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Partition functions in superstring theory and SQCD

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Declaration

Part of this dissertation is based on work that was carried out in collaboration with Dieter Lust, Noppadol Mekareeya and Oliver Schlotterer in 2012 and was published in the following paper:

- [7] D. Lust, N. Mekareeya, O. Schlotterer and A. Thomson, "Refined Partition Functions for Open Superstrings with 4, 8 and 16 Supercharges," Nucl. Phys. B **876** (2013) 55 [arXiv:1211.1018 [hep-th]].

The work above has not been submitted by me for any other degree, diploma or similar qualification. Furthermore, I herewith certify that, to the best of my knowledge, all of the material in this dissertation which is not my own work has been properly acknowledged.

Andrew James Thomson

Abstract

In this thesis we apply the methods of partition functions to massive superstring spectra and the moduli spaces, or spaces of zero-energy configurations, of supersymmetric QCD gauge theories.

In the first part of this thesis we consider the massive covariant perturbative superstring spectra of compactifications of the type I open superstring preserving 4, 8 or 16 supercharges. There are an enormous number of ways in which the required amount of symmetry can be obtained, but here we concentrate on the 'universal' states that are present in every possible compactification preserving that amount of supersymmetry. For each super-Poincaré representation we derive the multiplicity generating function, or the power series counting the number of times that representation occurs at each mass level, and from these we derive empirically the stable pattern or leading Regge trajectory that these multiplicity generating functions approach in the limit of large spin. For the mathematically tractable and phenomenologically relevant case of 4 supercharges we also derive these power series analytically and see that they agree with the empirical ones.

In the second part we introduce the type of partition functions called Hilbert series, which count the number of algebraically or linearly independent polynomials at each graded level of a graded algebraic structure such as a (graded) ring, module or ideal. In supersymmetric gauge theories the algebraic structure is the chiral ring which is generated by the gauge-invariant operators of the theory. The specific theories we consider are supersymmetric generalizations of QCD, or SQCD, with exceptional or related (by sequence or folding of the Dynkin diagram or Higgsing) gauge groups with specified numbers of flavours of matter in specific representations. We show, as for theories with classical gauge groups, that the moduli spaces are Calabi-Yau manifolds and also demonstrate relations between the Hilbert series of SQCD theories related by Higgsing on one or more flavours of matter in specific representations.

Contents

1 Introduction and Outline

In this section we will set the scene for this thesis by introducing string theory and supersymmetric gauge theory which are the central themes of the two main sections. We will not present a complete introduction to the subjects, but rather set the scene by describing some of the historical origins of the theories; this discussion will largely follow chapter 1, section 1.1 of [53]. We will then discuss how they flow into the more specific subjects discussed in this thesis.

The current understanding of physics is predicated on quantum field theory, which is quantum mechanics with observables being functions of the spacetime coordinates. It has been known since Maxwell's time that the electromagnetic interaction can be described by a quantum field theory, indeed a gauge theory mediated by the photon, which is massless and chargeless. This theory is called quantum electrodynamics, or QED. However, the weak and strong interactions presented further challenges to being described in such a way.

The weak interaction, as observed in beta decay (both β^- and β^+) and electron capture, was originally proposed as being described by the interaction of four fermions at the same spacetime point, or a current-current interaction. However, this is not renormalizable, because it requires a coupling constant of mass dimension -2. The difficulty with making it a gauge theory was that it is a short-range interaction and the gauge bosons would have to be massive; however massive gauge bosons could only be produced as a result of spontaneous symmetry breaking, where the vacuum state of the theory does not possess the full symmetry of the Lagrangian. In this specific example, a scalar doublet $\mathbf{H} = \binom{H^+}{H^0}$ (with H standing for Higgs) is introduced along with a potential $V(H)$ which has a minimum on the circle $|\mathbf{H}| = v$ for some v; we have to fix the vacuum, so we pick $\mathbf{H} = \binom{0}{v}$ and break the $SU(2) \times U(1)$ symmetry down to $U(1)$ following the procedure outlined in [41].

It was not known that renormalizability was preserved when the theory underwent spontaneous symmetry breaking, but 't Hooft proved that it was, making the theory consistent. Therefore the electromagnetic and weak interactions were unified, and both described by (the same) quantum field theory.

The 'strong interaction' originally referred to the interaction between protons and nucleons, together called nucleons, in atomic nuclei, which is mediated by pions, which have mass around 100 MeV; however this is now often referred to as the 'residual' strong interaction, with the strong interaction being that between quarks. Though the gauge group is the non-abelian $SU(3)$, the gauge bosons, or gluons, are massless (though six of the eight are charged) and the theory is easily described by a QFT, though since (6 of 8 of) the gluons are charged, *confinement* occurs and quarks are never observed free but only in combinations of 3 quarks (baryons), 3 antiquarks (antibaryons) and one of each (mesons). The theory of the strong interaction is called quantum chromodynamics (as the degrees of freedom are called *colours*), or QCD for short.

The electroweak and strong interactions were grouped together as the Standard Model. However, the scalar particle (or rather weak isospin $SU(2)$ doublet of particles) were still to be discovered, and the origin of fermion masses still needed explaining.

Neutrinos have only ever been observed left-handed, and antineutrinos right-handed, which is why they were originally assumed to be massless, because massive particles can always be Lorentz transformed into a frame in which their helicity would be reversed. Therefore, left-handed electrons and neutrinos were grouped together in an $SU(2)$ weak isospin doublet with hypercharge, which is the charge under the $U(1)$ of the Standard Model, $Y = -1$, while right-handed electrons form a weak isospin singlet with $Y = -2$ (we have $Q = I_3 + \frac{1}{2}Y$, where I_3 is the third component of isospin, weak or other). The theory is not *parity-invariant*.

Although quarks have always been considered to be massive, not that they have ever been observed free, the same construction was used, grouping the left-handed quarks (u, d), (c, s) and (t, b) into isospin (not weak here!) doublets with $Y = \frac{1}{3}$ and assigning the right-handed ones to singlets with hypercharge equal to twice their charge.

The kinetic term in the Dirac Lagrangian does not mix the left- and right-handed parts of the Dirac field, as we can see by the expansion of $\bar{\Psi_L}$. However, an explicit mass term would mix them and is therefore forbidden, since they transform in different representations of the (weak) isospin $SU(2)$.

$$\bar{\Psi_L} = \Psi_L^\dagger \gamma^0 = \frac{1}{2}\Psi^\dagger(1-\gamma_5)\gamma^0 = \frac{1}{2}\bar{\Psi}(1+\gamma_5) \tag{1.1}$$

Therefore, $\bar{\Psi_L}\Psi_L = 0$ and similarly for the right-handed part.

The Higgs field must form a doublet of the (weak) isospin $SU(2)$ so that its Yukawa-like interaction term with the left- and right-handed quark or lepton fields is a (weak) isospin singlet.

The Higgs mechanism, which breaks the $SU(2) \times U(1)$ gauge symmetry of the electroweak part of the Standard Model to the electromagnetic $U(1)$, which is a combination of the I_3 (the Cartan subalgebra $U(1)$ of (weak) isospin) and the hypercharge, also gives rise to fermion masses via spontaneous symmetry breaking with the masses proportional to the VEV of the Higgs field. The Standard Model was completed when the Higgs boson was discovered in 2012 at the LHC, having mass 126 GeV.

However, it was clear that the Standard Model could not describe all of fundamental physics. The first and most immediately obvious problem is that it does not include gravity, which is clearly an essential part of any fully unified theory. In any case, a quantum theory of gravity is not renormalizable, having short-distance divergences. We will leave this for now and discuss the other issues first.

Firstly, the theory is very arbitrary: why do these specific patterns, called multiplets, of masses and charges of particles occur? Indeed there are as many as 19 free parameters! Secondly, some parameters have values much smaller than they would be expected to; why this is is not known and it does not seem natural. A specific example is the difference of 17 orders of magnitude between the electroweak (Higgs) scale and the Planck scale at which quantum effects should be observed in gravitational interactions; this is known as the *hierarchy problem*. Other issues are the cosmological constant problem wherein the theory predicts a value an enormous 10^{120} times larger than the limits set by current observations, the fact that the theory does not account for either the 'dark matter' that makes up a quarter of the universe or the 'dark energy' that makes up another 70%, and the fact that the gauge couplings of the three interactions do not all meet at the same value at any energy, which is necessary for unification to occur.

A more specific problem is the 'solar neutrino problem', where only one third of the expected number of (electron) neutrinos were detected coming from the Sun in two separate experiments at detectors in South Dakota and in Japan. It was proposed that this could be resolved by the neutrinos 'oscillating' between the three types (electron, muon and tau neutrinos), however this is only possible if the neutrinos have mass. While this has been proven to be true by other experiments too, their masses are known to be extremely low, several orders of magnitude below those of the charged leptons, leaving the problem of explaining the origin and order of magnitude of their masses. It has been proposed that these low masses result from a 'see-saw' mechanism whereby a term in $-M\bar{\nu_R^c}\nu_R$, with M large and where the superscript c denotes charge conjugation, is added to the Lagrangian and this leads to two particles, one the neutrino with mass $\sim m_e^2 M^{-1}$, where m_e is the charged lepton mass, and one of mass $\sim M$, however owing to the low mass of the neutrino itself particles of mass M should be far too heavy to observe at the LHC.

Returning to more general (still non-gravitational) issues with the standard model, three ways have been proposed to resolve them. One is grand unification, in which the three gauge groups of the standard model are combined into one, usually $SU(5)$, $SO(10)$ or E_6 (it must have complex representations); this gave an accurate prediction of the weak mixing angle and the bottom/tau mass ratio. Another is extra dimensions, in which the theory is defined on a spacetime containing more than four dimensions. This was originally introduced by Kaluza and Klein to combine gravity and Maxwell's electromagnetic theory into one theory in which the fifth, and compact, dimension contained the electromagnetic information. In current theories there are usually more than one extra dimension, and they can be small or large. The reduction to the required four dimensions can be done in many different ways giving many different four-dimensional theories.

For the third way, we first note that the Coleman-Mandula 'no-go' theorem states that it is not possible to extend the Poincaré, or Lorentz plus translations, group or algebra to include an 'internal' symmetry group except in the trivial way; however it was discovered (actually by two groups in the USSR before Wess and Zumino's 'official' discovery) that this could be circumvented by allowing the algebra to be extended to a 'graded algebra', or 'superalgebra', in which fermionic generators and anticommutation relations were allowed. This graded algebra was called supersymmetry, or the super-Poincaré algebra.

In supersymmetric theories, each particle has a *superpartner* of opposite type (boson/fermion) but (if the symmetry is unbroken) the same mass. Supersymmetry provides a solution to the hierarchy problem, because the divergences from the Feynman diagrams cancel, at least partially accounts for dark matter (but not dark energy) as consisting of the lightest supersymmetric particle (LSP), which is necessarily stable owing to conservation of *R-parity* (standard model particles have even R-parity and their superpartners have odd R-parity), and reduces the discrepancy in the cosmological constant (though only to 10^{60}!).

However, supersymmetry creates its own difficulties. Firstly it must be a broken symmetry, since superpartners are not observed in nature. Secondly, it has been predicted to be broken at the TeV scale at which signatures should be visible at the LHC, but no superpartners have been observed yet, which suggests that either the breaking scale must be higher or that supersymmetry does not actually occur in nature. In the latter case, those current theories which make use of it, of which there are many, must be completely re-thought and alternative solutions sought to the hierarchy problem and other shortcomings of the (non-supersymmetric) standard model. We will however in this report assume that supersymmetry does occur in nature and ignore issues relating to its breaking.

We will now revisit the problem of the absence of gravity from the Standard Model.

We know that a quantum field theory of gravity is not renormalizable, because the graviton has spin 2 and that like the 4-fermion interaction originally proposed for the weak interaction it would have a coupling constant of mass dimension -2. As with the 4-fermion interaction theory, this is interpreted as though the current theory, where the Einstein-Hilbert action is the only term present, is an effective theory valid only below some scale, in this case the Planck scale, and there is a need for new physics at higher energies. One possibility is that the divergence is an artefact of the perturbative expansion about 0 and is absent when the theory is treated exactly. In renormalization group language of QFT, this would mean the theory would have a non-trivial UV fixed point. Another way is to soften the interaction by smoothing it out in space and time. We do not know if the theory has a UV fixed point, but there is a historical precedent for concentrating on the second option, namely the resolution of the 4-fermion weak interaction to a gauge theory mediated by massive bosons.

However, working out how to smooth out the theory is very complex, because by Lorentz invariance a smearing in space also means one in time and this could violate causality along with unitarity and other properties. There is only one way to smooth out the divergences while keeping Lorentz invariance, and this is string theory, with objects extended in one spacetime dimension, though they may be open, with ends, or closed, in a loop. This is the only case where both the spacetime and internal degrees of freedom can be kept under control, as quantizing membranes, with 2 extended spatial dimensions, gives rise to a continuous spectrum.

String theory, through the presence of the graviton, has (quantum) gravity built in, while in other theories, such as loop quantum gravity, it must be treated separately and bolted on piecemeal. String theory also gives rise to extra dimensions (as we will see, superstring theory is constrained to have 10 dimensions, with bosonic string theory having 26!) and GUT gauge groups (at least through heterotic string theory) and allows chiral gauge couplings, and also has no free parameters except the string scale. It is a unique theory, in which consistency forbids adding terms to the Lagrangian by hand. String theory also has the benefit that multiple Feynman diagrams in the various QFTs correspond to the same string interaction.

String theory has its issues too. Although it is a unique theory, it has a vast (10^{500}) set of possible vacua, called the *landscape*. Also, a non-perturbative formulation has not been fully described. However, progress has been made with the discovery of D-branes, which were initially sub-spaces on which open strings can end, and the various duality relations (S-, T- and U-duality, the last of which is the union of the first two) relating the five consistent superstring theories with each other and with an 11-dimensional theory called M-theory. S-duality relates strong and weak coupling, and T-duality relates compactification at large and small radii. (The D is for Dirichlet, which describes boundary conditions on the positions of the endpoints of open strings rather than their momenta; the latter are called Neumann. A D-brane with p spatial dimensions is called a Dp-brane.)

The extra dimensions also mean that to get the observed four dimensions of nature, one must *compactify* to small size or otherwise make invisible the other six (or 22!). One can do this in two ways; other than compactification, the other is the *brane-world scenario* where matter and the forces other than gravity are described by open strings and their ends are constrained to lie on D-branes, while gravity is described by closed strings which can escape from the branes, which could account for its great weakness relative to the other forces. These branes must have 3 spatial dimensions and are therefore D3-branes. We will not discuss brane-world scenarios further in this report but rather concentrate on compactification as the means to reduce the visible dimensionality of spacetime.

Another issue is that although in principle string theory amplitudes require fewer calculations than QFT ones, in practice they are very difficult to calculate. In 1997, Maldacena discovered [64] the AdS/CFT correspondence, which is a specific case of a more general gauge/gravity duality. In the general duality, a gravity theory in $d+1$ dimensions dual to gauge theory in d dimensions; normally we say that the gravity theory is in the *bulk* and the gauge theory is on its boundary. In the specific case of AdS/CFT, $AdS^d \times X^{10-d}$ is dual to a CFT in $d-1$ dimensions probing the singularity of the cone over X^{10-d}. The CFT is usually represented by D$(d-2)$-branes and X^{10-d} is a Sasaki-Einstein manifold, or one over which the cone is a singular Calabi-Yau (for d odd) preserving only some of the supersymmetry (one quarter, or 8 supercharges, for $d=5$), although it could be S^{10-d} with the cone being $\mathbb{R}^{11-d}$ preserving all 32 supercharges. The same structure exists with M-theory, with 11 instead of 10 and M2-branes corresponding to $AdS_4 \times X^7$.

Having introduced the generalities of string theory and supersymmetric gauge theory, we will now discuss the specifics of this thesis, which are the use of partition functions in both superstring theory and a type of supersymmetric gauge theory called supersymmetric QCD, or SQCD. (QCD without the ‘super’ is the theory of the strong interaction; here, as well as adding supersymmetry, we generalize it to allow any gauge group and any number of flavours of matter which can be in any representation of the gauge group as long as they do not give rise to an anomalous theory.)

Partition functions are a tool borrowed from statistical mechanics where they are used to derive expressions for quantities such as temperature and chemical potential in terms of derivatives the partition function; they have been applied to (super)string theory to obtain expressions for these quantities applied to black holes. They are like a trace of $\exp(-\beta H)$, where β is the reciprocal of the temperature (times a constant) and H is the Hamiltonian which has an expression in terms of raising and lowering operators, over all the states in the Fock space of the theory, which is built up by acting repeatedly with raising operators on the ground state.

In the supersymmetric gauge theories that we discuss in the second part of the thesis, if when the fully unrefined Hilbert series, the gauge theory name for a partition function (though, as we will see, not all partition functions are Hilbert series), is written as a rational function the numerator is palindromic, the moduli space is a Calabi-Yau manifold [1, 2, 3].

Most partition functions in the literature are *unrefined* and simply count the states at each level, which is specified by the mass, number of fields, etc. However, we can get more information about the states that comprise each level, and the representations of the characteristic group(s) of the theory in which they transform, by *refining* the spectra. We introduce new fugacities that distinguish the states from each other and, knowing the group and the map between Dynkin labels and fugacities, decompose each level into representations.

As well as in the contexts discussed here, refined partition functions, although for finite groups, are used in investigating moonshine conjectures, where they are variously known as twining characters [66, 67] and twisted elliptic genera [68, 69, 70]. There are also so-called McKay-Thompson series [65], which are not 'refined' series in the true sense as they are modular forms in a single variable, though they are obtained similarly to refined series by replacing the dimensions of finite group representations with the characters of a specific group element in each corresponding representation.

This thesis is divided into three parts, one short and two long. The first, short, section introduces some of the mathematical preliminaries that we will use in the other two sections. We discuss algebraic structures (rings, modules, ideals) with (possibly multi) gradings (different to those of graded Lie algebras in the sense of supersymmetry), Hilbert series which are partition functions counting the number of (algebraically or linearly) independent polynomials at each graded level, the different types of symmetric polynomials and the identities relating them, the plethystic formalism with the bosonic and fermionic plethystic exponential and logarithm, finite and Lie group characters, *Haar measures* that enable one to integrate over a whole Lie group manifold via the simpler integration over the maximal torus, invariant theory and Molien's sum formula for finite groups and the (Molien-)Weyl integral generalizing this sum formula to Lie groups.

The second part, or the first 'main' section, discusses perturbative string spectra. We begin this part of the thesis by introducing string theory from action principles, following [53, 54]. We quantize the string, concentrating on the light-cone method, though we also discuss two other methods of quantization called old covariant quantization and BRST quantization. We derive the zero-point energy and from that the condition on the number of dimensions using all three methods, demonstrating the last two because the derivation is more rigorous and less heuristic in these cases. Returning to light-cone quantization, we then introduce the use of plethystics to obtain refined string spectra based on [8], concentrating on the bosonic and type I superstring, without the Chan-Paton factors at the ends, though we do discuss closed type II superstring spectra, obtained by tensoring two type I superstrings together and imposing level matching, incorporating the Chan-Paton factors into type I superstrings, and the heterotic string, and (briefly) compactification of one spatial dimension on a circle as in [56]. Having demonstrated properties of the spectra, both bosonic and superstring, such as stable patterns, which we will define, we then move on to a systematic treatment of superstring spectra, following [7], concentrating on the open type I superstring, again without the Chan-Paton factors, in compactifications with 4, 8 and 16 preserved supercharges. We discuss methods by which those numbers of supercharges can be obtained, but we concentrate on the *universal* states present in all such compactifications.

The second part of the thesis relates to *Hilbert series* of supersymmetric QCD theories with exceptional gauge groups. Hilbert series are similar to partition functions but here they are used to count not string states for a given mass level but rather gauge-invariant quantities with a given number of fields of each type ((anti)fundamental, adjoint, spinor etc). One starts with the basic fields in the specified representations of the gauge and global symmetry groups, takes symmetric products (antisymmetric if they are fermionic) to arbitrary levels using a formalism called plethystics (here taking the plethystic exponential), imposes any *F-term relations* specified by a quantity called the superpotential (which is zero in SQCD hence there are no F-term relations, but most SUSY gauge theories do have them), and obtains an expression for the gauge-invariant quantities by integrating over the gauge group using the Haar measure. This gives the Hilbert series for the theory; one can then take the plethystic logarithm (here with only global symmetry group representations) to obtain explicit expressions for the generators, relations and *higher syzygies* of the theory, which determine whether the moduli space is freely generated (only generators), a complete intersection (generators and relations) or neither (there are higher syzygies). The Hilbert series itself, in unrefined form, can also be used to determine whether the moduli space is Calabi-Yau, if it has a palindromic numerator when expressed as a rational function in a specific form. We derive the Hilbert series for exceptional and related groups with specified numbers of flavours of matter in specified representations, and derive relations between the Hilbert series that relate to Higgsing of the group and/or folding of the Dynkin diagram. Some Hilbert series are harder to obtain than others, but it is often the case that two different SUSY gauge theories give the same Hilbert series, this is called duality (examples include *Seiberg duality*); it is often useful to use known dualities to conjecture new ones when it is known to be 'hard' to compute the Hilbert series for one theory and 'easier' to compute that for the actual or conjectured dual theory. We do not discuss Seiberg or other duality in this thesis, apart from a discussion at the end.

2 Symmetric polynomials and the plethystic programme

In this section we introduce the machinery that we use in the two main sections of this thesis. We start by introducing the algebraic preliminaries such as rings and modules, and then we discuss first the concept of symmetric polynomials and that they form an algebraic structure called a ring, and then their five different types, also introducing antisymmetric polynomials, in the latter part because of the need to use them to define Schur polynomials, which are the last type of symmetric polynomial to be introduced and the most difficult to visualize. We then introduce the concept of characters of group representations, which simplify representation theoretic computations greatly, and how they differ between Lie groups and finite groups, though they are conceptually the same. We then return to the symmetric polynomials and derive identities for those in two variables in terms of those in each variable separately, and then introduce the plethystic programme, a formalism for generalizing this (anti)symmetrization and enabling results to be obtained through other methods such as residues and (Taylor/Laurent) series expansion. We finish by introducing the *Haar measure* and *(Molien-)Weyl integral*, which are used to generalize Molien's sum formula for Hilbert series for finite groups to Lie groups, using the fact that any element of a Lie group is conjugate to an element of its maximal torus which is generated by the Cartan subalgebra and thus has the same character.

2.1 Preliminaries

In this section we will introduce some of the algebraic preliminaries that we will use in the rest of this thesis.

A ring R is an algebraic structure endowed with two binary operations:

addition, under which the elements of the ring form an abelian group, and multiplication (which is not necessarily commutative), with both left and right multiplication being distributive over addition:

$$a.(b + c) = a.b + a.c \quad \forall \quad a, b, c \in R \tag{2.1}$$

We normally suppress the . symbol for rings (but not for modules which we will describe later). If every non-zero element of the ring has a multiplicative inverse, the ring is a field.

A graded ring R is a ring with a grading:

$$R = \oplus_i R_i \tag{2.2}$$

$$r_i \in R_i, \quad r_j \in R_j \implies r_i r_j \in R_{i+j} \tag{2.3}$$

A module M over a ring R, called an R-module, is an algebraic structure endowed with addition and (left) multiplication by elements of the ring, which is again distributive over addition of elements of the module. A trivial example of an R-module is of course the ring R itself.

An ideal I is a subset of a ring with the following property: if $a, b \in I$ and $r, s \in R$, $ra + sb \in I$. An ideal is called finitely generated if every element can be written as a linear combination of finitely many basis elements, and principal if only one such element is required. It is freely generated if said linear combination is unique. The ring itself is trivially an ideal of itself.

Like rings, modules and ideals can be graded, with the multiplication being an element of the ring times an element of the module or ideal.

A Hilbert series is a power series that counts elements in a graded ring, module or ideal, weighted by the grading. For a freely generated structure, the Hilbert series, when expressed as a rational function of two polynomials, has numerator 1.

2.2 Symmetric polynomials

In this section we present a brief introduction to the symmetric polynomials. Much of this section and Section 2.4 is based on [23], which is a simplified version of the introductory parts of [45].

For a set of variables x_i, symmetric polynomials are those which remain invariant under the symmetric group S_n, where n is the number of variables, in particular they do not change under exchange of two variables x_i and x_j for $i \neq j$. The symmetric group is generated by these two-variable swappings which are called transpositions.

The symmetric polynomials form a ring which is called Λ in [23]. There are five types of symmetric polynomials, of which the first three are as follows:

- the complete (or full) symmetric polynomials $h_n(\mathbf{x})$, which are sums of every possible product of n of the x_i, not necessarily distinct:

$$h_n(\mathbf{x}) = \sum_{1 \leq i_1 \leq \ldots \leq i_n \leq m} \prod_{j=1}^{n} x_{i_j} \tag{2.4}$$

- the elementary symmetric polynomials $e_n(x)$, which are sums of every possible product of n distinct x_i:

$$e_n(\mathbf{x}) = \sum_{1 \leq i_1 < \ldots < i_n \leq m} \prod_{j=1}^{n} x_{i_j} \tag{2.5}$$

- the power sum symmetric polynomials (or Newton polynomials) $p_n(x)$, which are sums of the n-th powers of the x_i:

$$p_n(\mathbf{x}) = \sum_{i=1}^{m} x_i^n \tag{2.6}$$

All these polynomials can be generalised to a general partition λ, e.g.

$$h_\lambda(\mathbf{x}) = \prod_{i=1} h_{\lambda_i} \tag{2.7}$$

and the same for $e_\lambda(\mathbf{x})$ and $p_\lambda(\mathbf{x})$.

The fourth type of symmetric polynomials are the monomial symmetric polynomials, which can only be defined in terms of partitions rather than integers, though a partition can of course consist of only one integer. They are sums of every possible product of the x_i with a particular 'shape' specified by a partition λ, explicitly this can be written as

$$m_\lambda(\mathbf{x}) = \sum_{i_j \neq i_k, j \neq k} \prod_j x_{i_j}^{\lambda_j} \tag{2.8}$$

The final type of symmetric polynomials are the Schur polynomials, which can also only be defined in terms of partitions. They are denoted $s_\lambda(x)$. They are defined as

$$s_\lambda(\mathbf{x}) = \frac{\det(x_i^{\lambda_j+n-j-1})_{i,j=1}^n}{\det(x_i^{n-j-1})_{i,j=1}^n} \tag{2.9}$$

The denominator is called the Vandermonde determinant $\Delta(x)$. n denotes the number of x_i and λ must not have more than n non-zero entries here otherwise the matrix must be extended with zeroes and hence the denominator would be zero.

To visualise Schur polynomials, take the Young diagram corresponding to the partition λ and write down all possible semi-standard Young tableaux for that diagram, i.e. all tableaux with all entries between 1 and n and increasing weakly from left to right across a row and strictly from top to bottom down a column, and then take the sum of all terms which are the product of all the x_i for each entry i in the tableau.

As we will see in Section 2.3, Schur polynomials are characters for representations of unitary groups $U(n)$ where n is the number of x_i.

All five types of symmetric polynomial form a basis for the ring of symmetric polynomials Λ as defined in [23], with those where $|\lambda| = n$ forming a basis for the n-th graded piece Λ_n where $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$.

There are invertible matrices, not all both integer and with integer inverses, which allow one to convert between the types of symmetric polynomials. For example, we have

$$s_\lambda(\mathbf{x}) = \sum_{\mu, |\mu|=|\lambda|} m_{\lambda\mu} m_\mu(\mathbf{x}) \tag{2.10}$$

where $m_{\lambda\mu}$ are called the Kostka numbers.

We will return to symmetric polynomials in Section 2.4, but for now we will leave them behind and instead first introduce Lie groups and their representations and characters.

2.3 Lie groups, representations and characters

A Lie group is a continuous group with the structure of a manifold.

A Lie group G is generated by its corresponding Lie algebra $\mathfrak{g}$, which consists of elements $T^a, 1 \leq a \leq dim(G)$ with commutation relations $[T^a, T^b] = if_c^{ab}T^c$ where f_c^{ab} are called the structure constants. For an abelian group the structure constants vanish. The *dimension* $d = dim(G)$ is the total number of generators.

The *rank* $r = rank(G)$ is the dimension of the maximal torus of the group, which is generated by the maximal commuting subalgebra, which is called the Cartan subalgebra. The maximal torus is isomorphic to $U(1)^r$. The elements of the Cartan subalgebra can be relabelled as $H_i, 1 \leq i \leq rank(G)$, and the other elements of the group written in terms of *roots* E_α:

$$[H_i, H_j] = 0 \tag{2.11}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.12}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \tag{2.13}$$

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i \tag{2.14}$$

where $N_{\alpha\beta}$ vanishes if $\alpha + \beta$ is not a root.

The weights of a given root are determined by their commutators with the H_i. The roots corresponding to the Cartan subalgebra have zero weight in any basis. The non-zero roots, of which there are $d - r$ of them, can be divided into positive and negative roots, with the negative of a positive root being negative and vice versa. In a Cartesian basis, the positive roots are those for which the first non-zero entry is positive; one can then choose r of those positive roots to be the simple roots, in terms of which the positive roots can all be expressed using only positive (integer) coefficients.

A Lie group can be described by its Dynkin diagram, which consists of nodes linked by 0, 1, 2 or 3 lines, with an arrow pointing to the shorter root in the cases of 2 and 3 lines, which determine the relations between the roots. These are displayed in the *Cartan matrix*, whose entries are given by, in terms of the simple roots,

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{2.15}$$

with α_i and α_j the i- and j-th simple roots and $(.,.)$ an inner product to be defined later. It is usually the case however that the diagram is drawn first (which is especially useful when the Dynkin basis, as defined later, is used for the roots and weights), and the Cartan matrix written down later. The diagonal entries are all 2; the off-diagonal entries are 0 when the roots corresponding to the row and column are not linked by any lines, both -1 when linked by one line, and -1 and -(the number of lines) when lined by more than 1 line, with the more negative number in the row corresponding to the longer root.

The Dynkin diagram is constrained by the fact that the simple roots must be linearly independent, and this leads to a restriction to the observed families A_n, B_n, C_n, D_n and exceptional cases E_6, E_7, E_8, F_4, G_2. The subscript n, or the number of nodes in the Dynkin diagram, is equal to the rank of the group.

There are two bases commonly used to write down the roots of the group and weights of its representations:

- The Dynkin basis, where the i-th simple root α_i is specified by the i-th row of the Cartan matrix, the i-th fundamental weight ω_i by the Cartesian basis vector e_i and the inner product (in a weight basis) by $(\alpha, \beta) = \alpha_i G_{ij} \beta_j$ (where α_i is the i-th component of α, not the i-th simple root here, and the indices are summed over) where G_{ij} is given by
$$G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2} \tag{2.16}$$
where A_{ij} is the Cartan matrix.

- The Cartesian basis, where the inner product is the usual Cartesian one, the simple roots are chosen to fit the Cartan matrix and the fundamental weights are chosen so that
$$(\alpha_i, \omega_j) = \frac{\delta_{ij}(\alpha_j, \alpha_j)}{2} \tag{2.17}$$
where (α, β) is the usual Cartesian inner product.

In the Dynkin basis, the rows of the Cartan matrix represent the simple roots. The remaining positive roots can be built up iteratively; when the i-th entry of a positive (simple or other) root is negative, the roots obtained by progressively adding the i-th row of the Cartan matrix to the 'original' root are added to the list of positive roots (if not already present) and the process repeated with these new roots. The process stops for a given 'original' root when a root has no negative entries. The highest root, which is unique, is the one at the highest level, i.e. the one for which the number of simple roots that must be added to the zero root to arrive at it is the greatest.

The Cartesian basis is most commonly used with groups in the $U(N)$ (but not $SU(N)$), $SO(N)$ and $Sp(N)$ families.

A representation of a Lie group is specified uniquely by the Dynkin labels of its highest weight. There is one Dynkin label for each node of the Dynkin diagram. Each label represents the coefficient of the corresponding fundamental weight in the highest weight Λ of the representation:

$$\Lambda = \sum_{i=1}^{r} n_i \omega_i \tag{2.18}$$

where ω_i are the fundamental weights. In the Dynkin basis $\Lambda_i = n_i$.

Given the highest weight of a representation, all the weights, with their multiplicities, can be constructed by the reverse of the construction of the positive roots from the simple ones, progressively subtracting simple roots until no further subtractions are possible. In this construction, the number of times the i-th simple root α_i has to be subtracted from a given weight λ is given by the i-th entry in the weight in the Dynkin basis and $2\frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$, with the usual Cartesian inner product, in the Cartesian basis. All progressive subtractions are added to the list of weights (if not already present) and the process of subtraction of simple roots repeated on these new weights.

This construction only gives the weights, not their multiplicities; these can be calculated by assigning multiplicity 1 to the highest weight Λ and progressively calculating multiplicities of lower weights (as determined by 'level', i.e. the numbers of simple roots that must be subtracted from Λ to give a specified root λ; here, as opposed to the construction of the positive roots, lower roots have higher level) in terms of those of higher weights using Freudenthal's recursion formula, which is given by

$$((\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho))n_\lambda = 2 \sum_{\alpha \in \Delta_+} \sum_{k \geq 1} n_{\lambda + k\alpha}(\lambda + k\alpha, \alpha) \tag{2.19}$$

where n_λ is the multiplicity of weight λ (the highest weight Λ is understood here) and ρ is half the sum of all the positive roots, which is equal to the sum of the fundamental weights. This construction is outlined in [24].

In the next subsection we will introduce characters, which simplify calculations involving group representations and enable them to be manipulated algebraically.

2.3.1 Characters of group representations

The character of a group representation is taken as the trace of the matrices representing each element of the group. By cyclic invariance of traces, the character is the same for every element of a conjugacy class, i.e. the set of all group elements conjugate to a given element g, $[g] = \{hgh^{-1}; h \in G\}$.

There are a finite number of irreps of a finite group; by Schur's lemma the number of irreps is the same as the number of conjugacy classes of elements, and the squares of their dimensions add up to the dimension of the group.

The identity element is always in a conjugacy class by itself. For an abelian group, the same is true of every element.

For Lie groups, every element is conjugate to a (not necessarily unique) element of the maximal torus and hence the character can be expressed in terms of a number of parameters given by the rank of the group. These parameters are called chemical potentials in analogy to the term used in statistical mechanics; their exponentials are called fugacities. (Sometimes fugacities are called chemical potentials in an abuse of notation.) In this thesis and the papers on which this thesis is based and took its inspiration from, [7, 8, 1, 2, 3], fugacities are preferred, though chemical potentials are used directly in older literature such as [10].

The above construction of all the weights of a group representation, with their multiplicities, can be converted into a character by, for each weight, adding a term corresponding to each fugacity raised to the power of the corresponding entry in the weight, multiplied by the multiplicity. However, having defined fugacities, we can now introduce the *Weyl character formula*, which (at least in principle) simplifies the two-step method of constructing a character of a representation to one step:

$$\chi_G(\Lambda) = \frac{\sum_w (-1)^{|w|} \mathbf{z}^{w(\Lambda+\rho)}}{\mathbf{z}^\rho \prod_\alpha (1 - \mathbf{z}^{-\alpha})} \tag{2.20}$$

where z_i are fugacities, w is a Weyl group element (a product of Weyl reflections $w = w_{\alpha_1} \dots w_{\alpha_n}$ where w_α takes a general weight β to $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$), and we define $\mathbf{z}^\alpha = \prod_{i=1}^r z_i^{\alpha_i}$.

By the *Peter-Weyl theorem* [36], a class function for a Lie group can be decomposed uniquely in terms of the characters of its irreps.

For a finite group, the number of occurrences of each irrep R_i in a general representation R is given by the following formula:

$$n_i(R) = \frac{1}{|G|} \sum_{[g]} |[g]| \bar{\chi}_i([g]) \chi_R([g]) \tag{2.21}$$

where $[g]$ is the conjugacy class of elements containing element g (summed once per class, not per element!), $\chi_i([g])$ is the character of representation R_i taken over $[g]$ and $\chi_R([g])$ is the same for R. Since the number of irreps of a finite group is finite, the number of occurrences of each irrep in a general (reducible) representation can be easily calculated using this method. The decomposition theorem, both existence and uniqueness, follows from the (weighted, by the sizes of the conjugacy classes) orthogonality of characters of different representations.

For a Lie group G this generalizes to

$$n_i(R) = \int \mathrm{d}\mu_G(z_j) \bar{\chi}_i(z_j) \chi_R(z_j) \tag{2.22}$$

where $d\mu_G(z_j)$ is the integral over the group manifold. Because there are $dim(G)$ parameters defining a general element of the group, we use the fact that any element is conjugate to an element of the maximal torus to rewrite the integral in terms of a parametrization of this torus. We must therefore 'weight' the integral by the Jacobian of a general element of the adjoint representation in terms of the fugacities. This factor is called the Haar measure and is defined later in Section 2.6 and also in [9]. This decomposition is discussed in [19].

For a Lie group, the number of irreps is infinite and so this method cannot really be used to decompose a character of a general representation into those of the group's irreps. We instead decompose the representation by progressively finding the highest weight with the highest norm, calculating the character of the irrep with this highest weight by either the Weyl character formula or the previous construction, subtracting it (with its multiplicity) from the general reducible representation and repeating the process. We do, however, use this method to determine the number of singlets in a representation, as we do when we consider symmetrizations of representations of both a gauge symmetry group and a global one, which we do in the next section. (We integrate over the gauge group to find gauge singlets, and then decompose into representations of the global group by progressive subtraction.) We discuss finding gauge singlets, also called invariants, further in Section 2.7.

2.4 Symmetric polynomial identities for product groups

Sometimes one wishes to (anti)symmetrize representations of two (or more) different symmetry groups, usually a gauge group and a global group or a non-simple (not counting $U(1)$) gauge group. In this section we will discuss the case of two $U(N)$ symmetry groups (not necessarily the same N); we will leave the discussion of their decomposition into representations of other groups, called 'plethysm', to Section 2.7.

There are several very useful identities that express either the full or elementary symmetric polynomial in two (or more in one case) sets of variables as sums of products of those in the variables separately.

The first identity expresses a plethystic exponential of a product group representation in terms of the complete symmetric polynomials of one subgroup representation and the monomial symmetric polynomials of the other:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} h_{\lambda}(\mathbf{x}) m_{\lambda}(\mathbf{y}) \tag{2.23}$$

If counting of the number of fields in the product group representation is required, a t can easily be inserted in the denominator $(1 - t x_i y_j)$ and each summand on the RHS multiplied by $t^{|\lambda|}$ where $|\lambda|$ is the integer of which λ is a partition.

This is easy to see by inspection, for each λ_i fix the (different for each i) y_i that is raised to the power λ_i and sum all terms in the series expansion with that power of y_i. It is easy to see that this is $h_{\lambda_i}(\mathbf{x})$, and their product over all i is $h_{\lambda}(\mathbf{x})$. The function of y_i is $m_{\lambda}(\mathbf{y})$. This identity also holds with $\mathbf{x}$ and $\mathbf{y}$ reversed.

The second identity expresses the PE of the product group representation as a sum of products of Schur polynomials of the subgroup representations:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) \tag{2.24}$$

The proof of this is rather involved and described in [23]. It involves expanding the determinant $det(\frac{1}{1-x_i y_j})_{1\leq i,j\leq n}$, row and column reduction and factorizing out the Vandermonde determinant $\Delta(x) = \prod_{i<j}(x_i - x_j)$ for both $\mathbf{x}$ and $\mathbf{y}$.

To obtain an expression for $\prod_{i,j=1}^{n}(1 - x_i y_j)^{-1}$ in terms of Schur polynomials in both $\mathbf{x}$ and $\mathbf{y}$, one can first see easily by inspection that the determinant of the matrix of terms $(1 - x_i y_j)^{-1}$ with i and j between 1 and n must have denominator $\prod_{i,j=1}^{n}(1 - x_i y_j)$, which is symmetric in the x_i and y_j, and that the numerator must be antisymmetric in both and therefore proportional to the Vandermonde determinants $\Delta(\mathbf{x})$ and $\Delta(\mathbf{y})$, i.e. $\prod_{i<j}^{n}(x_i - x_j)$ and the same for y_i. By row and column manipulation, it is shown in [23] that

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{n} = \frac{\Delta(\mathbf{x})\Delta(\mathbf{y})}{\prod_{i,j=1}^{n}(1 - x_i y_j)} \tag{2.25}$$

i.e. the numerator is the product of the two Vandermonde determinants with no additional factor. To obtain an expression in terms of Schur polynomials, one must expand each term $(1 - x_i y_j)^{-1}$ as $\sum_{d_i=0}^{\infty}(x_i y_j)^{d_i}$, keeping the same exponent in each row of the matrix (we label rows by i and columns by j). Substituting this into the determinant, one gets

$$\begin{aligned} \det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{n} &= \sum_{d_i \geq 0} \det\left((x_i y_j)^{d_i}\right)_{i,j=1}^{n} \\ &= \sum_{d_i \geq 0} \prod_{i=1}^{n} x_i^{d_i} \det\left(y_j^{d_i}\right)_{i,j=1}^{n} \\ &= \Delta(\mathbf{x})\Delta(\mathbf{y}) \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) \end{aligned} \tag{2.26}$$

To explain the last derivation, firstly the d_i have to all be different to give a non-zero result in the determinant in the second row; then rearrange them into d'_i such that $d'_1 > \ldots > d'_n (\geq 0)$. The $\mathbf{y}$-determinant is therefore given by $(-1)^{sgn(\sigma)}\Delta(\mathbf{y})s_\lambda(\mathbf{y})$ where $\lambda_i = d'_i - n + i$ and σ is the permutation of the d'_i that gives the d_i. (One can see that λ, with trailing zeros removed, is a partition of some number $|\lambda| = \sum_{i=1}^n \lambda_i$.) One can then see that the coefficient of $\Delta(\mathbf{y})s_\lambda(\mathbf{y})$ is similarly $\Delta(\mathbf{x})s_\lambda(\mathbf{x})$, hence the formula. Dividing out by $\Delta(\mathbf{x})\Delta(\mathbf{y})$ we can put this into the form

$$\frac{1}{\prod_{i,j=1}^n (1 - x_i y_j)} = \sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) \tag{2.27}$$

which is the desired result.

There is a third identity which expands the PE of a product group representation in terms of the power sum symmetric (Newton) polynomials of the subgroup representations:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y}) \tag{2.28}$$

where $z_\lambda = \prod_k k^{i_k} i_k!$ when λ is rewritten as $1^{i_1} \ldots k^{i_k} \ldots$.

Unlike with the previous two expressions, this one can be generalized to representations of products of three or more groups rather than being restricted to two. It is easy to see that $p_\lambda(\mathbf{xy}) = p_\lambda(\mathbf{x})p_\lambda(\mathbf{y})$. However, we will have to leave its derivation till the next section once we have introduced the tools for doing so. These are grouped together into the *plethystic programme*, which we introduce now.

2.5 Introduction to plethystics

As well as the examples of massive (super)string partition functions and SQCD Hilbert series covered in this thesis, this finds applications in other quiver gauge theories including instanton moduli spaces [42, 71], brane tiling theories [39, 26, 28, 29, 30, 31], finding invariants of finite groups and converting between single- and multi-trace partition functions counting BPS operators [22, 19], etc.

The plethystic exponential (PE) is used to symmetrize (finite or infinite) power series to arbitrary orders.

Suppose first that the argument of the PE function is a polynomial (which is usually a generalized or Laurent polynomial when the gauge and/or global symmetry group is not $U(N)$, though in this case we have to treat it, and the PE, as a 'formal' power series rather than a 'real' polynomial) in one or more variables; that way each term is a monomial. In the first step of the derivation of the PE formula we temporarily replace each term with a generalized fugacity $X_i, 1 \leq i \leq n$ for a series of n terms. (To explicitly show symmetrization to each order separately, we introduce another fugacity t counting the number of terms; such counting fugacities, which denote $U(1)$ charges, are used in most if not all applications of plethystics. When used to go from single- to multi-trace partition functions counting BPS operators and back for finite numbers of fields [22], the 'counting' fugacity is called ν.)

The totally symmetric product of a sum of n terms X_i to order k is given simply by $h_k(X_i)$ as given above. This is the same as the Schur polynomial $s_\lambda(X_i)$ where λ is the single-row partition $[k]$ whence by the aforementioned visualization of Schur polynomials in terms of semi-standard Young tableaux the correspondence can be easily seen. The PE can thus be written as

$$PE\left[\sum_{i=1}^{n} tX_i\right] = \sum_{k=0}^{\infty} t^k h_k(X_i) \tag{2.29}$$

It is simple to see that

$$h_k(X_1, X_2, \ldots) = \sum_{j=0}^{k} X_1^j h_{k-j}(X_2, \ldots) \tag{2.30}$$

and substituting into the expression for the PE, one obtains

$$PE\left[\sum_{i=1}^{n} tX_i\right] = \sum_{k=0}^{\infty}\sum_{j=0}^{k} t^j X_1^j t^{k-j} h_{k-j}(X_2, \ldots) \tag{2.31}$$

Resumming j and $k-j$ (relabelling as k) from 0 to infinity, one then gets

$$\begin{aligned} PE\left[\sum_{i=1}^{n} tX_i\right] &= (1-tX_1)^{-1} \sum_{k=0}^{\infty} t^k h_k(X_2, \ldots) \\ &= (1-tX_1)^{-1}\, PE\left[\sum_{i=2}^{n} tX_i\right] \end{aligned} \tag{2.32}$$

and repeating the process one obtains the final expression

$$PE\left[\sum_{i=1}^{n} tX_i\right] = \prod_{i=1}^{n}(1-tX_i)^{-1} \tag{2.33}$$

This expression generalizes very simply to any function that can be expressed as a Taylor series, even an infinite one, in one or more variables. In some of the examples in [19, 22], it is a rational function and the explicit Taylor expansion is not used, though there we only calculate symmetrizations to low orders, not the full PE. As with the 'usual' exponential function, the exponential of the sum of two functions is the product of the exponentials of the functions by themselves.

To show the relations with power sum symmetric (Newton) polynomials (and their generalization to functions with possibly non-terminating Taylor series, the Adams operator $Adams^k(f(X_i)) = f(X_i^k)$) and also to derive the form of the inverse operation, the plethystic logarithm (PL), we take the log of the PE and Taylor expand:

$$\begin{aligned}\log PE\left[\sum_i tX_i\right] &= -\sum_i \log(1-tX_i) &(2.34)\\ &= \sum_i \sum_{k=1}^{\infty} \frac{t^k X_i^k}{k}\\ &= \sum_{k=1}^{\infty} \frac{t^k p_k(X_i)}{k}\\ PE\left[\sum_i tX_i\right] &= \exp\left(\sum_{k=1}^{\infty} \frac{t^k p_k(X_i)}{k}\right) &(2.35)\end{aligned}$$

For a general function $f(X_i)$, this can be written as:

$$\log PE\left[tf(X_i)\right] = \sum_{k=1}^{\infty} \frac{t^k f(X_i^k)}{k} \tag{2.36}$$

$$PE\left[tf(X_i)\right] = \exp(\sum_{k=1}^{\infty} \frac{t^k f(X_i^k)}{k}) \tag{2.37}$$

We wish to bring the log form to the desired form of simply $t\sum_i X_i$.

It is easy to see that for each prime p, one must subtract $\log PE\left[t^p p_p(X_i)\right]$; this removes all terms for which k is prime or a prime power (with only one prime factor), but 'over-corrects' in the case where k has more than one prime factor, indeed for $k = p_1 \dots p_r$ it leaves the terms in $p_k(X_i)$ as having coefficient $-(r-1)$.

One corrects this for products of two (distinct) primes p_1 and p_2 by adding $\log PE\left[t^{p_1 p_2} p_{p_1 p_2}(X_i)\right]$ back into the sum, but one sees that we then have to subtract back out terms in products of three distinct primes, and so on.

There is a function, the Möbius function $\mu(n)$, which returns $(-1)^r$ when n is a product of r distinct prime factors and 0 when n is divisible by a square of some number. Using this function, we can write the expression for the PL (still taken to be of the PE) as follows:

$$\begin{aligned} PL\left[PE\left[\sum_i tX_i\right]\right] &= -\sum_i \sum_{k=1}^{\infty} \mu(k) \log(1 - t^k X_i^k) \\ &= \sum_i \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mu(k) \frac{t^{kl} X_i^{kl}}{kl} \\ &= \sum_i \sum_{m=1}^{\infty} \sum_{k|m} \mu(k) \frac{t^m X_i^m}{m} = \sum_i tX_i \end{aligned} \tag{2.38}$$

We can now generalize the PL to take an arbitrary function of an arbitrary number of variables as an argument, as long as it takes the value 1 when all (or certain combinations of) fugacities are set to 0:

$$PL\left[f(t, X_i)\right] = \sum_{k=1}^{\infty} \frac{\mu(k) \log(f(t^k, X_i^k))}{k} \tag{2.39}$$

There is an analogue of the plethystic exponential for fermionic operators called the fermionic plethystic exponential (PE_F) which is defined in the same way as the PE except that only products of distinct X_i, which sum to totally antisymmetric products, occur in the final product. The totally antisymmetric product of a sum of n terms X_i to order k is given simply by $e_k(X_i)$ as given above. This is the same as the Schur polynomial $s_\lambda(X_i)$ where λ is the single-column partition $[1^k]$. Again this can be seen by visualizing the Schur polynomial in terms of semi-standard Young tableaux. The fermionic PE can thus be written as

$$\mathrm{PE}_F\left[\sum_{i=1}^{n} tX_i\right] = \sum_{k=0}^{\infty} t^k e_k(X_i) \tag{2.40}$$

By the same steps as for the PE, we obtain

$$\mathrm{PE}_F\left[\sum_{i=1}^{n} tX_i\right] = \prod_{i=1}^{n}(1+tX_i) \tag{2.41}$$

In exponential form, generalized to a general function argument $f(X_i)$ of a possibly infinite number of X_i, we have the same expression as for the 'ordinary' PE except for the insertion of a $(-1)^{k+1}$ factor:

$$\mathrm{PE}_F\left[f(t,X_i)\right] = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}f(t^k,X_i^k)}{k}\right) \tag{2.42}$$

The inverse is published in [36]. To derive the inverse, we do not work with the log form of the PE directly, but rather note that

$$PE\left[f(t,X_i)\right] = \prod_{r=0}^{\infty}\mathrm{PE}_F\left[f(t^{2^r}X_i^{2^r})\right] \tag{2.43}$$

and we therefore have for the inverse

$$\begin{aligned}\mathrm{PL}_F\left[f(t,X_i)\right] &= \sum_{r=0}^{\infty} PL\left[f(t,X_i)\right] \\ &= \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{\mu(k)\log(f(t^{2^r k},X_i^{2^r k}))}{k}\end{aligned} \tag{2.44}$$

The PL_F is not used in this thesis or indeed as yet in any other literature, but the other formulae find extensive use.

2.6 Haar measure

The Haar measure, as derived in [9], is the conversion factor that enables one to integrate over the whole group, of dimension $dim(G)$ for a group G, by the simpler integration over the maximal torus, of dimension $rank(G)$. It uses the fact that any group element is conjugate to an element of the maximal torus and they therefore have the same character. Schematically it is the Jacobian of the group manifold over the torus. For a group G

parametrized by fugacities $z_i, 1 \le i \le rank(G)$, it is given by

$$\int \mathrm{d}\mu_G = \frac{1}{|\mathfrak{W}|} \prod_{i=1}^{rank(G)} \oint_C \frac{dz_i}{2\pi i z_i} \prod_{\alpha \in \Delta} \left(1 - \prod_{i=1}^{rank(G)} z_i^{\alpha_i}\right) \tag{2.45}$$

where α denotes a root (i.e. a weight of the adjoint representation), α_i denotes the ith entry in the weight α in the Dynkin basis, $\mathfrak{W}$ is the Weyl group, or the subgroup of the isometry group of the root system generated by reflections in hyperplanes perpendicular to the roots, and C is the unit circle. (As mentioned before, a different (non-Dynkin) basis are used in [1, 2] in the case of $SO(N_c)$ and $Sp(N_c)$ gauge groups; however the method still works if the same basis is used throughout the integration.)

A simpler Haar measure is derived in [76] and used in [3] where the product is only over the positive roots and there is no division by the order of the Weyl group:

$$\int \mathrm{d}\mu_G = \prod_{i=1}^{rank(G)} \oint_C \frac{dz_i}{2\pi i z_i} \prod_{\alpha \in \Delta_+} \left(1 - \prod_{i=1}^{rank(G)} z_i^{\alpha_i}\right) \tag{2.46}$$

2.7 Invariant theory and the Molien-Weyl integral

Before we introduce invariant theory, we must recall that we often, as here, misuse the term ‘representation’ to mean the (vector) space on which elements of the group act, when in actual fact the word should instead be used to refer to the matrices representing the group elements.

Because antisymmetrization of a representation will always lead to a singlet at a level equal to the dimension of the representation, there will always be a completely antisymmetric invariant; this is not the case for symmetrization, so there will not always be a totally symmetric invariant, they are not present for the $A_n = SU(n+1)$ and $C_n = Sp(n)$ families.

A general invariant is specified by an object I with two or more indices which remains the same under multiplication by elements of the relevant representation(s) of the group, with each multiplication being by the same element. This definition holds for both Lie and finite groups. Simple examples of invariants in which each index represents an object of the same representation are the trace of $SO(N)$, the symplectic trace of $Sp(N)$ and the epsilon tensor of $SU(N)$ and $SO(N)$.

It is possible to have invariants where the indices do not all represent objects in the same representation, for example a fundamental and an antifundamental or (anti)fundamentals and an adjoint. In this case the elements of the group are not the 'same' as such but are the exponentials of the same linear combination of the generators of the group, which differ between representations but always have the same structure constants (i.e. commutation properties). These are called intertwiners and are discussed in [36]. The simplest example of an intertwiner is the delta function of $SU(N)$ and E_6 with one fundamental and one antifundamental index.

To get the spectrum of gauge-invariant operators (in terms of representations of the global symmetry group), one must project the representations of the gauge group generated by the plethystic exponential onto the trivial representation. There is a general formula from [20] which gives the number of occurrences of any given (irreducible) representation in a sum of representations; one multiplies the sum by the conjugate of the desired representation (which is as specified above) and integrates the product over the whole group; this uses the fact that the product of an irrep with its conjugate always contains exactly one singlet, while the product with an irrep other than the conjugate never produces a singlet. In this case, since the desired representation is the trivial one, so is the conjugate irrep and hence one can simply integrate over the whole group.

For a given irreducible representation R, the number of occurrences of R in a (possibly reducible) representation R' (which is usually a tensor product) is given by (as in [19]) (2.21) for a finite group and (2.22) for a Lie group. In this section, we are considering the number of invariants, so R is the singlet representation.

For a finite group with a representation given by matrices acting on a vector space of dimension n, the invariants can be obtained explicitly for a general polynomial argument $f((x))$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is an element of said vector space, using the *Reynolds operator*:

$$R(f(\mathbf{x})) = \frac{1}{|G|} \sum_{g \in G} f(g(\mathbf{x})) \qquad (2.47)$$

There is a theorem discussed in [22] and [6] that states that the maximum degree of a primitive invariant is the order of the group, meaning that we do not have to go to too high a degree to find all the invariants.

Molien's sum formula, quoted in [22] and derived in [6], gives the Hilbert series of the invariants of (a specific representation ρ of) a finite group:

$$H(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det\left(1 - \rho(g)t\right)} \tag{2.48}$$

where $\rho(g)$ is the matrix corresponding to element g in the representation ρ.

The plethystic logarithm of $H(t)$ gives the (number of) primitive invariants and (if present) relations and higher syzygies.

It is possible to 'refine' this series by replacing the multiplication by t by multiplying by $diag(t_1, \ldots, t_{dim(\rho)})$, which is outlined in [22]. This does not always give a PL in which every term is an integer, however (though they can when the moduli space is toric), so the invariants cannot always be assembled into representations of $SU(dim(\rho))$.

The (Molien-)Weyl integral is the generalization of Molien's sum formula to Lie groups. The plethystic exponential can be thought of as the determinant. Any matrix is conjugate to a diagonal matrix and such conjugation always leaves the identity invariant. We will not present a general formula here, but rather leave the details to the relevant sections.

3 Introduction to string spectra

3.1 String theory preliminaries

In this section we largely follow the procedure of light-cone quantization described in chapter 1 of [53] for the bosonic string and chapters 10 and 11 of [54] for the type I and heterotic superstrings respectively, though for space reasons we omit some details.

We start by considering the point-particle action. We could write this with X^0 fixed as being time, but we instead introduce some extra redundancy and write the action in a parametrization-invariant form:

$$S_{pp} = \int \mathrm{d}\tau (-\dot{X}^\mu \dot{X}_\mu)^{1/2} \tag{3.1}$$

This is not in a convenient form to work with, because of the square root. To remedy this, we introduce an auxiliary variable called a tetrad (so called because it was originally used in four-dimensional gravity theories, though it is now also used in other-dimensional theories, where it is more commonly called by its German name of D-bein for D dimensions, here a 1-bein or einbein). Our new action is

$$S_{pp'} = \int \mathrm{d}\tau \eta^{-1} \dot{X}^\mu \dot{X}_\mu - \eta m^2 \tag{3.2}$$

We see that solving the Euler-Lagrange equation for the tetrad brings us back to the original action.

Generalizing the point-particle action to an extended object with one spatial dimension, we have the Nambu-Goto action

$$S_{NG} = \int \mathrm{d}\tau \mathrm{d}\sigma (-\mathrm{det} h_{ab})^{1/2} \tag{3.3}$$
$$\text{where } h_{ab} = \partial_a X^\mu \partial_b X_\mu$$

This time, unlike in the point-particle case, we remove the square root by introducing not a tetrad (or 2-bein, zweibein) as such, but an auxiliary metric γ_{ab} (with inverse γ^{ab}), with Minkowski signature. This gives the Polyakov action (which was not discovered by Polyakov, though he worked out its properties):

$$S_P = \int \mathrm{d}\tau \mathrm{d}\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu \tag{3.4}$$

This has the symmetries of D-dimensional (spacetime) Poincaré invariance $X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu$ and 2-dimensional (world-sheet) diffeomorphism (diff) $\gamma'_{ab}(\sigma^c) = \gamma_{ab}(\sigma^c)$ and Weyl invariance $\gamma'_{ab} = \exp(2\omega(\tau,\sigma))\gamma_{ab}$.

Again, solving the Euler-Lagrange equation for the auxiliary metric brings us back to the Nambu-Goto action. However, we usually keep the extra redundancy, both for ease of use and also because the Polyakov action, and its point-particle analogue, can be used for massless particles, but the Nambu-Goto action and S_{pp} cannot.

In the light-cone quantization described in chapter 1, we rewrite indices 0,1 by +,-, defining new coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$, and denote the other indices by $i, 2 \leq i \leq D-1$. The Minkowski metric, with $(-+\ldots+)$ signature, now has components $\eta^{+-} = \eta^{-+} = -1$. We then set $X^+ = \tau$, which is easier to work with than keeping the original coordinate system and metric and setting $X^0 = \tau$. We show that we can fix the worldsheet metric to be Minkowski too. From chapter 2 onwards, we Wick rotate the worldsheet metric to Euclidean form, and work with complex coordinates.

Poincaré invariance forces $X^\mu(\tau,\sigma)$ to be periodic in σ. Other periodicities are possible if we relax Poincaré invariance in some spacetime dimensions, as occurs for constructions such as D-branes, orbifold compactifications etc. We do not consider those further here.

After a long calculation, and imposing the commutation relations $[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n}$, we arrive at the following expression for the mass of an open string state:

$$m^2 = \frac{1}{\alpha'} \sum_{i=2}^{D-1} \sum_{n=1}^{\infty} \left(N_{in} + \frac{1}{2} n \right) \quad \text{where} \quad N_{in}|\rangle = \alpha^i_{-n}\alpha^i_n|\rangle \tag{3.5}$$

We see that there is a divergent zero-point energy from the constant term.

In regular QFT, we usually discard this term, but in string theory we renormalize it. We do this using zeta-function regularization. The Riemann zeta function is defined as, for $Re(z) > 1$,

$$\zeta(z) = \sum_{i=1}^{\infty} n^{-z} \tag{3.6}$$

This has a simple pole at $z = 1$ with residue 1 but can otherwise be analytically continued into $Re(z) \leq 1$. By the following formula from [59], we have

$$\zeta(z) = 2^z \pi^{z-1} \sin(\frac{1}{2}\pi z)\Gamma(1-z)\zeta(1-z) \tag{3.7}$$

We see, knowing that $\zeta(2) = \frac{\pi^2}{6}$, that $\zeta(-1) = -\frac{1}{12}$. This must be multiplied by $\frac{1}{2}$, from the factor in (3.5), to give $-\frac{1}{24}$, and one term added for each of the $D-2$ oscillators from $2 \leq i \leq D-1$ to give a zero-point energy, which is the mass of the ground state of the theory in units of the string scale α'^{-1}, of $-\frac{D-2}{24}$. Bosonic string theory therefore has a tachyon, i.e. a particle of negative mass-squared, in more than 2 dimensions.

The states at the first excited level are given by $\alpha^i_{-1}|0;k\rangle$ for $2 \leq i \leq D-1$ and, by the commutation relations have mass $\frac{1}{\alpha'}\left(1 + \frac{2-D}{24}\right) = \frac{26-D}{24\alpha'}$. We know that for massless states, the spatial momentum cannot vanish so the little group, which leaves the momentum invariant, is $SO(D-2)$ (observe for the case of $p_\mu = (E, E, 0, \ldots)$), but for massive states we can have $p_\mu = (m, 0, \ldots)$ so the little group is $SO(D-1)$. There are only $D-2$ states at the first excited level, so they can only transform in a representation of $SO(D-2)$ and so they must be massless, and therefore D is constrained to be 26. The normal ordering constant, which is the mass of the ground state in units of α'^{-1}, is -1, taking it to be additive rather than subtractive.

This is a rather unrigorous derivation of the conditions on the number of dimensions of the bosonic string and the normal ordering constant. (In regular quantum field theory, the infinite zero-point energy is usually just discarded.) We will work with this method of quantization because the group(s) in which the raising operators, which build up the Fock space of quantum states, transform is most easily shown by this method, but first we will discuss more rigorous derivations of this condition based on more covariant methods of quantization. In all cases the actual commutation relations are the same, though with all D oscillators considered rather than just the

$D-2$ transverse ones and the spacetime metric in its original Minkowski form rather than the Euclidean restriction to the transverse space.

Firstly we have *old covariant* quantization. To do this, we recall that invariance of the action under variation of the (world-sheet) metric, whether Minkowski or Euclidean (as in the Wick-rotated theory described from chapter 2 onwards in [53]), gives rise to a conserved (world-sheet) energy-momentum tensor T^{ab} which is symmetric and traceless. Fixing the gauge means that the vanishing of T^{ab} does not hold as an operator equation, so instead it must be imposed as a constraint on the matrix elements between physical states.

This is equivalent to imposing that the Virasoro lowering operators $L_n + A\delta_{n,0}, n \geq 0$ annihilate physical states, with A being an as yet undefined (additive) constant. To see this, we now pass to complex coordinates on the world-sheet, defining $w = \sigma + i\tau$ and $\bar{w} = \sigma - i\tau$ and from them we obtain $z = e^{-iw}$ and $\bar{z} = e^{i\bar{w}}$. In this coordinate system the integral over σ from 0 to 2π is replaced by integration on a circle centred on the origin. In complex coordinates the traceless condition on T^{ab} means it has only zz and $\bar{z}\bar{z}$ components, and its conservation means these are respectively holomorphic and antiholomorphic, so we can write $T(z)$ and $\tilde{T}(\bar{z})$.

The Virasoro operators are defined in terms of the energy-momentum tensor as follows:

$$T(z) = T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{3.8}$$

$$L_n = \oint \mathrm{d}z \quad z^{n+1} T(z) \tag{3.9}$$

and similarly for $\tilde{L}_n$ in terms of $\tilde{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z})$ for the other half of the closed string. The condition that all Virasoro operators annihilate physical states is too restrictive, so by Hermiticity, $L_n^\dagger = L_{-n}$, we have

$$L_n |phys\rangle = 0, \quad n > 0 \tag{3.10}$$

$$(L_0 + A)|phys\rangle = 0 \tag{3.11}$$

States obtained by the action of L_{-n} on any state are called *spurious*, and are clearly orthogonal to any physical state by Hermitian conjugation; if they are themselves physical, they are called *null*. States that differ by a null state correspond to the same physical state, so the Fock space does not consist of 'states' as such but of equivalence classes or sets of states differing by a null state, with each class denoted by one representative state. We see that for us to obtain the same spectrum as in the light-cone case (not fixing zero-point energy or dimensions in the latter case), we must have $A = -1$ and 26 dimensions. We see that A is the zero-point energy.

Secondly we have *BRST* (Becchi-Rouet-Stora-Tyutin) quantization. In order to do BRST quantization, we must first fix the gauge using the Faddeev-Popov procedure borrowed from gauge theory. In string theory the gauge fixing introduces two new fields, one symmetric and traceless with two lower indices and the other with one upper index, which are then made fermionic ('ghosts'). This is described in [53], heuristically in chapter 3 and more rigorously in chapter 5. The ghosts also form a CFT, with their own energy-momentum tensor.

There are 'large' gauge transformations which cannot be specified in such a form and indeed are orthogonal to all those that can, these are called *moduli*. These are not the same as the massless scalars that define the parameters of a general string compactification, or those that parametrize the vacua of a supersymmetric gauge theory, which are also referred to as moduli.

In BRST quantization, physical and null states are called *closed* and *exact* respectively, and there is again a cohomology. Closed states are annihilated by the BRST charge, denoted Q_B, and exact states are given by $Q_B|\rangle$ for some state $|\rangle$ and are necessarily closed by the fact that $Q_B^2 = 0$. Each 'state' in the Fock space is again an equivalence class of closed states whose difference is exact.

We will now have a brief diversion into operator products, the central part of the machinery of CFT. Recalling that the path integral of a total (functional) derivative is zero, we see that taking a total derivative inside a path integral with insertions gives relations between the derivative of the insertion and the insertion multiplied by the derivative of the action. This may give rise to singularities in the product of two operators as their positions approach each other, as in chapter 2 of [53]. The energy-momentum tensor $T(z)$, and its antiholomorphic analogue $\tilde{T}(\bar{z})$, may be built out of the fundamental fields of the CFT, or they may be fundamental fields themselves, and their operator products are defined in terms of their constituent fields. The *central charge* is twice the coefficient of $(z-w)^{-4}$ in the operator product expansion of $T(z)T(w)$.

BRST quantization forces the dimension of the bosonic string to be 26 because this is the only case for which the BRST charge squares to zero, i.e. is nilpotent. We obtain this from the operator product of two BRST currents, which are built out of combinations of components of the ghost and matter CFTs.

The Weyl anomaly, which is a condition on the tracelessness of the energy-momentum tensor ($T_a^a = -\frac{c}{12}R$, for c the central charge of the theory and R the Ricci scalar, is derived in two ways, one from the variation of the energy-momentum tensor and the other from the vacuum partition function, in chapter 3 of [53]), requires the total central charge of the theory to vanish, this is 1 for every spacetime dimension and -26 for the ghost bc CFT giving $D-26$ in total.

We will now move on to the superstring. String amplitudes always contain an even number of fermions, so the periodicity conditions also allow the boundary conditions to be antiperiodic. The sector in which the fermions are periodic is called the Ramond (R) sector, and that in which they are antiperiodic is called the Neveu-Schwarz (NS) sector.

Taking θ as 0 in the R-sector and $\frac{1}{2}$ in the NS sector, we send $n \to n-\theta$ in the sum and obtain the answer $\frac{1}{24} - \frac{1}{8}(2\theta-1)^2$, again using zeta-function regularization, this time of the Hurwitz zeta function which is defined as

$$\zeta(z,a) = \sum_{i=0}^{\infty}(n+a)^{-z} \tag{3.12}$$

This formula reproduces Riemann's when $a=1$.

Expanding out the expression for the Hamiltonian, and hence the mass, we see that the sign of the zero-point energy is reversed from the bosonic case, so we have, again multiplying by $\frac{1}{2}$, $\frac{1}{24}$ for each periodic fermion and $-\frac{1}{48}$ for each antiperiodic fermion. (Antiperiodic bosons contribute $+\frac{1}{48}$.) In the R sector, the zero-point energy of the fermionic oscillators cancels that of the bosonic ones; in the NS sector, we get in total $-\frac{1}{16}$ for each of the $D-2$ directions transverse to the light cone.

In the NS sector, the states at the first excited level are given by $\psi^i_{-1/2}|0;k\rangle$ for $2 \leq i \leq D-1$ and, by the commutation relations have mass

$\frac{1}{\alpha'}\left(\frac{1}{2}+\frac{2-D}{16}\right)=\frac{10-D}{16\alpha'}$. Again, the little group is $SO(D-2)$ for massless states but $SO(D-1)$ for massive states. There are only $D-2$ states at the first excited level, constraining D to be 10, with zero-point energy $-\frac{1}{2}$. Again this can be determined by more rigorous methods, i.e. the vanishing of the Weyl anomaly and the nilpotency of the BRST charge. In this case we also have central charge $\frac{1}{2}$ from the fermionic oscillators in each spacetime dimension, and superconformal bosonic ghosts β and γ making up a SCFT with central charge 11. This reduces to $\frac{3}{2}D-26+11=0$ giving $D=10$.

In the R-sector we have zero modes which generate spinor representations of $SO(8)$. Picking one ground state which is annihilated, say, by $\psi^{2i}-i\psi^{2i+1}$ for $1\leq i\leq 4$, by acting with the $\psi^{2i}+i\psi^{2i+1}$ we obtain two spinor representations, the $\mathbf{8}$ obtained by acting an even number of times with these raising operators and the $\bar{\mathbf{8}}$ obtained by acting an odd number of times.

The GSO projection keeps states whose world-sheet *fermion number* is even. The NS ground state at mass level $-\frac{1}{2}$ is odd because of ghost modes, and in the R-sector we pick one $SO(8)$ spinor, usually the $\mathbf{8}$, to have even fermion number and the other odd. The fermionic oscillators are all odd, so their action flips the odd/even parity of a given state. In type IIA string theory, we keep the right-moving R-sector states whose world-sheet fermion number is odd, keeping the even condition on the NS sector states.

As regards old covariant quantization, we also have operators G_n with the moding of n being the same as that of the oscillators, i.e. half-odd integers in the NS sector and integers in the R sector. G_n must annihilate physical states for $n\geq 0$, reducing the two $SO(1,9)$ 16-dimensional spinors generated by the R-sector zero modes, here considering all 10 Minkowski modes, to the same 8-dimensional ones of $SO(8)$ as in the light-cone case.

The 6 internal dimensions can be replaced by a general superconformal field theory with $c=9$ ($\tilde{c}=9$ for the right-handed component of a closed string). The normal ordering constant as derived earlier remains the same. We do this in all three cases of the superstring that we consider, though we see that in the case of 8 supercharges two of the internal dimensions are toroidal so we consider 6-dimensional Minkowski spacetime with a $c=6$ SCFT on the remaining 4 internal dimensions, and in the case of 16 supercharges all 6 internal dimensions are toroidal so we treat this case as 10-dimensional Minkowski spacetime.

In these case, we must use BRST quantization to calculate its spectrum. This is done explicitly for the first massive level of the 4, 8 and 16-supercharge *universal* (to be defined later) superstring spectra in [15] and [37], and the partition functions are calculated in [12] in the 4-supercharge case and [11] in the 8-supercharge case.

We have derived these conditions from the worldsheet (S)CFT, without recourse to spacetime properties. The open bosonic string contains a massless vector (gauge) field, which must couple to a conserved current; similarly the closed bosonic string contains a massless second-rank symmetric traceless tensor field, which must couple to a conserved second-rank symmetric traceless tensor source, of which the only one present is the energy-momentum tensor. These conditions impose that the theory must have spacetime gauge and coordinate invariance respectively. The timelike and longitudinal oscillators of the string are removed by world-sheet coordinate (diff) invariance; from the spacetime point of view, it is the spacetime gauge and coordinate invariance that remove these two oscillators. Spacetime supersymmetry also appears in a rather round-about way, in the construction of null states in the R sector of the open type I superstring.

3.2 Introduction to string spectra

We will now return to the light-cone formalism.

We will start with [8], and then move on to the systematic treatment described in [7].

We denote the characters by the sums of the products of the fugacities raised to the power of the corresponding Dynkin labels. For the representations discussed in this section we have the following characters:

$$[1,0,0,0]_8(\vec{z}) = z_1 + \frac{z_2}{z_1} + \frac{z_3 z_4}{z_2} + \frac{z_3}{z_4} + \frac{z_4}{z_3} + \frac{z_2}{z_3 z_4} + \frac{z_1}{z_2} + \frac{1}{z_1} \quad (3.13)$$

$$[0,0,1,0]_8(\vec{z}) = z_3 + \frac{z_2}{z_3} + \frac{z_1 z_4}{z_2} + \frac{z_1}{z_4} + \frac{z_4}{z_1} + \frac{z_2}{z_1 z_4} + \frac{z_3}{z_2} + \frac{1}{z_3} \quad (3.14)$$

$$[0,0,0,1]_8(\vec{z}) = z_4 + \frac{z_2}{z_4} + \frac{z_1 z_3}{z_2} + \frac{z_1}{z_3} + \frac{z_3}{z_1} + \frac{z_2}{z_1 z_3} + \frac{z_4}{z_2} + \frac{1}{z_4} \quad (3.15)$$

$$[1,0,0,0]_9(\vec{y}) = y_1 + \frac{y_2}{y_1} + \frac{y_3}{y_2} + \frac{y_4^2}{y_3} + 1 + \frac{y_3}{y_4^2} + \frac{y_2}{y_3} + \frac{y_1}{y_2} + \frac{1}{y_1} \quad (3.16)$$

$$[0,0,0,1]_9(\vec{y}) = y_4 + \frac{y_3}{y_4} + \frac{y_2 y_4}{y_3} + \frac{y_1 y_4}{y_2} + \frac{y_2}{y_4} + \frac{y_4}{y_1} + \frac{y_1 y_3}{y_2 y_4} + \frac{y_3}{y_1 y_4}$$

$$+ \frac{y_1 y_4}{y_3} + \frac{y_2 y_4}{y_1 y_3} + \frac{y_1}{y_4} + \frac{y_4}{y_2} + \frac{y_2}{y_1 y_4} + \frac{y_3}{y_2 y_4} + \frac{y_4}{y_3} + \frac{1}{y_4} \quad (3.17)$$

$$[1,0,\ldots]_{24}(\vec{z}) = z_1 + \sum_{i=1}^{9} \frac{z_{i+1}}{z_i} + \frac{z_{11} z_{12}}{z_{10}} + \frac{z_{11}}{z_{12}}$$

$$+ \frac{z_{12}}{z_{11}} + \frac{z_{10}}{z_{11} z_{12}} + \sum_{i=1}^{9} \frac{z_i}{z_{i+1}} + \frac{1}{z_1} \quad (3.18)$$

$$[1,0,\ldots]_{25}(\vec{y}) = y_1 + \sum_{i=1}^{10} \frac{y_{i+1}}{y_i} + \frac{y_{12}^2}{y_{11}} + 1$$

$$+ \frac{y_{11}}{y_{12}^2} + \sum_{i=1}^{10} \frac{y_i}{y_{i+1}} + \frac{1}{y_1} \quad (3.19)$$

When going between $SO(2n) = D_n$ and $SO(2n+1) = B_n$ representations, as we do here for $n = \frac{D-2}{2}$ in both the bosonic case ($D = 26$) and the superstring ($D = 10$), there is a mapping from z_i to y_i:

$$y_i = z_i, \quad 1 \le i \le n-2, \quad y_{n-1} = z_{n-1} z_n, \quad y_n = z_n \quad (3.20)$$

and the inverse mapping is $z_{n-1} = \frac{y_{n-1}}{y_n}$ with $z_i = y_i$ for all other i.

We will start with the simplest case of the bosonic string in 26 dimensions, though our reference [8] begins with the open type I superstring.

There is one raising operator α^i_{-m} for each transverse direction $i, 2 \le i \le D-1$ and each level $m \in \mathbb{N}_{>0}$. These combine to give an argument for the plethystic exponential of $[1,0,\ldots]_{24}(z_i) \sum_{m=1}^{\infty} q^m = ([1,0,\ldots]_{25}(y_i) - 1)\frac{q}{1-q}$.

The zero-point energy, in units of α'^{-1}, is -1.

$$Z_{bos}(q, y_i) = \text{PE}\left[([1, 0, \ldots]_{25}(y_i) - 1)\frac{q}{1-q}\right] \tag{3.21}$$

There is a general argument, outlined in [10], that this series does not give negative coefficients of any representation of $SO(25)$ at massive levels, despite there being one in the PE.

It is possible to calculate each level using Mathematica, but we did it using self-written Java and LiE [5] programs.

