Almost Everywhere Convergence of Dyadic Partial Sums of Fourier Series for Almost Periodic Functions

by

Andrew David Bailey

A thesis submitted to
The University of Birmingham
for the degree of
Master of Philosophy

School of Mathematics
The University of Birmingham
September 2008

Abstract

It is a classical result that for a function $f \in L^p(\mathbb{T})$, dyadic partial sums of the Fourier series of f converge almost everywhere for $p \in (1, \infty)$. In 1968, E. A. Bredihina established an analogous result for the Stepanov spaces of almost periodic functions in the case $p = 2$. Here, a new proof of the almost everywhere convergence result for Stepanov spaces is presented by way of a bound on an appropriate maximal operator for $p = 2^k$, $k \in \mathbb{N}$. In the process of establishing this, a number of general results are obtained that will facilitate further work pertaining to operator bounds and convergence issues in Stepanov spaces.

Acknowledgements

The most inevitable words of thanks (though they are in no way any less sincere) go to Dr. Jonathan Bennett who has been my supervisor for the last twelve months. Jon has been consistently encouraging and patient, and our weekly supervision meetings have always left me feeling inspired and excited. His guidance has unfailingly been invaluable during my terrifying transition to the world of "real maths", and I look forward to continuing to work under his supervision over the next few years.

I would additionally like to express my gratitude to Dr. Neal Bez who has acted as co-supervisor to me. Our various mathematical discussions have always been extremely valuable, and I also appreciate his agreeing to proof-read this thesis.

I am indebted to a number of other people aside from Jon and Neal for their mathematical contributions to my research. In particular, I would like to thank Professor Michael Cowling for introducing me to amalgam spaces and suggesting their relevance to the Stepanov norms and the Hilbert transform and Professors Anthony Carbery and James Wright for some very interesting and insightful discussion about my work on my first visit to Edinburgh University.

As far as non-mathematical contributions are concerned, I would like to express my gratitude to Professor Igor Verbitsky and Dr. Rebecca Waldecker for their assistance with interpreting some of the Russian and German texts (respectively) that I consulted. I would also like to acknowledge financial support for my course received from the School of Mathematics, with particular thanks to Professor Chris Parker and Dr. Dirk Hermans for arranging this.

At this time, I would additionally like to express my sincere thanks to Mr. Dennis Barnett. Whilst I have been fortunate enough to benefit from the mathematical teachings of a great number of talented individuals both while at school and as an undergraduate, I feel particularly indebted to Mr. Barnett, as I feel that he, more than any other, created a genuine love for mathematics in me, motivating me to study it beyond high school level. I would probably not be writing this now had it not been for his tuition, and I am hence profoundly grateful.

Finally (!), I would like to thank my family, for keeping me anchored throughout and having absolute unwavering confidence in me.

Contents

Remarks on Notation

It should be mentioned that the notation used in the study of almost periodic functions has not been standardised, and varies wildly across the different texts on the subject. The notation introduced herein ascribes to personal preference and instinct on the part of the author, rather than conforming to the style of any particular published work.

The symbol $\mathcal{S}(\mathbb{R})$ is used to represent the Schwartz space of rapidly decaying functions, namely those functions $f \in C^\infty(\mathbb{R})$ satisfying the property that for any n, $m \in \mathbb{N} \cup \{0\}$, the quantity $\sup\limits_{x\in\mathbb{R}} |x^n f^{(m)}(x)|$ is finite. The circle group is denoted by $\mathbb{T}$ with functions on $\mathbb{T}$ being identifiable with 2π-periodic functions on $\mathbb{R}$. It will be implicitly equipped with the normalised Lebesgue measure throughout.

Throughout, the notation $\widehat{f}$ will be used interchangeably for the Fourier transform of a function f on $\mathbb{R}$, the function on $\mathbb{Z}$ defining the Fourier coefficients of a function f on $\mathbb{T}$ and the function on $\mathbb{R}$ defining the Fourier coefficients of an almost-periodic function f on $\mathbb{R}$. The meaning in each instance should be clear from the context.

For a number $p \in [1, \infty]$, p' will be used to denote its dual number given by the relationship $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Finally, where not otherwise introduced, the letter C will be used to denote a fixed positive constant. For an operator T acting on a function (or "object") f, the notation $Tf \lesssim f$ will be taken to mean $Tf \leqslant Cf$ with C independent of f.

Introduction

Whilst almost periodic functions attract a fair degree of interest from modern mathematicians for their applications to differential equations (see, for example, [26] and [12] for outlines of this theory), it seems that comparatively little has been done in re-examining classical results in Fourier analysis in the more general setting furnished by the almost periodic function spaces. This thesis will consider one such result, namely the question of almost everywhere convergence of dyadic partial sums of Fourier series in the setting of the Stepanov almost periodic function spaces (which originate from [32]). In the process, the continuing validity of many of the standard results that are used in the L^p spaces is examined, leading to some general framework for further work of this type.

The first chapter gives an outline of the properties of almost periodic functions that are relevant to the remainder of the work. Whilst the choice and presentation of the material is perhaps a little non-standard, the results themselves are, unless otherwise stated, well-understood results from the literature, though unattributed proofs are original. Probably the most comprehensive reference on this type of material is [25]. A reasonable (though certainly less thorough) English language alternative is [3].

In Chapter Two, a presentation of the dyadic almost everywhere convergence result for Fourier series of functions in $L^p(\mathbb{T})$, $p \in (1, \infty)$ is given. It is remarked that full presentations of a proof of this result are rarely found in the literature. Presentations of the analogous problem for Fourier integrals of functions in $L^p(\mathbb{R})$ (or indeed $L^p(\mathbb{R}^n)$, $n \in \mathbb{N}$) are easier to find, but still rare. The presentation given here takes a somewhat personal approach, with inspiration drawn from various sources, as referenced. The proof was ultimately enhanced with hindsight, using some of the ideas from the almost periodic case.

Chapter Three presents a proof of a bound on a maximal summation operator in the Stepanov spaces for $p = 2^k$, $k \in \mathbb{N}$, and develops theory necessary to show that this leads to the dyadic almost everywhere convergence result in this setting. The convergence result was previously established by E. A. Bredihina in [10] for $p = 2^*$. The proof given here takes a different approach, and it is the boundedness of the maximal summation operator established that is of most interest. Results established as part of the proof of the main result include an ℓ^2–valued bound on modified Hilbert transform operators in the Stepanov spaces for $p = 2^k$ and an almost periodic Littlewood–Paley style theorem for the Stepanov spaces for $p \in (1, \infty)$.

*It is remarked at this stage, that despite no mention of it being made there, the proof given in [10] actually provides the almost everywhere convergence for $p \geqslant 2$ by the nesting property of the Stepanov spaces given in Proposition 1.2.8.

Chapter 1

Almost Periodic Functions

1.1 Introduction and Definitions

The initial development of the concept of almost periodic functions is due to Harald Bohr in the 1920s in [5] and [6]. Whilst a periodic function has a fixed period over which it repeats, it is easiest to think of an almost periodic function (from an intuitive standpoint, at least) as a function that will repeat to within any desired level of accuracy over sufficiently long periods. A trivial example would be $f(x) := \sin(x) + \sin(\pi x)$ which, as a sum of two periodic functions, fails to be periodic owing to the two frequencies of the oscillations being incommensurate.

This discussion can be formalised with Bohr's original definition:

Definition 1.1.1 (Bohr Almost Periodicity) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is said to be almost periodic if for all $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0 + K_\varepsilon]$ satisfying $\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon$.*

The numbers τ are referred to as *translation numbers*, with the free choice in x_0 giving a *relative density* condition on them. This is a development of the fact that any integer multiple of the period of a periodic function is also a period, and is essential to the definition (otherwise any continuous function would satisfy the above).

Definition 1.1.2 (Bohr Space) *The normed vector space of all functions satisfying Definition 1.1.1 equipped with the uniform (supremum) norm will be referred to as the Bohr space of almost periodic functions and will be denoted by B.*

From a Fourier analytic point of view, B can be thought of as an almost-periodic analogue of the space $C(\mathbb{T})$ (also equipped with the uniform norm). It is clear that this is quite a restricted class of functions and that greater generality is desirable. This can be achieved by introducing various alternatives to the uniform norm and relaxing the continuity requirement:

Definition 1.1.3 (Stepanov, Weyl and Besicovitch Norms) *Let $f \in L^p_{loc}(\mathbb{R})$ for some $p \in [1, \infty)$. The following (semi-)norms on f can be defined:*

- *The Stepanov norms are given by $\|f\|_{S^p,r} := \sup_{x\in\mathbb{R}} \left(\frac{1}{r} \int_x^{x+r} |f(s)|^p \, ds \right)^{\frac{1}{p}}$ for any fixed $r > 0$.*
- *The Weyl semi-norm is given by $\|f\|_{W^p} := \lim_{r\to\infty} \|f\|_{S^p,r}$.*
- *The Besicovitch semi-norm is given by $\|f\|_{B^p} := \limsup_{x\to\infty} \left(\frac{1}{2x} \int_{-x}^{x} |f(s)|^p \, ds \right)^{\frac{1}{p}}$.*

Note that the Stepanov norms are norms (rather than semi-norms) under the usual (L^p–) convention that functions that differ only on a (Lebesgue) null set are equal. The fact that $\| \cdot \|_{S^p,r}$ satisfies the properties of a norm and that $\| \cdot \|_{W^p}$ and $\| \cdot \|_{B^p}$ satisfy the properties of semi-norms is easily proved and will not be presented here. The Stepanov norms also satisfy the following property:

Lemma 1.1.4 *Fix $p \in [1, \infty)$ and choose any distinct $r_1, r_2 \in \mathbb{R}^+$. Then $\| \cdot \|_{S^p,r_1}$ and $\|.\|_{S^p,r_2}$ are equivalent norms.*

Proof. Without loss of generality, assume $r_1 < r_2$ and choose any function $f \in L^p_{loc}(\mathbb{R})$

such that $\|f\|_{S^p,r_2}$ is finite. It follows that

$$\begin{aligned}
\|f\|_{S^p,r_1} &= \left(\sup_{x\in\mathbb{R}} \frac{1}{r_1}\int_x^{x+r_1} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&= \left(\sup_{x\in\mathbb{R}} \frac{r_2}{r_1}\frac{1}{r_2}\int_x^{x+r_2-(r_2-r_1)} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leqslant \left(\frac{r_2}{r_1}\right)^{\frac{1}{p}} \left(\sup_{x\in\mathbb{R}} \frac{1}{r_2}\int_x^{x+r_2} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&= \left(\frac{r_2}{r_1}\right)^{\frac{1}{p}} \|f\|_{S^p,r_2}.
\end{aligned}$$

Now, choose $N \in \mathbb{N}$ to be the least integer such that $N \geqslant \frac{r_2}{r_1}$. Then,

$$\begin{aligned}
\|f\|_{S^p,r_2} &= \left(\sup_{x\in\mathbb{R}} \frac{1}{r_2}\int_x^{x+r_2} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leqslant \left(\frac{r_1}{r_2}\right)^{\frac{1}{p}} \left(\sup_{x\in\mathbb{R}} \frac{1}{r_1}\int_x^{x+Nr_1} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\leqslant \left(\frac{r_1}{r_2}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{N}\sup_{x\in\mathbb{R}} \frac{1}{r_1}\int_{x+(j-1)r_1}^{x+jr_1} |f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&= \left(\frac{Nr_1}{r_2}\right)^{\frac{1}{p}} \|f\|_{S^p,r_1}.
\end{aligned}$$

It hence follows that $\|\cdot\|_{S^p,r_1}$ and $\|\cdot\|_{S^p,r_2}$ are equivalent. □

It is also necessary to show that the limit given in the definition of the Weyl norm exists, for which a proof similar to [3], pp. 72–3 is presented:

Lemma 1.1.5 *For $p \in [1,\infty)$ and $f \in L^p_{loc}(\mathbb{R})$, $\lim_{r\to\infty} \|f\|_{S^p,r}$ always exists if it is permitted to take the value ∞.*

Proof. By Lemma 1.1.4, if for some $r^* \in \mathbb{R}^+$, $\|f\|_{S^p,r^*} = \infty$, then $\|f\|_{S^p,r} = \infty$ for all $r \in \mathbb{R}^+$, and so the limit exists (and is infinite in this case).

Suppose $\|f\|_{S^p,r}$ is finite for all $r \in \mathbb{R}^+$. Choose any r_1, $r_2 \in \mathbb{R}^+$ and let $n \in \mathbb{N}$ be such that $(n-1)r_2 \leqslant r_1 \leqslant nr_2$. From this, $nr_2 \leqslant r_1 + r_2$ and so, using the proof of Lemma 1.1.4,

$$\begin{aligned} \|f\|_{S^p,r_1} &\leqslant \left(\frac{nr_2}{r_1}\right)^{\frac{1}{p}} \|f\|_{S^p,nr_2} \\ &\leqslant \left(\frac{r_1+r_2}{r_1}\right)^{\frac{1}{p}} \|f\|_{S^p,nr_2} \\ &\leqslant \left(1+\frac{r_2}{r_1}\right)^{\frac{1}{p}} \left(\frac{nr_2}{nr_2}\right)^{\frac{1}{p}} \|f\|_{S^p,r_2} \\ &= \left(1+\frac{r_2}{r_1}\right)^{\frac{1}{p}} \|f\|_{S^p,r_2}. \end{aligned}$$

It thus follows that $\limsup_{r_1\to\infty} \|f\|_{S^p,r_1} \leqslant \|f\|_{S^p,r_2}$.

By the arbitrary choice of r_1 and r_2, it may be concluded that $\limsup_{r\to\infty} \|f\|_{S^p,r} \leqslant \liminf_{r\to\infty} \|f\|_{S^p,r}$ and hence the limit exists as required. $\square$

The almost periodic function spaces corresponding to each norm can now be defined as follows:

Definition 1.1.6 (Stepanov, Weyl and Besicovitch Spaces) *Let $p \in [1,\infty)$. Then the S^p (Stepanov), W^p (Weyl) and B^p (Besicovitch) spaces are defined as the space of all functions, $f \in L^p_{loc}(\mathbb{R})$, satisfying the following definition with the appropriate (semi-) norm:*

For all $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0+K_\varepsilon]$ satisfying $\|f(\cdot+\tau) - f\| < \varepsilon$.

Note that the space B^1 should not be confused with B. There is no ambiguity in the definition of the S^p spaces, owing to the equivalence of the Stepanov norms given in Lemma 1.1.4. For convenience, $\|\cdot\|_{S^p}$ will be taken to mean $\|\cdot\|_{S^p,1}$ when using a Stepanov norm from hereon, and it is this norm that will be used almost exclusively.

1.2 Fundamental Properties

The following theorem is very significant:

Theorem 1.2.1 (The Fundamental Theorem) *Let* $p \in [1, \infty)$. *Define the set of trigonometric polynomials to be the set of all functions of the form*

$$f(x) = \sum_{n=-N}^{N} a_n e^{i\lambda_n x}$$

where $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$, $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $N \in \mathbb{N}$. *Then* B *is identically equal to the closure of the set of trigonometric polynomials in the space* $C(\mathbb{R})$ *equipped with the uniform norm, and* S^p, W^p *and* B^p *are all identically equal to the closure of the set of trigonometric polynomials in the spaces* $\{f \in L^p_{loc}(\mathbb{R}) : \|f\| < \infty\}$ *with respect to the appropriate norm.*

For a proof for B, see [7], pp. 80–88. For the other spaces, see [3], Chapter II*.

Another relevant property of the aforementioned spaces is that they are, in the order introduced, increasingly general. Specifically, the following holds:

Proposition 1.2.2 *Let* $p \in [1, \infty)$. *Then* $B \subset S^p \subset W^p \subset B^p$.

Proof. By looking at Definitions 1.1.1 and 1.1.6, it suffices to show that for any function $f \in L^p_{loc}(\mathbb{R})$, $\|f\|_{B^p} \leqslant \|f\|_{W^p} \leqslant \|f\|_{S^p} \leqslant \|f\|_B$, so these inequalities will be considered in turn:

*Note that in Chapter I of [3], the name "Fundamental Theorem" is used to refer to a different theorem than the one given here.

$$
\begin{aligned}
\|f\|_{B^p} &= \lim_{y\to\infty} \sup_{x\geqslant y} \left(\frac{1}{2x} \int_{-x}^{x} |f(s)|^p \, ds \right)^{\frac{1}{p}} \\
&\leqslant \lim_{y\to\infty} \sup_{x\geqslant y} \sup_{z\in\mathbb{R}} \left(\frac{1}{2x} \int_{z-x}^{z+x} |f(s)|^p \, ds \right)^{\frac{1}{p}} \\
&= \lim_{y\to\infty} \sup_{x\geqslant y} \sup_{z\in\mathbb{R}} \left(\frac{1}{2x} \int_{z}^{z+2x} |f(s)|^p \, ds \right)^{\frac{1}{p}} \\
&= \limsup_{r\to\infty} \|f\|_{S^p,r} \\
&= \|f\|_{W^p} \text{ by Lemma 1.1.5.}
\end{aligned}
$$

Now, from the proof of Lemma 1.1.4, $\|f\|_{W^p} \leqslant \lim_{r\to\infty} \left(\frac{N_r}{r}\right)^{\frac{1}{p}} \|f\|_{S^p}$, where N_r is the least positive integer such that $N_r > r$. But $\lim_{r\to\infty} \frac{N_r}{r} = 1$, hence $\|f\|_{W^p} \leqslant \|f\|_{S^p}$.

Finally,

$$
\begin{aligned}
\|f\|_{S^p} &= \left(\sup_{x\in\mathbb{R}} \int_{x}^{x+1} |f(s)|^p \, ds \right)^{\frac{1}{p}} \\
&\leqslant \left(\sup_{t\in\mathbb{R}} |f(t)|^p \sup_{x\in\mathbb{R}} \int_{x}^{x+1} \, ds \right)^{\frac{1}{p}} \\
&= \|f\|_{B}.
\end{aligned}
$$

□

The S^p spaces perhaps form the closest almost periodic analogue of the spaces $L^p(\mathbb{T})$ and it is these spaces with which the bulk of the discussion herein will be concerned. The fact that the W^p and B^p semi-norms fail to be norms actually holds to the extent that two functions f and g in either space may differ on a set of infinite measure and still satisfy $\|f - g\| = 0$. A brief demonstration of this fact for W^1 may be undertaken by means of the following:

Proposition 1.2.3 *There exists $f \in L^1_{loc}(\mathbb{R})$ such that $\|f\|_{W^1} = 0$, but $f(x) \neq 0$ for all*

$x \in \mathbb{R}$.

