(1+\varepsilon A_1+\varepsilon^2A_2)\frac{1}{r}-\varepsilon\{1+2\varepsilon A_1+\varepsilon^2(2A_2+A_1^2)\}\ln r\\ &-\varepsilon(1+2\varepsilon A_1)\frac{\ln r}{r}+\varepsilon^2\left[\frac{r}{2}-\frac{1}{2r}-\frac{\ln r}{r}-\frac{(\ln r)^2}{r}\right]+\frac{3\varepsilon^3A_1r}{2}\\ &+\varepsilon^3\left[-\frac{r^2}{12}-\frac{r\ln r}{2}+\frac{3r}{4}-\frac{(\ln r)^2}{2}+\frac{3}{2}\ln r\right]+O(\varepsilon^4). \end{aligned} \tag{2.23}$$

Equation (2.23) must match the four term inner expansion of $u=1-w$ with w given by (2.8). Here too, we retain terms through $O(\varepsilon^3)$ after expansion in the small variable $x=\varepsilon r$. In view of the $O(\varepsilon)$ magnitude of C, the following approximations that can be computed using (2.17), suffice:

$$E_2(x)=\frac{1}{x}+\ln x+\gamma-1-\frac{x}{2}+\frac{x^2}{12}+O(x^3), \tag{2.24}$$

$$F_2(x)=-\frac{\ln x}{x}-\frac{(\gamma+2)}{x}-3\ln x+5-3\gamma-4\ln 2-\frac{x\ln x}{2}+x\left(\frac{11}{4}-\frac{\gamma}{2}\right)+O(x^2\ln x), \tag{2.25}$$

$$\begin{aligned} H_2(x)=&\frac{1}{x}\left\{(\ln x)^2+(2\gamma+5)\ln x+\gamma^2+5\gamma+\frac{15}{2}+2\ln 2\right\}+\frac{1}{2}(\ln x)^2\\ &+\left(\frac{25}{2}+\gamma-2\ln 2\right)\ln x+\left\{\frac{\gamma^2+25\gamma-59+45\ln 3}{2}-2(\gamma+1)\ln 2-3I\right\}\\ &+O[x(\ln x)^2]. \end{aligned} \tag{2.26}$$

The constant I here is the known definite integral

$$I=\int_0^x e^{-t}[E_1(t)]^2\,dt=1.22855867 \tag{2.27}$$

([20], eq. 2.5.12.2 recomputed; the value quoted on p. 116 of [14] is inaccurate). When we expand and truncate, all terms are found to match, and from the matching of the different powers of ε we obtain the evaluations

$$A_1(\varepsilon)=-\ln\varepsilon-\gamma+1, \tag{2.28}$$

$$C_2(\varepsilon)=-2\ln\varepsilon-2\gamma-1, \tag{2.29}$$

$$A_2(\varepsilon)=2(\ln\varepsilon)^2+4(\gamma-1)\ln\varepsilon+2\gamma^2-4\gamma+4-4\ln 2, \tag{2.30}$$

$$C_3(\varepsilon)=5(\ln\varepsilon)^2+(10\gamma+1)\ln\varepsilon+5\gamma^2+\gamma+1-6\ln 2, \tag{2.31}$$

$$\begin{aligned} A_3(\varepsilon)=&-5(\ln\varepsilon)^3+(-15\gamma+15.5)(\ln\varepsilon)^2+(-15\gamma^2+31\gamma-26.5+24\ln 2)(\ln\varepsilon)\\ &-5\gamma^3+15.5\gamma^2-26.5\gamma+20.5+4(6\gamma+1)\ln 2-22.5\ln 3+3I. \end{aligned} \tag{2.32}$$

The solutions for $A_1(\varepsilon)$ and $C_2(\varepsilon)$ agree with those of Lagerstrom and Casten [15, § 5.4, § 5.8] after they have accounted fully for switchback. Our expression for $A_2(\varepsilon)$ is new, although it could have been derived from the three-term expansions that they give. The expressions for $C_3(\varepsilon)$ and $A_3(\varepsilon)$ are also new, but their determination does require the newly added final terms of expansions (2.8) and (2.19).

As a test of the usefulness of the matched expansions we now compare their predictions with results from the numerical integration of (1.1). We focus on two

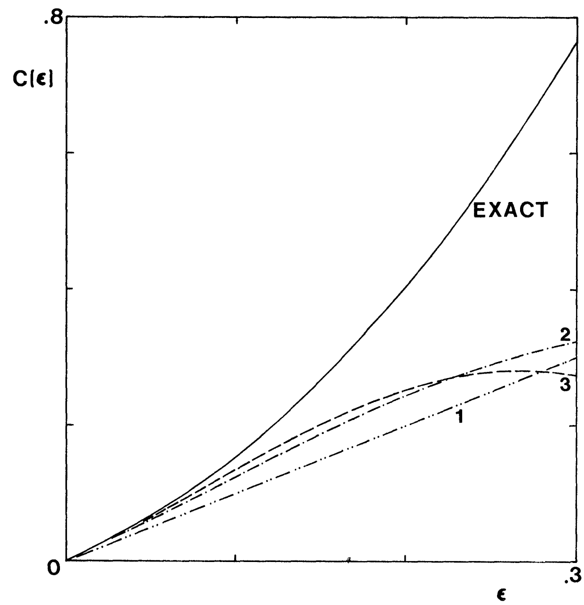

FIG. 1. $C(\varepsilon)$ *for* $n=2$, *compared with sums of one, two, and three terms, as labeled, of expansion* (2.22).