To generate a given level n in the bosonic case, or the bosonic part of a type I or heterotic superstring spectrum, we start by finding all partitions λ of n. We then rewrite each partition $\lambda = [\lambda_1, \ldots]$ as $1^{n_1} \ldots i^{n_i} \ldots$ where n_i is the number of occurrences of i in λ, and take the tensor product of the symmetrizations of $[1, 0, 0, 0]_8 = [1, 0, 0, 0]_9 - 1$ in the superstring cases or $[1, 0, \ldots]_{24} = [1, 0, \ldots]_{25} - 1$ in the bosonic case to orders n_i for each i. Summing over all partitions of n, we obtain the n'th level of the bosonic partition function, and we obtain the whole partition function by adding together each level weighted by q^n.

Up to mass level 9, the bosonic partition function is given by, assuming representations are of $SO(25)$ unless stated:

$$
\begin{aligned}
Z_{bos} &= \frac{1}{q} + [1,\ldots]_{24} + q[2,\ldots] \\
&+ q^2\left([3,\ldots] + [0,1,\ldots]\right) + q^3\left([4,\ldots] + [2,\ldots] + [1,1,\ldots] + 1\right) \\
&+ q^4\left([5,\ldots] + [3,\ldots] + [2,1,\ldots] + [1,1,\ldots] + [1,\ldots] + [0,1,\ldots]\right) \\
&+ q^5\left([6,\ldots] + [4,\ldots] + [3,1,\ldots] + [3,\ldots] + [2,1,\ldots] + 2[2,\ldots] + [1,1,\ldots]\right. \\
&\left. + [1,\ldots] + [0,2,\ldots] + [0,0,1,\ldots] + 1\right) \\
&+ q^6\left([7,\ldots] + [5,\ldots] + [4,1,\ldots] + [4,\ldots] + [3,1,\ldots] + 2[3,\ldots] + 2[2,1,\ldots]\right. \\
&\left. + [2,\ldots] + [1,2,\ldots] + 2[1,1,\ldots] + [1,0,1,\ldots] + 2[1,\ldots] + 2[0,1,\ldots]\right) \\
&+ q^7\left([8,\ldots] + [6,\ldots] + [5,1,\ldots] + [5,\ldots] + [4,1,\ldots] + 3[4,\ldots] + 2[3,1,\ldots]\right. \\
&+ 2[3,\ldots] + [2,2,\ldots] + 2[2,1,\ldots] + [2,0,1,\ldots] + 4[2,\ldots] + [1,2,\ldots] \\
&\left. + 3[1,1,\ldots] + [1,0,1,\ldots] + 2[1,\ldots] + 2[0,2,\ldots] + [0,1,\ldots] + [0,0,1,\ldots] + 2\right) \\
&+ q^8\left([9,\ldots] + [7,\ldots] + [6,1,\ldots] + [6,\ldots] + [5,1,\ldots] + 3[5,\ldots] + 2[4,1,\ldots]\right. \\
&+ 2[4,\ldots] + [3,2,\ldots] + 3[3,1,\ldots] + [3,0,1,\ldots] + 5[3,\ldots] + [2,2,\ldots] \\
&+ 5[2,1,\ldots] + [2,0,1,\ldots] + 3[2,\ldots] + 2[1,2,\ldots] + 4[1,1,\ldots] + 2[1,0,1,\ldots] \\
&\left. + 4[1,\ldots] + [0,3,\ldots] + [0,2,\ldots] + [0,1,1,\ldots] + 4[0,1,\ldots] + [0,0,1,\ldots]\right) \\
&+ q^9\left([10,\ldots] + [8,\ldots] + [7,1,\ldots] + [7,\ldots] + [6,1,\ldots] + 3[6,\ldots] + 2[5,1,\ldots]\right. \\
&+ 3[5,\ldots] + [4,2,\ldots] + 3[4,1,\ldots] + [4,0,1,\ldots] + 6[4,\ldots] + [3,2,\ldots] \\
&+ 5[3,1,\ldots] + [3,0,1,\ldots] + 5[3,\ldots] + 3[2,2,\ldots] + 6[2,1,\ldots] + 3[2,0,1,\ldots] \\
&+ 8[2,\ldots] + [1,3,\ldots] + 3[1,2,\ldots] + [1,1,1,\ldots] + 7[1,1,\ldots] + 2[1,0,1,\ldots] \\
&+ 4[1,\ldots] + 4[0,2,\ldots] + [0,1,1,\ldots] + 3[0,1,\ldots] + 2[0,0,1,\ldots] \\
&\left. + [0,0,0,1,\ldots] + 3\right)
\end{aligned}
$$

We see stable patterns emerging as the coefficients of $[n_1, \ldots]$ remain the same as both the mass level and n_1 are increased by 1. Fixing $n_2, \ldots$, the number of n_1 for which this is the case increases as the lowest level of agreement does. To explain them, we must first introduce *multiplicity generating functions*, which are the generating functions for the multiplicities at each mass level of a specific representation. To demonstrate stable patterns, we show the multiplicity generating functions for $[3, \ldots]$, $[4, \ldots]$ and $[5, \ldots]$:

$$
\begin{aligned}
Z_{bos} = \ldots &+ [3,\ldots](q^2+q^4+q^5+2q^6+2q^7+5q^8+5q^9+\ldots) \\
&+ [4,\ldots](q^3+q^5+q^6+3q^7+2q^8+6q^9+\ldots) \\
&+ [5,\ldots](q^4+q^6+q^7+3q^8+3q^9)+\ldots
\end{aligned}
$$

We see that the coefficients of q^n in the MGF for $[3,\ldots]$ agree with those of q^{n+1} in that for $[4,\ldots]$ up to the terms in q^6 and q^7 respectively (i.e.the first 4 coefficients), and the MGFs for $[4,\ldots]$ and $[5,\ldots]$ agree in the first 5 coefficients. This suggests that as $n \to \infty$, the MGFs of $[n,\ldots]$ approach q^{n-1} multiplied by a constant polynomial that we refer to as a stable pattern, with the first disagreement, which is subtracted from the stable pattern, occurring at q^{2n}. This is called the first subleading Regge trajectory.

Similar patterns occur when the second and subsequent Dynkin labels are not all zero. We will define stable patterns (or leading Regge trajectories), subleading Regge trajectories and multiplicity generating functions more rigorously in the next section.

For the open type I superstring, we start by defining the following functions in terms of the raising operators which transform in the vector representation of $SO(8)$ in each case:

$$
Z_B = \mathrm{PE}\left[\frac{q}{1-q}[1,0,0,0]_8\right] = \mathrm{PE}\left[\frac{q}{1-q}([1,0,0,0]_9-1])\right] \quad (3.22)
$$

$$
Z_F(f) = \mathrm{PE}_F\left[\frac{f}{1-q}[1,0,0,0]_8\right] = \mathrm{PE}_F\left[\frac{f}{1-q}([1,0,0,0]_9-1])\right] \quad (3.23)
$$

In the NS sector, we have $f = \pm q^{1/2}$, and in the R sector we have $f = \pm q$; we take appropriate linear combinations of the two in order to impose the GSO projection. In particular, in the NS sector the zero-point energy is $-\frac{1}{2}$ and we keep terms with integer powers of q (after multiplying by $q^{-1/2}$); in the R sector we keep the action of an even number of positive-energy oscillators on [0,0,0,1] and an odd number on [0,0,1,0] (Z is the full partition function with both bosonic and fermionic modes, and note that $Z_F(-q) = Z_B^{-1}$ and $[0,0,0,1]_8 + [0,0,1,0]_8 = [0,0,0,1]_9$):

$$Z_{NS} = \frac{1}{2q^{1/2}}\left(Z_F(q^{1/2}) - Z_F(-q^{1/2})\right) \tag{3.24}$$

$$\begin{aligned} Z_R &= \frac{1}{2}\left([0,0,0,1]_8\left(Z_F(q)+Z_F(-q)\right)\right. \\ &\quad \left. +[0,0,1,0]_8\left(Z_F(q)-Z_F(-q)\right)\right) \end{aligned} \tag{3.25}$$

$$Z = Z_B(Z_{NS}+Z_R) \tag{3.26}$$

$$\begin{aligned} &= \frac{1}{2}([0,0,0,1]_8 - [0,0,1,0]_8) \\ &\quad +Z_B\left(Z_{NS}+\frac{1}{2}[0,0,0,1]_9 Z_F(q)\right) \end{aligned} \tag{3.27}$$

We see that the product of Z_B and $Z_F(q)$ gives even coefficients of all representations at all massive levels, because the arguments of the two functions are the same [8], and hence we have no fractional terms at massive levels in Z. The argument from [10] that there are no negative coefficients at massive levels applies here too.

We calculate level n of the R-sector of the fermionic or heterotic part explicitly as we do the for the bosonic case but with antisymmetrization. In the NS-sector, we find all partitions into half-odd integers, or into odd integers. (Since the zero-point energy is $-\frac{1}{2}$, we find partitions of $n+\frac{1}{2}$ into half-odd integers or of $2n+1$ into odd integers to give an integer total level, though in the 4- and 8-supercharge cases we also need to find those of n or $2n$ respectively, since we must multiply by the partition function of the internal dimensions before taking the GSO projection. In the heterotic NS case we keep integer levels, since the zero-point energy is -1; we do not discuss heterotic cases with reduced supersymmetry in this thesis.)

To get the total bosonic and fermionic partition function at a given level n, which can be a half-odd integer in the NS case, we sum all tensor products of bosonic and fermionic levels with total level n. The total partition function is the sum of this for all levels weighted by q^n, or simply the tensor product of the two partition functions. The same applies when 'internal' partition functions are included.

To get the true level, we must add the zero-point energy, which is -1 for the bosonic string and the NS sector of the heterotic string, $-\frac{1}{2}$ for the NS sector of the type I superstring, 0 for the R sector of the type I superstring and 1 for the R sector of the heterotic string.

We see that the massive levels of the open type I superstring decompose into a product of the massive supermultiplet, $[2,0,0,0]_9 + [1,0,0,1]_9 + [0,0,1,0]_9$, with another term. The 'factored' spectrum, up to level 9, is listed in [8]. As in the bosonic case, we see stable patterns emerging as the coefficients of $[n_1, n_2, n_3, n_4]$ remain the same as both the mass level and n_1 are increased by 1. Fixing n_2, n_3 and n_4, the number of n_1 for which this is the case increases as the lowest level of agreement does.

We can tensor two open type I strings together and apply level matching to get the type II strings, in which the massive levels are the same in both the type IIA and type IIB cases. We can also easily obtain the complete closed and open type I spectrum, obtaining the closed (torus and Klein bottle) part by taking the graded symmetric square (symmetric for bosons, antisymmetric for fermions and simply the product of 'cross' terms) and the open (annulus and Möbius strip) part by incorporating the Chan-Paton factors of $SO(32)$ into the partition function, $[2,0,\ldots] + 1$ (symmetric) for odd mass levels and the adjoint $[0,1,0,\ldots]$ (antisymmetric) for even ones, following chapter 6 of [53]. The first few levels are shown in [8] in both cases.

We can do the same for the heterotic string, either the $SO(32)$ case where all 32 heterotic oscillators are in the same (R or NS) sector at the same time or the $E_8 \times E_8$ where there are two sets of 16 oscillators in each of which all oscillators must be in the same sector but the two sets can be in different sectors. Again GSO projections must be applied. Multiplying by the 10-dimensional bosonic partition gives the heterotic side of the partition function, and the overall closed partition function can be obtained by tensoring with the type I partition function and level-matching, similarly to the type II case. (It is usually easier to calculate heterotic spectra, especially in the $E_8 \times E_8$ case, using the bosonic construction outlined in section 11.6 of [54].)

The first few levels of the heterotic strings of both types are listed in [8].

3.3 Systematic treatment of type I superstring compactifications

In this section, and the next chapter, we return to the open type I superstring and its compactifications which preserve some or all of the supersymmetry.

There are many ways of compactifying the superstring to preserve a desired amount of supersymmetry.

The amount of supersymmetry in a 10D theory is reduced by a factor of four, or $\frac{3}{4}$ of the supersymmetry is broken, by compactification on a Calabi-Yau 3-fold. The spinors $\mathbf{4} = [0,1,0]$ and $\bar{\mathbf{4}} = [0,0,1]$ of the $SO(6)$ on the internal space decompose under the $SU(3)$ holonomy to $\mathbf{3}+\mathbf{1} = ([1,0]+[0,0])$ and $\bar{\mathbf{3}}+\mathbf{1} = ([0,1]+[0,0])$ respectively.

Similarly compactification on a CY 2-fold, of which there is only one non-trivial example, K3, reduces the spinors $[1,0]$ and $[0,1]$ of the $SO(4) = SU(2) \times SU(2)$ on the internal space decompose under the $SU(2)$ holonomy to $[1]$ and $2[0]$, preserving (or breaking) half of the supersymmetry. (It is a general rule that compactification on a CY n-fold, of which the only other relevant case is compactification of M-theory on a CY 4-fold down to 3 dimensions, preserves 2^{1-n} of the supersymmetry, here reducing the 32 supercharges down to 4 or $\mathcal{N}_{3d} = 2$, because the spinor and conjugate spinor of $SO(2n)$ together decompose under $SU(n)$ into the sum of all the antisymmetric k-th rank tensors with $0 \leq k \leq n$, and the first and last of these are singlets.)

We can also compactify superstring theory on orbifolds, which are quotients of compact manifolds (usually tori) by finite groups; these can be thought of as singular limits of Calabi-Yau manifolds. K3 necessarily has 2nd Betti number $b_2 = 22$ (or Hodge number $h_{1,1} = 20$) and we see in [74] that this is reproducible by compactification of 4 dimensions on T^2 orbifolded either by $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In [35], several examples, with the Euler number, given by $\chi = \sum_{i=0}^{2n}(-1)^i b_i = \sum_{i,j=0}^{n}(-1)^{i+j}h_{i,j} = 2(h_{1,1} - h2,1)$ with the last step coming from Hodge duality, equal to twice the number of generations of chiral matter in the fundamental of E_6 minus the number of generations of antichiral matter in the antifundamental.

Reducing the dimension of space is also possible using D-brane systems.

Non-geometric compactifications, such as Gepner models, are also possible as ways to reduce the supersymmetry; we see, again in [35], that the 3^5 Gepner model is equivalent to compactification on the quintic CY 3-fold.

However, in this thesis, we concentrate on the *universal* states that are present in all compactifications preserving the desired amount of supersymmetry.

Having seen both Regge-like behaviour, in that the first mass level at which the $[k, 0, \ldots]$ representation occurs in the spectrum is k plus a constant (-1 in the bosonic and 'unfactored' superstring cases and +1 for the 'factored' superstring), and manifest supersymmetry, in that the massive levels of the superstring can be written as a tensor product of the fundamental massive supermultiplet of the theory and another term, we now investigate this systematically for the cases of 4, 8 and 16 preserved supercharges. We start by using the methods we have described to derive the partition functions for the spacetime dimensions, then we form the appropriate products with the spectra of the internal dimensions obtained by conformal field theory methods.

The choice of 4, 6 and 10 dimensions is natural as these are the maximal dimensions in which it is possible to have theories with 4, 8 and 16 supercharges respectively. The dimension of the minimal spinor in d dimensions is $2^{\lfloor d/2 \rfloor + 1}$, divided by 2 if either a Majorana (not symplectic) or Weyl condition can be imposed or 4 if both can be simultaneously imposed. A Majorana condition can be imposed if $d \equiv 0, 1, 2, 3, 4 \mathrm{mod} 8$ and a Weyl condition if $d \equiv 2 \mathrm{mod} 4$.

Spinors of $SO(1,3)$ can have a Majorana property, but not Weyl, hence the R-symmetry group is $(S)U(\mathcal{N}_{4d})$, special for $\mathcal{N}_{4d} = 4$ (because there is no need for CPT conjugation), those of $SO(1,5)$ can be symplectic Majorana and Weyl so there are two types and the R-symmetry group is $Sp(\mathcal{N}_{6d,L}) \times Sp(\mathcal{N}_{6d,R})$, where $Sp(1) \cong SU(2)$ and $Sp(2) \cong SO(5)$, and those of $SO(1,9)$ are Majorana-Weyl so there are again two types and the R-symmetry is $SO(\mathcal{N}_{10d,L}) \times SO(\mathcal{N}_{10d,R})$, vanishing except in the Type IIB case.

When we compactify the $\mathcal{N}_{6d} = (1,0)$ theory on 2 dimensions, the little group $SO(5)$ decomposes to $SO(3) \times SO(2)$ and the $SO(2) \cong U(1)$ joins with the $SU(2) \cong Sp(1)$ R-symmetry to give $U(2)$, which is the R-symmetry in the $\mathcal{N}_{4d} = 2$ case. When we compactify the $\mathcal{N}_{10d} = 1$ theory on 2 dimensions, the little group $SO(9)$ decomposes to $SO(7) \times SO(2)$ and the latter is the $U(1)$ R-symmetry, $\mathcal{N}_{8d} = 1$. Compactifying on 4 dimensions, it decomposes to $SO(5) \times SO(4)$ and the latter is the $SU(2) \times SU(2) \cong Sp(1) \times Sp(1)$ R-symmetry, $\mathcal{N}_{6d} = (1,1)$; compactifying on 6 dimensions it decomposes to $SO(3) \times SO(6)$ and the latter is the $SU(4)$ R-symmetry, $\mathcal{N}_{4d} = 4$.

We see that the existence of 8 supercharges in 4 dimensions, or $\mathcal{N}_{4d} = 2$, gives rise to an internal worldsheet theory in two parts, one of which has $\mathcal{N}_{2d} = 2$ and $c = 3$ and the other has $\mathcal{N}_{2d} = 4$ and $c = 6$. The first of these corresponds to two toroidally-compactified dimensions. In the case of 16 supercharges in 4 dimensions, or $\mathcal{N}_{4d} = 4$, the internal worldsheet theory consists of three $\mathcal{N}_{2d} = 2$ and $c = 3$ parts, so all 6 compact dimensions are toroidal. Therefore we can calculate the spectra with 4, 6 and 10 spacetime dimensions and reduce via toroidal compactification if required. The original version of [7] treated all three theories as 4-dimensional theories and then assembled the higher-dimensional theories via decomposition of the 4-dimensional R-symmetry into $SO(d-4)$ and the residual R-symmetry, but the method used in the current version and in this thesis is more convenient.

In this thesis, we ignore the compactification-dependent Kaluza-Klein and winding modes. Thus, determining the lower dimensional spectra becomes a group theoretical problem of branching the associated Lorentz and R symmetry groups.

Since we can easily infer the spectra of type IIA/B closed superstring theories with twice the number of supercharges from our open string results by tensoring two copies together and level matching, we will not do this here.

We devote most attention to the $\mathcal{N}_{4d} = 1$ case, since this is both the most mathematically tractable and, since chiral matter is only possible in this case, the only phenomenologically relevant case. It is expected that this case would, after supersymmetry breaking, give rise to phenomenologically interesting string solutions at low energies with the spectrum of certain extensions of the supersymmetric Standard Model.

It is known [37] that amplitudes involving standard model gluons and either 0 or 2 quarks (the number must be even since they are fermions) are independent of the model of compactification and their signatures could be observed at the LHC if the string scale is low enough (in the TeV range), as is the case in compactifications with large extra dimensions. No such signatures have been observed so far, however. By contrast, those involving

4 or more quarks leave model-dependent signatures.

The involvement of massive (string) states in scattering amplitudes between massless states has a precedent in weak decay, where the W- and Z-bosons are, except in top quark decay, heavier than the decaying particles. This is permitted by Heisenberg's uncertainty principle and explains the short range of the interaction.

In our computations we do not grade the states by their fermion numbers, in other words we add the spectra of fermionic states to the bosonic ones rather than subtracting them. As we will see, this leads to partition functions which are not modular-invariant. As expected from supersymmetry, the spectra vanish if a grading by fermion number is introduced.

Open string states carry *Chan-Paton factors* at their endpoints, which are attached to D-branes. In oriented string theories, the Chan-Paton degrees of freedom transform in the adjoint representation of the group; in unoriented theories the representation depends on the mass level, as shown in chapter 6 of [53]. We do not include these factors in our partition functions.

We start, in Section 4.1, by developing the foundations for refined superstring partition functions. Using light-cone quantization, we compute the $SO(D-1)$-covariant spacetime partition functions in the bosonic, NS and R sectors. We then, in Section 4.2, describe the universal spectra of the internal dimensions, adapting those from [12] in the 4-supercharge $\mathcal{N}_{4d} = 1$ case and [11] in the 8-supercharge $\mathcal{N}_{6d} = (1, 0)$ case to our requirements, and in Sections 4.3, 4.4 and 4.5, tensor them together with the partition functions of the spacetime dimensions to give overall partition functions which are super Poincaré covariant. Factoring them into super-Poincaré multiplets, we can then obtain *multiplicity generating functions*, where we choose a representation and count, for each mass level, the number of times the representation occurs at each level, as opposed to grouping by mass level and counting how many of each representation there are in the spectrum at that level. These power series are what give rise to the stable patterns, also called (leading) Regge trajectories, which we derive both analytically and empirically, from the tabulated data up to mass level 25, in the $\mathcal{N}_{4d} = 1$ case, seeing that they agree, and empirically in the 8- and 16-supercharge cases.

4 The covariant perturbative superstring spectrum

4.1 Spacetime partition functions

This section reviews the construction of a refined partition function for the oscillator modes of the world-sheet fields $\partial X^\mu, \psi^\mu$ which form a supermultiplet on the world-sheet and carry an $SO(1, d-1)$ vector index $\mu = 0, 1, \ldots, d-1$. The quantization, whether lightcone, old covariant or BRST, removes the $\mu = 0, 1$ modes leaving $D-2$ modes which form a vector representation of the $SO(D-2)$ massless little group, although since massive particles with D-dimensional timelike momentum form representations of the massive little group $SO(D-1)$, the dependence on y_i necessarily arranges into characters of this group.

We introduce one fugacity y_i for each pair of oscillators $\partial X^{2i}, \partial X^{2i+1}$ and their (world-sheet) fermionic superpartners ψ^{2i}, ψ^{2i+1}, and we take two linear combinations $\partial X^{i\pm}, \psi^{i\pm}$ having charges ± 2 respectively under the Cartan generator corresponding to that fugacity and 0 under all other Cartan generators. This differs from the last section in which we used Dynkin labels for the powers of the fugacities in each term in the character expansion. We normalize the charge under the i-th Cartan generator to ± 2 rather than ± 1 so that the weights of spinor representations have (odd) integer charges under all the Cartan generators rather than half-odd integer charges.

Recall that we define the character of a representation of a group as the sum of terms consisting of products of the fugacities raised to powers specified by the charge of each weight under the Cartan subalgebra generator corresponding to that fugacity. We therefore see that the vector representation of $SO(D-1)$ decomposes into those of $SO(3)$ as follows:

$$[1,0,\ldots]_{SO(D-1)}(\vec{y}) = \sum_{i=1}^{(D-2)/2}(y_i^2+\frac{1}{y_i^2})+1 = \sum_{i=1}^{(D-2)/2}[2]_{SO(3)}(y_i) - \frac{1}{2}(D-4) \tag{4.1}$$

In the $SO(D-1)$ characters, $\vec{y}$ refers to the vector of all $\frac{1}{2}(D-2)$ fugacities y_i, and in the $SO(3)$ case to one specific $SO(D-1)$ fugacity, with the sum (in the character of the vector) or product (in the partition function and the character of the spinor representation) being taken over all $SO(D-1)$ fugacities. This would not be the case if we used Dynkin labels. Therefore, we will first calculate the spectra in the 4-dimensional case with 2 directions perpendicular to the lightcone (or not removed by the physical/null or closed/exact cohomology in old covariant or BRST quantization respectively), and then build those in the $D>4$ case as products of $\frac{1}{2}(D-2)$ copies of the $D=4$ spectra.

In the 4D case, we will first consider the contribution to the refined partition function made by the bosonic oscillators, then that from the fermionic ones in both the NS and R sectors. We will then multiply them together to get the full spacetime NS- and R-sector partition functions, and also quote these in unrefined form, and we will finish this section with a discussion of obtaining $D>4$ spectra from products of 4D ones.

We recall that the character of the $[n]$ representation of $SO(3)$ (or $SU(2)$), of dimension $n+1$, is given by:

$$[n]_y = \sum_{k=-n/2}^{n/2} y^{2k} \tag{4.2}$$

There is one bosonic raising operator for each positive integer mode and each direction transverse to the lightcone, so the contribution of the lightcone bosons to the refined partition function is given by

$$\begin{aligned}
\chi_B^{SO(3)}(q,y) &= \mathrm{P}E\left[([2]_y - 1)\,(q + q^2 + q^3 + q^4 + \ldots)\right] \\
&= \mathrm{P}E\left[([2]_y - 1)\frac{q}{1-q}\right] \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-y^2q^n)\,(1-y^{-2}q^n)} = \frac{1}{(qy^2;q)_\infty (qy^{-2};q)_\infty} \qquad (4.3) \\
&= -iq^{\frac{1}{12}}(y-y^{-1})\frac{\eta(q)}{\vartheta_1(y^2,q)} \ . \qquad (4.4)
\end{aligned}$$

where the q-Pochhammer symbol $(a,q)_n$ and $(a,q)_\infty$ are defined by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \ , \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad (4.5)$$

and our conventions for the Dedekind eta and the Jacobi theta functions are [1]

$$\begin{aligned}
\eta(q) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(q;q)_\infty \ , \qquad (4.6) \\
\vartheta_1(y,q) &= -iq^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \ , \qquad (4.7) \\
\vartheta_2(y,q) &= q^{\frac{1}{8}}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)\,(1+yq^n)(1+y^{-1}q^n) \ , \qquad (4.8) \\
\vartheta_3(y,q) &= \prod_{n=1}^{\infty}(1-q^n)\,(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2}) \ , \qquad (4.9) \\
\vartheta_4(y,q) &= \prod_{n=1}^{\infty}(1-q^n)\,(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2}) \ , \qquad (4.10)
\end{aligned}$$

where here and throughout the thesis we define

$$y = \exp(2\pi i u) \ , \qquad q = \exp(2\pi i \tau) \ . \qquad (4.11)$$

[1] These conventions are related to, for example, those adopted in Appendix A of [73] by $y = \exp(2\pi i v)$, $q = \exp(2\pi i\tau)$. We refer the reader to this reference for further properties of such functions.

In terms of an infinite sum, the Jacobi theta functions can be written as

$$\vartheta\left[{a \atop b}\right](y,q) = \sum_{m\in\mathbb{Z}} q^{\frac{1}{2}(m-a/2)^2}(e^{-i\pi b}y)^{(m-a/2)} \ , \tag{4.12}$$

where

$$\vartheta_1 = \vartheta\left[{1 \atop 1}\right] \ , \qquad \vartheta_2 = \vartheta\left[{1 \atop 0}\right] \ , \qquad \vartheta_3 = \vartheta\left[{0 \atop 0}\right] \ , \qquad \vartheta_4 = \vartheta\left[{0 \atop 1}\right] \ . \tag{4.13}$$

Explicitly, the first few terms in the power series of $\chi_B^{SO(3)}(q,y)$ can be written in terms of $SO(3)$ characters $[k]_y$ as

$$\begin{aligned}\chi_B^{SO(3)}(q,y) &= 1 + q([2]_y - 1) + q^2[4]_y + q^3([2]_y + [6]_y) \\ &+ q^4([0]_y + 2[4]_y + [8]_y) + q^5(2[2]_y + [4]_y + 2[6]_y + [10]_y) \\ &+ q^6(2[0]_y + [2]_y + 3[4]_y + 2[6]_y + 2[8]_y + [12]_y) \\ &+ q^7(4[2]_y + 3[4]_y + 4[6]_y + 2[8]_y + 2[10]_y + [14]_y) + \ldots \end{aligned} \tag{4.14}$$

We see that only even-labelled representations of $SO(3)$, i.e. those with integer spin, occur, at least up to this order, as we expect from the form of (4.3) and which is masked by the theta function expression.

From such a power series, we are motivated to rewrite (4.3) as an infinite sum of the form

$$\chi_B^{SO(3)}(q,y) = \sum_{k=0}^{\infty}[k]_y f_k(q) \ , \tag{4.15}$$

for some function $f_k(q)$ which depends only on q and not on y. We know that this is possible, because $\chi_B^{SO(3)}(q,y)$ is invariant under $y \leftrightarrow y^{-1}$ and the characters $[k]_y, k \geq 0$ form a complete basis of functions invariant under this interchange. The use of this form of the partition function will become clear later.

In order to do so, we rewrite (4.3) using the q-binomial theorem[2] as

$$\chi_B^{SO(3)}(q,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{y^{2(m-n)}}{(q;q)_m(q;q)_n}q^{m+n} =: \sum_{k=0}^{\infty}[k]_y f_k(q) \ . \tag{4.16}$$

The q-binomial theorem can be shown to be true combinatorically; the coefficient of z^n can be shown to consist of the contributions $q^{\sum_{i=1}^n \lambda_i}$ summed

[2]The version we use states that $\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_\infty}$.

over all partitions $\lambda = (\lambda_1, \ldots)$ consisting of no more than n terms, some of which may be zero, which is equal by transposition to the sum of the contributions for all partitions in which all the λ_i are less than or equal to n, which is given by $(q;q)_n^{-1}$. To obtain (4.16), we multiply together two such expansions, one with $z = qy^2$ and the other with $z = qy^{-2}$.

Before we proceed further, we state an identity that we are going to use many times later. From (4.2) and the residue theorem, we find that

$$\int \mathrm{d}\mu_{SO(3)}(y)\; y^m [n]_y = \begin{cases} \delta_{0,n} & \text{for } m = 0\;, \\ \frac{1}{2}(\delta_{|m|,n} - \delta_{|m|,n+2}) & \text{for } m \neq 0\;, \end{cases} \tag{4.17}$$

where the Haar measure $\mathrm{d}\mu_{SO(3)}$ is given by

$$\int \mathrm{d}\mu_{SO(3)}(y) = \int \mathrm{d}\mu_{SU(2)}(y) = \frac{1}{2}\frac{1}{2\pi i}\oint_{|y|=1} \frac{\mathrm{d}y}{y}(1-y^2)(1-y^{-2}) \tag{4.18}$$

(A simpler Haar measure, in which the division by 2, which is the order of the Weyl group of $SO(3)$, and the multiplication by $(1 - y^{-2})$ are omitted, was introduced in [3] and was used in old versions of [7], but we reverted to the more conventional form for the current version.) We prove (4.17) by expanding out the integrand:

$$\int \mathrm{d}\mu_{SO(3)}(y)\; y^m [n]_y = \frac{1}{2}\frac{1}{2\pi i}\oint_{|y|=1} \frac{\mathrm{d}y}{y}(-y^{m-n-2} + y^{m-n} + y^{m+n} - y^{m+n+2}) \tag{4.19}$$

There are no odd y powers in (4.16), so only integer spin representations occur, i.e. $f_{2k+1}(q) = 0$ for all k, and the expressions for $f_{2k}(q)$ are given as follows:

$$\begin{aligned} f_{2k}(q) &= \int \mathrm{d}\mu_{SO(3)}(y)\; \chi_B^{SO(3)}(q,y)[2k]_y \\ &= \sum_{n=0}^{\infty} \frac{q^{2n+k}}{(q;q)_n (q;q)_{n+k+1}} \left(1 - q - q^{n+k+1}\right) \\ &= (q;q)_\infty^{-2} \sum_{n=1}^{\infty} (-1)^{n-1}(1-q^n)^2 q^{nk+\frac{1}{2}n(n-1)}\;. \end{aligned} \tag{4.20}$$

Our $SO(3)$ character expansion of the bosonic partition function is thus as follows:

$$\chi_B^{SO(3)}(q,y) = (q;q)_\infty^{-2} \sum_{n=1}^{\infty} (-1)^{n-1}(1-q^n)^2 \sum_{k=0}^{\infty} q^{nk+\frac{1}{2}n(n-1)}[2k]_y \ . \quad (4.21)$$

Note that the pattern $\sum_{n=1}^{\infty}(-1)^{n-1}q^{nk}[2k]_y \ldots$, where the $\ldots$ ellipsis does not depend on y and k, is described in section 6 of [10] as an alternating sequence of additive and subtractive Regge trajectories of slope $\frac{1}{n}$, with the power of q on the x-axis and k on the y-axis. This is the source of the stable patterns as described in the previous section in bosonic string theory. We will calculate these explicitly later in this section for the spacetime spectra, both bosonic and combined bosonic and fermionic in both the NS and R sectors, and the main focus of this thesis is to investigate such patterns, if they occur, for the superstring spectra that we will consider later. (Our heuristic treatment of the $\mathcal{N}_{10d} = 1$ case with 16 supercharges shows that they do.) These patterns, or the leading Regge trajectory at least, are called stable because, as $m \to \infty$, more and more coefficients, i.e. the first m, of the multiplicity generating function for $[2m]$ match those of the stable pattern.

Multiplicities of representations $[2m]$ and their asymptotics

Let us determine the multiplicity of irreducible $SO(3)$ representations $[2m]$ at each mass level. Recall the orthogonality of characters with respect to the Haar measure:

$$\int \mathrm{d}\mu_{SO(3)}(y)[m]_y[n]_y = \delta_{mn} \ . \quad (4.22)$$

From (4.21), we find that the generating function of the multiplicity of $[2m]$ is equal to $f_{2m}(q)$:

$$\begin{aligned} M(\chi_B^{SO(3)}, [2m]; q) &= \int \mathrm{d}\mu_{SO(3)}(y)\ [2m]_y \chi_B^{SO(3)}(q,y) \\ &= (q;q)_\infty^{-2} \sum_{n=1}^{\infty} (-1)^{n-1}(1-q^n)^2 q^{\frac{1}{2}n(n-1)} q^{nm} \end{aligned} \quad (4.23)$$

Asymptotics as $m \to \infty$. The expression (4.23) found for multiplicity generating functions greatly simplifies in the limit $m \to \infty$ of large spin and mass level. In order to compute an asymptotic formula in this regime, we apply Laplace's method to our question. Since $0 < q < 1$, the terms in the series peak sharply near the $n = 1$ term as $m \to \infty$. Therefore, it is expected that for any $\epsilon > 0$

$$M(\chi_B^{SO(3)}, [2m]; q) \sim (q;q)_\infty^{-2} \sum_{n=1}^{1+\lfloor\epsilon\rfloor} (-1)^{n-1}(1-q^n)^2 q^{\frac{1}{2}n(n-1)} q^{nm}, \quad m \to \infty \ . \tag{4.24}$$

Now let us write $n = 1 + t$, where t is small compared with 1. Note that

$$q^{\frac{1}{2}n(n-1)} = 1 + \frac{1}{2}(\log q)t + O(t^2) \ , \tag{4.25}$$

Substituting the leading term of this power series into the right hand side of (4.24) and extending the region of summation to ∞, we find that the leading behaviour of $M(\chi_B^{SO(3)}, [2m]; q)$ is given by

$$\begin{aligned} M(\chi_B^{SO(3)}, [2m]; q) &\sim (q;q)_\infty^{-2} \sum_{t=0}^{\infty} (-1)^t (1-q^{t+1})^2 q^{m(t+1)} \\ &= (q;q)_\infty^{-2} \frac{q^m(1-q)^2\left(1-q^{1+m}\right)}{(1+q^m)\,(1+q^{1+m})\,(1+q^{2+m})} \\ &= (q^2;q)_\infty^{-2} \frac{q^m\,(1-q^m)}{(1+q^m)^3} \ , \qquad m \to \infty \ . \end{aligned} \tag{4.26}$$

The higher order corrections can be computed by taking into account the subleading terms of (4.25). Note that the next to leading term of (4.26) is of order $O(q^{2m} \log q)$. Thus, asymptotic formula (4.26) reproduces the exact result up to $O(q^{2m-1})$.

Interpretation and stable pattern. We can extract some information about bosonic string states from (4.26).

- The representation $[2m]$ appears first time in the bosonic partition function $\chi_B^{SO(3)}(q, y)$ at mass level q^m.
- The multiplicities of $[2m]$ at levels $q^{m+\ell}$, for $0 \leq \ell \leq m-1$, are independent of m. We refer to a set of these numbers as a *stable pattern* for bosonic string theory. The generating function for such a stable pattern can be determined by taking a formal limit $m \to \infty$ in

(4.26):

$$\lim_{m\to\infty} q^{-m} M(\chi_B^{SO(3)}, [2m]; q) = (q^2; q)_\infty^{-2} = \prod_{k=2}^{\infty} (1-q^k)^{-2} \qquad (4.27)$$

$$= 1 + 2q^2 + 2q^3 + 5q^4 + 6q^5 + 13q^6 + 16q^7 + 30q^8 + 40q^9 + 66q^{10}$$
$$+ 90q^{11} + 142q^{12} + 192q^{13} + 290q^{14} + 396q^{15} + 575q^{16} + 782q^{17}$$
$$+ 1112q^{18} + 1500q^{19} + 2092q^{20} + 2808q^{21} + 3848q^{22} + 5132q^{23}$$
$$+ O(q^{24}) \ . \qquad (4.28)$$

We do not actually need to calculate the explicit asymptotic expression to derive this stable pattern, but can instead simply read it off as the $n = 1$ term in the series in (4.23). Note that terms with low orders in this power series are in agreement with the data presented in Table 6b of [10]. In actual fact terms match up to level q^{2m} inclusive, i.e. for $\ell = m$ as well as for $0 \leq \ell \leq m-1$, because the second Regge trajectory starts at level q^{2m+1} on account of $\frac{1}{2}n(n-1) = 1$ for $n = 2$. The exclusion of level q^{2m} occurs because of taking the asymptotic expression.

4.1.1 The NS sector in $d = 4$

Under NS boundary conditions, there is one raising operator for each positive half-odd integer and each transverse direction, so the worldsheet superpartners ψ^i of the lightcone bosons contribute

$$\begin{aligned} f_{\mathrm{NS}}(q;y) &= \mathrm{PE}_F\left[([2]_y - 1)\frac{q^{\frac{1}{2}}}{1-q}\right] \\ &= \prod_{n=1}^{\infty}(1+y^2q^{n-1/2})(1+y^{-2}q^{n-1/2}) \qquad (4.29) \\ &= q^{\frac{1}{24}}\frac{\vartheta_3(y^2,q)}{\eta(q)} \ . \qquad (4.30) \end{aligned}$$

to the spacetime partition functions. We shall rewrite this function as an infinite sum by means of Jacobi's triple product identity, which is another way of expressing (4.30) :

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^m \ . \qquad (4.31)$$

Applying identity (4.31) with $x = q^{1/2}$ and $z = y^2$ to (4.29), we obtain

$$\begin{aligned} f_{\rm NS}(q,y) &= (q;q)_\infty^{-1} \sum_{m=-\infty}^{+\infty} y^{2m} q^{m^2/2} &(4.32)\\ &= (q;q)_\infty^{-1} \sum_{m=0}^{\infty} q^{\frac{1}{2}m^2}(1-q^{m+\frac{1}{2}})[2m]_y \ , &(4.33)\end{aligned}$$

where (4.33) can be obtained by applying (4.17) and the orthogonality of the characters to (4.32) as follows:

$$\begin{aligned} &\int {\rm d}\mu_{SO(3)}(y)\ f_{\rm NS}(q,y)[2k]_y \\ &= (q;q)_\infty^{-1}\left[\sum_{m=0}^{\infty} q^{m^2/2}\delta_{m,k} - \sum_{m=-\infty}^{-1} q^{m^2/2}\delta_{-m,k+1}\right] \\ &= (q;q)_\infty^{-1}\left(q^{\frac{1}{2}k^2} - q^{\frac{1}{2}(k+1)^2}\right) = (q;q)_\infty^{-1} q^{\frac{1}{2}k^2}(1-q^{k+\frac{1}{2}}) \ . \qquad (4.34)\end{aligned}$$

Let us combine the bosonic partition function with the NS-sector contribution. Using (4.3), (4.33) and the multiplication rule $[2m]\cdot[2k] = \sum_{l=|k-m|}^{k+m}[2l]$, we find that

$$\begin{aligned} \chi_{\rm NS}^{SO(3)}(q,y) &:= \chi_B^{SO(3)}(q,y) f_{\rm NS}(q,y) = -iq^{1/8}(y-y^{-1})\frac{\vartheta_3(y^2,q)}{\vartheta_1(y^2,q)} &(4.35)\\ &= (q;q)_\infty^{-3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^{m+\frac{1}{2}})(1-q^n)^2 \\ &\times\ q^{\frac{1}{2}n(n-1)+\frac{1}{2}m^2}\sum_{k=0}^{\infty} q^{nk}\sum_{\ell=|k-m|}^{k+m}[2\ell] \ . &(4.36)\end{aligned}$$

We can manipulate this expression in order to rewrite it as $\sum_{k=0}^{\infty} f_{kmn}(q)[2k]$, for some function $f_{kmn}(q)$. In order to determine this function, we use the orthogonality of characters:

$$\begin{aligned} f_{kmn}(q) &= \int \mathrm{d}\mu_{SO(3)}(y) \sum_{k'=0}^{\infty} q^{nk'} \sum_{\ell=|k'-m|}^{k'+m} [2\ell]_y \; [2k]_y \\ &= \frac{q^{n|k-m|} - q^{n(k+m+1)}}{1-q^n} \; . \end{aligned} \tag{4.37}$$

We could alternatively do this by rearrangement of the inequalities to express the range of k in terms of ℓ and swapping the order of the sums over k and ℓ:

$$\begin{aligned} \ell \geq k - m &\implies k \leq \ell + m \\ \ell \geq m - k &\implies k \geq m - \ell \\ \ell \leq k + m &\implies k \geq \ell - m \end{aligned} \tag{4.38}$$

We then relabel $k \leftrightarrow \ell$, giving the same expression for $f_{kmn}(q)$. Therefore, we obtain, by either method,

$$\begin{aligned} \chi_{\mathrm{NS}}^{SO(3)}(q,y) &= (q;q)_\infty^{-3} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} (1-q^{m+\frac{1}{2}})(1-q^n) \\ &\times \; q^{\frac{1}{2}[n(n-1)+m^2]} \sum_{k=0}^{\infty} \left(q^{n|k-m|} - q^{n(k+m+1)}\right) [2k] \; . \end{aligned} \tag{4.39}$$

We emphasise that the $SO(3)$ irreducible representations with odd Dynkin labels do not appear in the partition function $\chi_{\mathrm{NS}}^{SO(3)}(q,y)$. This is as expected on account of their absence from both (4.3) and (4.29), but is masked by the theta function expression in (4.35).

In terms of a power series in q, the first few terms are given explicitly by

$$\begin{aligned}
\chi_{\rm NS}^{SO(3)}(q,y) = 1 &+ q^{1/2}([2]-1) + q[2] + q^{3/2}([0]+[4]) + q^2([0]+2[4]) \\
&+ q^{5/2}(2[2]+[4]+[6]) + q^3(3[2]+[4]+2[6]) \\
&+ q^{7/2}(2[0]+2[2]+4[4]+[6]+[8]) \\
&+ q^4(3[0]+3[2]+5[4]+2[6]+2[8]) \\
&+ q^{9/2}([0]+7[2]+4[4]+6[6]+[8]+[10]) \\
&+ q^5([0]+9[2]+7[4]+7[6]+2[8]+2[10]) \\
&+ q^{11/2}(6[0]+8[2]+13[4]+7[6]+6[8]+[10]+[12]) \\
&+ q^6(8[0]+11[2]+17[4]+11[6]+8[8]+2[10]+2[12]) \\
&+ q^{13/2}(4[0]+20[2]+19[4]+18[6]+9[8]+6[10]+[12]+[14]) \\
&+ q^7(6[0]+26[2]+25[4]+25[6]+13[8]+8[10]+2[12]+2[14]) \\
&+ \ldots \qquad (4.40)
\end{aligned}$$

As in the bosonic case, we see hints of a stable pattern emerging here too, one for the integer powers of q and one for half-odd integer powers.

Setting $y=1$, we obtain the unrefined partition function

$$\begin{aligned}
\chi_{\rm NS}^{SO(3)}(q,y=1) &= \chi_B^{SO(3)}(q,y) f_{\rm NS}(q,y) = \prod_{n=1}^{\infty}\left(\frac{1+q^{n-1/2}}{1-q^n}\right)^2 \\
&= (q;q)_\infty^{-3}\vartheta_3(1,q) = q^{-1/8}\frac{\vartheta_3(1,q)}{\eta(q)^3} \ . \qquad (4.41)
\end{aligned}$$

In a previous version of [7] we derived this by replacing the characters $[2m]$ by their dimensions $(2m+1)$ in the refined partition function, but in the current version and here we derive it more directly and simply.

Multiplicities of representations $[2j]$ and their asymptotics

Similarly to the bosonic partition function, we can read off the generating function for the multiplicities of the representations $[2j]$ at different mass levels of the NS superstring

$$\begin{aligned}
M(\chi_{\rm NS}^{SO(3)},[2j],q) &= (q;q)_\infty^{-3}\sum_{m=0}^{\infty}(1-q^{m+\frac{1}{2}})q^{\frac{1}{2}m^2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n) \\
&\times \ q^{\frac{1}{2}n(n-1)}(q^{n|j-m|}-q^{n(j+m+1)}) \ . \qquad (4.42)
\end{aligned}$$

Asymptotics as $j \to \infty$. In this limit, we have $|j - m| \sim j - m$ for a finite m. Futhermore, the summand as a function of n is sharply peaked near $n = 1$, and so we can determine the leading behaviour of the sum over n using Laplace's method as follows (where $\epsilon > 0$):

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)q^{\frac{1}{2}n(n-1)}(q^{n(j-m)} - q^{n(j+m+1)}) \\
&\sim (1-q)\sum_{n=1}^{1+\lfloor\epsilon\rfloor}\left[q^{n(j-m)} - q^{n(j+m+1)}\right] \qquad \text{for } \epsilon > 0 \\
&\sim (1-q)\sum_{n=1}^{1+\lfloor\epsilon\rfloor}\left[q^{n(j-m)} - q^{n(j+m+1)}\right] \qquad \text{for } \epsilon > 0 \\
&\sim (1-q)\sum_{t=0}^{\infty}\left[q^{(t+1)(j-m)} - q^{(t+1)(j+m+1)}\right] \\
&= q^{j-m}(1-q)\frac{1-q^{2m+1}}{(1-q^{1+j+m})(1-q^{j-m})} \ .
\end{aligned} \tag{4.43}
$$

Therefore, we find that

$$
\begin{aligned}
M(\chi_{\rm NS}^{SO(3)},[2j],q) \sim\ & (q;q)_\infty^{-3} q^j (1-q)\left[\sum_{m=0}^{\infty} q^{-m+\frac{m^2}{2}} \frac{\left(1-q^{2m+1}\right)\left(1-q^{\frac{1}{2}+m}\right)}{(1+q^{1+j-m})\,(1+q^{j-m})}\right] \\
\sim\ & (q;q)_\infty^{-3} q^j \frac{1-q}{(1-q^j)^2}\left[\sum_{m=0}^{\infty} q^{\frac{1}{2}(m-1)^2-\frac{1}{2}}\left(1-q^{2m+1}\right)\left(1-q^{\frac{1}{2}+m}\right)\right] \\
\sim\ & (q;q)_\infty^{-3} \sum_{m=0}^{\infty} (1-q^{m+\frac{1}{2}}) q^{\frac{1}{2}m^2}\left[\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)\left\{q^{n(j-m)}-q^{n(j+m+1)}\right\}\right] \\
=\ & (q;q)_\infty^{-3} q^j (1-q) \\
\times & \left[\sum_{m=0}^{\infty} \frac{q^{-m+\frac{m^2}{2}}\left(1-q^{\frac{1}{2}+m}\right)}{(1+q^{1+j-m})\,(1+q^{j-m})} - q\sum_{m=0}^{\infty}\frac{q^{m+\frac{m^2}{2}}\left(1-q^{\frac{1}{2}+m}\right)}{(1+q^{1+j+m})\,(1+q^{2+j+m})}\right] \\
\sim\ & (q;q)_\infty^{-3} \frac{q^j(1-q)}{(1+q^j)^2}\left[\sum_{m=0}^{\infty} q^{-m+\frac{m^2}{2}}\left(1-q^{\frac{1}{2}+m}\right) - q\sum_{m=0}^{\infty} q^{m+\frac{m^2}{2}}\left(1-q^{\frac{1}{2}+m}\right)\right] \\
=\ & (q;q)_\infty^{-3} \frac{q^j(1-q)}{(1+q^j)^2} \\
\times & \left[\frac{(1+2\sqrt{q}-q)+(1-q)\vartheta_3(0,\sqrt{q})}{2\sqrt{q}} - \frac{(1+2\sqrt{q}-q)-(1-q)\vartheta_3(0,\sqrt{q})}{2\sqrt{q}}\right] \\
=\ & (q;q)_\infty^{-3} q^{j-\frac{1}{2}}\left(\frac{1-q}{1+q^j}\right)^2 \vartheta_3(1,q)\ + O(q^{2j-1}) \qquad (4.44)
\end{aligned}
$$

Note that asymptotic formula (4.44) reproduces the exact result up to the order $q^{2j-\frac{3}{2}}$. We emphasise that the representation $[2j]$ appears first time at mass level $q^{j-\frac{1}{2}}$. One can also see this by observing that for $j \geq m$, the lowest term in the power series for given m, n is at mass level $nj + \frac{1}{2}\left[(m-n)^2 - n\right]$ and minimizing over m this gives a lowest term at level $n(j-\frac{1}{2})$ so we do get an alternating sequence of Regge trajectories, of the same slopes $\frac{1}{n}$, and a stable pattern, this time from $j-\frac{1}{2}$ to $2j-\frac{3}{2}$, in the same way as we do in the bosonic case.

In [10], the individual n summands of (4.42) are interpreted as an alternating sequence of additive and subtractive Regge trajectories of slope $\frac{1}{n}$. In the notation of equation (6.2) of that reference, the $M(\chi_{\rm NS}^{SO(3)},[2j],q)$ are expanded as

$$M(\chi_{\rm NS}^{SO(3)}, [2j], q) \;=\; q^j\,\tau_1^{\rm NS}(q) \;-\; q^{2j}\,\tau_2^{\rm NS}(q) \;+\; q^{3j}\,\tau_3^{\rm NS}(q) \;-\; \dots$$
$$= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,q^{\ell j}\,\tau_\ell^{\rm NS}(q)\ . \tag{4.45}$$

Setting $|j-m| = j-m$ in (4.42) leads to the following asymptotic expressions for the $\tau_\ell^{\rm NS}$:

$$\tau_\ell^{\rm NS}(q) = (q;q)_\infty^{-3}\,q^{-\frac{1}{2}\ell}\,(1-q^\ell)\sum_{m=0}^{\infty} q^{\frac{1}{2}(m-\ell)^2}\,(1-q^{m+\frac{1}{2}})\,(1-q^{2m\ell+\ell}) \tag{4.46}$$

The stable pattern. The generating function of the stable pattern can be determined by projecting the sum in (4.45) to the first term (or, equivalently, by taking the limit $j\to\infty$):

$$\lim_{j\to\infty} q^{-j}M(\chi_{\rm NS}^{SO(3)}, [2j], q) = \tau_1^{\rm NS}(q)$$
$$= (q;q)_\infty^{-3}q^{-1/2}(1-q)^2\vartheta_3(1,q) \tag{4.47}$$
$$= \Big(2 + 2q + 8q^2 + 14q^3 + 34q^4 + 58q^5 + 120q^6 + 204q^7 + 378q^8 + 632q^9$$
$$+ 1096q^{10} + 1786q^{11} + 2968q^{12} + \dots\Big)$$
$$+ \frac{1}{\sqrt{q}}\Big(1 + q + 6q^2 + 9q^3 + 24q^4 + 42q^5 + 88q^6 + 151q^7 + 287q^8$$
$$+ 480q^9 + 846q^{10} + 1388q^{11} + 2326q^{12} + \dots\Big)\ . \tag{4.48}$$

Note that terms with low orders in the power series (4.48) are in agreement with the data presented in Table 6c of [10].

4.1.2 The R sector in $d = 4$

The R sector fermionic oscillators have zero modes and so the ground state is degenerate, forming a spinor representation of the (massive) little group since the oscillators are $\frac{1}{\sqrt{2}}$ times the generators of the Clifford algebra. In the $D = 4$ case, the two weights in the spinor representation have charges ± 1 under the single generator of the Cartan subalgebra and so contribute $y + y^{-1}$ to the refined partition function. For the positive-energy modes, there is one mode for each positive integer and each direction transverse to the lightcone. Therefore, the R sector of the worldsheet superpartners ψ^i of the lightcone bosons contributes

$$
\begin{aligned}
f_{\mathrm{R}}(q,y) &= (y+y^{-1})\mathrm{PE}_F\left[([2]_y-1)\frac{q}{1-q}\right] \\
&= (y+y^{-1})\prod_{n=1}^{\infty}(1+y^2q^n)(1+y^{-2}q^n) \qquad (4.49)\\
&= q^{-\frac{1}{12}}\frac{\vartheta_2(y^2,q)}{\eta(q)} \qquad (4.50)
\end{aligned}
$$

to the spacetime partition function. Again, it will turn out to be beneficial to rewrite this function as an infinite sum. We proceed as follows. Replacing z by xz in (4.31), we obtain

$$
\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n}z)(1+x^{2n-2}z^{-1}) = \sum_{m=-\infty}^{+\infty} x^{m^2+m}z^m \ . \qquad (4.51)
$$

Using the identity

$$
\prod_{n=1}^{\infty}(1+x^{2n-2}z^{-1}) = (1+z^{-1})\prod_{n=1}^{\infty}(1+x^{2n}z^{-1}) \ , \qquad (4.52)
$$

we arrive at

$$
(z^{\frac{1}{2}}+z^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1+x^{2n}z)(1+x^{2n}z^{-1}) = \frac{\sum_{m=-\infty}^{+\infty} x^{m^2+m}z^{m+1/2}}{\prod_{n=1}^{\infty}(1-x^{2n})} \ . \qquad (4.53)
$$

Applying identity (4.53) to (4.49) with $x=q^{1/2}$ and $z=y^2$, we have

$$
\begin{aligned}
f_{\mathrm{R}}(q,y) &= (q;q)_\infty^{-1}\sum_{m=-\infty}^{+\infty} y^{2m+1}q^{m(m+1)/2} \\
&= (q;q)_\infty^{-1}\sum_{m=0}^{\infty} q^{\frac{1}{2}m(m+1)}(1-q^{m+1})[2m+1]_y \\
&= q^{-1/8}(q;q)_\infty^{-1}\sum_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} q^{\frac{1}{2}m^2}(1-q^{m+\frac{1}{2}})[2m]_y \ , \qquad (4.54)
\end{aligned}
$$

where the second equality follows from (4.17) and the orthogonality of the characters.

Let us combine the bosonic partition function with the NS-sector contribution. Using (4.3), (4.49) and the multiplication rule $[2m+1]\cdot[2k] = \sum_{l=|k-m-\frac{1}{2}|}^{k+m+\frac{1}{2}}[2l]$, where l sums over half-odd integers, we find that

$$\chi_{\mathrm{R}}^{SO(3)}(q,y) := \chi_B^{SO(3)}(q,y) f_{\mathrm{R}}(q,y) = -i(y-y^{-1})\frac{\vartheta_2(y^2,q)}{\vartheta_1(y^2,q)} \tag{4.55}$$

$$\begin{aligned} &= q^{-\frac{1}{8}}(q;q)_\infty^{-3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^{m+1})(1-q^n)^2 q^{\frac{1}{2}n(n-1)+\frac{1}{2}(m+\frac{1}{2})^2} \\ &\times\sum_{k=0}^{\infty}q^{nk}\sum_{\ell=|k-m-\frac{1}{2}|}^{k+m+\frac{1}{2}}[2\ell] \\ &= q^{-\frac{1}{8}}(q;q)_\infty^{-3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^{m+1})(1-q^n) q^{\frac{1}{2}[n(n-1)+(m+\frac{1}{2})^2]} \\ &\times\sum_{k=0}^{\infty}\left(q^{n|k-m|}-q^{n(k+m+2)}\right)[2k+1]\ . \end{aligned} \tag{4.56}$$

Again, this can be rewritten as $\sum_{k=0}^{\infty} f_{kmn}(q)[2k+1]$, for some function $f_{kmn}(q)$ (different to the one for the NS sector). In order to determine this function, we use the orthogonality of characters:

$$\begin{aligned} f_{kmn}(q) &= \int \mathrm{d}\mu_{SO(3)}(y)\sum_{k'=0}^{\infty}q^{nk'}\sum_{\ell=|k'-m|}^{k'+m}[2\ell]_y\ [2k+1]_y \\ &= \frac{q^{n|k-m|}-q^{n(k+m+2)}}{1-q^n}\ . \end{aligned} \tag{4.57}$$

We could alternatively do this by rearrangement of the inequalities to express the range of k in terms of ℓ and swapping the order of the sums over k and ℓ:

$$\begin{aligned} \ell \geq k-m-\frac{1}{2} &\implies k \leq \ell+m+\frac{1}{2} \\ \ell \geq m+\frac{1}{2}-k &\implies k \geq m+\frac{1}{2}-\ell \\ \ell \leq k+m+\frac{1}{2} &\implies k \geq \ell-m-\frac{1}{2} \end{aligned} \tag{4.58}$$

We then relabel $\ell \to \ell-\frac{1}{2}$ (taking $[2\ell]$ to $[2\ell+1]$ to emphasize the odd Dynkin label; the form with $\ell \in \mathbb{Z}_{>0}+\frac{1}{2}$ however makes rearrangement of the inequalities easier) followed by $k \leftrightarrow \ell$, giving the same expression for (this) $f_{kmn}(q)$. This resembles (4.39) up to a shift in the summations over m, k by $\pm\frac{1}{2}$. We emphasise that $SO(3)$ irreducible representation with even Dynkin labels do not appear in the R sector partition function $\chi_{\mathrm{R}}^{SO(3)}(q,y)$. This is easily seen from (4.49).

In terms of a power series, the first few powers of the partition function are explicitly given by

$$\begin{aligned}\chi_{\mathrm{R}}^{SO(3)}(q,y) = {} & [1] + 2[3]q + 2([1]+[3]+[5])q^2 + (4[1]+4[3]+4[5]+2[7])q^3 \\ & + (6[1]+10[3]+8[5]+4[7]+2[9])q^4 \\ & + (12[1]+18[3]+16[5]+10[7]+4[9]+2[11])q^5 \\ & + (22[1]+32[3]+30[5]+22[7]+10[9]+4[11]+2[13])q^6 \\ & + (36[1]+58[3]+56[5]+40[7]+24[9]+10[11]+4[13]+2[15])q^7 \\ & + \dots \end{aligned} \tag{4.59}$$

Again we see hints of a stable pattern arising.

Setting $y = 1$, we obtain the unrefined partition function

$$\chi_{\mathrm{R}}^{SO(3)}(q, y=1) = 2\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^2 = q^{-\frac{1}{8}}(q;q)_\infty^{-3}\vartheta_2(1,q) = \frac{\vartheta_2(1,q)}{\eta(q)^3} \tag{4.60}$$

Again for simplicity we derive this directly rather than via the refined partition function, which we did in an earlier version of [7].

Multiplicities of representations $[2j+1]$ and their asymptotics

The generating function for the multiplicities of the representations $[2j+1]$ at different mass levels are

$$\begin{aligned} & M(\chi_{\mathrm{R}}^{SO(3)}, [2j+1], q) \\ & = q^{-\frac{1}{8}}(q;q)_\infty^{-3}\sum_{m=0}^{\infty}(1-q^{m+1})q^{\frac{1}{2}(m+\frac{1}{2})^2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)q^{\frac{1}{2}n(n-1)} \\ & \times (q^{n|j-m|} - q^{n(j+m+2)}) \end{aligned} \tag{4.61}$$

in close analogy to (4.42). In fact, one can obtain the above formula by shifting $m \to m + \frac{1}{2}$ and $j \to j + \frac{1}{2}$ in (4.42) and multiply by an overall factor $q^{-\frac{1}{8}}$.

Asymptotics as $j \to \infty$. Similarly to the NS-sector, we find that the leading behaviour of $M(\chi_{\rm R}^{SO(3)}, [2j+1], q)$ is

$$\begin{aligned}
&M(\chi_{\rm R}^{SO(3)}, [2j+1], q) \\
&\sim\ q^{-\frac{1}{8}}(q;q)_\infty^{-3} q^{j+\frac{1}{2}} \frac{1-q}{(1+q^j)^2} \left[\sum_{m=0}^{\infty} q^{\frac{1}{2}(m-\frac{1}{2})^2-\frac{1}{2}} \left(1-q^{2m+2}\right)\left(1-q^{m+1}\right) \right] \\
&\sim (q;q)_\infty^{-3} \frac{q^{j-\frac{1}{8}}(1-q)}{(1+q^j)^2} \\
&\times \left[\sum_{m=0}^{\infty} q^{-m+\frac{1}{2}(m+\frac{1}{2})^2} \left(1-q^{m+1}\right) - q^2 \sum_{m=0}^{\infty} q^{m+\frac{1}{2}(m+\frac{1}{2})^2} \left(1-q^{m+1}\right) \right] \\
&= (q;q)_\infty^{-3} \frac{q^{j-\frac{1}{8}}(1-q)}{(1+q^j)^2} \\
&\times \left[\left\{ q^{\frac{1}{8}} + \frac{1}{2}(1-q)\vartheta_2(0,\tau) \right\} - \left\{ q^{\frac{1}{8}} - \frac{1}{2}(1-q)\vartheta_2(0,\tau) \right\} \right] \\
&=\ (q;q)_\infty^{-3} q^{j-\frac{1}{8}} \left(\frac{1-q}{1+q^j} \right)^2 \vartheta_2(1,q)\ + O(q^{2j})\ . \qquad (4.62)
\end{aligned}$$

Note that the representation $[2j+1]$ appears first time at mass level q^j and the asymptotic formula reproduces the exact result up to the order q^{2j-1}. This time, we observe that for $j \geq m$, the lowest term in the power series for given m, n is at mass level $nj + \frac{1}{2}(m-n)(m-n+1)$ this time having a minimum over m of nj so we again get an alternating sequence of Regge trajectories of slopes $\frac{1}{n}$ and a stable pattern, this time from j to $2j-1$.

The Regge trajectories can be obtained explicitly from the expansion of $M(\chi_{\rm R}^{SO(3)}, [2j+1], q)$ in powers of q^j as

$$\begin{aligned}
M(\chi_{\rm R}^{SO(3)}, [2j+1], q) \ &=\ q^j\, \tau_1^{\rm R}(q)\ -\ q^{2j}\, \tau_2^{\rm R}(q)\ +\ q^{3j}\, \tau_3^{\rm R}(q)\ -\ \ldots \\
&=\ \sum_{\ell=1}^{\infty} (-1)^{\ell-1}\, q^{\ell j}\, \tau_\ell^{\rm R}(q)\ . \qquad (4.63)
\end{aligned}$$

The $|j-m| = j-m$ asymptotics yield the following expressions for the ℓ'th Ramond trajectory $\tau_\ell^{\rm R}$:

$$\tau_\ell^{\rm R}(q) = (q;q)_\infty^{-3}\, q^{-\frac{1}{8}}\, (1-q^\ell) \sum_{m=0}^{\infty} q^{\frac{1}{2}(m+\frac{1}{2}-\ell)^2}\, (1-q^{m+1})\, (1-q^{2m\ell+2\ell}) \qquad (4.64)$$

The stable pattern. The generating function of the stable pattern can be determined by taking the limit $j \to \infty$:

$$\begin{aligned}
&\lim_{j\to\infty} q^{-j} M(\chi_{\mathrm{R}}^{SO(3)}, [2j+1], q) = \tau_1^{\mathrm{R}}(q) \\
&= (q;q)_\infty^{-3} q^{-1/8} (1-q)^2 \vartheta_2(1,q) \qquad (4.65)\\
&= 2 + 4q + 10q^2 + 24q^3 + 48q^4 + 96q^5 + 184q^6 + 336q^7 + 600q^8 + 1048q^9 \\
&+ 1784q^{10} + 2984q^{11} + 4912q^{12} + 7952q^{13} + 12704q^{14} + 20048q^{15} \\
&+ 31256q^{16} + O(q^{17}) \ . \qquad (4.66)
\end{aligned}$$

Note that terms with low orders in the power series (4.66) are in agreement with the data presented in Table 6d of [10].

4.1.3 Bosonic partition function in $d > 4$

The bosonic partition function in $d = 2n+2$ space-time dimensions can be written as

$$\chi_B^{SO(2n+1)}(q, \vec{y}) = \mathrm{PE}\left[([1,0,\ldots,0]_y^{SO(2n+1)} - 1)\frac{q}{1-q}\right] \ , \qquad (4.67)$$

where $\vec{y} = (y_1, \ldots, y_n)$. Because the character of the vector representation $[1,0,\ldots,0]$ of $SO(2n+1)$, after subtracting 1, is given by the sum of the $SO(3)$ characters $[2]_{y_A} - 1$ for $1 \le A \le n$, the PE can be written as the product of n copies of the 4D partition function as follows:

$$\begin{aligned}
\chi_B^{SO(2n+1)}(q, \vec{y}) &= \mathrm{PE}\left[\frac{q}{1-q}\sum_{k=1}^{n}([2]_{y_k} - 1)\right] \\
&= \prod_{A=1}^{n} \chi_B^{SO(3)}(y_A) \ . \qquad (4.68)
\end{aligned}$$

We substitute (4.21) into this:

$$\begin{aligned}
&\chi_B^{SO(2n+1)}(q, \vec{y}) \\
&= (q;q)_\infty^{-2n} \sum_{\vec{n}\in\mathbb{Z}^n_{>0}} \sum_{\vec{k}\in\mathbb{Z}^n_{\ge 0}} \prod_{A=1}^{n} (-1)^{n_A - 1}(1-q^{n_A})^2 q^{n_A k_A + \frac{1}{2}n_A(n_A-1)} [2k_A]_{y_A}
\end{aligned} \qquad (4.69)$$

For our purpose of resolving the $SO(2n+1)$ content of the partition function, the aim is to rewrite (4.69) in the form

$$\chi_B^{SO(2n+1)}(q,\vec{y}) = \sum_{\lambda_1\geq\cdots\geq\lambda_n\geq 0} (\lambda_1,\ldots,\lambda_n)_{\vec{y}}\; G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)\ , \tag{4.70}$$

where the summations run over highest weight vectors $\vec{\lambda} := (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ subject to inequalities $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. We can convert these into $SO(2n+1)$ Dynkin label notation $[a_1,\ldots,a_n]$ by

$$\begin{aligned} a_i &= \lambda_i - \lambda_{i+1}, \quad 1 \leq i < n \\ a_n &= 2\lambda_n \end{aligned} \tag{4.71}$$

or equivalently

$$\begin{aligned} \lambda_i &= \sum_{j=1}^{n-1} a_j + \frac{1}{2}a_n, \quad 1 \leq i < n \\ \lambda_n &= \frac{1}{2}a_n \end{aligned} \tag{4.72}$$

Since (4.67) involves only the vector representation and the PE generates symmetrizations of the representation, spinor representations of $SO(2n+1)$ do not appear in $\chi_B^{SO(2n+1)}(q,\vec{y})$:

$$G^{B,SO(2n+1)}_{\lambda_1+\frac{1}{2},\ldots,\lambda_n+\frac{1}{2}}(q) = 0, \lambda_k \in \mathbb{Z}\ . \tag{4.73}$$

In general, $G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)$ can be interpreted as a *generating function for the multiplicities* of the $SO(2n+1)$ representation $(\lambda_1,\ldots,\lambda_n)$ in the bosonic string partition function. In subsequent sections, unless stated otherwise, the λ_i may be all integers or all half-odd integers. This differs from the treatment in [7] where the two cases are treated separately.

Some useful relations between $SO(2n+1)$ and $SO(3)$ representations

In general, the character of the irreducible representation $(\lambda_1,\ldots,\lambda_n)$, whether the λ_i are all integers or all half-odd integers, of $SO(2n+1)$ is given by the

Weyl character formula, which in our basis is:

$$(\lambda_1,\ldots,\lambda_n)_y = \frac{\det\left(y_j^{2(\lambda_i+n-i+\frac{1}{2})} - y_j^{-2(\lambda_i+n-i+\frac{1}{2})}\right)_{i,j=1}^{n}}{\det\left(y_j^{2(n-i+\frac{1}{2})} - y_j^{-2(n-i+\frac{1}{2})}\right)_{i,j=1}^{n}} \ . \qquad (4.74)$$

We can derive this expression by observing that any single Weyl reflection, in this basis, now taking $(\lambda_1,\ldots,\lambda_n)$ to be a general weight rather than necessarily a highest weight, either sends $\lambda_i \to -\lambda_i$ for one i, swaps the positions of two λ_i, or swaps the positions of two λ_i and reverses the sign of both, which can be taken to be a combination of one of the second type of reflection and two of the first type. The Weyl vector $(\rho_1,\ldots,\rho_n) = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$, where Δ is the set of roots and Δ_+ is the set of positive roots, is given by $\rho_i = 2(n-i+\frac{1}{2})$. (We later use Δ for the function that converts between $SO(3)^n$ and $SO(2n+1)$ representations and ρ for the conversion factor between the Haar measures.) Substituting into the Weyl character formula, we obtain the determinants in both the numerator and denominator as in (4.74).

Also, the Haar measure for $SO(2n+1)$ can be written as

$$\int \mathrm{d}\mu_{SO(2n+1)}(\vec{y}) = \int \mathrm{d}\mu_{SO(3)}(y_1)\cdots\int \mathrm{d}\mu_{SO(3)}(y_n)\ \rho(\vec{y}) \ , \qquad (4.75)$$

where

$$\rho(\vec{y}) = \frac{1}{n!}\prod_{1\le i<j\le n}(1-y_i^2y_j^2)(1-y_i^{-2}y_j^{-2})\left(1-y_i^2y_j^{-2}\right)\left(1-y_i^{-2}y_j^2\right) \ . \ (4.76)$$

In order to obtain compact formulae for the multiplicity generating functions $G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)$, we have to convert the $SO(3)$ character products in (4.69) into a basis of $(\lambda_1,\ldots,\lambda_n)_{\vec{y}}$, i.e. we have to find the Δ coefficients in the basis transformation

$$\prod_{A=1}^{n}[2k_A]_{y_A} = \sum_{\lambda_1\ge\ldots\ge\lambda_n\ge 0}\Delta\left(\lambda_1,\ldots,\lambda_n;2k_1,\ldots,2k_n\right)(\lambda_1,\ldots,\lambda_n)_{\vec{y}} \ . (4.77)$$

In general, according to (5.10) of [10], it can be shown that the coefficients in this basis transformation are given by

$$
\begin{aligned}
&\Delta(\lambda_1,\dots,\lambda_n;2k_1,\dots,2k_n) := \int \mathrm{d}\mu_{SO(2n+1)}(\vec{y})\ (\lambda_1,\dots,\lambda_n)_{\vec{y}} \prod_{A=1}^{n} [2k_A]_{y_A} \\
&= \frac{1}{n!}\sum_{\sigma_1,\sigma_2\in S_n} \mathrm{sgn}(\sigma_2) \prod_{A=1}^{n} \theta^{2n+\lambda_A-A-\sigma_2(A)}_{|\lambda_A-A+\sigma_2(A)|}\left(k_{\sigma_1(A)}\right) \\
&= \frac{1}{n!}\sum_{\sigma\in S_n} \det\left(\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}\right)\right)^{n}_{A,B=1} \qquad (4.78)
\end{aligned}
$$

where the function $\theta^n_m(k)$ is defined as

$$
\theta^n_m(k) = \begin{cases} 1 & \text{if } m \leq k \leq n\ , \\ 0 & \text{otherwise}\ . \end{cases} \qquad (4.79)
$$

Following [10] but using fugacities y_i instead of chemical potentials θ_A, we prove this formula as follows. We note that the Haar measure of $SO(2n+1)$ is equal to the square of the denominator in the Weyl character formula (4.74), so we substitute it into the first line of (4.78) and rewrite it as an integral over products of $SO(3)$ Haar measures, which we re-obtain by factoring them out of the two determinants converting each matrix element into a character:

$$
\begin{aligned}
&\Delta(\lambda_1,\dots,\lambda_n;2k_1,\dots,2k_n) \qquad (4.80) \\
&= \frac{1}{n!}\prod_{A=1}^{n}\int \mathrm{d}\mu_{SO(3)}(y_A)[2k_A]_{y_A} \\
&\times \det\left([2(\lambda_i+n-i)]_{y_j}\right)^{n}_{i,j=1} \det\left([2(n-i)]_{y_j}\right)^{n}_{i,j=1} \\
&= \frac{1}{n!}\sum_{\sigma_1,\sigma_2\in S_n} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2) \\
&\times \prod_{A=1}^{n}\int \mathrm{d}\mu_{SO(3)}(y_A)[2k_A]_{y_A}[2(n-\sigma_1(A)+\lambda_{\sigma_1(A)})]_{y_A}[2(n-\sigma_2(A))]_{y_A} \\
&\qquad (4.81)
\end{aligned}
$$

This identity holds for λ_i and k_i either all integers or all half-odd integers; $\Delta(\dots)$ vanishes when λ_i are all integers and k_i all half-odd integers or vice versa.

Thus, Eq. (4.78) implies the following expansion rule for $SO(3)$ character products in terms of $SO(2n+1)$ characters in Dynkin label notation: (λ_i are all integers for $1 \leq i \leq n-1$, but λ_n can be either an integer or half an odd integer, in which case k_A are all integers or all half-odd integers respectively)

$$\prod_{A=1}^{n} [2k_A]_{y_A} = \sum_{\vec{\ell} \in \mathbb{Z}^n_{\geq 0}} [\ell_1, \ldots, \ell_{n-1}, 2\ell_n]_{\vec{y}}$$
$$\times \Delta \left(2k_1, \ldots, 2k_n; \ell_1 + \ell_2 + \ldots + \ell_n, \ell_2 + \ldots + \ell_n, \ldots, \ell_n\right) \qquad (4.82)$$

The inverse decomposition formula follows from the $SO(2n+1)$ Haar measure (4.75):

$$[\ell_1, \ldots, \ell_{n-1}, 2\ell_n]_{\vec{y}} = \frac{1}{\rho(\vec{y})} \sum_{\vec{k} \in \mathbb{Z}^n_{\geq 0}} \prod_{A=1}^{n} [2k_A]_{y_A}$$
$$\times \quad (\ell_1 + \ell_2 + \ldots + \ell_n, \ell_2 + \ldots + \ell_n, \ldots, \ell_n; 2k_1, \ldots, 2k_n) \ , \qquad (4.83)$$

where $\rho(\vec{y})$ is defined as in (4.76) and $\vec{\ell} = (\ell_1, \ldots, \ell_n)$. This can be derived by simple manipulation of (4.82) as follows, with ℓ_i and ℓ'_i all integers for $1 \leq i \leq n-1$ and the same conditions (integer or half-odd integer) on ℓ_n

and ℓ'_n as on the k_A:

$$\int \mathrm{d}\mu_{SO(2n+1)}(\vec{y}) \prod_{A=1}^{n} [2k_A]_{y_A} [\ell'_1, \ldots, \ell'_{n-1}, 2\ell'_n]_{\vec{y}}$$
$$= \int \mathrm{d}\mu_{SO(2n+1)}(\vec{y}) \sum_{\vec{\ell}} [\ell_1, \ldots, \ell_{n-1}, 2\ell_n]_{\vec{y}}$$
$$\times \Delta \left(2k_1, \ldots, 2k_n; \ell_1 + \ell_2 + \ldots + \ell_n, \ell_2 + \ldots + \ell_n, \ldots, \ell_n\right)$$
$$\times [\ell'_1, \ldots, \ell'_{n-1}, 2\ell'_n]_{\vec{y}}$$
$$= \Delta \left(2k_1, \ldots, 2k_n; \ell'_1 + \ell'_2 + \ldots + \ell'_n, \ell'_2 + \ldots + \ell'_n, \ldots, \ell'_n\right) \tag{4.84}$$

Multiply both sides by $\prod_{A=1}^{n} [2k_A]_{y_A}$ and sum over k_A:

$$\sum_{k_A} \left(\int \mathrm{d}\mu_{SO(2n+1)}(\vec{y}) \prod_{A=1}^{n} [2k_A]_{y_A} [\ell'_1, \ldots, \ell'_{n-1}, 2\ell'_n]_{\vec{y}} \right) \prod_{B=1}^{n} [2k_B]_{y_B}$$
$$= \sum_{k_A} \left(\prod_{A=1}^{n} \left(\int \mathrm{d}\mu_{SO(3)}(y_A) [2k_A]_{y_A} \right) \rho(\vec{y}) [\ell'_1, \ldots, \ell'_{n-1}, 2\ell'_n]_{\vec{y}} \right) \prod_{B=1}^{n} [2k_B]_{y_B}$$
$$= \rho(\vec{y}) [\ell'_1, \ldots, \ell'_{n-1}, 2\ell'_n]_{\vec{y}}$$
$$= \sum_{k_A} \Delta \left(2k_1, \ldots, 2k_n; \ell'_1 + \ell'_2 + \ldots + \ell'_n, \ell'_2 + \ldots + \ell'_n, \ldots, \ell'_n\right) \prod_{A=1}^{n} [2k_A]_{y_A} \tag{4.85}$$

and the final step is trivial, just divide by $\rho(\vec{y})$.