Proof. It suffices to consider the following example:

$$f(x) := \begin{cases} 1, & x \in [-1,1] \\ \frac{1}{x^2}, & x \in \mathbb{R} \setminus [-1,1] \end{cases}$$

Clearly $f(x) \neq 0$ for all $x \in \mathbb{R}$, so it remains to show that $\|f\|_{W^1} = 0$:

$$\begin{aligned} \|f\|_{W^1} &= \lim_{r\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{r} \int_x^{x+r} |f(s)|\, ds \\ &= \lim_{r\to\infty} \frac{1}{r} \int_{\frac{-r}{2}}^{\frac{r}{2}} |f(s)|\, ds \\ &= 0. \end{aligned}$$

□

Now, note that for functions f satisfying $\|f\|_{W^1} = 0$, it is necessarily the case that $f \in W^1$ as for any $\tau \in \mathbb{R}$, $\|f(\cdot+\tau) - f\|_{W^1} \leqslant \|f(\cdot+\tau)\|_{W^1} + \|f\|_{W^1} = 0$. As the trivial function $g \equiv 0$ also satisfies $\|g\|_{W^1} = 0$, it follows that f and g are two functions differing on the entire real line, but satisfying $\|f - g\|_{W^1} = 0$.

This problem for the Besicovitch spaces will briefly be returned to in Section 1.4.1.

A further difficult property of the W^p spaces is that they are incomplete. See [8]* for a proof of this fact (in general, it follows from - or is used to prove, depending on the method of proof considered - the fact that the space $(\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{W^p} < \infty\}, \|\cdot\|_{W^p})$ is incomplete).

Fortunately, the following does hold:

Theorem 1.2.4 *Let $p \in [1,\infty)$. Then B, S^p and B^p are Banach spaces.*

*Note that this paper uses the notations S^p–*a.p.*, W^p–*a.p.* and B^p–*a.p.* for the almost periodic functions spaces introduced here. The symbols S^p, W^p and B^p are used for another purpose.

The completeness of B follows trivially from the completeness of the space $C(\mathbb{R})$ and Theorem 1.2.1. For a proof of the completeness of B^p, see [8].

For S^p, it is noted that by the Fundamental Theorem (Theorem 1.2.1) it suffices to prove completeness of the space $(\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\}, \|\cdot\|_{S^p})$.

Given this, the proof of the completeness of L^p in [29], pp. 67–8 adapts almost exactly to give the desired result. See also [8].

The following result is critical to the remainder of this thesis:

Proposition 1.2.5 *$\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\} \neq S^p$, that is to say, there exist functions $f \in L^p_{loc}(\mathbb{R}) \setminus S^p$, $p \in [1, \infty)$ such that $\|f\|_{S^p} < \infty$.*

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) := \begin{cases} 1, & x \in [0,1] \\ 0, & x \in \mathbb{R} \setminus [0,1] \end{cases}$$

Then certainly $\|f\|_{S^p} = 1$, but f fails to satisfy Definition 1.1.6, as for sufficiently large $\tau \in \mathbb{R}$, $\|f(\cdot + \tau) - f\|_{S^p} = \|f\|_{S^p} = 1$. □

The following intuitive result does hold:

Lemma 1.2.6 *For any $f \in S^p$, $p \in [1, \infty)$, it follows that $\|f\|_{S^p} < \infty$.*

Proof. Whilst this result is clear by the Fundamental Theorem, technically, proving the Fundamental Theorem requires that this result is already established, so a direct proof will be presented here.

By Definition 1.1.6, as $f \in S^p$, there exists $K_1 > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [-x_0, -x_0 + K_1]$ such that $\|f(\cdot + \tau) - f\|_{S^p} < 1$. Using this, for any fixed $x \in \mathbb{R}$,

$$
\begin{aligned}
\left(\int_x^{x+1} |f(s)|^p \, ds\right)^{\frac{1}{p}} &\leqslant \|f(\cdot+\tau)-f\|_{S^p} + \left(\int_x^{x+1} |f(s+\tau)|^p \, ds\right)^{\frac{1}{p}} \\
&< 1 + \left(\int_{x+\tau}^{x+\tau+1} |f(s)|^p \, ds\right)^{\frac{1}{p}} \\
&\leqslant 1 + \left(\int_0^{K_1+1} |f(s)|^p \, ds\right)^{\frac{1}{p}} \\
&< \infty
\end{aligned}
$$

as $f \in L^p_{loc}(\mathbb{R})$. As K_1 is independent of x, it follows that $\|f\|_{S^p} < \infty$. □

It is worth developing some arithmetical properties for S^p functions, which will be of use later.

Proposition 1.2.7 *Take $p \in [1, \infty)$, and choose $f, g \in S^p$ and $h \in S^{p'}$, interpreting $S^{p'}$ to mean $L^\infty(\mathbb{R})$ when $p = 1$. Further, let $(f_i)_{i\in\mathbb{N}} \subset S^p$ be an S^p–convergent sequence with limit f_∞. Then the following hold:*

- *$(f+g) \in S^p$.*
- *$fh \in S^1$.*
- *$\lim\limits_{i\to\infty} f_i \in S^p$.*

Furthermore, these results continue to hold if the Stepanov spaces are replaced with the corresponding Weyl spaces, Besicovitch spaces, or uniformly with the Bohr class.

Proof. For ease of notation and relevance to the remainder of this thesis, the proof will be presented only for the S^p spaces. It adapts exactly to the W^p and B^p spaces, and almost exactly to the B space.

A direct proof of the first two facts using Definition 1.1.6 is more involved than it might initially appear. The complication arises in finding a relatively dense set of translation numbers ("τ"s) that is *common* to both functions (there are proofs in the literature that are erroneous because they fail to consider this fact). For an insight into these difficulties, the reader is referred to [7], pp. 36–9, where some direct proofs for the Bohr class are presented (which rely on powerful theory developed earlier in the book).

Fortunately, the Fundamental Theorem permits a somewhat more straightforward approach, and this is how the proof here will proceed.

Firstly, note that there exist sequences of trigonometric polynomials, $(r_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \|f - r_n\|_{S^p} = 0$ and $\lim\limits_{n\to\infty} \|g - s_n\|_{S^p} = 0$. As $(r_n + s_n)_{n\in\mathbb{N}}$ is a sequence of trigonometric polynomials, and further, $\|f + g - (r_n + s_n)\|_{S^p} \leqslant \|f - r_n\|_{S^p} + \|g - s_n\|_{S^p}$, it can be concluded that $f + g \in S^p$.

Now, there also exists a sequence of trigonometric polynomials $(t_n)_{n\in\mathbb{N}}$ approximating h ($\lim\limits_{n\to\infty} \|h - t_n\|_{S^{p'}} = 0$). Consider that

$$\begin{aligned}
\|fh - r_n t_n\|_{S^1} &= \sup_{x\in\mathbb{R}} \int_x^{x+1} |f(s)h(s) - r_n(s)t_n(s)|\, ds \\
&\leqslant \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f(s)||h(s) - t_n(s)|\, ds + \int_x^{x+1} |t_n(s)||f(s) - r_n(s)|\, ds \right) \\
&\leqslant \sup_{x\in\mathbb{R}} \left(\left(\int_x^{x+1} |f(s)|^p\, ds \right)^{\frac{1}{p}} \left(\int_x^{x+1} |h(s) - t_n(s)|^{p'}\, ds \right)^{\frac{1}{p'}} \right. \\
&+ \left. \left(\int_x^{x+1} |t_n(s)|^{p'}\, ds \right)^{\frac{1}{p'}} \left(\int_x^{x+1} |f(s) - r_n(s)|^p\, ds \right)^{\frac{1}{p}} \right) \\
&\quad \text{by Hölder's inequality} \\
&\leqslant \|f\|_{S^p} \|h - t_n\|_{S^{p'}} + \|t_n\|_{S^{p'}} \|f - r_n\|_{S^p}.
\end{aligned}$$

Given that $\|f\|_{S^p} < \infty$, and that $(r_n t_n)_{n\in\mathbb{N}}$ is a sequence of trigonometric polynomials, it suffices to observe that $\sup\limits_{n\in\mathbb{N}} \|t_n\|_{S^{p'}} < \infty$ (as $(t_n)_{n\in\mathbb{N}}$ is a convergent sequence) to conclude that $fh \in S^1$. It thus follows that

$$\sup_{n\in\mathbb{N}} \|t_n\|_{S^{p'}} \leqslant \max\{\|t_n\|_{S^{p'}} : n \in [1, N_1 - 1] \cap \mathbb{N}\} \cup \{1 + \|h\|_{S^{p'}}\}.$$

For the proof of the final fact, choose any $\varepsilon > 0$ and note that by hypothesis, there exists $N_\varepsilon \in \mathbb{N}$ such that $\|f_i - f_\infty\|_{S^p} < \dfrac{\varepsilon}{3}$ for all $i \geqslant N_\varepsilon$.

Now, by the almost periodicity of f_{N_ε}, there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0 + K_\varepsilon]$ satisfying $\|f_{N_\varepsilon}(\cdot + \tau) - f_{N_\varepsilon}\|_{S^p} < \dfrac{\varepsilon}{3}$. To conclude that $f_\infty \in S^p$, note that

$$\begin{aligned} & \|f_\infty(\cdot + \tau) - f_\infty\|_{S^p} \\ \leqslant\ & \|f_\infty(\cdot + \tau) - f_{N_\varepsilon}(\cdot + \tau)\|_{S^p} + \|f_{N_\varepsilon}(\cdot + \tau) - f_{N_\varepsilon}\|_{S^p} + \|f_{N_\varepsilon} - f_\infty\|_{S^p} \\ <\ & \varepsilon. \end{aligned}$$ □

It is remarked that, given the multiplication property and the fact that the function $f(x) \equiv 1$ is in all almost periodic function spaces (as it is a continuous purely periodic function, so is certainly in B), it follows that $S^p \subset S^1$, $W^p \subset W^1$ and $B^p \subset B^1$ for all $p \in (1, \infty)$. The space B^1 thus contains all other classes of almost periodic functions under consideration (by Lemma 1.2.2).

In fact, with a simple application of Hölder's inequality, a further property that is rarely mentioned in the literature can be deduced:

Proposition 1.2.8 *Let p_1, $p_2 \in [1, \infty)$ with $p_1 < p_2$. Then*

$$\begin{aligned} S^{p_2} &\subseteq S^{p_1}, \\ W^{p_2} &\subseteq W^{p_1}, \\ B^{p_2} &\subseteq B^{p_1}. \end{aligned}$$

Proof. Again, only the proof for the Stepanov spaces will be presented. The other cases proceed in exactly the same way.

Take any $f \in S^{p_2}$, $\varepsilon > 0$ and consider that by Definition 1.1.6, there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0 + K_\varepsilon]$ satisfying

$$\|f(\cdot + \tau) - f\|_{S^{p_2}} < \varepsilon.$$

Now, for any such τ,

$$\begin{aligned}
&\|f(\cdot + \tau) - f\|_{S^{p_1}} \\
= &\left(\sup_{x\in\mathbb{R}} \int_x^{x+1} |f(s+\tau) - f(s)|^{p_1}\, ds\right)^{\frac{1}{p_1}} \\
\leqslant &\left(\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f(s+\tau) - f(s)|^{p_1 \frac{p_2}{p_1}}\, ds\right)^{\frac{p_1}{p_2}} \left(\int_x^{x+1} \, ds\right)^{\frac{1}{\left(\frac{p_2}{p_1}\right)'}}\right)^{\frac{1}{p_1}} \\
= &\|f(\cdot + \tau) - f\|_{S^{p_2}}.
\end{aligned}$$

It thus follows, using the same choices of K_ε and τ as required, that $f \in S^{p_1}$. □

It is noted that in particular, this proof gives that $\|\cdot\|_{S^{p_1}} \leqslant \|\cdot\|_{S^{p_2}}$ for $p_1 \leqslant p_2$, and this property will be used implicitly from now on.

For a thorough survey of the hierarchy of the various almost periodic function spaces, the reader is referred to [1].

1.3 Fourier Series for Almost Periodic Functions

When considering how to define the Fourier series of an almost periodic function, it is clear that there are a few important differences from the case of periodic functions. Firstly, by considering the simple example of the function $B \ni f := e^{i\pi\cdot} + e^{i\cdot}$, it is immediately obvious that it is insufficient to consider only infinite trigonometric polynomials of the form $\sum_{n\in\mathbb{Z}} c_n e^{in\cdot}$ as in the periodic case. Indeed by simple modification of this trivial example, it is apparent that at least series consisting of multiples of polynomials in the set $\{e^{i\lambda\cdot} : \lambda \in \mathbb{R}\}$ must be considered. Secondly, when calculating Fourier coefficients, as almost periodic functions do not (generally) have a single "period", it is likely to be insufficient to consider integrals over intervals of finite length, as in the periodic case.

A complication that arises from the first issue is that the set $\{e^{i\lambda\cdot} : \lambda \in \mathbb{R}\}$ is clearly uncountable, which means that any series of terms multiplying its members may not be well-defined.

A simple averaging operation turns out to be the tool to solve these various issues:

Definition 1.3.1 *Define* $M(f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx.$

The first key result is as follows:

Theorem 1.3.2 *The set* $\{e^{i\lambda\cdot} : \lambda \in \mathbb{R}\}$ *is "orthonormal" with respect to the averaging operation* M*. That is:*

$$M(e^{i\lambda_1\cdot} e^{-i\lambda_2\cdot}) = \begin{cases} 1, & \lambda_1 = \lambda_2 \\ 0, & \lambda_1 \neq \lambda_2 \end{cases}$$

Proof. Normality is immediately clear. For orthogonality, note that for $\lambda_1 \neq \lambda_2$, elementary integration yields that $M(e^{i\lambda_1\cdot} e^{-i\lambda_2\cdot}) = \lim_{T\to\infty} \frac{1}{T(\lambda_1 - \lambda_2)} \sin((\lambda_1 - \lambda_2)T) = 0.$ □

The Fourier coefficients of an almost periodic function may now be defined:

Definition 1.3.3 *Let $f \in B^1$. Then for any $\lambda \in \mathbb{R}$, the λ^{th} Fourier coefficient of f is defined to be $\widehat{f}(\lambda) := M(fe^{-i\lambda\cdot})$. The numbers λ are referred to as Fourier exponents.*

To show that these Fourier coefficients are well-defined, the following lemma is required:

Lemma 1.3.4 *For $f \in B^1$, $p \in [1, \infty)$, the quantity $M(f)$ exists and is equal to $\|f\|_{B^1}$.*

For a simple proof, see [3], p. 93 (the equality is not stated, but holds trivially as a consequence of the existence). By the inclusions discussed at the end of Section 1.2, it follows that for an almost periodic function f in any of the spaces under consideration, $M(f)$ exists. Further, by the multiplication property (Proposition 1.2.7), $M(fe^{-i\lambda\cdot})$ exists for any $\lambda \in \mathbb{R}$ and hence the Fourier coefficients of any almost periodic function are well defined.

The aforementioned problem pertaining to uncountability can now be eliminated by means of the following:

Theorem 1.3.5 *For $f \in B^1$ there exists a countable set $E \subseteq \mathbb{R}$ such that $\widehat{f}(\lambda) = 0$ for all $\lambda \in \mathbb{R} \setminus E$.*

Proof. There is a presentation of a proof from first principles in [7], pp. 48–50 (the proof is stated for B, but holds for B^1 without adaptation). Here the Fundamental Theorem will once again be appealed to, permitting a more elegant solution.

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials such that $\lim_{n\to\infty} \|f - r_n\|_{B^1} = 0$. It then follows that $\lim_{n\to\infty} M(r_n e^{-i\lambda\cdot}) = M(fe^{-i\lambda\cdot})$.

Now by the "orthogonality" from Theorem 1.3.2 and the fact that r_n is a trigonometric polynomial, $M(r_n e^{-i\lambda\cdot})$ may differ from zero only for the finite number of $\lambda \in \mathbb{R}$ such that there is a term in $e^{i\lambda\cdot}$ in r_n. It thus follows that $M(fe^{-i\lambda\cdot}) = \lim_{n\to\infty} M(r_n e^{-i\lambda\cdot})$ may differ from zero for only countably many $\lambda \in \mathbb{R}$. $\square$

This allows the Fourier* series of an almost periodic function to be defined:

Definition 1.3.6 (Fourier Series) *Let $f \in B^1$ and enumerate its Fourier exponents with non-zero corresponding Fourier coefficients as $(\lambda_n)_{n\in\mathbb{N}}$. Then the Fourier series of f may be written as:*

$$f \sim \sum_{n\in\mathbb{N}} \widehat{f}(\lambda_n)e^{i\lambda_n \cdot}.$$

For purely periodic functions, the definition of the Fourier series given here practically reduces to the standard definition, excepting minor technical considerations such as the permissibility of zero coefficients in a standard Fourier series. The proof of this fact is slightly non-trivial, and the reader is referred to [7] pp. 50–51 for a proof and further discussion.

A potential problem with this definition that presents itself is that it gives no order in which to enumerate the Fourier exponents for a particular almost periodic function. Consequently, if the series is convergent, different arrangements could result in different sums. Discussion of this issue will be resumed shortly.

The following basic properties of Fourier series carry over from the periodic setting:

Proposition 1.3.7 *Let f, $g \in B^1$. Then the following hold:*

- $kf \sim \sum_{n\in\mathbb{N}} k\widehat{f}(\lambda_n)e^{i\lambda_n \cdot}$ *for* $k \in \mathbb{C}$.
- $\overline{f} \sim \sum_{n\in\mathbb{N}} \overline{\widehat{f}(\lambda_n)}e^{-i\lambda_n \cdot}$.
- $f \pm g \sim \sum_{n\in\mathbb{N}} (\widehat{f}(\lambda_n) \pm \widehat{g}(\lambda_n))e^{i\lambda_n \cdot}$.

The proof of these facts is elementary and will not be presented here.

The following key result holds for almost periodic functions as for periodic functions:

Theorem 1.3.8 (Uniqueness Theorem) *There is no $f \in B^1$ with $\|f\| \neq 0$ such that*

*It should be noted that some authors adopt the terminology *Bohr–Fourier* series (and correspondingly Bohr–Fourier coefficient and Bohr–Fourier exponent).

$\widehat{f}(\lambda) = 0$ *for all* $\lambda \in \mathbb{R}$*, where the norm* $\|\cdot\|$ *is chosen to correspond to any space that* f *belongs to.*

The proof of this is more involved than that in the periodic setting. The reader is referred to [3], p. 109 for the details.

The Uniqueness Theorem remedies the potential problem regarding the ordering of the Fourier series for an almost periodic function. In particular, two different arrangements of the Fourier series cannot converge to different functions (regarded as members of the appropriate almost periodic space), as their difference would result in a non-zero function with identically zero Fourier coefficients (using the additive property of Fourier series from Proposition 1.3.7).

There is a notion of convolution that holds for almost periodic functions, which can be defined as follows (as in [19], p. 164):

Definition 1.3.9 (Mean Convolution) *Let* f*,* $g \in B^1$*. Then the mean convolution of* f *and* g *is given by* $(f \underset{M}{*} g)(x) := M_y(f(x-y)g(y)) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x-y)g(y)\,dy$*.*

This has a similar property to regular convolution:

Lemma 1.3.10 *Let* $f \in B^1$ *and* $g(x) = \sum_{|n|\leqslant N} \widehat{g}(\lambda_n)e^{i\lambda_n x}$ *be a trigonometric polynomial. Then* $(g \underset{M}{*} f)(x) = \sum_{|n|\leqslant N} \widehat{f}(\lambda_n)\widehat{g}(\lambda_n)e^{i\lambda_n x}$ *for* $x \in \mathbb{R}$*.*

Proof.

$$
\begin{aligned}
(g \underset{M}{*} f)(x) &= \sum_{|n|\leqslant N} (\widehat{g}(\lambda_n)e^{i\lambda_n x}) \underset{M}{*} f \\
&= \sum_{|n|\leqslant N} \widehat{g}(\lambda_n) \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(y)e^{i\lambda_n(x-y)}\,dy \\
&= \sum_{|n|\leqslant N} \widehat{f}(\lambda_n)\widehat{g}(\lambda_n)e^{i\lambda_n x}. \qquad \square
\end{aligned}
$$

Another result that holds for almost periodic functions, and will be very useful later, is an analogue of Parseval's identity:

Theorem 1.3.11 (Parseval's Identity) *For any* $f \in B^2$, $M(|f|^2) = \sum_{n\in\mathbb{N}} |\widehat{f}(\lambda_n)|^2$.