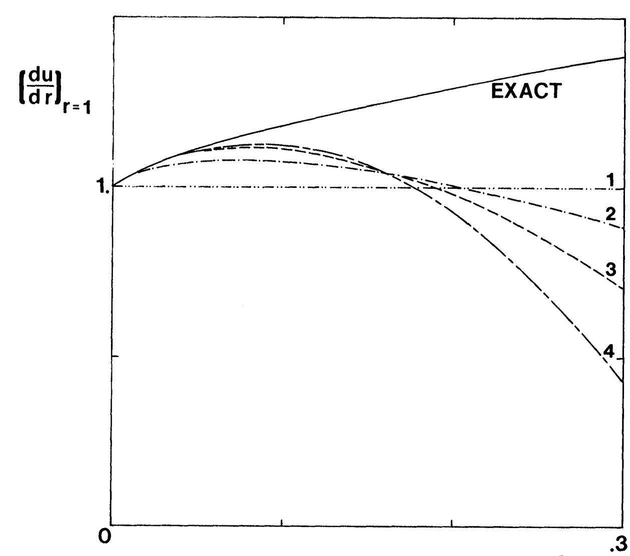

FIG. 2. $(du/dr)_{r=1}$ *for* $n=2$, *compared with sums of one, two, three, and four terms, as indicated, of expansion* (2.33).

fundamental scalar properties: the multiple $C(\varepsilon)$ of $-E_n(x)$ in u as $x\to\infty$, and $(du/dr)_{r=1}$. The latter is an analogue of the drag on the sphere, a quantity of considerable interest in the fluid dynamical problem [3], [19]. Differentiation of series (2.19) gives

$$\begin{aligned}\left(\frac{du}{dr}\right)_{r=1} &= A-2\varepsilon A^2-\frac{11}{4}\varepsilon^3A^4+O(\varepsilon^4A^5),\\ &= 1+\varepsilon(A_1-2)+\varepsilon^2(A_2-4A_1)+\varepsilon^3\left(A_3-4A_2-2A_1^2-\frac{11}{4}\right)\\ &\quad +O[\varepsilon^4(\ln\varepsilon)^4].\end{aligned} \tag{2.33}$$

Figures 1 and 2 show how poorly the various truncations of (2.22) and (2.33) represent the exact values of $C(\varepsilon)$ and $(du/dr)_{r=1}$, respectively, even over the limited range $0\leqq\varepsilon\leqq 0.3$. Extra terms do improve the fit in the $\varepsilon\to 0$ limit and in a remarkably small range near $\varepsilon=0$, but the price to be paid for a close fit near $\varepsilon=0$ is a worse fit at quite small values of ε.

2.2. The case $n=1$. The distinctive feature of this case [2], [13]-[15] is that the series developments of both A and C are in powers of $(\ln\varepsilon)^{-1}$:

$$A=\frac{-1}{(\ln\varepsilon)}+\frac{\alpha_2}{(\ln\varepsilon)^2}+\frac{\alpha_3}{(\ln\varepsilon)^3}+\frac{\alpha_4}{(\ln\varepsilon)^4}+O[(\ln\varepsilon)^{-5}], \tag{2.34}$$

$$C=\frac{-1}{(\ln\varepsilon)}+\frac{\beta_2}{(\ln\varepsilon)^2}+\frac{\beta_3}{(\ln\varepsilon)^3}+O[(\ln\varepsilon)^{-4}]. \tag{2.35}$$

The nonlinear term of (2.2) is then negligible as far as the matching can be carried out, and the inner expansion is simply

$$u=A\ln r+O(\varepsilon A^2). \tag{2.36}$$

The approximations needed in the outer solution are

$$E_1(x)=-\ln x-\gamma+O(x), \tag{2.37}$$

$$F_1(x)=-\ln x-2\ln 2-\gamma+O(x\ln x), \tag{2.38}$$

and

$$H_1(x)=(\ln 2-1.5)\ln x+(\gamma+2)\ln 2-1.5\gamma-4.5\ln 3+1.5I+O(x\ln x). \tag{2.39}$$

We have no choice but to match powers of logarithms, and when this is done, the values

$$\alpha_2=\gamma,\quad \alpha_3=-\gamma^2+2\ln 2,\quad \alpha_4=\gamma^3-6(\gamma+1)\ln 2+4.5\ln 3-1.5I, \tag{2.40}$$

$$\beta_2=\gamma+1,\quad \beta_3=-\gamma^2-2\gamma-0.5+\ln 2, \tag{2.41}$$

are obtained for the coefficients of expansions (2.34) and (2.35). The values of α_2, β_2, and α_3 agree with those of previous workers [2], [13], [15]. However, our α_4 does not agree with the A_3 given in equation (5.18b) of Lagerstrom [14], nor does the third term $\beta_3E_1(x)+2\beta_2F_1(x)-H_1(x)$ of our outer expansion agree with the $f_3(x)$ of his equation (5.18a). We therefore believe Lagerstrom's equations (5.18) to be in error.