Generating function for the multiplicities

According to (4.69), the bosonic spacetime partition function in $2n+2$ dimensions depends on Lorentz fugacities through the factor

$$\begin{aligned}
&\sum_{k_1,\ldots,k_n\geq 0} \Delta(\lambda_1,\ldots,\lambda_n;2k_1,\ldots,2k_n)q^{n_1k_1+\ldots+n_nk_n} \\
&= \sum_{k_1,\ldots,k_n\geq 0} \det(\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|}(k_A))^n_{A,B=1}q^{n_1k_1+\ldots+n_nk_n} \\
&= \det\left(\sum_{k_A\geq 0}\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|}(k_A)q^{n_Ak_A}\right)^n_{A,B=1} . \\
&= \prod_{C=1}^n q^{n_C(\lambda_C-C+1)}\det\left(\sum_{k_A=B-1}^{2n-B-1}q^{n_Ak_A}\right)^n_{A,B=1} . \qquad (4.86)
\end{aligned}$$

Let us apply this to (4.69) to compute $G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)$. For $\lambda_1\geq\cdots\geq\lambda_n\geq n-1$, the argument in the absolute value is non-negative and so

$$\begin{aligned}
&\sum_{k_1,\ldots,k_n\geq 0} \Delta(\lambda_1,\ldots,\lambda_n;2k_1,\ldots,2k_n)q^{n_1k_1+\ldots n_nk_n} \\
&= \prod_{A=1}^n q^{n_A(\lambda_A-A+1)}\prod_{1\leq B<C\leq n}(q^{n_C}-q^{n_B})(1-q^{n_C+n_B}) \qquad (4.87)
\end{aligned}$$

for $\lambda_1\geq\cdots\geq\lambda_n\geq n-1$.

This formula can be derived by column-reducing to get the (A,B)-th matrix element (for $B<n$) to equal $q^{n_A(B-1)}+q^{n_A(2n-B-1)}$ and then splitting the determinant into the sum of 2^{n-1} determinants, labelled by an integer $p, 0\leq p\leq 2^{n-1}-1$, with each element in the B-th column containing either the lower or higher power depending on whether the $B-1$-th bit (we choose from the right) of p in binary is 0 or 1. Each determinant factorizes into the product of the Vandermonde determinant and a (different for each term) Schur polynomial both with the specialization $y_A\to q^{k_A}$, weighted by the sign of the permutation required to bring the powers into downward ascending order. We can show that the sum of these gives rise to the product of $(1-q^{n_C+n_B})$; that of $(q^{n_C}-q^{n_B})$ is the Vandermonde determinant.

It is pointed out by [10] and can be checked directly that the contribution from $\lambda_n<n-1$ to the bosonic string partition function is zero. Therefore,

we have

$$
\begin{aligned}
&G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q) \\
&= (q;q)^{-2n}_\infty \sum_{\vec{n}\in\mathbb{Z}^n_+} \prod_{A=1}^{n} (-1)^{n_A-1}(1-q^{n_A})^2 q^{n_A(\lambda_A-A+1)+\frac{1}{2}n_A(n_A-1)} \\
&\times \prod_{1\leq B<C\leq n} (q^{n_C}-q^{n_B})(1-q^{n_C+n_B}) \ , \qquad (4.88)
\end{aligned}
$$

for all $\lambda_1,\ldots,\lambda_n \in \mathbb{Z}$ and $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$.

4.1.4 The contributions from the NS and R sectors in $d > 4$

The contribution from the NS sector can be obtained by taking a product of n copies of (4.39):

$$
\begin{aligned}
&\chi^{SO(2n+1)}_{\mathrm{NS}}(q,\vec{y}) = \prod_{A=1}^{n} \chi^{SO(3)}_{\mathrm{NS}}(q;y_A) \\
&= (q;q)^{-3n}_\infty \sum_{\vec{m}\in\mathbb{Z}^n_{\geq 0}} \sum_{\vec{n}\in\mathbb{Z}^n_+} \prod_{A=1}^{n} (-1)^{n_A+1}\left(1-q^{m_A+\frac{1}{2}}\right)(1-q^{n_A}) \times \\
&q^{\frac{1}{2}[n_A(n_A-1)+m_A^2]} \sum_{\vec{k}\in\mathbb{Z}^n_{\geq 0}}^{\infty} \prod_{A=1}^{n} (q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)})\ [2k_A]_{y_A} \ . \qquad (4.89)
\end{aligned}
$$

Similarly for the contribution from the R sector, the product of n copies of (4.56):

$$
\begin{aligned}
&\chi^{SO(2n+1)}_{\mathrm{R}}(q,\vec{y}) = \prod_{A=1}^{n} \chi^{SO(3)}_{\mathrm{R}}(q;y_A) \\
&= q^{-\frac{n}{8}}(q;q)^{-3n}_\infty \sum_{\vec{m}\in\mathbb{Z}^n_{\geq 0}} \sum_{\vec{n}\in\mathbb{Z}^n_+} \prod_{A=1}^{n} (-1)^{n_A+1}(1-q^{m_A+1})(1-q^{n_A}) \times \\
&q^{\frac{1}{2}[n_A(n_A-1)+\left(m_A+\frac{1}{2}\right)^2]} \sum_{\vec{k}\in\mathbb{Z}^n_{\geq 0}} \prod_{A=1}^{n} (q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+2)})\ [2k_A+1]_{y_A} \ .
\end{aligned}
\qquad (4.90)
$$

The unrefined partition functions can be written as

$$\chi_{\rm NS}^{SO(2n+1)}(q,\{y_i=1\}) \;=\; q^{-n/8}\frac{\vartheta_3(1,q)^n}{\eta(q)^{3n}}\;, \qquad (4.91)$$

$$\chi_{\rm R}^{SO(2n+1)}(q,\{y_i=1\}) \;=\; \frac{\vartheta_2(1,q)^n}{\eta(q)^{3n}}\;. \qquad (4.92)$$

4.2 Internal SCFT on the compact dimensions

We can replace the remaining 6 dimensions of spacetime by any worldsheet superconformal field theory with $c = 9$. The internal SCFT can be quantized using the BRST method.

It is shown explicitly in chapter 18 of [54] that the SCFT description of four- and six dimensional string compactifications with $\mathcal{N}_{4d} = 1, \mathcal{N}_{4d} = 2$ or $\mathcal{N}_{6d} = (1,0)$ spacetime SUSY, the last of which can be compactified on a 2-torus to the second, comprises universal sectors with enhanced $\mathcal{N}_{2d} = 2,4$ worldsheet SUSY [54, 15]. The purpose of this section is to collect the associated charged characters, starting from the expressions given in [12, 11] but adapting the dependence on fugacities s, x and z of the internal symmetries to the R symmetries of the spectrum.

4.2.1 $\mathcal{N}_{2d} = 2$ worldsheet superconformal algebra at $c = 9$

The internal SCFT universal to any four dimensional string compactification with $\mathcal{N}_{4d} = 1$ spacetime SUSY has $\mathcal{N}_{2d} = 2$ worldsheet SUSY. The resulting model independent partition function receives contributions from characters of the $\mathcal{N}_{2d} = 2$ superconformal algebra with central charge $c = 9$. Its representations are characterized by the conformal weight h and the $U(1)$ charge ℓ of their highest weight state. The representations needed to describe $\mathcal{N}_{4d} = 1$ compactifications have $(h,\ell) = (0,0)$ in the NS sector and $(h,\ell) = \left(\frac{3}{8},\frac{3}{2}\right)$ in the R sector. (In our partition functions we normalize the $U(1)$ charge to 1 in the R sector for the purpose of the power of the $U(1)$ fugacity in the partition functions, i.e. we multiply it by $\frac{2}{3}$.)

We will not discuss the details of the internal SCFT here; they can be found in [12], although we use a different $U(1)$ charge (more correctly the SCFT splits into two parts and there are therefore two $U(1)$ charges in the theory; [12] refines the spectrum with a fugacity s raised to the power of (a multiple of) only the first $U(1)$ charge, while we use a (diagonal) combination of the two which is BRST-invariant, which is required in order to generate spacetime symmetries, including the R-symmetry), and [15], where we obtain the first massive level explicitly via BRST quantization.

We first calculate what is called the Verma module, which is constructed by the action of raising operators of the (super)Virasoro algebra on the highest weight state, on which the action of lowering operators vanishes and L_0 has eigenvalue h as above. (This is different to string partition functions, where the states are built up .) From the (super)Virasoro algebra, outlined for the various cases in [53, 54, 12, 11], the action of any raising operator, bosonic or fermionic, wth mode $-n$ raises the eigenvalue of L_0 by n. The Verma module can be irreducible, or it can contain null states which are orthogonal to all other states in the module, so these must be removed. In [11] this is done by finding the lowest null state and then iteratively constructing null states from lower null states; this construction leads to an alternating sum as we ensure each null state is only subtracted out once from the Verma module. The action of raising operators on null states leads to further null states, so the total partition function can be written as this alternating sum times the partition function of the original Verma module.

The internal partition function is calculated by taking the trace of q^{L_0}, refined by s raised to the power of the $U(1)$ charge (or by $rank(G)$ fugacities each raised to the power of the corresponding Cartan subalgebra charge for a general R-symmetry group G). We do not incorporate the factor of $q^{-c/24}$ as in [53, 54, 12] because the total central charge of the theory, including spacetime dimensions and (super)ghosts, is zero.

The NS-sector

The internal character in this sector is given by

$$\begin{aligned}\chi^{\mathcal{N}_{2d}=2,c=9}_{\mathrm{NS},h=0,\ell=0}(q;s) &= (1-q)\chi^{SO(3)}_{\mathrm{NS}}(q,1)\sum_{p\in\mathbb{Z}}\frac{q^{p^2+p-\frac{1}{2}}s^{2p}}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}\\ &= (q;q)^{-3}_{\infty}(1-q)\vartheta_3(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2+p-\frac{1}{2}}s^{2p}}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})} \qquad (4.93)\\ &= (q;q)^{-3}_{\infty}(1-q)\vartheta_3(1,q)\sum_{p=0}^{\infty}s_{2p}\frac{q^{p^2+p-\frac{1}{2}}}{(1+q^{p+\frac{1}{2}})(1+q^{p-\frac{1}{2}})} \qquad (4.94)\end{aligned}$$

where we have introduced the notation

$$s_n = \begin{cases} s^n + s^{-n} & : \ n > 0 \\ 1 & : \ n = 0 \end{cases} \qquad (4.95)$$

to compactly represent the fugacity dependence. Explicitly to the first few orders we have

$$\begin{aligned}&1+q+(2+s_2)q^{3/2}+(3+s_2)q^2+(4+s_2)q^{5/2}+(6+2s_2)q^3\\ &+(10+4s_2)q^{7/2}+(15+6s_2)q^4+(20+8s_2)q^{9/2}+(28+12s_2)q^5\\ &+(42+19s_2+s_4)q^{11/2}+(59+27s_2+2s_4)q^6\\ &+(78+36s_2+2s_4)q^{13/2}+(107+51s_2+3s_4)q^7+O(q^{15/2})\ ,\end{aligned}$$

The *unrefined* internal character (*i.e.* setting s to unity) can be rewritten in terms of modular functions as follows:

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=2,c=9}_{\mathrm{NS},h=0,\ell=0}(q;s=1) &= (q;q)_\infty^{-3}\vartheta_3(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2+p-\frac{1}{2}}(1-q)}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})} \\
&= (q;q)_\infty^{-3}\vartheta_3(1,q)\sum_{p\in\mathbb{Z}} q^{p^2}\left(\frac{1}{1+q^{p+\frac{1}{2}}}-\frac{1}{1+q^{p-\frac{1}{2}}}\right) \\
&= (q;q)_\infty^{-3}\vartheta_3(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2}(1-q^{2p+1})}{1+q^{p+\frac{1}{2}}} \\
&= (q;q)_\infty^{-3}\vartheta_3(1,q)\sum_{p\in\mathbb{Z}}\left(q^{p^2}-q^{(p+\frac{1}{2})^2+\frac{1}{4}}\right) \\
&= q^{1/8}\frac{\vartheta_3(1,q)}{\eta(q)^3}\left[\vartheta_3(1,q^2)-q^{1/4}\vartheta_2(1,q^2)\right] \ . \qquad (4.96)
\end{aligned}$$

The R-sector

The internal character in this sector is given by

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=2,c=9}_{\mathrm{R},h=3/8,\ell=3/2}(q;s) &= (1-q)\chi^{SO(3)}_{\mathrm{R}}(q,1)\sum_{p\in\mathbb{Z}}\frac{q^{p^2-1}\,s^{2p-1}}{(1+q^p)(1+q^{p-1})} \\
&= (q;q)_\infty^{-3}(1-q)\vartheta_2(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2-\frac{9}{8}}s^{2p-1}}{(1+q^p)(1+q^{p-1})} \qquad (4.97) \\
&= (q;q)_\infty^{-3}(1-q)\vartheta_2(1,q)\sum_{p=0}^{\infty}s_{2p+1}\frac{q^{(p+\frac{1}{2})^2}-\frac{3}{8}}{(1+q^{p+1})(1+q^{-p})} \\
& \qquad (4.98)
\end{aligned}$$

Explicitly to the first few orders we have

$$\begin{aligned}
&s_1+2s_1q+6s_1q^2+(2s_3+14s_1)q^3+(4s_3+30s_1)q^4 \\
&+(10s_3+62s_1)q^5+(24s_3+122s_1)q^6+(50s_3+230s_1)q^7+O(q^8) \ .
\end{aligned}$$

The unrefined internal character can be rewritten in terms of modular functions as

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=2,c=9}_{\mathrm{R},h=3/8,\ell=3/2}(q;s=1) &= (q;q)^{-3}_{\infty}\vartheta_2(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2-\frac{9}{8}}(1-q)}{(1+q^p)(1+q^{p-1})} \\
&= (q;q)^{-3}_{\infty}\vartheta_2(1,q)\sum_{p\in\mathbb{Z}} q^{p^2-p-\frac{1}{8}}\left(\frac{1}{1+q^p}-\frac{1}{1+q^{p-1}}\right) \\
&= (q;q)^{-3}_{\infty}\vartheta_2(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2-p-\frac{1}{8}}(1-q^{2p}}{(1+q^p)} \\
&= (q;q)^{-3}_{\infty}\vartheta_2(1,q)\sum_{p\in\mathbb{Z}}\left(q^{(p-\frac{1}{2})^2-\frac{3}{8}}-q^{p^2-\frac{1}{8}}\right) \\
&= q^{-1/4}\frac{\vartheta_2(1,q)}{\eta(q)^3}\left[\vartheta_2(1,q^2)-q^{1/4}\vartheta_3(1,q^2)\right] \; .
\end{aligned} \tag{4.99}$$

Some features

Let us discuss some properties of the above internal characters.

- We have normalized the $U(1)_R$ differently from [12] such that all integer powers of s occur. According to the infinite sums within (4.93) and (4.97), even powers s_{2p} firstly occur along with $q^{p^2+p-1/2}$, i.e. in the NS sector at mass level p^2+p-1, except when $p=0$ when the levels are 0 and $-\frac{1}{2}$ respectively. (The difference comes from the zero point energy which is $-\frac{1}{2}$ for a 10D (total) theory in the NS sector, which is not incorporated into these characters). Odd powers s_{2p-1} of the $U(1)_R$ fugacity, on the other hand, firstly show up at power q^{p^2-1}, which is their mass level in the R sector.

- The unrefined internal R character (4.99) can be derived from the NS counterpart (4.96) by exchanging ϑ_2 and ϑ_3 and multiplying by an overall factor $q^{-3/8}$.

- Both of the unrefined internal characters (4.96) and (4.99) are *not* modular invariant. This can be seen from the modular transformation

$q \mapsto \tilde{q} = e^{-2\pi i/\tau}$,

$$\vartheta_2\left(1,\tilde{q}\right) = \vartheta_4(1,q)\sqrt{-i\tau}\ , \quad \eta\left(\tilde{q}\right) = \eta(q)\sqrt{-i\tau}\ ,$$
$$\vartheta_3(1,\tilde{q}) = \vartheta_3(1,q)\sqrt{-i\tau}\ . \qquad (4.100)$$

We will make use of these transformations later when we come to calculate the total numbers of states at each level in the combined spacetime and internal partition functions.

4.2.2 $\mathcal{N}_{2d} = 4$ worldsheet superconformal algebra at $c = 6$

As stated, but this time not derived, in chapter 18 of [54], the existence of eight supercharges in four or six dimensional spacetime implies that the universal part of the internal SCFT contains a sector with central charge $c = 6$, enhanced $\mathcal{N}_{2d} = 4$ worldsheet SUSY and $SU(2)$ Kac Moody symmetry at level 1. The $c = 6$ representations contributing to the NS sector and R sector of $\mathcal{N}_{4d} = 2$ and $\mathcal{N}_{6d} = (1,0)$ spectra are characterized by values $(h,\ell) = (0,0)$ and $(h,\ell) = \left(\frac{1}{4},\frac{1}{2}\right)$, respectively, of the conformal weight h and the spin ℓ with respect to the $SU(2)$ Kac Moody symmetry. (As with the $U(1)$ charge in the 4-supercharge case, our $SU(2)$ is not the same as that in [11]; again the CFT splits into two pieces each with their own $SU(2)$, [11] uses the first one, while we use a (diagonal) linear combination of the two, again because of the need for BRST invariance in order to generate the R-symmetry.) Again the first massive level is obtained explicitly by BRST quantization in [15].

The second sector of the internal SCFT describing $\mathcal{N}_{4d} = 2$ supersymmetric string compactifications has central charge $c = 3$ and $\mathcal{N}_{2d} = 2$ worldsheet SUSY. This corresponds to two toroidally compactified dimensions with spectrum the same as for two spacetime dimensions. In the $\mathcal{N}_{6d} = (1,0)$ case, they are two spacetime dimensions and the two cases are related by toroidal compactification on those two dimensions.

The NS-sector

The internal character in this sector is given by, written in the first line treating each $SU(2)$ weight separately (almost as though we are considering it as a $U(1)$), in the second line (using (4.17)) as a sum of $SU(2)$ representations times terms independent of r and in subsequent lines its explicit expansion to low orders:

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(q;r) &= \chi^{SO(3)}_{\mathrm{NS}}(q,1)\sum_{m\in\mathbb{Z}} q^{\frac{1}{2}m^2+\frac{1}{4}}\, r^{2m}\ \frac{q^{m-\frac{1}{2}}\ -\ r^{-2}}{1\ +\ q^{m-\frac{1}{2}}} \\
&= (q;q)_\infty^{-3}\vartheta_3(1,q)\sum_{k=0}^{\infty}[2k]_r \frac{(1-q)(1-q^{k+\frac{1}{2}})}{(1+q^{k-\frac{1}{2}})(1+q^{k+\frac{3}{2}})} q^{\frac{1}{2}k^2+k-\frac{1}{2}} \\
&= [0]_r+[2]_r q+([2]_r+[0]_r)q^{3/2}+([2]_r+2[0]_r)q^2+(2[2]_r+2[0]_r)q^{5/2} \\
&+(4[2]_r+2[0]_r)q^3+([4]_r+5[2]_r+4[0]_r)q^{7/2}+(2[4]_r+6[2]_r+7[0]_r)q^4 \\
&+(2[4]_r+10[2]_r+8[0]_r)q^{9/2}+(3[4]_r+16[2]_r+9[0]_r)q^5 \\
&+(6[4]_r+21[2]_r+15[0]_r)q^{11/2}+(9[4]_r+27[2]_r+23[0]_r)q^6 \\
&+(12[4]_r+39[2]_r+27[0]_r)q^{13/2}+([6]_r+17[4]_r+56[2]_r+33[0]_r)q^7 \\
&+O(q^{15/2})\ .
\end{aligned} \tag{4.101}$$

The unrefined internal character for the NS-sector can be written as

$$\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(q;r=1) \quad = q^{1/8}\tfrac{\vartheta_3(1,q)^2}{\eta(q)^3}\left[1-2iq^{1/8}\mu\left(\tfrac{1+\tau}{2},\tau\right)\right]\ , \tag{4.102}$$

where $\mu(u,\tau)$ is an Appell-Lerch sum defined in (4.104); for our purpose, we have

$$\mu\left(\frac{1+\tau}{2},\tau\right) \quad = -\tfrac{i}{\vartheta_3(1,q)}\sum_{m\in\mathbb{Z}}\tfrac{q^{\frac{1}{2}m^2-\frac{1}{8}}}{1+q^{m-\frac{1}{2}}}\ , \tag{4.103}$$

where we have used the fact that $\vartheta_1\left(e^{2\pi i(1+\tau)/2},q\right)=q^{-1/8}\vartheta_3(1,q)$.

The Appell-Lerch sum is defined as follows [16]:[3]

$$\mu(u,\tau) = -\frac{e^{i\pi u}}{\vartheta_1(y,q)}\sum_{m\in\mathbb{Z}}(-1)^m\frac{e^{\pi i m(m+1)\tau+2\pi i m u}}{1-e^{2\pi i m\tau+2\pi i u}}\ , \tag{4.104}$$

where

$$y=\exp(2\pi i u)\ , \qquad q=\exp(2\pi i\tau)\ . \tag{4.105}$$

[3] The notation in this thesis and that in Proposition 1.4 of [16] can be related as follows. Our notation is on the left hand sides of the following equalities: $\mu(u,q)=\mu(u,u,q)$, and $\vartheta_1(u,\tau)=-\vartheta(u,\tau)$.

The R-sector

The internal character in this sector is given as follows. Again the first line treats each $SU(2)$ weight separately, the second line is written (again using (4.17)) as a sum of $SU(2)$ representations times terms independent of r and the subsequent lines are its explicit expansion to low orders:

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{R},h=\frac{1}{4},\ell=\frac{1}{2}}(q;r) &= \chi^{SO(3)}_{\mathrm{R}}(q,1)\sum_{m\in\mathbb{Z}} r^{2m+1}\,\frac{q^m\ -\ r^{-2}}{1\ +\ q^m}\,q^{\frac{1}{2}m^2+\frac{1}{2}m} \\
&= q^{-\frac{1}{8}}(q;q)^{-3}_{\infty}\vartheta_2(1,q)\sum_{k=0}^{\infty}[2k+1]_r\frac{(1-q)(1-q^{k+1})}{(1+q^k)(1+q^{k+2})}q^{\frac{1}{2}k^2+\frac{3}{2}k} \qquad (4.106)\\
&= [1]_r + 2[1]_r q + (2[3]_r + 4[1]_r)q^2 + (4[3]_r + 10[1]_r)q^3 \\
&+ (10[3]_r + 20[1]_r)q^4 + (2[5]_r + 22[3]_r + 38[1]_r)q^5 \\
&+ (6[5]_r + 44[3]_r + 72[1]_r)q^6 + (14[5]_r + 86[3]_r + 130[1]_r)q^7 + O(q^8)\ .
\end{aligned}$$

The unrefined internal character for the R-sector can be written as

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{R},h=\frac{1}{4},\ell=\frac{1}{2}}(q;r=1) &= \frac{\vartheta_2(1,q)}{\eta(q)^3}\sum_{m\in\mathbb{Z}}\left(\frac{q^m-1}{1+q^m}\right)q^{\frac{1}{2}m(m+1)} \\
&= \frac{\vartheta_2(1,q)}{\eta(q)^3}\sum_{m\in\mathbb{Z}}\left[\left(1-\frac{2}{1+q^m}\right)q^{\frac{1}{2}m(m+1)}\right] \\
&= q^{-1/8}\frac{\vartheta_2(1,q)^2}{\eta(q)^3}\left[1-2iq^{1/8}\mu\left(1/2,\tau\right)\right]\ , \qquad (4.107)
\end{aligned}$$

where we have[4]

$$\mu\left(1/2,\tau\right) \ = -\tfrac{i}{\vartheta_2(1,q)}\textstyle\sum_{m\in\mathbb{Z}}\tfrac{q^{\frac{1}{2}m(m+1)}}{1+q^m}\ , \qquad (4.108)$$

where we have used the fact that $\vartheta_1(-1,q)=\vartheta_2(1,q)$.

Some features

- Characters $[n]_r$ of $SU(2)_R$ follow the same highest weight notation as for $SO(3)$, i.e. we have $[1]_r = r + r^{-1}$ for the fundamental representation and $[n]_r = \sum_{k=-n/2}^{+n/2} r^{2k}$ in the general spin $n/2$ case. Again, the infinite sum representations allow to read off the lowest level where individual $SU(2)_R$ representations contribute: Integer spin representa-

[4] This function is also closely related to the function $h_2(q)$ introduced in [11, 13, 14].

tions $[2k]_r$ firstly occur at $q^{k^2/2+k-1/2}$, i.e. at mass level $k^2/2+k-1/2$, except for $[0]_r$ first occurring at q^0 or mass level 0. (Again the actual mass level of first occurrence in the NS sector is this plus $-\frac{1}{2}$, the zero point energy in the NS sector which we have not incorporated into these characters.) Spinorial representations $[2k+1]_r$, on the other hand, firstly show up at $q^{k(k+3)/2}$, i.e. at mass level $k(k+3)/2$.

- Observe that the unrefined internal characters in both NS and R sectors involve Appell-Lerch sums, which are mock modular forms. Since the characters and are holomorphic in q, it is immediate that they are *not* modular invariant.

4.3 Spectrum in $\mathcal{N}_{4d} = 1$ supersymmetric compactifications

This section opens up the main body of this work where the SCFT ingredients introduced so far are applied to counting universal super Poincaré multiplets in the perturbative string spectrum. We start with the phenomenologically relevant and mathematically most tractable $\mathcal{N}_{4d} = 1$ supersymmetric scenario. Its SCFT description requires the internal sector with enhanced $\mathcal{N}_{2d} = 2$ worldsheet SUSY introduced in subsection 4.2.1, independently on the compactification details. The BRST invariant completion of the internal current takes the role of the $U(1)_R$ symmetry generator. Lorentz quantum numbers enter through the partition functions (4.39) and (4.56) of the spacetime SCFT for the ∂X^μ and ψ^μ oscillators, expressed in terms of characters of the massive little group $SO(3)$ in four dimension.

The universal part of the $\mathcal{N}_{4d} = 1$ spectrum is built from both spacetime oscillators and internal operators. On the level of its partition function $\chi^{\mathcal{N}_{4d}=1}(q; y, s)$, this amounts to forming a GSO projected product of NS- and R characters from the spacetime- and internal SCFT, see (4.93) and (4.97) for the latter. In a power series expansion in q, the coefficient of the n'th power q^n comprises characters for the $\mathcal{N}_{4d} = 1$ super Poincaré multiplets occurring at the n'th mass level with $m^2 = n/\alpha'$. The aforementioned massive supercharacters are functions of $SO(3)$ fugacity y and $U(1)_R$ fugacity s.

The fundamental $\mathcal{N}_{4d} = 1$ multiplet[5] consists of 2 real bosonic degrees of freedom and a Majorana fermion with 2 real fermonic on-shell degrees of freedom after taking the Dirac equation into account. The two real bosonic degrees of freedom can be complexified to yield a complex scalar and its complex conjugate; they transform as a singlet under the little group $SO(3)$ and each of them carries opposite R-charges $+1$ and -1. On the other hand, the two real fermonic degrees of freedom transform as a doublet under the little group $SO(3)$ and each of them carries zero R-charge. Thus, these 2+2 states yield the character

$$Z(\mathcal{N}_{4d} = 1) \;\; = \;\; [1]_y + (s + s^{-1}) \; . \tag{4.109}$$

Any other massive representation of $\mathcal{N}_{4d} = 1$ super Poincaré is specified by the little group $SO(3)$ quantum number n and the $U(1)_R$ charge Q of its highest weight state or Clifford vacuum. Its $SO(3) \times U(1)_R$ constituents follow from a tensor product:

$$\begin{aligned} [\![n, Q]\!] \;\; &:= \;\; Z(\mathcal{N}_{4d} = 1) \cdot s^Q [n]_y \;\; = \;\; s^Q [n]_y \big([1]_y + (s + s^{-1}) \big) \qquad (4.110) \\ &= \;\; \begin{cases} s^Q \left([n+1] \; + \; (s + s^{-1}) \, [n] \; + \; [n-1] \right) & \text{for } n \geq 1 \\ s^Q \left([1] \; + \; (s + s^{-1}) \, [0] \right) & \text{for } n = 0 \end{cases} \end{aligned}$$

The super-Poincaré character $[\![n, Q]\!]$ corresponds to $4(n+1)$ states of spin $\frac{n+1}{2}$, $\frac{n}{2}$ and (if $n \neq 0$) $\frac{n-1}{2}$ that can be generated from a Clifford vacuum with spin $n/2$ and $U(1)_R$ charge $Q + 1$[6]. Note that Q is even whenever the maximum spin quantum number $n + 1$ is.

In this setting, we find the (GSO projected) $\mathcal{N}_{4d} = 1$ partition function

$$\chi^{\mathcal{N}_{4d}=1}(q; y, s) \;\; := \;\; \chi_{\rm NS}^{\mathcal{N}_{4d}=1} \,|_{\rm GSO} \, (q; y, s) + \chi_{\rm R}^{\mathcal{N}_{4d}=1} \,|_{\rm GSO} \, (q; y, s) \; , \tag{4.111}$$

where GSO projection removes half odd integer mass levels $\alpha' m^2 \in \mathbb{Z} - \frac{1}{2}$ from the NS sector and interlocks spacetime chirality with $U(1)_R$ charges

[5] As we shall see below, the fundamental multiplet does not appear on its own in both massless and massive spectra. Representations appearing in the massive spectrum arise from certain non-trivial products with the fundamental multiplet.

[6] In this terminology, the first label of $[\![n, Q]\!]$ refers to the average spin of the $SO(3)$ irreducibles. We deviate from the common practice that supermultiplets are referred to through the highest spin therein. The supercharacter $[\![3, 0]\!] = [4] + [2] + (s + s^{-1})[3]$, for instance, describes $U(1)_R$ neutral bosons of spin two and one, and two massive gravitinos of opposite $U(1)_R$ charges.

in the R sector. We can capture this projection through the following [7]. (Note: before imposing the GSO projection, we must multiply the overall NS sector partition function by $q^{-\frac{1}{2}}$ corresponding to the zero-point energy in this sector.)

$$\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q) =$$
$$\frac{1}{2}\,q^{-\frac{1}{2}}\left[\chi_{\mathrm{NS}}^{SO(3)}(q;y)\,\chi_{\mathrm{NS},h=0,\ell=0}^{\mathcal{N}_{2d}=2,c=9}(q;s)\;-\;\chi_{\mathrm{NS}}^{SO(3)}(e^{2\pi i}q;y)\,\chi_{\mathrm{NS},h=0,\ell=0}^{\mathcal{N}_{2d}=2,c=9}(e^{2\pi i}q;s)\right],$$
$$\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q) = \frac{1}{2}\,\chi_{\mathrm{R}}^{SO(3)}(q;y)\,\chi_{\mathrm{R},h=3/8,\ell=3/2}^{\mathcal{N}_{2d}=2,c=9}(q;s)\;. \tag{4.112}$$

In order to compactly represent the leading terms in a power series expansion of the partition function $\chi^{\mathcal{N}_{4d}=1}$, let us introduce the shorthand

$$[\![n,\pm Q]\!] \;:=\; \begin{cases} [\![n,+Q]\!] \;+\; [\![n,-Q]\!] & :\; Q\neq 0 \\ [\![n,0]\!] & :\; Q=0 \end{cases} \tag{4.113}$$

which exploits that $U(1)_R$ charges $\pm Q$ always appear on symmetric footing. The pairing of supermultiplets with opposite (nonzero) $U(1)_R$ charges combines Majorana fermions as they appear in the fundamental multiplet

[7] The formula for the GSO projected R sector is reliable for positive powers $q^{\geq 1}$ only and inaccurate at the massless level: The coefficient of q^0 in $\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}$ is $\frac{1}{2}(y+y^{-1})(s+s^{-1})$ instead of the desired value $ys+(ys)^{-1}$. One can just add to the former $\frac{1}{2}(y-y^{-1})(s-s^{-1})$ to compensate this mismatch. This artifact of the mismatch between massive and massless little groups does not affect the main focus our analysis – the massive particle content. Indeed, the character ys corresponds to the left-handed gaugino and the character $(ys)^{-1}$ corresponds to the right-handed gaugino; they carry opposite R-charge $+1$ and -1 and opposite helicities $+1/2$ and $-1/2$.

(4.109) to Dirac fermions. The content of the first $\mathcal{N}_{4d} = 1$ levels reads

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=1}(q; y, s) \;&=\; \underbrace{\left(y^2 + y^{-2} \,+\, \frac{1}{2}\,(y + y^{-1})\,(s + s^{-1})\right) q^0}_{\text{4 massless states}} \\
&+\; \underbrace{\big(\, [\![3,0]\!] \;+\; [\![0,\pm1]\!] \,\big)\, q}_{\text{24 states at level 1}} \\
&+\; \underbrace{\big(\, [\![5,0]\!] \;+\; [\![3,0]\!] \;+\; 2\,[\![2,\pm1]\!] \;+\; 2\,[\![1,0]\!] \,\big)\, q^2}_{\text{104 states at level 2}} \\
&+\; \big(\, [\![7,0]\!] \;+\; [\![5,0]\!] \;+\; 3\,[\![4,\pm1]\!] \;+\; 5\,[\![3,0]\!] \;+\; 2\,[\![2,\pm1]\!] \\
&\qquad +\; [\![1,\pm2]\!] \;+\; 5\,[\![1,0]\!] \;+\; 3\,[\![0,\pm1]\!] \,\big)\, q^3 \;+\; \mathcal{O}(q^4)\;,
\end{aligned} \tag{4.114}$$

subleading orders up to mass level eight are summarized in Table 4.1. The explicit form of the vertex operators at mass level one[8] can be found in section 5 of [15] (equations (5.3) to (5.6) for bosons and equations (5.14) to (5.18) for fermions) in the RNS framework.

Character multiplicities up to mass level $\alpha' m^2 = 25$ are gathered in table 4.2 and in the tables of appendix 4.B.1.

4.3.1 The total number of states at a given mass level

In this subsection, we focus on the total number of states present at a given mass level and derive the novel asymptotic formula (4.125). These numbers can indeed be obtained by adding up the dimensions of representations presented in table 4.1. Our aim here is to compute such numbers analytically and asymptotically for large mass levels.

[8] Let us discuss about the states at the first mass level. The 24 total states consist of the following multiplets:

(1) **the massive spin 3/2 multiplet** $[\![3,0]\!]$: it contains a massive spin 2 field with 5 on-shell degrees of freedoms (OSDOFs), a massive spin 1 field with 3 OSDOFs, a massive spin 3/2 field with 4 OSDOFs, and a Dirac fermion with 4 OSDOFs; so we have 8+8 real OSDOFs in total

(2) **the massive spin 0 multiplet** $[\![0,\pm1]\!]$: the two constituents $[\![0,1]\!]$ and $[\![0,-1]\!]$ of the massive scalar multiplet correspond to two massless chiral fields, Φ and $\widetilde{\Phi}$ (not complex conjugate to each other) at $Q = \pm1$. The opposite Q-charges are necessary to form an invariant mass term $\Phi\widetilde{\Phi}$ in the superpotential. This multiplet contains 4 + 4 real OSDOFs coming from two complex scalars plus two Majorana fermions; the latter are equivalent to one massive Dirac fermion. Note that the spin 0 multiplet is also referred to as two spin 1/2 multiplets in [15].

$\alpha' m^2$	Representations of $\mathcal{N}_{4d} = 1$ super Poincaré
1	$[\![3,0]\!] + [\![0,\pm1]\!]$
2	$[\![5,0]\!] + [\![3,0]\!] + 2\,[\![2,\pm1]\!] + 2\,[\![1,0]\!]$
3	$[\![7,0]\!] + [\![5,0]\!] + 3\,[\![4,\pm1]\!] + 5\,[\![3,0]\!] + 2\,[\![2,\pm1]\!] + [\![1,\pm2]\!] + 5\,[\![1,0]\!] + 3\,[\![0,\pm1]\!]$
4	$[\![9,0]\!] + [\![7,0]\!] + 3\,[\![6,\pm1]\!] + 7\,[\![5,0]\!] + 4\,[\![4,\pm1]\!] + 2\,[\![3,\pm2]\!] + 12\,[\![3,0]\!]$ $+ 11\,[\![2,\pm1]\!] + 2\,[\![1,\pm2]\!] + 12\,[\![1,0]\!] + 3\,[\![0,\pm1]\!]$
5	$[\![11,0]\!] + [\![9,0]\!] + 3\,[\![8,\pm1]\!] + 7\,[\![7,0]\!] + 5\,[\![6,\pm1]\!] + 2\,[\![5,\pm2]\!] + 17\,[\![5,0]\!] + 18\,[\![4,\pm1]\!]$ $+ 6\,[\![3,\pm2]\!] + 31\,[\![3,0]\!] + 20\,[\![2,\pm1]\!] + 6\,[\![1,\pm2]\!] + 28\,[\![1,0]\!] + [\![0,\pm3]\!] + 15\,[\![0,\pm1]\!]$
6	$[\![13,0]\!] + [\![11,0]\!] + 3\,[\![10,\pm1]\!] + 7\,[\![9,0]\!] + 5\,[\![8,\pm1]\!] + 2\,[\![7,\pm2]\!] + 19\,[\![7,0]\!]$ $+ 21\,[\![6,\pm1]\!] + 8\,[\![5,\pm2]\!] + 45\,[\![5,0]\!] + 39\,[\![4,\pm1]\!] + 15\,[\![3,\pm2]\!] + 72\,[\![3,0]\!]$ $+ 3\,[\![2,\pm3]\!] + 58\,[\![2,\pm1]\!] + 17\,[\![1,\pm2]\!] + 64\,[\![1,0]\!] + 21\,[\![0,\pm1]\!]$
7	$[\![15,0]\!] + [\![13,0]\!] + 3\,[\![12,1]\!] + 7\,[\![11,0]\!] + 5\,[\![10,1]\!] + 2\,[\![9,2]\!] + 19\,[\![9,0]\!] + 22\,[\![8,1]\!]$ $+ 8\,[\![7,2]\!] + 51\,[\![7,0]\!] + 49\,[\![6,1]\!] + 22\,[\![5,2]\!] + 108\,[\![5,0]\!] + 4\,[\![4,3]\!] + 105\,[\![4,1]\!]$ $+ 43\,[\![3,2]\!] + 166\,[\![3,0]\!] + 5\,[\![2,3]\!] + 115\,[\![2,1]\!] + 38\,[\![1,2]\!] + 136\,[\![1,0]\!] + 6\,[\![0,3]\!]$ $+ 66\,[\![0,1]\!]$
8	$[\![17,0]\!] + [\![15,0]\!] + 3\,[\![14,1]\!] + 7\,[\![13,0]\!] + 5\,[\![12,1]\!] + 2\,[\![11,2]\!] + 19\,[\![11,0]\!]$ $+ 22\,[\![10,1]\!] + 8\,[\![9,2]\!] + 53\,[\![9,0]\!] + 52\,[\![8,1]\!] + 24\,[\![7,2]\!] + 125\,[\![7,0]\!] + 4\,[\![6,3]\!]$ $+ 135\,[\![6,1]\!] + 62\,[\![5,2]\!] + 254\,[\![5,0]\!] + 10\,[\![4,3]\!] + 223\,[\![4,1]\!] + 101\,[\![3,2]\!] + 357\,[\![3,0]\!]$ $+ 21\,[\![2,3]\!] + 274\,[\![2,1]\!] + [\![1,4]\!] + 89\,[\![1,2]\!] + 289\,[\![1,0]\!] + 7\,[\![0,3]\!] + 112\,[\![0,1]\!]$

Table 4.1: The content of the first eight $\mathcal{N}_{4d} = 1$ levels.

The starting point is the unrefined partition function obtained by setting the fugacities y and s in (4.111) to unity. The total number of states N_m at the mass level m can be read off from the coefficient of q^m in the power series of $\chi^{\mathcal{N}_{4d}=1}(q; y = 1, s = 1)$.

Supersymmetry implies that

$$\chi_{\rm NS}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y = 1, s = 1) = \chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y = 1, s = 1)\ . \quad (4.115)$$

which can, of course, be checked directly using (4.112), (4.41), (4.60), (4.96) and (4.99). Since the formula for the R sector is simpler, we proceed from there.

$$\begin{aligned} \chi^{\mathcal{N}_{4d}=1}(q; y = 1, s = 1) &= 2\chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y = 1, s = 1) \\ &= \chi_{\rm R}^{SO(3)}(q, y = 1)\chi_{{\rm R}, h=3/8, \ell=3/2}^{\mathcal{N}_{2d}=2, c=9}(q; s = 1) \\ &= q^{-1/4}\frac{\vartheta_2(1,q)^2}{\eta(q)^6}\left[\vartheta_2(1,q^2) - q^{1/4}\vartheta_3(1,q^2)\right]\ . \end{aligned} \quad (4.116)$$

Indeed, the power series of $\chi^{\mathcal{N}_{4d}=1}(q; y = 1, s = 1)$ in q reproduces the numbers presented in the first column of Table 4.13. We mention in passing that $\chi^{\mathcal{N}_{4d}=1}(q; y = 1, s = 1)$ is *not* a modular form.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can be computed from

$$N_m = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}q}{q^{m+1}}\ \chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)\ , \tag{4.117}$$

where $\mathcal{C}$ is a contour around the origin.

Let us compute the number of states N_m in the limit $m \to \infty$. Since the integrand of (4.117) is sharply peaked near $q = 1$, we need to examine the behaviour of $\chi^{\mathcal{N}_{4d}=1}(q; y = 1, s = 1)$ as $q \to 1^-$. The $q \to 1^-$ regime in question is related to the easily accessible $q \to 0$ limit

$$\eta(q) \sim q^{1/24}\ , \qquad \vartheta_3(1,q) \sim 1\ , \qquad \vartheta_4(1,q) \sim 1\ , \qquad q \to 0 \tag{4.118}$$

through modular transformation $q = e^{2\pi i \tau} \mapsto \widetilde{q} = e^{-2\pi i/\tau}$:

$$\begin{aligned}
&\vartheta_2\textbf{-function}: && \vartheta_4\left(1,\tilde{q}\right) = \vartheta_2(1,q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\vartheta_2(1,q) \\
&\Rightarrow && \vartheta_2(1,q) \sim \sqrt{2\pi}(1-q)^{-1/2}, \qquad q \to 1^-\ , && (4.119) \\
&\vartheta_3\textbf{-function}: && \vartheta_3\left(1,\tilde{q}\right) = \vartheta_3(1,q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\vartheta_3(1,q) \\
&\Rightarrow && \vartheta_3(1,q) \sim \sqrt{2\pi}(1-q)^{-1/2}, \qquad q \to 1^-\ , && (4.120) \\
&\eta\textbf{-function}: && \eta\left(\tilde{q}\right) = \eta(q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\eta(q) \\
&\Rightarrow && \eta(q) \sim \sqrt{2\pi}(1-q)^{-1/2}\exp\left(\frac{\pi^2}{6\log q}\right), \qquad q \to 1^-\ , \\
& && && (4.121)
\end{aligned}$$

Hence, we have

$$\vartheta_2(1,q^2) \sim \vartheta_3(1,q^2) \sim \sqrt{2\pi}(1-q^2)^{-1/2}\ , \quad q \to 1^-\ , \tag{4.122}$$

and so as $q \to 1^-$,

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1) \sim (2\pi)^{-3/2}(1-q)^2(1-q^{1/4})(1-q^2)^{-1/2} \\
\times \exp\left(-\frac{\pi^2}{\log q}\right)\ . \qquad (4.123)
\end{aligned}$$

Hence, as $m \to \infty$,

$$N_m \sim (2\pi)^{-3/2} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}q}{q} \, (1-q)^2 (1-q^{1/4})(1-q^2)^{-1/2} \times \exp\left(-\frac{\pi^2}{\log q} - m \log q \right) \; . \qquad (4.124)$$

Observe that the argument of the exponential function has a critical value at $q_0 = \exp(-\pi/\sqrt{m})$; this is the saddle point. The direction of steepest descent at this point is the imaginary direction in q. We deform the contour $\mathcal{C}$ such that it passes through $q = q_0$ and tangent to this direction. The leading contribution comes from expansions around $q = q_0$ in the steepest descent direction. Writing $q = q_0 e^{i\theta}$, we have

$$\begin{aligned} N_m &\sim (2\pi)^{-3/2} (1-q_0)^2 (1-q_0^{1/4})(1-q_0^2)^{-1/2} \\ &\quad \times \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \mathrm{d}\theta \exp\left(-\frac{\pi^2}{i\theta + \log q_0} - m(i\theta + \log q_0) \right) , \quad \epsilon > 0 \\ &\sim (2\pi)^{-3/2} (1-q_0)^2 (1-q_0^{1/4})(1-q_0^2)^{-1/2} \\ &\quad \times e^{2\pi\sqrt{m}} \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \mathrm{d}\theta \exp\left(-\frac{m^{3/2}}{\pi}\theta^2 + O(\theta^3) \right) , \qquad \epsilon > 0 \\ &\sim (2\pi)^{-3/2} (1-q_0)^2 (1-q_0^{1/4})(1-q_0^2)^{-1/2} e^{2\pi\sqrt{m}} \\ &\quad \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \exp\left(-\frac{m^{3/2}}{\pi}\theta^2 \right) \\ &\sim \frac{\pi}{32} m^{-2} \exp\left(2\pi\sqrt{m}\right) \; , \qquad m \to \infty \; . \end{aligned} \qquad (4.125)$$

4.3.2 The GSO projected NS- and R sectors

In what follows, we compute analytic expressions of the refined partition function $\chi^{\mathcal{N}_{4d}=1}(q; y, s)$ and discuss its asymptotic behaviour.

The NS sector

Let us write the partition function $\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1} |_{\mathrm{GSO}} \, (q; y, s)$, defined in (4.112), as

$$\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1} |_{\mathrm{GSO}} \, (q; y, s) = \sum_{k=0}^{\infty} \sum_{p=-\infty}^{\infty} [2k]_y s^{2p} \; F_{k,p}^{\mathrm{NS}}(q) \; , \qquad (4.126)$$

where the function $F^{\rm NS}_{k,p}(q)$ follows from (4.39), (4.93) and (4.112):

$$
\begin{aligned}
F^{\rm NS}_{k,p}(q) &= (q;q)_\infty^{-6}(1-q)q^{p^2+p-1}\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}} \\
&\times \sum_{m=0}^{\infty}(q^{n|k-m|}-q^{n(k+m+1)}) \\
&\times \frac{1}{2}q^{\frac{1}{2}m^2}\left[\frac{(1-q^{m+\frac{1}{2}})\vartheta_3(1,q)}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}+(-1)^{m^2}\frac{(1+q^{m+\frac{1}{2}})\vartheta_4(1,q)}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right] .
\end{aligned} \tag{4.127}
$$

This expression can be simplified further in the asymptotic limit $k \to \infty$. In this limit, $q^{n|k-m|} \sim q^{n(k-m)}$ and the dominant contribution in the summation over n comes from $n = 1$. The summation over n can be asymptotically evaluated as follows (assume that m is finite):

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}(q^{n|k-m|}-q^{n(k+m+1)}) \\
&\sim \sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{n(k-m)}(1-q^{n(2m+1)}) \\
&\sim \frac{q^k(1-q)\left(1-q^{2k}\right)}{\left(1+q^k\right)^4}\left\{q^{-m}\left(1-q^{2m+1}\right)\right\} .
\end{aligned} \tag{4.128}
$$

The summation over m can be evaluated by considering

$$
\sum_{m=0}^{\infty} q^{\frac{1}{2}m^2-m}\left(1-q^{m+\frac{1}{2}}\right)\left(1-q^{2m+1}\right) = q^{-\frac{1}{2}}(1-q)\vartheta_3(1,q) , \tag{4.129}
$$

$$
\sum_{m=0}^{\infty}(-1)^{m^2} q^{\frac{1}{2}m^2-m}\left(1+q^{m+\frac{1}{2}}\right)\left(1-q^{2m+1}\right) = -q^{-\frac{1}{2}}(1-q)\vartheta_4(1,q) . \tag{4.130}
$$

In such a limit, the function $F^{\rm NS}_{k,p}(q)$ becomes

$$
\begin{aligned}
F^{\rm NS}_{k,p}(q) &\sim \frac{1}{2}(q;q)^{-6}_{\infty}(1-q)^3 q^{p^2+p+k-\frac{3}{2}}\,\frac{1-q^{2k}}{(1+q^k)^4} \\
&\quad\times\left[\frac{\vartheta_3(1,q)^2}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}-\frac{\vartheta_4(1,q)^2}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right] \\
&\sim \frac{1}{2}(q;q)^{-6}_{\infty}(1-q)^3 q^{p^2+p+k-\frac{3}{2}} \\
&\quad\times\left[\frac{\vartheta_3(1,q)^2}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}-\frac{\vartheta_4(1,q)^2}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right]~, \qquad k\to\infty~.
\end{aligned}
\tag{4.131}
$$

The R sector

Similarly the partition function $\chi^{\mathcal{N}_{4d}=1}_{\rm R}\,|_{\rm GSO}\;(q;y,s)$, defined in (4.112), can be written as

$$
\chi^{\mathcal{N}_{4d}=1}_{\rm R}\,|_{\rm GSO}\;(q;y,s)=\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}[2k+1]_y s^{2p-1}\;F^{\rm R}_{k,p}(q)~, \tag{4.132}
$$

where the function $F^{\rm R}_{k,p}(q)$ follows from (4.56), (4.97) and (4.112):

$$
\begin{aligned}
F^{\rm R}_{k,p}(q) &= \frac{1}{2}(q;q)^{-6}_{\infty}(1-q)\frac{q^{p^2-\frac{5}{4}}}{(1+q^p)(1+q^{p-1})}\vartheta_2(1,q) \\
&\quad\times\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}} \\
&\quad\times\sum_{m=0}^{\infty}q^{\frac{1}{2}(m+\frac{1}{2})^2}(1-q^{m+1})(q^{n|k-m|}-q^{n(k+m+2)})~.
\end{aligned}
\tag{4.133}
$$

In the limit $k\to\infty$, this function can be simplified further. The summation over n can be asymptotically evaluated as follows (assume that m is finite):

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}(q^{n|k-m|}-q^{n(k+m+2)}) \\
&\sim\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{n(k-m)}(1-q^{n(2m+2)}) \\
&\sim\frac{q^k(1-q)\left(1-q^{2k}\right)}{\left(1+q^k\right)^4}\left\{q^{-m}\left(1-q^{2m+2}\right)\right\}~,
\end{aligned}
\tag{4.134}
$$

and the summation over m can be computed as follows:

$$\sum_{m=0}^{\infty} q^{\frac{1}{2}(m+\frac{1}{2})^2-m}\left(1-q^{m+1}\right)\left(1-q^{2m+2}\right) \quad = (1-q)\vartheta_2(1,q)\ . \tag{4.135}$$

Therefore, we have the following asymptotic formula:

$$\begin{aligned} F^{\mathrm{R}}_{k,p}(q) &\sim \frac{1}{2}(q;q)_\infty^{-6}\frac{q^{p^2+k-\frac{5}{4}}(1-q)^3\left(1-q^{2k}\right)}{(1+q^p)\left(1+q^{p-1}\right)\left(1+q^k\right)^4}\vartheta_2(1,q)^2 \\ &\sim \frac{1}{2}(q;q)_\infty^{-6}\frac{q^{p^2+k-\frac{5}{4}}(1-q)^3}{(1+q^p)\left(1+q^{p-1}\right)}\vartheta_2(1,q)^2\ , \qquad k\to\infty\ . \end{aligned} \tag{4.136}$$

Combining both sectors

Combining the NS- and R contributions from the previous subsections gives rise to the following $SO(3)\times U(1)_R$ covariant partition function

$$\begin{aligned} \chi^{\mathcal{N}_{4d}=1}(q;y,s) &= \chi^{\mathcal{N}_{4d}=1}_{\mathrm{NS}}\ |_{\mathrm{GSO}}\ (q;y,s) + \chi^{\mathcal{N}_{4d}=1}_{\mathrm{R}}\ |_{\mathrm{GSO}}\ (q;y,s) \\ &= \sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}\left([2k]_y s^{2p}\ F^{\mathrm{NS}}_{k,p}(q) + [2k+1]_y s^{2p-1}\ F^{\mathrm{R}}_{k,p}(q)\right) \\ &= \sum_{k=0}^{\infty}\left\{[2k]\left(F^{\mathrm{NS}}_{k,0}(q)+\sum_{p=1}^{\infty}s_{2p}F^{\mathrm{NS}}_{k,p}(q)\right)+[2k+1]\sum_{p=1}^{\infty}s_{2p-1}F^{\mathrm{R}}_{k,p}(q)\right\}\ , \end{aligned} \tag{4.137}$$

where s_m is defined by (4.95). Even though the $F^{\mathrm{NS}}_{k,p}$ and $F^{\mathrm{R}}_{k,p}$ functions are known, the representation (4.137) of the overall partition function does not make $\mathcal{N}_{4d}=1$ SUSY manifest to all mass levels. In order to do so, we have to combine $SO(3)\times U(1)_R$ representations to supermultiplets (4.110) and rewrite (4.137) as[9]

$$\chi^{\mathcal{N}_{4d}=1}(q;y,s) = \sum_{n=0}^{\infty}\sum_{Q=0}^{\infty}[\![n,\pm Q]\!]\,M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)\ . \tag{4.138}$$

This introduces a *multiplicity generating function* $M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)$ for the supermultiplet $[\![n,Q]\!]$ appearing in the partition function $\chi^{\mathcal{N}_{4d}=1}$. To

[9]The symmetry of (4.137) under $s\to s^{-1}$ guarantees that $M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q) = M(\chi^{\mathcal{N}_{4d}=1},[\![n,-Q]\!],q)$, so we shall henceforth assume that $Q\geq 0$.

lighten our notation in the subsequent steps, we shall use the shorthand

$$G_{n,Q}(q) := M(\chi^{\mathcal{N}_{4d}=1}, [\![n, Q]\!], q) \; . \tag{4.139}$$

By comparing (4.137) with (4.138), it is immediate that

$$G_{2n,2Q}(q) = G_{2n+1,2Q+1}(q) = 0 \; , \qquad \text{for all } n \geq 0 \text{ and } Q \geq 0 \; . \tag{4.140}$$

Recurrence relations

In order to relate the supersymmetric multiplicity generating functions $G_{n,Q}$ to their $SO(3) \times U(1)_R$ relatives $F^{\mathrm{NS}}_{k,p}$ and $F^{\mathrm{R}}_{k,p}$, we use (4.110) to rewrite (4.138) in terms of characters of irreducible $SO(3)$ characters and the fugacity s as

$$\begin{aligned}
&\chi^{\mathcal{N}_{4d}=1}(q; y, s) \\
&= [0]\Big[(G_{1,0} + 2G_{0,1}) + \sum_{Q=1}^{\infty} s_{2Q}\left(G_{0,2Q-1} + G_{1,2Q} + G_{0,2Q+1}\right)\Big] \\
&+ \sum_{k=1}^{\infty}[2k]\Big[\left(G_{2k-1,0} + 2G_{2k,1} + G_{2k+1,0}\right) \qquad\qquad (4.141) \\
&+ \sum_{Q=1}^{\infty} s_{2Q}\left(G_{2k-1,2Q} + G_{2k,2Q-1} + G_{2k,2Q+1} + G_{2k+1,2Q}\right)\Big] \\
&+ \sum_{k=0}^{\infty}[2k+1] \\
&\times \sum_{Q=1}^{\infty} s_{2Q-1}\left(G_{2k,2Q-1} + G_{2k+1,2Q-2} + G_{2k+1,2Q} + G_{2k+2,2Q-1}\right) \; , \qquad (4.142)
\end{aligned}$$

where $G_{n,Q}$ is a shorthand notation for $G_{n,Q}(q)$.

Comparing (4.137) with (4.141), we have the following relations:

$$2G_{0,1}(q) + G_{1,0}(q) = F^{\rm NS}_{0,0}(q) \ , \tag{4.143}$$

$$G_{2k-1,0}(q) + 2G_{2k,1}(q) + G_{2k+1,0}(q) = F^{\rm NS}_{k,0}(q) \ , \quad k \geq 1 \tag{4.144}$$

$$G_{0,2Q-1}(q) + G_{0,2Q+1}(q) + G_{1,2Q}(q) = F^{\rm NS}_{0,Q}(q) \ , \quad Q \geq 1 \tag{4.145}$$

$$G_{2k-1,2Q}(q) + G_{2k,2Q-1}(q) + G_{2k,2Q+1}(q) + G_{2k+1,2Q}(q) = F^{\rm NS}_{k,Q}(q) \ , \\ k, Q \geq 1 \tag{4.146}$$

$$G_{2k,2Q-1}(q) + G_{2k+1,2Q-2}(q) + G_{2k+1,2Q}(q) + G_{2k+2,2Q-1}(q) = F^{\rm R}_{k,Q}(q) \ , \\ k \geq 0, Q \geq 1 \ . \tag{4.147}$$

These relations are useful for computing a multiplicity generating function for a representation $[\![\text{odd}, \text{even}]\!]$ (or $[\![\text{even}, \text{odd}]\!]$) when the one for opposite parity is known. However, the recursion is not powerful enough to directly determine all the $G_{n,Q}$ in terms of $F^{\rm NS}_{k,p}$ and $F^{\rm R}_{k,p}$. The following subsection follows an alternative approach to determine the $G_{n,Q}$.

4.3.3 Multiplicities of representations in the $\mathcal{N}_{4d} = 1$ partition function

Our aim in this subsection is to factor out the fundamental $\mathcal{N}_{4d} = 1$ super Poincaré character $Z(\mathcal{N}_{4d} = 1) = [1]_y + s + s^{-1}$ and to compute explicitly the multiplicity generating functions $G_{n,Q}(q)$ for $[\![n, Q]\!]$ in

$$\chi^{\mathcal{N}_{4d}=1}(q; y, s) = \sum_{n=0}^{\infty} \sum_{Q=-\infty}^{\infty} [\![n, Q]\!] G_{n,Q}(q) \ . \tag{4.148}$$

Using the second equality of (4.110) and orthogonality of $SO(3) \times U(1)_R$ representations, we have

$$\begin{aligned} G_{n,Q}(q) &= M(\chi^{\mathcal{N}_{4d}=1}, [\![n, Q]\!], q) \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{{\rm d}s}{s} \int {\rm d}\mu_{SO(3)}(y) \ [n]_y s^{-Q} \frac{\chi^{\mathcal{N}_{4d}=1}(q; y, s)}{[1]_y + (s + s^{-1})} \ , \end{aligned} \tag{4.149}$$

where $\mathcal{C}$ is a contour in the complex s-plane enclosing the origin. In order to proceed, we use the geometric series expansion of the inverse $Z(\mathcal{N}_{4d}=1)$,[10]

$$\frac{1}{[1]_y+(s+s^{-1})} = \frac{1}{s+s^{-1}}\frac{1}{1+\frac{[1]_y}{s+s^{-1}}} = \sum_{m=0}^{\infty}(-1)^m \frac{[1]_y^m}{(s+s^{-1})^{m+1}} \ . \tag{4.151}$$

In what follows, we consider the contributions from $\chi_{\text{NS}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}\ (q;y,s)$ and $\chi_{\text{R}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}\ (q;y,s)$ separately and then add up these results to yield the overall multiplicity generating function defined by (4.148),

$$\begin{aligned} &M(\chi^{\mathcal{N}_{4d}=1}, [\![n,Q]\!], q) \\ &= M(\chi_{\text{NS}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}, [\![n,Q]\!], q) + M(\chi_{\text{R}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}, [\![n,Q]\!], q)\ , \end{aligned} \tag{4.152}$$

where $\chi_{\text{NS,R}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}$ are given by (4.126) and (4.132).

Multiplicities in the NS-sector

The series expansion of $(Z(\mathcal{N}_{4d}=1))^{-1}$ leads to the following NS sector contribution to the multiplicity generating function of the supermultiplet $[\![n,Q]\!]$

$$\begin{aligned} &M(\chi_{\text{NS}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}, [\![n,Q]\!], q) \\ &:= \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}s}{s}\int \mathrm{d}\mu_{SO(3)}(y)\ \frac{[n]_y}{s^Q}\times\frac{\chi_{\text{NS}}^{\mathcal{N}_{4d}=1}\ |_{\text{GSO}}\ (q;y,s)}{[1]_y+(s+s^{-1})} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}(-1)^m F_{k,p}^{\text{NS}}(q)\ \frac{1}{2\pi i}\oint_{|s|=1-\epsilon}\frac{\mathrm{d}s}{s}\frac{s^{2p}}{s^Q(s+s^{-1})^{m+1}} \\ &\times\int \mathrm{d}\mu_{SO(3)}(y)\ [n]_y[1]_y^m[2k]_y\ , \end{aligned} \tag{4.153}$$

We shall henceforth take $\mathcal{C}$ to be a circle centred at the origin with the radius $1-\epsilon$, with $0<\epsilon<1$. The quantities in the curly brackets can be

[10]Note that $\frac{1}{[1]_y+(s+s^{-1})}$ can also be written in another way as follows:

$$\frac{1}{[1]_y+(s+s^{-1})} = \sum_{m=0}^{\infty}(-1)^m s^{m+1}[m]_y\ . \tag{4.150}$$

However, we shall not take this approach, since otherwise this would lead to tensor products in (4.155) and (4.156) which are harder to evaluate in comparison with our current approach.

computed as follows:

$$\frac{1}{2\pi i}\oint_{|s|=1-\epsilon}\frac{\mathrm{d}s}{s}\frac{s^{2p}}{s^{Q}(s+s^{-1})^{m+1}}$$
$$=\begin{cases}(-1)^{\frac{1}{2}(Q-m-2p-1)}\binom{\frac{1}{2}(Q+m-2p-1)}{m} & \text{for } Q-m \text{ odd and } Q+m\geq 2p+1\\ 0 & \text{otherwise ,}\end{cases} \tag{4.154}$$

and

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n]_y[1]_y^m[2k]_y=\begin{cases}T_{2n+1}\left(m,\frac{1}{2}m+n-k\right) & \text{if } m \text{ is even}\\ 0 & \text{if } m \text{ is odd ,}\end{cases} \tag{4.155}$$

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n+1]_y[1]_y^m[2k]_y=\begin{cases}T_{2n+2}\left(m,\frac{1}{2}m+n+\frac{1}{2}-k\right) & \text{if } m \text{ is odd}\\ 0 & \text{if } m \text{ is even ,}\end{cases} \tag{4.156}$$

where

$$T_p(m,k)=\binom{m}{k}-\binom{m}{k-p}\ . \tag{4.157}$$

We can derive (4.155) as follows, using (4.17) in the first step:

$$\begin{aligned}&\int \mathrm{d}\mu_{SO(3)}(y)\ [2n]_y[1]_y^m[2k]_y\\ &=\int \mathrm{d}\mu_{SO(3)}\left(\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{l}-\binom{m}{l-1}\right)[m-2l]_y\right)\sum_{p=|n-k|}^{n+k}[2p]_y\\ &=\binom{m}{\frac{1}{2}m-|n-k|}-\binom{m}{\frac{1}{2}m-n-k-1}\\ &=\binom{m}{\frac{1}{2}m+n-k}-\binom{m}{\frac{1}{2}m-n-k-1}\end{aligned} \tag{4.158}$$

as required, and similarly for (4.156). Note that (4.154), (4.155) and (4.156) are in perfect agreement with the selection rule

$$M(\chi^{\mathcal{N}_{4d}=1},[\![2n,2Q]\!],q)=M(\chi^{\mathcal{N}_{4d}=1},[\![2n+1,2Q+1]\!],q)=0\ . \tag{4.159}$$

The nonzero multiplicities of $[\![2n, 2Q+1]\!]$ and $[\![2n+1, 2Q]\!]$ receive the following NS sector contributions:

$$
\begin{aligned}
&M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}, [\![2n, 2Q+1]\!], q) \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p} F_{k,p}^{\mathrm{NS}}(q) \binom{Q+m-p}{2m} T_{2n+1}(2m, m+n-k) \ ,
\end{aligned}
\tag{4.160}
$$

$$
\begin{aligned}
&M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}, [\![2n+1, 2Q]\!], q) \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p} F_{k,p}^{\mathrm{NS}}(q) \binom{Q+m-p}{2m+1} T_{2n+2}(2m+1, m+n+1-k) \ .
\end{aligned}
\tag{4.161}
$$

Multiplicities in the R-sector

Similarly to the NS-sector, the generating function for the multiplicity of the representation $[\![n, Q]\!]$ in the function $\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q; y, s)$ is given by

$$
\begin{aligned}
&M(\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}, [\![n, Q]\!], q) \\
&:= \frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s} \int \mathrm{d}\mu_{SO(3)}(y)\ \frac{[n]_y}{s^Q} \times \frac{\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q;y,s)}{[1]_y + (s+s^{-1})} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty} (-1)^m F_{k,p}^{\mathrm{R}}(q)\ \frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s}\, \frac{s^{2p-1}}{s^Q(s+s^{-1})^{m+1}} \\
&\times \int \mathrm{d}\mu_{SO(3)}(y)\ [n]_y [1]_y^m [2k+1]_y \ ,
\end{aligned}
\tag{4.162}
$$

with $0 < \epsilon < 1$,

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s}\, \frac{s^{2p-1}}{s^Q(s+s^{-1})^{m+1}} \\
&= \begin{cases} (-1)^{\frac{1}{2}(Q-m-2p)} \binom{\frac{1}{2}(Q+m-2p)}{m} & \text{for } Q-m \text{ even and } Q+m \geq 2p \\ 0 & \text{otherwise} \ , \end{cases}
\end{aligned}
\tag{4.163}
$$

and

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n]_y[1]_y^m[2k+1]_y = \begin{cases} T_{2n+1}\left(m, \frac{1}{2}m+n-k-\frac{1}{2}\right) & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}\ , \end{cases} \tag{4.164}$$

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n+1]_y[1]_y^m[2k+1]_y = \begin{cases} T_{2n+2}\left(m, \frac{1}{2}m+n-k\right) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd}\ , \end{cases} \tag{4.165}$$

where $T_p(m,k)$ is defined as above, and can again be derived similarly to (4.158), and the zeros once again confirm the selection rule (4.159).

The multiplicities of $[\![2n, 2Q+1]\!]$ are given by

$$\begin{aligned} &M(\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}|_{\mathrm{GSO}}, [\![2n, 2Q+1]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m}(-1)^{Q-m-p+1}F_{k,p}^{\mathrm{R}}(q)\binom{Q+m-p+1}{2m+1} \\ &\times T_{2n+1}(2m+1, m+n-k)\ . \end{aligned} \tag{4.166}$$

The multiplicities of $[\![2n+1, 2Q]\!]$ are given by

$$\begin{aligned} &M(\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}|_{\mathrm{GSO}}, [\![2n+1, 2Q]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m}(-1)^{Q-m-p}F_{k,p}^{\mathrm{R}}(q)\binom{Q+m-p}{2m}T_{2n+2}(2m, m+n-k)\ . \end{aligned} \tag{4.167}$$

Combining the NS and R sectors

Now we can assemble the NS- and R sector results to obtain the full multiplicities of the representation $[\![n, Q]\!]$ in $\chi^{\mathcal{N}_{4d}=1}(q; y, s)$. First, it is clear that

$$G_{2n,2Q}(q) = G_{2n+1,2Q+1}(q) = 0\ . \tag{4.168}$$

The nonzero multiplicities of $[\![2n, 2Q+1]\!]$ and $[\![2n+1, 2Q]\!]$ are most conveniently presented in terms of the shorthands

$$\begin{aligned}
&\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,p,k;q) \\
&:= (-1)^{Q-m-p}\Bigg[F^{\mathrm{NS}}_{k,p}(q)\binom{Q+m-p}{2m}T_{2n+1}(2m, m+n-k) \\
&- F^{\mathrm{R}}_{k,p}(q)\binom{Q+m-p+1}{2m+1}T_{2n+1}(2m+1, m+n-k)\Bigg] \qquad (4.169)
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) \\
&:= (-1)^{Q-m-p}\Bigg[F^{\mathrm{NS}}_{k,p}(q)\binom{Q+m-p}{2m+1}T_{2n+2}(2m+1, m+n+1-k) \\
&+ F^{\mathrm{R}}_{k,p}(q)\binom{Q+m-p}{2m}T_{2n+2}(2m, m+n-k)\Bigg] \qquad (4.170)
\end{aligned}$$

for the contributions $\mathfrak{M}_{[\![\cdot,\cdot]\!]}(m,p,k;q)$ of individual terms in the m,p,k triple sum to the multiplicity generating function. The result for $[\![2n, 2Q+1]\!]$ supermultiplets is

$$\begin{aligned}
G_{2n,2Q+1}(q) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m}\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,p,k;q) \\
= \quad &\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Bigg[\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n,2Q+1]\!]}(m+p,p,k;q)\Big\} \\
&+ \sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,m+p+1,k;q)\Bigg] . \qquad (4.171)
\end{aligned}$$

whereas the multiplicities of $[\![2n+1, 2Q]\!]$ are given by

$$\begin{aligned}
G_{2n+1,2Q}(q) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) \\
= \quad &\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Bigg[\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big\} \\
&+ \sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\Bigg] . \qquad (4.172)
\end{aligned}$$

4.3.4 Asymptotic analysis for the multiplicities

This subsection is devoted to the multiplicity generating function $G_{n,Q}(q)$ in the limit $n \to \infty$. We shall present analytic expressions for their $n \to \infty$ asymptotics whose derivation is defered to appendix 4.A. The method essentially relies on identifying the dominant contribution to the triple sums in (4.171) and (4.172). The end result for multiplicity generating functions $G_{n,Q}(q)$ reads

$$G_{2n+1,2Q}(q) \sim \frac{(1-q)^2 q^{n-\frac{3}{2}}}{2(q;q)_\infty^6} \mathcal{F}(q,Q) \ , \quad n \to \infty \ , \tag{4.173}$$

$$\begin{aligned} G_{2n,2Q+1}(q) \sim & \frac{(1-q)^2 q^{n-\frac{3}{2}}}{2(q;q)_\infty^6 (1+q)} \\ & \times \left[\frac{q^{(Q+1)^2+\frac{1}{4}}(1-q)}{(1+q^Q)\,(1+q^{Q+1})} \vartheta_2(1,q)^2 - \mathcal{F}(q,Q) - \mathcal{F}(q,Q+1) \right] \end{aligned} \tag{4.174}$$

with the function $\mathcal{F}(q,Q)$ given by

$$\begin{aligned} &\mathcal{F}(q,Q) \\ &= \vartheta_2(1,q)^2 \Big[q^{1-Q} u_1(\sqrt{q},Q) + (-1)^Q (1-q)(v_1(\sqrt{q},Q) + q^{-1/4} w_1(\sqrt{q},Q)) \Big] \\ &+ \vartheta_3(1,q)^2 \Big[- q^{1-Q} u_2(\sqrt{q},Q) + (-1)^Q (1-q)(v_2(\sqrt{q},Q) + q^2 w_2(\sqrt{q},Q)) \Big] \\ &+ \vartheta_4(1,q)^2 \Big[q^{1-Q} u_2(-\sqrt{q},Q) - (-1)^Q (1-q)(v_2(-\sqrt{q},Q) + q^2 w_2(-\sqrt{q},Q)) \Big] \ . \end{aligned} \tag{4.175}$$

The three pairs of functions u_i, v_i and w_i correspond to the three summations in (4.171) and (4.172):

$$u_1(q,Q) = \sum_{p=0}^{\infty} q^{2\left(p+\frac{3}{2}\right)^2} \frac{1-q^{4p+4Q+6}}{(1+q^{2p+2})(1+q^{2p+4})} \ ,$$
$$u_2(q,Q) = \sum_{p=0}^{\infty} q^{2(p+1)^2} \frac{1-q^{4p+4Q+4}}{(1+q^{2p+1})(1+q^{2p+3})} \ . \qquad (4.176)$$

$$v_1(q,Q) = \sum_{p=0}^{\lfloor Q/2 \rfloor} \frac{q^{2(p-\frac{1}{2})^2}(1+q^2)^{2p}}{(1+q^{2p-2})(1+q^{2p})} \binom{Q}{2p}$$
$$\times\, {}_3F_2 \left[\begin{matrix} 1,\ Q+1,\ 2p-Q \\ p+1/2,\ p+1 \end{matrix} ;\ \frac{(1+q)^2}{4q} \right] \ ,$$
$$v_2(q,Q) = \sum_{p=0}^{\lfloor Q/2 \rfloor} \frac{(1+q)q^{2p^2}(1+q^2)^{2p}}{(1+q^{2p-1})(1+q^{2p+1})} \binom{Q}{2p+1}$$
$$\times\, {}_3F_2 \left[\begin{matrix} 1,\ Q+1,\ 2p+1-Q \\ p+1,\ p+3/2 \end{matrix} ;\ \frac{(1+q)^2}{4q} \right] \ , \qquad (4.177)$$

$$w_1(q,Q) = \sum_{m=0}^{\infty} \sum_{p=0}^{Q-1} \frac{(-1)^{p+1} q^{1+2(1+m+p)^2-2m} \left(1+q^2\right)^{2m} \binom{Q-1-p}{2m}}{\left(1+q^{2(m+p)}\right)\left(1+q^{2(1+m+p)}\right)} \ ,$$
$$w_2(q,Q) = q^{-\frac{9}{2}} \sum_{m=0}^{\infty} \sum_{p=0}^{Q-1} \frac{(-1)^{p+1} q^{2\left(m+p+\frac{3}{2}\right)^2-2m} \left(1+q^2\right)^{2m+1} \binom{Q-1-p}{1+2m}}{(1+q^{1+2m+2p})(1+q^{3+2m+2p})} \ .$$
$$(4.178)$$

Note that the leading orders in the power series are

$$G_{2n+1,2Q}(q) \ \sim \ q^{n+Q(Q+2)} \ , G_{2n,2Q+1}(q) \ \sim \ q^{n+Q^2+3Q+1} \ , q \to 0 \ , \qquad (4.179)$$

i.e. the supermultiplet $[\![2n+1, 2Q]\!]$ firstly occurs at mass level $n+Q(Q+2)$ whereas the $[\![2n, 2Q+1]\!]$ multiplet firstly occurs at mass level $n+Q^2+3Q+1$.

For reference, we list the leading q powers for the $G_{n\to\infty,Q}$ regime for some small values of the $U(1)_R$ charge, obtained by expansion of (4.173) and (4.174): firstly for even values $Q \in 2\mathbb{N}_0$

$$\begin{aligned}
G_{2n+1,0}(q) &\sim q^n(1 + q + 7q^2 + 19q^3 + 53q^4 + 133q^5 + 328q^6 + 752q^7 \\
&\quad + 1689q^8 + 3635q^9 + O(q^{10})) \ , \\
G_{2n+1,2}(q) &\sim q^{n+3}(2 + 8q + 24q^2 + 73q^3 + 187q^4 + 467q^5 + 1090q^6 \\
&\quad + 2457q^7 + 5314q^8 + O(q^9)) \ , \\
G_{2n+1,4}(q) &\sim q^{n+8}(2 + 10q + 36q^2 + 110q^3 + 306q^4 + 773q^5 + 1861q^6 \\
&\quad + 4245q^7 + 9327q^8 + O(q^9)) \ , \\
G_{2n+1,6}(q) &\sim q^{n+15}(2 + 10q + 38q^2 + 124q^3 + 352q^4 + 928q^5 + 2282q^6 \\
&\quad + 5335q^7 + O(q^8)) \ ,
\end{aligned} \tag{4.180}$$

and secondly for odd values $Q \in 2\mathbb{N} - 1$

$$\begin{aligned}
G_{2n,1}(q) &\sim q^{n+1}(3 + 5q + 22q^2 + 53q^3 + 150q^4 + 345q^5 + 836q^6 + 1824q^7 \\
&\quad + 4011q^8 + O(q9)) \ , \\
G_{2n,3}(q) &\sim q^{n+5}(4 + 11q + 46q^2 + 117q^3 + 331q^4 + 784q^5 + 1876q^6 \\
&\quad + 4133q^7 + O(q^8)) \ , \\
G_{2n,5}(q) &\sim q^{n+11}(4 + 12q + 55q^2 + 150q^3 + 437q^4 + 1078q^5 + 2640q^6 \\
&\quad + 5951q^7 + O(q^8)) \ , \\
G_{2n,7}(q) &\sim q^{n+19}(4 + 12q + 56q^2 + 159q^3 + 474q^4 + 1197q^5 + 2994q^6 \\
&\quad + 6882q^7 + O(q^8)) \ .
\end{aligned} \tag{4.181}$$

Note that the general formula greatly simplifies at $U(1)_R$ charges $Q = 0$

and $Q=1$,

$$G_{2n+1,0}(q) \sim \frac{q^n}{(q;q)_\infty^6} \times$$
$$\left\{ \frac{1}{2}(1-q)^2 q^{-\frac{1}{2}} \Big(u_1(\sqrt{q})\vartheta_2(1,q)^2 - \big[u_2(\sqrt{q})\vartheta_3(1,q)^2 - u_2(-\sqrt{q})\vartheta_4(1,q)^2 \big] \Big) \right.$$
$$\left. + \frac{1}{4}\frac{(1-q)^3}{1+q} q^{-\frac{1}{4}} \vartheta_2(1,q)^2 \right\}, \qquad n \to \infty \tag{4.182}$$
$$G_{2n,1}(q) \sim \frac{(1-q)^3 q^{n+1}}{4(q;q)_\infty^6} \times$$
$$\left[q^{-\frac{5}{2}} \left\{ \frac{\vartheta_3(1,q)^2}{(1+q^{-\frac{1}{2}})(1+q^{\frac{1}{2}})} - \frac{\vartheta_4(1,q)^2}{(1-q^{-\frac{1}{2}})(1-q^{\frac{1}{2}})} \right\} - \frac{1}{2} q^{-\frac{9}{4}} \vartheta_2(1,q)^2 \right.$$
$$\left. - q^{-\frac{5}{2}} \frac{1+q}{1-q} \Big(u_1(\sqrt{q})\vartheta_2(1,q)^2 - \big[u_2(\sqrt{q})\vartheta_3(1,q)^2 - u_2(-\sqrt{q})\vartheta_4(1,q)^2 \big] \Big) \right] \tag{4.183}$$

where $u_i(q) \equiv u_i(q;0)$, see the first subsection of appendix 4.A.