For a proof, the reader is referred to [3], p. 109.

It is emphasised that by Lemma 1.2.2, Parseval's identity holds *a fortiori* for $f \in S^2$ and $f \in W^2$.

Before concluding this section, one final result will be developed that is an extension of the "Strengthened Mean Value Theorem" in [7], p. 44:

Theorem 1.3.12 *Take* $f \in S^p$, $p \in [1, \infty)$ *and* $\lambda \in \mathbb{R}$. *Then for any* $a \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)e^{-i\lambda x}\, dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T+a}^{T+a} f(x)e^{-i\lambda x}\, dx.$$

Proof. First, observe that

$$\begin{aligned} &\frac{1}{2T} \int_{-T+a}^{T+a} f(x)e^{-i\lambda x}\, dx \\ = \ &\frac{1}{2T} \int_{-T+a}^{-T} f(x)e^{-i\lambda x}\, dx + \frac{1}{2T} \int_{-T}^{T} f(x)e^{-i\lambda x} dx + \frac{1}{2T} \int_{T}^{T+a} f(x)e^{-i\lambda x}\, dx. \end{aligned}$$

Now, by an application of Hölder's inequality,

$$\begin{aligned} \left|\frac{1}{2T} \int_{T}^{T+a} f(x)e^{-i\lambda x}\, dx\right| &\leqslant \frac{1}{2T} \int_{T}^{T+a} |f(x)|\, dx \\ &\leqslant \frac{a^{\frac{1}{p'}}}{2T} \|f\|_{S^p,a}. \end{aligned}$$

By a similar treatment of the $\frac{1}{2T}\int_{-T+a}^{-T} f(x)e^{-i\lambda x}\,dx$ term,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{T}^{T+a} f(x)e^{-i\lambda x}\,dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{-T} f(x)e^{-i\lambda x}\,dx = 0.$$

It thus follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}\,dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a} f(x)e^{-i\lambda x}\,dx. \qquad \square$$

1.4 Other Developments

This self-contained section aims to give a brief outline of some further theory related to almost periodic functions, setting them in a broader context. It is not required for the remainder of the thesis.

1.4.1 The Bohr Compactification

The following definition will be given in the most abstract setting, in line with the presentation in [30]:

Definition 1.4.1 *Let G be a locally compact abelian group with dual group Γ and let Γ_d be Γ equipped with the discrete topology. Then the Bohr Compactification of G, G_B, is the compact abelian group which is the dual group of Γ_d.*

Associated with G_B is the map $i_B : G \to G_B$ defined so that for any $x \in G$, $\gamma \in \Gamma$, $\gamma(x) = (i_B(x))(\gamma)$. This satisfies the following property which ensures that G_B is indeed a compactification of G:

Theorem 1.4.2 *Regarded as a map onto its range, i_B is a continuous isomorphism, with $i_B(G)$ dense in G_B.*

For a proof of this fact, as well as verification that G_B is a compactification of G, see [30], pp. 30–31.

The significance of this definition in relation to almost periodic functions is encapsulated in the following result, as stated in [1], pp. 129–30 and p. 167:

Theorem 1.4.3 *The space B is isometrically isomorphic to $C(\mathbb{R}_B, \mathbb{R})$, and for $p \in [1, \infty)$, B^p is isometrically isomorphic to $L^p(\mathbb{R}_B, \mathbb{R})$, where $\mathbb{R}_B$ is equipped with its Haar measure.*

This is of interest as it allows certain properties of B and B^p to be deduced from the properties of continuous functions and L^p functions respectively. For example, it gives that the B^p spaces are reflexive, and that B^p is dual to $B^{p'}$.

Furthermore, it was mentioned earlier that functions f, $g \in B^p$ may differ on a set of infinite measure and still satisfy $\|f - g\|_{B^p} = 0$. The possibility of this may be deduced from the representation of B^p as $L^p(\mathbb{R}_B, \mathbb{R})$. In particular, it is noted that $\|f-g\|_{L^p(\mathbb{R}_B,\mathbb{R})} = 0$ if and only if f and g, considered as members of $L^p(\mathbb{R}_B, \mathbb{R})$, differ on a set of Haar measure zero. Now, $\mathbb{R}$ forms a set of Haar measure zero in $\mathbb{R}_B$ (see (33.28) in [18], p. 313), and thus it is feasible that f and g may differ on the entire of $\mathbb{R}$.

The above does not make it permissible to modify a B^p function anywhere on $\mathbb{R}$ such that the norm of the difference with the original function is zero (which would make the entire space trivial). This is because the function's values on $\mathbb{R}_B$ are determined from its values on $\mathbb{R}$ via isomorphism.

It is remarked that this perspective on B^p allows some sense to be made of B^p functions that can be represented by series that diverge to infinity at every point in $\mathbb{R}$! One example of such a function, as stated in [24], is $\displaystyle\sum_{n\in\mathbb{N}} \frac{\cos(2^{-n}\pi\cdot)}{n}$.

1.4.2 Almost Periodic Functions on Groups

Whilst all of the theory of almost periodic functions so far has been concerned with functions taking values on $\mathbb{R}$, there is a limited theory in the more abstract setting of $\mathbb{R}$–valued functions taking values from a group.

To motivate the definition, an alternative (reasonably intuitive) definition of Bohr almost periodicity due to Bochner will be considered:

Definition 1.4.4 *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be normal if for any $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, there exists a subsequence $(h_{n_i})_{i\in\mathbb{N}}$ such that the sequence $(f(\cdot + h_{n_i}))_{i\in\mathbb{N}}$ is uniformly pointwise convergent.*

This satisfies the following:

Theorem 1.4.5 *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is a member of the Bohr class B if and only if it is normal.*

There is a proof of this result in [3], pp. 11–12.

With this equivalent definition of Bohr almost periodicity in mind, the following definition of almost periodicity for a function taking values on a group can be considered analogously. In what follows, G will always be used to represent an arbitrary group (which does not need to have any topological structure).

Definition 1.4.6 *Let $f : G \to \mathbb{R}$ be an arbitrary function. Then f is said to be almost periodic on the right if for any $(a_n)_{n\in\mathbb{N}} \subseteq G$, there exists a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ such that the sequence $(f(\cdot\, a_{n_i}))_{i\in\mathbb{N}}$ is uniformly pointwise convergent. Analogously, f is said to be almost periodic on the left if for any $(a_n)_{n\in\mathbb{N}} \subseteq G$, there exists a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ such that the sequence $(f(a_{n_i}\cdot))_{i\in\mathbb{N}}$ is uniformly pointwise convergent.*

This definition originates from Von Neumann's paper [34]. It is noted that, in contrast to the case of Bohr almost periodicity, there is no continuity requirement on the function under consideration (indeed, without topological structure on G, this concept is not well-defined).

Independently of whether G is abelian, the following holds:

Theorem 1.4.7 *A function $f : G \to \mathbb{R}$ is almost periodic on the right if and only if it is almost periodic on the left.*

For a proof of this fact, see [12], p. 170.

The full theory of almost periodic functions on groups is quite rich, and a complete exposition is beyond the scope of this thesis. Through a use of representation theory, it is possible to develop the concept of a Fourier series, which leads, for example, to analogues of Parseval's identity (Theorem 1.3.11) and the Fundamental Theorem (Theorem 1.2.1). Von Neumann's original paper [34], along with its sequel with Bochner [4] offer a very comprehensive introduction to the subject. There is also a detailed exposition in Chapter VII of [12] and a discussion from the perspective of Banach space valued functions on a group in Chapter 13 of [35].

Chapter 2

The Classical Problem for $L^p(\mathbb{T})$

2.1 Introduction to the Classical Problem

The following theorem is classical:

Theorem 2.1.1 (The Classical Dyadic Convergence Problem) *Let $f \in L^p(\mathbb{T})$, $p \in (1, \infty)$. Then for almost every $x \in \mathbb{T}$, $f(x) = \lim_{k\to\infty} \sum_{|n|\leqslant 2^k} \widehat{f}(n)e^{inx}$.*

The special case of $p = 2$ originates from [23]. It is developing an analogue of this theorem for the S^p spaces of almost periodic functions by means of a bound on an appropriate maximal operator that is the main focus of this thesis. This chapter outlines a proof in the standard periodic case. An alternative proof to the one given here (using somewhat more old-fashioned complex methods) is given in [36], Chapter XV.

The approach taken will involve bounding the maximal operator

$$S^*f = \sup_{k\in\mathbb{N}} \Big| \sum_{|n|\leqslant 2^k} \widehat{f}(n)e^{in\cdot} \Big|$$

and appealing to the following, which is a special case of Theorem 2.2 in [13], p. 27:

Theorem 2.1.2 *Let $\{T_j\}$ be a family of linear operators on $L^p(\mathbb{T})$, $p \in [1, \infty)$. Then if* *the corresponding maximal operator,* $T^*f := \sup_j |T_j f|$ *is weak (p–q) for some* $q \in [1, \infty)$*, the set* $\{f \in L^p(\mathbb{T}) : \lim_{j\to\infty} T_j f(x) = f(x) \text{ a.e.}\}$ *is closed in* $L^p(\mathbb{T})$*.*

Note that Theorem 2.1.1 certainly holds for standard trigonometric polynomials ($\sum_{|n|\leqslant N} c_n e^{in\cdot}$, $N \in \mathbb{N}$, $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$) and that the set of trigonometric polynomials is dense in $L^p(\mathbb{T})$. Consequently, once the bound on the maximal operator S^* has been attained, Theorem 2.1.1 follows from Theorem 2.1.2.

To begin, for each $k \in \mathbb{N}$, define the summation operator

$$S_k f = \sum_{|n|\leqslant 2^k} \widehat{f}(n) e^{in\cdot}$$

for $f \in L^p(\mathbb{T})$, $p \in (1, \infty)$.

Furthermore, let $\phi \in \mathcal{S}(\mathbb{R})$ be such that $\operatorname{supp}(\widehat{\phi}) \subseteq [-1, 1]$ and $\widehat{\phi}(\xi) = 1$ for $\xi \in [-\frac{1}{2}, \frac{1}{2}]$. Let $\phi_k := 2^k \phi(2^k\cdot)$ so that $\widehat{\phi_k} := \widehat{\phi}(2^{-k}\cdot)$. Define $R_k f(x) = (\phi_k * f)(x)$ for $f \in L^p(\mathbb{T})$, where $*$ represents convolution on the *line*, that is

$$(\phi_k * f)(x) = \int_{\mathbb{R}} \phi_k(y) f(x - y)\, dy.$$

Throughout what follows, a function on $\mathbb{T}$ will be considered to be synonymous with a 2π–periodic function on $\mathbb{R}$, the version adopted appropriate to the context.

It is trivial to see that in spite of applying convolution on $\mathbb{R}$ to a function f on $\mathbb{T}$, $R_k f$ is a function on $\mathbb{T}$. Furthermore, for each $n \in \mathbb{Z}$,

$$\begin{aligned}
\widehat{R_k f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{\mathbb{R}}\phi_k(y)f(x-y)\,dy\right)e^{-inx}\,dx \\
&= \int_{\mathbb{R}}\phi_k(y)\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-y)e^{-inx}\,dx\,dy \\
&= \int_{\mathbb{R}}\phi_k(y)\left(\frac{1}{2\pi}\int_{-\pi-y}^{\pi-y}f(x)e^{-inx}\,dx\right)e^{-iny}\,dy \\
&= \int_{\mathbb{R}}\phi_k(y)\widehat{f}(n)e^{-iny}\,dy \\
&= \widehat{\phi_k}(n)\widehat{f}(n).
\end{aligned}$$

The interchange in integration in the above is justified by Fubini's theorem (as stated in [29], pp. 164–5) and the fact that $\phi_k \in L^1(\mathbb{R})$ for each $k \in \mathbb{N}$, and that $f \in L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$.

As discussed above, Theorem 2.1.1 will follow immediately from the following:

Theorem 2.1.3 *Let $p \in (1,\infty)$ and take $f \in L^p(\mathbb{T})$. Then* $\|\sup_{k\in\mathbb{N}}|S_k f|\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}$.

To prove this, make the estimate

$$\begin{aligned}
\sup_{k\in\mathbb{N}}|S_k f| &\leqslant \sup_{k\in\mathbb{N}}|(S_k - R_k)f| + \sup_{k\in\mathbb{N}}|R_k f| \\
&\leqslant (\sum_{k\in\mathbb{N}}|(S_k - R_k)f|^2)^{\frac{1}{2}} + \sup_{k\in\mathbb{N}}|R_k f|.
\end{aligned}$$

It is thus certainly sufficient to bound the two terms, $(\sum_{k\in\mathbb{N}}|(S_k - R_k)f|^2)^{\frac{1}{2}}$ and $\sup_{k\in\mathbb{N}}|R_k f|$, separately. In what follows, these two terms will be referred to as the "square function" and the "maximal function" respectively.

2.2 Bounding the Maximal Function

Consider the following preliminary result regarding the Hardy–Littlewood maximal function:

Lemma 2.2.1 *Let $f \in L^p(\mathbb{T})$ for $p \in (1, \infty)$ and make the definitions:*

$$\begin{aligned} \mathcal{M}_{\mathbb{R}} f(x) &:= \sup_{T>0} \frac{1}{2T} \int_{x-T}^{x+T} |f(y)| \, dy, \\ \mathcal{M}_{\mathbb{T}} f(x) &:= \sup_{T \in (0,\pi]} \frac{1}{2T} \int_{x-T}^{x+T} |f(y)| \, dy. \end{aligned}$$

Then for any $x \in \mathbb{T}$, $\mathcal{M}_{\mathbb{R}} f(x) = \mathcal{M}_{\mathbb{T}} f(x)$.

Proof. Fix $x \in \mathbb{T}$ and choose any $T \in \mathbb{R}^+$, writing $T = k\pi + d$, $k \in \mathbb{N} \cup \{0\}$, $d \in [0, \pi)$.

Then,

$$\begin{aligned} & \frac{1}{2T} \int_{x-T}^{x+T} |f(y)| \, dy \\ = & \frac{1}{2(k\pi + d)} \left(k \int_{x-\pi}^{x+\pi} |f(y)| \, dy + \int_{x-d}^{x+d} |f(y)| \, dy \right) \text{ by periodicity} \\ \leqslant & \frac{1}{2(k\pi + d)} \left(k(2\pi) \mathcal{M}_{\mathbb{T}} f(x) + 2d \mathcal{M}_{\mathbb{T}} f(x) \right) \\ = & \mathcal{M}_{\mathbb{T}} f(x). \end{aligned}$$

So $\mathcal{M}_{\mathbb{R}} f(x) = \mathcal{M}_{\mathbb{T}} f(x)$, as required. □

This allows the maximal function to be dealt with immediately. Noting that $f \in L^p(\mathbb{T})$, it follows that $f \in L^1_{loc}(\mathbb{R})$ (by Hölder's inequality). Further, ϕ is Schwartz, so certainly majorised by a radial, decreasing and integrable function. As R_k is a convolution operator of the appropriate form, it thus follows that $\sup_{k\in\mathbb{N}} |R_k f(x)| \lesssim \mathcal{M}_{\mathbb{R}} f(x) = \mathcal{M}_{\mathbb{T}} f(x)$. By the well-known L^p–boundedness of the Hardy–Littlewood maximal function (see, for example, [33] Chapter IV), $\| \sup_{k\in\mathbb{N}} |R_k f(x)| \|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}$.

2.3 Bounding the Square Function

This requires a little more work than the bound on the maximal function. To start with, consider the following definitions:

Definition 2.3.1 *For $f \in C^\infty(\mathbb{T})$, define the conjugate function of f to be the result of the (Hilbert) transformation:*

$$Hf(x) = -i \sum_{n\in\mathbb{Z}} sgn(n) \widehat{f}(n) e^{inx}.$$

Define the positive Riesz Projection to be the following operator:

$$P_+ f(x) = \sum_{n=1}^{\infty} \widehat{f}(n) e^{inx}.$$

The following result is well known:

Theorem 2.3.2 *Let $p \in (1, \infty)$. Then for any $f \in L^p(\mathbb{T})$, $\|Hf\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}$, where the domain of definition of H is continuously extended.*

For a proof, see, for example, [17] p. 212 (despite the result being stated for $f \in C^\infty(\mathbb{T})$, the result above follows by density).

This result has the following immediate corollary:

Corollary 2.3.3 *Let $p \in (1, \infty)$. Then for any $f \in L^p(\mathbb{T})$, $\|P_+ f\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}$, where the domain of definition of P_+ is continuously extended.*

Proof. This follows immediately from the previous theorem, density and the fact that for $f \in C^\infty(\mathbb{T})$, $P_+ f = \frac{1}{2}(f + iHf) - \frac{1}{2}\widehat{f}(0)$. $\square$

It is now possible to prove the following result in a similar fashion to [17], Proposition 3.5.5, p. 211:

Theorem 2.3.4 *Take $p \in (1, \infty)$. Then for any $f \in L^p(\mathbb{T})$, $\sup\limits_{k \in \mathbb{N}} \|S_k f\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}$.*

Proof. First note that for $x \in \mathbb{T}$,

$$S_k f(x) = \sum_{|n| \leqslant 2^k} \widehat{f}(n) e^{inx} = e^{-i2^k x} \sum_{n=0}^{2^{k+1}} (f e^{i2^k \cdot})\widehat{\ }(n) e^{inx}.$$

By the unimodular nature of the exponential factors, it follows that $\|S_k\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} = \|S_k'\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})}$, where $S_k' f(x) = \sum\limits_{n=0}^{2^{k+1}} \widehat{f}(n) e^{inx}$.

Now, consider that for $f \in C^\infty(\mathbb{T})$,

$$\begin{aligned} S_k' f(x) &= \sum_{n=0}^{\infty} \widehat{f}(n) e^{inx} - \sum_{n=2^{k+1}+1}^{\infty} \widehat{f}(n) e^{inx} \\ &= \sum_{n=0}^{\infty} \widehat{f}(n) e^{inx} - e^{i(2^{k+1}+1)x} \sum_{n=0}^{\infty} \widehat{f}(n + 2^{k+1} + 1) e^{inx} \\ &= P_+ f(x) - e^{i(2^{k+1}+1)x} P_+(e^{-i(2^{k+1}+1)\cdot} f)(x) + \widehat{f}(0) - e^{i(2^{k+1}+1)x} \widehat{f}(2^{k+1} + 1). \end{aligned}$$

In particular, using the fact that $|\widehat{f}(n)| \leqslant \|f\|_{L^1(\mathbb{T})} \leqslant \|f\|_{L^p(\mathbb{T})}$ for $n \in \mathbb{N}$, by the triangle inequality, $\sup\limits_{k \in \mathbb{N}} \|S_k'(f)\|_{L^p(\mathbb{T})} \leqslant (2\|P_+\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} + 2)\|f\|_{L^p(\mathbb{T})}$.