Expansions (2.34) and (2.35) become more compact if they are "telescoped" [11], [25] and rearranged, as in [15], in powers of $(\ln\varepsilon+\gamma)^{-1}$. Because this quantity becomes infinite at $\varepsilon=0.56$ whereas $(\ln\varepsilon)^{-1}$ does not become infinite until $\varepsilon=1$, this rearrangement is not helpful numerically. But the combination $(\ln\varepsilon+\gamma)^{-1}$ is the small-ε approximation to $[-E_1(\varepsilon)]^{-1}$ and, as we see in § 3 below, the latter is an effective parameter for expansion and hence for rearrangement.

Figure 3 compares $A=(du/dr)_{r=1}$ with various estimates, as functions of $-(\ln \varepsilon)^{-1}$. The same general features that were found in the $n=2$ case recur, although the three-term approximation here is an improvement over the two-term one for the range shown. The four-term approximation has begun to deviate markedly from A at $-1/(\ln \varepsilon)=0.4$ when ε is only 0.08. Again extra terms are of little use except close to the $\varepsilon \to 0$ limit.

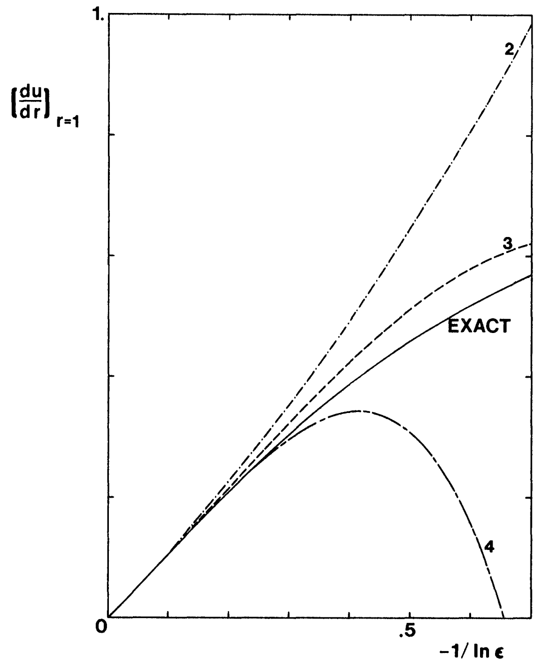

FIG. 3. $A=(du/dr)_{r=1}$ *for* $n=1$, *compared with sums of two, three, and four terms of expansion* (2.34), *as functions of* $(-\ln \varepsilon)^{-1}$. *The one-term expansion, a line through O of unit slope, is omitted to avoid cluttering the figure.*

3. The use of the outer expansion alone. As was first noted for the original fluid mechanical problem of low Reynolds number flow [12], [19], the outer expansion contains the inner one and so can be used throughout $x \geqq \varepsilon$ [22]. The outer expansion (2.8) can be chosen in such a way that it satisfies the boundary condition $w=1$ at $x=\varepsilon$. This is achieved to lowest order with the linearized solution (2.6) and the choice $C=1/E_n(\varepsilon)$. When C is chosen in this way, some rearrangement of expansion (2.8) is needed because multiples of $E_n(x)$ must then arise in the later terms in the expansion. The result is the following series:

$$w(x,\varepsilon)=\frac{E_n(x)}{E_n(\varepsilon)}+\frac{1}{[E_n(\varepsilon)]^2}\left\{\frac{F_n(\varepsilon)E_n(x)}{E_n(\varepsilon)}-F_n(x)\right\}+\frac{1}{[E_n(\varepsilon)]^3}$$
$$\times\left\{H_n(x)-\frac{2F_n(\varepsilon)F_n(x)}{E_n(\varepsilon)}+\frac{[2[F_n(\varepsilon)]^2-E_n(\varepsilon)H_n(\varepsilon)]}{[E_n(\varepsilon)]^2}E_n(x)\right\}$$
$$+O\{[E_n(\varepsilon)]^{-4}\}. \tag{3.1}$$

It gives the estimate

$$C(\varepsilon)\sim\frac{1}{E_n(\varepsilon)}+\frac{F_n(\varepsilon)}{[E_n(\varepsilon)]^3}+\frac{\{2[F_n(\varepsilon)]^2-E_n(\varepsilon)H_n(\varepsilon)\}}{[E_n(\varepsilon)]^5}, \tag{3.2}$$

and an estimate for $(du/dr)_{r=1}$ is obtained by differentiation of (3.1). Expansion (3.1) is formally in powers of $[E_n(\varepsilon)]^{-1}$, which is small when ε is small but exponentially large when ε is large. However, coefficients that are exponentially large in ε multiply