4.3.5 Empirical approach to $\mathcal{N}_{4d}=1$ asymptotic patterns

In the previous subsection, we have derived the large spin asymptotics for multiplicity generating functions $G_{k,Q}(q)$ of individual $\mathcal{N}_{4d}=1$ multiplets (at finite Q while $k \to \infty$), the main results being (4.173) and (4.174). The asymptotic formulae can be viewed as the supersymmetric generalization of truncating the infinite sum expression (4.23) for the $SO(3)$ multiplicity generating function in the $d=4$ bosonic partition function to its $n=1$ term. In [10], this $n=1$ term is interpreted as the leading (additive) Regge trajectory of unit slope, followed by an infinite tower of sister trajectories of fractional slope and alternating sign.

Let us borrow some notation from equation (6.2) of [10] and expand the

$G_{k,Q}(q)$ in an infinite series of trajectories τ_ℓ:

$$\begin{aligned} G_{2n+1,2Q}(q) &= q^n \, \tau_1^{2Q}(q) \;-\; q^{2n} \, \tau_2^{2Q}(q) \;+\; q^{3n} \, \tau_3^{2Q}(q) \;-\; \dots \\ &= \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \, q^{\ell n} \, \tau_\ell^{2Q}(q) \qquad (4.184) \\ G_{2n,2Q+1}(q) &= q^n \, \tau_1^{2Q+1}(q) \;-\; q^{2n} \, \tau_2^{2Q+1}(q) \;+\; q^{3n} \, \tau_3^{2Q+1}(q) \;-\; \dots \\ &= \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \, q^{\ell n} \, \tau_\ell^{2Q+1}(q) \end{aligned}$$

It is not obvious that the patterns observed in [10] for non-supersymmetric theories persist for the counting of super-Poincaré multiplets, i.e. that the spacetime partition functions of the reference preserve the nested structure in (4.184) after multiplication with the internal characters. At any rate, all our $\mathcal{N}_{4d} = 1$ data suggests that both of $\tau_\ell^{2Q}(q)$ and $\tau_\ell^{2Q+1}(q)$ are power series in q with non-negative coefficients. Our analytic results (4.173) and (4.174) identify the first coefficient functions $\tau_1(q)$ in (4.184):

$$\tau_1^{2Q}(q) \;=\; \frac{(1-q)^2 q^{-\frac{3}{2}}}{2(q;q)_\infty^6} \mathcal{F}(q,Q) \qquad (4.185)$$

$$\begin{aligned} \tau_1^{2Q+1}(q) \;&=\; \frac{(1-q)^2 q^{-\frac{3}{2}}}{2(q;q)_\infty^6 (1+q)} \\ &\times \left[\frac{q^{(Q+1)^2+\frac{1}{4}}(1-q)}{(1+q^Q)\,(1+q^{Q+1})} \vartheta_2(1,q)^2 - \mathcal{F}(q,Q) - \mathcal{F}(q,Q+1) \right] \end{aligned} \qquad (4.186)$$

The methods presented in appendix 4.A and applied in the previous subsection are not suitable to extract subleading Regge trajectories $\tau_{\ell \geq 2}(q)$, i.e. $\mathcal{N}_{4d} = 1$ analogues of $n \geq 2$ terms in the sum (4.23). Instead, we shall rely on an empirical approach, more specifically on explicit results obtained from a supercharacter expansion of the partition function (4.112) up to the 25th mass level.

As an illustrative example, let us first of all investigate the family of $Q = 0$ supermultiplets: The following table 4.2 gathers $[\![2n+1, 0]\!]$ multiplicities in the first 25 levels. Numbers marked in red directly correspond to the leading trajectory $\tau_1^0(q)$ whereas those in blue are additionally affected by the subleading trajectory $\tau_2^0(q)$. Given the leading trajectories (4.185), our data

$\alpha' m^2$	# $[\![1,0]\!]$	# $[\![3,0]\!]$	# $[\![5,0]\!]$	# $[\![7,0]\!]$	# $[\![9,0]\!]$	# $[\![11,0]\!]$	# $[\![13,0]\!]$	# $[\![15,0]\!]$	# $[\![17,0]\!]$	# $[\![19,0]\!]$	# $[\![21,0]\!]$
1	0	1	0								
2	2	1	1	0							
3	5	5	1	1	0						
4	12	12	7	1	1	0					
5	28	31	17	7	1	1	0				
6	64	72	45	19	7	1	1	0			
7	136	166	108	51	19	7	1	1	0		
8	289	357	254	125	53	19	7	1	1	0	
9	588	757	557	302	131	53	19	7	1	1	0
10	1175	1548	1200	675	320	133	53	19	7	1	1
11	2293	3100	2482	1479	726	326	133	53	19	7	1
12	4399	6053	5028	3106	1611	744	328	133	53	19	7
13	8267	11620	9910	6373	3422	1663	750	328	133	53	19
14	15325	21855	19173	12713	7098	3557	1681	752	328	133	53
15	27949	40496	36322	24856	14297	7428	3609	1687	752	328	133
16	50306	73846	67720	47539	28216	15061	7564	3627	1689	752	328
17	89367	132860	124161	89401	54430	29909	15394	7616	3633	1689	752
18	156930	235871	224479	165210	103182	58054	30687	15530	7634	3635	1689
19	272424	413879	400257	300837	192109	110702	59786	31021	15582	7640	3635
20	468130	717909	705032	539962	352279	207282	114437	60567	31157	15600	7642
21	796410	1232463	1227214	956883	636445	382179	215074	116183	60901	31209	15606
22	1342531	2094716	2113394	1674933	1134836	694090	398007	218848	116965	61037	31227
23	2243232	3527456	3602086	2899342	1997955	1243836	725457	405910	220597	117299	61089
24	3717405	5887668	6081317	4965411	3477396	2200438	1304682	741559	409698	221379	117435
25	6111615	9745995	10173766	8420331	5986079	3847540	2316123	1336712	749501	411448	221713

Table 4.2: $\mathcal{N}_{4d} = 1$ multiplets at $U(1)_R$ charge $Q = 0$

from table 4.2 can be used to determine the following subleading behaviour for $Q = 0$ multiplets:

$$
\begin{aligned}
G_{2n+1,0}(q) \quad \sim \quad & q^n \,(1 + q + 7q^2 + 19q^3 + 53q^4 + 133q^5 + 328q^6 + 752q^7 \\
& + 1689q^8 + 3635q^9 + 7642q^{10} + 15608q^{11} + 31235q^{12} \\
& + 61115q^{13} + 117513q^{14} + 221927q^{15} + 412778q^{16} + 756372q^{17} \\
& + 1367753q^{18} + 2441849q^{19} + 4309132q^{20} + 7520092q^{21} \\
& + 12989357q^{22} + 22216885q^{23} + 37651970q^{24} + 63252874q^{25} + \ldots) \\
- \ & q^{2n+1} \,(2 + 8q + 26q^2 + 78q^3 + 214q^4 + 548q^5 + 1330q^6 + 3080q^7 \\
& + 6872q^8 + 14832q^9 + 31102q^{10} + 63574q^{11} + 127020q^{12} \\
& + 248590q^{13} + 477504q^{14} + \ldots) \\
+ \ & q^{3n+1} \,(1 + 4q + 19q^2 + 61q^3 + 187q^4 + 503q^5 + 1294q^6 + 3113q^7 \\
& + 7217q^8 + 16036q^9 + 34584q^{10} + \ldots) \\
- \ & q^{4n+2} \,(2 + 10q + 38q^2 + 124q^3 + 364q^4 + 978q^5 + 2476q^6 + \ldots) \\
+ \ & q^{5n+2} \,(1 + 4q + 21q^2 + 72q^3 + \ldots) \ + \ \ldots \ , \qquad n \to \infty
\end{aligned} \tag{4.187}
$$

The first term linear in q^n simply reproduces (4.182) for $\tau_1^{Q=0}(q)$ whereas higher powers of q^n allow to read off subleading $\tau_{\ell\geq 2}^{Q=0}(q)$ to certain order in q:

$$
\begin{aligned}
\tau_2^{Q=0}(q) \quad = \quad & q\,(2 + 8q + 26q^2 + 78q^3 + 214q^4 + 548q^5 + 1330q^6 + 3080q^7 \\
& + 6872q^8 + 14832q^9 + 31102q^{10} + 63574q^{11} \\
& + 248590q^{13} + 477504q^{14} + \ldots)
\end{aligned} \tag{4.188}
$$

$$
\begin{aligned}
\tau_3^{Q=0}(q) \quad = \quad & q\,(1 + 4q + 19q^2 + 61q^3 + 187q^4 + 503q^5 + 1294q^6 + 3113q^7 \\
& + 7217q^8 + 16036q^9 + 34584q^{10} + \ldots)
\end{aligned} \tag{4.189}
$$

$$
\tau_4^{Q=0}(q) \quad = \quad q^2\,(2 + 10q + 38q^2 + 124q^3 + 364q^4 + 978q^5 + 2476q^6 + \ldots) \tag{4.190}
$$

$$
\tau_5^{Q=0}(q) \quad = \quad q^2\,(1 + 4q + 21q^2 + 72q^3 + \ldots) \tag{4.191}
$$

Determining higher order terms in the $\tau_{\ell\geq 2}^{Q=0}(q)$ would require $\mathcal{O}(q^{26})$ parts of (4.112), this is where we stopped the explicit evaluation.

Similarly, the $[\![2n+1,2]\!]$ and $[\![2n,1]\!]$ multiplicities up to level q^{25} as tabulated in appendix 4.B.1 determine the associated $\tau_\ell^{\cdot\cdot}(q)$ coefficients to the following orders:

$$\begin{aligned}
\tau_2^{Q=2}(q) &= q^3\,(2+11q+37q^2+114q^3+319q^4+822q^5+2000q^6+4645q^7 \\
&\quad +10354q^8+22317q^9+46702q^{10}+95210q^{11}+189656q^{12}+\ldots) \\
\tau_3^{Q=2}(q) &= q^3\,(2+8q+33q^2+104q^3+310q^4+826q^5+2093q^6+4991q^7 \\
&\quad +11454q^8+\ldots) \\
\tau_4^{Q=2}(q) &= q^3\,(1+5q+22q^2+77q^3+237q^4+664q^5+\ldots) \\
\tau_5^{Q=2}(q) &= q^4\,(3+12q+49q^2+\ldots) \qquad (4.192) \\
\tau_2^{Q=1}(q) &= 1+4q+15q^2+50q^3+143q^4+379q^5+947q^6+2244q^7+5103q^8 \\
&\quad +11196q^9+23804q^{10}+49252q^{11}+99465q^{12}+196522q^{13}+380719q^{14}+\ldots \\
\tau_3^{Q=1}(q) &= 1+5q+22q^2+70q^3+212q^4+568q^5+1458q^6+3496q^7 \\
&\quad +8093q^8+17936q^9+\ldots \\
\tau_4^{Q=1}(q) &= 1+6q+24q^2+83q^3+252q^4+698q^5+\ldots \\
\tau_5^{Q=1}(q) &= 1+6q+25q^2+\ldots \qquad (4.193)
\end{aligned}$$

Note that the analytic result (4.173) for $\tau_1^2(q), \tau_1^1(q)$ was used as an extra input, in addition to the explicit results for the first 25 mass level, to make a few more orders of the subleading $\tau_{\ell\geq 2}^{\cdots}(q)$ accessible. Some more leading and subleading τ_ℓ^Q for larger values of Q are given in (4.B.1).

4.4 Spectra in compactifications with 8 supercharges

In six dimensional Minkowski space, the minimal realization of SUSY involves eight supercharges. They form two left- handed Weyl spinors of $SO(6)$ which are related through an $SU(2)_R$ R symmetry. Our notation for such minimally supersymmetric theories in $d=6$ is $\mathcal{N}_{6d}=(1,0)$. Superstring compactification subject to $\mathcal{N}_{6d}=(1,0)$ SUSY are described by a universal SCFT sector with $c=6$ and $\mathcal{N}_{2d}=4$ SUSY on the worldsheet, see subsection 4.2.2 for details. In addition, the SCFT introduces $SO(5)$ quantum numbers for the massive string states through a six dimensional spacetime sector for which the methods of subsections 4.1.3 and 4.1.4 are applicable.

The fundamental multiplet of $\mathcal{N}_{6d} = (1,0)$ theories consists of 8+8 states

$$Z(\mathcal{N}_{6d} = (1,0)) \ := \ [1,0] \ + \ [2]_R \ + \ [1]_R \, [0,1] \ . \tag{4.194}$$

where $[p]_R$ is the character of the $p{+}1$ dimensional representation of $SU(2)_R$. Generic multiplets follow through the tensor product with some $SO(5) \times SU(2)_R$ representation with little group quantum numbers $[n_1, n_2]$ and R symmetry content $[k]_R$. This leads to the general supercharacter

$$[\![n_1, n_2; p]\!] \ := \ Z(\mathcal{N}_{6d} = (1,0)) \cdot [p]_R \, [n_1, n_2] \ . \tag{4.195}$$

The partition function capturing the universal spectrum of six dimensional $\mathcal{N}_{6d} = (1,0)$ compactifications is obtained thorugh a GSO projected product of internal $\chi^{\mathcal{N}_{2d}=4,c=6}_{\dots}(q;r)$ characters (with $SU(2)_R$ fugacity r) defined by (4.101) as well as (4.106) and $SO(5)$ spacetime characters (4.89) and (4.90). The GSO projection removes half odd integer mass leves from the NS sector and enforces the R spin field to be a left handed $SO(6)$ spinor, therefore (again needing to multiply by $q^{-\frac{1}{2}}$ in the NS case to incorporate the zero-point energy):

$$\begin{aligned}
\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r) \ &= \ \chi^{\mathcal{N}_{6d}=(1,0)}_{\mathrm{NS}} \, |_{\mathrm{GSO}} \, (q;\vec{y},r) \ + \ \chi^{\mathcal{N}_{6d}=(1,0)}_{\mathrm{R}} \, |_{\mathrm{GSO}} \, (q;\vec{y},r) \\
\chi^{\mathcal{N}_{6d}=(1,0)}_{\mathrm{NS}} \, |_{\mathrm{GSO}} \, (q;\vec{y},r) \ &= \ \frac{1}{2} \, q^{-\frac{1}{2}} \, \Big[\, \chi^{SO(5)}_{\mathrm{NS}}(q;\vec{y}) \, \chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(q;r) \\
&\qquad - \ \chi^{SO(5)}_{\mathrm{NS}}(e^{2\pi i}q;\vec{y}) \, \chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(e^{2\pi i}q;r) \, \Big] \\
\chi^{\mathcal{N}_{6d}=(1,0)}_{\mathrm{R}} \, |_{\mathrm{GSO}} \, (q;\vec{y},r) \ &= \ \frac{1}{2} \, \chi^{SO(5)}_{\mathrm{R}}(q;\vec{y}) \, \chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{R},h=1/4,\ell=1/2}(q;r) \ .
\end{aligned} \tag{4.196}$$

The power series expansion of (4.196) starts as[11]

$$\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r) \;=\; \underbrace{\left(y_1^2+y_1^{-2}+y_2^2+y_2^{-2} \;+\; \frac{1}{2}[1]_r \prod_{i=1}^{2}(y_i+y_i^{-1}) \right) q^0}_{\text{8 massless states}}$$
$$+ \;\; \underbrace{\llbracket 1,0;0 \rrbracket \, q}_{\text{80 states at level 1}} \;\; + \;\; \underbrace{\big(\llbracket 2,0;0 \rrbracket \;+\; \llbracket 0,2;0 \rrbracket \;+\; \llbracket 0,1;1 \rrbracket \big)\, q^2}_{\text{512 states at level 2}}$$
$$+ \;\; \big(\llbracket 3,0;0 \rrbracket \;+\; 2\,\llbracket 1,0;0 \rrbracket \;+\; \llbracket 0,0;0 \rrbracket$$
$$+ \; \llbracket 1,2;0 \rrbracket \;+\; \llbracket 0,2;0 \rrbracket \;+\; \llbracket 0,0;2 \rrbracket \;+\; 2\,\llbracket 1,1;1 \rrbracket \;+\; \llbracket 0,1;1 \rrbracket \big)\, q^3 \;+\; \mathcal{O}(q^4)\;. \tag{4.197}$$

The $q^{\leq 6}$ coefficients are listed in table 4.3, further information on the particle content up to level 25 is tabulated in appendix 4.B.2.

$\alpha' m^2$	representations of $\mathcal{N}_{6d}=(1,0)$ super Poincaré
1	$\llbracket 1,0;0 \rrbracket$
2	$\llbracket 2,0;0 \rrbracket + \llbracket 0,2;0 \rrbracket + \llbracket 0,1;1 \rrbracket$
3	$\llbracket 3,0;0 \rrbracket + 2\llbracket 1,0;0 \rrbracket + \llbracket 0,0;0 \rrbracket + \llbracket 1,2;0 \rrbracket + \llbracket 0,2;0 \rrbracket + \llbracket 0,0;2 \rrbracket + 2\llbracket 1,1;1 \rrbracket + \llbracket 0,1;1 \rrbracket$
4	$\llbracket 4,0;0 \rrbracket + 3\llbracket 2,0;0 \rrbracket + 2\llbracket 1,0;0 \rrbracket + 2\llbracket 0,0;0 \rrbracket + \llbracket 2,2;0 \rrbracket + 2\llbracket 1,2;0 \rrbracket + 4\llbracket 0,2;0 \rrbracket$ $+2\llbracket 1,0;2 \rrbracket + \llbracket 0,2;2 \rrbracket + 3\llbracket 1,1;1 \rrbracket + 4\llbracket 0,1;1 \rrbracket + 2\llbracket 2,1;1 \rrbracket$
5	$\llbracket 5,0;0 \rrbracket + 3\llbracket 3,0;0 \rrbracket + 4\llbracket 2,0;0 \rrbracket + 9\llbracket 1,0;0 \rrbracket + 3\llbracket 0,0;0 \rrbracket + \llbracket 3,2;0 \rrbracket + 2\llbracket 2,2;0 \rrbracket$ $+7\llbracket 1,2;0 \rrbracket + 6\llbracket 0,2;0 \rrbracket + \llbracket 0,4;0 \rrbracket + 3\llbracket 2,0;2 \rrbracket + 3\llbracket 1,0;2 \rrbracket + 3\llbracket 0,0;2 \rrbracket + \llbracket 1,2;2 \rrbracket$ $+3\llbracket 0,2;2 \rrbracket + 2\llbracket 3,1;1 \rrbracket + 4\llbracket 2,1;1 \rrbracket + 9\llbracket 1,1;1 \rrbracket + 8\llbracket 0,1;1 \rrbracket + \llbracket 1,3;1 \rrbracket + 4\llbracket 0,3;1 \rrbracket$ $+\llbracket 0,1;3 \rrbracket$
6	$\llbracket 6,0;0 \rrbracket + \llbracket 4,2;0 \rrbracket + 2\llbracket 4,1;1 \rrbracket + 3\llbracket 4,0;0 \rrbracket + 2\llbracket 3,2;0 \rrbracket + 4\llbracket 3,1;1 \rrbracket + 3\llbracket 3,0;2 \rrbracket$ $+5\llbracket 3,0;0 \rrbracket + \llbracket 2,3;1 \rrbracket + \llbracket 2,2;2 \rrbracket + 8\llbracket 2,2;0 \rrbracket + 12\llbracket 2,1;1 \rrbracket + 4\llbracket 2,0;2 \rrbracket + 14\llbracket 2,0;0 \rrbracket$ $+\llbracket 1,4;0 \rrbracket + 5\llbracket 1,3;1 \rrbracket + 6\llbracket 1,2;2 \rrbracket + 13\llbracket 1,2;0 \rrbracket + 2\llbracket 1,1;3 \rrbracket + 23\llbracket 1,1;1 \rrbracket + 9\llbracket 1,0;2 \rrbracket$ $+12\llbracket 1,0;0 \rrbracket + 4\llbracket 0,4;0 \rrbracket + 9\llbracket 0,3;1 \rrbracket + 9\llbracket 0,2;2 \rrbracket + 19\llbracket 0,2;0 \rrbracket + 3\llbracket 0,1;3 \rrbracket$ $+18\llbracket 0,1;1 \rrbracket + 4\llbracket 0,0;2 \rrbracket + 8\llbracket 0,0;0 \rrbracket$
7	$\llbracket 7,0;0 \rrbracket + \llbracket 5,2;0 \rrbracket + 2\llbracket 5,1;1 \rrbracket + 3\llbracket 5,0;0 \rrbracket + 2\llbracket 4,2;0 \rrbracket + 4\llbracket 4,1;1 \rrbracket + 3\llbracket 4,0;2 \rrbracket$ $+5\llbracket 4,0;0 \rrbracket + \llbracket 3,3;1 \rrbracket + \llbracket 3,2;2 \rrbracket + 8\llbracket 3,2;0 \rrbracket + 13\llbracket 3,1;1 \rrbracket + 5\llbracket 3,0;2 \rrbracket + 17\llbracket 3,0;0 \rrbracket$ $+\llbracket 2,4;0 \rrbracket + 5\llbracket 2,3;1 \rrbracket + 6\llbracket 2,2;2 \rrbracket + 16\llbracket 2,2;0 \rrbracket + 2\llbracket 2,1;3 \rrbracket + 31\llbracket 2,1;1 \rrbracket + 17\llbracket 2,0;2 \rrbracket$ $+24\llbracket 2,0;0 \rrbracket + 5\llbracket 1,4;0 \rrbracket + 15\llbracket 1,3;1 \rrbracket + 16\llbracket 1,2;2 \rrbracket + 38\llbracket 1,2;0 \rrbracket + 7\llbracket 1,1;3 \rrbracket$ $+51\llbracket 1,1;1 \rrbracket + \llbracket 1,0;4 \rrbracket + 20\llbracket 1,0;2 \rrbracket + 35\llbracket 1,0;0 \rrbracket + \llbracket 0,5;1 \rrbracket + 3\llbracket 0,4;2 \rrbracket + 9\llbracket 0,4;0 \rrbracket$ $+2\llbracket 0,3;3 \rrbracket + 26\llbracket 0,3;1 \rrbracket + 22\llbracket 0,2;2 \rrbracket + 34\llbracket 0,2;0 \rrbracket + 7\llbracket 0,1;3 \rrbracket + 39\llbracket 0,1;1 \rrbracket$ $+\llbracket 0,0;4 \rrbracket + 13\llbracket 0,0;2 \rrbracket + 13\llbracket 0,0;0 \rrbracket$

Table 4.3: $\mathcal{N}_{6d}=(1,0)$ multiplets occurring up to mass level 7

[11] Again, there is a subtlety in applying (4.196) to the massless R sector, see the footnote before (4.112). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y_1 - y_1^{-1})(y_2 - y_2^{-1})(r - r^{-1})$ to get the correct massless character in R sector.

4.4.1 The total number of states at a given mass level

In this subsection, we compute the total number of states present at a given mass level through the unrefined partition function, *i.e.* by setting the fugacities y_1, y_2 and r in (4.197) to unity. The total number of states N_m at the mass level m can be read off from the coefficient of q^m in the power series of $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$.

We follow the analysis presented in subsection 4.3.1. The unrefined partition function is given by

$$\begin{aligned}\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\}) &= 2\chi_{\mathrm{R}}^{\mathcal{N}_{6d}=(1,0)}\,|_{\mathrm{GSO}}\,(q;y=1,s=1)\\ &= \chi_{\mathrm{R}}^{SO(5)}(q;\{y_i=1\})\,\chi_{\mathrm{R},h=1/4,\ell=1/2}^{\mathcal{N}_{2d}=4,c=6}(q;r=1)\\ &= \chi_{\mathrm{R}}^{SO(3)}(q;\{y=1\})^2\,\chi_{\mathrm{R},h=1/4,\ell=1/2}^{\mathcal{N}_{2d}=4,c=6}(q;r=1)\\ &= q^{-1/8}\frac{\vartheta_2(1,q)^4}{\eta(q)^9}\left[1-2iq^{1/8}\mu\left(1/2,\tau\right)\right]\ .\end{aligned} \tag{4.198}$$

Indeed, the power series of $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$ in q reproduces the numbers presented in the second column of Table 4.13. Note that $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$ is *not* a modular form, since the Appell-Lerch sum is a mock modular form and it is not added by a suitable non-holomorphic component to be modular.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can also be computed from

$$N_m = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}q}{q^{m+1}}\ \chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})\ , \tag{4.199}$$

where $\mathcal{C}$ is a contour around the origin.

Now let us compute an asymptotic formula for the number of states N_m at a mass level m when $m\to\infty$. We focus on the limit $q\to 1^-$ and proceed in a similar way to subsection 4.3.1.

Let us first examine the leading behaviour of $\mu\left(1/2,\tau\right)$ as $q\to 1^-$ or $\tau\to 0$. Using the second point of Proposition 1.5 of [16], we find that

$$\frac{1}{\sqrt{-i\tau}}\mu\left(\frac{1}{2\tau},-\frac{1}{\tau}\right)+\mu\left(\frac{1}{2},\tau\right)=\frac{1}{2i}\ . \tag{4.200}$$

Let us consider $\mu\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right)$ as $q \to 1^-$ or equivalently $\tau = i\epsilon$ as $\epsilon \to 0^+$. It follows from the definition of Appell-Lerch sum that

$$\begin{aligned}
\mu\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) &= -\frac{e^{i\pi/(2\tau)}}{\vartheta_1(e^{2\pi i/(2\tau)}, e^{-2\pi i/\tau})} \sum_{m\in\mathbb{Z}} (-1)^m \frac{e^{-i\pi m^2/\tau}}{1 - e^{-2\pi i m/\tau + \pi i/\tau}} \\
&\sim -\frac{e^{\pi/(2\epsilon)}}{-ie^{\pi/(4\epsilon)}} \times (-2e^{-\pi/\epsilon}) , \qquad \tau = i\epsilon, \ \epsilon \to 0^+ \\
&= 2i \exp\left(-\frac{3\pi}{4\epsilon}\right) , \qquad (4.201)
\end{aligned}$$

where in the second 'equality' only $m = 0, 1$ in the infinite sum contribute to the leading behaviour and we have used the fact that $\vartheta_1(e^{2\pi i/(2\tau)}, e^{-2\pi i/\tau}) = -ie^{\pi/(4\epsilon)}$, as $\tau = i\epsilon$, $\epsilon \to 0^+$. Hence, to the leading order, one can neglect the first term in (4.200) in comparison with $1/(2i)$ on the right hand side and so

$$\mu\left(\frac{1}{2}, \tau\right) \sim \frac{1}{2i} , \qquad q \to 1^- . \qquad (4.202)$$

Therefore it follows from (4.198) that, as $q \to 1^-$,

$$\begin{aligned}
&\chi^{\mathcal{N}_{6d}=(1,0)}(q; \{y_i = 1, r = 1\}) \sim q^{-1/8} \frac{\vartheta_2(1, q)^4}{\eta(q)^9} \left(1 - q^{1/8}\right) \\
&\sim (2\pi)^{-5/2} (1 - q^{1/8})(1 - q)^{5/2} \exp\left(-\frac{3\pi^2}{2\log q}\right) , \qquad (4.203)
\end{aligned}$$

where we have used (4.121) and (4.119). Hence, as $m \to \infty$,

$$N_m \sim (2\pi)^{-5/2} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}q}{q} \ (1 - q^{1/8})(1 - q)^{5/2} \exp\left(-\frac{3\pi^2}{2\log q} - m \log q\right) \qquad (4.204)$$

The saddle point is at $q_0 = \exp\left(-\pi\sqrt{3}/\sqrt{2m}\right)$ and the steepest descent direction is the imaginary direction in q. We proceed in a similar way to (4.125) by writing $q = q_0 e^{i\theta}$ and using Laplace's method to obtain

$$\begin{aligned}
N_m &\sim (2\pi)^{-5/2} (1 - q_0^{1/8})(1 - q_0)^{5/2} e^{\pi\sqrt{6m}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \exp\left(-\frac{1}{\pi}\sqrt{\frac{2}{3}} m^{3/2} \theta^2\right) \\
&\sim \frac{9\pi}{2^{17/2}} m^{-5/2} \exp\left(\pi\sqrt{6m}\right) , \qquad m \to \infty . \qquad (4.205)
\end{aligned}$$

4.4.2 The GSO projected NS and R sectors

The NS sector

From (4.291), the partition function of the GSO projected NS sector is

$$\chi_{\rm NS}^{\mathcal{N}_{6d}=(1,0)}\,|_{\rm GSO}\,(q;y,s)=\sum_{k_1,k_2,p=0}^{\infty}[2k_1]_{y_1}[2k_2]_{y_2}[2p]_r\;F^{\rm NS}_{k_1,k_2,p}(q)\ , \quad (4.206)$$

where the function $F^{\rm NS}_{k,p}(q)$ is given by

$$\begin{aligned}
&F^{\rm NS}_{k_1,k_2,p}(q)=(q;q)_\infty^{-9}(1-q)q^{\frac{1}{2}p^2+p-1}\\
&\times\sum_{\vec{n}\in\mathbb{Z}_+^2}\sum_{\vec{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}(-1)^{n_A+1}(1-q^{n_A})q^{\frac{1}{2}m_A^2+\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)})\\
&\times\frac{1}{2}\left[\frac{(1-q^{p+\frac{1}{2}})\vartheta_3(1,q)}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{3}{2}})}\prod_{A=1}^{2}(1-q^{m_A+\frac{1}{2}})\right.\\
&\left.+(-1)^{m_1^2+m_2^2+p^2}\frac{(1+q^{p+\frac{1}{2}})\vartheta_4(1,q)}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{3}{2}})}\prod_{A=1}^{2}(1+q^{m_A+\frac{1}{2}})\right]\ . \quad (4.207)
\end{aligned}$$

Asymptotics. This expression can be simplified further in the asymptotic limit $k_1,k_2\to\infty$. Using (4.128), we have

$$\begin{aligned}
&\sum_{\vec{n}\in\mathbb{Z}_+^2}\prod_{A=1}^{2}(-1)^{n_A+1}(1-q^{n_A})q^{\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)})\\
&\sim(1-q)^2\prod_{A=1}^{2}\frac{q^{k_A}\left(1-q^{2k_A+2}\right)}{\left(1+q^{k_A}\right)^4}\left\{q^{-m_A}\left(1-q^{2m_A+1}\right)\right\}\ , \quad (4.208)
\end{aligned}$$

and using (4.130) we have

$$\sum_{\vec{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}q^{\frac{1}{2}m_A^2-m_A}\left(1-q^{m_A+\frac{1}{2}}\right)\left(1-q^{2m_A+1}\right)=q^{-1}(1-q)^2\vartheta_3(1,q)^2\ , \quad (4.209)$$

$$\begin{aligned}
&\sum_{\vec{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}(-1)^{m_A^2}q^{\frac{1}{2}m_A^2-m_A}\left(1+q^{m_A+\frac{1}{2}}\right)\left(1-q^{2m_A+1}\right)\\
&=q^{-1}(1-q)^2\vartheta_4(1,q)^2\ , \quad (4.210)
\end{aligned}$$

Therefore we arrive at an asymptotic formula for $F^{\rm NS}_{k_1,k_2,p}(q)$ when $k_1, k_2 \to \infty$:

$$F^{\rm NS}_{k_1,k_2,p}(q) \sim \frac{1}{2}(q;q)_\infty^{-9}(1-q)^5 q^{\frac{1}{2}p^2+p+k_1+k_2-2}\Bigg[\frac{(1-q^{p+\frac{1}{2}})}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{3}{2}})}\vartheta_3(1,q)^3 + (-1)^{p^2}\frac{(1+q^{p+\frac{1}{2}})}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{3}{2}})}\vartheta_4(1,q)^3\Bigg] , \qquad k_1,k_2\to\infty \ . \tag{4.211}$$

The R sector

The partition function of the GSO projected R sector is

$$\chi_{\rm R}^{\mathcal{N}_{6d}=(1,0)}|_{\rm GSO}\ (q;y,s) = \sum_{k_1,k_2,p=0}^{\infty}[2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p+1]_r\ F^{\rm R}_{k_1,k_2,p}(q)\ , \tag{4.212}$$

where $F^{\rm R}_{k_1,k_2,p}(q)$ is given by

$$F^{\rm R}_{k_1,k_2,p}(q) = (q;q)_\infty^{-9}(1-q)q^{\frac{1}{2}p^2+\frac{3}{2}p-\frac{3}{8}}\times\frac{1}{2}\frac{(1-q^{p+1})\vartheta_2(1,q)}{(1+q^p)(1+q^{p+2})} \times\sum_{\vec{n}\in\mathbb{Z}_+^2}\sum_{\vec{m}\in\mathbb{Z}^2_{\geq 0}}\prod_{A=1}^{2}(-1)^{n_A}(1-q^{n_A})q^{\frac{1}{2}(m_A+\frac{1}{2})^2+\binom{n_A}{2}} \times(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+2)})(1-q^{m_A+1}) \tag{4.213}$$

Similarly to the NS sector, an asymptotic formula for $F^{\rm NS}_{k_1,k_2,p}(q)$ when $k_1, k_2\to\infty$ is given by

$$F^{\rm R}_{k_1,k_2,p}(q) \sim \frac{1}{2}(q;q)_\infty^{-9}(1-q)^5 q^{\frac{1}{2}p^2+\frac{3}{2}p+k_1+k_2-\frac{3}{8}}\frac{(1-q^{p+1})}{(1+q^p)(1+q^{p+2})}\vartheta_2(1,q)^3\ . \tag{4.214}$$

4.4.3 Multiplicities of representations in the $\mathcal{N}_{6d} = (1,0)$ partition function

Combining the contributions from the NS and R sectors, we have

$$\begin{aligned}\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r) &= \chi_{\rm NS}^{\mathcal{N}_{6d}=(1,0)}\ |_{\rm GSO}\ (q;\vec{y},r)\ +\ \chi_{\rm R}^{\mathcal{N}_{6d}=(1,0)}\ |_{\rm GSO}\ (q;\vec{y},r) \\ &= \sum_{k_1,k_2,p=0}^{\infty} \Big([2k_1]_{y_1}[2k_2]_{y_2}[2p]_r\ F^{\rm NS}_{k_1,k_2,p}(q) \\ &+ [2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p+1]_r\ F^{\rm R}_{k_1,k_2,p}(q)\Big)\ . \end{aligned} \quad (4.215)$$

Making SUSY manifest amounts to rewriting the partition function as

$$\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r) \quad = \textstyle\sum_{n_1,n_2\geq 0}\sum_{p=0}^{\infty} [\![n_1,n_2;p]\!]\ G_{n_1,n_2,p}(q)\ , \quad (4.216)$$

and the aim is to compute explicitly a *multiplicity generating function* $G_{n_1,n_2,p}(q)$.

Before proceeding further, we observe the selection rule

$$G_{n_1,2n_2,2p+1}(q) = 0\ , \qquad G_{n_1,2n_2+1,2p}(q) = 0\ . \quad (4.217)$$

It follows from (4.215) that $[k_1]_{y_1}[k_2]_{y_2}[p]_r$ with odd (respectively even) values of p only enter with a product of two representations with both odd (resp. even) k_1 and k_2. According to (4.82), the product $[k_1]_{y_1}[k_2]_{y_2}$ with both odd (resp. even) k_1 and k_2 decomposes into only spin (resp. non-spin) representations of $SO(5)$. In other words, a spin (resp. non-spin) representation only comes with an odd (resp. even) value of p, and hence (4.217) follows.

The multiplicity of $[\![n_1,n_2;p]\!]$ appearing in $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r)$ can be determined as follows:

$$\begin{aligned} G_{n_1,n_2,p}(q) &= \int {\rm d}\mu_{SU(2)}(r)[p]_r \int {\rm d}\mu_{SO(5)}(\vec{y})[n_1,n_2]_{\vec{y}}\ \frac{\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r)}{Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)}\ , \\ &= G^{\rm NS}_{n_1,n_2,p}(q) + G^{\rm R}_{n_1,n_2,p}(q)\ , \\ &= \int {\rm d}\mu_{SU(2)}(r)[p]_r \Delta_{n_1+\frac{1}{2}n_2,\frac{1}{2}n_2;2k_1,2k_2} \frac{\chi^{\mathcal{N}_{6d}=(1,0)}(q;\vec{y},r)}{Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)} \end{aligned} \quad (4.218)$$

where

$$G^{\rm NS}_{n_1,n_2,p}(q) = \int \mathrm{d}\mu_{SU(2)}(r)[p]_r \int \mathrm{d}\mu_{SO(5)}(\vec{y})[n_1,n_2]_{\vec{y}}$$
$$\times \sum_{k_1,k_2,p'\geq 0} \frac{[2k_1]_{y_1}[2k_2]_{y_2}[2p']_r}{Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)} F^{\rm NS}_{k_1,k_2,p'}(q) \ , \tag{4.219}$$
$$G^{\rm R}_{n_1,n_2,p}(q) = \int \mathrm{d}\mu_{SU(2)}(r)[p]_r \int \mathrm{d}\mu_{SO(5)}(\vec{y})[n_1,n_2]_{\vec{y}}$$
$$\times \sum_{k_1,k_2,p'\geq 0} \frac{[2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p'+1]_r}{Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)} F^{\rm R}_{k_1,k_2,p'}(q) \tag{4.220}$$

and the inverse of the character of the fundamental multiplet in (4.194) can be written as a geometric series[12] [13] similar to (4.151)

$$\begin{aligned}[Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)]^{-1} &= \frac{r^2}{\left(1+\frac{r}{y_1y_2}\right)\left(1+\frac{ry_1}{y_2}\right)\left(1+\frac{ry_2}{y_1}\right)(1+ry_1y_2)} \\ &= \sum_{m_1,\ldots,m_4\geq 0} (-1)^{m_1+m_2+m_3+m_4} r^{2+m_1+m_2+m_3+m_4} \\ &\quad \times y_1^{-m_1+m_2-m_3+m_4} y_2^{-m_1-m_2+m_3+m_4} \ . \end{aligned} \tag{4.223}$$

[12]Note that this can also be rewritten as

$$\begin{aligned}[Z(\mathcal{N}_{6d}=(1,0))(\vec{y},r)]^{-1} &= r^2\,\mathrm{PE}\,[s[0,1]_{\vec{y}}] \quad \text{with } s=-r \\ &= \sum_{m=0}^{\infty}(-1)^m r^{m+2}[0,m]_{\vec{y}} \ . \end{aligned} \tag{4.221}$$

where in the last equality we have used the fact that $\mathrm{Sym}^m[0,1]=[0,m]$.

[13]After [7] was published, a new formula for the inverse of the fundamental super-Poincaré multiplet was found:

$$\begin{aligned} Z(\mathcal{N}_{6d}=(1,0))^{-1} &= \frac{1}{((r+r^{-1})+[1]_{y_1y_2})((r+r^{-1})+[1]_{\frac{y_1}{y_2}})} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(r+r^{-1})^{m+2}}\sum_{p=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{p}[m-2p]_{y_1}[m-2p]_{y_2} \end{aligned} \tag{4.222}$$

While it is more complex to use than (4.151) for the $\mathcal{N}_{4d}=1$ case, it shares its property of being symmetric under $r\leftrightarrow r^{-1}$.

Some useful identities

Before we proceed further, let us derive some useful identities for the elementary building blocks of $G_{n_1,n_2,p}$. The first one follows from (4.17):

$$\mathcal{I}_0(w;p_1,p_2) := \int \mathrm{d}\mu_{SO(3)}(r)\ r^w [p_1]_r [p_2]_r$$
$$= \begin{cases} \delta_{p_1,p_2} & \text{for } w=0 \\ \frac{1}{2}\sum_{p=0}^{\frac{1}{2}(p_1+p_2-|p_1-p_2|)} \left(\delta_{|w|,2p+|p_1-p_2|} - \delta_{|w|,2p+2+|p_1-p_2|}\right) & \text{for } w \neq 0 \end{cases} \tag{4.224}$$

Next, we are interested in the following integral:

$$\mathcal{I}(\vec{w};\vec{k};\vec{n}) := \int \mathrm{d}\mu_{SO(5)}(\vec{y})\ y_1^{w_1} y_2^{w_2} [k_1]_{y_1} [k_2]_{y_2} [n_1,n_2]_{\vec{y}}\ . \tag{4.225}$$

We compute this using the decomposition formula (4.83), together with (4.224). In what follows, we assume that $\vec{k},\vec{n} \in \mathbb{Z}^2_{\geq 0}$ and $\vec{w} \in \mathbb{Z}^2$. (Here, as i(4.78), the k_A, the k'_A and n_2 can be integers or half-odd integers (independently of each other, though the two k_A must be of the same type as must the two k'_A, while in [7] we consider the cases separately.)

$$\mathcal{I}(\vec{w};2k_1,2k_2;n_1,2n_2)$$
$$= \sum_{\vec{k}'} \Delta(n_1+n_2,n_2;2k'_1,2k'_2) \prod_{A=1}^{2} \mathcal{I}_0(w_A;2k_A,2k'_A) \tag{4.226}$$

where from (4.78)

$$\Delta(\lambda_1,\lambda_2;2k_1,2k_2) \quad = \tfrac{1}{2}\textstyle\sum_{\sigma\in S_2} \det\left(\theta^{4+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}\right)\right)^2_{A,B=1} \tag{4.227}$$

Multiplicity generating function

The NS- and R sector contributions to the multiplicity generating function for the representation $[\![n_1, n_2; p]\!]$ can be rewritten as

$$G^{\rm NS}_{n_1,n_2,p}(q) = \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \mathcal{I}_0(W_1(\vec{m}), p, 2p') \times \sum_{k_1,k_2\geq 0} \mathcal{I}(\vec{W}_2(\vec{m}); 2k_1, 2k_2; n_1, n_2)\ F^{\rm NS}_{k_1,k_2,p'}(q) \tag{4.228}$$

$$G^{\rm R}_{n_1,n_2,p}(q) = \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \mathcal{I}_0(W_1(\vec{m}), p, 2p'+1) \times \sum_{k_1,k_2\geq 0} \mathcal{I}(\vec{W}_2(\vec{m}); 2k_1+1, 2k_2+1; n_1, n_2)\ F^{\rm R}_{k_1,k_2,p'}(q)\ , \tag{4.229}$$

where we define

$$\begin{aligned} W_1(\vec{m}) &= 2 + m_1 + m_2 + m_3 + m_4\ , \\ \vec{W}_2(\vec{m}) &= (-m_1 + m_2 - m_3 + m_4, -m_1 - m_2 + m_3 + m_4)\ . \end{aligned} \tag{4.230}$$

As stated in (4.218), the multiplicity of the representation $[\![n_1, n_2; p]\!]$ in the $\mathcal{N}_{6d} = (1, 0)$ partition function is given by

$$\begin{aligned} G_{n_1,n_2,p}(q) &= G^{\rm NS}_{n_1,n_2,p}(q) + G^{\rm R}_{n_1,n_2,p}(q) \\ &= \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \Big[\mathcal{I}_0(W_1(\vec{m}); p, 2p') \\ &\times \sum_{k_1,k_2\geq 0} \mathcal{I}(\vec{W}_2(\vec{m}); 2k_1, 2k_2; n_1, n_2)\ F^{\rm NS}_{k_1,k_2,p'}(q) \\ &+ \mathcal{I}_0(W_1(\vec{m}), p, 2p'+1) \sum_{k_1,k_2\geq 0} \mathcal{I}(\vec{W}_2(\vec{m}); 2k_1+1, 2k_2+1; n_1, n_2)\ F^{\rm R}_{k_1,k_2,p'}(q)\Big]\ . \end{aligned} \tag{4.231}$$

4.4.4 Empirical approach to $\mathcal{N}_{6d} = (1, 0)$ asymptotic patterns

In this subsection, we follow the lines of subsection 4.3.5 and investigate the large spin asymptotics of multiplicity generating functions $G_{n,k,p}(q)$ for universal $\mathcal{N}_{6d} = (1, 0)$ supermultiplets $[\![n, k; p]\!]$. Similar to the $\mathcal{N}_{4d} = 1$ strategy, the $G_{n,k,p}(q)$ are expanded in powers of q^n where n denotes the first Dynkin label that we loosely identify with the spin. The coefficients

$\tau_\ell^{k,p}(q)$ of $(q^n)^\ell$ turn out to be power series with non-negative coefficients which enter with alternating sign $(-1)^{\ell-1}$:

$$\begin{aligned} G_{n,k,p}(q) &= q^n\,\tau_1^{k,p}(q) \;-\; q^{2n}\,\tau_2^{k,p}(q) \;+\; q^{3n}\,\tau_3^{k,p}(q) \;-\; \ldots \\ &= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,q^{\ell n}\,\tau_\ell^{k,p}(q) \end{aligned} \tag{4.232}$$

In spacetime dimensions higher than four, the analytic methods of subsection 4.3.4 are no longer efficiently applicable. We could not find an asymptotic formula for (4.231) resembling (4.173) and (4.174) for the large spin regime of the $\mathcal{N}_{4d} = 1$ multiplicity generating functions. Hence, we determine the $\tau_\ell^{k,p}(q)$ including the leading trajectory $\tau_1^{k,p}(q)$ from our data found by expanding the partition function (4.196) up to mass level 25. The multiplicities of $[\![n,0;0]\!]$ multiplets are shown in the following table 4.4, data for nonzero values $(k,p) = (2,0),(0,2)$ and $(1,1)$ can be found in appendix 4.B.2. Table entries marked in red are only affected by the stable pattern $\tau_{\ell=1}^{k,p}(q)$ whereas the blue numbers arise from $q^n\tau_1^{k,p}(q) - q^{2n}\tau_2^{k,p}(q)$, i.e. by including the (subtractive) subleading trajectory.

Levels of first appearance

Let us firstly determine the level of first appearance for various families $\{[\![n,k;p]\!],\ n = 0,1,\ldots\}$ of $\mathcal{N}_{6d} = (1,0)$ supermultiplets with second $SO(5)$ Dynkin label k and R symmetry quantum number p fixed. It is identical to the leading q power of the multiplicity generaing function $G_{0,k,p}(q)$ or its expansion coefficients $\tau_\ell^{k,p}(q)$ defined by (4.232). The following table 4.5 gathers the mass levels $\alpha' m^2 \leq 25$ where the first instance of a $\{[\![n,k;p]\!],\ n = 0,1,\ldots\}$ member can be found:

We observe that, roughly speaking, the level of first appearance for supermultiplets $[\![n,k;p]\!]$ depends linearly[14] on the $SO(5)$ Dynkin label k (with slope $\frac{3}{2}$) but quadratically on the R symmetry spin $p/2$, in agreement with the final remark in subsection 4.2.2.

[14] The linear k dependence can be partially understood from the $\lambda_{1,2}$ dependence in (4.88). However, the bosonic string suggests that an $SO(5)$ representation $[n,k]$ is delayed by *two* levels under $k \mapsto k+1$ whereas the observations from table 4.5 clearly show a delay of *three* levels per $k \mapsto k+1$. Even though we cannot give a detailed explanation on analytical grounds, it is clear that this extra delay in mass level must be due to the worldsheet fermions, see e.g. (4.89) and (4.90).

$\alpha' m^2$	# ⟦0,0;0⟧	# ⟦1,0;0⟧	# ⟦2,0;0⟧	# ⟦3,0;0⟧	# ⟦4,0;0⟧	# ⟦5,0;0⟧	# ⟦6,0;0⟧	# ⟦7,0;0⟧	# ⟦8,0;0⟧	# ⟦9,0;0⟧	# ⟦10,0;0⟧	# ⟦11,0;0⟧
1	0	1	0									
2	0	0	1	0								
3	1	2	0	1	0							
4	2	2	3	0	1	0						
5	3	9	4	3	0	1	0					
6	8	12	14	5	3	0	1	0				
7	13	35	24	17	5	3	0	1	0			
8	30	58	63	29	18	5	3	0	1	0		
9	53	135	116	82	32	18	5	3	0	1	0	
10	107	243	265	153	88	33	18	5	3	0	1	0
11	193	505	503	358	172	91	33	18	5	3	0	1
12	376	918	1044	696	403	178	92	33	18	5	3	0
13	670	1803	1975	1474	801	423	181	92	33	18	5	3
14	1246	3269	3887	2839	1711	846	429	182	92	33	18	5
15	2220	6136	7235	5687	3355	1824	866	432	182	92	33	18
16	4005	11015	13691	10754	6784	3605	1870	872	433	182	92	33
17	7025	20052	25041	20649	13021	7348	3718	1890	875	433	182	92
18	12407	35469	45971	38304	25243	14213	7606	3764	1896	876	433	182
19	21469	63030	82532	71226	47411	27774	14790	7720	3784	1899	876	433
20	37182	109838	147906	129443	89013	52547	29015	15048	7766	3790	1900	876
21	63492	191293	260818	234646	163536	99387	55177	29600	15162	7786	3793	1900
22	108142	328527	457957	418298	299140	183903	104797	56431	29859	15208	7792	3794
23	182254	562391	794256	741961	538495	338749	194850	107476	57016	29973	15228	7795
24	306007	952431	1369976	1299438	963344	613928	360467	200360	108738	57275	30019	15234
25	509309	1605996	2339762	2261945	1702039	1105604	656324	371692	203052	109324	57389	30039

Table 4.4: $\mathcal{N}_{6d} = (1,0)$ multiplets with $SO(5)$ quantum numbers $[n, 0]$ and $SU(2)_R$ spin 0

$\downarrow p,\ \overrightarrow{k}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1		2		5		8		11		14		17		20		23	
1		2		4		7		10		13		16		19		22		25
2	3		4		7		10		13		16		19		22		25	
3		5		7		10		13		16		19		22		25		
4	7		8		10		13		16		19		22		25			
5		9		11		14		17		20		23						
6	11		12		15		18		21		24							
7		14		16		19		22		25								
8	17		18		20		23											
9		20		22		25												
10	23		24															

Table 4.5: Mass level where the $[\![0,k;p]\!]$ multiplet of $\mathcal{N}_{6d} = (1,0)$ firstly occurs. Empty spaces indicate that the representations in question do not occur at levels ≤ 25.

Explicit formulae for the $\tau_\ell^{k,p}(q)$

Let us now list the leading terms in $\tau_\ell^{0,0}(q), \tau_\ell^{2,0}(q), \tau_\ell^{0,2}(q)$ and $\tau_\ell^{1,1}(q)$, obtained through the entries of table 4.4 and its $(k,p) \neq (0,0)$ relatives displayed in appendix 4.B.2. This allows to reconstruct the large spin asymptotics of the multiplicity generating functions $G_{n,k,p}(q)$ via (4.232).

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[0]$

$$
\begin{aligned}
\tau_1^{0,0}(q) &= 1 + 0q + 3q^2 + 5q^3 + 18q^4 + 33q^5 + 92q^6 + 182q^7 + 433q^8 \\
&\quad + 876q^9 + 1900q^{10} + 3794q^{11} + 7796q^{12} + 15238q^{13} + 30049q^{14} \\
&\quad + 57465q^{15} + 109773q^{16} + 205349q^{17} + 382249q^{18} + 700520q^{19} + \ldots \\
\tau_2^{0,0}(q) &= q\,(1 + 4q^1 + 10q^2 + 30q^3 + 76q^4 + 190q^5 + 449q^6 + 1035q^7 \\
&\quad + 2298q^8 + 4999q^9 + 10580q^{10} + 21976q^{11} + 44727q^{12} + 89543q^{13} + \ldots) \\
\tau_3^{0,0}(q) &= q\,(1 + q + 10q^2 + 23q^3 + 81q^4 + 194q^5 + 531q^6 + 1232q^7 + \\
&\quad + 2967q^8 + 6586q^9 + \ldots) \\
\tau_4^{0,0}(q) &= q^2\,(1 + 5q + 16q^2 + 53q^3 + 153q^4 + 417q^5 + \ldots) \\
\tau_5^{0,0}(q) &= q^2\,(1 + q + 11q^2 + \ldots)
\end{aligned}
\tag{4.233}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[0]$

$$
\begin{aligned}
\tau_1^{2,0}(q) &= q^2\,(1 + 2q + 8q^2 + 17q^3 + 52q^4 + 117q^5 + 293q^6 + 645q^7 \\
&\quad + 1468q^8 + 3119q^9 + 6667q^{10} + 13674q^{11} + 27913q^{12} + 55446q^{13} \\
&\quad + 109165q^{14} + 210717q^{15} + 402714q^{16} + 757889q^{17} + 1412208q^{18} + \ldots) \\
\tau_2^{2,0}(q) &= q^3\,(1 + 4q + 14q^2 + 41q^3 + 118q^4 + 306q^5 + 764q^6 + 1818q^7 \\
&\quad + 4191q^8 + 9344q^9 + 20318q^{10} + 43083q^{11} + 89493q^{12} + 182239q^{13} + \ldots) \\
\tau_3^{2,0}(q) &= q^5\,(3 + 9q + 40q^2 + 114q^3 + 345q^4 + 890q^5 + 2297q^6 + 5481q^7 \\
&\quad + 12871q^8 + \ldots) \\
\tau_4^{2,0}(q) &= q^6\,(1 + 5q + 23q^2 + 79q^3 + 251q^4 + 717q^5 + \ldots) \\
\tau_5^{2,0}(q) &= q^8\,(3 + 10q + 48q^2 + \ldots)
\end{aligned}
\tag{4.234}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[2]$

$$\begin{aligned}
\tau_1^{0,2}(q) &= q^3\,(3 + 5q + 20q^2 + 46q^3 + 128q^4 + 288q^5 + 696q^6 + 1513q^7 \\
&\quad + 3354q^8 + 7025q^9 + 14707q^{10} + 29736q^{11} + 59679q^{12} + 116933q^{13} \\
&\quad + 226900q^{14} + 432515q^{15} + 816089q^{16} + \ldots) \\
\tau_2^{0,2}(q) &= q^2\,(1 + 3q^1 + 13q^2 + 37q^3 + 109q^4 + 285q^5 + 727q^6 + 1737q^7 \\
&\quad + 4050q^8 + 9075q^9 + 19868q^{10} + 42302q^{11} + 88278q^{12} + \ldots) \\
\tau_3^{0,2}(q) &= q^2\,(1 + 2q + 13q^2 + 37q^3 + 124q^4 + 331q^5 + 906q^6 + 2233q^7 \\
&\quad + 5456q^8 + \ldots) \\
\tau_4^{0,2}(q) &= q^3\,(2 + 7q + 29q^2 + 92q^3 + 282q^4 + \ldots) \\
\tau_5^{0,2}(q) &= q^3\,(1 + 3q + 18q^2 + \ldots)
\end{aligned} \tag{4.235}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation $[1]$

$$\begin{aligned}
\tau_1^{1,1}(q) &= q^2\,(2 + 4q + 13q^2 + 35q^3 + 89q^4 + 216q^5 + 508q^6 + 1145q^7 \\
&\quad + 2521q^8 + 5402q^9 + 11320q^{10} + 23238q^{11} + 46856q^{12} + 92850q^{13} \\
&\quad + 181217q^{14} + 348612q^{15} + 661792q^{16} + 1240786q^{17} + \ldots) \\
\tau_2^{1,1}(q) &= q^2\,(1 + 4q + 13q^2 + 43q^3 + 122q^4 + 323q^5 + 814q^6 + 1962q^7 \\
&\quad + 4550q^8 + 10233q^9 + 22370q^{10} + 47718q^{11} + 99574q^{12} + \ldots) \\
\tau_3^{1,1}(q) &= q^3\,(1 + 5q + 21q^2 + 70q^3 + 211q^4 + 584q^5 + 1529q^6 + 3798q^7 \\
&\quad + \ldots) \\
\tau_4^{1,1}(q) &= q^4\,(1 + 6q + 24q^2 + 85q^3 + \ldots) \\
\tau_5^{1,1}(q) &= q^5\,(1 + \ldots)
\end{aligned} \tag{4.236}$$

Further $\tau_\ell^{k,p}(q)$ are listed in (4.B.2). They suggest that the $\tau_\ell^{k,p}(q)$ expansion (4.232) coverges more quickly with larger values of k and smaller values of p.

4.4.5 Four dimensional $\mathcal{N}_{4d} = 2$ spectra

In order to determine universal string spectra with $\mathcal{N}_{4d} = 2$ SUSY, we shall now compactify two dimensions of minimally supersymmetric $\mathcal{N}_{6d} = (1,0)$ theories on a T^2. This preserves all the eight supercharges and the internal rotation symmetry becomes an R symmetry factor of $SO(2)_R \cong U(1)_R$.

Hence, the dimensionally reduced theory in $d = 4$ spacetime dimensions enjoys $\mathcal{N}_{4d} = 2$ SUSY and R symmetry $SU(2)_R \times U(1)_R = U(2)_R$. The fundamental $\mathcal{N}_{4d} = 2$ super Poincaré multiplet encompasses 8+8 states,

$$Z(\mathcal{N}_{4d} = 2) = [2]_y + [2]_r[0]_y + (z^2 + z^{-2})[0]_y + (z + z^{-1})[1]_r[1]_y \tag{4.237}$$

where z denotes the $U(1)_R$ fugacity. The tensor product of (4.237) with a Clifford vacuum in some $SO(3) \times SU(2)_R \times U(1)_R$ representation yields a family of supermultiplets characterized by three quantum numbers – n for $SO(3)$ spin, m for $SU(2)_R$ spin and p for $U(1)_R$ charge. The resulting $16(n+1)(m+1)$ states are described by the supercharacter [15]

$$[\![n;m,p]\!] \quad := \quad Z(\mathcal{N}_{4d} = 2) \cdot z^p \, [m]_r \, [n]_y \ . \tag{4.243}$$

The position of the semicolon in the arguments of the supercharacter allows to distinguish $\mathcal{N}_{4d} = 2$ multiplets $[\![\cdot;\cdot,\cdot]\!]$ from $\mathcal{N}_{6d} = (1,0)$ multiplets $[\![\cdot,\cdot;\cdot]\!]$.

The universal partition function of $\mathcal{N}_{4d} = 2$ scenarios is obtained through

[15] The simplicity of the $SO(3)$ tensor product $[2m]\cdot[2k] = \sum_{l=|k-m|}^{k+m}[2l]$ allows for compact closed formulae for the $SO(3) \times SU(2)_R \times U(1)_R$ decomposition of a general $\mathcal{N}_{4d} = 2$ supercharacter:

$$\begin{aligned}[\![n;m,p]\!] = z^p \big\{ & [m]_r \, [n+2] \ + \ [m]_r \, [n-2] \ + \ [m+2]_r \, [n] \ + \ [m-2]_r \, [n] \ + \ 2 \, [m]_r \, [n] \\ & + \ (z^2 + z^{-2}) \, [m]_r \, [n] \ + \ (z + z^{-1}) \big([m+1]_r \ + \ [m-1]_r \big) \big([n+1] \ + \ [n-1] \big) \big\}\end{aligned} \tag{4.238}$$

This generic character formula (4.238) holds for values $n, m \geq 2$ of the Clifford vacuum's $SO(3) \times SU(2)_R$ spin quantum numbers and specializes otherwise:

$$\begin{aligned}[\![n;0,p]\!] =& z^p \big\{ [n+2] \ + \ [n-2] \ + \ [2]_r \, [n] \ + \ (1 + z^2 + z^{-2}) \, [n] \\ & + \ (z + z^{-1}) \, [1]_r \big([n+1] \ + \ [n-1] \big) \big\} \ , \qquad n \geq 2 \qquad (4.239) \\ [\![0;m,p]\!] =& z^p \big\{ [m]_r \, [2] \ + \ [m]_r \, [0] \ + \ [m+2]_r \, [0] \ + \ [m-2]_r \, [0] \ + \ (z^2 + z^{-2}) \, [m]_r \, [0] \\ & + \ (z + z^{-1}) \big([m+1]_r \ + \ [m-1]_r \big) \, [1] \big\} \ , \qquad m \geq 2 \qquad (4.240) \\ [\![0;0,p]\!] =& z^p \big\{ [2] \ + \ [2]_r \, [0] \ + \ (z^2 + z^{-2}) \, [0] \ + \ (z + z^{-1}) \, [1]_r \, [1] \big\} \qquad (4.241) \\ [\![1;1,p]\!] =& z^p \big\{ [1]_r \, [3] \ + \ [3]_r \, [1] \ + \ (2 + z^2 + z^{-2}) \, [1]_r \, [1] \\ & (z + z^{-1}) \big([2]_r \, [2] \ + \ [2]_r \, [0] \ + \ [2] \ + \ [0] \big) \big\} \qquad (4.242)\end{aligned}$$

We observe the general selection rule that either none or all of n, m, p are odd, hence, there is no need to consider $[\![1;0,p]\!]$ or $[\![0;1,p]\!]$.

GSO projection of the following character products:

$$\chi^{\mathcal{N}_{4d}=2}(q;y,r,z) = \chi_{\rm NS}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z) \;+\; \chi_{\rm R}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z)$$
$$\chi_{\rm NS}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z) = \frac{1}{2}\,q^{-\frac{1}{2}}\,\Big[\,\chi_{\rm NS}^{SO(3)}(q;y)\,\chi_{{\rm NS},h=0,\ell=0}^{\mathcal{N}_{2d}=4,c=6}(q;r)\,\chi_{\rm NS}^{SO(3)}(q;z)$$
$$-\;\chi_{\rm NS}^{SO(3)}(e^{2\pi i}q;y)\,\chi_{{\rm NS},h=0,\ell=0}^{\mathcal{N}_{2d}=4,c=6}(e^{2\pi i}q;r)\,\chi_{\rm NS}^{SO(3)}(e^{2\pi i}q;z)\,\Big]$$
$$\chi_{\rm R}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z) = \frac{1}{2}\,\chi_{\rm R}^{SO(3)}(q;y)\,\chi_{{\rm R},h=1/4,\ell=1/2}^{\mathcal{N}_{2d}=4,c=6}(q;r)\,\chi_{\rm R}^{SO(3)}(q;z) \tag{4.244}$$

Its symmetry under reversal $p \mapsto -p$ of $U(1)_R$ charges motivates the definition

$$[\![n;m,\pm p]\!] \;:=\; \begin{cases} [\![n;m,p]\!] \;+\; [\![n;m,-p]\!] & : \; p \neq 0 \\ [\![n;m,0]\!] & : \; p = 0 \end{cases}\;, \tag{4.245}$$

then the power series expansion of (4.244) starts like[16]

$$\chi^{\mathcal{N}_{4d}=2}(q;y,r,z) \;=\; \underbrace{\left(y^2+y^{-2}+z^2+z^{-2} \;+\; \frac{1}{2}(y+y^{-1})[1]_z[1]_r\right)q^0}_{8\ \text{massless states}}$$
$$+\underbrace{\big(\,[\![2;0,0]\!] \;+\; [\![0;0,\pm 2]\!]\,\big)\,q}_{80\ \text{states at level 1}}$$
$$+\;\underbrace{\big([\![4;0,0]\!] \;+\; 2\,[\![2;0,\pm2]\!] \;+\; [\![2;0,0]\!] \;+\; [\![1;1,\pm1]\!] \;+\; [\![0;0,\pm4]\!] \;+\; 2\,[\![0;0,0]\!]\,\big)\,q^2}_{512\ \text{states at level 2}}$$
$$+\;\big([\![6;0,0]\!] \;+\; 2\,[\![4;0,\pm2]\!] \;+\; [\![4;0,0]\!] \;+\; 2\,[\![3;1,\pm1]\!] \;+\; 2\,[\![2;0,\pm4]\!] \;+\; 2\,[\![2;0,\pm2]\!]$$
$$+\;6\,[\![2;0,0]\!] + 2\,[\![1;1,\pm3]\!] \;+\; 3\,[\![1;1,\pm1]\!] \;+\; [\![0;2,0]\!] \;+\; [\![0;0,\pm6]\!]$$
$$+\;4\,[\![0;0,\pm2]\!] \;+\; 2\,[\![0;0,0]\!]\,\big)\,q^3 \;+\; \mathcal{O}(q^4) \tag{4.246}$$

The vertex operators occurring in the three multiplets of the first mass level have been constructed in [15], see equations (6.3) to (6.11) of that reference for bosons and equations (6.22) to (6.30) for fermions. The content of the first five levels is summarized in table 4.6:

[16] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.112). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y-y^{-1})(z-z^{-1})(r-r^{-1})$ to get the correct massless character in R sector.

$\alpha' m^2$	representations of $\mathcal{N}_{4d} = 2$ super Poincaré
1	$[\![2;0,0]\!] + [\![0;0,\pm 2]\!]$
2	$[\![4;0,0]\!] + 2\,[\![2;0,\pm 2]\!] + [\![2;0,0]\!] + [\![1;1,\pm 1]\!] + [\![0;0,\pm 4]\!] + 2\,[\![0;0,0]\!]$
3	$[\![6;0,0]\!] + 2\,[\![4;0,\pm 2]\!] + [\![4;0,0]\!] + 2\,[\![3;1,\pm 1]\!] + 2\,[\![2;0,\pm 4]\!] + 2\,[\![2;0,\pm 2]\!] + 6\,[\![2;0,0]\!]$ $+ 2\,[\![1;1,\pm 3]\!] + 3\,[\![1;1,\pm 1]\!] + [\![0;2,0]\!] + [\![0;0,\pm 6]\!] + 4\,[\![0;0,\pm 2]\!] + 2\,[\![0;0,0]\!]$
4	$[\![8;0,0]\!] + 2\,[\![6;0,\pm 2]\!] + [\![6;0,0]\!] + 2\,[\![5;1,\pm 1]\!] + 2\,[\![4;0,\pm 4]\!] + 3\,[\![4;0,\pm 2]\!] + 8\,[\![4;0,0]\!]$ $+ 3\,[\![3;1,\pm 3]\!] + 6\,[\![3;1,\pm 1]\!] + [\![2;2,\pm 2]\!] + 3\,[\![2;2,0]\!] + 2\,[\![2;0,\pm 6]\!] + 3\,[\![2;0,\pm 4]\!]$ $+ 12\,[\![2;0,\pm 2]\!] + 11\,[\![2;0,0]\!] + 2\,[\![1;1,\pm 5]\!] + 5\,[\![1;1,\pm 3]\!] + 10\,[\![1;1,\pm 1]\!] + 2\,[\![0;2,\pm 2]\!]$ $+ [\![0;2,0]\!] + [\![0;0,\pm 8]\!] + 5\,[\![0;0,\pm 4]\!] + 4\,[\![0;0,2]\!] + 11\,[\![0;0,0]\!]$
5	$[\![10;0,0]\!] + 2\,[\![8;0,\pm 2]\!] + [\![8;0,0]\!] + 2\,[\![7;1,\pm 1]\!] + 2\,[\![6;0,\pm 4]\!] + 3\,[\![6;0,\pm 2]\!] + 8\,[\![6;0,0]\!]$ $+ 3\,[\![5;1,\pm 3]\!] + 7\,[\![5;1,\pm 1]\!] + [\![4;2,\pm 2]\!] + 4\,[\![4;2,0]\!] + 2\,[\![4;0,\pm 6]\!] + 4\,[\![4;0,\pm 4]\!]$ $+ 16\,[\![4;0,\pm 2]\!] + 17\,[\![4;0,0]\!] + 3\,[\![3;1,\pm 5]\!] + 11\,[\![3;1,\pm 3]\!] + 21\,[\![3;1,\pm 1]\!] + [\![2;2,\pm 4]\!]$ $+ 7\,[\![2;2,\pm 2]\!] + 8\,[\![2;2,0]\!] + 2\,[\![2;0,\pm 8]\!] + 3\,[\![2;0,\pm 6]\!] + 15\,[\![2;0,\pm 4]\!] + 23\,[\![2;0,\pm 2]\!]$ $+ 38\,[\![2;0,0]\!] + [\![1;3,\pm 1]\!] + 2\,[\![1;1,\pm 7]\!] + 6\,[\![1;1,\pm 5]\!] + 16\,[\![1;1,\pm 3]\!] + 28\,[\![1;1,\pm 1]\!]$ $+ 3\,[\![0;2,\pm 4]\!] + 4\,[\![0;2,\pm 2]\!] + 9\,[\![0;2,0]\!] + [\![0;0,\pm 10]\!] + 5\,[\![0;0,\pm 6]\!] + 6\,[\![0;0,\pm 4]\!]$ $+ 21\,[\![0;0,\pm 2]\!] + 16\,[\![0;0,0]\!]$

Table 4.6: $\mathcal{N}_{4d} = 2$ multiplets occurring up to mass level 5

Comparison with the partition function (4.197) of the $\mathcal{N}_{6d} = (1,0)$ ancestor theory (and table 4.3) clearly demonstrates that the six dimensional viewpoint gives a more streamlined handle on the spectrum in terms of fewer supermultiplets. This is why we do not provide an asymptotic analysis and data tables for the universal $\mathcal{N}_{4d} = 2$ spectrum like we did for the $d = 6$ ancestor in subsection 4.4.4 and appendix 4.B.2.

4.5 Spectra in compactifications with 16 supercharges

This section is devoted to maximally supersymmetric type I superstring compactifications on even dimensional tori where all the sixteen supercharges are preserved. The methods introduced in subsections 4.1.3 and 4.1.4 are applied to decompose the partition function of the $(\partial X^i, \psi^i)$ CFT describing $d = 10, 8, 6, 4$ spacetime dimensions into characters of the little group $SO(d-1)$. The $d = 10$ case takes the role of the ancestor theory for 16 supercharges, so its spectrum will be analyzed in particular detail. In the remaining cases $d = 8, 6, 4$, dimensional reduction converts part of the higher dimensional Lorentz symmetry into an internal R symmetry, i.e. we branch the ten dimensional little group into $SO(9) \to SO(d-1) \times SO(10-d)_R$. In this process, individual Lorentz fugacities y_k with $k > \frac{1}{2}(d-2)$ are reinterpreted as R symmetry fugacities r_k.

Before looking at individual dimensionalities in detail, let us fix the notation for describing supersymmetric spectra with R symmetries: Characters of the spacetime little group $SO(d-1)$ are denoted by $[a_1, \ldots, a_n]$ with fugacities $y_1, \ldots, y_n$ and $n = \frac{1}{2}(d-2)$ whereas those of the R symmetry $SO(10-d)_R$ receive an extra subscript $[b_1, \ldots, b_\ell]_R$ with fugacities $r_1, \ldots, r_\ell$ and $\ell = 5 - \frac{d}{2}$. Our notation for supercharacters makes use of double brackets $[\![a_1, \ldots, a_n; b_1, \ldots, b_\ell]\!]$ enclosing the $SO(d-1) \times SO(10-d)_R$ quantum numbers of the highest weight state. The semicolon between a_n and b_1 separates spacetime from R symmetry Dynkin labels and eliminates any ambiguity about the spacetime dimension under consideration.

4.5.1 Ten dimensional $\mathcal{N}_{10d} = 1$ spectra

In this subsection, we want to revisit the results of [8] on $SO(9)$ covariant partition functions for ten dimensional open string excitations and examine further symmetry patterns. The minimal massive $\mathcal{N}_{10d} = 1$ SUSY multiplet encompasses $SO(9)$ representations of a spin two tensor, a three-form and a massive gravitino[17]

$$Z(\mathcal{N}_{10d} = 1) \ := \ [2,0,0,0] \ + \ [0,0,1,0] \ + \ [1,0,0,1] \ . \tag{4.247}$$

This is precisely the particle content of the first mass level, its vertex operators can for instance be found in equations (2.8), (2.9) and (2.22) of [15].

The generic multiplet is obtained as a tensor product of $Z(\mathcal{N}_{10d} = 1)$ with some $SO(9)$ representation and therefore described by the following $\mathcal{N}_{10d} = 1$ supercharacter:

$$[\![a_1, a_2, a_3, a_4]\!] \ := \ Z(\mathcal{N}_{10d} = 1) \cdot [a_1, a_2, a_3, a_4] \tag{4.248}$$

This is the basic building blocks of the refined ten dimensional partition function. The latter can be obtained through standard GSO projection of

[17]Note that $Z(\mathcal{N}_{10d} = 1)$ is denoted by Z_Q in [8].

the spacetime CFT

$$
\begin{aligned}
\chi^{\mathcal{N}_{10d}=1}(q;\vec{y}) &= \chi_{\rm NS}^{\mathcal{N}_{10d}=1}|_{\rm GSO}\,(q;\vec{y}) \;+\; \chi_{\rm R}^{\mathcal{N}_{10d}=1}|_{\rm GSO}\,(q;\vec{y}) \\
\chi_{\rm NS}^{\mathcal{N}_{10d}=1}|_{\rm GSO}\,(q;\vec{y},r) &= \frac{1}{2}\,q^{-\frac{1}{2}}\left[\,\chi_{\rm NS}^{SO(9)}(q;\vec{y}) \;-\; \chi_{\rm NS}^{SO(9)}(e^{2\pi i}q;\vec{y})\,\right] \\
\chi_{\rm R}^{\mathcal{N}_{10d}=1}|_{\rm GSO}\,(q;\vec{y},r) &= \frac{1}{2}\,\chi_{\rm R}^{SO(9)}(q;\vec{y})\;, \qquad (4.249)
\end{aligned}
$$

where $\chi_{\rm NS}^{SO(9)}(q;\vec{y})$ and $\chi_{\rm R}^{SO(9)}(q;\vec{y})$ are given by (4.89) and (4.90).

In a power series expansion in q, the coefficient of the n'th power q^n comprises the super Poincaré characters of the n'th mass level $m^2 = n/\alpha'$:[18]

$$
\begin{aligned}
\chi^{\mathcal{N}_{10d}=1}(q;\vec{y}) = &\underbrace{\left(\sum_{j=1}^{4}(y_j^2+y_j^{-2}) \;+\; \frac{1}{2}\prod_{j=1}^{4}(y_j+y_j^{-1})\right)}_{16\text{ massless states}} q^0 + \underbrace{[\![0,0,0,0]\!]\,q}_{256\text{ states at level }1} \\
&+ \underbrace{[\![1,0,0,0]\!]\,q^2}_{2304\text{ states at level }2} + \underbrace{\big(\,[\![2,0,0,0]\!] + [\![0,0,0,1]\!]\,\big)\,q^3}_{15360\text{ states at level }3} \\
&+ \big(\,[\![3,0,0,0]\!] + [\![1,0,0,1]\!] + [\![1,0,0,0]\!] + [\![0,1,0,0]\!]\,\big)\,q^4 \;+\; \mathcal{O}(q^5)\;. \qquad (4.250)
\end{aligned}
$$

The supermultiplets up to level eight are listed in table 4.7 and the complete first 25 mass levels can be found in table 4.8 and appendix 4.B.3.

The total number of states at a given mass level

The total number of states at a given mass level m can be read off from the coefficient of q^m in the partition function $\chi^{\mathcal{N}_{10d}=1}(q;\vec{y})$ when the $SO(9)$ fugacities $y_1,\ldots,y_4$ are set to unity. The function $\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\})$ is referred to as the *unrefined partition function.* From (4.91), (4.249) and

[18] Note the usual subtlety about the massless R sector which was explained in the footnote before (4.112). One can simply fix this by adding $\frac{1}{2}([0,0,0,1]_{SO(8)} - [0,0,1,0]_{SO(8)}) = \frac{1}{2}\prod_{i=1}^{4}(y_i - y_i^{-1})$ to the present result and obtain the correct answer; see also (3.16) of [8]. The $\frac{1}{2}[1,0,0,0]_9$ factor in the massive sector of the aforementioned (3.16) exactly matches our formula at any positive q power.