By density, this holds for all $f \in L^p(\mathbb{T})$. Corollary 2.3.3 now completes the proof. $\square$

This result can be used to show that dyadic partial sums of Fourier series of functions in $L^p(\mathbb{T})$ for $p \in (1, \infty)$ converge in norm (the proof is also easily adapted to regular partial sums), and is not strictly necessary for proving the boundedness of the square function. However, it is included as some of the methods used in the above proof will be useful in what is to come.

The following is now needed:

Lemma 2.3.5 (Khintchine's Inequality) *For $p \in (0,\infty)$, there exist positive constants A_p, B_p, such that for every $(c_j)_{j\in\mathbb{N}} \in \ell^2$, $\sum_{j\in\mathbb{N}} c_j r_j \in L^p[0,1]$ with*

$$A_p\Big(\sum_{j\in\mathbb{N}} |c_j|^2\Big)^{\frac{1}{2}} \leqslant \|\sum_{j\in\mathbb{N}} c_j r_j\|_{L^p[0,1]} \leqslant B_p\Big(\sum_{j\in\mathbb{N}} |c_j|^2\Big)^{\frac{1}{2}}$$

where $r_j : [0,1] \to \{1,-1\}$ are the Rademacher functions, defined by partitioning $[0,1]$ into equal intervals of length 2^{-j} and assigning alternating signs to consecutive intervals, where $r_j(0) = 1$.

For a proof, see Appendix D in [31].

This can be used to prove the following vector-valued extension theorem for linear operators:

Theorem 2.3.6 *For $p \in (0,\infty)$, suppose that T is a bounded linear operator on $L^p(\mathbb{T})$. Then T has an ℓ^2-valued extension. That is to say that for all $(f_j)_{j\in\mathbb{N}} \subseteq L^p(\mathbb{T})$,*

$$\|\Big(\sum_{j\in\mathbb{N}} |T(f_j)|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \lesssim \|T\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})}\|\Big(\sum_{j\in\mathbb{N}} |f_j|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{T})}.$$

Proof.

$$\begin{aligned}
\|(\sum_{j\in\mathbb{N}}|T(f_j)|^2)^{\frac{1}{2}}\|^p_{L^p(\mathbb{T})} &\lesssim \|\|\sum_{j\in\mathbb{N}}T(f_j)(x)r_j(t)\|_{L^p_t[0,1]}\|^p_{L^p_x(\mathbb{T})} \text{ by Khintchine's inequality}\\
&= \int_0^1 \|\sum_{j\in\mathbb{N}}T(f_j)(x)r_j(t)\|^p_{L^p_x(\mathbb{T})}\,dt \text{ by Fubini's theorem}\\
&= \int_0^1 \|T(\sum_{j\in\mathbb{N}}f_j(x)r_j(t))\|^p_{L^p_x(\mathbb{T})}\,dt\\
&\leqslant \|T\|^p\int_0^1 \|\sum_{j\in\mathbb{N}}f_j(x)r_j(t)\|^p_{L^p_x(\mathbb{T})}\,dt\\
&= \|T\|^p\|\|\sum_{j\in\mathbb{N}}f_j(x)r_j(t)\|_{L^p_t[0,1]}\|^p_{L^p_x(\mathbb{T})} \text{ by Fubini's theorem}\\
&\lesssim \|T\|^p\|(\sum_{j\in\mathbb{N}}|f_j|^2)^{\frac{1}{2}}\|^p_{L^p(\mathbb{T})} \text{ by Khintchine's inequality.}
\end{aligned}$$ □

It is noted that with a longer proof, the "$\lesssim$" in Theorem 2.3.6 may be sharpened to "$\leqslant$" (see [17], pp. 311–12).

One final result is now needed:

Theorem 2.3.7 (Littlewood–Paley) *Let $\psi \in C^1(\mathbb{R})$ be an integrable function with mean value zero, such that there exists a constant $C > 0$ so that for any $x \in \mathbb{R}$,*

$$|\psi(x)| + |\psi^{'}(x)| \leqslant \frac{C}{(1+|x|)^2}.$$

Furthermore, define $\psi_j := 2^j\psi(2^j\cdot)$ so that $\widehat{\psi_j} = \widehat{\psi}(2^{-j}\cdot)$. Then for $f \in L^p(\mathbb{T})$, $p \in (1,\infty)$,

$$\|(\sum_{j\in\mathbb{Z}}|f*\psi_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}.$$

Proof. First, define $\widetilde{f} := f - \int_{\mathbb{T}} f$ and observe that $\widetilde{f}$ has mean value zero. Now, note that

$$\begin{aligned}
& \|(\sum_{j\in\mathbb{Z}} |f * \psi_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
= \ & \|(\sum_{j\in\mathbb{Z}} |\left(\widetilde{f} + \int_{\mathbb{T}} f\right) * \psi_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
= \ & \|(\sum_{j\in\mathbb{Z}} |\widetilde{f} * \psi_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})}
\end{aligned}$$

as ψ_j has mean value zero and $\displaystyle\int_{\mathbb{T}} f$ is constant.

Consequently, it can be assumed without loss of generality that f has mean value zero.

This being given, the proof of the analogous result on $\mathbb{R}$, as given as Theorem 5.1.2 in [17], pp. 339–41, along with the results it depends on from [17], can be adapted, stage-by-stage, to give the above theorem. The assumption that f has mean value zero is used to permit the Calderón–Zygmund decomposition at arbitrary height $\lambda > 0$ in the proof of Theorem 4.6.1, pp. 326–7.

For discussion of the Calderón–Zygmund decomposition of a function on $\mathbb{T}$, see [33], Chapter IV. $\square$

The necessary framework now provided, bounding of the square function may proceed as follows:

Define ψ such that $\widehat{\psi} = \widehat{\phi}(\frac{1}{2}\cdot) - \widehat{\phi}$ and note that trivially $\psi \in \mathcal{S}(\mathbb{R})$. Also, as $\mathrm{supp}(\widehat{\phi}(\frac{1}{2}\cdot)) \subseteq [-2, 2]$ and $\widehat{\phi}(\frac{\xi}{2}) = 1$ for $\xi \in [-1, 1]$, it follows that $\chi_{[-1,1]}\widehat{\phi}(\frac{1}{2}\cdot) = \chi_{[-1,1]}$. Trivially, $\chi_{[-1,1]}\widehat{\phi} = \widehat{\phi}$, so it follows that $\chi_{[-1,1]} - \widehat{\phi} = \chi_{[-1,1]}\widehat{\psi}$.

Now, define ψ_k in the usual way, by setting $\psi_k = 2^k\psi(2^k\cdot)$, so that $\widehat{\psi_k} = \widehat{\psi}(2^{-k}\cdot)$. Using this and the ideas of the proof of Theorem 2.3.4,

$$
\begin{aligned}
& \|(\sum_{k\in\mathbb{N}} |(S_k - R_k)f(x)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
= \ & \|(\sum_{k\in\mathbb{N}} |S_k(f*\psi_k)(x)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
= \ & \|(\sum_{k\in\mathbb{N}} |S_k'(e^{i2^k\cdot}(f*\psi_k))(x)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
= \ & \|\Bigg(\sum_{k\in\mathbb{N}} |P_+(e^{i2^k\cdot}(f*\psi_x))(x) - e^{i(2^{k+1}+1)x}P_+(e^{-i(2^k+1)\cdot}(f*\psi_k))(x) + \\
& (e^{i2^k\cdot}(f*\psi_k))\widehat{\ }(0) - e^{i(2^{k+1}+1)x}(e^{i2^k\cdot}(f*\psi_k))\widehat{\ }(2^{k+1}+1)|^2\Bigg)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
\leqslant \ & \|(\sum_{k\in\mathbb{N}} |P_+(e^{i2^k\cdot}(f*\psi_k))(x) + (e^{i2^k\cdot}(f*\psi_k))\widehat{\ }(0)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} + \\
& \|(\sum_{k\in\mathbb{N}} |P_+(e^{-i(2^k+1)\cdot}(f*\psi_k))(x) + (e^{i2^k\cdot}(f*\psi_k))\widehat{\ }(2^{k+1}+1)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \\
\lesssim \ & \|(\sum_{k\in\mathbb{N}} |(f*\psi_k)(x)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{T})} \text{ by Theorem 2.3.6 and Corollary 2.3.3} \\
\lesssim \ & \|f\|_{L^p(\mathbb{T})} \text{ by Theorem 2.3.7.}
\end{aligned}
$$

This concludes the proof of Theorem 2.1.3.

It is mentioned at this point that boundedness of the square function can be established much more easily in the case of $p = 2$. In particular, consider that

$$
\begin{aligned}
& \|(\sum_{k\in\mathbb{N}} |(S_k - R_k)f|^2)^{\frac{1}{2}}\|_{L^2(\mathbb{T})}^2 \\
= \ & \sum_{k\in\mathbb{N}} \|(S_k - R_k)f\|_{L^2(\mathbb{T})}^2 \text{ by the Lebesgue Monotone Convergence theorem} \\
= \ & \sum_{k\in\mathbb{N}} \sum_{2^{k-1}\leqslant|n|\leqslant 2^k} |(1-\widehat{\phi}_k(n))\widehat{f}(n)|^2 \text{ by Parseval's identity} \\
\lesssim \ & \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \text{ using that } \widehat{\phi} \text{ is bounded} \\
= \ & \|f\|_{L^2(\mathbb{T})}^2 \text{ by Parseval's identity.}
\end{aligned}
$$

In closing, it is noted that the analogous problem to Theorem 2.1.1 for Fourier integrals can be considered in dimensions higher than 1. In particular, the following can be said:

Theorem 2.3.8 (The Full Classical Dyadic Convergence Problem) *For $k \in \mathbb{N} \cup \{0\}$, $f \in L^p(\mathbb{R}^d)$, $d \in \mathbb{N}$ and $p \in [1, \infty]$, define the following operators:*

$$\begin{aligned} \widehat{S_k f} &:= m_k \widehat{f}, \\ \widehat{\widetilde{S_k} f} &:= \widetilde{m_k} \widehat{f}, \end{aligned}$$

where the multipliers are given by:

$$\begin{aligned} m_k &:= \chi_{\{x \in \mathbb{R}^d : |x| < 2^k\}}, \\ \widetilde{m_k} &:= \chi_{\{x \in \mathbb{R}^d : |x_i| < 2^k, i \in [1,d] \cap \mathbb{N}\}}. \end{aligned}$$

Then for $p \in [2, \frac{2d}{d-1})$, $\lim\limits_{k \to \infty} S_k f(x) = f(x)$ for almost every $x \in \mathbb{R}^d$, and for $p \in (1, \infty)$, $\lim\limits_{k \to \infty} \widetilde{S_k} f(x) = f(x)$ for almost every $x \in \mathbb{R}^d$.

The added difficulty for the spherical summation operator, S_k over the rectangular one, $\widetilde{S_k}$, resulting in a more restricted range of p, pertains to the failure of norm boundedness of S_k in $L^p(\mathbb{R}^d)$ for $d > 1$, $p \neq 2$. The proof of this originates from C. Fefferman's ground-breaking paper, [14].

For a proof of the result for S_k, see [11]. For $\widetilde{S_k}$, see [17], pp. 368–71. It is noted that it is erroneously claimed in [17] that a result for S_k identical to the one for $\widetilde{S_k}$ is proved. It is also stated in the list of known errata for [17] that only the first five lines of the proof of Theorem 5.3.2 are required, and that the remainder follows as an easy consequence of the vector-valued operator bound on the summation operator from Exercise 4.6.1(b). These errors are to be corrected in the forthcoming second edition.

Chapter 3

The Almost Periodic Case

3.1 Validity of the Classical Approach and Issues of Interpretation

For the remainder of this thesis, for the sake of simplicity, certain conventions regarding Fourier series of almost periodic functions will be adopted. Firstly, for any particular $f \in S^p$, $p \in [1, \infty)$, the corresponding Fourier exponents $(\lambda_n)_{n\in\mathbb{N}}$, enumerated by means of Theorem 1.3.5, will be extended to range over $\mathbb{Z}$ by setting $\lambda_{-n} = -\lambda_n$, $n \in \mathbb{Z}$, applying a Fourier coefficient of 0 to any newly introduced Fourier exponents. Furthermore, the Fourier exponents will be ordered such that they are increasing in n.

The reader is reminded that, as per the discussion in Section 1.3 in relation to the Uniqueness Theorem, reordering the Fourier series of an almost periodic function cannot result in a different sum.

The theorem that will be proved is the following:

Theorem 3.1.1 (The Almost Periodic Dyadic Convergence Problem) *Let $f \in S^{2^k}$, $k \in \mathbb{N}$ be such that there exists $\alpha > 0$ such that $\lambda_{n+1} - \lambda_n > \alpha$ for all $n \in \mathbb{Z}$, where $(\lambda_n)_{n\in\mathbb{Z}}$*

are the Fourier exponents of f defined by the convention above. Then for almost every $x \in \mathbb{R}$, $f(x) = \lim_{k \to \infty} \sum_{|\lambda_n| \leqslant 2^k} \widehat{f}(\lambda_n) e^{i\lambda_n x}$.

As discussed before, this result will be obtained by bounding a maximal summation operator. The result is stated for S^{2^k}, $k \in \mathbb{N}$ to emphasise that the bound will be obtained for each of these spaces. It is noted, however, that by Proposition 1.2.8, it suffices to consider the S^2 space alone for the convergence result, as $S^{2^k} \subseteq S^2$ for all $k \in \mathbb{N}$.

As mentioned in the introduction to this thesis, E. A. Bredihina has obtained a convergence result of a form similar to the above in [10] by taking a somewhat different approach to the one that will be considered in this chapter. As discussed earlier, the forthcoming boundedness of a maximal summation operator should be considered to be the most important result of this thesis.

It is entirely reasonable to suggest that an alternative analogous problem to Theorem 2.1.1 would be to consider sums of the form $\sum_{|n| \leqslant 2^k} \widehat{f}(\lambda_n) e^{i\lambda_n x}$. This has not been pursued here, as it loses the aspect of dyadic support for the Fourier exponents. There is a result of this form for $p = 2$ considered in [10], but it places more stringent conditions on f than the result analogous to the above formulation.

In approaching the proof of this result, the first question that must be answered is whether bounding an appropriate maximal operator gives the desired almost everywhere convergence as in the periodic case. This breaks down into three sub-questions:

1) Is there an analogue of Theorem 2.1.2? That is to say, does "weak boundedness" of the appropriate maximal operator in Stepanov norm lead to closure of a set like the one given in that theorem?

2) It was implicitly used in the periodic case that a strong bound on the maximal operator automatically gave the weak bound required for Theorem 2.1.2. Does "strong boundedness" of an operator in Stepanov norm imply "weak boundedness"?

3) Is there a dense subspace of S^p where Theorem 3.1.1 is known to hold, so that closure of the set from Theorem 2.1.2 will provide the desired almost everywhere convergence on the whole of S^p?

These are most easily addressed in reverse order.

3) is trivially answered by the Fundamental Theorem (Theorem 1.2.1). The S^p spaces are identically equal to the closure of the trigonometric polynomials in $\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\}$, and Theorem 3.1.1 certainly holds for the trigonometric polynomials.

2) can be answered negatively if the definition of "weak boundedness" is taken to be the same as for $L^p(\mathbb{T})$; that is that an operator $T : S^p \to S^p$ is weakly bounded $(p - -p)$ if for all $\lambda > 0$, $f \in S^p$, $|\{x \in \mathbb{R} : |Tf(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_{S^p}}{\lambda}\right)^p$.

To see this, consider the identity operator acting on S^p. This is clearly (strongly) bounded in the regular sense. Note also that the function $\sin(x)$ is in S^p as it is a continuous periodic function, so certainly almost periodic in all senses. However, for any $\lambda \in (0, 1)$, the one dimensional Lebesgue measure of the set $\{x \in \mathbb{R} : |\sin(x)| > \lambda\}$ is infinite, so there is no hope of a weak-type bound in the conventional sense.

Fortunately, this problem can be resolved by introducing a new definition of "weak boundedness" that is more appropriate for the Stepanov norms:

Definition 3.1.2 (Stepanov Operator Boundedness) *Let $T : S^p \to S^q$ for $p, q \in [1, \infty)$. Then T is said to be strongly bounded (p–q) if $\|Tf\|_{S^q} \lesssim \|f\|_{S^p}$. T is said to be weakly bounded (p–q) if $\sup_{x \in \mathbb{R}} |\{s \in [x, x+1] : |Tf(s)| > \lambda\}| \lesssim \left(\frac{\|f\|_{S^p}}{\lambda}\right)^q$ for all $\lambda \geqslant 0$.*

This now behaves in the usual way:

Theorem 3.1.3 (Strong $\Rightarrow$ Weak) *Let $T : S^p \to S^q$ for $p, q \in [1, \infty)$ be strongly* *bounded (p–q). Then it is weakly bounded (p–q).*

Proof. Suppose T is strongly bounded, fix any positive λ and let $E_x := \{s \in [x, x+1] : |Tf(s)| > \lambda\}$. Then it follows that

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |\{s \in [x, x+1] : |Tf(s)| > \lambda\}| &= \sup_{x\in\mathbb{R}} \int_{E_x} ds \\
&\leqslant \sup_{x\in\mathbb{R}} \int_{E_x} \left(\frac{|Tf(s)|}{\lambda}\right)^q ds \\
&\leqslant \sup_{x\in\mathbb{R}} \int_x^{x+1} \left(\frac{|Tf(s)|}{\lambda}\right)^q ds \\
&= \left(\frac{\|Tf\|_{S^q}}{\lambda}\right)^q \\
&\lesssim \left(\frac{\|f\|_{S^p}}{\lambda}\right)^q .
\end{aligned}$$

□

This leaves *1)*, which, given Definition 3.1.2 can be answered in the affirmative with the proof of Theorem 2.2 in [13] adapted appropriately:

Theorem 3.1.4 *Let $\{T_j\}_{j\in\mathbb{N}}$ be a family of linear operators on S^p, $p \in [1, \infty)$ and assume that $T^*f := \sup_{j\in\mathbb{N}} |T_j f|$ is weakly bounded (p–q) for some $q \in [1, \infty)$. Then the set $E := \{f \in S^p : \lim_{j\to\infty} T_j f(x) = f(x) \text{ a.e.}\}$ is closed in S^p.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be such that $\|f_n - f\|_{S^p} \to 0$ as $n \to \infty$ for some $f \in S^p$.