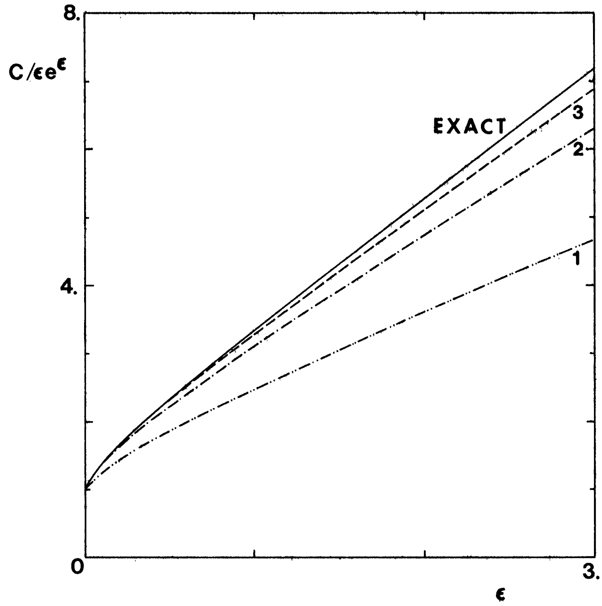

FIG. 4. *$C(\varepsilon)$ for $n=2$ compared with sums of one, two, and three terms of expansion* (3.2) *after scaling. The scaling is needed because $C(\varepsilon)$ grows asymptotically as $\varepsilon^n e^\varepsilon$ for large ε. This asymptotic behavior is well established at $\varepsilon=3$, by which stage each curve is fairly straight.*

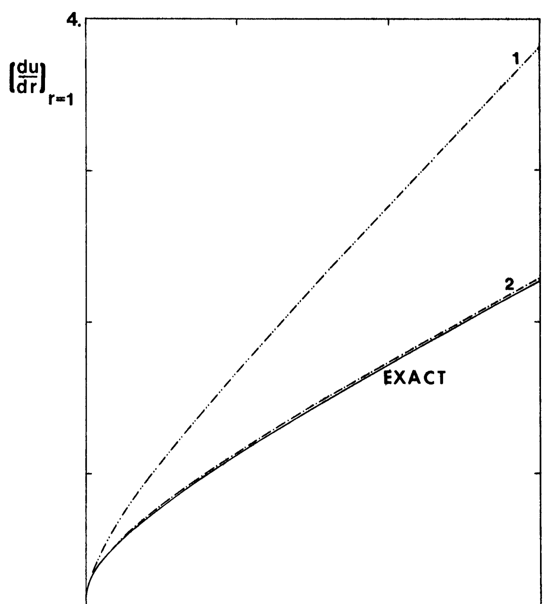

FIG. 5. *$(du/dr)_{r=1}$ for $n=1$, shown here by the full curve, compared with estimates obtained from the derivative of expansion* (3.1). *The relative error of the three-term estimate is only* 0.16 *percent at $\varepsilon=3$ and so is invisible on the scale of the figure.*

terms that are exponentially small in x in expansion (3.1) so that the second and third terms of (3.2) are never large relative to the leading term.

Figures 4 and 5, in which the ε-range is 10 times that of Figs. 1 and 2, display how well and consistently approximations obtained by this approach match exact values for both $n=1$ and $n=2$. To avoid repetitive figures, one curve only is plotted for each case. The only significant differences occur near $\varepsilon=0$ where the $n=1$ curves turn down sharply because of the $(-\ln\varepsilon)^{-1}$ singularity. The approximations for $(du/dr)_{r=1}$ are especially good once the second term is included and some account is taken of the nonlinear term of (2.4). Consequently the graphs of $w^{(2)}$ and w in Fig. 6 are close near $x=\varepsilon=2$. The approximations of this section are not as good at predicting the magnitude C of the exponential tail of w, but, because w is of small magnitude there, this effect is less noticeable in Fig. 6.

4. An iterative scheme that converges for all ε. The excellent approximations that were obtained in § 3 when the outer expansion is used everywhere suggest the following iterative scheme for solving (2.4):

Compute the sequence $\{w^{(j)}(x,\varepsilon)\}$ of functions that are defined iteratively as solutions of the equations

$$\frac{d^2w^{(j+1)}}{dx^2}+\left(\frac{n}{x}+1\right)\frac{dw^{(j+1)}}{dx}=w^{(j)}\frac{dw^{(j)}}{dx},\qquad j\geqq 0, \tag{4.1}$$

with boundary conditions

$$w^{(j+1)}(\varepsilon,\varepsilon)=1,\qquad w^{(j+1)}(\infty,\varepsilon)=0. \tag{4.2}$$

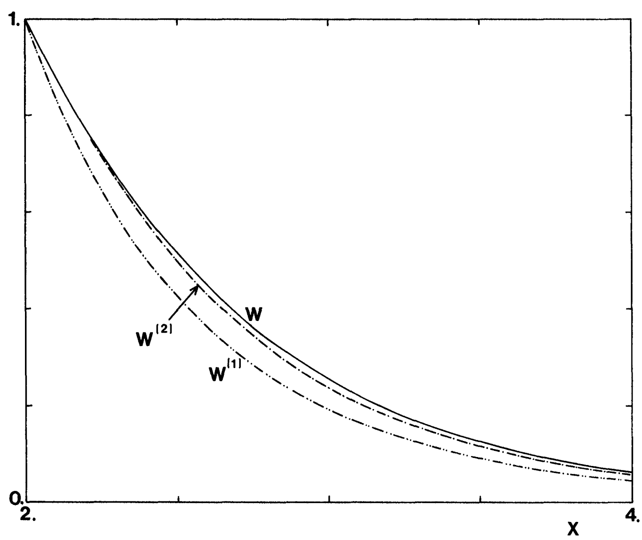

FIG. 6. *One- and two-term approximations* $w^{(1)}$ *and* $w^{(2)}$ *as defined in* (4.3), *and* w, *for* $\varepsilon=2$ *and* $n=2$. *The graph of* $w^{(3)}$ *has been omitted because it is so close to that of* w. *It lies above* $w^{(2)}$ *and below* w, *and its largest deviation from* w *is of magnitude* 0.004 *and occurs near* $x=3$.