$\alpha' m^2$	representations of $\mathcal{N}_{10d} = 1$ super Poincaré
1	$\llbracket 0,0,0,0 \rrbracket$
2	$\llbracket 1,0,0,0 \rrbracket$
3	$\llbracket 2,0,0,0 \rrbracket + \llbracket 0,0,0,1 \rrbracket$
4	$\llbracket 3,0,0,0 \rrbracket + \llbracket 1,0,0,1 \rrbracket + \llbracket 1,0,0,0 \rrbracket + \llbracket 0,1,0,0 \rrbracket$
5	$\llbracket 4,0,0,0 \rrbracket + \llbracket 2,0,0,1 \rrbracket + \llbracket 2,0,0,0 \rrbracket + \llbracket 1,1,0,0 \rrbracket + \llbracket 1,0,0,1 \rrbracket + \llbracket 0,1,0,0 \rrbracket$ $+ \llbracket 0,0,1,0 \rrbracket + \llbracket 0,0,0,1 \rrbracket + \llbracket 0,0,0,0 \rrbracket$
6	$\llbracket 5,0,0,0 \rrbracket + \llbracket 3,0,0,1 \rrbracket + \llbracket 3,0,0,0 \rrbracket + \llbracket 2,1,0,0 \rrbracket + \llbracket 2,0,0,1 \rrbracket + \llbracket 2,0,0,0 \rrbracket + 2\llbracket 1,1,0,0 \rrbracket$ $+ \llbracket 1,0,1,0 \rrbracket + 2\llbracket 1,0,0,1 \rrbracket + 2\llbracket 1,0,0,0 \rrbracket + \llbracket 0,1,0,1 \rrbracket + \llbracket 0,1,0,0 \rrbracket + \llbracket 0,0,0,2 \rrbracket$ $+ 2\llbracket 0,0,0,1 \rrbracket$
7	$\llbracket 6,0,0,0 \rrbracket + \llbracket 4,0,0,1 \rrbracket + \llbracket 4,0,0,0 \rrbracket + \llbracket 3,1,0,0 \rrbracket + \llbracket 3,0,0,1 \rrbracket + \llbracket 3,0,0,0 \rrbracket + 2\llbracket 2,1,0,0 \rrbracket$ $+ \llbracket 2,0,1,0 \rrbracket + 3\llbracket 2,0,0,1 \rrbracket + 3\llbracket 2,0,0,0 \rrbracket + \llbracket 1,1,0,1 \rrbracket + 2\llbracket 1,1,0,0 \rrbracket + \llbracket 1,0,1,0 \rrbracket$ $+ \llbracket 1,0,0,2 \rrbracket + 4\llbracket 1,0,0,1 \rrbracket + 2\llbracket 1,0,0,0 \rrbracket + \llbracket 0,2,0,0 \rrbracket + 2\llbracket 0,1,0,1 \rrbracket + 2\llbracket 0,1,0,0 \rrbracket$ $+ 3\llbracket 0,0,1,0 \rrbracket + \llbracket 0,0,0,2 \rrbracket + 2\llbracket 0,0,0,1 \rrbracket + 2\llbracket 0,0,0,0 \rrbracket$
8	$\llbracket 7,0,0,0 \rrbracket + \llbracket 5,0,0,1 \rrbracket + \llbracket 5,0,0,0 \rrbracket + \llbracket 4,1,0,0 \rrbracket + \llbracket 4,0,0,1 \rrbracket + \llbracket 4,0,0,0 \rrbracket + 2\llbracket 3,1,0,0 \rrbracket$ $+ \llbracket 3,0,1,0 \rrbracket + 3\llbracket 3,0,0,1 \rrbracket + 4\llbracket 3,0,0,0 \rrbracket + \llbracket 2,1,0,1 \rrbracket + 3\llbracket 2,1,0,0 \rrbracket + \llbracket 2,0,1,0 \rrbracket$ $+ \llbracket 2,0,0,2 \rrbracket + 5\llbracket 2,0,0,1 \rrbracket + 3\llbracket 2,0,0,0 \rrbracket + \llbracket 1,2,0,0 \rrbracket + 3\llbracket 1,1,0,1 \rrbracket + 5\llbracket 1,1,0,0 \rrbracket$ $+ 4\llbracket 1,0,1,0 \rrbracket + 2\llbracket 1,0,0,2 \rrbracket + 7\llbracket 1,0,0,1 \rrbracket + 5\llbracket 1,0,0,0 \rrbracket + \llbracket 0,2,0,0 \rrbracket + \llbracket 0,1,1,0 \rrbracket$ $+ 4\llbracket 0,1,0,1 \rrbracket + 5\llbracket 0,1,0,0 \rrbracket + \llbracket 0,0,1,1 \rrbracket + 2\llbracket 0,0,1,0 \rrbracket + 3\llbracket 0,0,0,2 \rrbracket + 4\llbracket 0,0,0,1 \rrbracket$ $+ \llbracket 0,0,0,0 \rrbracket$

Table 4.7: $\mathcal{N}_{10d} = 1$ multiplets occurring up to mass level eight

SUSY[19], we have

$$
\begin{aligned}
\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\}) &= 2\chi_{\mathrm{R}}^{\mathcal{N}_{10d}=1}\,|_{\mathrm{GSO}}\,(q;\{y_i=1\}) \\
&= \frac{\vartheta_2(1,q)^4}{\eta(q)^{12}} = 16\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8 \; . \qquad (4.252)
\end{aligned}
$$

The coefficients in the power series of this formula reproduces the third column of Table 4.13. It also agrees with (5.3.37) of [57]. Note that $\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\})$ is *not* a modular form.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can be determined by

$$
N_m = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}q}{q^{m+1}}\; \chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\}) \; , \qquad (4.253)
$$

where $\mathcal{C}$ is a contour around the origin.

Now let us compute an asymptotic formula for the number of states N_m

[19] The agreement of GSO projected partition functions for NS and R sectors follows from Jacobi's abstruse identity:

$$
\vartheta_3(1,q)^4 - \vartheta_4(1,q)^4 - \vartheta_2(1,q)^4 = 0 \; . \qquad (4.251)
$$

at mass level m when $m \to \infty$. Note that a similar discussion can be found in subsections 4.3.3 and 5.3.1 of [57]. For completeness, let us go over some details here. We focus on the limit $q \to 1^-$ and proceed in a similar way to subsection 4.3.1. The asymptotic behaviour (4.119) and (4.121) of $\vartheta_2(1,q)$ and $\eta(q)$, respectively, leads to

$$\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\}) \sim \frac{1}{(2\pi)^4}(1-q)^4 \exp\left(-\frac{2\pi^2}{\log q}\right) \ , \quad q \to 1^- \ . \tag{4.254}$$

Let us now combine (4.253) with (4.254). As $m \to \infty$,

$$N_m \sim \frac{1}{(2\pi)^4}\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}q}{q}\ (1-q)^4 \exp\left(-\frac{2\pi^2}{\log q} - m\log q\right) \ . \tag{4.255}$$

The saddle point is at $q_0 = \exp\left(-\pi\sqrt{\frac{2}{m}}\right)$ and the steepest descent direction is the imaginary direction in q. We proceed in a similar way to (4.125) by writing $q = q_0 e^{i\theta}$ and using Laplace's method to obtain

$$\begin{aligned} N_m &\sim \frac{1}{(2\pi)^4}(1-q_0)^4\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\mathrm{d}\theta \exp\left(-\frac{2\pi^2}{i\theta+\log q_0} - m(i\theta+\log q_0)\right), \quad \epsilon > 0 \\ &\sim \frac{1}{4}m^{-2}\exp\left(2\pi\sqrt{2m}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta \exp\left(-\frac{m^{3/2}}{\pi\sqrt{2}}\theta^2\right) \\ &= \frac{1}{4}m^{-2}\exp\left(2\pi\sqrt{2m}\right)\frac{1}{2\pi}\frac{2^{1/4}\pi}{m^{3/4}} \\ &\sim \frac{1}{2^{11/4}}m^{-11/4}e^{2\pi\sqrt{2m}} \ , \qquad m \to \infty \ . \end{aligned} \tag{4.256}$$

For example, for $m = 100$, the exact value for N_{100} is 1.59×10^{32} and the value from (4.256) is 1.83×10^{32} ; the error is approximately 15 %.

The GSO projected NS and R sectors

In this section we compute the contributions from the NS and R sectors to the partition function given in (4.249). Here we consider the refined partition function, *i.e.* the fugacities y's are kept explicit.

The NS sector. From (4.249) and (4.89), the partition function of the GSO projected NS sector has the structure

$$\chi_{\rm NS}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;y) \quad = \textstyle\sum_{\vec{k}\in\mathbb{Z}^4_{\geq 0}} F^{\rm NS}_{k_1,\ldots,k_4}(q)\prod_{A=1}^4 [2k_A]_{y_A}\ , \qquad (4.257)$$

where the functions $F^{\rm NS}_{k_1,\ldots,k_4}(q)$ are given by

$$\begin{aligned} F^{\rm NS}_{k_1,\ldots,k_4}(q) &= (q;q)_\infty^{-12}\sum_{\vec{n}\in\mathbb{Z}^4_+}\sum_{\vec{m}\in\mathbb{Z}^4_{\geq 0}} \\ &\times\prod_{A=1}^4 (-1)^{n_A+1}(1-q^{n_A})q^{\frac{1}{2}m_A^2+\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)}) \\ &\times\frac{1}{2}\left[\prod_{A=1}^4(1-q^{m_A+\frac{1}{2}})+(-1)^{m_1^2+m_2^2+m_3^2+m_4^2}\prod_{A=1}^4(1+q^{m_A+\frac{1}{2}})\right]\ . \qquad (4.258)\end{aligned}$$

The R sector. From (4.249) and (4.90), the partition function of the GSO projected R sector is

$$\chi_{\rm R}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;y,s) \quad = \textstyle\sum_{\vec{k}\in\mathbb{Z}^4_{\geq 0}} F^{\rm R}_{k_1,\ldots,k_4}(q)\prod_{A=1}^4 [2k_A+1]_{y_A}\ , \qquad (4.259)$$

where the function $F^{\rm R}_{k_1,\ldots,k_4}(q)$ is given by

$$\begin{aligned} F^{\rm R}_{k_1,\ldots,k_4}(q) &= \frac{1}{2}q^{-\frac{1}{2}}(q;q)_\infty^{-12} \\ &\times\sum_{\vec{m}\in\mathbb{Z}^4_{\geq 0}}\sum_{\vec{n}\in\mathbb{Z}^4_+}\prod_{A=1}^4(-1)^{n_A+1}(1-q^{m_A+1})(1-q^{n_A})q^{\frac{1}{2}\left(m_A+\frac{1}{2}\right)^2+\binom{n_A}{2}} \\ &\times\prod_{A=1}^4(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+2)})\ . \qquad (4.260)\end{aligned}$$

Multiplicities of representations in the $\mathcal{N}_{10d}=1$ partition function

Combining the contributions from the NS and R sectors, we have

$$\begin{aligned} \chi^{\mathcal{N}_{10d}=1}(q;\vec{y}) &= \chi_{\rm NS}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;\vec{y}) \ + \ \chi_{\rm R}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;\vec{y}) \\ &= \sum_{\vec{k}\in\mathbb{Z}^4_{\geq 0}}\left(F^{\rm NS}_{\vec{k}}(q)\prod_{A=1}^4[2k_A]_{y_A}+F^{\rm R}_{\vec{k}}\prod_{A=1}^4[2k_A+1]_{y_A}\right)\ . \qquad (4.261)\end{aligned}$$

Supersymmetry implies that this partition function can be rewritten as

$$\chi^{\mathcal{N}_{10d}=1}(q;\vec{y}) \quad = \sum_{\vec{n}\in\mathbb{Z}^4_{\geq 0}} [\![n_1,n_2,n_3,n_4]\!]\; G_{n_1,n_2,n_3,n_4}(q)\ , \qquad (4.262)$$

and the aim is to compute explicitly a *multiplicity generating function* $G_{n_1,n_2,n_3,n_4}(q)$.

The multiplicity of $[\![n_1,n_2,n_3,n_4]\!]$ appearing in $\chi^{\mathcal{N}_{10d}=1}(q;\vec{y})$ can be determined as follows:

$$\begin{aligned} G_{n_1,n_2,n_3,n_4}(q) &= \int \mathrm{d}\mu_{SO(9)}(\vec{y})[n_1,n_2,n_3,n_4]_{\vec{y}}\ \frac{\chi^{\mathcal{N}_{10d}=1}(q;\vec{y})}{Z(\mathcal{N}_{10d}=1)(\vec{y})}\ , \\ &= G^{\mathrm{NS}}_{n_1,n_2,n_3,n_4}(q) + G^{\mathrm{R}}_{n_1,n_2,n_3,n_4}(q)\ , \end{aligned} \qquad (4.263)$$

where

$$\begin{aligned} G^{\mathrm{NS}}_{n_1,n_2,n_3,n_4}(q) &= \int \mathrm{d}\mu_{SO(9)}(\vec{y})[n_1,n_2,n_3,n_4]_{\vec{y}} \\ &\quad \times \sum_{\vec{k}\in\mathbb{Z}^4_{\geq 0}} \frac{\prod_{A=1}^4 [2k_A]_{y_A}}{Z(\mathcal{N}_{10d}=1)(\vec{y})} F^{\mathrm{NS}}_{k_1,\ldots,k_4}(q)\ , \qquad (4.264) \\ G^{\mathrm{R}}_{n_1,n_2,n_3,n_4}(q) &= \int \mathrm{d}\mu_{SO(9)}(\vec{y})[n_1,n_2,n_3,n_4]_{\vec{y}} \\ &\quad \times \sum_{\vec{k}\in\mathbb{Z}^4_{\geq 0}} \frac{\prod_{A=1}^4 [2k_A+1]_{y_A}}{Z(\mathcal{N}_{10d}=1)(\vec{y})} F^{\mathrm{R}}_{k_1,\ldots,k_4}(q)\ . \qquad (4.265) \end{aligned}$$

The inverse of the character of the fundamental multiplet in (4.247) can be written as a geometric series[20] similar to (4.151) and (4.223)

$$
\begin{aligned}
&[Z(\mathcal{N}_{10d}=1)(\vec{y},r)]^{-1} \\
&= \frac{y_4^4}{\left(1+\frac{y_4}{y_1y_2y_3}\right)\left(1+\frac{y_1y_4}{y_2y_3}\right)\left(1+\frac{y_2y_4}{y_1y_3}\right)\left(1+\frac{y_1y_2y_4}{y_3}\right)\left(1+\frac{y_3y_4}{y_1y_2}\right)\left(1+\frac{y_1y_3y_4}{y_2}\right)} \\
&\times \frac{1}{\left(1+\frac{y_2y_3y_4}{y_1}\right)(1+y_1y_2y_3y_4)} \\
&= \sum_{\vec{m}\in\mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} y_1^{\sum_{j=1}^8 (-1)^j m_j} y_2^{\sum_{j=1}^8 (-1)^{\lfloor (j+1)/2 \rfloor} m_j} \\
&\times y_3^{\sum_{j=1}^8 (-1)^{\lfloor (j+3)/4 \rfloor} m_j} y_4^{4+\sum_{j=1}^8 m_j} \ . \qquad (4.267)
\end{aligned}
$$

Some useful identities

In this section, we derive some useful identities that will be put into use later. Once we plug the series expansion (4.267) of the inverse $Z(\mathcal{N}_{10d}=1)$ into the integrand of (4.263), the elementary contributions to multiplicity generating functions G_{n_1,n_2,n_3,n_4} are integrals of type

$$
\mathcal{J}(\vec{w};\vec{k};\vec{n}) := \int \mathrm{d}\mu_{SO(9)}(\vec{y})[n_1,n_2,n_3,n_4]_{\vec{y}} \prod_{A=1}^{4} y_A^{w_A}[k_A]_{y_A} \ . \qquad (4.268)
$$

As usual, we consider the cases of k_A, k'_A, w_A and n_4 (independently) integer or half-integer together, which are treated separately in [7]:

$$
\begin{aligned}
\mathcal{J}(\vec{w};\ 2k_1,\dots,2k_4;\ n_1,\dots,2n_4) = &\sum_{\vec{k}'} \Delta(\vec{\lambda}_{ns};2k'_1,\dots,2k'_4) \\
&\times \prod_{A=1}^{4} \mathcal{J}_0(w_A;2k_A,2k'_A) \qquad (4.269)
\end{aligned}
$$

[20]Note that this can also be rewritten as

$$
[Z(\mathcal{N}_{10d}=1)(\vec{y})]^{-1} = \lim_{s\to -1}(\mathrm{PE}\,[s[0,0,0,1]_{\vec{y}}])^{1/2} = \left[\sum_{m=0}^{\infty}(-1)^m \mathrm{Sym}^m[0,0,0,1]_{\vec{y}}\right]^{1/2} . \qquad (4.266)
$$

where $\vec{\lambda}_{ns} = (n_1 + n_2 + n_3 + n_4, n_2 + n_3 + n_4, n_3 + n_4, n_4)$. Recall from (4.78) that

$$\Delta(\vec{\lambda}; 2k_1, \ldots, 2k_4) \quad = \tfrac{1}{4!} \textstyle\sum_{\sigma \in S_4} \det \left(\theta^{8+\lambda_A - A - B}_{|\lambda_A - A + B|} \left(k_{\sigma(A)}\right)\right)^4_{A,B=1} \tag{4.270}$$

Multiplicity generating function

The NS- and R sector contributions to the multiplicity generating function for the representation $[\![n_1, n_2, n_3, n_4]\!]$ can be rewritten as

$$\begin{aligned} G^{\rm NS}_{n_1,\ldots,n_4}(q) &= \textstyle\sum_{\vec{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\vec{k} \in \mathbb{Z}^4_{\geq 0}} \\ &\times \mathcal{J}(\vec{W}(\vec{m}); 2k_1, \ldots, 2k_4; \vec{n}) \ F^{\rm NS}_{k_1,\ldots,k_4}(q) \ , \end{aligned} \tag{4.271}$$

$$\begin{aligned} G^{\rm R}_{n_1,\ldots,n_4}(q) &= \textstyle\sum_{\vec{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\vec{k} \in \mathbb{Z}^4_{\geq 0}} \\ &\times \mathcal{J}(\vec{W}(\vec{m}); 2k_1 + 1, \ldots, 2k_4 + 1; \vec{n}) \ F^{\rm R}_{k_1,\ldots,k_4}(q) \ , \end{aligned} \tag{4.272}$$

where

$$\begin{aligned} \vec{W}(\vec{m}) = \Bigg(&\sum_{j=1}^{8} (-1)^j m_j, \ \sum_{j=1}^{8} (-1)^{\lfloor (j+1)/2 \rfloor} m_j, \\ &\sum_{j=1}^{8} (-1)^{\lfloor (j+3)/4 \rfloor} m_j, \ 4 + \sum_{j=1}^{8} m_j \Bigg) \ . \end{aligned} \tag{4.273}$$

As stated in (4.249), the multiplicity of the representation $[\![n_1, n_2, n_3, n_4]\!]$ in the $\mathcal{N}_{10d} = 1$ partition function is given by

$$\begin{aligned} &G_{n_1,n_2,n_3,n_4}(q) \\ &= \sum_{\vec{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\vec{k} \in \mathbb{Z}^4_{\geq 0}} \Big[\mathcal{J}(\vec{W}(\vec{m}); 2k_1, \ldots, 2k_4; \vec{n}) \ F^{\rm NS}_{k_1,\ldots,k_4}(q) \\ &+ \mathcal{J}(\vec{W}(\vec{m}); 2k_1 + 1, \ldots, 2k_4 + 1; \vec{n}) \ F^{\rm R}_{k_1,\ldots,k_4}(q) \Big] \ . \end{aligned} \tag{4.274}$$

4.5.2 Empirical approach to $\mathcal{N}_{10d} = 1$ asymptotic patterns

In this subsection, we proceed like in subsections 4.3.5 and 4.4.4 to obtain large spin asymptotics of multiplicity generating functions $G_{n,x,y,z}(q)$ for $\mathcal{N}_{10d} = 1$ supermultiplet $[\![n, x, y, z]\!]$. The supermultiplet content of the first 25 mass levels is used to determine the q expansion of the leading coefficients

$\tau_\ell^{x,y,z}(q)$ defined by:

$$\begin{aligned} G_{n,x,y,z}(q) &= q^n\,\tau_1^{x,y,z}(q) \;-\; q^{2n}\,\tau_2^{x,y,z}(q) \;+\; q^{3n}\,\tau_3^{x,y,z}(q) \;-\; \ldots \\ &= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,q^{\ell n}\,\tau_\ell^{x,y,z}(q) \end{aligned} \tag{4.275}$$

Again, the $\tau_\ell^{x,y,z}(q)$ are found to be power series in q with non-negative coefficients.

Having $d > 4$ spacetime dimensions makes the analytic methods of subsection 4.3.4 inefficient, i.e. we did not find a manageable asymptotic formula for (4.274). Hence, we compute the $\tau_\ell^{x,y,z}(q)$ at $\ell \leq 5$ on the basis of an $\mathcal{O}(q^{25})$ expansion of the partition function (4.249). The multiplicities of $[\![n,0,0,0]\!]$ multiplets are shown in the following table 4.8, and analogous data tables for $[\![n,x,y,z]\!]$ at nonzero values of x,y,z can be found in appendix 4.B.3. The numbers marked in red match with the leading trajectory contribution $q^n\tau_1^{x,y,z}(q)$ whereas blue numbers correspond to $q^n\tau_1^{x,y,z}(q) - q^{2n}\tau_2^{x,y,z}(q)$ including one subleading trajectory.

Levels of first appearance

The mass level where some $[\![0,x,y,z]\!]$ multiplet firstly occurs can be studied by inspecting the leading power of the multiplicity generating function $G_{0,x,y,z}(q)$ and therefore $\tau_\ell^{x,y,z}(q)$. The following table 4.9 gives an overview of this mass level threshold for various values of x,y,z.

For all supermultiplets $[\![0,x,y,z]\!]$ considered in table 4.9, the level of first appearance is delayed by three whenever the second Dynkin label is incremented as $x \mapsto x+1$. This suggests to look for a similar linear effect of $y \mapsto y+1$ and $z \mapsto z+1$. Up to the two exceptions $[\![0,0,0,0]\!]$ and $[\![0,0,0,1]\!]$, the data in the tables shows that the value y of the third Dynkin label increases the level of first appearance by $6y$.

The influence of the last Dynkin label z is much more difficult to probe without any explicit multiplicities beyond level 25 at hand. If an asymptotically linear relation between z and the level of first appearance of $[\![0,x,y,z]\!]$ exists, then it certainly admits even more exceptions than in the $y \mapsto y+1$ case. The onset of $[\![n,0,0,4]\!]$, $[\![n,0,0,5]\!]$ and $[\![n,0,0,6]\!]$ multiplets at levels 14, 19 and 24, respectively, suggests that an increment $z \mapsto z+1$ delays the $[\![0,x,y,z]\!]$ multiplet by five levels – at least in the regime of sufficiently

$\alpha' m^2$	$\#[\![0,0,0,0]\!]$	$\#[\![1,0,0,0]\!]$	$\#[\![2,0,0,0]\!]$	$\#[\![3,0,0,0]\!]$	$\#[\![4,0,0,0]\!]$	$\#[\![5,0,0,0]\!]$	$\#[\![6,0,0,0]\!]$	$\#[\![7,0,0,0]\!]$	$\#[\![8,0,0,0]\!]$	$\#[\![9,0,0,0]\!]$	$\#[\![10,0,0,0]\!]$	$\#[\![11,0,0,0]\!]$	$\#[\![12,0,0,0]\!]$	$\#[\![13,0,0,0]\!]$	$\#[\![14,0,0,0]\!]$
1	1	0													
2	0	1	0												
3	0	0	1	0											
4	0	1	0	1	0										
5	1	0	1	0	1	0									
6	0	2	1	1	0	1	0								
7	2	2	3	1	1	0	1	0							
8	1	5	3	4	1	1	0	1	0						
9	3	5	9	4	4	1	1	0	1	0					
10	3	12	10	11	5	4	1	1	0	1	0				
11	8	15	23	14	12	5	4	1	1	0	1	0			
12	8	30	31	31	16	13	5	4	1	1	0	1	0		
13	19	41	61	45	36	17	13	5	4	1	1	0	1	0	
14	22	77	89	87	53	38	18	13	5	4	1	1	0	1	0
15	41	109	164	132	104	58	39	18	13	5	4	1	1	0	1
16	57	190	245	244	162	113	60	40	18	13	5	4	1	1	0
17	100	282	426	378	299	179	118	61	40	18	13	5	4	1	1
18	138	471	656	657	473	332	188	120	62	40	18	13	5	4	1
19	235	710	1097	1040	830	532	350	193	121	62	40	18	13	5	4
20	336	1153	1699	1751	1333	938	565	359	195	122	62	40	18	13	5
21	544	1750	2778	2769	2263	1523	1000	583	364	196	122	62	40	18	13
22	799	2785	4309	4561	3630	2600	1635	1034	592	366	197	122	62	40	18
23	1261	4237	6907	7201	6025	4212	2803	1697	1052	597	367	197	122	62	40
24	1860	6634	10700	11637	9629	7034	4567	2918	1731	1061	599	368	197	122	62
25	2895	10082	16893	18301	15694	11337	7662	4774	2981	1749	1066	600	368	197	122

Table 4.8: $\mathcal{N}_{10d} = 1$ multiplets with $SO(9)$ quantum numbers $[n, 0, 0, 0]$

$\downarrow y, \overrightarrow{z}$	0	1	2	3	4	5	6	7
0	$1+3x$	$3+3x$	$6+3x$	$10+3x$	$14+3x$	$19+3x$	$24+3x$	
1	$5+3x$	$8+3x$	$12+3x$	$16+3x$	$20+3x$	$25+3x$		
2	$11+3x$	$14+3x$	$18+3x$	$22+3x$				
3	$17+3x$	$20+3x$	$24+3x$					
4	$23+3x$							

Table 4.9: First mass level where supermultiplets $[\![0, x, y, z]\!]$ of $\mathcal{N}_{10d} = 1$ firstly occur. Empty spaces indicate that the representations in question do not occur at levels ≤ 25.

large values of x, y, z.

On the basis of this reasonning, we conjecture that sufficiently high mass levels of first occurrence for general supermultiplets $[\![n, x, y, z]\!]$ are determined by the following overall prefactor in their multiplicity generating function:

$$G_{n,x,y,z}(q) \quad \sim \quad q^{n+3x+6y+5z-6} \times \mathcal{O}(1) \ , \qquad x, y, z \text{ large} \tag{4.276}$$

Note that also the six dimensional $\mathcal{N}_{6d} = (1,0)$ spectrum exhibis an asymptotic linear relation between the second $SO(5)$ label k and the level of first appearance: Table 4.5 shows that sufficiently high levels of first appearance for $[\![n, k; p]\!]$ are delayed by three under $k \mapsto k+2$.

Explicit formulae for the $\tau_\ell^{x,y,z}(q)$

We shall now give the explicit results for a large class of $\tau_\ell^{x,y,z}(q)$, obtained through the entries of table 4.8 and its generalizations to $(x, y, z) \neq (0, 0, 0)$ gathered in appendix 4.B.3. This reflects large spin information on the multiplicity generating functions $G_{n,x,y,z}(q)$ via (4.275).

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 0]$

$$\begin{aligned}
\tau_1^{0,0,0}(q) \quad &= \quad q^1 \, (1 + 0q + 1q^2 + 1q^3 + 4q^4 + 5q^5 + 13q^6 + 18q^7 + 40q^8 \\
&\quad + 62q^9 + 122q^{10} + 197q^{11} + 368q^{12} + 601q^{13} + 1070q^{14} + 1767q^{15} \\
&\quad + 3051q^{16} + 5022q^{17} + 8489q^{18} + 13897q^{19} + \ldots) \\
\tau_2^{0,0,0}(q) \quad &= \quad q^1 \, (1 + 2q + 4q^2 + 9q^3 + 18q^4 + 36q^5 + 70q^6 + 133q^7 \\
&\quad + 249q^8 + 460q^9 + 836q^{10} + 1503q^{11} + 2672q^{12} + 4699q^{13} + \ldots) \\
\tau_3^{0,0,0}(q) \quad &= \quad q^1 \, (1 + 1q + 5q^2 + 9q^3 + 26q^4 + 48q^5 + 112q^6 + 211q^7 \\
&\quad + 439q^8 + 818q^9 + \ldots) \\
\tau_4^{0,0,0}(q) \quad &= \quad q^1 \, (1 + 3q + 8q^2 + 20q^3 + 48q^4 + 106q^5 + \ldots) \\
\tau_5^{0,0,0}(q) \quad &= \quad q^1 \, (1 + 1q + 6q^2 + \ldots)
\end{aligned} \tag{4.277}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 0, 0]$

$$\begin{aligned}
\tau_1^{1,0,0}(q) &= q^4\,(1 + 2q + 3q^2 + 7q^3 + 14q^4 + 28q^5 + 53q^6 + 103q^7 \\
&+ 189q^8 + 352q^9 + 634q^{10} + 1146q^{11} + 2026q^{12} + 3578q^{13} + 6209q^{14} \\
&+ 10752q^{15} + 18378q^{16} + 31279q^{17} + \ldots) \\
\tau_2^{1,0,0}(q) &= q^5\,(1 + 2q + 5q^2 + 11q^3 + 26q^4 + 54q^5 + 114q^6 + 227q^7 \\
&+ 449q^8 + 863q^9 + 1639q^{10} + 3050q^{11} + 5618q^{12} + 10187q^{13} + \ldots) \\
\tau_3^{1,0,0}(q) &= q^8\,(2 + 5q + 15q^2 + 35q^3 + 86q^4 + 185q^5 + 403q^6 + 825q^7 \\
&+ \ldots) \\
\tau_4^{1,0,0}(q) &= q^{10}\,(1 + 3q + 11q^2 + 30q^3 + \ldots) \qquad (4.278)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 1, 0]$

$$\begin{aligned}
\tau_1^{0,1,0}(q) &= q^5\,(1 + 1q + 5q^2 + 8q^3 + 22q^4 + 40q^5 + 90q^6 + 165q^7 \\
&+ 338q^8 + 619q^9 + 1190q^{10} + 2149q^{11} + 3969q^{12} + 7048q^{13} + 12630q^{14} \\
&+ 22060q^{15} + 38603q^{16} + \ldots) \\
\tau_2^{0,1,0}(q) &= q^6\,(1 + 2q + 7q^2 + 17q^3 + 41q^4 + 91q^5 + 199q^6 + 412q^7 \\
&+ 841q^8 + 1665q^9 + 3241q^{10} + 6178q^{11} + 11611q^{12} + \ldots) \\
\tau_3^{0,1,0}(q) &= q^8\,(1 + 2q + 11q^2 + 25q^3 + 71q^4 + 160q^5 + 381q^6 + 809q^7 \\
&+ \ldots) \\
\tau_4^{0,1,0}(q) &= q^{11}\,(2 + 7q + 23q^2 + \ldots) \qquad (4.279)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 2]$

$$\begin{aligned}
\tau_1^{0,0,2}(q) &= q^6\,(1 + 2q + 7q^2 + 13q^3 + 33q^4 + 66q^5 + 143q^6 + 277q^7 \\
&+ 559q^8 + 1053q^9 + 2019q^{10} + 3715q^{11} + 6859q^{12} + 12338q^{13} \\
&+ 22156q^{14} + 39043q^{15} + \ldots) \\
\tau_2^{0,0,2}(q) &= q^7\,(1 + 4q^1 + 11q^2 + 28q^3 + 68q^4 + 155q^5 + 339q^6 + 716q^7 \\
&+ 1469q^8 + 2938q^9 + 5755q^{10} + 11054q^{11} + \ldots) \\
\tau_3^{0,0,2}(q) &= q^9\,(2 + 5q + 19q^2 + 48q^3 + 130q^4 + 301q^5 + 703q^6 + 1518q^7 \\
&+ \ldots) \\
\tau_4^{0,0,2}(q) &= q^{11}\,(1 + 4q + 16q^2 + 49q^3 + \ldots) \qquad (4.280)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 1]$

$$\begin{aligned}
\tau_1^{0,0,1}(q) &= q^3\,(1 + 1q + 3q^2 + 6q^3 + 12q^4 + 24q^5 + 48q^6 + 90q^7 + 171q^8 \\
&\quad + 317q^9 + 579q^{10} + 1045q^{11} + 1870q^{12} + 3299q^{13} + 5777q^{14} + 10017q^{15} \\
&\quad + 17222q^{16} + 29370q^{17} + \ldots) \\
\tau_2^{0,0,1}(q) &= q^4\,(1 + 2q^1 + 5q^2 + 13q^3 + 29q^4 + 62q^5 + 130q^6 + 263q^7 \\
&\quad + 520q^8 + 1008q^9 + 1916q^{10} + 3583q^{11} + 6609q^{12} + \ldots) \\
\tau_3^{0,0,1}(q) &= q^6\,(1 + 3q^1 + 10q^2 + 26q^3 + 63q^4 + 143q^5 + 315q^6 + 664q^7 \\
&\quad + \ldots) \\
\tau_4^{0,0,1}(q) &= q^8\,(1 + 4q + 12q^2 + 35q^3 + \ldots) \\
\tau_5^{0,0,1}(q) &= q^{10}\,(1 + \ldots) \qquad (4.281)
\end{aligned}$$

Further $\tau_\ell^{x,y,z}(q)$ listed in (4.B.3) support the trend that the $\tau_\ell^{x,y,z}(q)$ expansion (4.275) converges more quickly at higher value of x, y, z.

4.5.3 Eight dimensional $\mathcal{N}_{8d} = 1$ spectra

Starting from this subsection, we consider even dimensional type I superstring compactifications on T^2 tori preserving all the sixteen supercharges. The highest dimensional example is $\mathcal{N}_{8d} = 1$ SUSY in eight spacetime dimensions.

Let r denote the fugacity with respect to the R symmetry $SO(2)_R \cong U(1)_R$ and y_i the fugacities of the massive little group $SO(7)$, then the fundamental $\mathcal{N}_{8d} = 1$ super Poincaré multiplet is described by the supercharacter

$$\begin{aligned}
Z(\mathcal{N}_{8d} = 1) \;&:=\; (r^4 + r^{-4})\,[0,0,0] \;+\; (r^3 + r^{-3})\,[0,0,1] \\
&\quad +\; (r^2 + r^{-2})\,\big([0,1,0] \;+\; [1,0,0]\big) \;+\; (r + r^{-1})\,\big([1,0,1] \;+\; [0,0,1]\big) \\
&\quad +\; [2,0,0] \;+\; [0,0,2] \;+\; [1,0,0] \;+\; [0,0,0] \qquad (4.282)
\end{aligned}$$

which is obtained by branching the $SO(9)$ representations contributing to the $\mathcal{N}_{10d} = 1$ analogue (4.247) to $SO(7) \times U(1)_R$. The minimal multiplet (4.282) can be generated from a scalar Clifford vacuum of $U(1)_R$ charge $+4$, and the generic $\mathcal{N}_{8d} = 1$ multiplet follows from a Clifford vacuum with nontrivial $SO(7) \times U(1)_R$ quantum numbers[21]. This gives rise to the supercharacter

$$[\![a_1, a_2, a_3; Q]\!] \quad := \quad Z(\mathcal{N}_{8d} = 1) \cdot r^Q \, [a_1, a_2, a_3] \ . \tag{4.283}$$

The eight dimensional partition function is obtained from its ten dimensional ancestor (4.249) by singling out an internal factor $\chi^{SO(3)}_{\text{NS,R}}$ within $\chi^{SO(9)}_{\text{NS,R}}(\vec{y}) = \prod_{k=1}^4 \chi^{SO(3)}_{\text{NS,R}}(y_k)$ and reinterpreting its argument as an R-symmetry fugacity:

$$\begin{aligned}
\chi^{\mathcal{N}_{8d}=1}(q; \vec{y}, r) \quad &= \quad \chi^{\mathcal{N}_{8d}=1}_{\text{NS}} |_{\text{GSO}} \, (q; \vec{y}, r) \ + \ \chi^{\mathcal{N}_{8d}=1}_{\text{R}} |_{\text{GSO}} \, (q; \vec{y}, r) \\
\chi^{\mathcal{N}_{8d}=1}_{\text{NS}} |_{\text{GSO}} \, (q; \vec{y}, r) \quad &= \quad \frac{1}{2} \, q^{-\frac{1}{2}} \left[\chi^{SO(7)}_{\text{NS}}(q; \vec{y}) \, \chi^{SO(3)}_{\text{NS}}(q; r) \right. \\
&\qquad \left. - \ \chi^{SO(7)}_{\text{NS}}(e^{2\pi i} q; \vec{y}) \, \chi^{SO(3)}_{\text{NS}}(e^{2\pi i} q; r) \right] \\
\chi^{\mathcal{N}_{8d}=1}_{\text{R}} |_{\text{GSO}} \, (q; \vec{y}, r) \quad &= \quad \frac{1}{2} \, \chi^{SO(7)}_{\text{R}}(q; \vec{y}) \, \chi^{SO(3)}_{\text{R}}(q; r)
\end{aligned} \tag{4.284}$$

As with the products of the spacetime and internal partition functions in the 4- and 8-supercharge cases (4.112) and (4.196), we have to take the product before imposing the GSO projection, in both NS and R sectors.

Let us display the first four coefficients of the power series expansion in q:[22]

$$\begin{aligned}
&\chi^{\mathcal{N}_{8d}=1}(q; \vec{y}, r) \\
&= \underbrace{\left(\sum_{j=1}^{3} (y_j^2 + y_j^{-2}) \ + \ r^2 \ + \ r^{-2} \ + \ \frac{1}{2} \prod_{j=1}^{3} (y_j + y_j^{-1}) \, (r + r^{-1}) \right) q^0}_{16 \text{ massless states}} \\
&+ \underbrace{[\![0,0,0;0]\!] \, q}_{256 \text{ states at level } 1} \ + \ \underbrace{\big([\![0,0,0;\pm 2]\!] \ + \ [\![1,0,0;0]\!] \big) \, q^2}_{2304 \text{ states at level } 2} \\
&+ \ \big([\![0,0,0;\pm 4]\!] \ + \ [\![1,0,0;\pm 2]\!] \ + \ [\![0,0,1;\pm 1]\!] \\
&\qquad + \ [\![2,0,0;0]\!] \ + \ [\![0,0,0;0]\!] \big) \, q^3 \ + \ \mathcal{O}(q^4) \ .
\end{aligned} \tag{4.285}$$

---

[21] Recall that the semicolon in $[\![a_1, a_2, a_3; b]\!]$ separating the $U(1)_R$ quantum number b from the $SO(7)$ Dynkin labels a_1, a_2, a_3 eliminates potential confusion with $\mathcal{N}_{10d} = 1$ supercharacters (4.248).

[22] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.112). However, this can be fixed easily: one can simply add to it $\frac{1}{2} \prod_{j=1}^{3} (y_j - y_j^{-1}) \, (r - r^{-1})$ to get the correct massless character in R sector.

The pairing of opposite $U(1)_R$ charges $\pm Q$ motivates the following shorthand:

$$\llbracket a_1, a_2, a_3; \pm Q \rrbracket \quad := \quad \begin{cases} \llbracket a_1, a_2, a_3; Q \rrbracket + \llbracket a_1, a_2, a_3; -Q \rrbracket & \text{for } Q \neq 0 \ , \\ \llbracket a_1, a_2, a_3; 0 \rrbracket & \text{for } Q = 0 \ . \end{cases} \tag{4.286}$$

The supermultiplets up to level six are listed in table 4.10. The branching process obviously increases the number and diversity of multiplets compared to the ten dimensional analogue, cf. table 4.7. This is why we do not repeat the higher level analysis carried out for the $d = 10$ ancestor in dimensionally reduced settings.

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d} = 1$ partition function (4.250) into $SO(7) \times U(1)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(7) \times U(1)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = y_1, \qquad z_2 = y_2, \qquad z_3 = y_3, \qquad z_4 = s \ , \tag{4.287}$$

where $z_1, \ldots, z_4$ are fugacities of $SO(9)$, y_1, y_2, y_3 are fugacities of $SO(7)$ and s is a fugacity of $U(1)_R$. For example,

$$\begin{aligned} [1,0,0,0]_{\vec{z}} &= 1 + \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + z_2^2 + \frac{1}{z_3^2} + z_3^2 + \frac{1}{z_4^2} + z_4^2 \\ &= 1 + \frac{1}{y_1^2} + y_1^2 + \frac{1}{y_2^2} + y_2^2 + \frac{1}{y_3^2} + y_3^2 + \frac{1}{s^2} + s^2 \\ &= [1,0,0;0]_{\vec{y};s} + [0,0,0;+2]_{\vec{y};s} + [0,0,0;-2]_{\vec{y};s} \ , \end{aligned} \tag{4.288}$$

where the notation $[b_1, b_2, b_3; Q]$ denotes the $SO(7) \times U(1)_R$ representation.

4.5.4 Six dimensional $\mathcal{N}_{6d} = (1,1)$ spectra

Six dimensional type I compactifications with sixteen supercharges are said to possess $\mathcal{N}_{6d} = (1,1)$ SUSY. The spacetime symmetry branches to $SO(9) \to SO(5) \times SO(4)_R$, i.e. two Cartan generators of ten dimensional Lorentz group take the role of R symmetry generators probing fugacities r_1, r_2 of

$\alpha' m^2$	representations of $\mathcal{N}_{8d} = 1$ super Poincaré
1	$[\![0,0,0;0]\!]$
2	$[\![0,0,0;\pm 2]\!] + [\![1,0,0;0]\!]$
3	$[\![0,0,0;\pm 4]\!] + [\![1,0,0;\pm 2]\!] + [\![0,0,1;\pm 1]\!] + [\![2,0,0;0]\!] + [\![0,0,0;0]\!]$
4	$[\![0,0,0;\pm 6]\!] + [\![1,0,0;\pm 4]\!] + [\![0,0,1;\pm 3]\!] + [\![2,0,0;\pm 2]\!] + [\![1,0,0;\pm 2]\!] + 2\,[\![0,0,0;\pm 2]\!]$ $+ [\![1,0,1;\pm 1]\!] + [\![0,0,1;\pm 1]\!] + [\![3,0,0;0]\!] + 2\,[\![1,0,0;0]\!] + [\![0,1,0;0]\!] + [\![0,0,0;0]\!]$
5	$[\![0,0,0;\pm 8]\!] + [\![1,0,0;\pm 6]\!] + [\![0,0,1;\pm 5]\!] + [\![2,0,0;\pm 4]\!] + [\![1,0,0;\pm 4]\!] + 2\,[\![0,0,0;\pm 4]\!]$ $+ [\![1,0,1;\pm 3]\!] + 2\,[\![0,0,1;\pm 3]\!] + [\![3,0,0;\pm 2]\!] + [\![2,0,0;\pm 2]\!] + 3\,[\![1,0,0;\pm 2]\!] + 2\,[\![0,1,0;\pm 2]\!]$ $+ [\![0,0,0;\pm 2]\!] + [\![2,0,1;\pm 1]\!] + 2\,[\![1,0,1;\pm 1]\!] + 3\,[\![0,0,1;\pm 1]\!] + [\![4,0,0;0]\!] + 2\,[\![2,0,0;0]\!]$ $+ [\![1,1,0;0]\!] + 3\,[\![1,0,0;0]\!] + [\![0,1,0;0]\!] + [\![0,0,2;0]\!] + 4\,[\![0,0,0;0]\!]$
6	$[\![0,0,0;\pm 10]\!] + [\![1,0,0;\pm 8]\!] + [\![0,0,1;\pm 7]\!] + [\![2,0,0;\pm 6]\!] + [\![1,0,0;\pm 6]\!] + 2\,[\![0,0,0;\pm 6]\!]$ $+ [\![1,0,1;\pm 5]\!] + 2\,[\![0,0,1;\pm 5]\!] + [\![3,0,0;\pm 4]\!] + [\![2,0,0;\pm 4]\!] + 3\,[\![1,0,0;\pm 4]\!] + 2\,[\![0,1,0;\pm 4]\!]$ $+ 2\,[\![0,0,0;\pm 4]\!] + [\![2,0,1;\pm 3]\!] + 3\,[\![1,0,1;\pm 3]\!] + 3\,[\![0,0,1;\pm 3]\!] + [\![4,0,0;\pm 2]\!] + [\![3,0,0;\pm 2]\!]$ $+ 3\,[\![2,0,0;\pm 2]\!] + 2\,[\![1,1,0;\pm 2]\!] + 5\,[\![1,0,0;\pm 2]\!] + [\![0,1,0;\pm 2]\!] + 2\,[\![0,0,2;\pm 2]\!]$ $+ 4\,[\![0,0,0;\pm 2]\!] + [\![3,0,1;\pm 1]\!] + 2\,[\![2,0,1;\pm 1]\!] + 4\,[\![1,0,1;\pm 1]\!] + [\![0,1,1;\pm 1]\!]$ $+ 5\,[\![0,0,1;\pm 1]\!] + [\![5,0,0;0]\!] + 2\,[\![3,0,0;0]\!] + [\![2,1,0;0]\!] + 4\,[\![2,0,0;0]\!] + [\![1,1,0;0]\!]$ $+ [\![1,0,2;0]\!] + 5\,[\![1,0,0;0]\!] + 5\,[\![0,1,0;0]\!] + [\![0,0,2;0]\!] + 3\,[\![0,0,0;0]\!]$

Table 4.10: $\mathcal{N}_{8d} = 1$ multiplets occurring up to mass level six

$SO(4)_R \cong SU(2)_R \times SU(2)_R$. The fundamental supermultiplet of the $\mathcal{N}_{6d} = (1,1)$ super Poincaré group has the following $SO(5) \times SU(2)_R \times SU(2)_R$ particle content:

$$\begin{aligned} Z(\mathcal{N}_{6d} = (1,1)) \;\; := \;\; & [2,0] \cdot [0,0]_R \; + \; [0,2] \cdot [0,0]_R \; + \; [0,2] \cdot [1,1]_R \\ & + \; [1,0] \cdot [1,1]_R + \; [1,0] \cdot \big([2,0]_R \; + \; [0,2]_R \big) \; + \; [0,0] \cdot [2,2]_R \\ & + \; [0,0] \cdot [1,1]_R \; + \; [0,0] \cdot [0,0]_R + \; [1,1] \cdot \big([1,0]_R \; + \; [0,1]_R \big) \\ & + \; [0,1] \cdot \big([2,1]_R \; + \; [1,2]_R \; + \; [1,0]_R \; + \; [0,1]_R \big) \end{aligned} \tag{4.289}$$

Note that the R-symmetry characters $[\ldots]_R$ carry a subscript to avoid confusion with the Lorentz symmetry of identical rank.

The most general multiplet follows from (4.289) by taking tensor products with $SO(5) \times SU(2)_R \times SU(2)_R$ representations, this leads to the supercharacter

$$[\![a_1, a_2; b_1, b_2]\!] \;\; := \;\; Z(\mathcal{N}_{6d} = (1,1)) \cdot [a_1, a_2] \cdot [b_1, b_2]_R \tag{4.290}$$

The six dimensional partition function is obtained from its ten dimensional ancestor (4.249) by singling out two internal factor $\chi^{SO(3)}_{\text{NS,R}}$ within $\chi^{SO(9)}_{\text{NS,R}}(\vec{y}) = \prod_{k=1}^{4} \chi^{SO(3)}_{\text{NS,R}}(y_k)$ and reinterpreting their second argument as an R-symmetry fugacity:

$$\begin{aligned}\chi^{\mathcal{N}_{6d}=(1,1)}(q;\vec{y},\vec{r}) &= \chi_{\rm NS}^{\mathcal{N}_{6d}=(1,1)}\,|_{\rm GSO}\,(q;\vec{y},\vec{r}) \;+\; \chi_{\rm R}^{\mathcal{N}_{6d}=(1,1)}\,|_{\rm GSO}\,(q;\vec{y},\vec{r}) \\ \chi_{\rm NS}^{\mathcal{N}_{6d}=(1,1)}\,|_{\rm GSO}\,(q;\vec{y},\vec{r}) &= \frac{1}{2}\,q^{-\frac{1}{2}}\,\big[\,\chi_{\rm NS}^{SO(5)}(q;\vec{y})\,\chi_{\rm NS}^{SO(5)}(q;\vec{r}) \\ &\quad -\; \chi_{\rm NS}^{SO(5)}(e^{2\pi i}q;\vec{y})\,\chi_{\rm NS}^{SO(5)}(e^{2\pi i}q;\vec{r})\,\big] \\ \chi_{\rm R}^{\mathcal{N}_{6d}=(1,1)}\,|_{\rm GSO}\,(q;\vec{y},\vec{r}) &= \frac{1}{2}\,\chi_{\rm R}^{SO(5)}(q;\vec{y})\,\chi_{\rm R}^{SO(5)}(q;\vec{r}) \end{aligned} \tag{4.291}$$

Its q expansion starts like[23]

$$\begin{aligned}&\chi^{\mathcal{N}_{6d}=(1,1)}(q;\vec{y},\vec{r}) \\ &= \underbrace{\left(\sum_{j=1}^{2}(y_j^2+y_j^{-2}) \;+\; \sum_{j=1}^{2}(r_j^2+r_j^{-2}) \;+\; \frac{1}{2}\prod_{j=1}^{2}(y_j+y_j^{-1})\prod_{j=1}^{2}(r_j+r_j^{-1})\right) q^0}_{\text{16 massless states}} \\ &+ \underbrace{[\![0,0;0,0]\!]\,q}_{\text{256 states at level 1}} \;+\; \underbrace{\big(\,[\![0,0;1,1]\!] \;+\; [\![1,0;0,0]\!]\,\big)\,q^2}_{\text{2304 states at level 2}} \\ &+ \big(\,[\![0,0;2,2]\!] \;+\; [\![1,0;1,1]\!] \;+\; [\![0,1;1,0]\!] \\ &\qquad +\; [\![0,1;0,1]\!] \;+\; [\![2,0;0,0]\!] \;+\; [\![0,0;0,0]\!]\,\big)\,q^3 \;+\; \mathcal{O}(q^4)\;, \end{aligned} \tag{4.292}$$

and supermultiplets at higher levels ≤ 5 are listed in table 4.11.

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d} = 1$ partition function (4.250) into $SO(5) \times SU(2)_R \times SU(2)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(5) \times SU(2)_R \times SU(2)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = y_1, \qquad z_2 = y_2, \qquad z_3 = r_1 r_2, \qquad z_4 = r_1 r_2^{-1}\;, \tag{4.293}$$

where $z_1, \ldots, z_4$ are fugacities of $SO(9)$, y_1, y_2 are fugacities of $SO(5)$, and

[23] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.112). However, this can be fixed easily: one can simply add to it $\frac{1}{2}\prod_{j=1}^{2}(y_j - y_j^{-1})\prod_{j=1}^{2}(r_j - r_j^{-1})$ to get the correct massless character in R sector.

r_1, r_2 are fugacities for the two $SU(2)_R$ factors. For example,

$$\begin{aligned}
[1,0,0,0]_{\vec{z}} &= 1 + \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + z_2^2 + \frac{1}{z_3^2} + z_3^2 + \frac{1}{z_4^2} + z_4^2 \\
&= 1 + \frac{1}{y_1^2} + y_1^2 + \frac{1}{y_2^2} + y_2^2 + (r_1 + r_1^{-1})(r_2 + r_2^{-1}) \\
&= [1,0;0,0]_{\vec{y};\vec{r}} + [0,0;1,1]_{\vec{y};\vec{r}} \ , \qquad (4.294)
\end{aligned}$$

where the notation $[a_1, a_2; b_1, b_2]$ denotes the $SO(5) \times SU(2)_R \times SU(2)_R$ representation.

$\alpha' m^2$	representations of $\mathcal{N}_{6d} = (1,1)$ super Poincaré
1	$[\![0,0;0,0]\!]$
2	$[\![0,0;1,1]\!] + [\![1,0;0,0]\!]$
3	$[\![0,0;2,2]\!] + [\![1,0;1,1]\!] + [\![0,1;1,0]\!] + [\![0,1;0,1]\!] + [\![2,0;0,0]\!] + [\![0,0;0,0]\!]$
4	$[\![0,0;3,3]\!] + [\![1,0;2,2]\!] + [\![0,1;2,1]\!] + [\![0,0;2,0]\!] + [\![0,1;1,2]\!] + [\![2,0;1,1]\!] + [\![1,0;1,1]\!]$ $+ 2\,[\![0,0;1,1]\!] + [\![1,1;1,0]\!] + [\![0,1;1,0]\!] + [\![0,0;0,2]\!] + [\![1,1;0,1]\!] + [\![0,1;0,1]\!]$ $+ [\![3,0;0,0]\!] + 2\,[\![1,0;0,0]\!] + [\![0,2;0,0]\!]$
5	$[\![0,0;4,4]\!] + [\![1,0;3,3]\!] + [\![0,1;3,2]\!] + [\![0,0;3,1]\!] + [\![0,1;2,3]\!] + [\![2,0;2,2]\!] + [\![1,0;2,2]\!]$ $+ 2\,[\![0,0;2,2]\!] + [\![1,1;2,1]\!] + 2\,[\![0,1;2,1]\!] + 2\,[\![1,0;2,0]\!] + [\![0,0;2,0]\!] + [\![0,0;1,3]\!]$ $+ [\![1,1;1,2]\!] + 2\,[\![0,1;1,2]\!] + [\![3,0;1,1]\!] + [\![2,0;1,1]\!] + 3\,[\![1,0;1,1]\!] + 2\,[\![0,2;1,1]\!]$ $+ 2\,[\![0,0;1,1]\!] + [\![2,1;1,0]\!] + 2\,[\![1,1;1,0]\!] + 3\,[\![0,1;1,0]\!] + 2\,[\![1,0;0,2]\!] + [\![0,0;0,2]\!]$ $+ [\![2,1;0,1]\!] + 2\,[\![1,1;0,1]\!] + 3\,[\![0,1;0,1]\!] + [\![4,0;0,0]\!] + 2\,[\![2,0;0,0]\!] + [\![1,2;0,0]\!]$ $+ [\![1,0;0,0]\!] + 2\,[\![0,2;0,0]\!] + 3\,[\![0,0;0,0]\!]$

Table 4.11: $\mathcal{N}_{6d} = (1,1)$ multiplets occurring up to mass level five

4.5.5 Four dimensional $\mathcal{N}_{4d} = 4$ spectra

Finally, four dimensional theories with maximal $\mathcal{N}_{4d} = 4$ SUSY follow from the ten dimensional ancestor through compactification on T^6. The internal rotation group is identified with the R symmetry $SO(6)_R$, its characters are denoted by $[b_1, b_2, b_3]_R$. The universal partition function decomposes into characters of the $\mathcal{N}_{4d} = 4$ super Poincaré algebra, the fundamental one being

$$\begin{aligned}
Z(\mathcal{N}_{4d} = 4) \ = \ & [0]\,\big([0,0,2]_R + [0,2,0]_R + [2,0,0]_R + 2\big) + [2]\,[0,1,1]_R \\
+ \ & 2\,[2]\,[1,0,0]_R + [4] + [1]\,\big([0,0,1]_R + [0,1,0]_R + [1,0,1]_R + [1,1,0]_R\big) \\
+ \ & [3]\,\big([0,0,1]_R + [0,1,0]_R\big) \, . \qquad (4.295)
\end{aligned}$$

Any other supermultiplet follows by taking a tensor product of (4.295) with the $SO(3) \times SO(6)_R$ representation $[n]\,[b_1, b_2, b_3]_R$ of the the Clifford vacuum,

$$[\![n; b_1, b_2, b_3]\!] \quad := \quad Z(\mathcal{N}_{4d} = 4) \cdot [n]\,[b_1, b_2, b_3]_R \ . \tag{4.296}$$

The four dimensional partition function is obtained through the usual procedure from the ten dimensional ancestor (4.249), this time we have to interpret three factors of $\chi^{SO(3)}_{\rm NS,R}$ as carrying R-symmetry fugacities r_j:

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=4}(q; y, \vec{r}) \;&=\; \chi^{\mathcal{N}_{4d}=4}_{\rm NS} \,|_{\rm GSO}\, (q; y, \vec{r}) \;+\; \chi^{\mathcal{N}_{4d}=4}_{\rm R} \,|_{\rm GSO}\, (q; y, \vec{r}) \\
\chi^{\mathcal{N}_{4d}=4}_{\rm NS} \,|_{\rm GSO}\, (q; y, \vec{r}) \;&=\; \frac{1}{2}\, q^{-\frac{1}{2}} \left[\, \chi^{SO(3)}_{\rm NS}(q; y)\, \chi^{SO(7)}_{\rm NS}(q; \vec{r}) \right. \\
&\quad \left. -\; \chi^{SO(3)}_{\rm NS}(e^{2\pi i} q; y)\, \chi^{SO(7)}_{\rm NS}(e^{2\pi i} q; \vec{r}) \,\right] \\
\chi^{\mathcal{N}_{4d}=4}_{\rm R} \,|_{\rm GSO}\, (q; y, \vec{r}) \;&=\; \frac{1}{2}\, \chi^{SO(3)}_{\rm R}(q; y)\, \chi^{SO(7)}_{\rm R}(q; \vec{r})
\end{aligned} \tag{4.297}$$

The power series in q starts with[24]

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=4}(q; y, r_j) \;&=\; \underbrace{\left(y^2 + y^{-2} + \sum_{j=1}^{3} (r_j^2 + r_j^{-2}) \;+\; \frac{1}{2}\, [1]_y \prod_{j=1}^{3} (r_j + r_j^{-1}) \right) q^0}_{16 \text{ massless states}} \\
&+\; \underbrace{[\![0; 0, 0, 0]\!]\, q}_{256 \text{ states at level } 1} \;+\; \underbrace{\big([\![0; 1, 0, 0]\!] + [\![2; 0, 0, 0]\!] \big)\, q^2}_{2304 \text{ states at level } 2} \;+\; \big([\![0; 0, 0, 0]\!] + [\![0; 2, 0, 0]\!] \\
&+ [\![1; 0, 0, 1]\!] + [\![1; 0, 1, 0]\!] + [\![2; 1, 0, 0]\!] + [\![4; 0, 0, 0]\!] \big)\, q^3 \;+\; \mathcal{O}(q^4)\ ,
\end{aligned} \tag{4.298}$$

the coefficients of q^4 and q^5 can be found in table 4.12. The explicit vertex operators from the first level are listed in section 4 of [15].

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d} = 1$ partition function (4.250) into $SO(3) \times SO(6)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(3) \times SO(6)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = r_1, \qquad z_2 = r_2, \qquad z_3 = r_3, \qquad z_4 = y\ , \tag{4.299}$$

[24]Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.112). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y - y^{-1}) \prod_{j=1}^{3} (r_j - r_j^{-1})$ to get the correct massless character in R sector.

where $z_1, \ldots, z_4$ are fugacities of $SO(9)$, r_1, r_2, r_3 are fugacities of $SO(6)_R$ and y is a fugacity of $SO(3)$. For example,

$$
\begin{aligned}
[1,0,0,0]_{\vec{z}} &= 1 + \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + r_2^2 + \frac{1}{z_3^2} + z_3^2 + \frac{1}{z_4^2} + z_4^2 \\
&= \frac{1}{r_1^2} + r_1^2 + \frac{1}{r_2^2} + r_2^2 + \frac{1}{r_3^2} + r_3^2 + \left(1 + \frac{1}{y^2} + y^2\right) \\
&= [0;1,0,0]_{\vec{r};y} + [2;0,0,0]_{\vec{r};y} \ , \qquad (4.300)
\end{aligned}
$$

where the notation $[a; b_1, b_2, b_3]$ denotes the $SO(3) \times SO(6)_R$ representation for which the $SO(3)$ representation is $[a]$ and $SO(6)_R$ representation is $[b_1, b_2, b_3]_R$.

$\alpha' m^2$	representations of $\mathcal{N}_{4d} = 4$ super Poincaré
1	$[\![0;0,0,0]\!]$
2	$[\![0;1,0,0]\!] + [\![2;0,0,0]\!]$
3	$[\![0;0,0,0]\!] + [\![0;2,0,0]\!] + [\![1;0,0,1]\!] + [\![1;0,1,0]\!] + [\![2;1,0,0]\!] + [\![4;0,0,0]\!]$
4	$[\![0;0,1,1]\!] + 2[\![0;1,0,0]\!] + [\![0;3,0,0]\!] + [\![1;0,0,1]\!] + [\![1;0,1,0]\!] + [\![1;1,0,1]\!]$ $+ [\![1;1,1,0]\!] + 3[\![2;0,0,0]\!] + [\![2;1,0,0]\!] + [\![2;2,0,0]\!] + [\![3;0,0,1]\!] + [\![3;0,1,0]\!]$ $+ [\![4;1,0,0]\!] + [\![6;0,0,0]\!]$
5	$4[\![0;0,0,0]\!] + [\![0;0,0,2]\!] + [\![0;0,1,1]\!] + [\![0;0,2,0]\!] + [\![0;1,0,0]\!] + [\![0;1,1,1]\!]$ $+ 2[\![0;2,0,0]\!] + [\![0;4,0,0]\!] + 3[\![1;0,0,1]\!] + 3[\![1;0,1,0]\!] + 2[\![1;1,0,1]\!]$ $+ 2[\![1;1,1,0]\!] + [\![1;2,0,1]\!] + [\![1;2,1,0]\!] + 2[\![2;0,0,0]\!] + 2[\![2;0,1,1]\!] + 5[\![2;1,0,0]\!]$ $+ [\![2;2,0,0]\!] + [\![2;3,0,0]\!] + 2[\![3;0,0,1]\!] + 2[\![3;0,1,0]\!] + [\![3;1,0,1]\!] + [\![3;1,1,0]\!]$ $+ 3[\![4;0,0,0]\!] + [\![4;1,0,0]\!] + [\![4;2,0,0]\!] + [\![5;0,0,1]\!] + [\![5;0,1,0]\!] + [\![6;1,0,0]\!]$ $+ [\![8;0,0,0]\!]$

Table 4.12: $\mathcal{N}_{4d} = 4$ multiplets occurring up to mass level 5

4.6 Conclusion

We have investigated model independent superstring states common to all type I compactifications that preserve $\mathcal{N}_{4d} = 1$ and $\mathcal{N}_{6d} = (1,0)$ SUSY, respectively, and identified the underlying super Poincaré multiplets at individual mass levels. Part of our results are the associated unrefined partition functions together with their asymptotics for large mass levels, see (4.116)–(4.125) and (4.198)–(4.205). The refined versions of the universal partition functions are given by (4.112) and (4.196) and rewritten in terms of super Poincaré characters in (4.148), (4.171), (4.172), (4.216) and (4.231). Moreover, we have presented dimensional reductions of the universal $\mathcal{N}_{6d} = (1,0)$

and $\mathcal{N}_{10d} = 1$ spectra to even dimensions $d \geq 4$ in subsections 4.4.5, 4.5.3, 4.5.4 and 4.5.5.

Multiplicity generating functions for individual supermultiplets tend to stabilize in the regime where the spin j (or more generally the first $SO(d-1)$ Dynkin label) is comparable to the mass level $M = \alpha' m^2$. More specifically, the validity for the stable pattern roughly ranges between $\frac{1}{2}(M - M_0) \lessapprox j \lessapprox M - M_0$ where the offset M_0 depends on the remaining super Poincaré quantum numbers of the multiplets beyond the spin. In the mathematically most tractable $\mathcal{N}_{4d} = 1$ case, we have derived closed formulae (4.173) and (4.174) for the leading Regge trajectory. In the highest dimensional scenarios with given number of supercharges – $\mathcal{N}_{4d} = 1$, $\mathcal{N}_{6d} = (1, 0)$ and $\mathcal{N}_{10d} = 1$ – we extracted both leading and subleading Regge trajectories from explicitly computed multiplicities up to level $\alpha' m^2 = 25$, see subsections 4.3.5, 4.4.4 and 4.5.2.

4.6.1 The number of universal open string states

The following table 4.13 summarizes their numbers at low levels ≤ 9 in scenarios with 4, 8 and 16 supercharges, respectively. They are obtained by expanding the associated unrefined partition functions. For the cases of 4, 8 and 16 supercharges, the exact generating functions are respectively given by (4.116), (4.198), (4.252) and their asymptotics at large mass levels are respectively given by (4.125), (4.205), (4.256). Roughly speaking, the number of states increases exponentially with respect to the square root of the mass level.

$\alpha' m^2$	# states for 4 supercharges	# states for 8 supercharges	# states for 16 supercharges
0	4	8	16
1	24	80	256
2	104	512	2.304
3	384	2.576	15.360
4	1.240	11.008	84.224
5	3.648	41.792	400.896
6	9.992	144.784	1.711.104
7	25.792	465.856	6.690.816
8	63.392	1.409.792	24.332.544
9	149.464	4.050.112	83.219.712

Table 4.13: The number of model independent open string states in compactifications with 4, 8 and 16 supercharges, respectively, up to mass level $\alpha' m^2 = 9$.

4.A Deriving the asymptotic formulae for $\mathcal{N}_{4d} = 1$ multiplicity generating functions

In this appendix, we derive the asymptotic results on multiplicity generating function $G_{n,Q}(q)$ in the limit $n \to \infty$ presented in subsection 4.3.4.

In what follows, we will exploit the $n \to \infty$ behaviour of objects $T_p(m,k) := \binom{m}{k} - \binom{m}{k-p}$,

$$\begin{aligned} T_{2n+2}(2m+1, m+n+1-k) &\sim \tbinom{2m+1}{m+n+1-k} \ , \\ T_{2n+2}(2m, m+n-k) &\sim \tbinom{2m}{m+n-k} \ . \end{aligned} \tag{4.301}$$

assuming that $m, k \geq 0$

4.A.1 Warm-up: Multiplicities of $[\![2n+1, 0]\!]$ and $[\![2n, 1]\!]$ as $n \to \infty$

In order to get familiar with the asymptotic methods in the $\mathcal{N}_{4d} = 1$ context, we shall first of all discuss the large spin regime of supermultiplets with $U(1)_R$ neutral Clifford vacuum.

The multiplicity generating function for the representation $[\![2n+1, 0]\!]$ can be written as

$$\begin{aligned} G_{2n+1,0}(q) = &\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \mathfrak{M}_{[\![2n+1,0]\!]}(m, -p-1, k; q) \\ &+ \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \mathfrak{M}_{[\![2n+1,0]\!]}(p, p, k; q) \ , \end{aligned} \tag{4.302}$$

where the function $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ and $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ are defined in (4.169) and (4.170) and, as $n \to \infty$,

$$\begin{aligned} \mathfrak{M}_{[\![2n+1,0]\!]}(m, p, k; q) \sim (-1)^{-m-p} \Bigg[&F^{\mathrm{NS}}_{k,p}(q) \tbinom{m-p}{2m+1} \tbinom{2m+1}{m+n+1-k} \\ &+ F^{\mathrm{R}}_{k,p}(q) \tbinom{m-p}{2m} \tbinom{2m}{m+n-k} \Bigg] \ . \end{aligned} \tag{4.303}$$

Note that the binomial coefficient $\binom{\alpha}{\beta}$ increases as β increases from 0 to $\lfloor \alpha/2 \rfloor$ and then decreases as β increases from $\lfloor \alpha/2 \rfloor + 1$ to α.

Observe that $\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q)$ is sharply peaked near $(m,p,k) = (0,0,n)$ for n large. Therefore, the dominant contribution to the first set of summations in (4.302) comes from

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q) \\
&\sim \sum_{m=0}^{\lceil\epsilon_1\rceil}\sum_{p=0}^{\lceil\epsilon_2\rceil}\sum_{k=\lfloor n(1-\epsilon_3)\rfloor}^{\lceil n(1+\epsilon_3)\rceil}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q) \\
&\text{any } \epsilon_1,\epsilon_2,\epsilon_3>0, \quad n\to\infty \\
&\sim \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,n+\delta;q), \qquad n\to\infty\ . \qquad (4.304)
\end{aligned}$$

In the limit of large k, we can use asymptotic formulae (4.131) and (4.136) for $F^{\mathrm{NS}}_{k,p}(q)$ and $F^{\mathrm{R}}_{k,p}(q)$. The summation over δ from $-\infty$ to ∞ can be readily computed using the fact that

$$\begin{aligned}
\sum_{\delta=-\infty}^{\infty} q^{\delta}\binom{2m}{m-\delta} &= \sum_{\delta=-m}^{m} q^{\delta}\binom{2m}{m-\delta} = q^{-m}(1+q)^{2m}\ , \\
\sum_{\delta=-\infty}^{\infty} q^{\delta}\binom{2m+1}{m-\delta+1} &= \sum_{\delta=-(m+1)}^{m+1} q^{\delta}\binom{2m+1}{m-\delta+1} = q^{-m}(1+q)^{2m+1}\ .
\end{aligned} \qquad (4.305)$$

Next, the summation over m from 0 to ∞ can be computed using the following identities:

$$\begin{aligned}
\sum_{m=0}^{\infty}(-q)^{-m}(1+q)^{2m}\binom{1+m+p}{2m} &= (-q)^{-p-1}\frac{1-q^{2p+3}}{1-q}\ , \\
\sum_{m=0}^{\infty}(-q)^{-m}(1+q)^{2m+1}\binom{1+m+p}{1+2m} &= (-q)^{-p}\frac{1-q^{2p+2}}{1-q}\ . \qquad (4.306)
\end{aligned}$$

Thus, from (4.304), we find that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q) = \frac{(1-q)^2 q^{n-\frac{1}{2}}}{2(q,q)_\infty^6} \\
&\times\ \Big\{u_1(\sqrt{q})\vartheta_2(1,q)^2 - \big[u_2(\sqrt{q})\vartheta_3(1,q)^2 - u_2(-\sqrt{q})\vartheta_4(1,q)^2\big]\Big\}\ , \qquad (4.307)
\end{aligned}$$

where the functions $u_1(q)$ and $u_2(q)$ are defined as follows:

$$u_1(q) = \sum_{p=0}^{\infty} q^{2\left(p+\frac{3}{2}\right)^2} \frac{1-q^{4p+6}}{(1+q^{2p+2})(1+q^{2p+4})} \ ,$$
$$u_2(q) = \sum_{p=0}^{\infty} q^{2(p+1)^2} \frac{1-q^{4p+4}}{(1+q^{2p+1})(1+q^{2p+3})} \ . \tag{4.308}$$

It remains unclear whether $u_1(q)$ and $u_2(q)$ can be written in terms of known functions (if this is useful at all). In practice, it is easy to compute the power series $u_1(q)$ and $u_2(q)$ up to a high order in q. Moreover, their asymptotic formulae can be easily derived in the limit $q \to 0$. We shall come back to this point later.

Let us now examine the second set of summations in (4.302). The function $\mathfrak{M}_{[\![2n+1,0]\!]}(p,p,k;q)$ is sharply peaked near $(p,k) = (0,n)$ for large n. Thus,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \mathfrak{M}_{[\![2n+1,0]\!]}(p,p,k;q) \sim \mathfrak{M}_{[\![2n+1,0]\!]}(0,0,n;q) \ , \qquad n \to \infty$$
$$= \frac{1}{4(q;q)_\infty^6} \frac{(1-q)^3}{1+q} q^{n-\frac{1}{4}} \vartheta_2(1,q)^2 \ . \tag{4.309}$$

From (4.302), we simply add (4.304) and (4.309) together and obtain the expression (4.182) for $Q_{2n+1,0}$, in agreement with the stable pattern in table 4.2.

From recurrence relation (4.144) for $G_{n,Q}$, the asymptotic behaviour of multiplicity generating functions $U(1)_R$ charge $Q = 1$ is given by

$$G_{2n,1}(q) = \frac{1}{2}\left[F^{\mathrm{NS}}_{n,0}(q) - G_{2n-1,0}(q) - G_{2n+1,0}(q)\right] \ . \tag{4.310}$$

Using the asymptotics $G_{2n-1,Q} \sim q^{-1} G_{2n+1,Q}$ as well as (4.182) for $G_{2n+1,Q}$ and (4.131) for $F^{\mathrm{NS}}_{n,0}$, we arrive at (4.183). This also agrees with the stable pattern tabulated in appendix 4.B.1.

4.A.2 Multiplicities of $[\![2n+1, 2Q]\!]$ and $[\![2n, 2Q+1]\!]$ as $n \to \infty$, $Q = \mathcal{O}(1)$

This subsection generalizes the asymptotic results from the $Q = 0$ (or $Q = 1$) sector to generic $U(1)_R$ charges. The multiplicity generating function for

$[\![2n+1,2Q]\!]$ can be written as

$$G_{2n+1,2Q}(q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big\} + \sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\right] . \tag{4.311}$$

where the $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ function follows the following $n\to\infty$ behaviour:

$$\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) = (-1)^{Q-m-p} \times\left[F^{\mathrm{NS}}_{k,p}(q)\binom{Q+m-p}{2m+1}\binom{2m+1}{m+n+1-k} + F^{\mathrm{R}}_{k,p}(q)\binom{Q+m-p}{2m}\binom{2m}{m+n-k}\right] \tag{4.312}$$

The dominant contribution to $G_{2n+1,2Q}(q)$ comes from

$$\begin{aligned} G_{2n+1,2Q}(q) &\sim \sum_{m=0}^{\infty}\sum_{p=0}^{\lceil\epsilon_2\rceil}\sum_{k=\lfloor n(1-\epsilon_1)\rfloor}^{\lceil n(1+\epsilon_1)\rceil}\Big[\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big] \\ &+ \sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{k=\lfloor n(1-\epsilon_1)\rfloor}^{\lceil n(1+\epsilon_1)\rceil}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\ , \qquad \epsilon_1,\epsilon_2>0\ , n\to\infty \\ &\sim \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\Big[\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,n+\delta;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,n+\delta;q)\Big] \\ &+ \sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,n+\delta;q)\ , \qquad n\to\infty\ . \end{aligned} \tag{4.313}$$

The first set of summations can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,n+\delta;q)=\frac{(1-q)^2q^{n-Q-\frac{1}{2}}}{2(q;q)_\infty^6}$$
$$\times\left\{u_1(\sqrt{q},Q)\vartheta_2(1,q)^2-\left[u_2(\sqrt{q},Q)\vartheta_3(1,q)^2-u_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}, \tag{4.314}$$

where

$$u_1(q,Q)=\sum_{p=0}^{\infty}q^{2\left(p+\frac{3}{2}\right)^2}\frac{1-q^{4p+4Q+6}}{(1+q^{2p+2})(1+q^{2p+4})},$$
$$u_2(q,Q)=\sum_{p=0}^{\infty}q^{2(p+1)^2}\frac{1-q^{4p+4Q+4}}{(1+q^{2p+1})(1+q^{2p+3})}. \tag{4.315}$$

The next set of summations in (4.172) can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,n+\delta;q)=\frac{(-1)^Q(1-q)^3q^{n-\frac{3}{2}}}{2(q;q)_\infty^6}$$
$$\times\left\{v_1(\sqrt{q},Q)\vartheta_2(1,q)^2+\left[v_2(\sqrt{q},Q)\vartheta_3(1,q)^2-v_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}, \tag{4.316}$$

where[25]

$$v_1(q,Q)=\sum_{p=0}^{\lfloor Q/2\rfloor}\frac{q^{2(p-\frac{1}{2})^2}(1+q^2)^{2p}}{(1+q^{2p-2})(1+q^{2p})}\binom{Q}{2p}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p-Q\\ p+1/2,\ p+1\end{matrix};\ \frac{(1+q)^2}{4q}\right],$$
$$v_2(q,Q)=\sum_{p=0}^{\lfloor Q/2\rfloor}\frac{(1+q)q^{2p^2}(1+q^2)^{2p}}{(1+q^{2p-1})(1+q^{2p+1})}\binom{Q}{2p+1}$$
$$\times\,{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p+1-Q\\ p+1,\ p+3/2\end{matrix};\ \frac{(1+q)^2}{4q}\right], \tag{4.318}$$

[25]Upon obtaining the hypergeometric functions, we make use of the following identities for $p\geq 0$:

$$\sum_{m=0}^{Q}(-1)^mq^{-m}(1+q)^{2m}\binom{Q+m}{2p+2m}=\binom{Q}{2p}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p-Q\\ p+1/2,\ p+1\end{matrix};\ \frac{(1+q)^2}{4q}\right],$$
$$\sum_{m=0}^{Q}(-1)^mq^{-m}(1+q)^{2m+1}\binom{Q+m}{1+2p+2m}=\binom{Q}{2p+1}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p+1-Q\\ p+1,\ p+3/2\end{matrix};\ \frac{(1+q)^2}{4q}\right]. \tag{4.317}$$

The last set of summations in (4.172) can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,n+\delta;q)=\frac{(-1)^Q(1-q)^3q^{n-\frac{7}{4}}}{2(q;q)_\infty^6}$$
$$\times\ \left\{w_1(\sqrt{q},Q)\vartheta_2(1,q)^2+q^{\frac{9}{4}}\left[w_2(\sqrt{q},Q)\vartheta_3(1,q)^2-w_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}\ , \tag{4.319}$$

where

$$w_1(q,Q)=\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{1+2(1+m+p)^2-2m}\left(1+q^2\right)^{2m}\binom{Q-1-p}{2m}}{\left(1+q^{2(m+p)}\right)\left(1+q^{2(1+m+p)}\right)}\ ,$$
$$w_2(q,Q)=q^{-\frac{9}{2}}\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{2\left(m+p+\frac{3}{2}\right)^2-2m}\left(1+q^2\right)^{2m+1}\binom{Q-1-p}{1+2m}}{(1+q^{1+2m+2p})\,(1+q^{3+2m+2p})}\ . \tag{4.320}$$

Combining the three sets of summations into (4.172), we have

$$G_{2n+1,2Q}(q)=\frac{(1-q)^2q^n}{2q^{\frac{3}{2}}(q;q)_\infty^6}\times$$
$$\Big\{\vartheta_2(1,q)^2\Big[q^{1-Q}u_1(\sqrt{q},Q)+(-1)^Q(1-q)(v_1(\sqrt{q},Q)+q^{-1/4}w_1(\sqrt{q},Q))\Big]$$
$$+\vartheta_3(1,q)^2\Big[-q^{1-Q}u_2(\sqrt{q},Q)+(-1)^Q(1-q)(v_2(\sqrt{q},Q)+q^2w_2(\sqrt{q},Q))\Big]$$
$$+\vartheta_4(1,q)^2\Big[q^{1-Q}u_2(-\sqrt{q},Q)-(-1)^Q(1-q)(v_2(-\sqrt{q},Q)+q^2w_2(-\sqrt{q},Q))\Big]\Big\} \tag{4.321}$$

which exactly (4.173) with the definition (4.175) for the function $\mathcal{F}(q,Q)$ in the curly brackets. Note that this formula reproduces (4.182) when $Q=0$.