Take any $x \in \mathbb{R}$, $\lambda > 0$. Then,

$$
\begin{aligned}
& |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j f(s) - f(s)| > \lambda\}| \\
= \quad & |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j f(s) - f(s)| - |T_j f_n(s) - f_n(s)| > \lambda\}| \\
& \text{for any particular } n \in \mathbb{N} \text{ (as } \lim_{j\to\infty} |T_j f_n(s) - f_n(s)| = 0 \text{ almost everywhere in } \mathbb{R}) \\
\leqslant \quad & |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j(f - f_n)(s) - (f - f_n)(s)| > \lambda\}| \\
\leqslant \quad & |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j(f - f_n)(s)| + |(f - f_n)(s)| > \lambda\}| \\
\leqslant \quad & |\{s \in [x, x+1] : T^*(f - f_n)(s) + |(f - f_n)(s)| > \lambda\}| \\
\leqslant \quad & |\{s \in [x, x+1] : T^*(f - f_n)(s) > \tfrac{\lambda}{2}\}| + |\{s \in [x, x+1] : |(f - f_n)(s)| > \tfrac{\lambda}{2}\}| \\
\leqslant \quad & |\{s \in [x, x+1] : T^*(f - f_n)(s) > \tfrac{\lambda}{2}\}| + \int_x^{x+1} \frac{2^p |(f - f_n)(s)|^p}{\lambda^p}\, ds.
\end{aligned}
$$

By uniformity in x, it follows that

$$
\begin{aligned}
& \sup_{x\in\mathbb{R}} |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j f(s) - f(s)| > \lambda\}| \\
\leqslant \quad & \sup_{x\in\mathbb{R}} |\{s \in [x, x+1] : T^*(f - f_n)(s) > \tfrac{\lambda}{2}\}| + \left(\frac{2}{\lambda}\|f - f_n\|_{S^p}\right)^p \\
\leqslant \quad & C\left(\frac{2\|f - f_n\|_{S^p}}{\lambda}\right)^q + \left(\frac{2}{\lambda}\|f - f_n\|_{S^p}\right)^p
\end{aligned}
$$

by the assumption that T^* is (p–q) weak. Now taking the limit as $n \to \infty$, it follows that

$$
\sup_{x\in\mathbb{R}} |\{s \in [x, x+1] : \limsup_{j\to\infty} |T_j f(s) - f(s)| > \lambda\}| = 0 \text{ for all } \lambda > 0.
$$

It can hence be deduced that $|\{x \in \mathbb{R} : \limsup_{j\to\infty} |T_j f(x) - f(x)| > 0\}| = 0$ and so $\lim_{j\to\infty} T_j f(x) = f(x)$ for almost every $x \in \mathbb{R}$. Consequently, $f \in E$ and hence E is closed.□

It has thus been shown that bounding the appropriate maximal operator does still give the desired almost everywhere convergence as in the periodic case.

3.2 Forming the Decomposition

Define $S_kf(x) := \sum_{|\lambda_n|\leqslant 2^k} \widehat{f}(\lambda_n)e^{i\lambda_n x}$, $S^*f(x) := \sup_{k\in\mathbb{N}} |S_kf(x)|$. Then, as just established, Theorem 3.1.1 will follow immediately from the following:

Theorem 3.2.1 *Let $f \in S^{2^k}$, $k \in \mathbb{N}$ be such that there exists $\alpha > 0$ such that $\lambda_{n+1}-\lambda_n > \alpha$ for all $n \in \mathbb{N}$. Then $\|S^*f\|_{S^{2^k}} \lesssim \|f\|_{S^{2^k}}$.*

An approach analogous to the periodic case will be taken. In particular, let R_k be some 'smoothed-out' summation operator of a similar form, the exact definition of which will be determined shortly. Consider:

$$\begin{aligned} S^*f &= \sup_{k\in\mathbb{N}} |S_kf| \\ &\leqslant \sup_{k\in\mathbb{N}} |S_kf - R_kf| + \sup_{k\in\mathbb{N}} |R_kf| \\ &\leqslant (\sum_{k\in\mathbb{N}} |S_kf - R_kf|^2)^{\frac{1}{2}} + \sup_{k\in\mathbb{N}} |R_kf|. \end{aligned}$$

Again, it suffices to bound each of these two terms separately. As in Chapter 2, they will be referred to as the "square function" and the "maximal function".

While the idea of defining R_k as a mean convolution operator may seem the most intuitive way to proceed, it does present a difficulty in that the Fourier exponents of different S^p functions may be taken to be any countable subset of $\mathbb{R}$. Hence to define R_k as a mean convolution operator that would be suitable for all S^p functions would require that the operator kernel of R_k had non-zero Fourier coefficients for uncountably many Fourier exponents, which is not permissible. Another approach would be to define different R_k operators for different functions, dependent on where their Fourier exponents are located.

Fortunately, these complications can be avoided by introducing convolution on the line (as in Chapter 2). This technique works as a consequence of the following result, which is an extension of the lemma in [9] (where it is proved for the Bohr class):

Theorem 3.2.2 *Let $f \in S^p$, $p \in [1, \infty)$ be such that the Fourier exponents of f have no finite limit points. Let ζ_μ be a real-valued continuous function on $\mathbb{R}$ dependent on parameter $\mu \in \mathbb{R}$ which satisfies:*

$$\begin{aligned} &supp(\zeta_\mu) \subseteq [-\mu, \mu] \\ &\psi_\mu \in L^1(\mathbb{R}), \ \textit{where } \zeta_\mu(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi_\mu(t) e^{-iut} \, dt = \widehat{\psi}_\mu(u). \end{aligned}$$

Then $f * \psi_\mu \in S^p$ and $f * \psi_\mu(x) = \displaystyle\sum_{|\lambda_n| < \mu} \widehat{f}(\lambda_n) \zeta_\mu(\lambda_n) e^{i\lambda_n x}$, where $*$ represents convolution on the line, that is $f * \psi_\mu(x) = \displaystyle\int_{\mathbb{R}} f(x-u) \psi_\mu(u) \, du$.

Proof. Define $f_\mu(x) := f * \psi_\mu(u)$. First, to show that $f_\mu \in S^p$, consider the following:

$$
\begin{aligned}
\|f_\mu(\cdot+\tau)-f_\mu\|_{S^p} &= \left(\sup_{x\in\mathbb{R}}\int_x^{x+1}|f_\mu(s+\tau)-f_\mu(s)|^p\,ds\right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\left|\int_{\mathbb{R}}f(s+\tau-u)\psi_\mu(u)\,du-\int_{\mathbb{R}}f(s-u)\psi_\mu(u)\,du\right|^p ds\right)^{\frac{1}{p}} \\
&\leqslant \sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\left(\int_{\mathbb{R}}|f(s+\tau-u)-f(s-u)||\psi_\mu(u)|\,du\right)^p ds\right)^{\frac{1}{p}} \\
&\leqslant \sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\left(\int_x^{x+1}|f(s+\tau-u)-f(s-u)|^p\,ds\right)^{\frac{1}{p}}|\psi_\mu(u)|\,du \\
&\quad\ \text{by Minkowski's integral inequality} \\
&\leqslant \int_{\mathbb{R}}\left(\sup_{x\in\mathbb{R}}\int_x^{x+1}|f(s+\tau)-f(s)|^p\,ds\right)^{\frac{1}{p}}|\psi_\mu(u)|\,du \\
&= \|f(\cdot+\tau)-f\|_{S^p}\|\psi_\mu\|_{L^1}.
\end{aligned}
$$

It can hence be seen that $\|f_\mu(\cdot+\tau)-f_\mu\|_{S^p}\leqslant\|\psi_\mu\|_{L^1}\|f(\cdot+\tau)-f\|_{S^p}$, thus $f_\mu\in S^p$ by Definition 1.1.6.

Now, for $\lambda\in\mathbb{R}$, the λ^{th} Fourier coefficient of f_μ can be calculated as follows:

$$
\begin{aligned}
& \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_\mu(x)e^{-i\lambda x}\,dx \\
= {} & \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{\mathbb{R}}f(x-u)\psi_\mu(u)\,du\right)e^{-i\lambda x}\,dx \\
= {} & \lim_{T\to\infty}\int_{\mathbb{R}}\psi_\mu(u)\left(\frac{1}{2T}\int_{-T}^{T}f(x-u)e^{-i\lambda x}\,dx\right)du \\
= {} & \lim_{T\to\infty}\int_{\mathbb{R}}\psi_\mu(u)\left(\frac{1}{2T}\int_{-T+u}^{T+u}f(x)e^{-i\lambda x}\,dx\right)e^{-i\lambda u}\,du \\
= {} & \left(\int_{\mathbb{R}}\psi_\mu(u)e^{-i\lambda u}\,du\right)\widehat{f}(\lambda) \\
& \text{by the Lebesgue Dominated Convergence theorem and Theorem 1.3.12} \\
= {} & \widehat{f}(\lambda)\zeta_\mu(\lambda).
\end{aligned}
$$

To justify the change of order in integration above, it is noted that for any fixed positive T,

$$
\begin{aligned}
&\int_{\mathbb{R}} |\psi_\mu(u)| \left(\frac{1}{2T} \int_{-T}^{T} |f(x-u)|\, dx \right) du \\
\leqslant\ & \sup_{v\in\mathbb{R}} \left(\frac{1}{2T} \int_{-T}^{T} |f(x-v)|\, dx \right) \int_{\mathbb{R}} |\psi_\mu(u)|\, du \\
=\ & \|f\|_{S^1,2T} \|\psi_\mu\|_{L^1} \\
\leqslant\ & C\|f\|_{S^p,2T} \|\psi_\mu\|_{L^1}.
\end{aligned}
$$

Consequently, Fubini's theorem is applicable.

Furthermore, to justify the use of the Lebesgue Dominated Convergence theorem, it is noted that from the proof of Lemma 1.1.4, for $T > 1$, if $N \in \mathbb{N}$ is the least number such that $N \geqslant T$, then $\|f\|_{S^p,T} \leqslant \left(\frac{N}{T}\right)^{\frac{1}{p}} \|f\|_{S^p} \leqslant 2\|f\|_{S^p}$. It then suffices to consider similar reasoning to the above.

Given the conditions on ζ_μ, it follows that $f_\mu(x) = \sum_{|\lambda_n|<\mu} \widehat{f}(\lambda_n)\zeta_\mu(\lambda_n)e^{i\lambda_n x}$. $\square$

With this result proved, R_k may now be defined as follows:

Choose any $\phi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\phi}$ is continuous and satisfies $\mathrm{supp}(\widehat{\phi}) \subseteq [-1,1]$, $\widehat{\phi}(\xi) \in [0,1]$ for $\xi \in \mathbb{R}$ and $\widehat{\phi}(\xi) = 1$ for $\xi \in [-\frac{1}{2}, \frac{1}{2}]$. For each $k \in \mathbb{N}$, define $\phi_k = 2^k\phi(2^k\cdot)$, so that $\widehat{\phi_k} = \widehat{\phi}(2^{-k}\cdot)$.

Now, for $f \in S^p$, $p \in [1,\infty)$, define $R_k(f) := f * \phi_k$ (where convolution is taking place on $\mathbb{R}$). From Theorem 3.2.2, $R_k f = \sum_{n\in\mathbb{N}} \widehat{f}(\lambda_n)\widehat{\phi_k}(\lambda_n)e^{i\lambda_n\cdot}$.

With this decided, work with proving Theorem 3.2.1 may now proceed.

3.3 Bounding the Square Function for $p = 2$

Whilst it is by no means as trivial as in the periodic case, bounding the square function for $p = 2$ is a sensible place to start in the almost periodic setting, owing to the validity of Parseval's identity in S^2.

The following is a well-known inequality of Hilbert and Schur:

Lemma 3.3.1 *For $(a_j)_{j\in\mathbb{Z}} \in \ell^2$, define the operator $H : \ell^2 \to \ell^2$ by $(H(a_j))_j = \sum_{k\neq j} \frac{a_k}{j-k}$. Then $\|H(a_j)\|_{\ell^2} \leqslant \pi\|(a_j)\|_{\ell^2}$.*

For a proof, see [16]. Here, something stronger is needed, and will be constructed in the spirit of that proof. The approach is similar to that given in [27], pp. 138–40. See also [28].

Lemma 3.3.2 *Let $(\lambda_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ be an increasing sequence such that there exists $\alpha > 0$ so that $\lambda_{k+1} - \lambda_k > \alpha$ for all $k \in \mathbb{N}$. For $(a_j)_{j\in\mathbb{Z}} \in \ell^2$, define the operator $T : \ell^2 \to \ell^2$, $(T(a_j))_j := \sum_{k\in\mathbb{Z}\setminus\{j\}} \frac{a_k}{\lambda_j - \lambda_k}$. Then $\|T(a_j)\|_{\ell^2} \leqslant \frac{\pi}{\alpha}\|(a_j)\|_{\ell^2}$.*

Proof. To begin, assume that $(a_j)_{j\in\mathbb{Z}}$ is a compactly supported sequence. Given this, T has matrix representation

$$A := \begin{pmatrix} 0 & \frac{1}{\lambda_1-\lambda_2} & \frac{1}{\lambda_1-\lambda_3} & \cdots \\ \frac{1}{\lambda_2-\lambda_1} & 0 & \frac{1}{\lambda_2-\lambda_3} & \cdots \\ \frac{1}{\lambda_3-\lambda_1} & \frac{1}{\lambda_3-\lambda_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As A is a skew-Hermitian matrix, it follows that the operator norm of T is equal to the spectral radius of A (see [27] pp. 134–137). Consequently, $\frac{\|T(a_j)\|_{\ell^2}}{\|(a_j)\|_{\ell^2}} \leqslant \max_i(|\gamma_i|) = \frac{\|T(b_j)\|_{\ell^2}}{\|(b_j)\|_{\ell^2}}$, where the γ_i are the eigenvalues of A and $(b_j)_{j\in\mathbb{Z}}$ represents the eigenvector corresponding to the maximal eigenvalue. It thus suffices to assume that $(a_j)_{j\in\mathbb{Z}}$ represents an arbitrary eigenvector.

$$
\begin{aligned}
\|T(a_j)\|_{l^2}^2 &= \sum_{j\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}\setminus\{j\}} \frac{a_n}{\lambda_j-\lambda_n}|^2 \\
&= \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}\setminus\{j\}} \sum_{m\in\mathbb{Z}\setminus\{j\}} a_n\overline{a_m}\frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)} \\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_n\overline{a_m} \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)} \\
&= \sum_{n\in\mathbb{Z}} |a_n|^2 \sum_{j\in\mathbb{Z}\setminus\{n\}} \frac{1}{(\lambda_j-\lambda_n)^2} + \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{n\}} a_n\overline{a_m} \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)} \\
&\leqslant \sum_{n\in\mathbb{Z}} |a_n|^2 \sum_{j\in\mathbb{Z}\setminus\{n\}} \frac{1}{(\alpha(j-n))^2} + \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{n\}} a_n\overline{a_m} \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)} \\
&= \frac{\pi^2}{3\alpha^2} \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{n\}} a_n\overline{a_m} \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)}
\end{aligned}
$$

as $\displaystyle\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = \frac{\pi^2}{3}$.

A decomposition into a "diagonal term" and an "off-diagonal term" has been formed. The latter requires more work:

$$
\begin{aligned}
& \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \frac{1}{(\lambda_j-\lambda_n)(\lambda_j-\lambda_m)} \\
= & \frac{1}{\lambda_m-\lambda_n} \sum_{j\in\mathbb{Z}\setminus\{n,m\}} \left(\frac{1}{\lambda_j-\lambda_m} - \frac{1}{\lambda_j-\lambda_n}\right) \\
= & \frac{1}{\lambda_m-\lambda_n} \left(\sum_{j\in\mathbb{Z}\setminus\{m\}} \frac{1}{\lambda_j-\lambda_m} - \frac{1}{\lambda_n-\lambda_m} - \sum_{j\in\mathbb{Z}\setminus\{n\}} \frac{1}{\lambda_j-\lambda_n} + \frac{1}{\lambda_m-\lambda_n} \right) \\
= & \frac{2}{(\lambda_m-\lambda_n)^2} + \frac{1}{\lambda_m-\lambda_n} \left(\sum_{j\in\mathbb{Z}\setminus\{m\}} \frac{1}{\lambda_j-\lambda_m} - \sum_{j\in\mathbb{Z}\setminus\{n\}} \frac{1}{\lambda_j-\lambda_n} \right).
\end{aligned}
$$

Consequently, the off-diagonal term is given by:

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{2a_n\overline{a_m}}{(\lambda_m-\lambda_n)^2}+\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{a_n\overline{a_m}}{\lambda_m-\lambda_n}\left(\sum_{j\in\mathbb{Z}\setminus\{m\}}\frac{1}{\lambda_j-\lambda_m}-\sum_{j\in\mathbb{Z}\setminus\{n\}}\frac{1}{\lambda_j-\lambda_n}\right).$$

The first term here is easily dealt with and will be returned to shortly. Consider the problematic remainder and rewrite it as S_2-S_1, where

$$\begin{aligned} S_1 &:= \sum_{n\in\mathbb{Z}}a_n\left(\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{\overline{a_m}}{\lambda_m-\lambda_n}\right)\left(\sum_{j\in\mathbb{Z}\setminus\{n\}}\frac{1}{\lambda_j-\lambda_n}\right),\\ S_2 &:= \sum_{m\in\mathbb{Z}}\overline{a_m}\left(\sum_{n\in\mathbb{Z}\setminus\{m\}}\frac{-a_n}{\lambda_n-\lambda_m}\right)\left(\sum_{j\in\mathbb{Z}\setminus\{m\}}\frac{1}{\lambda_j-\lambda_m}\right).\end{aligned}$$

As A is skew-Hermitian it has purely imaginary eigenvalues, so given that $(a_j)_{j\in\mathbb{Z}}$ represents an eigenvector of A, it follows that $\displaystyle\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{\overline{a_m}}{\lambda_m-\lambda_n}=\overline{i\gamma a_n}=-i\gamma\overline{a_n}$ for some real γ. Hence $S_1=-i\gamma\displaystyle\sum_{n\in\mathbb{Z}}|a_n|^2\sum_{j\in\mathbb{Z}\setminus\{n\}}\frac{1}{\lambda_j-\lambda_n}$. Considering S_2 similarly, it follows that $S_2=-i\gamma\displaystyle\sum_{m\in\mathbb{Z}}|a_m|^2\sum_{j\in\mathbb{Z}\setminus\{m\}}\frac{1}{\lambda_j-\lambda_m}$. In particular, it follows that $S_2-S_1=0$.

Given this, the "off-diagonal" term is given by $\displaystyle\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{2a_n\overline{a_m}}{(\lambda_m-\lambda_n)^2}$. Now, using that this term must be real-valued and the fact that for $a,b\in\mathbb{C}$, $2|a\bar{b}|\leqslant|a|^2+|b|^2$,

$$\begin{aligned} &\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{2a_n\overline{a_m}}{(\lambda_m-\lambda_n)^2}\\ =\ &|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{2a_n\overline{a_m}}{(\lambda_m-\lambda_n)^2}|\\ \leqslant\ &\sum_{n\in\mathbb{Z}}|a_n|^2\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{(\alpha(m-n))^2}+\sum_{m\in\mathbb{Z}}|a_m|^2\sum_{n\in\mathbb{Z}\setminus\{m\}}\frac{1}{(\alpha(n-m))^2}\\ =\ &\frac{2\pi^2}{3\alpha^2}\sum_{n\in\mathbb{Z}}|a_n|^2.\end{aligned}$$

From before, it now follows that $\|T(a_j)\|_{\ell^2}^2 \leqslant \frac{\pi^2}{3\alpha^2}\sum_{n\in\mathbb{Z}}|a_n|^2 + \frac{2\pi^2}{3\alpha^2}\sum_{n\in\mathbb{Z}}|a_n|^2$ and so, as required, $\|T(a_j)\|_{\ell^2} \leqslant \frac{\pi}{\alpha}\|(a_j)\|_{\ell^2}$.