Take the starting member of this sequence to be $w^{(0)}(x, \varepsilon)=0$. Although it does not satisfy the boundary condition (4.2) at $x=\varepsilon$, subsequent members do. The first and second iterates are:

$$(4.3)\quad w^{(1)}(x, \varepsilon)=\frac{E_n(x)}{E_n(\varepsilon)},\quad w^{(2)}(x, \varepsilon)=\frac{E_n(x)}{E_n(\varepsilon)}+\frac{1}{[E_n(\varepsilon)]^2}\left\{\frac{F_n(\varepsilon)E_n(x)}{E_n(\varepsilon)}-F_n(x)\right\}.$$

They are simply one- and two-term truncations of series (3.1). The two approaches differ thereafter because $w^{(2)}\,dw^{(2)}/dx$ includes some terms that are ignored in the calculation of the three-term outer expansion.

This scheme was noted by Lagerstrom and Reinelt [16] who named it Oseen iteration, but a fuller discussion of it was given earlier by Rosenblat and Shepherd [21]. The latter showed that it converges to the true solution of (2.4) for sufficiently small ε for any positive integer n. We will show that it, in fact, converges for all ε, and in a simple monotonic manner, for *any* real $n \geqq 0$. Our convergence proof, like theirs, is based on the use of the Green function:

$$(4.4)\qquad G(x, s, \varepsilon)=\begin{cases} s^n\, e^s E_n(x)\left[\dfrac{E_n(s)}{E_n(\varepsilon)}-1\right], & \varepsilon \leqq s \leqq x < \infty,\\[2ex] s^n\, e^s\, E_n(s)\ \left[\dfrac{E_n(x)}{E_n(\varepsilon)}-1\right], & \varepsilon \leqq x \leqq s < \infty,\end{cases}$$

to represent the solution of the inhomogeneous equation (4.1) as

$$(4.5)\qquad w^{(j+1)}(x, \varepsilon)=w^{(1)}(x, \varepsilon)+\int_\varepsilon^\infty G(x, s, \varepsilon)w^{(j)}(s, \varepsilon)w'^{(j)}(s, \varepsilon)\, ds.$$

Important properties of this Green function are

$$(4.6\text{a})\qquad G(\varepsilon, s, \varepsilon)=G(\infty, s, \varepsilon)=0,$$

$$(4.6\text{b})\qquad G(x, \varepsilon, \varepsilon)=0,$$

$$(4.6\text{c})\qquad G(x, \infty, \varepsilon)=\left[\frac{E_n(x)}{E_n(\varepsilon)}-1\right]\lim_{s\to\infty} s^n\, e^s\, E_n(s)=\left[\frac{E_n(x)}{E_n(\varepsilon)}-1\right].$$

Its derivative with respect to s is

$$(4.7)\qquad \frac{\partial G(x, s, \varepsilon)}{\partial s}=\begin{cases}\left\{s^n\, e^s\left(\dfrac{n}{s}+1\right)\left[\dfrac{E_n(s)}{E_n(\varepsilon)}-1\right]-\dfrac{1}{E_n(\varepsilon)}\right\}E_n(x), & s<x,\\[2ex] \left[s^n\, e^s\left(\dfrac{n}{s}+1\right)E_n(s)-1\right]\left[\dfrac{E_n(x)}{E_n(\varepsilon)}-1\right], & x<s,\end{cases}$$

and is negative throughout $\varepsilon \leqq s < \infty$. The negativity of the first line of (4.7) follows from the fact that $E_n(s)$ decreases monotonically in $\varepsilon \leqq s < \infty$, while the negativity of the second line follows, for any real $n>0$, from inequality (5.1.19) of [1]. The derivative (4.7) is discontinuous at $s=x$ where it increases by one. An integration by parts of (4.5) gives the following alternative formula for the iteration:

$$(4.8)\qquad w^{(j+1)}(x, \varepsilon)=w^{(1)}(x, \varepsilon)-\frac{1}{2}\int_\varepsilon^\infty \frac{\partial G(x, s, \varepsilon)}{\partial s}[w^{(j)}(s, \varepsilon)]^2\, ds, \qquad j \geqq 1$$

that we now use to establish basic properties of the iteration.