This allows to quickly infer asymptotic $[\![2n,2Q+1]\!]$ multiplicities through the recursion (4.147) and the asymptotic relations $G_{2n+2,2Q+1}(q)\sim qG_{2n,2Q+1}(q)$ as $n\to\infty$:

$$G_{2n,2Q+1}(q)\sim\frac{1}{1+q}\left[F^{\mathrm{R}}_{n,Q+1}(q)-G_{2n+1,2Q}(q)-G_{2n+1,2Q+2}(q)\right] \tag{4.322}$$

The asymptotic formula (4.136) for $F^{\mathrm{R}}_{n,Q+1}(q)$ and the definition (4.175) for the function $\mathcal{F}(q,Q)$ then leads to (4.174).

4.B Data tables for super Poincaré multiplicities

This appendix contains data tables for multiplicities of super Poincaré representations up to mass level $\alpha' m^2 = 25$. We only display tables for the ancestor theories with 4, 8 and 16 supercharges, respectively, since these highest dimensional theories organize the states in the most economic number of supermultiplets. Particular attention is paid to stable patterns, i.e. to the asymptotics of multiplicity generating functions for large spins and mass levels.

Each of the following tables is devoted to family of supermultiplets whose quantum numbers differ in the first $SO(d-1)$ Dynkin label and match in the remaining $SO(d-1)$ and R symmetry quantum numbers. Rows are associated with mass levels, and columns are associated with the value of the first $SO(d-1)$ Dynkin label to which we loosely refer to as the spin. Independently of spacetime dimensions and supercharges, the multiplicity generating functions $G_{\ldots}(q)$ tend to stabilize for large values of the spin and the mass level in the limit where both of them are uniformly increased. This leading Regge trajectory (corresponding to the $\tau_1^{\cdots}(q)$ contribution in (4.184), (4.232) and (4.275)) is exact when numbers occur repeatedly along diagonal lines in the tables, these entries are marked in red.

Moreover, once the asymptotic numbers in red are subtracted from the data outside the first stable region, further subleading trajectories emerge. The leftover after this subtraction tends to stabilize along lines where the mass level grows twice as fast as the spin. This can be understood as the second Regge trajectory (corresponding to the $\tau_2^{\cdots}(q)$ contribution in (4.184), (4.232) and (4.275)) with slope $\frac{1}{2}$ and subtractive sign. Its region of exact validity is highlighted in blue.

4.B.1 4 supercharges in four dimensions

The tables in this subsection are based on the $\mathcal{N}_{4d} = 1$ partition function (4.112), organized in terms of multiplicity generating functions $G_{n,Q}(q)$, see (4.148).

$\alpha' m^2$	[[1;2]]	[[3;2]]	[[5;2]]	[[7;2]]	[[9;2]]	[[11;2]]	[[13;2]]	[[15;2]]	[[17;2]]	[[19;2]]	[[21;2]]	[[23;2]]
1	0											
2	0											
3	1	0										
4	2	2	0									
5	6	6	2	0								
6	17	15	8	2	0							
7	38	43	22	8	2	0						
8	89	101	62	24	8	2	0					
9	195	233	152	71	24	8	2	0				
10	411	512	361	176	73	24	8	2	0			
11	843	1089	803	430	185	73	24	8	2	0		
12	1694	2231	1734	978	456	187	73	24	8	2	0	
13	3302	4483	3602	2146	1053	465	187	73	24	8	2	0
14	6336	8758	7304	4525	2343	1079	467	187	73	24	8	2
15	11919	16795	14402	9300	4997	2420	1088	467	187	73	24	8
16	22053	31582	27835	18548	10383	5200	2446	1090	467	187	73	24
17	40173	58428	52685	36227	20921	10878	5277	2455	1090	467	187	73
18	72204	106359	98044	69217	41236	22068	11083	5303	2457	1090	467	187
19	128014	191004	179419	129896	79473	43785	22569	11160	5312	2457	1090	467
20	224337	338384	323661	239545	150345	84906	44955	22774	11186	5314	2457	1090
21	388651	592391	575773	435174	279322	161591	87520	45458	22851	11195	5314	2457
22	666314	1025226	1011672	779119	510970	301946	167204	88696	45663	22877	11197	5314
23	1131024	1755809	1756589	1377070	920804	555389	313632	169841	89199	45740	22886	11197
24	1902209	2976969	3017219	2404087	1637411	1006121	579053	319310	171019	89404	45766	22888
25	3170935	5000934	5129359	4150179	2874993	1798156	1052851	590920	321953	171522	89481	45775

$\alpha' m^2$	[[0; 1]]	[[2; 1]]	[[4; 1]]	[[6; 1]]	[[8; 1]]	[[10; 1]]	[[12; 1]]	[[14; 1]]	[[16; 1]]	[[18; 1]]	[[20; 1]]
1	1	0									
2	0	2	0								
3	3	2	3	0							
4	3	11	4	3	0						
5	15	20	18	5	3	0					
6	21	58	39	21	5	3	0				
7	66	115	105	49	22	5	3	0			
8	112	274	223	135	52	22	5	3	0		
9	267	543	521	296	146	53	22	5	3	0	
10	487	1159	1066	698	330	149	53	22	5	3	0
11	1027	2248	2258	1467	786	341	150	53	22	5	3
12	1872	4483	4465	3133	1682	821	344	150	53	22	5
13	3684	8456	8874	6300	3637	1774	832	345	150	53	22
14	6654	16077	16929	12629	7413	3868	1809	835	345	150	53
15	12430	29505	32174	24376	15014	7960	3961	1820	836	345	150
16	22104	54085	59444	46663	29304	16246	8195	3996	1823	836	345
17	39831	96778	109017	86997	56583	31974	16809	8288	4007	1824	836
18	69495	172263	195931	160521	106459	62184	33250	17045	8323	4010	1824
19	121751	301246	348996	290518	197927	117845	64978	33817	17138	8334	4011
20	208588	523209	612069	520208	360936	220529	123748	66270	34053	17173	8337
21	356951	896281	1063839	917434	650566	404759	232640	126586	66838	34146	17184
22	601090	1524153	1825894	1601735	1154779	733851	428967	238668	127882	67074	34181
23	1008432	2562971	3106955	2761714	2027692	1310137	781160	441385	241522	128450	67167
24	1670909	4278549	5231334	4717314	3515675	2312784	1400641	806110	447457	242819	128686
25	2755277	7075262	8737282	7973033	6035514	4030732	2482787	1449609	818653	450315	243387

$\alpha' m^2$	⟦1; 4⟧	⟦3; 4⟧	⟦5; 4⟧	⟦7; 4⟧	⟦9; 4⟧	⟦11; 4⟧	⟦13; 4⟧	⟦15; 4⟧	⟦17; 4⟧	⟦19; 4⟧	⟦21; 4⟧	⟦23; 4⟧	⟦25; 4⟧	⟦27; 4⟧
7	0													
8	1	0												
9	3	2	0											
10	9	8	2	0										
11	25	24	10	2	0									
12	63	65	34	10	2	0								
13	145	166	96	36	10	2	0							
14	327	387	251	108	36	10	2	0						
15	701	870	600	292	110	36	10	2	0					
16	1455	1868	1375	716	304	110	36	10	2	0				
17	2935	3884	2994	1676	759	306	110	36	10	2	0			
18	5784	7830	6304	3717	1804	771	306	110	36	10	2	0		
19	11124	15422	12839	7947	4058	1847	773	306	110	36	10	2	0	
20	21013	29656	25499	16409	8787	4188	1859	773	306	110	36	10	2	0
21	38962	55955	49404	32977	18350	9140	4231	1861	773	306	110	36	10	2
22	71109	103656	93817	64563	37270	19232	9270	4243	1861	773	306	110	36	10
23	127858	188982	174756	123758	73674	39339	19587	9313	4245	1861	773	306	110	36
24	226848	339385	320180	232485	142472	78301	40233	19717	9325	4245	1861	773	306	110
25	397364	601382	577497	429191	269832	152411	80412	40588	19760	9327	4245	1861	773	306

$\alpha' m^2$	⟦1; 6⟧	⟦3; 6⟧	⟦5; 6⟧	⟦7; 6⟧	⟦9; 6⟧	⟦11; 6⟧	⟦13; 6⟧	⟦15; 6⟧	⟦17; 6⟧	⟦19; 6⟧	⟦21; 6⟧	⟦23; 6⟧	⟦25; 6⟧
14	0												
15	1	0											
16	3	2	0										
17	10	8	2	0									
18	29	26	10	2	0								
19	73	76	36	10	2	0							
20	178	195	110	38	10	2	0						
21	406	474	294	122	38	10	2	0					
22	888	1086	733	338	124	38	10	2	0				
23	1876	2382	1711	868	350	124	38	10	2	0			
24	3845	5028	3815	2075	914	352	124	38	10	2	0		
25	7657	10304	8160	4716	2222	926	352	124	38	10	2	0	

$\alpha' m^2$	[[0;3]]	[[2;3]]	[[4;3]]	[[6;3]]	[[8;3]]	[[10;3]]	[[12;3]]	[[14;3]]	[[16;3]]	[[18;3]]	[[20;3]]	[[22;3]]
4	0											
5	1	0										
6	0	3	0									
7	6	5	4	0								
8	7	21	10	4	0							
9	29	44	37	11	4	0						
10	50	122	84	45	11	4	0					
11	135	254	227	108	46	11	4	0				
12	249	588	498	294	116	46	11	4	0			
13	569	1191	1136	668	322	117	46	11	4	0		
14	1061	2504	2359	1546	747	330	117	46	11	4	0	
15	2184	4885	4938	3278	1756	775	331	117	46	11	4	0
16	4044	9638	9770	6932	3790	1839	783	331	117	46	11	4
17	7804	18183	19255	13918	8113	4013	1867	784	331	117	46	11
18	14160	34268	36625	27663	16509	8671	4096	1875	784	331	117	46
19	26159	62704	69034	53180	33151	17810	8898	4124	1876	784	331	117
20	46461	114071	126973	100951	64405	36059	18381	8981	4132	1876	784	331
21	82968	203202	231136	187165	123324	70634	37407	18608	9009	4133	1876	784
22	144356	359209	413075	342732	230632	136240	73668	37982	18691	9017	4133	1876
23	250925	624938	730729	616388	425446	256624	142806	75029	38209	18719	9018	4133
24	428144	1078397	1274031	1095794	770702	476487	270343	145887	75604	38292	18727	9018
25	727755	1837377	2199827	1920245	1378855	868644	504339	277036	147252	75831	38320	18728

$\alpha' m^2$	[[0;5]]	[[2;5]]	[[4;5]]	[[6;5]]	[[8;5]]	[[10;5]]	[[12;5]]	[[14;5]]	[[16;5]]	[[18;5]]	[[20;5]]	[[22;5]]	[[24;5]]	[[26;5]]
10	0													
11	1	0												
12	0	3	0											
13	7	6	4	0										
14	10	26	11	4	0									
15	37	58	46	12	4	0								
16	70	163	111	54	12	4	0							
17	188	355	305	141	55	12	4	0						
18	359	832	696	394	149	55	12	4	0					
19	821	1726	1616	931	428	150	55	12	4	0				
20	1574	3664	3429	2198	1035	436	150	55	12	4	0			
21	3240	7267	7266	4762	2489	1069	437	150	55	12	4	0		
22	6100	14444	14582	10210	5493	2597	1077	437	150	55	12	4	0	
23	11809	27539	28985	20800	11934	5800	2631	1078	437	150	55	12	4	0
24	21646	52203	55668	41719	24651	12729	5908	2639	1078	437	150	55	12	4
25	40108	96213	105581	80976	49997	26553	13040	5942	2640	1078	437	150	55	12

$\alpha' m^2$	[[0; 7]]	[[2; 7]]	[[4; 7]]	[[6; 7]]	[[8; 7]]	[[10; 7]]	[[12; 7]]	[[14; 7]]
18	0							
19	1	0						
20	0	3	0					
21	7	6	4	0				
22	11	27	11	4	0			
23	41	63	47	12	4	0		
24	78	180	120	55	12	4	0	
25	214	402	336	150	56	12	4	0

4.B.2 8 supercharges in six dimensions

The tables in this subsection are based on the $\mathcal{N}_{6d} = (1, 0)$ partition function (4.196), organized in terms of multiplicity generating functions $G_{n_1,n_2,p}(q)$, see (4.216).

$\alpha' m^2$	[[0; 2; 0]]	[[1; 2; 0]]	[[2; 2; 0]]	[[3; 2; 0]]	[[4; 2; 0]]	[[5; 2; 0]]	[[6; 2; 0]]	[[7; 2; 0]]	[[8; 2; 0]]	[[9; 2; 0]]	[[10; 2; 0]]	[[11; 2; 0]]
1	0											
2	1	0										
3	1	1	0									
4	4	2	1	0								
5	6	7	2	1	0							
6	19	13	8	2	1	0						
7	34	38	16	8	2	1	0					
8	81	79	48	17	8	2	1	0				
9	156	184	103	51	17	8	2	1	0			
10	332	378	252	113	52	17	8	2	1	0		
11	636	813	530	279	116	52	17	8	2	1	0	
12	1276	1623	1171	604	289	117	52	17	8	2	1	0
13	2404	3290	2395	1350	631	292	117	52	17	8	2	1
14	4614	6386	4962	2816	1427	641	293	117	52	17	8	2
15	8537	12406	9823	5912	3001	1454	644	293	117	52	17	8
16	15853	23445	19436	11896	6361	3078	1464	645	293	117	52	17
17	28748	44075	37346	23836	12913	6549	3105	1467	645	293	117	52
18	52034	81247	71315	46446	26104	13368	6626	3115	1468	645	293	117
19	92579	148705	133388	89732	51295	27149	13556	6653	3118	1468	645	293
20	163950	268145	247448	169908	99935	53631	27607	13633	6663	3119	1468	645
21	286638	479693	451900	318623	190744	104983	54682	27795	13660	6666	3119	1468
22	498178	848018	818105	588270	360520	201413	107347	55140	27872	13670	6667	3119
23	856969	1487396	1462590	1075628	670688	382510	206529	108401	55328	27899	13673	6667
24	1465054	2583018	2592572	1942043	1235427	715151	393379	208899	108859	55405	27909	13674
25	2483037	4452127	4547623	3474093	2246578	1323605	737611	398523	209953	109047	55432	27912

$\alpha' m^2$	$[\![0,0;2]\!]$	$[\![1,0;2]\!]$	$[\![2,0;2]\!]$	$[\![3,0;2]\!]$	$[\![4,0;2]\!]$	$[\![5,0;2]\!]$	$[\![6,0;2]\!]$	$[\![7,0;2]\!]$	$[\![8,0;2]\!]$	$[\![9,0;2]\!]$	$[\![10,0;2]\!]$	$[\![11,0;2]\!]$
2	0											
3	1	0										
4	0	2	0									
5	3	3	3	0								
6	4	9	4	3	0							
7	13	20	17	5	3	0						
8	20	50	34	19	5	3	0					
9	53	101	93	43	20	5	3	0				
10	93	224	192	115	45	20	5	3	0			
11	203	449	446	252	125	46	20	5	3	0		
12	369	924	903	589	275	127	46	20	5	3	0	
13	743	1798	1920	1241	659	285	128	46	20	5	3	0
14	1355	3523	3792	2664	1405	683	287	128	46	20	5	3
15	2585	6673	7601	5410	3071	1476	693	288	128	46	20	5
16	4662	12617	14601	10981	6311	3245	1500	695	288	128	46	20
17	8585	23303	28083	21538	13007	6741	3317	1510	696	288	128	46
18	15272	42800	52540	41953	25810	13982	6916	3341	1512	696	288	128
19	27351	77315	97864	79808	50933	28012	14422	6988	3351	1513	696	288
20	47902	138661	178789	150444	97964	55666	29010	14598	7012	3353	1513	696
21	83950	245476	324415	278690	186802	107982	57944	29451	14670	7022	3354	1513
22	144814	431357	580136	511315	349601	207363	112896	58952	29627	14694	7024	3354
23	249137	750026	1029661	925300	648055	391117	217862	115197	59394	29699	14704	7025
24	423589	1294613	1806340	1658994	1183895	730037	412771	222852	116206	59570	29723	14706
25	717200	2214733	3145140	2940833	2142556	1343353	774118	423453	225163	116648	59642	29733

$\alpha' m^2$	$[\![0,1;1]\!]$	$[\![1,1;1]\!]$	$[\![2,1;1]\!]$	$[\![3,1;1]\!]$	$[\![4,1;1]\!]$	$[\![5,1;1]\!]$	$[\![6,1;1]\!]$	$[\![7,1;1]\!]$	$[\![8,1;1]\!]$	$[\![9,1;1]\!]$	$[\![10,1;1]\!]$	$[\![11,1;1]\!]$
1	0											
2	1	0										
3	1	2	0									
4	4	3	2	0								
5	8	9	4	2	0							
6	18	23	12	4	2	0						
7	39	51	31	13	4	2	0					
8	82	114	76	34	13	4	2	0				
9	165	249	174	85	35	13	4	2	0			
10	333	519	391	203	88	35	13	4	2	0		
11	652	1064	843	465	212	89	35	13	4	2	0	
12	1260	2137	1776	1024	495	215	89	35	13	4	2	0
13	2396	4202	3645	2203	1102	504	216	89	35	13	4	2
14	4499	8128	7330	4609	2399	1132	507	216	89	35	13	4
15	8321	15488	14450	9428	5080	2478	1141	508	216	89	35	13
16	15236	29063	28022	18898	10511	5280	2508	1144	508	216	89	35
17	27556	53844	53451	37201	21297	10997	5359	2517	1145	508	216	89
18	49336	98540	100527	71985	42376	22425	11198	5389	2520	1145	508	216
19	87449	178260	186521	137212	82828	44899	22915	11277	5398	2521	1145	508
20	153595	319063	341843	257835	159430	88321	46042	23116	11307	5401	2521	1145
21	267352	565412	619252	478197	302417	171054	90889	46533	23195	11316	5402	2521
22	461595	992485	1109824	876142	565992	326453	176672	92036	46734	23225	11319	5402
23	790578	1726764	1968850	1587104	1046065	614658	338400	179255	92527	46813	23234	11320
24	1343972	2979088	3459778	2844391	1910959	1142740	639492	344063	180403	92728	46843	23237
25	2268336	5098709	6025145	5046950	3452679	2099666	1193279	651564	346650	180894	92807	46852

$\alpha' m^2$	$[[0,4;0]]$	$[[1,4;0]]$	$[[2,4;0]]$	$[[3,4;0]]$	$[[4,4;0]]$	$[[5,4;0]]$	$[[6,4;0]]$	$[[7,4;0]]$	$[[8,4;0]]$	$[[9,4;0]]$	$[[10,4;0]]$	$[[11,4;0]]$	$[[12,4;0]]$
4	0												
5	1	0											
6	4	1	0										
7	9	5	1	0									
8	25	13	5	1	0								
9	61	38	14	5	1	0							
10	142	95	42	14	5	1	0						
11	312	238	108	43	14	5	1	0					
12	681	536	276	112	43	14	5	1	0				
13	1415	1216	642	289	113	43	14	5	1	0			
14	2909	2595	1482	680	293	113	43	14	5	1	0		
15	5804	5486	3235	1592	693	294	113	43	14	5	1	0	
16	11416	11186	6961	3511	1630	697	294	113	43	14	5	1	0
17	21988	22514	14456	7644	3621	1643	698	294	113	43	14	5	1
18	41816	44165	29554	16043	7924	3659	1647	698	294	113	43	14	5
19	78176	85560	58907	33146	16736	8034	3672	1648	698	294	113	43	14
20	144486	162571	115712	66723	34776	17016	8072	3676	1648	698	294	113	43
21	263440	305182	222926	132356	70428	35473	17126	8085	3677	1648	698	294	113
22	475248	564283	423773	257348	140501	72068	35753	17164	8089	3677	1648	698	294
23	847638	1031812	793186	493656	274795	144249	72765	35863	17177	8090	3677	1648	698
24	1497518	1863142	1466875	931993	530067	283053	145893	73045	35901	17181	8090	3677	1648
25	2619670	3330628	2677934	1738092	1006402	547844	286811	146590	73155	35914	17182	8090	3677

$\alpha' m^2$	$[\![0,6;0]\!]$	$[\![1,6;0]\!]$	$[\![2,6;0]\!]$	$[\![3,6;0]\!]$	$[\![4,6;0]\!]$	$[\![5,6;0]\!]$	$[\![6,6;0]\!]$	$[\![7,6;0]\!]$	$[\![8,6;0]\!]$	$[\![9,6;0]\!]$	$[\![10,6;0]\!]$	$[\![11,6;0]\!]$	$[\![12,6;0]\!]$	$[\![13,6;0]\!]$
7	0													
8	1	0												
9	4	1	0											
10	13	5	1	0										
11	35	17	5	1	0									
12	101	48	18	5	1	0								
13	238	140	52	18	5	1	0							
14	575	350	153	53	18	5	1	0						
15	1285	860	389	157	53	18	5	1	0					
16	2834	1983	976	402	158	53	18	5	1	0				
17	5972	4467	2279	1015	406	158	53	18	5	1	0			
18	12413	9647	5213	2395	1028	407	158	53	18	5	1	0		
19	24997	20422	11410	5513	2434	1032	407	158	53	18	5	1	0	
20	49629	41963	24476	12167	5629	2447	1033	407	158	53	18	5	1	0
21	96355	84692	50910	26287	12467	5668	2451	1033	407	158	53	18	5	1
22	184497	167219	103990	55095	27048	12583	5681	2452	1033	407	158	53	18	5
23	347237	324945	207612	113323	56917	27348	12622	5685	2452	1033	407	158	53	18
24	645476	620525	407840	227879	117556	57678	27464	12635	5686	2452	1033	407	158	53
25	1183084	1168737	786848	450666	237343	119382	57978	27503	12639	5686	2452	1033	407	158

$\alpha' m^2$	[[0, 2; 2]]	[[1, 2; 2]]	[[2, 2; 2]]	[[3, 2; 2]]	[[4, 2; 2]]	[[5, 2; 2]]	[[6, 2; 2]]	[[7, 2; 2]]	[[8, 2; 2]]	[[9, 2; 2]]	[[10, 2; 2]]	[[11, 2; 2]]
3	0											
4	1	0										
5	3	1	0									
6	9	6	1	0								
7	22	16	6	1	0							
8	54	47	19	6	1	0						
9	122	114	57	19	6	1	0					
10	269	282	147	60	19	6	1	0				
11	570	628	372	157	60	19	6	1	0			
12	1182	1397	867	408	160	60	19	6	1	0		
13	2384	2944	1973	966	418	160	60	19	6	1	0	
14	4720	6137	4285	2249	1002	421	160	60	19	6	1	0
15	9164	12349	9114	4962	2351	1012	421	160	60	19	6	1
16	17509	24540	18781	10746	5247	2387	1015	421	160	60	19	6
17	32937	47598	37992	22468	11461	5349	2397	1015	421	160	60	19
18	61121	91162	75102	46159	24208	11749	5385	2400	1015	421	160	60
19	111963	171440	146106	92470	50163	24932	11851	5395	2400	1015	421	160
20	202707	318632	279173	182328	101434	51941	25220	11887	5398	2400	1015	421
21	362956	583695	526058	352627	201679	105547	52668	25322	11897	5398	2400	1015
22	643253	1057824	976881	672443	393429	210967	107334	52956	25358	11900	5398	2400
23	1129052	1894240	1792109	1262534	756265	413603	215118	108061	53058	25368	11900	5398
24	1963846	3359194	3247454	2341077	1431348	799141	423000	216908	108349	53094	25371	11900
25	3386710	5896540	5821871	4284997	2674272	1520012	819640	427160	217635	108451	53104	25371

$\alpha' m^2$	$[\![0,4;2]\!]$	$[\![1,4;2]\!]$	$[\![2,4;2]\!]$	$[\![3,4;2]\!]$	$[\![4,4;2]\!]$	$[\![5,4;2]\!]$	$[\![6,4;2]\!]$	$[\![7,4;2]\!]$	$[\![8,4;2]\!]$	$[\![9,4;2]\!]$	$[\![10,4;2]\!]$	$[\![11,4;2]\!]$	$[\![12,4;2]\!]$
6	0												
7	3	0											
8	9	3	0										
9	33	12	3	0									
10	81	45	12	3	0								
11	218	126	48	12	3	0							
12	504	345	138	48	12	3	0						
13	1169	849	393	141	48	12	3	0					
14	2525	2025	989	405	141	48	12	3	0				
15	5415	4556	2426	1037	408	141	48	12	3	0			
16	11115	9997	5574	2569	1049	408	141	48	12	3	0		
17	22527	21139	12502	5988	2617	1052	408	141	48	12	3	0	
18	44383	43734	26921	13577	6131	2629	1052	408	141	48	12	3	0
19	86277	88152	56723	29598	13994	6179	2632	1052	408	141	48	12	3
20	164309	174452	116181	63019	30686	14137	6191	2632	1052	408	141	48	12
21	308983	338438	233542	130513	65753	31103	14185	6194	2632	1052	408	141	48
22	571846	646421	459542	264959	136982	66844	31246	14197	6194	2632	1052	408	141
23	1046250	1215097	889787	526615	279815	139729	67261	31294	14200	6194	2632	1052	408
24	1889540	2253670	1693826	1029156	559415	286341	140820	67404	31306	14200	6194	2632	1052
25	3377343	4124779	3179821	1977217	1099765	574444	289091	141237	67452	31309	14200	6194	2632

$\alpha' m^2$	$[\![0,0;4]\!]$	$[\![1,0;4]\!]$	$[\![2,0;4]\!]$	$[\![3,0;4]\!]$	$[\![4,0;4]\!]$	$[\![5,0;4]\!]$	$[\![6,0;4]\!]$	$[\![7,0;4]\!]$	$[\![8,0;4]\!]$	$[\![9,0;4]\!]$	$[\![10,0;4]\!]$	$[\![11,0;4]\!]$	$[\![12,0;4]\!]$	$[\![13,0;4]\!]$
6	0	0												
7	1	1	0											
8	2	2	1	0										
9	5	10	4	1	0									
10	12	20	15	4	1	0								
11	30	58	38	18	4	1	0							
12	61	125	104	44	18	4	1	0						
13	135	296	245	132	47	18	4	1	0					
14	273	613	575	313	139	47	18	4	1	0				
15	555	1320	1260	766	343	142	47	18	4	1	0			
16	1087	2639	2719	1704	846	350	142	47	18	4	1	0		
17	2115	5333	5628	3792	1926	877	353	142	47	18	4	1	0	
18	3999	10325	11477	7967	4333	2008	884	353	142	47	18	4	1	0
19	7521	19947	22744	16616	9280	4568	2039	887	353	142	47	18	4	1
20	13858	37496	44413	33421	19571	9854	4651	2046	887	353	142	47	18	4
21	25303	70043	84963	66421	39975	20993	10091	4682	2049	887	353	142	47	18
22	45553	128294	160356	128808	80349	43201	21580	10174	4689	2049	887	353	142	47
23	81270	233155	297815	246711	157849	87619	44657	21818	10205	4692	2049	887	353	142
24	143279	417523	546529	463836	305575	173443	90956	45246	21901	10212	4692	2049	887	353
25	250518	741533	989832	861982	581093	338524	180996	92425	45484	21932	10215	4692	2049	887

$\alpha' m^2$	[[0, 2; 4]]	[[1, 2; 4]]	[[2, 2; 4]]	[[3, 2; 4]]	[[4, 2; 4]]	[[5, 2; 4]]	[[6, 2; 4]]	[[7, 2; 4]]	[[8, 2; 4]]	[[9, 2; 4]]	[[10, 2; 4]]	[[11, 2; 4]]	[[12, 2; 4]]
7	0												
8	3	0											
9	6	4	0										
10	25	13	4	0									
11	57	47	14	4	0								
12	152	128	57	14	4	0							
13	338	338	159	58	14	4	0						
14	782	808	447	169	58	14	4	0					
15	1644	1886	1098	481	170	58	14	4	0				
16	3493	4153	2657	1219	491	170	58	14	4	0			
17	7041	8937	5997	2996	1253	492	170	58	14	4	0		
18	14124	18564	13258	6912	3120	1263	492	170	58	14	4	0	
19	27439	37778	28108	15522	7263	3154	1264	492	170	58	14	4	0
20	52817	74981	58430	33506	16489	7387	3164	1264	492	170	58	14	4
21	99411	146275	118038	70651	35926	16843	7421	3165	1264	492	170	58	14
22	185238	279950	234313	144914	76519	36905	16967	7431	3165	1264	492	170	58
23	339430	527948	455350	291435	158361	78991	37259	17001	7432	3165	1264	492	170
24	615770	980532	871500	573877	321433	164388	79973	37383	17011	7432	3165	1264	492
25	1102442	1798020	1640298	1111406	638384	335362	166872	80327	37417	17012	7432	3165	1264

$\alpha' m^2$	[[0, 0; 6]]	[[1, 0; 6]]	[[2, 0; 6]]	[[3, 0; 6]]	[[4, 0; 6]]	[[5, 0; 6]]	[[6, 0; 6]]	[[7, 0; 6]]	[[8, 0; 6]]	[[9, 0; 6]]	[[10, 0; 6]]	[[11, 0; 6]]	[[12, 0; 6]]	[[13, 0; 6]]
10	0													
11	1	0												
12	0	2	0											
13	5	5	3	0										
14	8	16	7	3	0									
15	27	42	30	8	3	0								
16	50	110	74	34	8	3	0							
17	129	253	212	93	35	8	3	0						
18	255	581	490	264	97	35	8	3	0					
19	565	1258	1184	648	286	98	35	8	3	0				
20	1101	2674	2587	1580	706	290	98	35	8	3	0			
21	2258	5480	5674	3580	1768	728	291	98	35	8	3	0		
22	4314	11042	11782	7961	4056	1829	732	291	98	35	8	3	0	
23	8389	21690	24263	16956	9193	4251	1851	733	291	98	35	8	3	0
24	15646	41956	48269	35421	19829	9701	4312	1855	733	291	98	35	8	3
25	29297	79620	94929	71854	42078	21153	9899	4334	1856	733	291	98	35	8

$\alpha' m^2$	$[\![0,3;1]\!]$	$[\![1,3;1]\!]$	$[\![2,3;1]\!]$	$[\![3,3;1]\!]$	$[\![4,3;1]\!]$	$[\![5,3;1]\!]$	$[\![6,3;1]\!]$	$[\![7,3;1]\!]$	$[\![8,3;1]\!]$	$[\![9,3;1]\!]$	$[\![10,3;1]\!]$	$[\![11,3;1]\!]$
3	0											
4	1	0										
5	4	1	0									
6	9	5	1	0								
7	26	15	5	1	0							
8	61	42	16	5	1	0						
9	140	109	48	16	5	1	0					
10	311	261	127	49	16	5	1	0				
11	669	604	318	133	49	16	5	1	0			
12	1387	1343	756	336	134	49	16	5	1	0		
13	2833	2883	1726	815	342	134	49	16	5	1	0	
14	5638	6031	3797	1887	833	343	134	49	16	5	1	0
15	11026	12313	8123	4213	1946	839	343	134	49	16	5	1
16	21191	24598	16912	9138	4376	1964	840	343	134	49	16	5
17	40119	48224	34431	19284	9563	4435	1970	840	343	134	49	16
18	74828	92924	68660	39746	20332	9726	4453	1971	840	343	134	49
19	137838	176248	134437	80231	42221	20759	9785	4459	1971	840	343	134
20	250749	329537	258807	158890	85837	43278	20922	9803	4460	1971	840	343
21	451108	608030	490719	309257	171219	88345	43705	20981	9809	4460	1971	840
22	802990	1108150	917317	592528	335580	176928	89404	43868	20999	9810	4460	1971
23	1415399	1996715	1692631	1118817	647375	348202	179445	89831	43927	21005	9810	4460
24	2471579	3559576	3085506	2084291	1230561	674467	353944	180504	89994	43945	21006	9810
25	4278524	6282467	5561480	3834679	2307511	1287320	687192	356463	180931	90053	43951	21006

$\alpha' m^2$	$[\![0,5;1]\!]$	$[\![1,5;1]\!]$	$[\![2,5;1]\!]$	$[\![3,5;1]\!]$	$[\![4,5;1]\!]$	$[\![5,5;1]\!]$	$[\![6,5;1]\!]$	$[\![7,5;1]\!]$	$[\![8,5;1]\!]$	$[\![9,5;1]\!]$	$[\![10,5;1]\!]$	$[\![11,5;1]\!]$	$[\![12,5;1]\!]$
6	0												
7	1	0											
8	6	1	0										
9	17	7	1	0									
10	54	23	7	1	0								
11	138	73	24	7	1	0							
12	341	202	79	24	7	1	0						
13	797	518	221	80	24	7	1	0					
14	1795	1254	584	227	80	24	7	1	0				
15	3879	2912	1441	603	228	80	24	7	1	0			
16	8183	6485	3410	1507	609	228	80	24	7	1	0		
17	16780	14008	7731	3599	1526	610	228	80	24	7	1	0	
18	33692	29414	16985	8239	3665	1532	610	228	80	24	7	1	0
19	66268	60280	36213	18272	8428	3684	1533	610	228	80	24	7	1
20	128089	120877	75329	39321	18782	8494	3690	1533	610	228	80	24	7
21	243471	237770	153142	82512	40618	18971	8513	3691	1533	610	228	80	24
22	456134	459491	305209	169218	85661	41128	19037	8519	3691	1533	610	228	80
23	842758	873960	597152	340066	176532	86960	41317	19056	8520	3691	1533	610	228
24	1537763	1638041	1149250	670793	356528	179691	87470	41383	19062	8520	3691	1533	610
25	2773038	3028963	2178141	1301158	706690	363883	180990	87659	41402	19063	8520	3691	1533

$\alpha' m^2$	$[\![0,1;3]\!]$	$[\![1,1;3]\!]$	$[\![2,1;3]\!]$	$[\![3,1;3]\!]$	$[\![4,1;3]\!]$	$[\![5,1;3]\!]$	$[\![6,1;3]\!]$	$[\![7,1;3]\!]$	$[\![8,1;3]\!]$	$[\![9,1;3]\!]$	$[\![10,1;3]\!]$	$[\![11,1;3]\!]$
4	0											
5	1	0										
6	3	2	0									
7	7	7	2	0								
8	19	20	9	2	0							
9	44	53	27	9	2	0						
10	100	130	76	29	9	2	0					
11	215	303	195	84	29	9	2	0				
12	454	675	472	223	86	29	9	2	0			
13	925	1453	1084	552	231	86	29	9	2	0		
14	1854	3036	2403	1302	581	233	86	29	9	2	0	
15	3630	6184	5144	2948	1387	589	233	86	29	9	2	0
16	6990	12327	10721	6442	3183	1416	591	233	86	29	9	2
17	13233	24088	21797	13674	7043	3269	1424	591	233	86	29	9
18	24712	46250	43391	28292	15133	7283	3298	1426	591	233	86	29
19	45490	87411	84717	57218	31670	15751	7369	3306	1426	591	233	86
20	82763	162815	162618	113413	64772	33187	15992	7398	3308	1426	591	233
21	148802	299261	307244	220754	129748	68318	33810	16078	7406	3308	1426	591
22	264749	543354	572296	422630	255152	137754	69852	34051	16107	7408	3308	1426
23	466300	975347	1051966	797014	493286	272632	141358	70476	34137	16115	7408	3308
24	813740	1732302	1910295	1482317	939075	530438	280808	142897	70717	34166	16117	7408
25	1407443	3046334	3429687	2721679	1762389	1016082	548377	284429	143521	70803	34174	16117

$\alpha' m^2$	[[0,3;3]]	[[1,3;3]]	[[2,3;3]]	[[3,3;3]]	[[4,3;3]]	[[5,3;3]]	[[6,3;3]]	[[7,3;3]]	[[8,3;3]]	[[9,3;3]]	[[10,3;3]]	[[11,3;3]]
6	0											
7	2	0										
8	7	2	0									
9	24	10	2	0								
10	63	38	10	2	0							
11	163	109	41	10	2	0						
12	385	295	124	41	10	2	0					
13	879	736	351	127	41	10	2	0				
14	1915	1740	902	366	127	41	10	2	0			
15	4066	3931	2202	959	369	127	41	10	2	0		
16	8365	8576	5105	2378	974	369	127	41	10	2	0	
17	16851	18124	11412	5604	2435	977	369	127	41	10	2	0
18	33194	37328	24640	12713	5781	2450	977	369	127	41	10	2
19	64238	75100	51777	27847	13222	5838	2453	977	369	127	41	10
20	122171	148039	106067	59296	29185	13399	5853	2453	977	369	127	41
21	228951	286468	212660	123042	62633	29695	13456	5856	2453	977	369	127
22	422965	545251	417987	249674	130948	63981	29872	13471	5856	2453	977	369
23	771624	1022124	807305	496442	267714	134322	64491	29929	13474	5856	2453	977
24	1390866	1889717	1534140	969373	536185	275750	135671	64668	29944	13474	5856	2453
25	2479819	3449211	2873001	1861540	1054472	554615	279134	136181	64725	29947	13474	5856

$\alpha' m^2$	[[0,1;5]]	[[1,1;5]]	[[2,1;5]]	[[3,1;5]]	[[4,1;5]]	[[5,1;5]]	[[6,1;5]]	[[7,1;5]]	[[8,1;5]]	[[9,1;5]]	[[10,1;5]]	[[11,1;5]]	[[12,1;5]]	[[13,1;5]]
8	0													
9	1	0												
10	3	2	0											
11	9	8	2	0										
12	26	25	10	2	0									
13	62	73	34	10	2	0								
14	148	188	105	36	10	2	0							
15	332	457	283	116	36	10	2	0						
16	721	1056	717	322	118	36	10	2	0					
17	1511	2343	1708	839	333	118	36	10	2	0				
18	3097	5020	3902	2053	880	335	118	36	10	2	0			
19	6181	10457	8566	4793	2183	891	335	118	36	10	2	0		
20	12114	21231	18249	10747	5170	2224	893	335	118	36	10	2	0	
21	23284	42177	37794	23329	11740	5302	2235	893	335	118	36	10	2	0
22	44053	82157	76466	49173	25807	12125	5343	2237	893	335	118	36	10	2
23	82070	157249	151421	101106	55044	26833	12257	5354	2237	893	335	118	36	10
24	150888	296196	294293	203277	114478	57629	27220	12298	5356	2237	893	335	118	36
25	273843	549904	562169	400661	232669	120665	58663	27352	12309	5356	2237	893	335	118

4.B.3 16 supercharges in ten dimensions

The tables in this subsection are based on the $\mathcal{N}_{10d} = 1$ partition function (4.249), organized in terms of multiplicity generating functions $G_{n_1,n_2,n_3,n_4}(q)$, see (4.262).

$\alpha' m^2$	$[0,1,0,0]$	$[1,1,0,0]$	$[2,1,0,0]$	$[3,1,0,0]$	$[4,1,0,0]$	$[5,1,0,0]$	$[6,1,0,0]$	$[7,1,0,0]$	$[8,1,0,0]$	$[9,1,0,0]$	$[10,1,0,0]$	$[11,1,0,0]$	$[12,1,0,0]$	$[13,1,0,0]$	$[14,1,0,0]$
3	0														
4	1	0													
5	1	1	0												
6	1	2	1	0											
7	2	2	2	1	0										
8	5	5	3	2	1	0									
9	7	9	6	3	2	1	0								
10	13	17	12	7	3	2	1	0							
11	21	29	23	13	7	3	2	1	0						
12	37	54	42	26	14	7	3	2	1	0					
13	60	90	77	48	27	14	7	3	2	1	0				
14	101	159	137	92	51	28	14	7	3	2	1	0			
15	165	268	243	163	98	52	28	14	7	3	2	1	0		
16	274	457	422	298	178	101	53	28	14	7	3	2	1	0	
17	441	760	732	522	326	184	102	53	28	14	7	3	2	1	0
18	717	1276	1248	924	580	341	187	103	53	28	14	7	3	2	1
19	1149	2088	2121	1592	1032	608	347	188	103	53	28	14	7	3	2
20	1847	3443	3551	2750	1801	1092	623	350	189	103	53	28	14	7	3
21	2928	5585	5929	4656	3134	1912	1120	629	351	189	103	53	28	14	7
22	4647	9060	9790	7886	5361	3351	1972	1135	632	352	189	103	53	28	14
23	7310	14538	16095	13160	9148	5762	3464	2000	1141	633	352	189	103	53	28
24	11482	23301	26221	21906	15414	9894	5982	3524	2015	1144	634	352	189	103	53
25	17908	36995	42535	36063	25846	16754	10303	6095	3552	2021	1145	634	352	189	103

$\alpha' m^2$	$[\![0,0,1,0]\!]$	$[\![1,0,1,0]\!]$	$[\![2,0,1,0]\!]$	$[\![3,0,1,0]\!]$	$[\![4,0,1,0]\!]$	$[\![5,0,1,0]\!]$	$[\![6,0,1,0]\!]$	$[\![7,0,1,0]\!]$	$[\![8,0,1,0]\!]$	$[\![9,0,1,0]\!]$	$[\![10,0,1,0]\!]$	$[\![11,0,1,0]\!]$	$[\![12,0,1,0]\!]$	$[\![13,0,1,0]\!]$	$[\![14,0,1,0]\!]$
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8	2	4	1	1	0										
9	7	6	5	1	1	0									
10	10	15	7	5	1	1	0								
11	22	24	20	8	5	1	1	0							
12	30	51	33	21	8	5	1	1	0						
13	64	85	73	38	22	8	5	1	1	0					
14	97	164	125	83	39	22	8	5	1	1	0				
15	179	276	249	148	88	40	22	8	5	1	1	0			
16	282	502	431	297	158	89	40	22	8	5	1	1	0		
17	496	842	803	529	321	163	90	40	22	8	5	1	1	0	
18	784	1473	1379	993	578	331	164	90	40	22	8	5	1	1	0
19	1335	2449	2462	1748	1099	602	336	165	90	40	22	8	5	1	1
20	2117	4164	4181	3153	1951	1149	612	337	165	90	40	22	8	5	1
21	3497	6853	7238	5454	3559	2058	1173	617	338	165	90	40	22	8	5
22	5546	11401	12131	9549	6218	3770	2108	1183	618	338	165	90	40	22	8
23	8981	18557	20509	16261	10990	6637	3878	2132	1188	619	338	165	90	40	22
24	14141	30342	33931	27794	18890	11791	6849	3928	2142	1189	619	338	165	90	40
25	22570	48846	56288	46628	32585	20406	12218	6957	3952	2147	1190	619	338	165	90

$\alpha' m^2$	[[0,0,0,2]]	[[1,0,0,2]]	[[2,0,0,2]]	[[3,0,0,2]]	[[4,0,0,2]]	[[5,0,0,2]]	[[6,0,0,2]]	[[7,0,0,2]]	[[8,0,0,2]]	[[9,0,0,2]]	[[10,0,0,2]]	[[11,0,0,2]]	[[12,0,0,2]]	[[13,0,0,2]]	[[14,0,0,2]]
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9	4	6	2	1	0										
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11	16	22	12	7	2	1	0								
12	32	40	29	13	7	2	1	0							
13	52	80	55	32	13	7	2	1	0						
14	98	141	115	62	33	13	7	2	1	0					
15	160	267	211	132	65	33	13	7	2	1	0				
16	286	463	409	249	139	66	33	13	7	2	1	0			
17	469	835	733	491	266	142	66	33	13	7	2	1	0		
18	805	1431	1351	900	531	273	143	66	33	13	7	2	1	0	
19	1314	2489	2375	1685	985	548	276	143	66	33	13	7	2	1	0
20	2199	4199	4218	3018	1864	1025	555	277	143	66	33	13	7	2	1
21	3558	7131	7270	5438	3378	1951	1042	558	277	143	66	33	13	7	2
22	5837	11842	12571	9530	6148	3560	1991	1049	559	277	143	66	33	13	7
23	9361	19709	21279	16701	10888	6520	3647	2008	1052	559	277	143	66	33	13
24	15106	32300	35990	28688	19266	11624	6704	3687	2015	1053	559	277	143	66	33
25	23999	52855	59966	49138	33418	20692	11999	6791	3704	2018	1053	559	277	143	66

$\alpha' m^2$	[[0,0,0,1]]	[[1,0,0,1]]	[[2,0,0,1]]	[[3,0,0,1]]	[[4,0,0,1]]	[[5,0,0,1]]	[[6,0,0,1]]	[[7,0,0,1]]	[[8,0,0,1]]	[[9,0,0,1]]	[[10,0,0,1]]	[[11,0,0,1]]	[[12,0,0,1]]	[[13,0,0,1]]	[[14,0,0,1]]
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8	4	7	5	3	1	1	0								
9	8	12	10	6	3	1	1	0							
10	12	22	19	11	6	3	1	1	0						
11	20	38	35	22	12	6	3	1	1	0					
12	34	66	62	43	23	12	6	3	1	1	0				
13	54	113	112	77	46	24	12	6	3	1	1	0			
14	89	190	197	142	85	47	24	12	6	3	1	1	0		
15	147	318	342	256	158	88	48	24	12	6	3	1	1	0	
16	233	532	587	452	288	166	89	48	24	12	6	3	1	1	0
17	376	877	1001	792	517	304	169	90	48	24	12	6	3	1	1
18	603	1438	1686	1376	916	550	312	170	90	48	24	12	6	3	1
19	954	2345	2823	2354	1610	983	566	315	171	90	48	24	12	6	3
20	1511	3795	4684	4003	2789	1740	1016	574	316	171	90	48	24	12	6
21	2383	6105	7716	6745	4795	3037	1808	1032	577	317	171	90	48	24	12
22	3727	9775	12620	11265	8164	5260	3169	1841	1040	578	317	171	90	48	24
23	5821	15552	20513	18678	13782	9019	5514	3237	1857	1043	579	317	171	90	48
24	9050	24624	33121	30757	23075	15332	9498	5647	3270	1865	1044	579	317	171	90
25	13998	38797	53183	50273	38366	25850	16217	9754	5715	3286	1868	1045	579	317	171

$\alpha' m^2$	⟦0,2,0,0⟧	⟦1,2,0,0⟧	⟦2,2,0,0⟧	⟦3,2,0,0⟧	⟦4,2,0,0⟧	⟦5,2,0,0⟧	⟦6,2,0,0⟧	⟦7,2,0,0⟧	⟦8,2,0,0⟧	⟦9,2,0,0⟧	⟦10,2,0,0⟧	⟦11,2,0,0⟧	⟦12,2,0,0⟧	⟦13,2,0,0⟧	⟦14,2,0,0⟧
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9	4	2	1	0											
10	5	5	2	1	0										
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12	21	21	10	6	2	1	0								
13	45	38	25	11	6	2	1	0							
14	74	78	46	26	11	6	2	1	0						
15	143	141	98	50	27	11	6	2	1	0					
16	240	269	178	106	51	27	11	6	2	1	0				
17	437	477	349	198	110	52	27	11	6	2	1	0			
18	731	870	629	389	206	111	52	27	11	6	2	1	0		
19	1280	1515	1170	713	409	210	112	52	27	11	6	2	1	0	
20	2126	2673	2067	1335	753	417	211	112	52	27	11	6	2	1	0
21	3619	4576	3709	2394	1422	773	421	212	112	52	27	11	6	2	1
22	5952	7867	6438	4328	2563	1462	781	422	212	112	52	27	11	6	2
23	9908	13251	11235	7604	4668	2650	1482	785	423	212	112	52	27	11	6
24	16128	22320	19168	13377	8250	4840	2690	1490	786	423	212	112	52	27	11
25	26386	37038	32718	23070	14611	8594	4927	2710	1494	787	423	212	112	52	27

$\alpha' m^2$	⟦0,3,0,0⟧	⟦1,3,0,0⟧	⟦2,3,0,0⟧	⟦3,3,0,0⟧	⟦4,3,0,0⟧	⟦5,3,0,0⟧	⟦6,3,0,0⟧	⟦7,3,0,0⟧	⟦8,3,0,0⟧	⟦9,3,0,0⟧	⟦10,3,0,0⟧	⟦11,3,0,0⟧	⟦12,3,0,0⟧	⟦13,3,0,0⟧	⟦14,3,0,0⟧
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12	4	2	1	0											
13	8	5	2	1	0										
14	18	12	6	2	1	0									
15	34	26	13	6	2	1	0								
16	73	55	30	14	6	2	1	0							
17	135	112	63	31	14	6	2	1	0						
18	261	222	133	67	32	14	6	2	1	0					
19	479	428	264	141	68	32	14	6	2	1	0				
20	885	815	520	285	145	69	32	14	6	2	1	0			
21	1577	1512	996	562	293	146	69	32	14	6	2	1	0		
22	2822	2776	1881	1091	583	297	147	69	32	14	6	2	1	0	
23	4922	5005	3482	2067	1133	591	298	147	69	32	14	6	2	1	0
24	8567	8930	6366	3865	2162	1154	595	299	147	69	32	14	6	2	1
25	14672	15706	11460	7105	4054	2204	1162	596	299	147	69	32	14	6	2

$\alpha' m^2$	[[0, 1, 1, 0]]	[[1, 1, 1, 0]]	[[2, 1, 1, 0]]	[[3, 1, 1, 0]]	[[4, 1, 1, 0]]	[[5, 1, 1, 0]]	[[6, 1, 1, 0]]	[[7, 1, 1, 0]]	[[8, 1, 1, 0]]	[[9, 1, 1, 0]]	[[10, 1, 1, 0]]	[[11, 1, 1, 0]]	[[12, 1, 1, 0]]	[[13, 1, 1, 0]]	[[14, 1, 1, 0]]
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13	43	36	18	8	2	1	0								
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15	162	157	91	44	19	8	2	1	0						
16	312	307	194	98	45	19	8	2	1	0					
17	554	591	385	208	99	45	19	8	2	1	0				
18	1010	1110	763	423	215	100	45	19	8	2	1	0			
19	1764	2041	1453	844	437	216	100	45	19	8	2	1	0		
20	3105	3701	2741	1636	882	444	217	100	45	19	8	2	1	0	
21	5310	6608	5043	3111	1718	896	445	217	100	45	19	8	2	1	0
22	9113	11636	9178	5810	3297	1756	903	446	217	100	45	19	8	2	1
23	15325	20254	16405	10673	6191	3379	1770	904	446	217	100	45	19	8	2
24	25728	34873	29035	19314	11467	6378	3417	1777	905	446	217	100	45	19	8
25	42607	59411	50676	34509	20876	11851	6460	3431	1778	905	446	217	100	45	19

$\alpha' m^2$	[[0, 2, 1, 0]]	[[1, 2, 1, 0]]	[[2, 2, 1, 0]]	[[3, 2, 1, 0]]	[[4, 2, 1, 0]]	[[5, 2, 1, 0]]	[[6, 2, 1, 0]]	[[7, 2, 1, 0]]	[[8, 2, 1, 0]]	[[9, 2, 1, 0]]	[[10, 2, 1, 0]]	[[11, 2, 1, 0]]	[[12, 2, 1, 0]]	[[13, 2, 1, 0]]	[[14, 2, 1, 0]]
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16	70	50	21	9	2	1	0								
17	160	109	57	22	9	2	1	0							
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19	629	497	283	130	59	22	9	2	1	0					
20	1176	1016	583	297	131	59	22	9	2	1	0				
21	2259	1983	1219	623	304	132	59	22	9	2	1	0			
22	4119	3837	2400	1306	637	305	132	59	22	9	2	1	0		
23	7570	7206	4727	2606	1346	644	306	132	59	22	9	2	1	0	
24	13461	13400	8972	5157	2693	1360	645	306	132	59	22	9	2	1	0
25	23950	24383	16923	9892	5364	2733	1367	646	306	132	59	22	9	2	1

$\alpha' m^2$	[[0,1,0,2]]	[[1,1,0,2]]	[[2,1,0,2]]	[[3,1,0,2]]	[[4,1,0,2]]	[[5,1,0,2]]	[[6,1,0,2]]	[[7,1,0,2]]	[[8,1,0,2]]	[[9,1,0,2]]	[[10,1,0,2]]	[[11,1,0,2]]	[[12,1,0,2]]	[[13,1,0,2]]	[[14,1,0,2]]
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14	76	61	30	12	4	1	0								
15	153	133	71	31	12	4	1	0							
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17	548	547	336	168	75	31	12	4	1	0					
18	1003	1058	687	361	171	75	31	12	4	1	0				
19	1819	2012	1365	752	371	172	75	31	12	4	1	0			
20	3227	3732	2646	1511	777	374	172	75	31	12	4	1	0		
21	5674	6825	5017	2973	1576	787	375	172	75	31	12	4	1	0	
22	9821	12252	9337	5702	3121	1601	790	375	172	75	31	12	4	1	0
23	16851	21737	17080	10752	6035	3186	1611	791	375	172	75	31	12	4	1
24	28565	38015	30794	19888	11457	6183	3211	1614	791	375	172	75	31	12	4
25	48036	65800	54747	36281	21354	11792	6248	3221	1615	791	375	172	75	31	12

$\alpha' m^2$	[[0,0,2,0]]	[[1,0,2,0]]	[[2,0,2,0]]	[[3,0,2,0]]	[[4,0,2,0]]	[[5,0,2,0]]	[[6,0,2,0]]	[[7,0,2,0]]	[[8,0,2,0]]	[[9,0,2,0]]	[[10,0,2,0]]	[[11,0,2,0]]	[[12,0,2,0]]	[[13,0,2,0]]	[[14,0,2,0]]
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15	43	30	17	4	2	0									
16	78	75	34	17	4	2	0								
17	169	150	91	36	17	4	2	0							
18	297	325	185	95	36	17	4	2	0						
19	593	622	414	201	97	36	17	4	2	0					
20	1043	1236	812	451	205	97	36	17	4	2	0				
21	1935	2296	1656	904	467	207	97	36	17	4	2	0			
22	3369	4316	3139	1863	941	471	207	97	36	17	4	2	0		
23	6003	7793	6029	3594	1957	957	473	207	97	36	17	4	2	0	
24	10261	14093	11090	6972	3804	1994	961	473	207	97	36	17	4	2	0
25	17753	24813	20426	13020	7444	3898	2010	963	473	207	97	36	17	4	2

$\alpha' m^2$	[[0,1,0,1]]	[[1,1,0,1]]	[[2,1,0,1]]	[[3,1,0,1]]	[[4,1,0,1]]	[[5,1,0,1]]	[[6,1,0,1]]	[[7,1,0,1]]	[[8,1,0,1]]	[[9,1,0,1]]	[[10,1,0,1]]	[[11,1,0,1]]	[[12,1,0,1]]	[[13,1,0,1]]	[[14,1,0,1]]
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11	29	28	16	7	3	1	0								
12	53	55	34	17	7	3	1	0							
13	96	107	70	36	17	7	3	1	0						
14	171	201	138	76	37	17	7	3	1	0					
15	300	369	268	153	78	37	17	7	3	1	0				
16	520	671	506	301	159	79	37	17	7	3	1	0			
17	891	1195	939	578	316	161	79	37	17	7	3	1	0		
18	1512	2101	1710	1089	611	322	162	79	37	17	7	3	1	0	
19	2541	3654	3071	2012	1163	626	324	162	79	37	17	7	3	1	0
20	4233	6280	5439	3663	2167	1196	632	325	162	79	37	17	7	3	1
21	6999	10680	9518	6573	3978	2241	1211	634	325	162	79	37	17	7	3
22	11481	18008	16466	11648	7199	4135	2274	1217	635	325	162	79	37	17	7
23	18704	30086	28203	20395	12861	7519	4209	2289	1219	635	325	162	79	37	17
24	30270	49864	47842	35340	22696	13500	7676	4242	2295	1220	635	325	162	79	37
25	48683	82031	80451	60618	39634	23943	13822	7750	4257	2297	1220	635	325	162	79

$\alpha' m^2$	[[0,2,0,1]]	[[1,2,0,1]]	[[2,2,0,1]]	[[3,2,0,1]]	[[4,2,0,1]]	[[5,2,0,1]]	[[6,2,0,1]]	[[7,2,0,1]]	[[8,2,0,1]]	[[9,2,0,1]]	[[10,2,0,1]]	[[11,2,0,1]]	[[12,2,0,1]]	[[13,2,0,1]]
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14	62	46	22	9	3	1	0							
15	125	98	52	23	9	3	1	0						
16	241	204	114	54	23	9	3	1	0					
17	460	408	242	120	55	23	9	3	1	0				
18	855	798	493	258	122	55	23	9	3	1	0			
19	1561	1522	982	531	264	123	55	23	9	3	1	0		
20	2806	2848	1904	1069	547	266	123	55	23	9	3	1	0	
21	4977	5233	3621	2094	1107	553	267	123	55	23	9	3	1	0
22	8706	9473	6754	4020	2181	1123	555	267	123	55	23	9	3	1
23	15067	16902	12404	7571	4212	2219	1129	556	267	123	55	23	9	3
24	25791	29782	22437	14033	7976	4299	2235	1131	556	267	123	55	23	9
25	43720	51867	40062	25611	14867	8168	4337	2241	1132	556	267	123	55	23

$\alpha' m^2$	[[0,0,1,1]]	[[1,0,1,1]]	[[2,0,1,1]]	[[3,0,1,1]]	[[4,0,1,1]]	[[5,0,1,1]]	[[6,0,1,1]]	[[7,0,1,1]]	[[8,0,1,1]]	[[9,0,1,1]]	[[10,0,1,1]]	[[11,0,1,1]]	[[12,0,1,1]]	[[13,0,1,1]]	[[14,0,1,1]]
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11	12	9	4	1	0										
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13	47	47	26	10	4	1	0								
14	90	98	58	27	10	4	1	0							
15	169	195	125	62	27	10	4	1	0						
16	304	378	258	136	63	27	10	4	1	0					
17	547	713	516	286	140	63	27	10	4	1	0				
18	966	1322	1001	584	297	141	63	27	10	4	1	0			
19	1677	2402	1903	1151	612	301	141	63	27	10	4	1	0		
20	2887	4299	3540	2226	1220	623	302	141	63	27	10	4	1	0	
21	4916	7584	6475	4207	2381	1248	627	302	141	63	27	10	4	1	0
22	8274	13215	11659	7808	4542	2450	1259	628	302	141	63	27	10	4	1
23	13822	22755	20706	14260	8510	4698	2478	1263	628	302	141	63	27	10	4
24	22889	38785	36301	25672	15681	8850	4767	2489	1264	628	302	141	63	27	10
25	37594	65459	62931	45588	28475	16395	9006	4795	2493	1264	628	302	141	63	27

$\alpha' m^2$	[[0,1,1,1]]	[[1,1,1,1]]	[[2,1,1,1]]	[[3,1,1,1]]	[[4,1,1,1]]	[[5,1,1,1]]	[[6,1,1,1]]	[[7,1,1,1]]	[[8,1,1,1]]	[[9,1,1,1]]	[[10,1,1,1]]	[[11,1,1,1]]	[[12,1,1,1]]	[[13,1,1,1]]	[[14,1,1,1]]
10	0														
11	1	0													
12	4	1	0												
13	11	5	1	0											
14	28	15	5	1	0										
15	65	40	16	5	1	0									
16	141	99	44	16	5	1	0								
17	292	224	111	45	16	5	1	0							
18	587	483	259	115	45	16	5	1	0						
19	1143	1007	572	271	116	45	16	5	1	0					
20	2176	2023	1216	607	275	116	45	16	5	1	0				
21	4056	3959	2495	1306	619	276	116	45	16	5	1	0			
22	7420	7580	4977	2710	1341	623	276	116	45	16	5	1	0		
23	13361	14206	9692	5467	2800	1353	624	276	116	45	16	5	1	0	
24	23720	26160	18474	10762	5683	2835	1357	624	276	116	45	16	5	1	0
25	41558	47429	34562	20726	11258	5773	2847	1358	624	276	116	45	16	5	1

$\alpha' m^2$	$[\![0,0,0,3]\!]$	$[\![1,0,0,3]\!]$	$[\![2,0,0,3]\!]$	$[\![3,0,0,3]\!]$	$[\![4,0,0,3]\!]$	$[\![5,0,0,3]\!]$	$[\![6,0,0,3]\!]$	$[\![7,0,0,3]\!]$	$[\![8,0,0,3]\!]$	$[\![9,0,0,3]\!]$	$[\![10,0,0,3]\!]$	$[\![11,0,0,3]\!]$	$[\![12,0,0,3]\!]$	$[\![13,0,0,3]\!]$	$[\![14,0,0,3]\!]$
9	0														
10	2	0													
11	3	2	0												
12	7	5	2	0											
13	16	13	5	2	0										
14	32	30	15	5	2	0									
15	62	65	36	15	5	2	0								
16	121	135	82	38	15	5	2	0							
17	222	272	176	88	38	15	5	2	0						
18	406	525	368	193	90	38	15	5	2	0					
19	731	997	732	412	199	90	38	15	5	2	0				
20	1291	1848	1431	836	429	201	90	38	15	5	2	0			
21	2247	3367	2722	1662	880	435	201	90	38	15	5	2	0		
22	3879	6033	5078	3218	1769	897	437	201	90	38	15	5	2	0	
23	6601	10664	9300	6100	3457	1813	903	437	201	90	38	15	5	2	0
24	11134	18593	16784	11343	6620	3564	1830	905	437	201	90	38	15	5	2
25	18612	32056	29830	20770	12428	6862	3608	1836	905	437	201	90	38	15	5

4.C Large spin asymptotics of super Poincaré multiplicities

This appendix contains some more data on the large spin asymptotics of $\mathcal{N}_{4d} = 1$, $\mathcal{N}_{6d} = (1,0)$ and $\mathcal{N}_{10d} = 1$ spectra. The leading and subleading Regge trajectories $\tau_\ell^Q(q), \tau_\ell^{k,p}(q)$ and $\tau_\ell^{x,y,z}(q)$ are defined through the expansion (4.184), (4.232) and (4.275) of super Poincaré multiplicity generating functions in terms of q^n powers (with n denoting the first $SO(d-1)$ Dynkin label). They have been computed on the basis of the $\alpha' m^2 \leq 25$ data given in Section 4.B.1

4.C.1 $\mathcal{N}_{4d} = 1$ multiplets at $SO(3)$ Dynkin label $[n \to \infty]$

This appendix contains more data on the asymptotics of universal $\mathcal{N}_{4d} = 1$ multiplets of $U(1)_R$ charge $Q \geq 2$. The $[\![2n+1, 2Q]\!]$ multiplicities up to level q^{25} determine the associated $\tau_\ell^{2Q}(q)$ coefficients for low charges Q to the following orders:

- $U(1)_R$ charge $Q = 2$:

$$\begin{aligned}
\tau_2^{Q=2}(q) &= q^3\,(2 + 11q + 37q^2 + 114q^3 + 319q^4 + 822q^5 + 2000q^6 \\
&\quad + 4645q^7 + 10354q^8 + 22317q^9 + 46702q^{10} + 95210q^{11} \\
&\quad + 189656q^{12} + \ldots) \\
\tau_3^{Q=2}(q) &= q^3\,(2 + 8q + 33q^2 + 104q^3 + 310q^4 + 826q^5 + 2093q^6 \\
&\quad + 4991q^7 + 11454q^8 + \ldots) \\
\tau_4^{Q=2}(q) &= q^3\,(1 + 5q + 22q^2 + 77q^3 + 237q^4 + 664q^5 + \ldots) \\
\tau_5^{Q=2}(q) &= q^4\,(3 + 12q + 49q^2 + \ldots) \qquad (4.323)
\end{aligned}$$

- $U(1)_R$ charge $Q = 4$:

$$\begin{aligned}
\tau_2^{Q=4}(q) &= q^8\,(2 + 14q + 57q^2 + 187q^3 + 542q^4 + 1438q^5 + 3563q^6 \\
&\quad + 8376q^7 + 18846q^8 + 40866q^9 + \ldots) \\
\tau_3^{Q=4}(q) &= q^8\,(2 + 14q + 58q^2 + 200q^3 + 591q^4 + 1612q^5 + \ldots) \\
\tau_4^{Q=4}(q) &= q^8\,(2 + 13q + 53q^2 + \ldots) \qquad (4.324)
\end{aligned}$$

- $U(1)_R$ charge $Q = 6$:

$$\begin{aligned}
\tau_2^{Q=6}(q) &= q^{15}\,(2 + 14q + 60q^2 + 209q^3 + 633q^4 + \ldots) \\
\tau_3^{Q=6}(q) &= q^{15}\,(2 + 14q + 64q^2 + \ldots) \qquad (4.325)
\end{aligned}$$

Also in the $[\![2n, 2Q+1]\!]$ sector, we can expand the subleading trajectories $\tau_{\geq 2}^{2Q+1}(q)$:

- $U(1)_R$ charge $Q = 1$:

$$\begin{aligned}
\tau_2^{Q=1}(q) &= 1 + 4q + 15q^2 + 50q^3 + 143q^4 + 379q^5 + 947q^6 + 2244q^7 \\
&\quad + 5103q^8 + 11196q^9 + 23804q^{10} + 49252q^{11} + 99465q^{12} \\
&\quad + 196522q^{13} + 380719q^{14} + \ldots \\
\tau_3^{Q=1}(q) &= 1 + 5q + 22q^2 + 70q^3 + 212q^4 + 568q^5 + 1458q^6 + 3496q^7 \\
&\quad + 8093q^8 + 17936q^9 + \ldots \\
\tau_4^{Q=1}(q) &= 1 + 6q + 24q^2 + 83q^3 + 252q^4 + 698q^5 + \ldots \\
\tau_5^{Q=1}(q) &= 1 + 6q + 25q^2 + \ldots \qquad (4.326)
\end{aligned}$$

- $U(1)_R$ charge $Q=3$:

$$\begin{aligned}
\tau_2^{Q=3}(q) &= q^4\,(1+9q+37q^2+120q^3+347q^4+922q^5+2287q^6 \\
&\quad +5385q^7+12142q^8+26395q^9+55605q^{10}+113973q^{11}+\ldots) \\
\tau_3^{Q=3}(q) &= q^4\,(4+17q+68q^2+208q^3+603q^4+1573q^5+3919q^6 \\
&\quad +9195q^7+\ldots) \\
\tau_4^{Q=3}(q) &= q^3\,(1+7q+28q^2+99q^3+304q^4+851q^5+\ldots) \\
\tau_5^{Q=3}(q) &= q^3\,(2+9q+38q^2+\ldots)
\end{aligned} \tag{4.327}$$

- $U(1)_R$ charge $Q=5$:

$$\begin{aligned}
\tau_2^{Q=5}(q) &= q^{10}\,(1+9q+43q^2+151q^3+462q^4+1277q^5+3264q^6 \\
&\quad +7865q^7+\ldots) \\
\tau_3^{Q=5}(q) &= q^{10}\,(4+20q+89q^2+292q^3+\ldots) \\
\tau_4^{Q=5}(q) &= q^9\,(1+9q+\ldots)
\end{aligned} \tag{4.328}$$

- $U(1)_R$ charge $Q=7$:

$$\tau_2^{Q=7}(q) = q^{18}\,(1+9q+\ldots) \tag{4.329}$$

Note that for all values of the $U(1)_R$ charge Q considered here, the leading q powers of the $\tau_\ell^Q(q)$ at fixed Q hardly vary with ℓ (at $Q=2$, for instance, we can read off $\tau_1^2,\tau_2^2,\tau_3^2,\tau_4^2 \sim \mathcal{O}(q^3)$ and $\tau_5^2 \sim \mathcal{O}(q^4)$ from (4.323)). In particular, the approximate agreement of the leading q powers of $\tau_1(q)$ and $\tau_2(q)$ supports our claim in the introduction that half of the nonzero multiplicities exactly match with the stable patterns.

4.C.2 $\mathcal{N}_{6d}=(1,0)$ multiplets at $SO(5)$ Dynkin labels $[n\to\infty,k]$

For the universal $\mathcal{N}_{6d}=(1,0)$ multiplets $[\![n\to\infty,k;p]\!]$ we display some $\tau_{\ell\le 5}^{k,p}(q)$ associated with super Poincaré quantum numbers (k,p) beyond the examples of subsection 5.4. Bosonic multiplets are characterized by the following asymptotic behaviour:

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[2]$

$$\begin{aligned}
\tau_1^{2,2}(q) &= q^4\,(1 + 6q + 19q^2 + 60q^3 + 160q^4 + 421q^5 + 1015q^6 + 2400q^7 \\
&+ 5398q^8 + 11900q^9 + 25371q^{10} + 53107q^{11} + 108500q^{12} + 218074q^{13} \\
&+ 430116q^{14} + 836194q^{15} + 1600889q^{16} + \ldots) \\
\tau_2^{2,2}(q) &= q^5\,(3 + 13q + 49q^2 + 151q^3 + 439q^4 + 1166q^5 + 2956q^6 \\
&+ 7119q^7 + 16566q^8 + 37224q^9 + 81414q^{10} + 173493q^{11} + \ldots) \\
\tau_3^{2,2}(q) &= q^6\,(3 + 12q + 53q^2 + 171q^3 + 537q^4 + 1486q^5 + 3960q^6 \\
&+ 9876q^7 + \ldots) \\
\tau_4^{2,2}(q) &= q^7\,(1 + 8q + 35q^2 + 134q^3 + 434q^4 + \ldots) \\
\tau_5^{2,2}(q) &= q^9\,(4 + \ldots)
\end{aligned} \tag{4.330}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 4]$ and $SU(2)_R$ representation $[0]$

$$\begin{aligned}
\tau_1^{4,0}(q) &= q^5\,(1 + 5q + 14q^2 + 43q^3 + 113q^4 + 294q^5 + 698q^6 + 1648q^7 \\
&+ 3677q^8 + 8090q^9 + 17182q^{10} + 35919q^{11} + 73211q^{12} + 147036q^{13} \\
&+ 289598q^{14} + 562694q^{15} + 1076373q^{16} + \ldots) \\
\tau_2^{4,0}(q) &= q^6\,(1 + 5q + 18q^2 + 56q^3 + 166q^4 + 446q^5 + 1143q^6 + 2787q^7 \\
&+ 6549q^8 + 14864q^9 + 32811q^{10} + 70532q^{11} + 148268q^{12} + \ldots) \\
\tau_3^{4,0}(q) &= q^9\,(4 + 14q + 61q^2 + 184q^3 + 561q^4 + 1495q^5 + 3896q^6 \\
&+ 9478q^7 + \ldots) \\
\tau_4^{4,0}(q) &= q^{11}\,(1 + 8q + 36q^2 + 131q^3 + \ldots)
\end{aligned} \tag{4.331}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[4]$

$$\begin{aligned}
\tau_1^{0,4}(q) &= q^6\,(1 + 4q + 18q^2 + 47q^3 + 142q^4 + 353q^5 + 887q^6 + 2049q^7 \\
&+ 4692q^8 + 10215q^9 + 21942q^{10} + 45608q^{11} + 93377q^{12} + 186790q^{13} \\
&+ 368341q^{14} + \ldots) \\
\tau_2^{0,4}(q) &= q^6\,(3 + 10q + 41q^2 + 124q^3 + 362q^4 + 952q^5 + 2424q^6 \\
&+ 5811q^7 + 13526q^8 + 30317q^9 + \ldots) \\
\tau_3^{0,4}(q) &= q^5\,(1 + 3q + 17q^2 + 53q^3 + 179q^4 + 501q^5 + 1392q^6 + \ldots) \\
\tau_4^{0,4}(q) &= q^5\,(1 + 3q + 16q^2 + 53q^3 + \ldots)
\end{aligned} \tag{4.332}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 4]$ and $SU(2)_R$ representation $[2]$

$$
\begin{aligned}
\tau_1^{4,2}(q) &= q^7\,(3 + 12q + 48q^2 + 141q^3 + 408q^4 + 1052q^5 + 2632q^6 \\
&\quad + 6194q^7 + 14200q^8 + 31309q^9 + 67467q^{10} + 141443q^{11} \\
&\quad + 290805q^{12} + 585447q^{13} + 1159182q^{14} + \ldots) \\
\tau_2^{4,2}(q) &= q^8\,(3 + 15q + 63q^2 + 206q^3 + 623q^4 + 1714q^5 + 4464q^6 \\
&\quad + 11006q^7 + 26108q^8 + 59679q^9 + 132452q^{10} + \ldots) \\
\tau_3^{4,2}(q) &= q^{10}\,(3 + 16q + 76q^2 + 262q^3 + 847q^4 + 2427q^5 + 6599q^6 \\
&\quad + \ldots) \\
\tau_4^{4,2}(q) &= q^{12}\,(1 + 11q + 52q^2 + \ldots) \qquad (4.333)
\end{aligned}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[4]$

$$
\begin{aligned}
\tau_1^{2,4}(q) &= q^8\,(4 + 14q + 58q^2 + 170q^3 + 492q^4 + 1264q^5 + 3165q^6 \\
&\quad + 7432q^7 + 17012q^8 + 37428q^9 + 80496q^{10} + 168377q^{11} \\
&\quad + 345433q^{12} + \ldots) \\
\tau_2^{2,4}(q) &= q^8\,(1 + 11q + 45q^2 + 169q^3 + 523q^4 + 1505q^5 + 3992q^6 \\
&\quad + 10086q^7 + 24241q^8 + \ldots) \\
\tau_3^{2,4}(q) &= q^9\,(3 + 15q + 70q^2 + 241q^3 + 781q^4 + \ldots) \\
\tau_4^{2,4}(q) &= q^{10}\,(3 + 15q + \ldots) \qquad (4.334)
\end{aligned}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 6]$ and $SU(2)_R$ representation $[0]$

$$
\begin{aligned}
\tau_1^{6,0}(q) &= q^8\,(1 + 5q + 18q^2 + 53q^3 + 158q^4 + 407q^5 + 1033q^6 + 2452q^7 \\
&\quad + 5686q^8 + 12640q^9 + 27521q^{10} + 58151q^{11} + 120616q^{12} + 244647q^{13} \\
&\quad + \ldots) \\
\tau_2^{6,0}(q) &= q^9\,(1 + 5q + 18q^2 + 57q^3 + 173q^4 + 473q^5 + 1234q^6 + 3060q^7 \\
&\quad + 7308q^8 + 16835q^9 + \ldots) \\
\tau_3^{6,0}(q) &= q^{13}\,(4 + 15q + 67q^2 + 209q^3 + \ldots) \qquad (4.335)
\end{aligned}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation [6]

$$
\begin{aligned}
\tau_1^{0,6}(q) &= q^{11}\,(3 + 8q + 35q^2 + 98q^3 + 291q^4 + 733q^5 + 1856q^6 + 4339q^7 \\
&\quad + 9987q^8 + 21954q^9 + \ldots) \\
\tau_2^{0,6}(q) &= q^{10}\,(1 + 5q^1 + 27q^2 + 88q^3 + 286q^4 + 804q^5 + 2171q^6 + \ldots) \\
\tau_3^{0,6}(q) &= q^{10}\,(3 + 10q + 46q^2 + 148q^3 + \ldots) \\
\tau_4^{0,6}(q) &= q^9\,(1 + 3q + \ldots)
\end{aligned} \tag{4.336}
$$

In addition, let us display some $\tau_\ell^{k,p}(q)$ associated with fermionic supermultiplets:

- $SO(5)$ Dynkin labels $[n \to \infty, 3]$ and $SU(2)_R$ representation [1]

$$
\begin{aligned}
\tau_1^{3,1}(q) &= q^4\,(1 + 5q + 16q^2 + 49q^3 + 134q^4 + 343q^5 + 840q^6 + 1971q^7 \\
&\quad + 4460q^8 + 9810q^9 + 21006q^{10} + 43952q^{11} + 90078q^{12} + 181178q^{13} \\
&\quad + 358196q^{14} + 697195q^{15} + 1337468q^{16} + \ldots) \\
\tau_2^{3,1}(q) &= q^5\,(1 + 7q + 25q^2 + 84q^3 + 247q^4 + 674q^5 + 1733q^6 + 4252q^7 \\
&\quad + 10005q^8 + 22774q^9 + 50306q^{10} + 108276q^{11} + \ldots) \\
\tau_3^{3,1}(q) &= q^7\,(2 + 11q + 46q^2 + 158q^3 + 486q^4 + 1369q^5 + 3622q^6 + \ldots) \\
\tau_4^{3,1}(q) &= q^9\,(2 + 13q + 57q^2 + \ldots)
\end{aligned} \tag{4.337}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation [3]

$$
\begin{aligned}
\tau_1^{1,3}(q) &= q^5\,(2 + 9q + 29q^2 + 86q^3 + 233q^4 + 591q^5 + 1426q^6 + 3308q^7 \\
&\quad + 7408q^8 + 16117q^9 + 34176q^{10} + 70842q^{11} + 143887q^{12} + 286959q^{13} \\
&\quad + 562767q^{14} + 1086923q^{15} + \ldots) \\
\tau_2^{1,3}(q) &= q^5\,(2 + 10q + 39q^2 + 125q^3 + 366q^4 + 990q^5 + 2530q^6 + 6157q^7 \\
&\quad + 14414q^8 + 32604q^9 + 71640q^{10} + 153380q^{11} + \ldots) \\
\tau_3^{1,3}(q) &= q^5\,(1 + 6q + 24q^2 + 87q^3 + 275q^4 + 799q^5 + 2168q^6 + 5570q^7 \\
&\quad + 13669q^8 + \ldots) \\
\tau_4^{1,3}(q) &= q^6\,(2 + 9q + 38q^2 + 135q^3 + 428q^4 + \ldots) \\
\tau_5^{1,3}(q) &= q^7\,(2 + 11q + \ldots)
\end{aligned} \tag{4.338}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 5]$ and $SU(2)_R$ representation [1]

$$\begin{aligned}
\tau_1^{5,1}(q) &= q^7\,(1 + 7q + 24q^2 + 80q^3 + 228q^4 + 610q^5 + 1533q^6 + 3691q^7 \\
&+ 8520q^8 + 19063q^9 + 41409q^{10} + 87751q^{11} + 181781q^{12} + 369134q^{13} \\
&+ 735899q^{14} + \ldots) \\
\tau_2^{5,1}(q) &= q^8\,(1 + 7q + 26q^2 + 92q^3 + 281q^4 + 791q^5 + 2090q^6 + 5251q^7 \\
&+ 12618q^8 + 29264q^9 + 65731q^{10} + \ldots) \\
\tau_3^{5,1}(q) &= q^{11}\,(2 + 12q + 55q^2 + 196q^3 + 625q^4 + 1808q^5 + \ldots) \\
\tau_4^{5,1}(q) &= q^{14}\,(2 + 15q + \ldots) \qquad (4.339)
\end{aligned}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 3]$ and $SU(2)_R$ representation [3]

$$\begin{aligned}
\tau_1^{3,3}(q) &= q^7\,(2 + 10q + 41q^2 + 127q^3 + 369q^4 + 977q^5 + 2453q^6 + 5856q^7 \\
&+ 13474q^8 + 29947q^9 + 64743q^{10} + 136433q^{11} + 281245q^{12} + 568184q^{13} \\
&+ 1127435q^{14} + \ldots) \\
\tau_2^{3,3}(q) &= q^8\,(3 + 18q + 75q^2 + 252q^3 + 762q^4 + 2111q^5 + 5496q^6 \\
&+ 13580q^7 + 32188q^8 + 73580q^9 + 163122q^{10} + \ldots) \\
\tau_3^{3,3}(q) &= q^9\,(1 + 11q + 49q^2 + 189q^3 + 617q^4 + 1841q^5 + 5079q^6 + \ldots) \\
\tau_4^{3,3}(q) &= q^{11}\,(3 + 19q + 84q^2 + \ldots) \qquad (4.340)
\end{aligned}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation [5]

$$\begin{aligned}
\tau_1^{1,5}(q) &= q^9\,(2 + 10q + 36q^2 + 118q^3 + 335q^4 + 893q^5 + 2237q^6 \\
&+ 5356q^7 + 12311q^8 + 27406q^9 + 59236q^{10} + 124892q^{11} + \ldots) \\
\tau_2^{1,5}(q) &= q^9\,(2 + 13q + 54q^2 + 186q^3 + 573q^4 + 1609q^5 + 4237q^6 \\
&+ 10575q^7 + \ldots) \\
\tau_3^{1,5}(q) &= q^9\,(2 + 10q + 45q^2 + 161q^3 + 518q^4 + \ldots) \\
\tau_4^{1,5}(q) &= q^9\,(1 + 6q + 26q^2 + \ldots) \qquad (4.341)
\end{aligned}$$

As mentioned in subsection 5.4, the $\tau_\ell^{k,p}(q)$ expansion (5.42) of $G_{n,k,p}(q)$ converges more rapidly at large vales of the second Dynkin label k and small values of the $SU(2)_R$ spin p.