To extend this result to any sequence $(a_j)_{j\in\mathbb{Z}} \in \ell^2$, it suffices to take a limit of compactly supported sequences. □

This leads to another result:

Lemma 3.3.3 *Let* $P(x) = \sum_{|n|\leqslant N} a_n e^{i\lambda_n x}$ *with* $\alpha > 0$ *such that* $\lambda_{n+1} - \lambda_n > \alpha$ *for all* $n \in [-N, N] \cap \mathbb{Z}$. *Then for any* $x \in \mathbb{R}$,

$$\int_x^{x+1} |P(s)|^2 ds \leqslant \left(\frac{2\pi}{\alpha} + 1\right) \sum_{|n|\leqslant N} |a_n|^2.$$

Proof.

Consider the following:

$$\begin{aligned}
& \int_x^{x+1} |\sum_{|n|\leqslant N} a_n e^{i\lambda_n s}|^2 \, ds \\
= \;& \int_x^{x+1} (\sum_{|n|\leqslant N} a_n e^{i\lambda_n s})(\sum_{|m|\leqslant N} \overline{a_m} e^{-i\lambda_m s}) ds \\
= \;& \sum_{|n|\leqslant N} \sum_{|m|\leqslant N} \int_x^{x+1} a_n \overline{a_m} e^{i(\lambda_n - \lambda_m)s} ds \\
= \;& \sum_{|n|\leqslant N} \sum_{\substack{|m|\leqslant N \\ m\neq n}} a_n \overline{a_m} \frac{1}{i(\lambda_n - \lambda_m)} (e^{i(\lambda_n-\lambda_m)(x+1)} - e^{i(\lambda_n-\lambda_m)x}) + \sum_{|n|\leqslant N} |a_n|^2
\end{aligned}$$

Now, this quantity is bounded above by

$$
\begin{aligned}
& |\sum_{|n|\leqslant N}\sum_{\substack{|m|\leqslant N\\ m\neq n}} a_n\overline{a_m}\frac{1}{i(\lambda_n-\lambda_m)}e^{i(\lambda_n-\lambda_m)(x+1)}| + |\sum_{|n|\leqslant N}\sum_{\substack{|m|\leqslant N\\ m\neq n}} a_n\overline{a_m}\frac{1}{i(\lambda_n-\lambda_m)}e^{i(\lambda_n-\lambda_m)x}| + \\
& \sum_{|n|\leqslant N}|a_n|^2 \\
\leqslant\; & 2\sum_{|n|\leqslant N}\sum_{\substack{|m|\leqslant N\\ m\neq n}}|\frac{a_n\overline{a_m}}{\lambda_n-\lambda_m}| + \sum_{|n|\leqslant N}|a_n|^2 \\
=\; & 2\sum_{|n|\leqslant N}|a_n\overline{(T(a_m))_n}| + \sum_{|n|\leqslant N}|a_n|^2 \text{ where } T \text{ is as in Lemma 3.3.2} \\
\leqslant\; & 2\|(a_n)\|_{\ell^2}\|T(a_m)\|_{\ell^2} + \sum_{|n|\leqslant N}|a_n|^2 \\
\leqslant\; & \left(\frac{2\pi}{\alpha}+1\right)\|(a_n)\|^2_{\ell^2} \text{ by Lemma 3.3.2.}
\end{aligned}
$$

□

There is now sufficient background material to prove the boundedness of the square function:

Theorem 3.3.4 *For any $f \in S^2$ with $\alpha > 0$ such that $\lambda_{n+1} - \lambda_n > \alpha$ for all $n \in \mathbb{Z}$,*

$$\|(\sum_{k\in\mathbb{N}}|S_k f - R_k f|^2)^{\frac{1}{2}}\|_{S^2} \leqslant \left(\frac{2\pi}{\alpha}+1\right)^{\frac{1}{2}}\|f(x)\|_{S^2}.$$

Proof.

$$
\begin{aligned}
& \|(\sum_{k\in\mathbb{N}}|S_kf-R_kf|^2)^{\frac{1}{2}}\|_{S^2}^2 \\
= \;& \sup_{x\in\mathbb{R}}\int_x^{x+1}\sum_{k\in\mathbb{N}}|\sum_{2^{k-1}<|\lambda_n|\leqslant 2^k}(1-\widehat{\phi_k}(\lambda_n))\widehat{f}(\lambda_n)e^{i\lambda_n s}|^2\,ds \\
= \;& \sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{N}}\int_x^{x+1}|\sum_{2^{k-1}<|\lambda_n|\leqslant 2^k}(1-\widehat{\phi_k}(\lambda_n))\widehat{f}(\lambda_n)e^{i\lambda_n s}|^2\,ds \\
& \text{by the Lebesgue Monotone Convergence theorem} \\
\leqslant \;& \left(\frac{2\pi}{\alpha}+1\right)\sum_{k\in\mathbb{N}}\sum_{2^{k-1}<|\lambda_n|\leqslant 2^k}|(1-\widehat{\phi_k}(\lambda_n))\widehat{f}(\lambda_n)|^2 \\
& \text{by Lemma 3.3.3} \\
\leqslant \;& \left(\frac{2\pi}{\alpha}+1\right)\sum_{n\in\mathbb{Z}}|\widehat{f}(\lambda_n)|^2 \\
= \;& \left(\frac{2\pi}{\alpha}+1\right)M(|f(x)|^2) \text{ by Parseval's identity (Theorem 1.3.11)} \\
\leqslant \;& \left(\frac{2\pi}{\alpha}+1\right)\|f(x)\|_{S^2}^2 \text{ by Proposition 1.2.2.}
\end{aligned}
$$

□

This establishes boundedness of the square function for the case $p = 2$.

3.4 Bounding the Maximal Function for $p \in (1, \infty)$

Boundedness of the maximal function can be established for all $p \in (1, \infty)$ by means of the following theorem:

Theorem 3.4.1 *For* $p \in (1, \infty)$, $\|\sup_{k\in\mathbb{N}}|R_k(f)|\|_{S^p} \lesssim \|f\|_{S^p}$.

Proof. For fixed $k \in \mathbb{N}$, make the following decompositions of f and ϕ_k for $l \in \mathbb{N}$, $j \in \mathbb{Z}$:

$$
\begin{aligned}
\phi_k^{(l)} &:= \phi_k \chi_{(-2^l,-2^{l-1}]\cup[2^{l-1},2^l)};\\
\phi_k^{(0)} &:= \phi_k \chi_{(-1,1)};\\
f_j &:= f\chi_{[j,j+1)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|\sup_{k\in\mathbb{N}}|R_k(f)|\|_{S^p} &= \|\sup_{k\in\mathbb{N}}|(\sum_{l=0}^{\infty}\phi_k^{(l)})*(\sum_{j\in\mathbb{Z}}f_j)|\|_{S^p}\\
&\leqslant \sum_{l=0}^{\infty}\|\sup_{k\in\mathbb{N}}|\phi_k^{(l)}*(\sum_{j\in\mathbb{Z}}f_j)|\|_{S^p}.
\end{aligned}
$$

Now, fix any $x\in\mathbb{R}$, $l\in\mathbb{N}\cup\{0\}$ and observe that

$$
\int_x^{x+1}\sup_{k\in\mathbb{N}}|\phi_k^{(l)}*(\sum_{j\in\mathbb{Z}}f_j)(s)|^p\,ds \leqslant \int_x^{x+1}(\sum_{j\in\mathbb{Z}}\sup_{k\in\mathbb{N}}|\phi_k^{(l)}*f_j(s)|)^p\,ds.
$$

Also, noting that $\mathrm{supp}(\phi_k^{(l)})\subseteq[-2^l,-2^{l-1}]\cup[2^{l-1},2^l]$ for $l\in\mathbb{N}$ and $\mathrm{supp}(f_j)\subseteq[j,j+1]$ for $j\in\mathbb{Z}$, it follows that

$$
\mathrm{supp}(\phi_k^{(l)}*f_j)\subseteq[-2^l+j,-2^{l-1}+(j+1)]\cup[2^{l-1}+j,2^l+(j+1)] \text{ for } l\in\mathbb{N},\ j\in\mathbb{Z}.
$$

Also, $\mathrm{supp}(\phi_k^{(0)})\subseteq[-1,1]$, hence

$$
\mathrm{supp}(\phi_k^{(0)}*f_j)\subseteq[-1+j,1+(j+1)]=[j-1,j+2] \text{ for } j\in\mathbb{Z}.
$$

For $l \in \mathbb{N}$, define

$$\begin{aligned} K_l &:= \{j \in \mathbb{Z} : \left|([-2^l + j, -2^{l-1} + (j+1)] \cup [2^{l-1} + j, 2^l + (j+1)]) \cap [x, x+1]\right| \neq 0\}; \\ K_0 &:= \{j \in \mathbb{Z} : |[j-1, j+2] \cap [x, x+1]| \neq 0\}. \end{aligned}$$

By a simple counting argument, $|K_0| \leqslant 4$, and for $l \in \mathbb{N}$, $|K_l| \leqslant 2(2^l - 2^{l-1} + 2) = 2^l + 4$, hence in general it is certainly true that for $l \in \mathbb{N} \cup \{0\}$, $|K_l| \leqslant 2^l + 4$.

Now, an application of Hölder's inequality for sums (noting that these are finite sums) gives that

$$\sum_{j \in K_l} \sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j| \leqslant (\sum_{j \in K_l} (\sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j|)^p)^{\frac{1}{p}} (\sum_{j \in K_l} 1)^{\frac{1}{p'}}.$$

It hence follows that

$$\begin{aligned} \int_x^{x+1} \sup_{k \in \mathbb{N}} |\phi_k^{(l)} * (\sum_{j \in \mathbb{Z}} f_j)(s)|^p \, ds &\leqslant \int_x^{x+1} (\sum_{j \in K_l} \sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j(s)|^p)(2^l + 4)^{\frac{p}{p'}} \, ds \\ &= \sum_{j \in K_l} \int_x^{x+1} (\sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j(s)|^p)(2^l + 4)^{\frac{p}{p'}} \, ds \\ &\leqslant (2^l + 4)^{\frac{p}{p'} + 1} \sup_{j \in K_l} \int_x^{x+1} \sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j(s)|^p \, ds. \text{ (*)} \end{aligned}$$

Now ϕ_k satisfies $\phi_k = 2^k \phi(2^k \cdot)$. Given that ϕ is Schwartz, for any $N \in \mathbb{N}$, there exists a positive constant C_N independent of k such that

$$|(2^k x)^N \phi_k(x)| = |(2^k x)^N 2^k \phi(2^k x)| \leqslant 2^k C_N.$$

It thus follows that $|\phi_k(x)| \leqslant \dfrac{C_N}{2^{(N-1)k} |x|^N}$. In particular, as $\text{supp}(\phi_k^{(l)}) \subseteq [-2^l, -2^{l-1}] \cup$

$[2^{l-1}, 2^l]$ for any $l \in \mathbb{N}$, for $x \in \mathbb{R}$,

$$
\begin{aligned}
|\phi_k^{(l)}(x)| &\leqslant \frac{C_N}{2^{(N-1)k+N(l-1)}} \\
&\leqslant \frac{C_N}{2^{N(l-1)}}
\end{aligned}
$$

as $k \in \mathbb{N}$.

Using this,

$$
\begin{aligned}
& (2^l+4)^{\frac{p}{p'}+1} \sup_{j \in K_l} \int_x^{x+1} \sup_{k \in \mathbb{N}} |\phi_k^{(l)} * f_j(s)|^p \, ds \\
\leqslant\ & (2^l+4)^{\frac{p}{p'}+1} \sup_{j \in K_l} \int_x^{x+1} \left(\frac{C_N}{2^{N(l-1)}} \int_{\mathbb{R}} |f_j(t)| \, dt \right)^p \, ds \\
\leqslant\ & \frac{C_N^p (2^l+4)^{\frac{p}{p'}+1}}{2^{Np(l-1)}} \sup_{j \in K_l} \int_x^{x+1} \|f_j\|^p_{L^1[j,j+1]} \, ds \text{ as } \operatorname{supp}(f_j) \subseteq [j, j+1] \\
=\ & \frac{C_N^p (2^l+4)^{\frac{p}{p'}+1}}{2^{Np(l-1)}} \sup_{j \in K_l} \|f\|^p_{L^1[j,j+1]} \\
\leqslant\ & \frac{C_N^p (2^l+4)^{\frac{p}{p'}+1}}{2^{Np(l-1)}} \|f\|^p_{S^1}.
\end{aligned}
$$

Consider now the case of $l = 0$. Note that since ϕ_k is Schwartz, $\sup_{k \in \mathbb{N}} ||\phi_k| * |f_j|(s)|$ is pointwise almost everywhere dominated by the Hardy–Littlewood maximal function, $\mathcal{M}_{\mathbb{R}}$ for any $j \in \mathbb{Z}$. Using this fact, the well-known L^p–boundedness of $\mathcal{M}_{\mathbb{R}}$ and (*), it follows that

$$\begin{aligned}
\left(\int_x^{x+1} \sup_{k\in\mathbb{N}} |\phi_k^{(0)} * (\sum_{j\in\mathbb{Z}} f_j)(s)|^p \, ds\right)^{\frac{1}{p}} &\leqslant \left((2^0+4)^{\frac{p}{p'}+1} \sup_{j\in K_0} \int_x^{x+1} \sup_{k\in\mathbb{N}} |\phi_k^{(0)} * f_j(s)|^p \, ds\right)^{\frac{1}{p}} \\
&\leqslant 5\left(\sup_{j\in K_0} \int_{\mathbb{R}} \sup_{k\in\mathbb{N}} ||\phi_k| * |f_j|(s)|^p \ ds\right)^{\frac{1}{p}} \\
&\leqslant 5 \sup_{j\in K_0} \|\mathcal{M}_{\mathbb{R}}(f_j)(s)\|_{L^p(\mathbb{R})} \\
&\leqslant 5 \sup_{j\in K_0} \|f_j\|_{L^p(\mathbb{R})} \\
&= 5 \sup_{j\in K_0} \|f\|_{L^p([j,j+1])} \\
&\leqslant 5\|f\|_{S^p}.
\end{aligned}$$

Now, using the fact that the choice of x was arbitrary and drawing everything together,

$$\begin{aligned}
\|\sup_{k\in\mathbb{N}} |R_k(f)|\|_{S^p} &\leqslant \sum_{l=1}^{\infty} \left(\frac{C_N^p(2^l+4)^{\frac{p}{p'}+1}}{2^{Np(l-1)}}\right)^{\frac{1}{p}} \|f\|_{S^1} + 5\|f\|_{S^p} \\
&= \sum_{l=1}^{\infty} \frac{C_N(2^l+4)}{2^{N(l-1)}} \|f\|_{S^1} + 5\|f\|_{S^p} \\
&= 2^N C_N \|f\|_{S^1} \sum_{l=1}^{\infty} \left((\frac{1}{2})^{l(N-1)} + 4(\frac{1}{2})^{Nl}\right) + 5\|f\|_{S^p}.
\end{aligned}$$

As $\|f\|_{S^1} \leqslant \|f\|_{S^p}$, it follows by fixing $N \geqslant 2$ that

$$\|\sup_{k\in\mathbb{N}} |R_k(f)|\|_{S^p} \lesssim \|f\|_{S^p}. \qquad \square$$

This gives the bound on the maximal function for $p \in (1,\infty)$, and completes the proof of Theorem 3.1.1 in the case of $p = 2$.

3.5 Bounding the Square Function for $p = 2^k$, $k \in \mathbb{N}$

In order to establish Theorem 3.1.1 in generality, it remains to consider the general boundedness of the square function. To start with, it will be useful to consider some results regarding the Hilbert transform on S^p. This can be defined exactly as in the standard settings:

Definition 3.5.1 *For $f \in S^p$, $p \in [1, \infty)$, the Hilbert transform is defined as*

$$Hf(x) := p.v.\ \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Integration in the complex plane over a suitable indented contour shows that this is equivalent to considering the operator such that for each $n \in \mathbb{Z}$, $\widehat{Hf}(\lambda_n) = -i\mathrm{sgn}(\lambda_n)\widehat{f}(\lambda_n)$. The conclusion of Theorem 3.2.2 thus holds in this case in spite of the failure of the hypotheses.