The negativity of $\partial G/\partial s$ immediately shows that

$$(4.9)\qquad w^{(j+1)}(x, \varepsilon)>w^{(1)}(x, \varepsilon)\quad \text{in } \varepsilon<x<\infty,\quad j \geqq 1.$$

The sequence $\{w^{(j)}(x, \varepsilon)\}_{j=1}^{\infty}$ is therefore bounded below. It is also bounded above because we can show by induction that

$$w^{(j)}(x, \varepsilon) < \tfrac{1}{2}[1 + w^{(1)}(x, \varepsilon)] < 1 \quad \text{in } \varepsilon < x < \infty, \quad j \geqq 1, \tag{4.10}$$

as follows. Inequality (4.10) is true for $j = 1$ from the explicit expression (4.3) for $w^{(1)}$. Next, when we use the inequality $w^{(j)} < 1$ in (4.8), we get

$$\begin{aligned} w^{(j+1)}(x, \varepsilon) &< w^{(1)}(x, \varepsilon) - \frac{1}{2}\int_{\varepsilon}^{\infty} \frac{\partial G(x, s, \varepsilon)}{\partial s}\, ds, \\ &= w^{(1)}(x, \varepsilon) - \frac{1}{2} G(x, \infty, \varepsilon) + \frac{1}{2} G(x, \varepsilon, \varepsilon), \\ &= \frac{1}{2}[1 + w^{(1)}(x, \varepsilon)] \quad \text{for } \varepsilon < x < \infty. \end{aligned} \tag{4.11}$$

Note that the resulting upper bound (4.10) is not close for large x, where each iterate decays in the same manner as $E_n(x)$.

It can also be shown by induction that the sequence $w^{(j)}$ is increasing at each point x in $\varepsilon < x < \infty$; that is,

$$w^{(j+1)}(x, \varepsilon) > w^{(j)}(x, \varepsilon) \quad \text{in } \varepsilon < x < \infty, \quad j \geqq 1. \tag{4.12}$$

The validity of the first $j = 1$ case of (4.12) follows either from the $j = 1$ case of (4.9) or from the explicit expressions (4.3). For the general case, we use (4.8) to obtain

$$\begin{aligned} &w^{(j+2)}(x, \varepsilon) - w^{(j+1)}(x, \varepsilon) \\ &\quad = -\frac{1}{2}\int_{\varepsilon}^{\infty} \frac{\partial G(x, s, \varepsilon)}{\partial s}[w^{(j+1)}(s, \varepsilon) - w^{(j)}(s, \varepsilon)][w^{(j+1)}(s, \varepsilon) + w^{(j)}(s, \varepsilon)]\, ds. \end{aligned} \tag{4.13}$$

From this we can deduce the $j \to j+1$ case of (4.12) from itself, as required. This monotonicity and the upper bound established earlier are sufficient to guarantee the pointwise convergence of the sequence of $w^{(j)}$. However, we will establish some further properties of this sequence before giving a formal theorem and its proof.

The differential equation (4.1) can be used to show that all the iterates have negative slopes throughout the interval; that is

$$\frac{dw^{(j)}(x, \varepsilon)}{dx} < 0, \quad \varepsilon \leqq x < \infty, \quad j \geqq 1. \tag{4.14}$$

This inequality is seen to be true for $j = 1$ from the explicit expression for $w^{(1)}$, while its validity for general values of j can be established inductively as follows. Suppose (4.14) is true. Then the right-hand side of (4.1) is negative throughout $\varepsilon \leqq x < \infty$. The derivative $dw^{(j+1)}(x, \varepsilon)/dx$ must be negative at $x = \varepsilon$ from (4.11). It either remains negative throughout the interval or vanishes at some interior point P. Equation (4.1) shows that, in the latter case, $d^2 w^{(j+1)}/dx^2$ is negative at P, which is therefore a local maximum of $w^{(j+1)}(x, \varepsilon)$. This is geometrically impossible, so that $dw^{(j+1)}/dx$ must remain negative throughout.

Equation (4.13) can be developed further to show that the iteration (4.1) defines a contraction mapping. We will use the metric

$$\rho_\varepsilon[f, g] = \sup_{\varepsilon \leqq x < \infty} |f(x, \varepsilon) - g(x, \varepsilon)|, \tag{4.15}$$

to define the distance between two continuous functions f and g. Inequalities (4.12) and (4.10) combined with (4.13) give

$$|w^{(j+2)}(x,\varepsilon)-w^{(j+1)}(x,\varepsilon)|$$

$$<-\frac{1}{2}\int_\varepsilon^\infty \frac{\partial G(x,s,\varepsilon)}{\partial s}[w^{(j+1)}(s,\varepsilon)-w^{(j)}(s,\varepsilon)][1+w^{(1)}(s,\varepsilon)]\,ds$$

$$(4.16)\qquad <\rho_\varepsilon[w^{(j+1)},w^{(j)}]\int_\varepsilon^\infty -\frac{1}{2}\frac{\partial G(x,s,\varepsilon)}{\partial s}[1+w^{(1)}(s,\varepsilon)]\,ds$$

$$=\rho_\varepsilon[w^{(j+1)},w^{(j)}]\left\{\frac{1}{2}\left[1-\frac{E_n(x)}{E_n(\varepsilon)}\right]+\frac{1}{2}\int_\varepsilon^\infty G(x,s,\varepsilon)\frac{dw^{(1)}(s,\varepsilon)}{ds}\,ds\right\},$$

the last line being achieved after an integration by parts and use of (4.6c). The integral that remains can be evaluated because, by our Green function formulation, it represents the solution $W(x,\varepsilon)$ of the boundary value problem

$$(4.17)\qquad \frac{d^2W}{dx^2}+\left(\frac{n}{x}+1\right)\frac{dW}{dx}=\frac{dw^{(1)}}{dx}=-\frac{1}{x^n\,e^x E_n(\varepsilon)},\qquad W(\varepsilon,\varepsilon)=W(\infty,\varepsilon)=0.$$