4.C.3 $\mathcal{N}_{10d} = 1$ multiplets at $SO(9)$ Dynkin labels $[n \rightarrow \infty, x, y, z]$

Also for $\mathcal{N}_{10d} = 1$ multiplets $[\![n \rightarrow \infty, x, y, z]\!]$ we would like to list some more $\tau^{x,y,z}_{\ell \leq 5}(q)$ beyond those of subsection 6.2. We focus on six bosonic families

- $SO(9)$ Dynkin labels $[n \rightarrow \infty, 2, 0, 0]$

$$\begin{aligned}
\tau^{2,0,0}_1(q) \ &= \ q^7\,(1 + 2q + 6q^2 + 11q^3 + 27q^4 + 52q^5 + 112q^6 + 212q^7 \\
&\quad + 423q^8 + 787q^9 + 1496q^{10} + 2724q^{11} + 5001q^{12} + 8927q^{13} \\
&\quad + 15950q^{14} + \ldots) \\
\tau^{2,0,0}_2(q) \ &= \ q^8\,(1 + 2q^1 + 6q^2 + 14q^3 + 34q^4 + 74q^5 + 161q^6 + 333q^7 \\
&\quad + 680q^8 + 1346q^9 + 2627q^{10} + \ldots) \\
\tau^{2,0,0}_3(q) \ &= \ q^{12}\,(3 + 7q + 23q^2 + 54q^3 + 138q^4 + \ldots) \\
\tau^{2,0,0}_4(q) \ &= \ q^{15}\,(1 + \ldots)
\end{aligned} \tag{4.342}$$

- $SO(9)$ Dynkin labels $[n \rightarrow \infty, 1, 1, 0]$

$$\begin{aligned}
\tau^{1,1,0}_1(q) \ &= \ q^8\,(1 + 2q + 8q^2 + 19q^3 + 45q^4 + 100q^5 + 217q^6 + 446q^7 \\
&\quad + 905q^8 + 1779q^9 + 3440q^{10} + 6521q^{11} + 12181q^{12} + 22396q^{13} + \ldots) \\
\tau^{1,1,0}_2(q) \ &= \ q^9\,(1 + 2q + 9q^2 + 23q^3 + 61q^4 + 143q^5 + 330q^6 + 715q^7 \\
&\quad + 1524q^8 + 3128q^9 + \ldots) \\
\tau^{1,1,0}_3(q) \ &= \ q^{12}\,(1 + 4q^1 + 16q^2 + 46q^3 + 125q^4 + \ldots)
\end{aligned} \tag{4.343}$$

- $SO(9)$ Dynkin labels $[n \rightarrow \infty, 1, 0, 2]$

$$\begin{aligned}
\tau^{1,0,2}_1(q) \ &= \ q^9\,(1 + 4q + 12q^2 + 31q^3 + 75q^4 + 172q^5 + 375q^6 + 791q^7 \\
&\quad + 1615q^8 + 3225q^9 + 6287q^{10} + 12044q^{11} + 22652q^{12} + \ldots) \\
\tau^{1,0,2}_2(q) \ &= \ q^{10}\,(1 + 4q + 14q^2 + 39q^3 + 104q^4 + 252q^5 + 587q^6 + 1300q^7 \\
&\quad + 2794q^8 + \ldots) \\
\tau^{1,0,2}_3(q) \ &= \ q^{13}\,(2 + 8q + 30q^2 + 87q^3 + \ldots)
\end{aligned} \tag{4.344}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 3, 0, 0]$

$$\begin{aligned}
\tau_1^{3,0,0}(q) &= q^{10}\,(1 + 2q + 6q^2 + 14q^3 + 32q^4 + 69q^5 + 147q^6 + 299q^7 \\
&\quad + 597q^8 + 1168q^9 + 2239q^{10} + 4226q^{11} + 7854q^{12} + \ldots) \\
\tau_2^{3,0,0}(q) &= q^{11}\,(1 + 2q + 6q^2 + 14q^3 + 35q^4 + 77q^5 + 172q^6 + 361q^7 \\
&\quad + 752q^8 + 1513q^9 + \ldots) \\
\tau_3^{3,0,0}(q) &= q^{16}\,(3 + 8q + 25q^2 + 63q^3 + \ldots) \qquad (4.345)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 2, 0]$

$$\begin{aligned}
\tau_1^{0,2,0}(q) &= q^{11}\,(2 + 4q + 17q^2 + 36q^3 + 97q^4 + 207q^5 + 473q^6 + 963q^7 \\
&\quad + 2016q^8 + 3957q^9 + 7809q^{10} + 14838q^{11} + \ldots) \\
\tau_2^{0,2,0}(q) &= q^{12}\,(2 + 6q + 22q^2 + 59q^3 + 153q^4 + 365q^5 + 842q^6 + 1842q^7 \\
&\quad + \ldots) \\
\tau_3^{0,2,0}(q) &= q^{14}\,(2 + 5q + 24q^2 + 62q^3 + \ldots) \qquad (4.346)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 2, 1, 0]$

$$\begin{aligned}
\tau_1^{2,1,0}(q) &= q^{11}\,(1 + 2q + 9q^2 + 22q^3 + 59q^4 + 132q^5 + 306q^6 + 646q^7 \\
&\quad + 1369q^8 + 2756q^9 + 5514q^{10} + 10682q^{11} + \ldots) \\
\tau_2^{2,1,0}(q) &= q^{12}\,(1 + 2q + 9q^2 + 23q^3 + 63q^4 + 150q^5 + 357q^6 + 791q^7 \\
&\quad + 1728q^8 + \ldots) \\
\tau_3^{2,1,0}(q) &= q^{16}\,(1 + 4q + 18q^2 + 51q^3 + \ldots) \qquad (4.347)
\end{aligned}$$

and five fermionic families of supermultiplets:

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 0, 1]$

$$\begin{aligned}
\tau_1^{1,0,1}(q) &= q^6\,(1 + 3q + 7q^2 + 17q^3 + 37q^4 + 79q^5 + 162q^6 + 325q^7 \\
&+ 635q^8 + 1220q^9 + 2298q^{10} + 4266q^{11} + 7807q^{12} + 14110q^{13} \\
&+ 25197q^{14} + 44530q^{15} + \ldots) \\
\tau_2^{1,0,1}(q) &= q^7\,(1 + 3q + 9q^2 + 24q^3 + 57q^4 + 131q^5 + 288q^6 + 610q^7 \\
&+ 1256q^8 + 2523q^9 + 4957q^{10} + 9557q^{11} + \ldots) \\
\tau_3^{1,0,1}(q) &= q^{10}\,(2 + 7q + 22q^2 + 61q^3 + 155q^4 + 367q^5 + 835q^6 + \ldots) \\
\tau_4^{1,0,1}(q) &= q^{13}\,(2 + 9q + 31q^2 + \ldots) \qquad (4.348)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 1, 1]$

$$\begin{aligned}
\tau_1^{0,1,1}(q) &= q^8\,(1 + 4q + 10q^2 + 27q^3 + 63q^4 + 141q^5 + 302q^6 + 628q^7 \\
&+ 1264q^8 + 2494q^9 + 4811q^{10} + 9119q^{11} + 17005q^{12} + 31260q^{13} + \ldots) \\
\tau_2^{0,1,1}(q) &= q^9\,(1 + 5q + 16q^2 + 44q^3 + 113q^4 + 269q^5 + 610q^6 + 1330q^7 \\
&+ 2804q^8 + 5748q^9 + \ldots) \\
\tau_3^{0,1,1}(q) &= q^{11}\,(1 + 6q + 19q^2 + 59q^3 + 160q^4 + 404q^5 + \ldots) \\
\tau_4^{0,1,1}(q) &= q^{14}\,(2 + 9q + \ldots) \qquad (4.349)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 2, 0, 1]$

$$\begin{aligned}
\tau_1^{2,0,1}(q) &= q^9\,(1 + 3q + 9q^2 + 23q^3 + 55q^4 + 123q^5 + 267q^6 + 556q^7 \\
&+ 1132q^8 + 2244q^9 + 4362q^{10} + 8318q^{11} + 15616q^{12} + 28873q^{13} + \ldots) \\
\tau_2^{2,0,1}(q) &= q^{10}\,(1 + 3q + 9q^2 + 25q^3 + 63q^4 + 150q^5 + 342q^6 + 749q^7 \\
&+ 1591q^8 + 3289q^9 + 6640q^{10} + \ldots) \\
\tau_3^{2,0,1}(q) &= q^{14}\,(2 + 8q + 27q^2 + 77q^3 + 204q^4 + \ldots) \qquad (4.350)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 3]$

$$\begin{aligned}
\tau_1^{0,0,3}(q) &= q^{10}\,(2 + 5q + 15q^2 + 38q^3 + 90q^4 + 201q^5 + 437q^6 + 905q^7 \\
&\quad + 1838q^8 + 3633q^9 + 7038q^{10} + 13374q^{11} + \ldots) \\
\tau_2^{0,0,3}(q) &= q^{11}\,(2 + 8q + 25q^2 + 69q^3 + 176q^4 + 418q^5 + 949q^6 + 2069q^7 \\
&\quad + \ldots) \\
\tau_3^{0,0,3}(q) &= q^{13}\,(3 + 11q + 38q^2 + 109q^3 + \ldots) \\
\tau_4^{0,0,3}(q) &= q^{15}\,(1 + \ldots)
\end{aligned} \qquad (4.351)$$

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 1, 1]$

$$\begin{aligned}
\tau_1^{1,1,1}(q) &= q^{11}\,(1 + 5q + 16q^2 + 45q^3 + 116q^4 + 276q^5 + 624q^6 + 1358q^7 \\
&\quad + 2852q^8 + 5825q^9 + 11616q^{10} + 22669q^{11} + \ldots) \\
\tau_2^{1,1,1}(q) &= q^{12}\,(1 + 5q + 17q^2 + 52q^3 + 142q^4 + 358q^5 + 855q^6 + 1950q^7 \\
&\quad + 4279q^8 + \ldots) \\
\tau_3^{1,1,1}(q) &= q^{15}\,(1 + 7q + 26q^2 + 84q^3 + 243q^4 + \ldots)
\end{aligned} \qquad (4.352)$$

These results confirm that the $\tau_\ell^{x,y,z}(q)$ expansion (6.34) of multiplicity generating functions $G_{n,x,y,z}(q)$ converges more quickly at higher values of the Dynkin labels x, y, z.

5 Hilbert series of SQCD with exceptional gauge groups

In this section we will discuss Hilbert series, which are another type of partition function, for supersymmetric QCD theories with exceptional and related gauge groups. We will begin with a short introduction to supersymmetric gauge theories as a whole, with derivations of the F-term and D-term constraints, and then specialize to quiver gauge theories, and their cousins brane tilings, and finally to SQCD. We will leave it till then to discuss the transition to Hilbert series, although they apply equally to other SUSY gauge theories, in particular to those with a nonzero (classical) superpotential which is not the case in SQCD. We will next give a short review of the currently known results for classical gauge groups with matter in (anti)fundamental representations, both with and without an adjoint field, and also for the simplest exceptional group, G_2, with an adjoint field present, before proceeding to the meat of the discussion about the other exceptional gauge groups, which we introduce and put into context, and other groups related to the exceptional ones either by sequence of Dynkin diagrams, the Higgs mechanism, and folding of the Dynkin diagrams.

5.1 Supersymmetric gauge theories

In 4-dimensional non-supersymmetric field theories, the mass dimension of a scalar field is 1, because the Lagrangian density must have dimension 4 to match the -4 of the integral ($length = mass^{-1}$) and the derivative operator, and of course the mass itself, have dimension 1. By the same argument, that of a gauge field is also 1, and that of a fermionic field is $\frac{3}{2}$.

Supersymmetry is a symmetry between bosons, whether they be scalar fields (of spin - or helicity for massless fields - 0) or gauge (vector) fields (1) - or gravitons, with spin 2, which are not part of the gauge theories we consider here - and fermions, whether their spin is $\frac{1}{2}$ or $\frac{3}{2}$ (again we do not deal with the latter, which are called gravitinos, here, or in gauge theories in general), so in supersymmetric field theories, one must incorporate both bosonic and fermionic fields, of whichever type, into supersymmetry multiplets which can be represented by so-called superfields.

We must therefore introduce new 'superspace' fermionic (Grassmann) coordinates θ^α, and its hermitian conjugate $\bar{\theta}^{\dot{\alpha}}$, with mass dimension $-\frac{1}{2}$. (In extended supersymmetry, i.e. $\mathcal{N} > 1$, these have a subscript A from 1 to $\mathcal{N}$.)

The summation convention for undotted indices is top left to bottom right, and for dotted indices it is from bottom left to top right. In $\mathcal{N} = 1$ supersymmetry, a (complex) scalar field and a (Dirac) fermion can be combined into a so-called chiral superfield; however since off-shell a complex scalar has 2 degrees of freedom and a Dirac fermion 4, and also for consistency with supersymmetric variations, an 'auxiliary' complex scalar F must be incorporated into the superfield as the highest (i.e. $\theta\theta$) term. An antichiral superfield can be constructed in a similar way.

A general superfield can be expressed as [40]

$$S(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = \phi + \theta\psi + \bar{\theta}\bar{\chi} + \theta\theta M + \bar{\theta}\bar{\theta}N + \theta\sigma^\mu\bar{\theta}V_\mu + \bar{\theta}\bar{\theta}\theta\rho + \theta\theta\bar{\theta}\bar{\lambda} + \theta\theta\bar{\theta}\bar{\theta}D \tag{5.1}$$

Certain conditions must exist on the coefficients of each power of θ and $\bar{\theta}$, which are functions of x^μ, and their derivatives with respect to x^μ, for this to be a superfield. For example, if there is only the ϕ term, it must be a constant.

This is not an irreducible representation of the supersymmetry algebra; chiral and antichiral superfields are irreps, and the other types are vector superfields, which we will consider here, and linear ones, which we will not. A vector superfield is real, so ϕ, V_μ and D are real and $\psi = \chi$, $\rho = \lambda$ and $M = N^*$ (we usually write $M \pm iN$). There are 8 bosonic and 8 fermionic degrees of freedom.

The matter fields Φ_i, which are chiral multiplets (in $\mathcal{N} = 1$) or hypermultiplets (in $\mathcal{N} = 2$), can transform in any representation of the gauge and global symmetry groups; the gauge field V^a is a vector multiplet and transforms in the adjoint representation of the gauge group and as a singlet of the global group.

Supersymmetry transformations are generated by supercharges $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ which are defined as follows in terms of partial derivatives and superspace coordinates as follows:

$$\mathcal{Q}_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{5.2}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = i\partial_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{5.3}$$

If ∂_μ acts on a superfield, we still have a superfield, but acting with ∂_α or $\partial_{\dot{\alpha}}$ do not give one; one must therefore define new 'chiral' derivatives D_α and $D_{\dot{\alpha}}$ which do give a superfield when acting on one. They are defined as follows:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{5.4}$$

$$\bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{5.5}$$

In these expressions we are using a spacetime coordinate x^μ; we can define a new spacetime coordinate y^μ under which the chirality is made manifest and $\bar{D}_{\dot{\alpha}}$ is equivalent to $\partial_{\dot{\alpha}}$:

$$y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} \tag{5.6}$$

In this basis, chiral superfields have an expansion solely in powers of θ and are independent of $\bar{\theta}$. Antichiral superfields can be defined similarly in terms of $\bar{y}^\mu = x^\mu - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}$ and depend solely on $\bar{\theta}$ in the new basis.

Chiral and antichiral superfields can be written as follows:

$$\Phi_i = \phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i \tag{5.7}$$

$$\Phi_i^\dagger = \phi_i^* + \sqrt{2}\bar{\theta}\bar{\psi}_i + \bar{\theta}\bar{\theta}F_i^* \tag{5.8}$$

In both cases there are 4 bosonic and 4 fermionic degrees of freedom. Vector superfields have 8 of both types.

A Lagrangian density for a supersymmetric gauge theory, just as for a non-supersymmetric theory, must consist of terms whose variation under supersymmetry transformations is a total derivative. We know that the variation of the F-term of a chiral superfield is a total derivative, and the same for the D-term of a general superfield, because they (excluding the superspace coordinates) are the terms with the highest mass dimension and to get terms of the same dimension from terms of lower dimension, including a parameter ϵ with mass dimension $-\frac{1}{2}$, we must take a derivative (raising the dimension by 1), and so the Lagrangian density must consist only of such terms. The Lagrangian density of a general supersymmetric gauge theory is specified by three quantities:

- The Kähler potential $K(\Phi, \Phi^\dagger)$, which is a real function of Φ and $\Phi^\dagger$
- The superpotential $W(\Phi)$, which is a holomorphic function of Φ and does not depend on $\Phi^\dagger$
- The gauge field strength term $\mathcal{W}^a_\alpha \mathcal{W}^{a\alpha}$

The Kähler potential is integrated over all of superspace as it is the D-term whose variation is a total derivative; the superpotential is integrated over half of superspace as it is an F-term. Technically, the gauge field strength term is a D-term, because it is a second antichiral derivative, though it occurs in the Lagrangian as an integral over half of superspace.

Solving the Euler-Lagrange equations for the auxiliary fields which form the highest component of each superfield, viz. the F-term in a chiral superfield and the D-term in a vector superfield, gives the F-term and D-term constraints respectively. The F-term constraints come from the interaction between the FF^* term in the Kähler potential and the Taylor expansion of the superpotential $W(\Phi)$ about the scalar part ϕ, and the D-term ones from that between the gauge field strength term, which contains the $D^a D^a$ term, and the $\Phi^\dagger V^a \Phi$ term from the expansion of $\Phi^\dagger e^{gV^aT^a}\Phi$ in the Kähler potential.

Like non-supersymmetric gauge theories, supersymmetric gauge theories exhibit gauge invariance. Taking Φ to $e^{ig\Lambda^a T^a}\Phi$ (and similarly $\Phi^\dagger$ to $\Phi^\dagger e^{-ig\Lambda^{a\dagger}T^a}$), where Λ^a is a chiral superfield in the adjoint representation of the gauge group (so $e^{ig\Lambda^a}\Phi$ is still chiral), One can choose Λ^a so that the components of V^a that do not contain both θ and $\bar{\theta}$ can be gauged away. This is called the Wess-Zumino gauge and simplifies the expansion of $e^{igV^aT^a}$ to go only up to order g^2. This is not supersymmetric because taking the variation under supersymmetry transformations produces terms dependent on θ but not $\bar{\theta}$ and vice versa, and also because there are 5 bosonic degrees of freedom but only 4 fermionic ones; however the variation can be gauged away by a 'compensating' gauge transformation.

The importance of supersymmetric gauge theories increased exponentially in 1997 on the proposal of the so-called AdS/CFT correspondence [64] - a special case of the more general gauge/gravity duality - by which a theory on anti-de Sitter (or other) space in $d+1$ dimensions containing gravity is conjectured to be related to a conformal (or other) field theory in d dimensions without gravity. Specifically, string theory (type IIA or IIB) on $AdS_{d+1} \times X^{9-d}$ is mapped to a conformal (or more general gauge) theory in d dimensions (usually on a Dp-brane where $p = d-1$) probing a Calabi-Yau singularity on the cone over X^{9-d}.

The two principal cases in current literature are type IIB string theory on $AdS_5 \times X^5$ where X^5 is a Sasaki-Einstein manifold which is mapped to a gauge theory on D3-branes probing a Calabi-Yau singularity on the cone over X^5 which is a (singular) Calabi-Yau 3-fold (6 real dimensions) - in the special case where X^5 is simply the 5-sphere S^5, the cone is simply $\mathbb{R}^6$ and so the theory is simply 4-dimensional $\mathcal{N} = 4$ super-Yang-Mills - and M-theory on $AdS_4 \times X^7$ which is mapped to a Chern-Simons theory on M2-branes probing a Calabi-Yau singularity on the cone over X^7 which is a CY 4-fold. The former (with a non-trivial SE/CY manifold, so $\mathcal{N} = 1$) are investigated - on the gauge side of the correspondence - in [29] and [27], and the latter in [31, 30, 28, 26, 39], and with the special case of X^7 being an orbifold of the 7-sphere S^7 by a finite subgroup G of $SU(2)$, with the cone being $\mathbb{R}^8/G$, in [50]. G is one of the following groups:

- $A_k = \mathbb{Z}_k$,
- $D_{k+2} = Dih(k)$, the dihedral group of order $2k$,
- the 'exceptional' subgroups of $SU(2)$ called E_6, E_7 and E_8.

In all cases the order of the group is the same as the sum of the squares of the (dual) Coxeter labels of the nodes of the extended Dynkin diagram of the Lie group of the same name. [1]

[1] The Coxeter label of a node is the coefficient of the simple root corresponding to the node in the linear independence relation between the simple roots, normalized to have greatest common factor 1. The 'lowest' root, i.e. the negative of the highest positive root, corresponds to the extended node and always has coefficient 1. To get the dual Coxeter numbers, one multiplies by the norm squared of the root and divides by that

Recent work in [49, 48, 47, 46] also links gauge amplitudes in d dimensions with gravity ones in $d+1$ dimensions by showing that supergravity, the low-energy limit of string theory, is the 'square' of super-Yang-Mills theory in some way and that in the cases of 3, 4, 6 and 10-dimensional SYM the number of dimensions of the squared theory can be raised by one and $\mathcal{N}$ be halved back to its original value.

The usefulness of this correspondence is that it is (generally) much easier to calculate scattering amplitudes in gauge theory than in string theory. There are few examples of direct comparisons of the two methods of calculation, but it is done in [75], where the $SO(9)$ string spectra (as calculated in the 16-supercharge section of the last chapter) are decomposed into $SO(4) \times SO(5)$ spectra and each of the latter representations 'lifted' to the Kaluza-Klein 'tower' of $SO(6)$ representations which contain them in their $SO(5)$ decompositions, and the gauge calculations done using Polya's enumeration theorem which is similar to the method of calculating invariants for finite groups described in Section 2.6.

Supersymmetric gauge theories are different from non-supersymmetric ones in that they always have flat directions in their potential, whether that comes from the F-terms, the D-terms or both [44]. These correspond to massless scalars, which are called moduli and span what is known as a moduli space, and are parametrized by gauge-invariant combinations of the fundamental fields in the theory. For example, in instanton theories [42], an instanton of a specific gauge group G is parametrized by its position (4 scalars - it corresponds to the Higgs branch of Dp-branes confined to D$(p+4)$-branes), its size (1), its orientation within $SU(2)$ (3) and its orientation as an $SU(2)$ instanton within G ($dim(G) - dim(H) - 3$ where H is the subgroup of G normal to $SU(2)$). For example, when $G = E_8$, $H = E_7$ and $dim(G) - dim(H) - 3 = 112 = 4(30 - 2)$ where 30 is the dual Coxeter number of E_8. This relation holds for all choices of G.

Moduli spaces also occur in other contexts, for example in string theory where several dimensions are compactified to finite size [53], the moduli specify quantities like the sizes of the dimensions and the angles at which they interact. Moduli spaces in supersymmetric gauge theories form an

of the longest (simple) root; for simply laced groups the Coxeter and dual Coxeter numbers are therefore the same. The (dual) Coxeter number of the group as a whole is the sum of those for the nodes, including the extended node.

algebraic structure called a (chiral, (multi)graded) ring and the dimension of each (multi)graded piece can be counted in a partition function-like quantity called a Hilbert series. Our treatment of supersymmetric gauge theories in this thesis is concerned with these Hilbert series. The dimension of a moduli space is given by the degree of the pole at $t = 1$ in the Hilbert series when all $U(1)$ counting fugacities are identified with each other.

Moduli spaces can be irreducible, or they can have more than one branch. In $\mathcal{N} = 2$ theories (in 4D), the branch parametrized by the scalars in hypermultiplets is called the Higgs branch, while that parametrized by those in vector multiplets is called the Coulomb branch. In the master space theories investigated in [51] and [52], the Higgs branch is the one for which the fields generically take non-zero values, while the Coulomb branch generally has some fields fixed at zero and thus has lower dimension. In these cases the Higgs branch is itself reducible into a 'coherent component' and other (linear) branches, which may include the Coulomb branch, and the coherent component can be split off from the linear branches using a technique called 'surgery'; in other cases such as the one-instanton moduli spaces in [42], it is reducible into a coherent component and a centre-of-mass part which spans $\mathbb{C}^2$.

We know from [1, 2] that the moduli spaces for SQCD theories with classical gauge groups and matter in (anti)fundamental representations are irreducible, and we expect the same to be true for those for exceptional gauge groups and for those for classical gauge groups with matter in non-(anti)fundamental representations such as spinors. (It is stated (without proof) in [1] that the (vacuum/mesonic) moduli space of an $SU(N_c)$ supersymmetric QCD theory is irreducible, since it is the symplectic quotient of the irreducible manifold $\mathbb{C}^n$ by a continuous (gauge) group.)

If the numerator of the Hilbert series is palindromic, the moduli space is Calabi-Yau. We leave it to [1, 2, 3] for an explanation. The full (irreducible) moduli spaces in these cases, and the coherent components of the master spaces in [51, 52] and the one-instanton moduli spaces in [42], are Calabi-Yau.

We emphasize that in this thesis we are considering classical moduli spaces. It is noted in [1] that quantum effects cause a dynamically generated runaway superpotential, called the ADS superpotential (after Affleck, Dine and Seiberg) to emerge for $SU(N_c)$ supersymmetric gauge theories with N_f

flavours where $N_f < N_c$, meaning that there is no classical supersymmetric vacuum; when $N_f = N_c$ the form of the gauge-invariant operators and relations are modified by quantum corrections (and the singularity removed) but their numbers are unaffected, and for $N_f > N_c$ the two vacuum moduli spaces coincide. The derivation, following [25], involves first Higgsing on one flavour, secondly (separately) giving that flavour a mass, comparing the results and extending to the case of more Higgsed or massive flavours.

The same effect also occurs for $SO(N_c)$ and $Sp(N_c)$ (with $N_f < N_c + 1$) gauge groups; following [44], we see that it also occurs for E_6 theories with the critical number of flavours (not considering antiflavours) being 4.

In the next section we introduce a specific type of supersymmetric gauge theories called quiver gauge theories, and related structures called brane tilings.

5.2 Quiver gauge theories and diagrams

Quiver gauge theories were originally used to describe the low-energy effective theories of stacks of branes probing orbifold singularities in type IIA or IIB string theory, with the nodes denoting stacks of 'fractional' branes at the various 'states' of the fixed point and the lines being open fundamental string states connecting two such stacks, being in the untwisted sector of the string spectrum if both endpoints lie on the same stack and the twisted sector if they lie on different stacks. The gauge groups are determined by the number of fractional branes and the presence and charge of an orientifold planes; if there is no such plane then all gauge groups are $U(N_i)$ for node i, if there is one then the gauge group corresponding to the stack which is stuck to the orientifold plane is $SO(N_i)$ if the charge is positive and $Sp(N_i)$ if it is negative, with all others remaining as $U(N_i)$.

In the non-SQCD quiver gauge theories considered in [42], all matter fields transform in the fundamental (or vector) representation of the 'from' gauge group and the antifundamental (or vector) representation of the 'to' gauge group, however once we get away from requiring brane pictures we can generalize quiver gauge theories to any symmetry groups and representations.

Spin representations of $SO(N)$ gauge groups can arise out of the fermionic zero modes of the group. Exceptional groups have no brane interpretation and, at least for the E-type groups and fundamental representations, are a pure M-theory effect [35]; with other exceptional groups and representations and non-fundamental and non-spinor representations of classical groups it is not known if they arise even through M-theory.

Quiver gauge theories are described by quiver diagrams, which consist of lines and nodes. In this thesis, and most of the literature, nodes denote vector multiplets in the adjoint representations of symmetry groups (usually circular nodes for gauge groups and square nodes for global groups) and lines denote hypermultiplets (for $\mathcal{N} = 2$) or chiral multiplets ($\mathcal{N} = 1$) in bifundamental or other (not a singlet of either group) representations of the symmetry groups of the two nodes which they join; if a line joins a node to itself, the corresponding multiplet transforms in the adjoint of the gauge group associated with that node (usually - it may transform in the (anti)symmetric 2nd-rank tensor representation instead, as in the $SO(N)$ and $Sp(N)$ cases in [42]) and as a singlet of all other gauge groups. In the $\mathcal{N} = 1$ case the multiplets (treated solely as SUSY multiplets without gauge group indices) are not self-conjugate before CPT conjugates are added and so the lines have a direction (except in the adjoint case where it is irrelevant).

In the (4D) $\mathcal{N} = 4$ case the only multiplet with maximum helicity 1 or less (as is required for gauge theories - gravity theories allow helicities up to 2) is the vector one with maximum helicity 1, so we can only have nodes in the quiver diagram, there cannot be any lines - the only theory is therefore pure SYM. (We recall that $\mathcal{N}_{3d} = 2\mathcal{N}_{4d}$.)

In the $\mathcal{N} = 2$ case we can have both vector- and hypermultiplets with maximum helicity 1 and $\frac{1}{2}$ respectively, so we can have both nodes and lines in the diagram. An $\mathcal{N} = 4$ vector multiplet, necessarily transforming in the adjoint of the gauge group, decomposes into a vector multiplet and a hypermultiplet (including CPT conjugates), of course also both in the adjoint, in $\mathcal{N} = 2$, so a $\mathcal{N} = 4$ quiver diagram, consisting by necessity of a node by itself, becomes an $\mathcal{N} = 2$ quiver diagram consisting of a node and a line linking the node to itself. However, a $\mathcal{N} = 2$ diagram which does not arise from a $\mathcal{N} = 4$ one can have multiple nodes linked by lines.

$\mathcal{N} = 1$ quiver diagrams also consist of both nodes and lines, but this time the lines have direction. An $\mathcal{N} = 2$ vector multiplet decomposes into a vector multiplet and an adjoint chiral multiplet in $\mathcal{N} = 1$ and a hypermultiplet decomposes into two chiral multiplets, one transforming in the same bifundamental representation as the original $\mathcal{N} = 2$ hypermultiplet and the other in its complex conjugate when this is different, so a node becomes a node and a line linking the node to itself and a line becomes two lines in opposite directions. An $\mathcal{N} = 4$ quiver diagram becomes a node with three lines all linking the node to itself.

In both $\mathcal{N} > 1$ cases the quiver gauge theory is uniquely described by the diagram; this is not the case for $\mathcal{N} = 1$ quiver gauge theories, which may have a nonzero superpotential, but this is not specified, at least uniquely, by the diagram. It can only consist of contributions from closed paths in the diagram, because these are the only combinations of fields which are gauge-invariant. The decomposition of an $\mathcal{N} = 2$ quiver diagram into an $\mathcal{N} = 1$ one leads to a unique superpotential which consists of the contributions from all closed paths; however a general $\mathcal{N} = 1$ quiver gauge theory can have any superpotential as long as every term is the contribution from a closed path, including a vanishing superpotential (as in the SQCD theories which are the main focus of this chapter and which we describe later, although there are no closed paths in these diagrams) which is trivially the sum of zero such terms. The superpotential gives rise to F-term constraints.

Quiver gauge theories can also arise through systems of interacting branes, with the amount of supersymmetry preserved dependent on the types and orientations (extended/pointlike directions and angles of intersection) of the D- and NS-branes present in the theory and the gauge groups dependent on the number and separation of each type and orientation of brane and the presence (and charge if present) or absence of an orientifold plane. For there to be any supersymmetry preserved at all, the number of 'Neumann-Dirichlet' directions (those in which one type of brane is extended and the other type pointlike) must be a multiple of 4 for all pairs of different types of branes, and if there are branes of the same type at angles to each other, certain combinations of the angles must vanish [54].

If every field occurs exactly twice in the superpotential, once with a +ve sign and once with a -ve sign, then the superpotential is called 'toric'. In the next subsection we will discuss structures related to and that can arise from quiver diagrams if they meet certain conditions, these are called brane tilings. In these theories the superpotential is always nonzero and is toric by construction.

5.2.1 Brane tilings

In quiver diagrams, a node represents a gauge group and a line a field transforming in the fundamental (or other) representation of the 'from' node and the antifundamental (or other) of the 'to' node, with superpotentials being closed loops in the diagrams, but there is no requirement either to have or not to have the term corresponding to a specific closed loop in the superpotential.

By contrast, a brane tiling is like a 'dual' of a quiver diagram, with faces representing gauge groups (closed loops in the quiver diagram can be thought of as 'faces' for the purpose of this dualization), lines fields (as in quiver diagrams) but thought of as joining two faces rather than two nodes, and nodes superpotential terms, with white nodes corresponding to positive terms in the superpotential with the (assumed) trace taken clockwise round the node, and black nodes negative terms with the trace taken anticlockwise. Because there are two types of nodes, brane tilings are called 'bipartite', and because all lines must join a white node to a black node, all faces have an even number of sides.

The physical interpretation of a brane tiling theory is different, but related, to that of a quiver gauge theory, as described in [38]. There are two types of NS5-branes, one extended in the 012345 directions and the other in 012367, both wrapping a 2-cycle on a 4-torus in directions 4567, and D5-branes (hence they only exist in type IIB string theory) extended in 0123 and wrapping the torus in the 46 directions and a holomorphic curve in the 57 directions. The NS5-branes intersect with the D5-branes, reducing the $\mathcal{N} = 4$ supersymmetry to a chiral $\mathcal{N} = 1$ theory. This picture can be T-dualized on the 4- and 6-directions turning the D5-brane into a D3-brane and the NS5-branes into pure geometry (specifically a (singular) Calabi-Yau 3-fold), which is where the relation to quiver diagrams comes in. The fundamental (F1) strings stretch between two 'stacks' of D5-branes which are actually the same stack but are separated by their intersection with an NS5-brane. All gauge groups are the same by anomaly cancellation, $U(N)$ for N D5-branes. We also see in [38] that the normalization of the R-charge of the superpotential to 0 and the requirement that the theory be superconformal (the beta function vanish) relates elegantly to the Euler number of the brane tiling vanish as must occur on a torus.

Only quiver diagrams that can be made 'periodic' correspond to brane tilings, and then the periodic quiver has to be able to tile a plane for it to be able to be converted into a brane tiling, this arises naturally from the physical interpretation. The diagram can then be dualized to give a brane tiling. All brane tilings with up to 8 terms in the superpotential are enumerated in [29] using a method that first finds all periodic and irreducible quiver diagrams and then dualizes them to brane tilings. (To extend this to tilings with 10 or more terms within a reasonable timescale would take a significant improvement to the algorithm finding them.)

A brane tiling always has a repeated unit, called the fundamental domain; this is easy to see because the D5-branes are wrapped on a torus and the NS5-branes intersect this torus.

The Hilbert series of brane tiling theories are not calculated in the same way as for quiver diagrams, but rather by a different method involving first calculating 'perfect matchings', which are groupings of the edges in the fundamental domain so that each perfect matching contains each node exactly once. Both the F-term and D-term constraints are expressed in terms of the matrix of perfect matchings; the 'master space' or combined mesonic and baryonic moduli space can be obtained by modding out the entire space of possible field values by the F-term constraints, and the mesonic one by modding it out by both the F-term and D-term constraints.

In [31, 30, 28, 26, 39], brane tilings are adapted to M-theory. The physical interpretation is again discussed in [38]. Starting with type IIA string theory and replacing the D5-branes in the type IIB brane tilings with D4-branes, again compactified on a torus in two of the four spatial dimensions, we then go to strong coupling where the theory 'grows' an extra dimension, and as this grows to infinite size the theory becomes M-theory, and the D2-branes that result from T-dualizing on the two toroidal dimensions of the D4-branes become M2-branes with the geometry from the T-dual of the NS5-branes which intersect the torus becoming a Calabi-Yau 4-fold with 8 real dimensions. The Hilbert series is calculated similarly to that for a 'normal' brane tiling, apart from the added detail of integer Chern-Simons levels for each gauge group.

In [27] and [32], brane tilings consisting solely of hexagons are related to orbifolds of $\mathbb{C}^3$ by $\mathbb{Z}_n$ and it is shown that counting them in this way gives the same results as other methods such as toric diagrams with area $\frac{1}{2}n$ and

Hermite normal forms with determinant n.

We will not discuss brane tilings, either the type IIB or M-theory case, further in this thesis, as they are not relevant to the SQCD quiver gauge theories whose discussion forms the bulk of the remainder of this section.

5.3 Supersymmetric QCD

The quiver diagram for SQCD theories is very simple. There is one circular node corresponding to the gauge group and this is joined to one or more $(S)U(N)$ global symmetry groups corresponding to the number of 'flavours' of matter transforming in a specific (irreducible or other) representation of the gauge group. There may also be a line from the gauge group node to itself symbolizing an adjoint field.

There are no closed loops in the diagram (not counting the adjoint field linking the gauge node to itself if present), so the classical superpotential is zero.

Since our focus is on Hilbert series, we will first give an introduction to these, and then discuss how to transition from a (general, not necessarily SQCD) supersymmetric gauge theory to a Hilbert series.

5.3.1 Introduction to Hilbert series

A Hilbert series enumerates elements in a graded algebraic structure such as a ring, module or ideal by grading, with each graded component of grading i being the number of (algebraically or linearly, which are often the same thing) independent (Laurent) polynomials of total degree i in the algebraic structure. In SQCD and other supersymmetric gauge theories the algebraic structure is the chiral ring consisting of gauge invariant operators.

To define a Hilbert series, one must first define a generating function: given a function f from $\mathbb{Z}$ to $\mathbb{Z}$, the generating function $G(z)$ for f is specified by

$$G(z) = \sum_{i=-\infty}^{\infty} f(i) z^i \tag{5.9}$$

For a graded algebraic structure $\mathcal{M}$, the Hilbert series is given by

$$\mathcal{M} = \bigoplus_i \mathcal{M}_i \implies H(t, \mathcal{M}) = \sum_{i=-\infty}^{\infty} dim(\mathcal{M}_i) t^i \tag{5.10}$$

The dimension is understood to be taken over the field over which the algebraic structure acts; in most cases it is the field of complex numbers, $\mathbb{C}$, so the dimension is the complex dimension. In the case of the one-instanton moduli space covered in [42] quaternionic dimensions are also discussed.

The terms generating function and Hilbert series, and also partition function, are often used interchangeably, but they are not identical in meaning and a given generating function or partition function is not necessarily a Hilbert series. In particular, the generating function counting the number of partitions of each positive integer is both a generating function and a partition function, but it is not a Hilbert series, because the coefficients tend to infinity at an exponential rate. The unrefined bosonic string partition function is just this function raised to the 24th power, divided by the nome q, so is again not a Hilbert series.

A characteristic property of a Hilbert series is that it is usually (in particular, in all cases expounded on here) expressible as a rational function, the denominator of which can be written in 'Euler form', i.e. as in the following (showing the form of the whole function):

$$H(t, \mathcal{M}) = \frac{Q(t)}{\prod_i (1 - t^i)^{a_i}} \tag{5.11}$$

[2] There are only a finite number of values of i for which $a_i > 0$; this has the consequence that the coefficients of a Hilbert series never tend to infinity at faster than polynomial rate. The term is not used in physics literature, where such series are referred to as partition functions. Hilbert series are also referred to as Hilbert-Poincaré or simply Poincaré series, however the naming after Hilbert alone is preferred here [19].

In fact Hilbert series can be expressed in such a form in two different ways both of which emphasize the fact that they are singular at $t = 1$, these are

[2] We will see, however, that in some cases to be seen later the expression of the Hilbert series with the denominator in Euler form is not in its lowest terms and that the locations and degrees of the poles are better seen from the expression of the denominator in which rather than factors of the form $(1 - t^n)$ we have the minimum polynomial of $e^{2\pi i/n}$, for example $(1 + t + t^2)$ instead of $(1 - t^3)$. This minimum polynomial is of degree $\phi(n)$ where ϕ is Euler's totient function. In most cases, however, the form with the denominator in Euler form is in its lowest terms and it is certainly easier to visualize. In this section we will assume this is the case and consider cases where it is not individually in the relevant sections later.

called Hilbert series of the first and second kind:

$$\text{First kind:} \qquad H(t,\mathcal{M}) = \frac{Q(t)}{\prod_i (1-t^i)^{a_i}} = \frac{Q(t)}{(1-t)^k(1+\dots)} \tag{5.12}$$

$$\text{Second kind:} \qquad H(t,\mathcal{M}) = \frac{P(t)}{\prod_i (1-t^i)^{b_i}} = \frac{P(t)}{(1-t)^{dim(\mathcal{M})}(1+\dots)} \tag{5.13}$$

$$\text{where} \qquad k = \sum_i a_i \qquad dim(\mathcal{M}) = \sum_i b_i$$

$$\text{In the first form } (1+\dots) = \prod_i \left(1+t+\dots+t^{i-1}\right)^{a_i} = \prod_i \left(\frac{1-t^i}{1-t}\right)^{a_i}$$

The same is true of the second form replacing a_i with b_i. Both $P(t)$ and $Q(t)$ are polynomials with integer coefficients; in the first form, k is the dimension of the embedding space and in the second, $dim(\mathcal{M})$ is that of the manifold.

$P(1)$ must be strictly greater than 0; its quotient by the second $(1+\dots)$ gives the coefficient of the leading pole at $t = 1$ and is the degree of the algebraic variety, which is the total number of degrees of freedom; $Q(1)$ however will be 0 if $k > dim(\mathcal{M})$ strictly as is usually the case.

It is noted in [1] that the expressions often make much more sense if the powers of t in the $(1-t^i)$ terms in the denominator are not all 1 (or the same), because such expressions give much more information about the geometry and other properties of the moduli space; in these cases H is said to be a Hilbert series over a weighted projective space rather than over an ordinary projective variety. For example, in supersymmetric QCD with $SU(N_c)$ gauge group and N_f flavours, mesons are usually weighted by 2 and baryons (if they exist) by N_f, because they are multiplicative combinations of that number of fundamental fields (weighted by 1).

In the next section we will discuss Hilbert series within the context of supersymmetric gauge theories.

5.3.2 SUSY gauge theories: transition to Hilbert series

To go from a supersymmetric gauge theory to a Hilbert series, the fundamental fields of the theory are converted into products of characters of the global and gauge symmetry groups, including the 'counting' $U(1)$s either as a separate weighting or embedded into the global group. Because we need to take symmetrized products (antisymmetrized if the fields are fermionic, in which case we consider the sum separately for bosonic and fermionic fundamental fields), the sum of all products of characters becomes the argument of a plethystic exponential.

If a field is 'frozen' to zero on a specific branch of the moduli space, such as the Φ field denoting the position of the D3-branes in [42], it is not included in the argument of the PE.

The F-term constraints, if they are present, are similarly converted into arguments of a PE to be divided by.

Constraints containing frozen fields are not incorporated into the Hilbert series, similarly to the frozen fields themselves, but constraints resulting from differentiating the superpotential with respect to the frozen fields are included, again as in [42].

This gives the Hilbert series for the F-flat moduli space, or the 'master space' of the theory. Master spaces are investigated for brane tiling theories in [51] and [52], though here they are evaluated using a different method involving 'perfect matchings', and in simpler cases by simple imposition of the F-term constraints and inspection of which fundamental fields are still independent and the relations between those which are not.

Imposing the Wess-Zumino gauge on the vector superfield V^a makes it non-supersymmetric since there are now 5 bosonic degrees of freedom including the auxiliary field D^a and only 4 fermionic ones; this is rectified by imposing the D-term conditions.

The 'mesonic' moduli space is the symplectic quotient of the F-flat moduli space by the gauge group, or its 'ordinary' quotient by the complexification of the gauge group; dividing out by the imaginary part of this corresponds to imposing the D-term conditions, and by the real part to imposing gauge invariance. (The mesonic moduli space is generated solely by mesonic moduli, or those built out of traces of the superfields; the master space is generated both by these and by baryonic moduli, which are built out of determinants of the superfields.)

Imposition of gauge invariance is done by integration over the gauge group manifold, i.e. over the maximal torus with weighting by the Haar measure. This gives the mesonic moduli space (although, in the SQCD cases investigated here, though not in the master spaces covered in [51] and [52], they may contain baryons).

We will first review the results for the classical groups, originally published in [1], [2] and (with an adjoint field, and also for G_2) [3], and then present the as-yet unpublished ones for the exceptional groups, including the case of G_2 without an adjoint. We also present results for groups related to exceptional groups either by following the sequence of removing the rightmost node in the Dynkin diagram ($E_5 = D_5$), Higgsing and/or folding, and aim to show relations between the corresponding Hilbert series. In all cases we compute series for theories both with and without an adjoint field, but we concentrate on the latter case.

In this work, we do not particularly work with fugacities directly, except for those of the $U(1)$ charges counting numbers of fields, except when we, following [1, 2, 3], work with Mathematica and calculate Hilbert series, both refined and unrefined, by residue methods. When this is the case, we simply have each fugacity correspond to a fundamental weight and set the power of the fugacity in each term to be the Dynkin label of that weight as is done for G_2 (with an adjoint) in the last of those papers. For $SO(N)$ groups, we would use the Cartesian basis when working solely with fundamental or adjoint representations, but the Dynkin basis when working with spinor representations, as opposed to in Chapter 4, following [7], where we use the Cartesian basis, but with all weights doubled (or fugacities squared), to avoid using half-integer weights, but to maintain the symmetry between weights and make conversion between $SO(3)^{(D-2)/2}$ and $SO(D-1)$ easier.

In principle the use of the plethystic programme allows one to calculate the whole Hilbert series, either refined or unrefined, analytically using either the residue method as in [1], [2] and [3], an alternative method independent of the number of flavours expounded on in [6], or some other method. However in practice this is often not the case, owing to memory or time constraints when either or both of the gauge group and the number of flavours is large. For the residue method, it can be shown that in the refined case the number of residues increases as $O((\frac{1}{2}dim(R))^{N_f})$ for matter in representation R, and in the unrefined case a similar mushrooming of the number of terms to be summed occurs because of the need to differentiate $N_f - 1$ times (or more even).

An alternative, flavour-independent, method is discussed in [6] where the plethystic exponential and Haar measure were expanded as power series and the coefficient of $\prod_{i=1}^{r} z_i^{-1}$ for all the gauge group fugacities z_i found and expanded in terms of products of complete symmetric polynomials $h_i(t)$ in the global symmetry group fugacities t_i (treating them as $U(N)$ fugacities which can then be unrefined to an $U(1)$ fugacity t or split off into t and $SU(N)$ fugacities). This method reproduces the $SU(N)$ results in [1] and, using the Cartesian basis, the $SO(N)$ and $Sp(N)$ results in and [2], and calculates series for G_2 to high order (up to 40, though the fully refined series is not shown in the paper) and F_4, E_6 and E_7 up to 3, 4 (antiflavours of E_6 are not considered) and 3 flavours respectively where the moduli space is a complete intersection, showing the character expansion up to order 8 in each case. Even this method is limited for the higher exceptional groups, however, with the problem coming from the dimension, i.e. the number of power series that have to be expanded, and the size of the Weyl group, or the number of terms for which coefficients of $\prod_{i=1}^{r} z_i^{-1}$ have to be found, rather than the rank of the group.

However, in this paper, at least for F_4, E_6 and E_7 (and B_3, D_4 and D_5, which are not exceptional), we will first generate the refined series using a program written in LiE [5], as in [4], and then convert them to unrefined series by replacing each character by the dimension of the corresponding representation. The Haar measure is very complicated and thus a large number of terms would have to be evaluated if using either of the previous methods, especially for the refined series of the exceptional groups. In this method, we see from Section 2.4 that the symmetrization of a product of representations of two different groups to a given order is given by the sum of the direct products of the plethysms of the two representations over all Young tableaux with number of boxes equal to the order. (The antisymmetrization of the product representation is given by the sum of the direct products of the plethysm of the first representation over a given Young tableau by the plethysm of the second representation over its transpose.) A sample program is shown in 5.10. We can only, because of time and memory constraints, evaluate the refined series up to some finite order. In this paper, we have gone up to level 24 in the G_2 case, 21 in the F_4 and E_6 cases and 20 for E_7.

We initially computed series with restrictions on the number of flavours, i.e. the maximum height of the Young tableau over which the plethysm of the matter representation is taken, but subsequently switched to a flavour-independent enumeration of the singlets as in [4] and [6]. The first method makes finding invariants easier as one only has to consider representations of $SU(N_f)$ which can be specified by Young tableaux with no more than N_f rows, so we convert the results from the second method into the first form before doing so.

We then use a 'trial and error' method to determine the generators, relations and higher syzygies, also using LiE, finding the lowest ones by inspection and obtaining the rest by repeatedly generating the whole Hilbert series generated by the known generators, comparing with the 'actual' Hilbert series, adding in the new generators and re-generating the Hilbert series; however this is often adequate to determine them fully when the moduli space is either freely generated or a complete intersection and the generators and relations are of lower order than that up to which the series is calculated, and when it is not it is still useful to obtain a great deal of information about them as is done up to order 18 for E_6 (with no antiflavours) and E_7 and order 13 for G_2 in [4]. It is a hard problem, however, to know exactly how high an order one has to go to to be sure to know the whole Hilbert series, and it is not known at the moment.

Invariants, relations and higher syzygies that arise at a certain number of flavours remain in the spectrum as the number of flavours increases, because they are specified by a Young tableau with N_f rows (where N_f is the number of flavours of first occurrence) and this is still a valid Young tableau for higher numbers of flavours.

When performing the plethystic exponential, because the Young tableaux corresponding to the global symmetry group representation and the plethysm of the matter representation of the gauge group are the same, the Young tableaux appearing in the Hilbert series cannot have more rows than the dimension of the representation of the gauge group in which the matter fields transform, however many flavours there are. However, as is seen in [4] for G_2, [1] for $SU(N)$ and [2] for $SO(N)$ and $Sp(N)$, it is possible for relations and higher syzygies to transform in representations of the flavour group corresponding to Young tableaux with more rows than the dimension of the matter representation.

To obtain the plethysm of a general representation R of a group G over a partition λ, one follows the procedure explained in [37] and also described below. This is done in LiE using the *plethysm* function, which takes a partition λ, a character χ (described either by a highest weight or a sum of highest weights with their multiplicities) and a group G; an example program is shown in Section 5.10.

We consider the representation R temporarily as a representation of

$U(dim(R))$ and relabel the terms in the character as $X_i, 1 \leq i \leq dim(R)$. If any weights in R have multiplicity greater than 1, we must introduce separate temporary fugacities in this step and later on map them back to the same weight of R. We take the Schur polynomial $s_\lambda(X)$ of these temporary fugacities, map them back to products of powers of the original fugacities (of G) and lastly decompose this character into characters of irreducible representations of G. This is shown for $\lambda = [2]$ and R the vector of $SO(5)$ in [37], and as follows.

The character expansion of the vector of $SO(5)$ is as follows:

$$\chi_{[1,0]}^{SO(5)}(z_i) = z_1 + \frac{z_2^2}{z_1} + 1 + \frac{z_1}{z_2^2} + \frac{1}{z_1} \tag{5.14}$$

Assigning temporary fugacities $X_i, 1 \leq i \leq 5$ to the terms in this series, we recall that the Schur polynomial over X_i for the partition [2], $s_{[2]}(X_i)$, which is the same as the complete symmetric polynomial $h_2(X_i)$ because this partition has only one row, writing out all terms explicitly, is

$$\begin{aligned} s_{[2]}(X_i) \quad = \quad & X_1^2 + X_1X_2 + X_1X_3 + X_1X_4 + X_1X_5 + X_2^2 + X_2X_3 + X_2X_4 \\ & + X_2X_5 + X_3^2 + X_3X_4 + X_3X_5 + X_4^2 + X_4X_5 + X_5^2 \end{aligned} \tag{5.15}$$

Substituting the terms in the character into this expression, we obtain

$$\begin{aligned} s_{[2]}(\chi_{[1,0]}^{SO(5)}(z_i)) \quad = \quad & z_1^2 + z_2^2 + z_1 + \frac{z_1^2}{z_2^2} + 1 + \frac{z_2^4}{z_1^2} + \frac{z_2^2}{z_1} + 1 \\ & + \frac{z_2^2}{z_1^2} + 1 + \frac{z_1}{z_2^2} + \frac{1}{z_1} + \frac{z_1^2}{z_2^4} + \frac{1}{z_2^2} + \frac{1}{z_1^2} \end{aligned} \tag{5.16}$$

We see that the highest weight in the plethysm is [2,0] in Dynkin label notation; calculating the character of this representation, either via the one-step Weyl character formula or the two-step construction of the weights and Freudenthal's recursion formula, we see that the plethysm decomposes into the character of the [2,0] representation and a singlet. We write this in

Dynkin label form as follows:

$$s_{[2]}(\chi^{SO(5)}_{[1,0]}(z_i)) = \chi^{SO(5)}_{[2,0]}(z_i) + \chi^{SO(5)}_{[0,0]}(z_i) \tag{5.17}$$

The dimension of the moduli space, which is the dimension of the pole at $t = 1$ in the unrefined Hilbert series, is given by the number of matter degrees of freedom that are not 'eaten' when the gauge group G is broken down to its 'unbroken' subgroup H by the Higgs mechanism. This is given by $(\sum_R N_R dim(R)) - dim(G) + dim(H)$, where R sums over all possible matter representations and N_R is the number of 'flavours' of representation R. Usually there are only one or two types of matter fields in a theory, though cases with 3 types have had their Hilbert series computed in the case of $SU(N)$ with an adjoint, N_f fundamentals and the same number of antifundamentals as covered in [3] and the cases covered in this thesis of $SO(8)$ with vectors, spinors and conjugate spinors or one adjoint and both types of spinor, and are also discussed in [34].

Since the gauge group is broken progressively for each added flavour of a given matter representation R until it is broken completely at the number of flavours at which the moduli space becomes a complete intersection (or one fewer) and henceforth remains completely broken, the dimension of the moduli space increases at an increasing rate until this number of flavours is reached and subsequently increases by $dim(R)$ at each step. This 'critical' number of flavours is, except in the case of $SO(N)$ and flavours in the fundamental where it is N (and taking a 'flavour' to mean both a fundamental and an antifundamental in the case of $SU(N)$), given by

$$N_f^{crit} = \frac{I^2(Ad)}{I^2(R_{mat})} \tag{5.18}$$

where $I^2(R)$ is the second Dynkin index for representation R and the matter transforms in representation R_{mat}. The second Dynkin index of a representation R of a group G is given by

$$Tr_R(T^a T^b) = f(G) I^2(R) \delta^{ab} \tag{5.19}$$

where Tr_R is the trace taken over the representation R and T^a (where a is an adjoint index) is the a-th generator (with the representation understood).

The trace over the adjoint is usually denoted simply Tr and that over the (anti)fundamental of $SU(N)$ or $Sp(N)$ or the vector of $SO(N)$ is usually denoted tr. The factor $f(G)$ is $\frac{1}{2}$ for $SU(N)$ groups to normalize $I(R)$ to be 1 for the (anti)fundamental; for $Sp(N)$ it is chosen to set $I(R)$ for the fundamental to 1, and similarly for $SO(N)$ it is 1, chosen to set $I(R)$ for the vector to 2. For other groups and representations the Dynkin indices, along with the dimensions, are shown in Table 5.1. The normalization is chosen so that the adjoint has as its Dynkin index twice the dual Coxeter number. (The conventions of [24] and [72], rather than [5], are used to order the entries. For A_n groups the Dynkin index and dimension remain the same when the order of all entries is reversed.)

The sum of the Dynkin indices of the matter representations given the aforementioned normalization must be even so that the $\mathbb{Z}_2$ anomaly is not violated [3]. This constrains the total number of fundamentals and antifundamentals in $SU(N)$ SQCD, and the number of fundamentals in $Sp(N)$ SQCD, to be even, because the Dynkin index of these representations is 1. We usually only consider $SU(N)$ theories with the same number of fundamentals and antifundamentals as in [1] and [3]. The only other common matter representations that could have odd Dynkin indices are the 2nd-order symmetric and antisymmetric tensors of $SU(N)$, which have indices $N + 2$ and $N - 2$ respectively and are thus odd for odd N, though since there are no invariants of the antisymmetric tensor in this case we do not consider it. The numbers of flavours in theories with $SO(N)$ or exceptional gauge groups is not constrained by this anomaly, because the Dynkin index of any representation is even.

As the number of flavours is increased, the gauge group is further broken (when the Higgs mechanism is applied on all possible flat directions) and eventually the number of flavours is such that the gauge group is broken completely, this normally happens at the aforementioned 'critical' number of flavours (or one less flavour in the case of G_2), and from then on the dimension of the moduli space increases by the same amount, i.e. the dimension of the matter representation(s), with each added flavour. We see that the same applies for the dimension of each individual pole and thus can write a general expression for the unrefined series as a polynomial of degree linear in the number of flavours divided by a product of terms of the form $(1 - t^n)^m$ where m is also linear in the number of flavours for each n, or

G	R	$dim(R)$	$I^2(R)$
A_{n-1}	[1,0,...,0]	n	1
A_{n-1}	[2,0,...,0]	$\frac{n(n+1)}{2}$	$n+2$
A_{n-1}	[0,1,0,...,0]	$\frac{n(n-1)}{2}$	$n-2$
A_{n-1}	[1,0,...,0,1]	n^2-1	$2n$
B_n	[1,0,...,0]	$2n+1$	2
B_n	[0,1,0,...,0]	$n(2n+1)$	$4n-2$
B_n	[0,...,0,1]	2^n	2^{n-2}
C_n	[1,0,...,0]	$2n$	1
C_n	[2,0,...,0]	$n(2n+1)$	$2n+2$
C_n	[0,1,0,...,0]	$n(2n-1)-1$	$2n-2$
D_n	[1,0,...,0]	$2n$	2
D_n	[0,1,0,...,0]	$n(2n-1)$	$4n-4$
D_n	[0,...,0,1]	2^{n-1}	2^{n-3}
D_n	[0,...,0,1,0]	2^{n-1}	2^{n-3}
E_6	[1,0,0,0,0,0]	27	6
E_6	[0,0,0,0,1,0]	27	6
E_6	[0,0,0,0,0,1]	78	24
E_7	[0,0,0,0,0,1,0]	56	12
E_7	[1,0,0,0,0,0,0]	133	36
F_4	[0,0,0,1]	26	6
F_4	[1,0,0,0]	52	18
G_2	[0,1]	7	2
G_2	[1,0]	14	8

Table 5.1: Dynkin indices and dimensions for groups and representations discussed in this thesis

sometimes a product of terms of the form $(1-\omega_{n,k}^{-1}t)^m$ where $\omega_{n,k} = e^{2\pi ik/n}$ and k and n are coprime, or of powers of products of such terms which have integer coefficients but are irreducible over $\mathbb{Z}$. [3]

We use the following notation to denote the fundamental fields of the theories:

- Q_a^i for quark fields in the fundamental representation of $SU(N)$, G_2, F_4, E_6 or E_7 or the vector representation of $SO(N)$ or $Sp(N)$
- $\tilde{Q}_i^a$ for antiquark fields where the fundamental is complex and hence the antifundamental is not the same representation, i.e. $SU(N)$, E_6. We use tildes rather than bars because this is not the complex conjugate of the quark field but rather a separate independent field. (We do use bars for the complex conjugate, i.e. when discussing D-term constraints and Higgsing.)
- S_a^i for (quark) fields in the spinor representation of $SO(N)$ and the 'conjugate' spinor representation of $SO(4n)$ for which both spinor representations are self-conjugate
- S_i^a for (antiquark) fields in the conjugate spinor representation of $SO(4n+2)$ where the conjugate spinor is the complex conjugate of the spinor
- ϕ_{ab} for fields in the adjoint representation of $SO(N)$, $Sp(N)$, G_2, F_4 and E_7 in which the fundamental is self-conjugate, i.e. real ($SO(N)$, G_2 and F_4) or pseudo-real ($Sp(N)$ and E_7)
- ϕ_b^a for fields in the adjoint representation of $SU(N)$ and E_6 in which the fundamental is complex.

In all cases $i, j, \dots$ denote global symmetry group indices and $a, b, \dots$ gauge group ones.

We use the following symbols to denote 'counting' (i.e. $U(1)$) fugacities:

- t to count quark fields

[3] For example, $(1-t^6)$ is expressible as the product $(1-t)(1+t)(1+t+t^2)(1-t+t^2)$, while we also have $1-t^3 = (1-t)(1+t+t^2)$ and $1-t^2 = (1-t)(1+t)$, and sometimes the rational function would not be in its lowest terms if expressed in the first form. For examples see Section 5.6.

- u for antiquarks; this differs from [1] and [3] where $\tilde{t}$ is used
- v for conjugate spinors when there are all 3 types of field (vector, spinor, conjugate spinor; in this thesis we will only do this for $SO(8)$) in the theory (when there are only 2 types of (non-adjoint) matter field, still for $SO(8)$, we consider them to be spinors and conjugate spinors but still use t and u, when there is only one type we consider it to be a vector)
- s to count adjoint fields.

If we 'merge' the counting fugacity with a set of $SU(N)$ fugacities to give a set of $U(N)$ fugacities, we append the subscript i to the counting fugacities.

5.4 Review of results for classical gauge groups

In this section we will set the scene for the results for exceptional and related gauge groups to come in the next sections with a short review of those for classical gauge groups with matter in (anti)fundamental representations both with and without an adjoint field and G_2 with fundamental matter and an adjoint field as discussed in [1], [2] and [3]. We show how to obtain the character expansion, and then discuss how relations and higher syzygies arise from the character expansion in the cases which are not freely generated.

5.4.1 $SU(N)$ gauge group without adjoint

The gauge group is $SU(N_c)$ and the global symmetry group is $U(N_f) \times U(N_f)$, where N_c is the number of colours and N_f the number of flavours. There are two global groups because there are two types of fields: quarks transforming in the fundamental of the global symmetry group and the antifundamental of the gauge group and antiquarks which transform in the antifundamental of the global group and the fundamental of the gauge group, although there are N_f of each.

As we will do with all the cases we consider, we use two methods of describing the refined Hilbert series; firstly using t_i for i between 1 and N_f, the number of flavours, and secondly splitting the $U(N_f)$ into a $U(1)$ part, described by one counting fugacity t, and a $SU(N)$ part, which we describe using Dynkin labels. We usually use the second notation, which is more succinct and explicit in terms of group representations, when describing character expansions.

In [1], the Hilbert series were first obtained using methods from algebraic geometry, and later re-derived using the plethystic programme, where we see that the two methods produce the same results in the cases where the moduli space is either freely generated or a complete intersection, so we use the second method to derive the Hilbert series when it is neither.

The moduli space, which is the (chiral) ring whose graded pieces are counted in the Hilbert series, is parametrized by the gauge-invariant operators of the theory. For $N_f < N_c$, the only possible gauge-invariant operators are the mesons, defined as $M^i_j = Q^i_a \tilde{Q}^a_j$ with summation over the gauge indices; there are N_f^2 of them and they transform in the $[1, 0, \ldots, 0; 0, \ldots, 0, 1]$ representation of the global symmetry group, and there are no relations between them so the moduli space is freely generated. (In other theories, such as instanton theories, there are relations between the mesons, which come from the superpotential, e.g. in the one-instanton case the meson matrix is traceless and squares to zero, i.e. is nilpotent of order 2.)

For $N_f \geq N_c$, we also have baryons, defined by
$B^{i_1 \ldots i_{N_c}} = \epsilon^{a_1 \ldots a_{N_c}} Q^{i_1}_{a_1} \ldots Q^{i_{N_c}}_{a_{N_c}}$, and antibaryons, defined similarly as
$\tilde{B}_{i_1 \ldots i_{N_c}} = \epsilon_{a_1 \ldots a_{N_c}} \tilde{Q}^{a_1}_{i_1} \ldots \tilde{Q}^{a_{N_c}}_{i_{N_c}}$. (The nomenclature echoes the usage in standard particle physics, with mesons consisting of a quark and an antiquark and (anti)baryons of a number of (anti)quarks equal to the number of colours; also mesons are contracted with traces and baryons with determinants or fully antisymmetric tensors.) There are $\binom{N_f}{N_c}$ of each, giving $N_f^2 + 2\binom{N_f}{N_c}$ generators in total.

There are relations between mesons and (anti)baryons even in this SQCD case with no superpotential. Firstly, because of the properties of products of the epsilon tensors, the baryon and antibaryon multiply to give a product of mesons; there are $\binom{N_f}{N_c}^2$ such relations: $B^{i_1 \ldots i_{N_c}} \tilde{B}_{j_1 \ldots j_{N_c}}$. Secondly, for $N_f > N_c$ strictly, dualizing the baryon using the epsilon tensor of the global group and contracting with a meson vanishes on account of antisymmetrizing over $N_c + 1$ flavour indices, and therefore by necessity over the same number of colour indices, so it must vanish because there are only N_c of the latter: $M.{}^*B = 0$, and the same for antibaryons. There are $2N_f\binom{N_f}{N_c+1}$ such relations. For $N_f = N_c$, the dual baryon and antibaryon (there is only one of each) are scalars, so they have no 'inner' product with mesons, but their product is the determinant of the meson matrix: $^*B^*\tilde{B} = det(M)$.

These gauge invariant operators could be seen simply by inspection, and as we will see in Section 5.6, and has been explained in [25], at least for the case of $N_f < N_c$, by consideration of how many generators of the original $SU(N_c)$ gauge group were broken by giving the quark fields vacuum expectation values (VEVs) and therefore how many of the 'original' quark fields were not 'eaten' (one field is eaten per broken generator) but rather left as massless fields in the 'new' theory and basically 'guess' the representations of the residual gauge groups in which they transform. For $N_f < N_c$, the gauge group is broken to $SU(N_c - N_f)$, the number of broken generators is $N_c^2 - 1 - ((N_c - N_f)^2 - 1) = 2N_f N_c - N_f^2$ and therefore the number of fields left massless is N_f^2, which matches exactly the number of mesons and is consistent with the fact that these are the only generators and there are no relations between them. A similar construction applies for the other classical groups as is seen in [2].

For $N_f \geq N_c$, again following [25], the gauge group is broken completely so the number of gauge fields left over is $2N_f N_c - N_c^2 + 1$. For $N_f = N_c$, this equals $N_f^2 + 1$ and is equal to the N_f^2 mesons plus the baryon and antibaryon minus the one relation, but for $N_f > N_c$ there are too many relations to exactly cancel out the extra generators, so we need further back-relations between the primitive relations and the primitive generators, these are called higher syzygies.

These higher syzygies are difficult to calculate, so we cannot simply 'observe' them; we need some way of determining them systematically, which is where the plethystic programme comes in. The plethystic exponential (PE) is a generator for the symmetrization of the fundamental fields to arbitrary orders, and the gauge invariant Hilbert series, which can be refined or unrefined (or partially refined!), is then obtained by integrating over the Haar measure for the gauge group. The plethystic logarithm (PL) can then be used to extract the generators, relations and higher syzygies. When we generalize to the case of exceptional gauge groups, especially the higher ones, or non-(anti)fundamental representations of classical groups, we even need to use plethystics to determine the primitive invariants and relations, and then deduce the residual gauge groups.

As is described in Section 5.3.2, the argument for the plethystic exponential is the sum of all fundamental fields expressed in terms of characters of both the gauge and global groups and weighted by any counting fugacities.

Like the 'usual' exponential, the plethystic exponential of the sum of two arguments is the product of the plethystic exponentials of the arguments taken separately. We will see how this gives the character expansion, and observe how relations and higher syzygies arise in the case of $N_f \geq N_c$, for the refined Hilbert series for $SU(N_c)$ gauge group and $SU(N_f) \times SU(N_f)$ global group.

We recall from Section 2.4 that the symmetrization of the product of two $U(N)$ fundamentals, here $U(N_f)$ and $U(N_c)$ (actually $SU(N_c)$ but we will treat it as $U(N_c)$ for now), written in the form of simple sums of fugacities $t_i, 1 \leq i \leq N_f$ and $z_j, 1 \leq j \leq N_c$, to general order k is given by the sum of the products of the Schur polynomials in the two sets of fugacities separately with the partition being the same in both cases:

$$Sym^k(\sum_{i=1}^{N_f}\sum_{j=1}^{N_c} t_i z_j) = \sum_{|\lambda|=k} s_\lambda(t_i) s_\lambda(z_j) \tag{5.20}$$

When working with character expansions, we decompose the $U(N)$ global group(s) to $U(1) \times SU(N)$ where the $U(1)$ fugacity counts the number of fields. We usually split off the $U(1)$ counting fugacity by setting $t = (\prod_{i=1}^{N-1} t_i)^{1/N}$. This can be done by defining new $SU(N)$ fugacities $z_i = t_i/t$ and seeing that $z_N = (\prod_{i=1}^{N-1} z_i)^{-1}$, or using the Dynkin labels of the weights to determine the powers of the fugacities in each term, with the mapping being

$$\begin{array}{rcl} t_1 & \to & t z_1 \\ t_i & \to & t\dfrac{z_i}{z_{i-1}}, 1 < i < N \\ t_N & \to & t/z_{N-1} \end{array} \tag{5.21}$$

Both methods are explored in [1]. [4]

The reverse mapping obtains the $(t_1, \ldots, t_N)$ powers from those of

[4]A third approach, which keeps characters 'symmetric' in the fugacities, is used in [43]. It treats the $U(N)$ fugacities z_i as independent and imposes the $SU(N)$ condition via a delta function, using the fact that the latter can be expanded as an infinite Laurent power series: $\delta(\prod_{i=1}^N z_i - 1) = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}(\prod_{i=1}^N z_i)^n$. This method is not convenient here though.

$(z_1,\ldots,z_{N-1};t)$ via the matrix $(a_{ij})_{1\leq i,j\leq N}$, where the a_{ij} are given by

$$\begin{aligned} a_{ij} &= \begin{cases} \frac{N-i}{N}, & 1\leq j\leq i \\ \frac{-i}{N}, & i+1\leq j\leq N \end{cases} \\ a_{Ni} &= \frac{1}{N}, \qquad 1\leq i\leq N \end{aligned} \tag{5.22}$$

We do the same thing with the gauge group, though here as the gauge group is $SU(N_c)$ we either absorb the temporary 'counting' fugacity into that for the global group or we ignore it entirely. (We could also describe this process as simply replacing the temporary $(S)U(N)$ fugacities with the terms in the character of the global or gauge group representation, which is called a 'specialization', and multiplying by a counting fugacity if the group is $U(N)$.)

For a group $SU(N)$, the Schur polynomial s_λ, with the fugacities specialized to the terms in the character of the fundamental as above, is just the character of the representation with Dynkin labels $[a_1,a_2,\ldots,a_{N-1}]$ where $a_i=\lambda_i-\lambda_{i+1}$ for $1\leq i\leq N-1$. (This is not the case for other gauge groups, where the plethysm is non-trivial.) There must not be more than N rows in the Young tableau corresponding to the partition. When the field transforms in the antifundamental of that particular (gauge or global) $SU(N)$, the plethysm is given by the complex conjugate of that of the fundamental, i.e. $[a_{N-1},\ldots,a_1]$.

The quark fields transform in the fundamental representation of the first $SU(N_f)$ in the global symmetry group, the antifundamental of the $SU(N_c)$ gauge group and a singlet of the second $SU(N_f)$ in the global group and are counted by the $U(1)$ fugacity t. The antiquarks transform as a singlet of the first $SU(N_f)$, the fundamental of $SU(N_c)$, the antifundamental of the second $SU(N_f)$ and are counted by u.