It turns out that for the present purposes, it will be useful to consider slightly modified versions of the Hilbert transform:

Definition 3.5.2 *For $f \in S^p$, $p \in [1, \infty)$ and $\theta \in \{1, 2\}$, define H_θ to be the operator such that $\widehat{H_\theta f}(\lambda) = -isgn_\theta \widehat{f}(\lambda)$ for any $\lambda \in \mathbb{R}$, where $sgn_\theta : \mathbb{R} \to \{-1, 1\}$ are given by*

$$\begin{aligned} sgn_1(x) &:= \begin{cases} 1, & x \in [0, \infty) \\ -1, & x \in (-\infty, 0) \end{cases} \\ sgn_2(x) &:= \begin{cases} 1, & x \in (0, \infty) \\ -1, & x \in (-\infty, 0] \end{cases} \end{aligned}$$

The following lemma shows that a well-known identity for the regular Hilbert transform adapts well for the modified Hilbert transforms given above, acting on almost periodic trigonometric polynomials:

Lemma 3.5.3 *Let f be a trigonometric polynomial, $f(x) = \sum_{|n|\leqslant N} \widehat{f}(\lambda_n)e^{i\lambda_n x}$. Then for $\theta \in \{1,2\}$,*

$$(H_\theta(f))^2 = f^2 + 2H_\theta(fH_\theta(f)).$$

Proof. First note that

$$\begin{aligned}
& 2H_\theta(fH_\theta(f)) \\
= \;& 2H_\theta((\sum_{|n|\leqslant N} \widehat{f}(\lambda_n)e^{i\lambda_n x})(\sum_{|m|\leqslant N}(-i\mathrm{sgn}_\theta(\lambda_m))\widehat{f}(\lambda_m)e^{i\lambda_m x})) \\
= \;& 2H_\theta(\sum_{|n|\leqslant N}\sum_{|m|\leqslant N}(-i\mathrm{sgn}_\theta(\lambda_m))\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x}) \\
= \;& 2\sum_{|n|\leqslant N}\sum_{|m|\leqslant N}(-\mathrm{sgn}_\theta(\lambda_m)\mathrm{sgn}_\theta(\lambda_n+\lambda_m)\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x}) \\
= \;& 2\sum_{|n|\leqslant N}\sum_{|m|\leqslant N}(-\mathrm{sgn}_\theta(\lambda_n)\mathrm{sgn}_\theta(\lambda_n+\lambda_m)\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x}) \\
& \text{by symmetry in } n \text{ and } m.
\end{aligned}$$

Averaging these last two lines gives that

$$2H_\theta(fH_\theta(f)) = \sum_{|n|\leqslant N}\sum_{|m|\leqslant N} -(\mathrm{sgn}_\theta(\lambda_n)+\mathrm{sgn}_\theta(\lambda_m))\mathrm{sgn}_\theta(\lambda_n+\lambda_m)\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x}.$$

Now, $f^2 = \sum_{|n|\leqslant N}\sum_{|m|\leqslant N}\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x}$, so it follows that

$$f^2+2H_\theta(fH_\theta(f)) = \sum_{|n|\leqslant N}\sum_{|m|\leqslant N}(1-(\mathrm{sgn}_\theta(\lambda_n)+\mathrm{sgn}_\theta(\lambda_m))\mathrm{sgn}_\theta(\lambda_n+\lambda_m))\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)\cdot}$$

It is trivial to see that for either $\theta \in \{1, 2\}$,

$$\mathrm{sgn}_\theta(\lambda_n + \lambda_m)(\mathrm{sgn}_\theta(\lambda_n) + \mathrm{sgn}_\theta(\lambda_m)) = 1 + \mathrm{sgn}_\theta(\lambda_n)\mathrm{sgn}_\theta(\lambda_m).$$

Equivalently,

$$1 - (\mathrm{sgn}_\theta(\lambda_n) + \mathrm{sgn}_\theta(\lambda_m))\mathrm{sgn}_\theta(\lambda_n + \lambda_m) = -\mathrm{sgn}_\theta(\lambda_n)\mathrm{sgn}_\theta(\lambda_m).$$

Consequently, it follows that

$$\begin{aligned} f^2 + 2H_\theta(fH_\theta(f)) &= \sum_{|n|\leqslant N}\sum_{|m|\leqslant N}(-\mathrm{sgn}_\theta(\lambda_n)\mathrm{sgn}_\theta(\lambda_m))\widehat{f}(\lambda_n)\widehat{f}(\lambda_m)e^{i(\lambda_n+\lambda_m)x} \\ &= (\sum_{|n|\leqslant N}(-i\mathrm{sgn}_\theta(\lambda_n))\widehat{f}(\lambda_n)e^{i\lambda_n x})^2 \\ &= (H_\theta(f))^2. \end{aligned}$$

□

Using this, the following vector-valued operator bound for the modified Hilbert transforms may be established:

Theorem 3.5.4 *For any $k \in \mathbb{N}$, let $(f_j)_{j\in\mathbb{N}} \subseteq S^{2^k}$ be such that there exists $\alpha > 0$ such that for every $j \in \mathbb{N}$, the Fourier exponents of f_j are separated by α. Then for $\theta \in \{1, 2\}$,*

$$\|(\sum_{j\in\mathbb{N}}|H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2^k}} \lesssim \|(\sum_{j\in\mathbb{N}}|f_j|^2)^{\frac{1}{2}}\|_{S^{2^k}}.$$

Proof. To start with, fix any $\varepsilon > 0$ and for each $j \in \mathbb{N}$, choose $N_j \in \mathbb{N}$ such that $f_j^{(N_j)}$ is a trigonometric polynomial of degree N_j approximating f_j in the sense that

$\|f_j^{(N_j)} - f_j\|_{S^{2^k}} < \dfrac{\varepsilon}{2^{\frac{j}{2}}}$. Now, note that

$$\begin{aligned}
& \|(\sum_{j\in\mathbb{N}} |f_j^{(N_j)}|^2)^{\frac{1}{2}} - (\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2^k}} \\
\leqslant\ & \|(\sum_{j\in\mathbb{N}} |f_j^{(N_j)} - f_j|^2)^{\frac{1}{2}}\|_{S^{2^k}} \\
=\ & \|\sum_{j\in\mathbb{N}} |f_j^{(N_j)} - f_j|^2\|_{S^{2^{k-1}}}^{\frac{1}{2}} \\
\leqslant\ & (\sum_{j\in\mathbb{N}} \|f_j^{(N_j)} - f_j\|_{S^{2^k}}^2)^{\frac{1}{2}} \text{ by the triangle inequality.}
\end{aligned}$$

Consequently, by density, it suffices to assume that each f_j is a trigonometric polynomial. Also, place the assumption that f_j is real valued for every $j \in \mathbb{N}$. The result then extends to complex valued functions by splitting f_j into its real and imaginary parts and applying the triangle inequality, using linearity of H_θ. Using the convolution definition of H, it is clear that Hf_j is real-valued when f_j is. As $H_\theta f_j = Hf_j - (-1)^\theta \widehat{f_j}(0)$, it follows that $H_\theta f_j$ is also real-valued when f_j is.

The proof will proceed by induction.

The case $k = 1$:

To begin, assume that $\sum_{j\in\mathbb{N}} |f_j|^2$ is a finite sum. The below will then be complete by appealing to the Lebesgue Monotone Convergence theorem.

$$\begin{aligned}
& \|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^2}^2 \\
= \quad & \sup_{x\in\mathbb{R}} \sum_{j\in\mathbb{N}} \int_x^{x+1} |H_\theta f_j|^2 \\
\leqslant \quad & \left(\frac{2\pi}{\alpha}+1\right) \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{N}} |-i\mathrm{sgn}_\theta(\lambda_n)\widehat{f}_j(\lambda_n)|^2 \text{ by Lemma 3.3.3} \\
= \quad & \left(\frac{2\pi}{\alpha}+1\right) \sum_{j\in\mathbb{N}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_j|^2 \text{ by Parseval's identity} \\
= \quad & \left(\frac{2\pi}{\alpha}+1\right) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{j\in\mathbb{N}} |f_j|^2 \\
\leqslant \quad & \left(\frac{2\pi}{\alpha}+1\right) \|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^2}^2 \text{ by Proposition 1.2.2.}
\end{aligned}$$

Inductive step: Assume the result for $p := 2^k$ and note that it follows *a fortiori* that $\|H_\theta f\|_{S^p} \lesssim \|f\|_{S^p}$. Now using Lemma 3.5.3,

$$\begin{aligned}
& \|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}} \\
= \quad & \|(\sum_{j\in\mathbb{N}} f_j^2 + 2H_\theta(f_j H_\theta(f_j)))\|_{S^p}^{\frac{1}{2}} \\
\leqslant \quad & (\|\sum_{j\in\mathbb{N}} f_j^2\|_{S^p} + 2\|\sum_{j\in\mathbb{N}} H_\theta(f_j H_\theta(f_j))\|_{S^p})^{\frac{1}{2}} \\
= \quad & (\|(\sum_{j\in\mathbb{N}} f_j^2)^{\frac{1}{2}}\|_{S^{2p}}^2 + 2\|\sum_{j\in\mathbb{N}} H_\theta(f_j H_\theta(f_j))\|_{S^p})^{\frac{1}{2}}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\|\sum_{j\in\mathbb{N}} H_\theta(f_j H_\theta(f_j))\|_{S^p}\\
= \;& \|H_\theta(\sum_{j\in\mathbb{N}} f_j H_\theta(f_j))\|_{S^p}\\
\leqslant \;& \|H_\theta\|_{S^p\to S^p}\|\sum_{j\in\mathbb{N}} |f_j H_\theta(f_j)|\|_{S^p}\\
\leqslant \;& \|H_\theta\|_{S^p\to S^p}\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}(\sum_{j\in\mathbb{N}} |H_\theta(f_j)|^2)^{\frac{1}{2}}\|_{S^p}\\
= \;& \|H_\theta\|_{S^p\to S^p}\left(\sup_{x\in\mathbb{R}}\int_x^{x+1}(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{p}{2}}(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\
\leqslant \;& \|H_\theta\|_{S^p\to S^p}\left(\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}(\sum_{j\in\mathbb{N}} |f_j|^2)^{p}\right)^{\frac{1}{2}}\left(\int_x^{x+1}(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{p}\right)^{\frac{1}{2}}\right)^{\frac{1}{p}}\\
\leqslant \;& \|H_\theta\|_{S^p\to S^p}\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}.
\end{aligned}$$

It thus follows that

$$\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}} \leqslant \left(\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}^2 + 2\|H_\theta\|_{S^p\to S^p}\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}\right)^{\frac{1}{2}}.$$

Using this,

$$\left(\frac{\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}}{\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}}\right)^2 \leqslant 1 + 2\|H_\theta\|_{S^p\to S^p}\frac{\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}}{\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}}.$$

Consequently,

$$\|(\sum_{j\in\mathbb{N}} |H_\theta f_j|^2)^{\frac{1}{2}}\|_{S^{2p}} \leqslant \left(\|H_\theta\|_{S^p\to S^p} + \sqrt{\|H_\theta\|_{S^p\to S^p}^2 + 1}\right)\|(\sum_{j\in\mathbb{N}} |f_j|^2)^{\frac{1}{2}}\|_{S^{2p}}. \qquad \square$$

It is noted at this point that there is a certain degree of subtlety concerning the problem of boundedness of the Hilbert transform in the setting of Stepanov norms. In particular, the following is true:

Theorem 3.5.5 *The Hilbert transform fails to be bounded in the space* $(\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\}, \|\cdot\|_{S^p})$.

To prove this result, the following simple lemma is required:

Lemma 3.5.6 *The Stepanov norm,* $\|\cdot\|_{S^p}$*, is equivalent to the "amalgam" norm defined by* $\|\cdot\|_{(L^p,\ell^\infty)} := \left(\sup_{x\in\mathbb{Z}} \int_x^{x+1} |\cdot|^p\right)^{\frac{1}{p}}$.

Proof. It is clear that for $f \in L^p_{loc}(\mathbb{R})$, $\left(\sup_{x\in\mathbb{Z}} \int_x^{x+1} |f(s)|^p\,ds\right)^{\frac{1}{p}} \leqslant \|f\|_{S^p}$. Now for any $x \in \mathbb{R}$,

$$\int_x^{x+1} |f(s)|^p\,ds \leqslant \int_{[x]}^{[x+1]} |f(s)|^p\,ds + \int_{[x+1]}^{[x+2]} |f(s)|^p\,ds$$

where $[x]$ indicates the integer part of x.

It hence follows that $\|f\|_{S^p} \leqslant 2^{\frac{1}{p}} \left(\sup_{x\in\mathbb{Z}} \int_x^{x+1} |f(s)|^p\,ds\right)^{\frac{1}{p}}$. ☐

By the equivalence of these norms, it follows that $\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\}$ is equal to the "amalgam space" (L^p, ℓ^∞) of functions f such that $\|f\|_{(L^p,\ell^\infty)} < \infty$, as defined in [15].

Now, from Theorem 2.6 in [15], the pre-dual space of (L^p, ℓ^∞) is $(L^{p'}, \ell^1)$ (which is defined in the obvious way: $\|\cdot\|_{(L^{p'},\ell^1)} := \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |\cdot|^{p'}\right)^{\frac{1}{p'}}$). From this fact, if the Hilbert transform is bounded on (L^p, ℓ^∞), it is also bounded on $(L^{p'}, \ell^1)$. This can be shown to be false by the following:

By an elementary calculation, $H(\chi_{[0,1]})(x) = \frac{1}{\pi} \log \frac{|x|}{|x-1|}$. Consequently,

$$\|H(\chi_{[0,1]})\|_{(L^{p'},\ell^1)} = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} \left(\log \frac{|x|}{|x-1|} \right)^{p'} dx \right)^{\frac{1}{p'}}.$$

Observing that $\log\left(\frac{|x|}{|x-1|}\right) \sim \frac{1}{|x|}$ as $|x| \to \infty$ and that for $n \in \mathbb{N}$, $\left(\int_n^{n+1} \frac{1}{x^{p'}}\, dx \right)^{\frac{1}{p'}} \geqslant \left(\frac{1}{(n+1)^{p'}} \right)^{\frac{1}{p'}} = \frac{1}{n+1}$, it follows that $\|H(\chi_{[0,1]})\|_{(L^{p'},\ell^1)} = \infty$.

In line with the proof of the periodic result, the next theorem to be developed is an analogue of the Littlewood–Paley result from Theorem 2.3.7. To begin with, a standard Littlewood–Paley theorem for $\mathbb{R}$ is stated. This result is a special case of Theorem 5.1.2 from [17], p. 339.

Theorem 3.5.7 (Littlewood–Paley on $L^p(\mathbb{R})$) *Let $\Psi \in C^1(\mathbb{R})$ be an integrable function with mean value zero, such that there exists a positive constant C so that for all $x \in \mathbb{R}$,*

$$|\Psi(x)| + |\Psi'(x)| \leqslant \frac{C}{(1+|x|)^2}.$$

Define $\Psi_k = 2^k \Psi(2^k \cdot)$ so that $\widehat{\Psi_k} = \widehat{\Psi}(2^{-k}\cdot)$. Then for all $f \in L^p(\mathbb{R})$, $p \in (1,\infty)$

$$\|(\sum_{k\in\mathbb{N}} |\Psi_k * f|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

For a proof, see [17], pp. 339–41.

With this stated, an analogous result for S^p, $p \in (1,\infty)$ may be developed.

Theorem 3.5.8 (Littlewood–Paley on S^p) *Let $\psi \in \mathcal{S}(\mathbb{R})$ have mean value zero and* *define* $\psi_k := 2^k\psi(2^k\cdot)$ *so that* $\widehat{\psi_k} = \widehat{\psi}(2^{-k}\cdot)$*. Then for* $f \in S^p$*,* $p \in (1,\infty)$*,*

$$\|(\sum_{k\in\mathbb{N}} |f * \psi_k|^2)^{\frac{1}{2}}\|_{S^p} \lesssim \|f\|_{S^p}.$$

Proof. The proof here will follow a similar scheme to the proof of Theorem 3.4.1.

For each $k \in \mathbb{N}$, make the following decompositions for $l \in \mathbb{N}$, $j \in \mathbb{Z}$:

$$\begin{aligned}
\psi_k^{(l)} &:= \psi_k\chi_{(-2^l,-2^{l-1}]\cup[2^{l-1},2^l)};\\
\psi_k^{(0)} &:= \psi_k\chi_{(-1,1)};\\
f_j &:= f\chi_{[j,j+1)}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\|(\sum_{k\in\mathbb{N}} |f * \psi_k|^2)^{\frac{1}{2}}\|_{S^p}\\
= \;& \|(\sum_{k\in\mathbb{N}} |(\sum_{j\in\mathbb{Z}} f_j) * (\sum_{l=0}^{\infty} \psi_k^{(l)})|^2)^{\frac{1}{2}}\|_{S^p}\\
\leqslant \;& \sum_{l=0}^{\infty} \|(\sum_{k\in\mathbb{N}} |\psi_k^{(l)} * (\sum_{j\in\mathbb{Z}} f_j)|^2)^{\frac{1}{2}}\|_{S^p}.
\end{aligned}$$

For fixed $x \in \mathbb{R}$, $l \in \mathbb{N}\cup\{0\}$, define

$$I(x,l) := \int_x^{x+1} (\sum_{k\in\mathbb{N}} |\psi_k^{(l)} * (\sum_{j\in\mathbb{Z}} f_j)|^2)^{\frac{p}{2}}$$

and note that $\|(\sum_{k\in\mathbb{N}}|f*\psi_k|^2)^{\frac{1}{2}}\|_{S^p} \leqslant \sum_{l=0}^{\infty}\sup_{x\in\mathbb{R}}(I(x,l))^{\frac{1}{p}}$.

Observe that

$$I(x,l) \leqslant \int_x^{x+1} (\sum_{j\in\mathbb{Z}}(\sum_{k\in\mathbb{N}}|\psi_k^{(l)} * f_j|^2)^{\frac{1}{2}})^p.$$

As in the proof of Theorem 3.4.1, for $j \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{supp}(\psi_k^{(l)} * f_j) &\subseteq [-2^l + j, -2^{l-1} + (j+1)] \cup [2^{l-1} + j, 2^l + (j+1)] \text{ for } l \in \mathbb{N}; \\
\mathrm{supp}(\psi_k^{(0)} * f_j) &\subseteq [-1 + j, 1 + (j+1)] = [j-1, j+2].
\end{aligned}$$

As before, for $l \in \mathbb{N}$, define

$$\begin{aligned}
K_l :=& \{j \in \mathbb{Z} : |([-2^l + j, -2^{l-1} + (j+1)] \cup [2^{l-1} + j, 2^l + (j+1)]) \cap [x, x+1]| \neq 0\}, \\
K_0 :=& \{j \in \mathbb{Z} : |[j-1, j+2] \cap [x, x+1]| \neq 0\}
\end{aligned}$$

and note that $|K_l| \leqslant 2^l + 4$ for any $l \in \mathbb{N} \cup \{0\}$.