After multiplication by the integrating factor $x^n\,e^x$ and integration, we obtain

$$(4.18)\qquad W(x,\varepsilon)=\int_\varepsilon^\infty G(x,s,\varepsilon)\frac{dw^{(1)}(s,\varepsilon)}{ds}\,ds=\frac{E_{n-1}(x)}{E_n(\varepsilon)}-\frac{E_{n-1}(\varepsilon)E_n(x)}{[E_n(\varepsilon)]^2}.$$

We use this evaluation in (4.16) and maximize with respect to x to obtain

$$(4.19)\qquad \rho_\varepsilon[w^{(j+2)},w^{(j+1)}]\le k(\varepsilon)\rho_\varepsilon[w^{(j+1)},w^{(j)}]$$

where

$$(4.20)\qquad k(\varepsilon)=\frac{1}{2}\sup_{\varepsilon\le x<\infty}\left\{1-\frac{E_n(x)}{E_n(\varepsilon)}+\frac{E_{n-1}(x)}{E_n(\varepsilon)}-\frac{E_{n-1}(\varepsilon)E_n(x)}{[E_n(\varepsilon)]^2}\right\}.$$

The expression in the curly brackets here vanishes at $x=\varepsilon$, and tends to one as $x\to\infty$. Differentiation shows that it attains an interior maximum at

$$(4.21)\qquad x=x_0(\varepsilon)=1+\frac{E_{n-1}(\varepsilon)}{E_n(\varepsilon)}>1+\varepsilon.$$

The fact that this maximum is less than two and hence that the mapping is a contraction is most easily seen using the inequality $w^{(1)}<1$ in the second line of (4.16) to obtain

$$(4.22)\qquad k(\varepsilon)<\left[1-\frac{E_n[x_0(\varepsilon)]}{E_n(\varepsilon)}\right]<1.$$

The right-hand side here is always less than one because $x_0(\varepsilon)$ is finite for any finite ε. Also, as $\varepsilon\to\infty$, asymptotic expansions for $E_n(x)$ [1, eq. (5.1.51)] show that $x_0(\varepsilon)\sim \varepsilon+2-n/\varepsilon$ while $k(\varepsilon)\to 0.5(1+e^{-2})=0.568$. (Numerically, it is found that $k(\varepsilon)$ is an increasing function of ε, and is less than this limit for smaller ε.)

Results that have already been established provide several of the essential steps for a constructive proof of the following existence and uniqueness theorem.

THEOREM. *The differential equation* (2.4) *with* $n>0$ *has, for all* $\varepsilon>0$ *and boundary conditions* $w(x=\varepsilon)=1$, $w(x=\infty)=0$, *a unique continuous solution in the space X as defined below.*

Proof. For any fixed $\varepsilon > 0$, let X be the set of all functions $f(x)$ that are continuous on $[\varepsilon, \infty)$ and are such that

$$0 \leqq f(x) \leqq \tfrac{1}{2}[1 + w^{(1)}(x, \varepsilon)] \quad \text{for all } x \in [\varepsilon, \infty). \tag{4.23}$$

With the metric ρ_ε of (4.15), X is a complete metric space. The mapping T that is defined by

$$Tf = w^{(1)}(x, \varepsilon) - \frac{1}{2}\int_\varepsilon^\infty \frac{\partial G(x, s, \varepsilon)}{\partial s} f^2(s)\, ds, \tag{4.24}$$

maps some arbitrary member f of X into some other member Tf of X, because (4.24) shows Tf to be a continuous function, while the argument used in deriving (4.11) can be applied to show that Tf also satisfies (4.23). The mapping T is a contraction on X as is seen by repeating the analysis between (4.16) and (4.19) to obtain

$$\rho_\varepsilon[Tf, Tg] \leqq k(\varepsilon)\rho_\varepsilon[f, g], \tag{4.25}$$

for an arbitrary pair of functions f and g of X. Hence the Contraction Mapping Theorem (e.g., [17, p. 27]) can be applied to show that there is, in the space X, a unique continuous solution of the nonlinear integral equation

$$w = Tw, \tag{4.26}$$

to which the sequence $\{w^{(j)}(x, \varepsilon)\}_{j=1}^{\infty}$ will tend. This solution satisfies the required boundary conditions. One differentiation of (4.26) with respect to x shows w to be differentiable, while a second differentiation shows that this w does indeed satisfy the differential equation (2.4) from which the integral equation (4.26) was originally derived. □

4.1. Other proofs of existence. Although Rosenblat and Shepherd [21] used the same iterative procedures as ours, their analysis was based on a pair of integral equations for $w(x, \varepsilon)/w^{(1)}(x, \varepsilon)$ and $w'(x, \varepsilon)/w'^{(1)}(x, \varepsilon)$. They proved convergence for sufficiently small ε only. A key quantity in their proof is $h_n(\varepsilon)$ that is defined as

$$h_n(\varepsilon) = \sup_{\varepsilon \leqq x < \infty} \{I_A(x, \varepsilon), I_B(x, \varepsilon)\}. \tag{4.27}$$