Now, applying Hölder's inequality to the sum in j and using the above,

$$\begin{aligned}
\left(\sum_{j\in K_l}(\sum_{k\in\mathbb{N}}|\psi_k^{(l)} * f_j|^2)^{\frac{1}{2}}\right)^p &\leqslant \left(\left(\sum_{j\in K_l}(\sum_{k\in\mathbb{N}}|\psi_k^{(l)} * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \left(\sum_{j\in K_l} 1^{p'}\right)^{\frac{1}{p'}}\right)^p \\
&\leqslant \left(\sum_{j\in K_l}(\sum_{k\in\mathbb{N}}|\psi_k^{(l)} * f_j|^2)^{\frac{p}{2}}\right)(2^l+4)^{\frac{p}{p'}}.
\end{aligned}$$

It thus follows that for $x \in \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} I(x,l) &\leqslant \int_x^{x+1} \left(\sum_{j\in K_l} (\sum_{k\in\mathbb{N}} |\psi_k^{(l)} * f_j|^2)^{\frac{p}{2}} \right) (2^l+4)^{\frac{p}{p'}} \\ &\leqslant (2^l+4)^{\frac{p}{p'}+1} \sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |\psi_k^{(l)} * f_j|^2)^{\frac{p}{2}}. \text{ (*)} \end{aligned}$$

Using the fact that ψ is Schwartz, proceeding as in the proof of Theorem 3.4.1, it follows that for $l \in \mathbb{N}$, for each $N \in \mathbb{N}$, there exists some fixed positive constant C_N such that for any $k \in \mathbb{N}$,

$$|\psi_k^{(l)}| \leqslant \frac{C_N}{2^{(N-1)k+N(l-1)}}.$$

So, for $x \in \mathbb{R}$, $l \in \mathbb{N}$,

$$\begin{aligned} I(x,l) &\leqslant (2^l+4)^{\frac{p}{p'}+1} \sup_{j\in K_l} \int_x^{x+1} \left(\sum_{k\in\mathbb{N}} |\frac{C_N}{2^{(N-1)k+N(l-1)}} \int_{\mathbb{R}} f_j|^2 \right)^{\frac{p}{2}} \\ &\leqslant (2^l+4)^{\frac{p}{p'}+1} \left(\frac{C_N}{2^{N(l-1)}} \right)^p \sup_{j\in K_l} \left(\sum_{k\in\mathbb{N}} (\frac{1}{2^{(N-1)k}} \|f\|_{L^1([j,j+1])})^2 \right)^{\frac{p}{2}} \\ &= (2^l+4)^{\frac{p}{p'}+1} \left(\frac{C_N}{2^{N(l-1)}} \right)^p \sup_{j\in K_l} \|f\|^p_{L^1([j,j+1])} \left(\sum_{k\in\mathbb{N}} \frac{1}{4^{(N-1)k}} \right)^{\frac{p}{2}}. \end{aligned}$$

Choose $N \geqslant 2$ to make the sum in k convergent and denote its sum by D_N^2. Letting $C'_N := C_N D_N$, it now follows that

$$\begin{aligned} I(x,l) &\leqslant (2^l+4)^{\frac{p}{p'}+1} \left(\frac{C'_N}{2^{N(l-1)}} \right)^p \sup_{j\in K_l} \|f\|^p_{L^1([j,j+1])} \\ &\leqslant (2^l+4)^{\frac{p}{p'}+1} \left(\frac{C'_N}{2^{N(l-1)}} \right)^p \|f\|^p_{S^1} \\ &\leqslant (2^l+4)^{\frac{p}{p'}+1} \left(\frac{C'_N}{2^{N(l-1)}} \right)^p \|f\|^p_{S^p}. \end{aligned}$$

Consider the remaining case of $l = 0$. Continuing from **(*)**,

$$\begin{aligned}
(I(x,0))^{\frac{1}{p}} &\leqslant \left(5^{\frac{p}{p'}+1} \sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |\psi_k^{(0)} * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leqslant 5\left(\sup_{j\in K_l} \int_x^{x+1} \left((\sum_{k\in\mathbb{N}} |\psi_k * f_j|^2)^{\frac{1}{2}} + (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{1}{2}}\right)^p\right)^{\frac{1}{p}} \\
&\leqslant 5\left(\sup_{j\in K_l} \int_{\mathbb{R}} (\sum_{k\in\mathbb{N}} |\psi_k * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} + 5\left(\sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&= 5\sup_{j\in K_l} \|(\sum_{k\in\mathbb{N}} |\psi_k * f_j|^2)^{\frac{1}{2}}\|_{L^p} + 5\left(\sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leqslant C\sup_{j\in K_l} \|f_j\|_{L^p} + 5\left(\sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\quad \text{by Theorem 3.5.7} \\
&= C\sup_{j\in K_l} \|f\|_{L^p([j,j+1])} + 5\left(\sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leqslant C\|f\|_{S^p} + 5\left(\sup_{j\in K_l} \int_x^{x+1} (\sum_{k\in\mathbb{N}} |(\psi_k - \psi_k^{(0)}) * f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, consider that

$$
\begin{aligned}
& \left(\sup_{j\in K_l}\int_x^{x+1}(\sum_{k\in\mathbb{N}}|(\psi_k-\psi_k^{(0)})*f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
= & \left(\sup_{j\in K_l}\int_x^{x+1}(\sum_{k\in\mathbb{N}}|(\sum_{l=1}^{\infty}\psi_k^{(l)})*f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
\leqslant & \sum_{l=1}^{\infty}\left(\sup_{j\in K_l}\int_x^{x+1}(\sum_{k\in\mathbb{N}}|\psi_k^{(l)}*f_j|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
\leqslant & \sum_{l=1}^{\infty}\left(\sup_{j\in K_l}\int_x^{x+1}(\sum_{k\in\mathbb{N}}\left(\frac{C_N}{2^{(N-1)k+N(l-1)}}\right)^2\|f\|^2_{L^1([j,j+1])})^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
\leqslant & \sum_{l=1}^{\infty}\left(\frac{C_N}{2^{N(l-1)}}\right)\sup_{j\in K_l}\|f\|_{L^1([j,j+1])}\left(\sum_{k\in\mathbb{N}}\frac{1}{4^{(N-1)k}}\right)^{\frac{1}{2}} \\
\leqslant & C_N^{'}\left(\sum_{l=1}^{\infty}\frac{1}{2^{N(l-1)}}\right)\|f\|_{S^p}.
\end{aligned}
$$

It thus follows that

$$
(I(x,0))^{\frac{1}{p}} \leqslant \left(C+5C_N^{'}\left(\sum_{l=1}^{\infty}\frac{1}{2^{N(l-1)}}\right)\right)\|f\|_{S^p}.
$$

Drawing everything together,

$$
\begin{aligned}
& \|(\sum_{k\in\mathbb{N}}|f*\psi_k|^2)^{\frac{1}{2}}\|_{S^p} \\
\leqslant & \sum_{l=0}^{\infty}\sup_{x\in\mathbb{R}}(I(x,l))^{\frac{1}{p}} \\
\leqslant & \left(C+5C_N^{'}\left(\sum_{l=1}^{\infty}\frac{1}{2^{N(l-1)}}\right)\right)\|f\|_{S^p}+\sum_{l=1}^{\infty}(2^l+4)\left(\frac{C_N'}{2^{N(l-1)}}\right)\|f\|_{S^p} \\
\lesssim & \|f\|_{S^p}
\end{aligned}
$$

as $N \geqslant 2$. □

One final, simple result is required:

Lemma 3.5.9 *For $f \in S^p$, $p \in [1, \infty)$,*

$$S_k f(x) = \frac{i}{2}(e^{-2\pi i 2^k x} H_1(e^{2\pi i 2^k \cdot} f)(x) - e^{2\pi i 2^k x} H_2(e^{-2\pi i 2^k \cdot} f)(x))$$

for $x \in \mathbb{R}$.

Proof.

$$\begin{aligned}
& \widehat{(S_k f)}(\lambda_n) \\
= \;& \chi_{[-2^k, 2^k]}(\lambda_n) \widehat{f}(\lambda_n) \\
= \;& \frac{1}{2}(\mathrm{sgn}_1(\lambda_n + 2^k)\widehat{f}(\lambda_n) - \mathrm{sgn}_2(\lambda_n - 2^k)\widehat{f}(\lambda_n)) \\
= \;& \frac{i}{2}(-i\mathrm{sgn}_1(\lambda_n + 2^k)(\widehat{e^{2\pi i 2^k \cdot}} f)(\lambda_n + 2^k) + i\mathrm{sgn}_2(\lambda_n - 2^k)(\widehat{e^{-2\pi i 2^k \cdot}} f)(\lambda_n - 2^k)).
\end{aligned}$$

□

With this established, the boundedness of the square function may now be proved:

Theorem 3.5.10 *Let $f \in S^p$, $p = 2^k$, $k \in \mathbb{N}$ be such that there exists $\alpha > 0$ such that $\lambda_{n+1} - \lambda_n > \alpha$ for all $n \in \mathbb{N}$. Then*

$$\|(\sum_{k \in \mathbb{N}} |S_k f - R_k f|^2)^{\frac{1}{2}}\|_{S^p} \lesssim \|f\|_{S^p}.$$

Proof. Firstly, as in the proof of the periodic result, define $\psi \in \mathcal{S}(\mathbb{R})$ so that $\widehat{\psi} = \widehat{\phi}(\frac{1}{2}\cdot) - \widehat{\phi}$

and note that $\chi_{[-1,1]} - \widehat{\phi} = \chi_{[-1,1]}\widehat{\psi}$. It thus follows by Theorem 3.2.2 that

$$\begin{aligned}
& \|(\sum_{k\in\mathbb{N}} |S_k f - R_k f|^2)^{\frac{1}{2}}\|_{S^p} \\
= & \|(\sum_{k\in\mathbb{N}} |S_k(f * \psi_k)|^2)^{\frac{1}{2}}\|_{S^p} \\
= & \|(\sum_{k\in\mathbb{N}} |e^{-2\pi i 2^k \cdot} H_1(e^{2\pi i 2^k \cdot}(f * \psi_k)) - e^{2\pi i 2^k \cdot} H_2(e^{-2\pi i 2^k \cdot}(f * \psi_k))|^2)^{\frac{1}{2}}\|_{S^p} \\
& \text{by Lemma 3.5.9} \\
\leqslant & \|(\sum_{k\in\mathbb{N}} |H_1(e^{2\pi i 2^k \cdot}(f * \psi_k))|^2)^{\frac{1}{2}}\|_{S^p} + \|(\sum_{k\in\mathbb{N}} |H_2(e^{-2\pi i 2^k \cdot}(f * \psi_k))|^2)^{\frac{1}{2}}\|_{S^p} \\
\lesssim & \|(\sum_{k\in\mathbb{N}} |f * \psi_k|^2)^{\frac{1}{2}}\|_{S^p} \text{ by Theorem 3.5.4} \\
\lesssim & \|f\|_{S^p} \text{ by Theorem 3.5.8.}
\end{aligned}$$

□

This completes the proof of Theorem 3.1.1.

Concluding Remarks

The principal difficulty in establishing convergence results and operator bounds in S^p is the great lack of "standard" results in this setting, requiring well-understood theory to be constantly reviewed in a new context. Whilst many results pertaining to Fourier series have been long established to transfer from the setting of $L^p(\mathbb{T})$, albeit often with additional complications (for example, Parseval's identity), fundamental results pertaining to operator theory have been subject to very little study by the mathematical community.

Given a Marcinkiewicz-style interpolation result on S^p, the ℓ^2–valued boundedness of the modified Hilbert transforms on S^{2^k} given in Theorem 3.5.4 would immediately extend itself to S^p for $p \geqslant 2$. However, such an interpolation result seems unlikely, as classical abstract interpolation theory (as in [2], for example) establishes Marcinkiewicz's result in spaces that have re-arrangement* invariant norms. The supremum present in the Stepanov norm immediately negates this property.

Furthermore, if it were additionally the case that S^p and $S^{p'}$ were dual in the same way as for the L^p spaces, ℓ^2-valued boundedness of the modified Hilbert transforms on S^p, $p \geqslant 2$ would extend to all S^p with $p \in (1, \infty)$. However, the fact that $\{f \in L^p_{loc}(\mathbb{R}) : \|f\|_{S^p} < \infty\} \neq S^p$ would seem to immediately destroy any chance of the S^p spaces dualising in some "nice" way. It is even commented in [24] that the space S^p "does not appear to be

*In line with Chapter Two of [2], two $\mathbb{R}$-valued functions f and g on a measure space (X, μ) are understood to be re-arrangements of one another if $\mu\{x \in X : |f(x)| > \lambda\} = \mu\{x \in X : |g(x)| > \lambda\}$ for all $\lambda > 0$.

dual space".

All of this notwithstanding, the ℓ^2–valued boundedness of the modified Hilbert transforms is the only result presented in this thesis that is required for establishing boundedness of the maximal operator S^* that has not been proved for general $p \in (1, \infty)$. Consequently the natural first direction for further research on this subject would be to attempt to generalise Theorem 3.5.4.

There has been some study of the Hilbert transform in the context of Stepanov norms undertaken by Sumiyuki Koizumi. He introduces a "generalised Hilbert transform" of his own in [20] defined as follows:

Definition (Koizumi's Generalised Hilbert Transform) *For $p \in [1, \infty)$, let W_p denote the class of measurable functions such that $\int_{\mathbb{R}} \frac{|f(x)|^p}{1+x^2}\,dx < \infty$. Then for $f \in W_p$, the generalised Hilbert transform is defined as:*

$$\widetilde{H}f(x) := p.v.\ \frac{x+i}{\pi} \int_{\mathbb{R}} \frac{f(t)}{t+i} \frac{dt}{x-t}.$$

Koizumi studies this from the perspective of Stepanov norms in [21] and [22] (noting that for any $f \in L^p_{loc}(\mathbb{R})$ such that $\|f\|_{S^p} < \infty$, it is necessarily the case that $f \in W_p$), and in particular, claims in [22] to have proved a theorem equivalent to the following:

"Theorem" *For $p \in (1, \infty)$, let $f \in L^p_{loc}(\mathbb{R})$ be such that $\|f\|_{S^p} < \infty$. Then,*

$$\|\widetilde{H}f\|_{S^p} \leqslant A_p \|f\|_{S^p}$$

where $A_p = O\left(\frac{1}{p-1}\right)$ as $p \to 1$.

Given this result, it is possible to show that the regular Hilbert transform is bounded for $f \in S^p$, by observing that for such f, $\widetilde{H}f = Hf + M(\widetilde{H}f)$ (where M is the averaging operation from Definition 1.3.1). Noting that $M(\widetilde{H}f)$ is finitely determined by Koizumi's "Theorem" and Lemma 1.3.4, the boundedness of the regular Hilbert transform follows. This can trivially be used to show (scalar) boundedness of the modified Hilbert transforms considered in Theorem 3.5.4, and noting that the inductive part of the proof of that theorem only requires scalar boundedness, Theorem 3.5.4 would be extended to $p \geqslant 2$.

Unfortunately, the author believes that the proof of the above in [22] contains an error*, which he has been unable to repair to date.

Another potentially interesting generalisation of this research is the possibility of extending the various results to functions on $\mathbb{R}^d$, for $d > 1$ (or even a general Banach space). Whilst the concept of Bohr almost periodic functions in higher dimensions has been well studied (see, for example, [3] pp. 59–66), there has been practically no study of higher dimensional analogues of the more general almost periodic function spaces. In the case of the Besicovitch norms, there is perhaps a certain degree of ambiguity over the sense in which the limit should be taken, though in the Stepanov and Weyl spaces, an intuitive way of defining a higher-dimensional norm would seem to be clear. However, the extent to which the various results presented in Chapter One would transfer to this context would need to be subject to careful study.

*In particular, when dealing with the "J_3" term, the triangle inequality is used "backwards" on moving from the second line to the third.

Bibliography

[1] Andres, J., Bersani, A. M., and Grande, R. F., Hierarchy of Almost-Periodic Function Spaces, *Rendiconti di Matematica e delle sue Applicazioni (Serie VII)* **26**, *no. 2* (2006), pp. 121–88.

[2] Bennett, C., and Sharpley, R., *Interpolation of Operators* (Pure and Applied Mathematics, vol. 129), Academic Press, London, 1988.

[3] Besicovitch, A. S., *Almost Periodic Functions*, Dover, New York, 1954.

[4] Bochner, S., and Von Neumann, J., Almost Periodic Functions in a Group II, *Transactions of the American Mathematical Society* **37**, *no. 1* (1935), pp. 21–50.

[5] Bohr, H., Zur Theorie Der Fast Periodischen Funktionen (On a Theory of Almost Periodic Functions - in German), *Acta Mathematica* **45** (1924), pp. 29–127.

[6] Bohr, H., Zur Theorie Der Fastperiodischen Funktionen II (On a Theory of Almost Periodic Functions II - in German), *Acta Mathematica* **46** (1925), pp. 101–214.

[7] Bohr, H., *Almost Periodic Functions*, Chelsea, New York, 1947.

[8] Bohr, H., and Følner, E., On Some Types of Functional Spaces, *Acta Mathematica* **76** (1945), pp. 31–155.

[9] Bredihina, E. A., Some Problems in Summation of Fourier Series of Almost Periodic Functions, *American Mathematical Society Translations (Series 2)* **26** (1963), pp. 253–61.

[10] Bredihina, E. A., *К Теореме А. Н. Колмогорова О Лакунарных Частных Суммах Ряда Фурье* (Concerning A. N. Kolmogorov's Theorem on Lacunary Partial Sums of Fourier Series - in Russian), Сибирский Математический Журнал **9**, *no. 2* (1968), pp. 456–61.

[11] Carbery, A., Rubio de Francia, J. L., and Vega, L., Almost Everywhere Summability of Fourier Integrals, *Journal of the London Mathematical Society, Second Series* **38**, *no. 3* (1988), pp. 513–524.

[12] Corduneanu, C., *Almost Periodic Functions* (Interscience Tracts in Pure and Applied Mathematics, no. 22), Interscience, New York, 1968.

[13] Duoandikoetxea, J., *Fourier Analysis* (Graduate Studies in Mathematics, vol. 29), American Mathematical Society, Providence, Rhode Island, 2000.

[14] Fefferman, C., The Multiplier Problem for the Ball, *Annals of Mathematics. Second Series* **94** (1971), pp. 330–36.

[15] Fournier, J. J. F., and Stewart, J., Amalgams of L^p and ℓ^q, *Bulletin of the American Mathematical Society* **13***, no. 1* (1985), pp. 1–21.

[16] Grafakos, L., An Elementary Proof of the Square Summability of the Discrete Hilbert Transform, *American Mathematical Monthly* **101***, no. 5* (1994), pp. 456–8.

[17] Grafakos, L., *Classical and Modern Fourier Analysis*, Pearson Education, Upper Saddle River, New Jersey, 2004.

[18] Hewitt, E., and Ross, K. A., *Abstract Harmonic Analysis II* (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 152), Springer-Verlag, Berlin, 1970.

[19] Katznelson, Y., *An Introduction to Harmonic Analysis*, Dover, New York, 1976.

[20] Koizumi, S., On the Singular Integrals V, *Proceedings of the Japan Academy* **35***, no. 1* (1959), pp. 1–6.

[21] Koizumi, S., Hilbert Transforms in the Stepanoff Space, *Proceedings of the Japan Academy* **38** (1962), pp. 735–40.

[22] Koizumi, S., On an Integral Inequality of the Stepanoff Type and its Applications, *Proceedings of the Japan Academy* **42** (1966), pp. 896–900.

[23] Kolmogorov, A. N., Une Contribution à l'Étude de la Convergence des Séries de Fourier (A Contribution to the Study of the Convergence of Fourier Series - in French), *Fundamenta Mathematicae* **5** (1924), pp. 96–7.

[24] LaVarnway, G. T., and Cooke, R., A Characterization of Fourier Series of Stepanov Almost-Periodic Functions, *The Journal of Fourier Analysis and Applications* **7***, no. 2* (2001), pp. 127–42.

[25] Levitan, B. M., *Почти Периодические Функции (Almost Periodic Functions - in Russian)*, Государственное Издательство Технико-Теоретической Литературы, Moscow, 1953.

[26] Levitan, B. M., and Zhikov, V. V., *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

[27] Montgomery, H. L., *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis* (Regional Conference Series in Mathematics, no. 84), American Mathematical Society, Providence, Rhode Island, 1990.

[28] MONTGOMERY, H. L., AND VAUGHAN, R. C., Hilbert's Inequality, *Journal of the London Mathematical Society, Second Series* **8**, *no. 1* (1974), pp. 73–82.

[29] RUDIN, W., *Real and Complex Analysis*, third ed., McGraw-Hill, Singapore, 1987.

[30] RUDIN, W., *Fourier Analysis on Groups*, Wiley-Interscience, New York, 1990.

[31] STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[32] STEPANOV, V. V., Sur Quelques Généralisations des Fonctions Presque Périodiques (On Some Generalisations of Almost Periodic Functions - in French), *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* **181** (1925), pp. 90–92.

[33] TORCHINSKY, A., *Real-Variable Methods in Harmonic Analysis*, Dover, New York, 1986.

[34] VON NEUMANN, J., Almost Periodic Functions in a Group I, *Transactions of the American Mathematical Society* **36**, *no. 3* (1934), pp. 445–92.

[35] ZAIDMAN, S., *Almost-Periodic Functions in Abstract Spaces* (Research Notes in Mathematics, vol. 126), Pitman, Marshfield, 1985.

[36] ZYGMUND, A., *Trigonometric Series*, (vol. 1 and 2), second ed., Cambridge University Press, Cambridge, 1959.