The I's here are integrals whose subscripts match the integrands of [21]. These integrals in our notation are

$$I_A(x, \varepsilon) = \int_\varepsilon^\infty \frac{E_n(s)|G(x, s, \varepsilon)|}{s^n\, e^s\, E_n(x) E_n(\varepsilon)}\, ds, \tag{4.28}$$

$$I_B(x, \varepsilon) = \int_\varepsilon^\infty \frac{x^n\, e^x}{s^n\, e^s} \frac{E_n(s)}{E_n(\varepsilon)} \left|\frac{\partial G(x, s, \varepsilon)}{\partial x}\right| ds. \tag{4.29}$$

Both can be evaluated explicitly in terms of exponential integrals. The former, because G is everywhere negative, is simply

$$I_A(x, \varepsilon) = \frac{w^{(2)}(x, \varepsilon) - w^{(1)}(x, \varepsilon)}{w^{(1)}(x, \varepsilon)}. \tag{4.30}$$

It increases monotonically with x to the limit $F_n(\varepsilon)/[E_n(\varepsilon)]^2$ as $x \to \infty$. The integral $I_B(x, \varepsilon)$ has the same limit as $x \to \infty$. Although it is an increasing function of x for large x, it has a single interior minimum and decreases for smaller x. However, its $x \to \infty$ limit is larger than its value at $x = \varepsilon$ provided that $F_n(\varepsilon)/[E_n(\varepsilon)]^2$ increases monotonically with ε. Although we have no formal proof of the latter, numerical calculation shows it to be true. The optimal choice for their $h_n(\varepsilon)$ is then

$$h_n(\varepsilon)=F_n(\varepsilon)/[E_n(\varepsilon)]^2, \tag{4.31}$$

rather than that given by their equation (3.15), and this choice tends to 0.5 as $\varepsilon\to\infty$. The success of Rosenblat and Shepherd's method of proof requires that $h_n(\varepsilon)<0.25$, and so it applies for moderate values of ε, e.g., for $\varepsilon<0.93$ when $n=2$, but not for all ε. Rosenblat and Shepherd also prove the existence of a single generalized asymptotic expansion of equation (1.1) based on the outer expansion.

Others have proved the existence of a solution by different and less direct methods. Hsiao's [10] proof applies to the case $n=1$ for sufficiently small ε. Smith [23] proves existence and uniqueness for sufficiently large ε. Tam [24] and Cohen, Fokas, and Lagerstrom [4] both consider a more general equation than that considered here. Both prove existence and Cohen, Fokas, and Lagerstrom also prove uniqueness. In both cases, the behavior of the two-point boundary value problem (1.1)-(1.2) is derived from an analysis of the initial value problem.

5. Discussion. It is not difficult to understand the manner in which the approximation (3.1) is so effective. Its leading term correctly reproduces the true behavior of solutions of Lagerstrom's equation (1.1) with w decreasing monotonically and dw/dx increasing monotonically from $x=\varepsilon$ to $x=\infty$. Subsequent terms correct the error of the leading term in a uniform way. A more complex behavior must arise when logarithms occur as they do in the $n=2$ solution of § 2.1; the fact that $\ln\varepsilon$ changes sign at $\varepsilon=1$ causes the effects of any term with $\ln\varepsilon$ as a factor to reverse at $\varepsilon=1$. Thus it is not surprising in particular that a truncated power series in which terms in $\ln\varepsilon$ are prominent has difficulty in describing a monotonic solution, and adding further terms with successively more logarithms is unlikely to help. This argument also suggests that the occurrence of switchback in matched asymptotic expansions may more generally be a signal that the resulting approximation are of limited use for moderate values of ε. The problem basically is that the arrangement in a power series in ε is then a poor one. Terms in $\ln\varepsilon$ would appear in the successful approximation (3.1) if it were expanded in powers of ε. But in (3.1) they are safely contained in the monotonic gauge function $[E_n(\varepsilon)]^{-1}$ for which no sign reversals occur.

The problem of the slow incompressible flow of a viscous fluid past a rigid symmetric obstacle is, of course, a considerably more complicated problem than that considered here. Nevertheless there are similarities. The matched asymptotic expansions for flow past a cylinder lead to expansions in powers of $(\ln R)^{-1}$, where R is the Reynolds number [11]-[13], [19], [25]. Switchback terms in $\ln R$ complicate the series in powers of R for flow past a sphere [3], [19], [25], although they are then less prevalent than in the analysis of § 2.1. The drag on a sphere as predicted by the matched asymptotic expansions is found to be good for a small range of values of R only [3], [6] with little to show for the hard work entailed in extending the expansions. This situation is reminiscent of that shown in our Figs. 1 and 2. Proudman [18] has suggested that expansion of the drag in powers of R is then a poor arrangement. His diagnosis is correct as far as the model problem (1.1) is concerned. The remedy in the case of the model problem is simply to use the outer expansion throughout. Our analysis therefore suggests that iterative improvement of the outer Oseen expansion may also be the approach to use for the fluid dynamical problem. Finn [7] and Finn and Smith [8] have proved that such an iterative approach to the Navier-Stokes equations converges for sufficiently small flow velocities. This iteration might have a more significant range of usefulness, and be effective so long as the solution of Oseen's equations are qualitatively correct as a first approximation to the solution of the Navier-Stokes equations. Iteration converges fairly rapidly for the model problem even when the first term of (3.1) is not accurate. Whether it also does so for the Navier-Stokes equations remains to be seen.

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