The plethystic exponential of $t[1,0,\ldots,0]_{SU(N_f)_1}[0,\ldots,0,1]_{SU(N_c)}$, which denotes the quark fields, is given by, in character expansion,

$$\sum_{n_i\geq 0, 1\leq i\leq min(N_f,N_c)} t^{\sum_{i=1}^{min(N_f,N_c)} in_i}[n_1,n_2,\ldots]_{SU(N_f)_1}[\ldots,n_2,n_1]_{SU(N_c)} \tag{5.23}$$

The summation to $min(N_f,N_c)$ occurs because Young tableaux with more rows than this contain antisymmetrizations of either the gauge or global group representation to order more than their dimension and therefore vanish. (One goes from a partition λ to an $SU(N)$ representation as before, i.e. $n_i = \lambda_i - \lambda_{i+1}$.) When $N_f < N_c$, columns with N_f boxes do not contribute to the $SU(N_f)$ Dynkin labels of (half) the global group, but they do to the $U(1)$ charge to which power t is raised, which is the integer of which the corresponding λ is a partition, and also to the $(N_c - N_f)$-th (from the left, N_f-th from the right) $SU(N_c)$ Dynkin labels of the gauge group; the same applies the other way round (the N_c-th Dynkin label) for $N_f > N_c$. When they are equal, columns of length $N_f = N_c$ contribute only to the power of t. (This comes from the fact that the epsilon invariant of $SU(N)$ is not an invariant of $U(N)$, because the determinant of a $U(N)$ matrix is not fixed to be 1, therefore columns of N boxes can contribute to a $U(N)$ tableau, but not an $SU(N)$ one, so they must contribute to the $U(1)$ charge instead.)

The PE of the combined gauge/global representation denoting the antiquark fields is given by the complex conjugate of this expression, counted by u and transforming non-trivially in the second $SU(N_f)$:

$$\sum_{m_i \geq 0, 1 \leq i \leq min(N_f, N_c)} u^{\sum_{i=1}^{min(N_f,N_c)} i m_i} [m_1, m_2, \ldots]_{SU(N_c)} [\ldots, m_2, m_1]_{SU(N_f)_2} \tag{5.24}$$

When the two expansions are tensored together, gauge singlets occur when the gauge group representations in the two terms are conjugates of each other, i.e. they have the same Dynkin labels but with the order reversed.

When $N_f < N_c$, $n_i = m_i = 0$ for $i > N_f$ necessarily, and gauge singlets arise when $n_i = m_i$ for $1 \leq i \leq N_f$. These correspond to a character expansion of

$$\sum_{n_i \geq 0, 1 \leq i \leq N_f} (tu)^{\sum_{i=1}^{N_f} i n_i} [n_1, \ldots, n_{N_f - 1}]_{SU(N_f)_1} [n_{N_f - 1}, \ldots, n_1]_{SU(N_f)_2} \tag{5.25}$$

We see that n_{N_f} does not contribute to the $SU(N_f)$ Dynkin labels, but only to the overall power of tu, so we can factor the dependence on it out as follows:

$$\frac{1}{(1 - (tu)^{N_f})} \sum_{n_i \geq 0, 1 \leq i \leq N_f - 1} (tu)^{\sum_{i=1}^{N_f - 1} i n_i} \times [n_1, \ldots, n_{N_f - 1}]_{SU(N_f)_1} [n_{N_f - 1}, \ldots, n_1]_{SU(N_f)_2} \tag{5.26}$$

When $N_f \geq N_c$, $n_i = m_i$ for $1 \leq i \leq N_c - 1$, n_{N_c} and m_{N_c} can take non-zero values but do not have to be equal (because two $U(N_c)$ Young tableaux with different numbers of leftmost columns of N_c boxes correspond to the same $SU(N_c)$ tableau), and $n_i = m_i = 0$ for $i > N_c$. This leads to a character expansion

$$\begin{aligned} \sum_{n_i \geq 0, 1 \leq i \leq N_c - 1, n_{N_c}, m_{N_c} \geq 0} & (tu)^{\sum_{i=1}^{N_f} i n_i} t^{N_c n_{N_c}} u^{N_c m_{N_c}} \\ & \times [n_1, \ldots, n_{N_c - 1}, n_{N_c}, 0, \ldots, 0]_{SU(N_f)_1} \\ & \times [0, \ldots, 0, m_{N_c}, n_{N_c - 1}, \ldots, n_1]_{SU(N_f)_2} \end{aligned} \tag{5.27}$$

When $N_f = N_c$, this simplifies to

$$\begin{aligned} & \frac{1}{(1 - t^{N_c})(1 - u^{N_c})} \sum_{n_i \geq 0, 1 \leq i \leq N_c - 1} (tu)^{\sum_{i=1}^{N_c} i n_i} \\ & \times [n_1, \ldots, n_{N_c - 1}]_{SU(N_c)_1} [n_{N_c - 1}, \ldots, n_1]_{SU(N_c)_2} \end{aligned} \tag{5.28}$$

We have shown how to obtain the character expansion. We will now discuss how relations and higher syzygies arise from the character expansion in the case of $N_f \geq N_c$.

For $N_f < N_c$, we recall that the character expansion is generated by the mesonic generators $[1, 0, \ldots, 0; 0, \ldots, 0, 1]tu$ (with the semicolon separating representations of the two $SU(N_f)$ representations); we know that there are no others, and no relations, but it is easy to check explicitly that these generate the full expansion. These are also generators for $N_f \geq N_c$, but we also have two generators $[0, \ldots, 0, 1_{N_c}, 0, \ldots, 0; 0, \ldots, 0]t^{N_c}$ and $[0, \ldots, 0; 0, \ldots, 0, 1_{N_c}, 0, \ldots, 0]u^{N_c}$, with the notation denoting that the 1 is in the N_c-th position from the left if before the semicolon and from the right if after, so they are singlets in the case of $N_f = N_c$.

We can easily observe that the coefficient of $[\ldots, 1_{N_c}, \ldots; \ldots, 1_{N_c}, \ldots]t^{N_c}u^{N_c}$ (or the singlet $t^{N_c}u^{N_c}$ in the $N_f = N_c$ case) (here we simplify the notation; all unspecified Dynkin labels are zero) in the character expansion is 1; however it can be constructed from the generators in two ways, firstly as one of the terms in the N_c-th symmetrization of $[1, \ldots; \ldots, 1]tu$, and secondly as a product of the two baryonic generators, so we must subtract one back out as a relation. This is the ${}^*B^*\tilde{B} = det(M)$ relation when $N_f = N_c$ and its generalization to $N_f > N_c$ strictly.

We also know that, for $N_f > N_c$ strictly, the character expansion does not contain any terms with a non-zero Dynkin index in the $N_c + 1$-th or any higher position, either from the left if before the semicolon or from the right if after. However, the product of the meson and the baryon contains a term $[\ldots, 1_{N_c+1}, \ldots; \ldots, 1]t^{N_c+1}u$, which must also be removed as a relation. This is the $M.^*B = 0$ relation (where the star denotes the Hodge dual). The $M.^*\tilde{B} = 0$ relation is similar but with the antibaryon. These are the two primary relations, as we saw earlier.

Following on from the second invariant above, we see at order $t^{N_c+2}u^2$ we have only two representations in the character expansion: $[2, \ldots, 1_{N_c}, \ldots; \ldots, 2]$ and $[0, 1, \ldots, 1_{N_c}, \ldots; \ldots, 1, 0]$. However, products of two symmetrized mesonic generators and one baryonic one give the following sum (understanding the order $t^{N_c+2}u^2$):

$$[2, \ldots, 1_{N_c}, \ldots; \ldots, 2] + [1, \ldots, 1_{N_c+1}, \ldots; \ldots, 2] + [0, 1, \ldots, 1_{N_c}, \ldots; \ldots, 1, 0] \\ +[1, \ldots, 1_{N_c+1}, \ldots; \ldots, 1, 0] + [\ldots, 1_{N_c+2}, \ldots; \ldots, 1, 0]$$

The product of the $M.^*B = 0$ relation with another mesonic generator subtracts the following terms back out (order again understood):

$$[1, \ldots, 1_{N_c+1}, \ldots; \ldots, 2] + [1, \ldots, 1_{N_c+1}, \ldots; \ldots, 1, 0] + [\ldots, 1_{N_c+2}, \ldots; \ldots, 2] \\ +[\ldots, 1_{N_c+2}, \ldots; \ldots, 1, 0]$$

We see that we must add back in $[\ldots, 1_{N_c+2}, \ldots; \ldots, 2]$ to get the desired character expansion. This is our first higher syzygy (only for $N_f > N_c + 1$). The others can be obtained in similar fashion, order by order, though they can also be obtained, along with the primitive invariants and relations, by taking the plethystic logarithm of the Hilbert series.

Returning to the expression of the Hilbert series as rational functions, here are the partially (un)refined and fully unrefined series, calculated using

Mathematica, for $A_2 = SU(3)$ for 3, 4 and 5 flavours:

$$g^{(3,A_2)}(t,u) = \frac{1 + tu + t^2u^2}{1 - t^3)(1 - u^3)(1 - tu)^8} = \frac{1 - t^3u^3}{1 - t^3)(1 - u^3)(1 - tu)^9}$$
$$g^{(3,A_2)}(t,t) = \frac{1 - t + t^2}{(1 - t)^{10}(1 + t)^8(1 + t + t^2)} = \frac{1 + t^3}{(1 - t^2)^9(1 - t^3)}$$
$$= \frac{1 + t^2 + t^4}{(1 - t^2)^8(1 - t^3)^2}$$

$$\begin{aligned} g^{(4,A_2)}(t,u) = (1 - t^3)^{-4}(1 - u^3)^{-4}(1 - tu)^{-12} \times \\ (1 + 4tu - 4t^4u + 10t^2u^2 - 16t^5u^2 + 6t^8u^2 + \\ 4t^3u^3 - 16t^6u^3 + 8t^9u^3 - 4tu^4 + 2t^4u^4 - \\ 4t^7u^4 + 6t^{10}u^4 - 16t^2u^5 + 20t^5u^5 - 4t^8u^5 - \\ 16t^3u^6 + 38t^6u^6 - 16t^9u^6 - 4t^4u^7 + 20t^7u^7 - \\ 16t^{10}u^7 + 6t^2u^8 - 4t^5u^8 + 2t^8u^8 - 4t^{11}u^8 + \\ 8t^3u^9 - 16t^6u^9 + 4t^9u^9 + 6t^4u^{10} - 16t^7u^{10} + \\ 10t^{10}u^{10} - 4t^8u^{11} + 4t^{11}u^{11} + t^{12}u^{12}) \end{aligned}$$

$$g^{(4,A_2)}(t,t) = \frac{(1 + t^2)(1 + 3t^2 + 4t^3 + 7t^4 + 4t^5 + 7t^6 + 4t^7 + 3t^8 + t^{10})}{(1 - t^2)^{12}(1 - t^3)^4}$$

$$
\begin{aligned}
g^{(5,A_2)}(t,u) = {} & (1-t^3)^{-7}(1-u^3)^{-7}(1-tu)^{-16} \times \\
& (1+3t^3+t^6+9tu+2t^4u-16t^7u+45t^2u^2- \\
& 75t^5u^2+15t^8u^2+15t^{11}u^2+3u^3+74t^3u^3- \\
& 292t^6u^3+245t^9u^3-65t^{12}u^3+2tu^4+t^4u^4- \\
& 313t^7u^4+595t^{10}u^4-300t^{13}u^4+50t^{16}u^4- \\
& 75t^2u^5-45t^5u^5+150t^8u^5+210t^{11}u^5- \\
& 315t^{14}u^5+75t^{17}u^5+u^6-292t^3u^6+731t^6u^6- \\
& 210t^9u^6-140t^{12}u^6-35t^{15}u^6+50t^{18}u^6- \\
& 16tu^7-313t^4u^7+1634t^7u^7-2090t^{10}u^7+ \\
& 715t^{13}u^7-35t^{16}u^7+15t^2u^8+150t^5u^8+ \\
& 675t^8u^8-2175t^{11}u^8+1650t^{14}u^8-315t^{17}u^8+ \\
& 245t^3u^9-210t^6u^9-725t^9u^9+100t^{12}u^9+ \\
& 715t^{15}u^9-300t^{18}u^9+595t^4u^{10}-2090t^7u^{10}+ \\
& 1775t^{10}u^{10}+100t^{13}u^{10}-140t^{16}u^{10}-65t^{19}u^{10}+ \\
& 15t^2u^{11}+210t^5u^{11}-2175t^8u^{11}+3900t^{11}u^{11}- \\
& 2175t^{14}u^{11}+210t^{17}u^{11}+15t^{20}u^{11}-65t^3u^{12}- \\
& 140t^6u^{12}+100t^9u^{12}+1775t^{12}u^{12}-2090t^{15}u^{12}+ \\
& 595t^{18}u^{12}-300t^4u^{13}+715t^7u^{13}+100t^{10}u^{13}- \\
& 725t^{13}u^{13}-210t^{16}u^{13}+245t^{19}u^{13}-315t^5u^{14}+ \\
& 1650t^8u^{14}-2175t^{11}u^{14}+675t^{14}u^{14}+150t^{17}u^{14}+ \\
& 15t^{20}u^{14}-35t^6u^{15}+715t^9u^{15}-2090t^{12}u^{15}+ \\
& 1634t^{15}u^{15}-313t^{18}u^{15}-16t^{21}u^{15}+50t^4u^{16}- \\
& 35t^7u^{16}-140t^{10}u^{16}-210t^{13}u^{16}+731t^{16}u^{16}- \\
& 292t^{19}u^{16}+t^{22}u^{16}+75t^5u^{17}-315t^8u^{17}+ \\
& 210t^{11}u^{17}+150t^{14}u^{17}-45t^{17}u^{17}-75t^{20}u^{17}+ \\
& 50t^6u^{18}-300t^9u^{18}+595t^{12}u^{18}-313t^{15}u^{18}+ \\
& t^{18}u^{18}+2t^{21}u^{18}-65t^{10}u^{19}+245t^{13}u^{19}- \\
& 292t^{16}u^{19}+74t^{19}u^{19}+3t^{22}u^{19}+15t^{11}u^{20}+ \\
& 15t^{14}u^{20}-75t^{17}u^{20}+45t^{20}u^{20}-16t^{15}u^{21}+ \\
& 2t^{18}u^{21}+9t^{21}u^{21}+t^{16}u^{22}+3t^{19}u^{22}+t^{22}u^{22})
\end{aligned}
$$

$$
\begin{aligned}
g^{(5,A_2)}(t,t) = (1-t^2)^{-15}(1+t)^{-1}(1-t^3)^{-7} \times \\
(1+t+10t^2+23t^3+68t^4+135t^5+281t^6+446t^7+ \\
695t^8+895t^9+1090t^{10}+1115t^{11}+1090t^{12}+895t^{13}+ \\
695t^{14}+446t^{15}+281t^{16}+135t^{17}+68t^{18}+23t^{19}+ \\
10t^{20}+t^{21}+t^{22})
\end{aligned}
$$

We see that, when the series are partially or fully unrefined, that the fractions simplify. For example, there are 25 generators at order tu and 10 each at orders t^3 and u^3 for the 5-flavour series, but the powers of $(1-tu)$, $(1-t^3)$ and $(1-u^3)$ in the denominator are 16, 7 and 7 respectively.

We see immediately that the totally unrefined series, when written in lowest terms, do not have their denominators in Euler form when the number of flavours is not 4. We also observe that the difference between the degree of the denominator and that of the numerator is equal to the number of degrees of freedom in the fundamental fields, in both the partially refined (counting t and u degrees of freedom separately) and fully unrefined cases:

$$
\begin{aligned}
g^{(3,A_2)}(t,u)(t) &: 1.3+1.0+8.1 = 2+3.3 \\
g^{(3,A_2)}(t,t) &: 10.1+8.1+1.2 = 2+6.3 \\
g^{(4,A_2)}(t,u)(t) &: 4.3+4.0+12.1 = 12+4.3 \\
g^{(4,A_2)}(t,t) &: 16.1+12.1+4.2 = 12+8.3 \\
g^{(5,A_2)}(t,u)(t) &: 7.3+7.0+16.1 = 22+5.3 \\
g^{(5,A_2)}(t,t) &: 15.2+1.1+7.3 = 22+10.3
\end{aligned}
$$

This can be explained in terms of the form of the Molien-Weyl integral, where the plethystic exponential contains one factor of a given fugacity in the denominator for every degree of freedom in a matter field counted by that fugacity and the Haar measure contains no factors of any global symmetry group fugacities. We expect it to also be the case when dealing with fully refined series. (As explained in [1], this also explains why the numerator is palindromic in the non-freely generated cases; in the freely generated cases it is trivially palindromic because it is 1.)

We now move on to the other infinite families of (classical) Lie groups, $SO(N)$ and $Sp(N)$, which were discussed in [2].

5.4.2 $SO(N)$ gauge group without adjoint

In the theory with gauge group $SO(N_c)$, there is only one type of field, transforming in the vector (or fundamental) of the gauge group and the fundamental of the global symmetry group, so the latter is simply $U(N_f)$. The number of flavours can take any value, because the second Dynkin index of the vector of $SO(N_c)$ is 2, which is even.

This time, when using the plethystic exponential and Molien-Weyl integration analytically, we usually use the Cartesian basis for the weights of the fundamental, with the weights being $\pm e_i$, $1 \leq i \leq n$ for $N_c = 2n$ (denoted D_n) and with an extra zero weight for $N_c = 2n+1$ (denoted B_n). The positive roots are $e_i \pm e_j$ for $1 \leq i < j \leq n$, and for B_n, e_i for $1 \leq i \leq n$. (All roots, positive and negative, are expressible as the difference of two weights of the fundamental, i.e. it is the antisymmetric square.)

Returning to character expansions, the plethysm of the vector of $SO(N)$ over a given partition λ contains exactly one singlet when all the λ_i for $1 \leq i \leq N-1$ are even and those for $i > N$ are zero; λ_N can be odd or even, because columns of length N reduce to singlets and can be cancelled out; the column of N boxes is an invariant. This leads to a character expansion of the form, for $N_f < N_c$,

$$\sum_{n_i \geq 0, 1 \leq i \leq N_f} t^{\sum_{i=1}^{N_f} 2in_i} [2n_1, \ldots, 2n_{N_f-1}]_{SU(N_f)} \tag{5.29}$$

and for $N_f = N_c$,

$$\begin{aligned} &\textstyle\sum_{n_i \geq 0, 1 \leq i \leq N_f} t^{\sum_{i=1}^{N_f-1} 2in_i + N_f i_{N_f}} [n_1, \ldots, n_{N_f-1}]_{SU(N_f)} \\ = &\textstyle\frac{1}{(1-t^{N_f})} \sum_{n_i \geq 0, 1 \leq i \leq N_f-1} t^{\sum_{i=1}^{N_f-1} 2in_i} [n_1, \ldots, n_{N_f-1}]_{SU(N_f)} \end{aligned} \tag{5.30}$$

and for $N_f > N_c$,

$$\sum_{n_i \geq 0, 1 \leq i \leq N_c} t^{\sum_{i=1}^{N_c-1} 2in_i + N_c i_{N_c}} [2n_1, \ldots, 2n_{N_c-1}, n_{N_c}, 0, \ldots, 0]_{SU(N_f)} \tag{5.31}$$

We can again derive the generators - $[2, \ldots]t^2$ and (for $N_f \geq N_c$) $[\ldots, 1_{N_c}, \ldots]t^{N_c}$ - the relations, $[\ldots, 2_{N_c}, \ldots]t^{2N_c}$ (for $N_f \geq N_c$) and $[1, \ldots, 1_{N_c+1}, \ldots]t^{N_c+2}$ (for $N_f \geq N_c{+}1$), and the higher syzygies, the first being $[2, \ldots, 1_{N_c+2}, \ldots]t^{N_c+4}$

(for $N_f \geq N_c + 2$), from the character expansion. The generators correspond to $M^{ij} = \delta^{ab} Q^i_a Q^j_b$ (the mesons) and $B^{i_1 \dots i_{N_c}} = \epsilon^{a_1 \dots a_{N_c}} Q^{i_1}_{a_1} \dots Q^{i_{N_c}}_{a_{N_c}}$ (the baryons), and the relations to $BB = M \dots M$ (schematically) and $M.^*B = 0$, similar to the $SU(N)$ case but with differences in detail because there is only one type of fundamental field. For $N_f < N_c$ the number of generators is the number of fields left massless by the Higgsing construction as in the $SU(N)$ case, and again for $N_f > N_c$ the need for higher syzygies can be seen from the over-cancellation of the generators exceeding the dimension of the moduli space by the relations, and so on...

We observe that the $SU(N_f)$ representations in the character expansion of the $SO(N_c)$ SQCD theory can be obtained by adding the Dynkin labels of the two $SU(N_f)$ groups in the $SU(N_c)$ theory, reversing the order of the second set of labels, and setting $u = t$, although only one term is kept when multiple terms in the $SU(N_c)$ expansion coalesce to the same one in the $SO(N_c)$ one, and the same applies with the generators, relations and higher syzygies (again keeping only one if there are multiples). This relates to the fact that the quiver diagram can be formed by 'folding' that of the $SU(N_c)$ theory, taking the symmetric (orientifold) projection and restricting the global symmetry group to its diagonal $SU(N_f)$ subgroup.

Again returning to the expressions of the Hilbert series of rational functions, here are the partially (un)refined and fully unrefined series for $SO(3)$

for 3, 4, 5, 6 and 7 flavours, calculated using Mathematica:

$$
\begin{aligned}
g^{(3,SO(3))}(t) &= \frac{1-t+t^2}{(-1+t)^6(1+t)^5} = \frac{1+t^3}{(1-t^2)^6} \\
g^{(4,SO(3))}(t) &= \frac{1-2t+4t^2-2t^3+t^4}{(1-t)^9(1+t)^7)} = \frac{1+t^2+4t^3+t^4+t^6}{(1-t^2)^9} \\
g^{(5,SO(3))}(t) &= \frac{1-3t+9t^2-9t^3+9t^4-3t^5+t^6}{(1-t)^{12}(1+t)^9} \\
&= \frac{1+3t^2+10t^3+6t^4+6t^5+10t^6+3t^7+t^9}{(1-t^2)^{12}} \\
g^{(6,SO(3))}(t) &= \frac{1-4t+16t^2-24t^3+36t^4-24t^5+16t^6-4t^7+t^8}{(1-t)^15(1+t)^11} \\
&= (1-t^2)^{-15}(1+6t^2+20t^3+21t^4+36t^5+56t^6+36t^7+21t^8+ \\
&\quad 20t^9+6t^{10}+t^{12}) \\
g^{(7,SO(3))}(t) &= \frac{1-5t+25t^2-50t^3+100t^4-100t^5+100t^6-50t^7+25t^8-5t^9+t^{10}}{(1-t)^{18}(1+t)^{13}} \\
&= (1-t^2)^{-18}(1+10t^2+35t^3+55t^4+126t^5+220t^6+225t^7+ \\
&\quad 225t^8+220t^9+126t^{10}+55t^{11}+35t^{12}+10t^{13}+t^{15})
\end{aligned}
$$

Here they are for $B_2 = SO(5)$ with 5, 6 and 7 flavours:

$$
\begin{aligned}
g^{(5,B_2)}(t) &= \frac{1-t+t^2-t^3+t^4}{(1-t)^{15}(1+t)^{14})} = \frac{1+t^5}{(1-t^2)^{15}} \\
g^{(6,B_2)}(t) &= \frac{1-2t+4t^2-6t^3+9t^4-6t^5+4t^6-2t^7+t^8}{(1-t)^{20}(1+t)^{18}} \\
&= \frac{1+t^2+t^4+6t^5+t^6+t^8+t^{10}}{(1-t^2)^{20}} \\
g^{(7,B_2)}(t) &= (1-t)^{-25}(1+t)^{-22}(1-3t+9t^2-19t^3+39t^4-48t^5+56t^6- \\
&\quad 48t^7+39t^8-19t^9+9t^{10}-3t^{11}+t^{12}) \\
&= (1-t^2)^{-25}(1+3t^2+6t^4+21t^5+10t^6+15t^7+15t^8+10t^9+ \\
&\quad 21t^{10}+6t^{11}+3t^{13}+t^{15})
\end{aligned}
$$

Again we observe that the difference between the degree of the denominator and that of the numerator is equal to the number of degrees of freedom in the fundamental fields, and again it is the case that when the Hilbert series is written as a rational function in lowest terms for $N_f \geq N_c$, the denominator is not in Euler form.

5.4.3 $Sp(N)$ gauge group without adjoint

Again there is only one type of field, transforming in the vector (or fundamental) of the gauge group $Sp(N_c)$ and the fundamental of the global symmetry group $U(N_f)$, where N_f must be even (and is often written as $2N_f$) because the second Dynkin index of the fundamental of $Sp(N_c)$ is 1, which is odd. We again use the Cartesian basis for the weights of the fundamental, with the weights being $\pm e_i$, $1 \leq i \leq N_c$ for $Sp(N_c)$. The positive roots are $e_i \pm e_j$ for $1 \leq i \leq j \leq N_c$ and $2e_i$ for $1 \leq i \leq N_c$. In this case all roots, positive and negative, are expressible as the sum of two weights of the fundamental, i.e. it is the symmetric square.

Returning to character expansions, the plethysm of the vector of $Sp(N)$ over a given partition λ contains exactly one singlet when $\lambda_{2i} = \lambda_{2i-1}$ for all integer i, or alternatively when all the λ_i^T are even where λ_T is the transpose of λ. The character expansion is as follows:

$$\sum_{n_i \geq 0, 1 \leq i \leq min(N_f, N_c)} t^{\sum_{i=1}^{min(N_f,N_c)} 2in_i} [0, n_1, 0, \ldots, 0, n_{N_f-1}, 0]_{SU(2N_f)} \quad (5.32)$$

with all odd-indexed Dynkin labels being zero.

This also results from folding the $SU(N_c)$ quiver diagram, but keeping the antisymmetric orientifold projection; the global symmetry group is enhanced to $SU(2N_f)$.

The generators are $[0, 1, 0, \ldots]t^2$; there are no baryons since they break up into products of N_c mesons (which are contracted using a symplectic, i.e. antisymmetric, trace). By inspection, when $N_f \geq N_c$, there is a relation $[\ldots, 1_{2N_c+2}, \ldots]t^{2N_c+2}$ and a second-order syzygy $[1, \ldots, 1_{2N_c+3}, \ldots]t^{2N_c+4}$.

Here are analytic expressions for the Hilbert series for $C_3 = Sp(3)$ with

4, 5 and 6 flavours:

$$
\begin{aligned}
g^{(4,C_3)}(t) &= \frac{1+t^2+t^4+t^6}{(1-t^2)^{27}} = \frac{1-t^8}{(1-t^2)^{28}} \\
g^{(5,C_3)}(t) &= \frac{1+6t^2+21t^4+56t^6+81t^8+81t^{10}+56t^{12}+21t^{14}+6t^{16}+t^{18}}{(1-t^2)^{39}} \\
&= \frac{1-45t^8+99t^{10}-55t^{12}-55t^{18}+99t^{20}-45t^{22}+t^{30}}{(1-t^2)^{45}} \\
g^{(6,C_3)}(t) &= (1-t^2)^{-51}(1+15t^2+120t^4+680t^6+2565t^8+6777t^{10}+ \\
&\quad 12965t^{12}+17775t^{14}+17775t^{16}+12965t^{18}+6777t^{20}+2565t^{22}+ \\
&\quad 680t^{24}+120t^{26}+15t^{28}+t^{30})
\end{aligned}
$$

From the second form of the 5-flavour series, we see the 45 generators at order 2 in the $[0,1,0,\ldots]$ representation, the 45 relations at order 8 in the $[\ldots,0,1,0]$ representation and the 99 second-order syzygies in the $[1,0,\ldots,0,1]$ representation as required, and also the 55 third-order syzygies in the $[2,0,\ldots]$ representation.

Again we observe that the difference between the degree of the denominator and that of the numerator is equal to the number of degrees of freedom in the fundamental fields; however here the denominator is in Euler form.

5.4.4 $SU(N)$, $SO(N)$, $Sp(N)$ and G_2 gauge groups with adjoint

We only give a simple statement of the results here: the $SO(N)$ series with one flavour are freely generated (though still with the gauge group broken completely), those with two flavours and the $SU(N)$, $Sp(N)$ and G_2 series with one flavour are complete intersections, though with the number of relations being N rather than 1 in the $Sp(N)$ case, and all series with higher numbers of flavours are non-complete intersections. We again observe that the difference between the degree of the denominator and that of the numerator, in both the (anti)fundamental and adjoint fugacities, is once again equal to the number of degrees of freedom in the fundamental fields of that type.

Having set the scene, we now move on to the main part of this chapter of this thess, which is the Hilbert series of supersymmetric QCD theories with exceptional gauge groups.

5.5 Hilbert series for exceptional gauge groups

Having set the scene with an overview of the already-published results for the classical Lie groups, we will now present new results detailing Hilbert series of exceptional gauge groups.

We will begin by introducing the exceptional Lie groups, firstly in terms of composition algebras which we will define, and secondly in terms of Dynkin diagrams.

A composition algebra $\mathbb{A}$ is an algebra with a function $n : \mathbb{A} \to \mathbb{R}$ such that

$$\forall a, b \in \mathbb{A}, n(ab) = n(a)n(b) \tag{5.33}$$

When the composition algebra is such that $ab = 0 \implies a = 0$ or $b = 0$, the function n is called the norm and the algebra is called a (normed) division algebra [47]. (A division algebra is also one in which division is possible, for any a and b in $\mathbb{A}$ there are unique elements x and y for which $a = bx$ and $a = yb$. Unless the algebra is commutative, x and y are not necessarily the same.) There are four normed division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, the first being ordered, commutative and associative, the second losing ordering, the third commutativity and the last associativity, though it is still alternative (i.e. associative when two of the three arguments are the same). (At the next level up we have the sedenions, which do not form a division algebra because there are cases of two non-zero sedenions multiplying to give zero, and there are no Lie groups based on them.)

For any division algebra $\mathbb{A}$, the projective space $\mathbb{AP}^n$ is defined as the set of points in $\mathbb{A}^{n+1} - 0^{n+1}$ identified under $(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1})$ for nonzero λ and z_i not all zero in $\mathbb{A}$.

Weighted projective spaces, which are often more useful than 'ordinary' ones when describing moduli spaces and Hilbert series as in [1], are defined similarly: $\mathbb{WAP}^n_{[a_1:\ldots:a_{n+1}]}$ for positive integers a_i is the set of points in $\mathbb{A}^{n+1} - 0^{n+1}$ identified under $(z_1, \ldots, z_{n+1}) \sim (\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+1}} z_{n+1})$ for nonzero λ and z_i not all zero in $\mathbb{A}$.

The simple Lie groups, except for G_2 which is the automorphism group of the imaginary octonions, are defined as the isometry groups (preserving distances, i.e. norms (or their square roots), between points) of projective spaces as follows:

$$\begin{aligned} A_n &= Isom(\mathbb{CP}^{n+1}) \\ B_n &= Isom(\mathbb{RP}^{2n+1}) \\ C_n &= Isom(\mathbb{HP}^n) \\ D_n &= Isom(\mathbb{RP}^{2n}) \\ G_2 &= Aut(Im\mathbb{O}) \\ F_4 &= Isom(\mathbb{OP}^2) \\ E_6 &= Isom((\mathbb{O}\times\mathbb{C})\mathbb{P}^2) \\ E_7 &= Isom((\mathbb{O}\times\mathbb{H})\mathbb{P}^2) \\ E_8 &= Isom((\mathbb{O}\times\mathbb{O})\mathbb{P}^2) \end{aligned}$$

Because the octonions are not associative, one can only go up to two levels in the projective space, thereby giving only a finite number of groups, the exceptional groups.

Another way of defining the exceptional groups is in terms of Dynkin diagrams. Fully connected Dynkin diagrams are constrained by the requirement that all roots be linearly independent. If there is a triple line between two nodes, there can be no other nodes, giving G_2. If there is a double line, there can be any number of nodes all joined by single lines and without branches attached to either node giving B_n or C_n when attached to the long root or the short root respectively, but there can only be one node attached to each by a single line if there are nodes attached to both, giving F_4. If there are only single lines, there can only be one node (if any) attached to three other nodes, and the reciprocals of the number of nodes attached to the central node (including the central node itself) must add up to greater than 1, giving D_n, E_6, E_7 and E_8.

The E_n groups arise in string theory in two ways:

- as the U-duality groups when type II (A or B) string theory or M-theory is compactified down to $(11-n)$ dimensions; while the T-duality group affects only NS-NS fields (the metric, dilaton and Kalb-Ramond fields), the U-duality group mixes NS-NS and R-R fields. When n is 6 or greater, the extra vector fields formed by dualization of the $A_{\mu\nu\rho}$ fields in M-theory or the Kalb-Ramond ($B_{\mu\nu}$) and R-R fields give rise to (complex) representations of the (non-compact forms of) the exceptional groups, and the scalars form representations of their cosets by their maximal compact subgroups. (The 28 vectors in the $n = 7$ case are self-dual giving rise to the **56** fundamental representation of E_7; for $n = 8$ vectors are dual to scalars so there are no 'vectors' as such, and this seems to correspond to the fact that there is no 'fundamental' separate from the adjoint for E_8.) [5]

- $E_8 \times E_8$ is one of the two possible gauge groups for the heterotic string. In the fermionic construction, it arises through allowing one set of 16 oscillators to have (NS or R) boundary conditions independently of the other 16; in terms of $SO(16)$ representations this gives two massless combinations of **120**, the adjoint of SO(16), and **128**, the spinor. Since the massless bosons must be in the adjoint of the gauge group (whatever that is), there must be a group containing $SO(16)$ whose adjoint decomposes to give this; this group is E_8, and there are two copies. In the bosonic construction it is simply one of the only two even self-dual lattices in 16 dimensions, the other being $Spin(32)/\mathbb{Z}_2$.

E_6 has been considered as a possible gauge group for a grand unified theory (GUT), since it contains the standard model gauge group $SU(3) \times SU(2) \times U(1)$ (via $SU(5)$ then $SO(10)$ which are also candidate GUT gauge groups) and has chiral representations. When the $E_8 \times E_8$ heterotic string theory is compactified on a Calabi-Yau 3-fold, one of the E_8 groups is broken down to $E_6 \times SU(3)$ and then to E_6 by imposition of the holonomy.

We will now discuss Hilbert series for the exceptional gauge groups, starting with G_2 and then moving on to F_4, E_6 and E_7. (We do not work with E_8 because it has no fundamental other than its adjoint.)

5.5.1 G_2 gauge group

Because the second Dynkin index of the fundamental is even, having the value 2, $\mathbb{Z}_2$ anomaly cancellation does not require the number of flavours to be even, unlike in the $SU(N)$ case, where it is the total number of quark and antiquark fields that must be even (in all cases investigated so far they

[5] For n between 4 and 7, the vectors transform in the $[0, \ldots, 0, 1, 0]$ representation (using conventions in [24] and [72]); in all cases the scalars transform in the adjoint of E_n cosetted out by its maximal compact subgroup.

have been equal so their sum is necessarily even) and the $Sp(N)$ case where the second Dynkin index of the fundamental is 1.

We reproduced the results of [6] for up to 4 flavours (the complete intersection case). As for $SU(N)$ and $Sp(N)$ (but not $SO(N)$, at least with matter in the vector representation), the first relation occurred at the number of flavours given by $I^2(Ad)/I^2(R_{mat})$, in this case 4, and at the order given by $I^2(Ad)$, which is twice the dual Coxeter number of G_2, in this case 8.

The refined series are as follows:

$$\begin{aligned}
PL(g^{(1,G_2)}(t)) &= t^2 \\
PL(g^{(2,G_2)}(t)) &= [2]t^2 \\
PL(g^{(3,G_2)}(t)) &= [2,0]t^2 + [0,0]t^3 \\
PL(g^{(4,G_2)}(t)) &= [2,0,0]t^2 + [0,0,1]t^3 + [0,0,0]t^4 - [0,0,0]t^8
\end{aligned}$$

The unrefined series for up to 10 flavours (we have calculated them up to

16 flavours) are as follows:

$$
\begin{aligned}
g^{(1,G_2)}(t) &= \frac{1}{1-t^2} \\
g^{(2,G_2)}(t) &= \frac{1}{(1-t^2)^3} \\
g^{(3,G_2)}(t) &= \frac{1}{(1-t^2)^6(1-t^3)} \\
g^{(4,G_2)}(t) &= \frac{1-t^8}{(1-t^2)^{10}(1-t^3)^4(1-t^4)} = \frac{1+t^4}{(1-t^2)^{10}(1-t^3)^4} \\
g^{(5,G_2)}(t) &= (1-t^2)^{-14}(1-t^3)^{-7}(1+t^2+3t^3+6t^4+3t^5+7t^6+8t^7+7t^8+3t^9+ \\
&\quad 6t^{10}+3t^{11}+t^{12}+t^{14}) \\
g^{(6,G_2)}(t) &= (1-t^2)^{-18}(1-t^3)^{-10}\times \\
&\quad (1+3t^2+10t^3+21t^4+30t^5+75t^6+120t^7+165t^8+ \\
&\quad 220t^9+315t^{10}+330t^{11}+330t^{12}+330t^{13}+315t^{14}+ \\
&\quad 220t^{15}+165t^{16}+120t^{17}+75t^{18}+30t^{19}+21t^{20}+ \\
&\quad 10t^{21}+3t^{22}+t^{24}) \\
g^{(7,G_2)}(t) &= (1-t^2)^{-22}(1-t^3)^{-13}\times \\
&\quad (1+6t^2+22t^3+56t^4+132t^5+379t^6+792t^7+ \\
&\quad 1539t^8+2912t^9+5146t^{10}+7902t^{11}+11641t^{12}+ \\
&\quad 16220t^{13}+20727t^{14}+24178t^{15}+27111t^{16}+28308t^{17}+ \\
&\quad 27111t^{18}+24178t^{19}+20727t^{20}+16220t^{21}+11641t^{22}+ \\
&\quad 7902t^{23}+5146t^{24}+2912t^{25}+1539t^{26}+792t^{27}+ \\
&\quad 379t^{28}+132t^{29}+56t^{30}+22t^{31}+6t^{32}+t^{34}) \\
g^{(8,G_2)}(t) &= (1-t^2)^{-26}(1-t^3)^{-16}\times \\
&\quad (1+10t^2+40t^3+125t^4+400t^5+1320t^6+3440t^7+ \\
&\quad 8565t^8+20296t^9+44146t^{10}+87760t^{11}+165885t^{12}+ \\
&\quad 293760t^{13}+484152t^{14}+749168t^{15}+1098065t^{16}+ \\
&\quad 1510640t^{17}+1953290t^{18}+2388256t^{19}+2762723t^{20}+ \\
&\quad 3006160t^{21}+3088820t^{22}+3006160t^{23}+2762723t^{24}+ \\
&\quad 2388256t^{25}+1953290t^{26}+1510640t^{27}+1098065t^{28}+ \\
&\quad 749168t^{29}+484152t^{30}+293760t^{31}+165885t^{32}+ \\
&\quad 87760t^{33}+44146t^{34}+20296t^{35}+8565t^{36}+3440t^{37}+ \\
&\quad 1320t^{38}+400t^{39}+125t^{40}+40t^{41}+10t^{42}+t^{44})
\end{aligned}
$$

$$
\begin{aligned}
g^{(9,G_2)}(t) &= (1-t^2)^{-30}(1-t^3)^{-19} \times \\
& (1 + 15t^2 + 65t^3 + 246t^4 + 975t^5 + 3665t^6 + 11490t^7 + \\
& 34605t^8 + 97745t^9 + 254106t^{10} + 614670t^{11} + \\
& 1406060t^{12} + 3015525t^{13} + 6072051t^{14} + 11549384t^{15} + \\
& 20789430t^{16} + 35353485t^{17} + 56945075t^{18} + 87075099t^{19} + \\
& 126401415t^{20} + 174199340t^{21} + 228323595t^{22} + \\
& 284797560t^{23} + 337968425t^{24} + 381719832t^{25} + \\
& 410741685t^{26} + 420953780t^{27} + 410741685t^{28} + \\
& 381719832t^{29} + 337968425t^{30} + 284797560t^{31} + \\
& 228323595t^{32} + 174199340t^{33} + 126401415t^{34} + \\
& 87075099t^{35} + 56945075t^{36} + 35353485t^{37} + 20789430t^{38} + \\
& 11549384t^{39} + 6072051t^{40} + 3015525t^{41} + 1406060t^{42} + \\
& 614670t^{43} + 254106t^{44} + 97745t^{45} + 34605t^{46} + \\
& 11490t^{47} + 3665t^{48} + 975t^{49} + 246t^{50} + 65t^{51} + 15t^{52} + t^{54}) \\
g^{(10,G_2)}(t) &= (1-t^2)^{-34}(1-t^3)^{-22} \times \\
& (1 + 21t^2 + 98t^3 + 441t^4 + 2058t^5 + 8722t^6 + 31998t^7 + \\
& 112497t^8 + 368138t^9 + 1114707t^{10} + 3162468t^{11} + \\
& 8463202t^{12} + 21284768t^{13} + 50484807t^{14} + 113363042t^{15} + \\
& 241238152t^{16} + 486846282t^{17} + 933833944t^{18} + \\
& 1704845582t^{19} + 2964447333t^{20} + 4914491846t^{21} + \\
& 7776829413t^{22} + 11754525288t^{23} + 16979480803t^{24} + \\
& 23456748996t^{25} + 31009542807t^{26} + 39241573086t^{27} + \\
& 47552486211t^{28} + 55200676926t^{29} + 61398005196t^{30} + \\
& 65438823594t^{31} + 66842005296t^{32} + 65438823594t^{33} + \\
& 61398005196t^{34} + 55200676926t^{35} + 47552486211t^{36} + \\
& 39241573086t^{37} + 31009542807t^{38} + 23456748996t^{39} + \\
& 16979480803t^{40} + 11754525288t^{41} + 7776829413t^{42} + \\
& 4914491846t^{43} + 2964447333t^{44} + 1704845582t^{45} + \\
& 933833944t^{46} + 486846282t^{47} + 241238152t^{48} + \\
& 113363042t^{49} + 50484807t^{50} + 21284768t^{51} + 8463202t^{52} + \\
& 3162468t^{53} + 1114707t^{54} + 368138t^{55} + 112497t^{56} + \\
& 31998t^{57} + 8722t^{58} + 2058t^{59} + 441t^{60} + 98t^{61} + 21t^{62} + t^{64})
\end{aligned}
$$

N_f	$7N_f$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge group
1	7	1	0	6	8	A_2
2	14	3	0	11	3	A_1
3	21	7	0	14	0	$\emptyset$
4	28	15	1	14	0	$\emptyset$

Table 5.2: Numbers of invariants, relations and broken and unbroken generators and unbroken gauge groups for G_2 SQCD theories with N_f flavours of quarks in the fundamental representation

Those up to 8 flavours could be calculated from the unrefined series up to order t^{24} obtained from LiE, since we knew the powers to which $(1-t^2)$ and $(1-t^3)$ were raised in the denominator given the arithmetic progression started at 4 flavours; they agreed with those calculated in [6]. Those for 9 and 10 flavours were calculated, with the patterns continuing, and those for up to 8 flavours checked further, using Mathematica.

As with the $SU(3)$, $SO(3)$ and $SO(5)$ cases checked earlier, the difference between the degree (as a polynomial) of the denominator and that of the numerator is equal to the number of degrees of freedom in the matter fields when the moduli space is not freely generated, i.e. the numerator is not just 1. Since the power of $(1-t^2)$ increases by 4 for each extra flavour, that of $(1-t^3)$ by 3 and the number of matter d.o.f. by 7, that should give an increase of the degree of the numerator by $4.2+3.3-7=10$, and this is indeed the case.

One can express the unrefined series in the following general form:

$$g^{(N_f,G_2)}(t) = \frac{P_{10N_f-36}(t)}{(1-t^2)^{4N_f-6}(1-t^3)^{3N_f-8}}, N_f \geq 4 \qquad (5.34)$$

where $P_N(t)$ denotes an as yet unconstrained polynomial of degree N. It turns out that this polynomial is palindromic, which means the moduli space of the theory is Calabi-Yau. We were able to calculate the Hilbert series exactly in Mathematica up to 16 flavours, and in all cases this formula was satisfied.

The numbers of invariants, relations and broken and unbroken generators and unbroken gauge groups for a given number of flavours are given in Table 5.2:

N_f	$d(2)$	$d(3)$	deg $P(t)$	$dim(\mathcal{M})$
1	1	0	0	1
2	3	0	0	3
3	6	1	0	7
4	10	4	4	14
$N_f \geq 4$	$4N_f - 6$	$3N_f - 8$	$10N_f - 36$	$7N_f - 14$

Table 5.3: Powers of $(1 - t^n)$ in denominator of unrefined Hilbert series for G_2 SQCD theories with N_f flavours with $1 \leq N_f \leq 3$ and upper and lower bounds for 4-flavour case

The dimension of the moduli space, also known as the Krull dimension, is given by the degree of the pole at $t = 1$ when the Hilbert series is written in the form

$$HS(t) = \frac{P(t)}{\prod_n (1 - t^n)^{d(n)}} \tag{5.35}$$

where $P(t)$ is a polynomial with a non-zero value at $t = 1$ and $d(n)$ are functions of N_f (or the various N_R in the case of multiple types of matter field) for each n, taking positive values only for a finite number of positive n and zero otherwise. The dimension of the moduli space is given by the sum of the $d(n)$, i.e. $\sum_n d(n)$.

The rate of increase of the $d(n)$ with the number of flavours follow the same pattern as that of the dimension of the moduli space, increasing at an increasing rate until the 'critical' number of flavours is reached and at a constant rate thereafter. We summarize this information in Table 5.3:

We also, following [4], calculated invariants for the case of arbitrary numbers of flavours using the 'trial and error' approach noted earlier. We found all the invariants, relations and higher syzygies up to order 24 and for up to 20 flavours, i.e. with no more than 20 rows in the Young tableau. We found that they agreed with those found in [4] up to order 11, but were different for orders 12 and 13, which is where [4] stopped. There are over 3000 different invariants (including relations and higher syzygies) up to order 24, some of which occur over 8000 times, so we do not list them here, but we summarize their numbers in Tables 5.4 and 5.5, the first for invariants and even-order higher syzygies and the second for relations and odd-order higher syzygies:

	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[10]	1	4	3	3	1	0	0	0	0	0	0	0	0	0	0	0
[11]	2	5	5	4	4	1	0	0	0	0	0	0	0	0	0	0
[12]	1	2	0	2	0	1	1	0	0	0	0	0	0	0	0	0
[13]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15]	0	1	2	3	3	1	1	0	0	0	0	0	0	0	0	0
[16]	1	22	79	127	127	107	64	30	5	0	0	0	0	0	0	0
[17]	7	109	361	576	628	530	373	210	100	32	6	0	0	0	0	0
[18]	15	184	586	999	1133	1008	728	470	258	125	44	11	1	0	0	0
[19]	14	90	147	133	123	105	89	71	61	46	41	26	9	1	0	0
[20]	3	2	1	0	0	0	0	1	1	1	0	0	0	0	1	0
[21]	0	3	15	28	32	29	23	15	8	4	2	0	0	0	0	0
[22]	1	36	188	519	827	1037	1070	958	609	305	81	11	0	0	0	0
[23]	6	870	11641	44343	85488	108049	103996	81940	55263	31879	15809	6362	1992	372	7	0
[24]	58	5249	54336	199420	391732	515095	511480	418421	293013	181439	98887	48048	20090	7242	2039	429

Table 5.4: Invariants and even-order higher syzygies of G_2 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12]	0	1	1	2	0	0	0	0	0	0	0	0	0	0	0	0
[13]	1	11	20	20	17	10	3	0	0	0	0	0	0	0	0	0
[14]	4	26	45	54	44	34	17	7	1	0	0	0	0	0	0	0
[15]	6	25	43	49	46	33	22	14	7	1	0	0	0	0	0	0
[16]	2	2	0	0	0	0	0	0	0	1	1	0	0	0	0	0
[17]	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
[18]	0	2	6	7	9	6	4	2	1	0	0	0	0	0	0	0
[19]	1	21	136	332	463	494	423	263	118	25	2	0	0	0	0	0
[20]	9	392	2361	5685	7976	8183	6657	4612	2656	1315	501	147	21	0	0	0
[21]	33	1049	6086	15016	22485	23980	20453	14626	9210	5038	2433	966	318	69	9	0
[22]	52	1182	5774	13322	19621	21443	18625	13961	9276	5728	3245	1721	794	301	79	14
[23]	31	141	81	13	1	0	0	2	4	4	3	2	0	5	21	34
[24]	3	11	41	105	145	146	123	95	61	35	18	7	2	0	0	0

Table 5.5: Relations and odd-order higher syzygies of G_2 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

5.5.2 F_4 gauge group

Because the second Dynkin index of the fundamental is again even, this time 6, $\mathbb{Z}_2$ anomaly cancellation does not require the number of flavours to be even.

We reproduced the results of [6] for up to 3 flavours (the complete intersection case). As for $SU(N)$, $Sp(N)$ and G_2 (but not $SO(N)$, at least with matter in the vector representation), the first relation occurred at the number of flavours given by $I^2(Ad)/I^2(R_{mat})$, in this case 3, and at the order given by $I^2(Ad)$, which is twice the dual Coxeter number of F_4, in this case 18.

The refined series are as follows:

$$\begin{aligned}
PL(g^{(1,F_4)}(t)) &= t^2+t^3\\
PL(g^{(2,F_4)}(t)) &= [2]t^2+[3]t^3+[0]t^4\\
PL(g^{(3,F_4)}(t)) &= [2,0]t^2+[3,0]t^3+[0,2]t^4+[0,1]t^5+[0,0]t^6+[0,0]t^9-[0,0]t^{18}
\end{aligned}$$

The unrefined series are as follows:

$$\begin{aligned}
g^{(1,F_4)}(t) &= \frac{1}{(1-t^2)(1-t^3)}\\
g^{(2,F_4)}(t) &= \frac{1}{(1-t^2)^3(1-t^3)^4(1-t^4)}\\
g^{(3,F_4)}(t) &= \frac{1-t^{18}}{(1-t^2)^6(1-t^3)^{10}(1-t^4)^6(1-t^5)^3(1-t^6)(1-t^9)}\\
&= \frac{1+t^9}{(1-t^2)^6(1-t^3)^{10}(1-t^4)^6(1-t^5)^3(1-t^6)}
\end{aligned}$$

Again, the difference between the degree (as a polynomial) of the denominator and that of the numerator is equal to the number of degrees of freedom in the matter fields when the moduli space is not freely generated (here for 3 flavours): $6.2+10.3+6.4+3.5+1.6-3.26=9$ as required.

The numbers of invariants, relations and broken and unbroken generators and unbroken gauge groups for a given number of flavours are given in Table 5.6:

For the case of one adjoint matter field and nothing else, H is always given by $U(1)^{rank(G)}$. This is always completely broken by the addition of more matter fields whatever the representation(s), even in the case of 1 flavour

N_f	$26N_f$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge group
1	26	2	0	24	28	D_4
2	52	8	0	44	8	A_2
3	78	27	1	52	0	$\emptyset$

Table 5.6: Numbers of invariants, relations and broken and unbroken generators and unbroken gauge groups for F_4 SQCD theories with N_f flavours of quarks in the fundamental representation

of matter in the vector representation of $SO(N)$ in which case the moduli space is freely generated, and hence the dimension of the moduli space is given by the sum of those of the non-adjoint matter representations as in [3].

We revisit the empirical observation from the cases of G_2 and classical groups: when the Hilbert series has a non-trivial numerator, i.e. the moduli space is not freely generated but is instead either a complete intersection or a non-complete intersection, the degree of $P(t)$ (as a polynomial, i.e. the highest exponent of t rather than the value of $P(t)$ at $t = 1$) is given by the degree of the denominator as a polynomial minus the number of degrees of freedom, i.e.

$$degP(t) = (\sum_{n=1}^{\infty} nd(n)) - (\sum_{R} N_R dim(R)) \tag{5.36}$$

This is also the case for the Hilbert series of the one-instanton moduli spaces found in [42], and also those for finite groups in [19] with the number of 'degrees of freedom' in the latter case being replaced by the value of N for which the finite group is a subgroup of $SU(N)$.

In the one-instanton case with classical gauge and global symmetry groups, the number of degrees of freedom in bifundamental fields is always given by $2kN$ with the 2 coming from the existence of two types of bifundamental which are complex conjugate to each other in the $(S)U(k)$ (with gauge group $SU(N)$, the $U(1)$ from $U(N)$ being absorbed into the gauge $U(k)$) case and from the global group being $Sp(N)$ (or C_N) if the gauge group is $SO(k)$ and vice versa in the other cases. In the $U(k)$ case there were also two adjoint matter fields with $2k^2$ degrees of freedom, but this is projected down to the symmetric 2nd-rank tensor in the $SO(k)$ gauge group case (because it contains the centre-of-mass piece), giving $k(k+1)$ d.o.f., and the antisymmetric 2nd-rank tensor in the $Sp(k)$ case, giving $2k(2k-1)$ d.o.f. We then subtract twice the dimension of the adjoint, because we have constraints at order t^2 transforming in the adjoint; this gives overall differences in degrees of $2kN$ for the $U(k)$ case, $2k(N+1)$ in the $SO(k)$ case and $2k(N-2)$ in the $Sp(k)$ case. (These are also the dimension of the moduli spaces of these theories.) In all cases the difference in degree is k times the dual Coxeter number of the global symmetry group. It is also shown in the same paper that the same applies, at least for one instanton, for exceptional groups.

The finite subgroups of $SU(2)$ covered were:

- $A_k = \mathbb{Z}_k$,
- $D_{k+2} = Dih(k)$, the dihedral group of order $2k$, with elements representing reflections multiplied by i from their Euclidean forms to give them determinant 1, and
- the exceptional cases E_6, E_7 and E_8, for which the degrees of the numerators were the same as the (dual - for simply laced groups they are the same) Coxeter number of the corresponding Lie group and the orders of the group were the same as the sum of the squares of the Coxeter labels of each node in the extended Dynkin diagram of the group.

Those of $SU(3)$ covered were the infinite families $\mathbb{Z}_n \times \mathbb{Z}_n$, $\Delta(3n^2)$ (including the Valentiner group $\Delta(27)$, or the case $n=3$, covered in [22]) and $\Delta(6n^2)$ and exceptional cases which we will not list here. In all cases listed the moduli space was a complete intersection.

Because of the computational difficulty involved in calculating Hilbert series, whether refined or unrefined, in Mathematica by either the residue method or the method described in [6], we initially tried to calculate it via the same method by which we calculated the unrefined Hilbert series for D_4 with up to 12 vectors (no spinors or conjugate spinors) and G_2 with up to 8 flavours, starting with the character expansion up to some order (24 for G_2 and 22 for D_4), converting each representation to its dimension in the $SU(N)$ group where N is the number of flavours and summing them for each order to get an unrefined series, remembering that the powers of the $(1-t^n)$ terms in the denominator increase at the same rate and so does the highest power of t in the numerator, multiplying the unrefined series by the denominator, taking the series up to half the required power and completing the palindrome, observing that any terms between half the required power and the order to which terms were taken do fit the palindromic pattern. This method is limited by the order to which terms were taken, the highest power in the numerator must not exceed twice this order, hence we could only calculate up to 12 vectors in D_4 and 8 flavours in G_2.

We tried to calculate the Hilbert series for F_4 with 4 flavours by this method, but were unsuccessful. We will see why this was the case in the next few paragraphs.

Following (5.35), we define the $d(n)$ as in the G_2 case, and they have the same properties as in that case. Because the rate of increase of the powers of $(1 - t^n)$ (or at least the minimum polynomials of $e^{2\pi i/n}$, as in the $SU(N)$ and $SO(N)$ cases discussed earlier) increases until the complete intersection is reached, the values of the various $d(n)$ for gauge group F_4 with 4 fundamentals must be bounded below by

$$d(n)|_{N_f=4} \geq 2d(n)|_{N_f=3} - d(n)|_{N_f=2} \tag{5.37}$$

However, they must sum to the dimension of the moduli space, which is $26N_f - 52 = 52$. The remaining 8 must be distributed across the various values of $d(n)$ in some way. They must also be bounded above by the number of primitive invariants at that order which are relevant to the case of that number of flavours, i.e. whose Young tableaux (possibly including leftmost columns of N_f boxes) have N_f rows or fewer, and the upper bound is the sum of the dimensions of all relevant tableaux with n total boxes. For orders (i.e. number of boxes) up to 6, these are summarized in Table 5.7:

One sees that the upper bounds for $d(n), 2 \leq n \leq 6$ must be 10, 20, 20, 20 and 20 (from both primitive invariants at order 6) respectively. Calculating the lower bounds from the values at 2 and 3 flavours, we summarize this information, and the dimension of the moduli space, in Table 5.8:

Knowing that the $d(n)$ must sum to 52, the combination giving the lowest possible degree for the polynomial in the denominator, and hence for the numerator, has $d(n)$ respectively 10, 20, 14, 6 and 2, giving a numerator of degree $10.2 + 20.3 + 14.4 + 6.5 + 2.6 - 26.4 = 74$. Half of this is 37, which is greater than the maximum order to which we calculated the series order by

Order	Young tableau	$SU(4)$ representation	Dimension
2	[2,...]	[2,0,0]	10
3	[3,...]	[3,0,0]	20
4	[0,2,...]	[0,2,0]	20
5	[0,1,1,...]	[0,1,1]	20
6	[0,0,2,...]	[0,0,2]	10
6	[2,0,0,1,...]	[2,0,0]	10

Table 5.7: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of F_4 SQCD theories and the corresponding representations and dimensions in the case $N_f = 4$

N_f	$d(2)$	$d(3)$	$d(4)$	$d(5)$	$d(6)$	deg $P(t)$	$dim(\mathcal{M})$
1	1	1	0	0	0	0	2
2	3	4	1	0	0	0	8
3	6	10	6	3	1	9	26
4	≥ 9 ≤ 10	≥ 16 ≤ 20	≥ 11 ≤ 20	≥ 6 ≤ 20	≥ 2 ≤ 20	?	52
$N_f \geq 3$	?	?	?	?	?	?	$26N_f - 52$

Table 5.8: Powers of $(1 - t^n)$ in denominator of unrefined Hilbert series for F_4 SQCD theories with N_f flavours with $1 \leq N_f \leq 3$ and upper and lower bounds for 4-flavour case

order (21), so we could not use this method to obtain the unrefined series.

We did in the end calculate the unrefined Hilbert series for F_4 with 4 fundamentals using Mathematica (it took 7 days!), and obtained the following expression:

$$
\begin{aligned}
g^{(4,F_4)}(t) = \quad & ((1-t^2)^{10}(1-t^3)^{16}(1-t^4)^{14}(1-t^5)^8(1-t^6)^4)^{-1} \times \\
& (1 + 4t^3 + 6t^4 + 12t^5 + 26t^6 + 44t^7 + 89t^8 + 176t^9 + \\
& 314t^{10} + 556t^{11} + 980t^{12} + 1648t^{13} + 2758t^{14} + 4544t^{15} + \\
& 7243t^{16} + 11344t^{17} + 17460t^{18} + 26244t^{19} + 38812t^{20} + \\
& 56332t^{21} + 80090t^{22} + 111820t^{23} + 153365t^{24} + 206328t^{25} + \\
& 272824t^{26} + 354492t^{27} + 452314t^{28} + 567224t^{29} + \\
& 699270t^{30} + 846968t^{31} + 1008792t^{32} + 1181428t^{33} + \\
& 1360194t^{34} + 1540076t^{35} + 1715048t^{36} + 1877856t^{37} + \\
& 2022566t^{38} + 2142856t^{39} + 2232850t^{40} + 2288704t^{41} + \\
& 2307904t^{42} + 2288704t^{43} + 2232850t^{44} + 2142856t^{45} + \\
& 2022566t^{46} + 1877856t^{47} + 1715048t^{48} + 1540076t^{49} + \\
& 1360194t^{50} + 1181428t^{51} + 1008792t^{52} + 846968t^{53} + \\
& 699270t^{54} + 567224t^{55} + 452314t^{56} + 354492t^{57} + \\
& 272824t^{58} + 206328t^{59} + 153365t^{60} + 111820t^{61} + 80090t^{62} + \\
& 56332t^{63} + 38812t^{64} + 26244t^{65} + 17460t^{66} + 11344t^{67} + \\
& 7243t^{68} + 4544t^{69} + 2758t^{70} + 1648t^{71} + 980t^{72} + \\
& 556t^{73} + 314t^{74} + 176t^{75} + 89t^{76} + 44t^{77} + 26t^{78} + \\
& 12t^{79} + 6t^{80} + 4t^{81} + t^{84})
\end{aligned}
$$

As we see, the $d(n)$ are 10, 16, 14, 8 and 4 respectively, giving a numerator of degree 84.

Recalling the unrefined series for F_4 with 3 flavours, the case of the complete intersection, we obtained the following general expression for the un-

N_f	$d(2)$	$d(3)$	$d(4)$	$d(5)$	$d(6)$	deg $P(t)$	$dim(\mathcal{M})$
$N_f \geq 3$	$4N_f - 6$	$6N_f - 8$	$8N_f - 18$	$5N_f - 12$	$3N_f - 8$	$75N_f - 216$	$26N_f - 52$

Table 5.9: Powers of $(1 - t^n)$ in denominator of unrefined Hilbert series for F_4 SQCD theories with $N_f \geq 3$ flavours

refined Hilbert series of F_4 with more than 3 fundamentals:

$$g^{(N_f,F_4)}(t) = \frac{P_{75N_f-216}(t)}{(1-t^2)^{4N_f-6}(1-t^3)^{6N_f-8}(1-t^4)^{8N_f-18}(1-t^5)^{5N_f-12}(1-t^6)^{3N_f-8}} \quad (5.38)$$

and we can complete the last row of Table 5.8, which we do in Table 5.9: The refined series would have the $d(n)$ equal to the numbers of invariants at each value of n, giving a numerator of degree $10.2+20.3+20.4+20.5+20.6-26.4 = 276$, or 69 in each flavour fugacity. (The $[2, 0, 0, 1, \ldots]$ invariant, which has 4 rows and therefore does not occur in the 3-flavour case, may not be required, which would reduce the degree of the denominator, and therefore the numerator, by 60 to 216 (the numerator), or 54 in each flavour; we will see later that this is indeed the case.)

We also calculated invariants for the case of arbitrary flavour numbers using the 'trial and error' approach noted earlier. This was done for G_2, E_6 and E_7 in [4], but not for F_4.

- The first order at which constraints occur is 11, which again is equal to the sum of the highest (9) and lowest (2) orders of generators in the case of the complete intersection (here 3 flavours). Again, constraints appear at order 11 at the next flavour up (here 4 flavours), and there are no invariants at this order with 4 flavours (indeed we do not find new ones here until we get to 7 flavours).
- There are single-column invariants with 9 and 17 boxes. The column of 26 boxes is not an independent invariant, but rather a product of these two invariants.

Table 5.10 shows the number of invariants of F_4 (including second- and higher even-order syzygies) and Table 5.11 the number of relations (including higher odd-order syzygies) for a specific 'mass' level (i.e. number of fields) having a specific number of rows (i.e. the minimum number of flavours at which they appear) in their Young tableaux. The levels are specified in the first column of the table and the (minimum) number of flavours in the top row.

We see from the first table that the 'primary' invariants follow a diagonal pattern, with the minimum number of flavours at which a new invariant appears with a given number of boxes increasing with the number of boxes until we get the 17-box single column invariant and the 18-box invariant with a column of 16 boxes; up to order 21 there are no more invariants with Young tableaux as much as 13 boxes deep! We expect that these latter 'invariants' may be higher syzygies and therefore that the former ones may be all the primary invariants, and therefore the weighted projective space can be determined in similar fashion to those for $SU(N)$ gauge groups in [1], though it is necessarily more complicated, however the determination of the primary relations or first-order syzygies still has some way to go. The emergence of the relations does begin to follow a diagonal pattern too after order 18 (twice the dual Coxeter number and the order of the relation in the complete intersection case at 3 flavours), however. The first second-order syzygy (linear dependence between primary invariants and relations) occurs at 6 flavours at order 16 and at orders 17 and above (up to 21 at least) they occur at 4 flavours which is the lowest number for which the moduli space is not a complete intersection.

5.5.3 E_6 gauge group

Unlike G_2 and F_4, the Dynkin diagram of E_6 has a symmetry about the axis containing the 3 and 6 nodes (in the notation of [24] and [72]; in that of [5] they are the 4 and 2 nodes) and it has complex representations [6], the $[1,0,0,0,0,0]$ and $[0,0,0,0,1,0]$ being complex conjugates of each other. In this section, we call the first representation the fundamental and fields transforming in it quarks or flavours (denoted by Q_a^i and counted by t) and the second the antifundamental and fields transforming in it antiquarks or antiflavours (denoted by $\tilde{Q}_i^a$ and counted by u).

Because the second Dynkin index of both the fundamental and the an-

[6] This does not always follow from the Dynkin diagram having a symmetry, i.e. for $SO(4n)$ the two spinor representations are inequivalent but they are self-conjugate, as is the vector representation of $SO(8)$ where there is a triality between all three.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
[2]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	1	0	2	3	1	0	1	0	0	0	0	0	0	0	0
[10]	0	0	0	0	0	3	3	1	1	0	0	0	0	0	0	0	0
[11]	0	0	0	0	0	0	5	4	2	0	0	0	0	0	0	0	0
[12]	0	0	0	0	0	0	0	9	5	1	0	0	0	0	0	0	0
[13]	0	0	0	0	0	0	0	0	10	3	1	0	0	0	0	0	0
[14]	0	0	0	0	0	0	0	0	0	5	3	1	0	0	0	0	0
[15]	0	0	0	0	0	0	0	0	0	0	3	2	1	0	0	0	0
[16]	0	0	0	0	0	1	0	0	0	0	0	2	1	1	0	0	0
[17]	0	0	0	2	7	4	3	0	0	0	0	0	0	1	1	0	1
[18]	0	0	0	17	220	331	86	15	0	0	0	0	0	0	0	1	0
[19]	0	0	0	52	1068	3271	3262	1265	106	0	1	0	0	0	0	0	0
[20]	0	0	0	113	2960	12836	20182	16186	7206	902	2	0	0	0	0	0	0
[21]	0	0	0	191	6455	36345	75147	81796	57695	26850	5681	2	0	0	0	0	0

Table 5.10: Invariants and even-order higher syzygies of F_4 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	2	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	5	11	10	0	0	0	0	0	0	0	0	0	0	0	0	0
[13]	0	0	0	7	31	36	25	0	0	0	0	0	0	0	0	0	0	0	0
[14]	0	0	0	10	60	98	93	50	5	0	0	0	0	0	0	0	0	0	0
[15]	0	0	0	14	96	220	258	201	99	19	0	0	0	0	0	0	0	0	0
[16]	0	0	0	16	124	395	584	580	397	179	37	3	0	0	0	0	0	0	0
[17]	0	0	0	7	78	463	1026	1281	1143	711	286	66	10	0	0	0	0	0	0
[18]	0	0	1	2	3	142	930	2002	2430	2000	1139	428	105	18	1	0	0	0	0
[19]	0	0	0	0	0	0	227	1271	3033	3930	3134	1656	592	143	25	3	1	1	0
[20]	0	0	0	0	0	0	0	326	1465	3230	5272	4390	2208	763	185	38	7	3	0
[21]	0	0	0	0	0	0	1	0	452	1381	2202	5336	5389	2733	942	245	58	10	1

Table 5.11: Relations and odd-order higher syzygies of F_4 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

tifundamental is 6, $\mathbb{Z}_2$ anomaly cancellation does not require either the number of each or the total number to be even. Without loss of generality, we take the number of flavours to be greater than or less than the number of antiflavours (we can easily permute t and u if required).

Both the refined and unrefined series are calculated in [6], but only for the case with no antiflavours. Those for up to 4 total flavours are known, however.

For the case with no antiflavours, and up to 4 flavours, the refined series, calculated by inspection of the character expansion, are as follows:

$$
\begin{aligned}
PL(g^{(1,0,E_6)}(t,u)) &= t^3 \\
PL(g^{(2,0,E_6)}(t,u)) &= [3]t^3 \\
PL(g^{(3,0,E_6)}(t,u)) &= [3,0]t^3 + [0,0]t^6 \\
PL(g^{(4,0,E_6)}(t,u)) &= [3,0,0]t^3 + [0,0,2]t^6 + [0,0,0]t^{12} - [0,0,0]t^{24}
\end{aligned}
$$

The unrefined series are as follows:

$$
\begin{aligned}
g^{(1,0,E_6)}(t,u) &= \frac{1}{1-t^3} \\
g^{(2,0,E_6)}(t,u) &= \frac{1}{(1-t^3)^4} \\
g^{(3,0,E_6)}(t,u) &= \frac{1}{(1-t^3)^{10}(1-t^6)} \\
g^{(4,0,E_6)}(t,u) &= \frac{1-t^{24}}{(1-t^3)^{20}(1-t^6)^{10}(1-t^{12})} \\
&= \frac{1+t^{12}}{(1-t^3)^{20}(1-t^6)^{10}}
\end{aligned}
$$

Again, the difference between the degree (as a polynomial) of the denominator and that of the numerator is equal to the number of degrees of freedom in the matter fields when the moduli space is not freely generated (here for 4 flavours): $20.3 + 10.6 - 4.27 = 12$ as required.

The numbers of invariants, relations and broken and unbroken generators of the gauge group and the unbroken gauge groups are listed in Table 5.12:

Because of memory constraints, we were unable to calculate the unrefined Hilbert series for the 5-flavour case. However, as with the case of F_4 with 4 flavours where we were able to calculate the unrefined series, we can calculate lower and upper bounds for the degree of the numerator. As for

(N_f, N_a)	$27N_f$ $+27N_a$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge gp
(1,0)	27	1	0	26	52	F_4
(2,0)	54	4	0	50	28	D_4
(3,0)	81	11	0	70	8	A_2
(4,0)	108	31	1	78	0	$\emptyset$

Table 5.12: Numbers of invariants, relations, broken and unbroken generators and unbroken gauge groups for E_6 SQCD theories with up to 4 flavours and no antiflavours

Order	Young tableau	$SU(5)$ representation	Dimension
3	[3,...]	[3,0,0,0]	35
6	[0,0,2,...]	[0,0,2,0]	50

Table 5.13: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of E_6 SQCD theories and the corresponding representations and dimensions in the case of 5 flavours and no antiflavours

F_4, we have, shifting the number of flavours up by one, and defining the $d(n)$ as in (5.35):

$$d(n)|_{N_f=5} \geq 2d(n)|_{N_f=4} - d(n)|_{N_f=3} \tag{5.39}$$

Again they must sum to the dimension of the moduli space, which is

$27N_f - 78 = 57$. The remaining 8 must be distributed across the various values of $d(n)$ in some way. They must also be bounded above by the number of primitive invariants at that order whose Young tableaux have N_f rows or fewer. Only orders divisible by 3 are relevant to the case of E_6 with no antiflavours; for the series up to 4 flavours the only orders to occur are 3 and 6 (though there is an invariant of order 12 that is absorbed into the relation, and one of order 9 that arises in the 5-flavour case), and these are as in Table 5.13:

One sees that the upper bounds for $d(3)$ and $d(6)$ must be 35 and 50 respectively. Calculating the lower bounds from the values at 3 and 4 flavours, we summarize this information in Table 5.14:

We see by inspection that the case giving the numerator of lowest degree has $(1-t^3)$ raised to power 35 and $(1-t^6)$ to power 22 in the denominator, and this gives a numerator of degree $3.35 + 6.22 - 5.27 = 102$. The highest possible numerator, assuming $(1-t^3)$ and $(1-t^6)$ are the only factors in

N_f	$d(3)$	$d(6)$	deg $P(t)$	$dim(\mathcal{M})$
1	1	0	0	1
2	4	0	0	4
3	10	1	0	11
4	20	10	12	30
5	≥ 30 ≤ 35	≥ 19 ≤ 50	?	57
$N_f \geq 4$	?	?	?	$27N_f - 78$

Table 5.14: Powers of $(1-t^n)$ in denominator of unrefined Hilbert series for E_6 SQCD theories with N_f flavours with $1 \leq N_f \leq 4$ and upper and lower bounds for 5-flavour case

the denominator, occurs for the case of denominator $(1-t^3)^{30}(1-t^6)^{27}$, giving a numerator of degree $3.30 + 6.27 - 5.27 = 117$. The refined series would have degree $3.35 + 6.50 - 5.27 = 270$, or 54 in each flavour fugacity. (We will see in Section 5.6 that this gives an upper bound of 54 too on the degree of the numerator in each flavour fugacity in the F_4 refined series with 4 flavours, meaning that the $[2,0,0,1,\ldots]$ invariant plays no role in the PE/denominator term in this series.)

When we introduce antiflavours, we get the following refined series for up to 4 total flavours:

$$
\begin{aligned}
PL(g^{(1,1,E_6)}(t,u)) &= tu + t^3 + u^3 + t^2u^2 \\
PL(g^{(2,1,E_6)}(t,u)) &= [1]tu + [3]t^3 + [0]u^3 + [2]t^2u^2 + [0]t^4u \\
PL(g^{(3,1,E_6)}(t,u)) &= [1,0]tu + [3,0]t^3 + [0,0]u^3 + [2,0]t^2u^2 + [0,2]t^4u \\
&\quad +[0,1]t^5u^2 + [0,0]t^6 + [0,0]t^9u^3 - [0,0]t^{18}u^6 \\
PL(g^{(2,2,E_6)}(t,u)) &= [1;1]tu + [3;0]t^3 + [0;3]u^3 + [2;2]t^2u^2 + [0;1]t^4u \\
&\quad +[1;0]tu^4 + [1;1]t^3u^3 + [0;0]t^4u^4 + [0;0]t^6u^6 - [0;0]t^{12}u^{12}
\end{aligned}
$$

$(N_f,\overline{N_f})$	$27N_f$ $+27\overline{N_f}$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge gp
(1,1)	54	4	0	50	28	D_4
(2,1)	81	11	0	70	8	A_2
(2,2), (3,1)	108	31	1	78	0	$\emptyset$

Table 5.15: Numbers of invariants, relations, broken and unbroken generators and unbroken gauge groups for E_6 SQCD theories with up to 4 total flavours including at least one antiflavour

The unrefined series are as follows:

$$
\begin{aligned}
g^{(1,1,E_6)}(t,u) &= \frac{1}{(1-tu)(1-t^3)(1-u^3)(1-t^2u^2)} \\
g^{(2,1,E_6)}(t,u) &= \frac{1}{(1-tu)^2(1-t^3)^4(1-u^3)(1-t^2u^2)^3(1-t^4u)} \\
g^{(3,1,E_6)}(t,u) &= (1-t^{18}u^6)(1-tu)^{-3}(1-t^3)^{-10}(1-u^3)^{-1}(1-t^2u^2)^{-6}(1-t^4u)^{-6} \\
&\quad \times(1-t^5u^2)^{-3}(1-t^6)^{-1}(1-t^9u^3)^{-1} \\
g^{(2,2,E_6)}(t,u) &= (1-t^{12}u^{12})(1-tu)^{-4}(1-t^3)^{-4}(1-u^3)^{-4}(1-t^2u^2)^{-9}(1-t^4u)^{-2} \\
&\quad \times(1-tu^4)^{-2}(1-t^3u^3)^{-4}(1-t^4u^4)^{-1}(1-t^6u^6)^{-1}
\end{aligned}
$$

Again, the difference between the degree (as a polynomial) of the denominator and that of the numerator is equal to the number of degrees of freedom in the matter fields when the moduli space is not freely generated, as in the (3,1) and (2,2) cases here:

$$
\begin{aligned}
g^{(3,1,E_6)}(t,u)(t) &: 3.1+10.3+1.0+6.2+6.4+3.5+1.6+1.9 = 18+3.27 \\
g^{(3,1,E_6)}(t,u)(u) &: 3.1+10.0+1.3+6.2+6.1+3.2+1.0+1.3 = 6+1.27 \\
g^{(2,2,E_6)}(t,u)(t) &: 4.1+4.3+4.0+9.2+2.4+2.1+4.3+1.4+1.6 = 12+2.27
\end{aligned}
$$

The numbers of invariants, relations and broken and unbroken generators of the gauge group and the unbroken gauge groups are listed in Table 5.15:

In Table 5.16 we show the invariants corresponding to each term in the denominator and their dimensions in the (4,1) case:

We will revisit the issue of the form of partially refined series later in Section 5.6, but for now we will still present, in Table 5.17, the lower and upper bounds for the powers $d(n,m)$, defined similarly to in (5.35), of the factors in the denominator as in the cases of the fully unrefined series with

Order(t,u)	Young tableau	$SU(4)$ representation	Dimension
(2,2)	[2,...]	[2,0,0]	10
(3,0)	[3,...]	[3,0,0]	20
(4,1)	[0,2,...]	[0,2,0]	20
(5,2)	[0,1,1,...]	[0,1,1]	20
(6,0)	[0,0,2,...]	[0,0,2]	10

Table 5.16: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of E_6 SQCD theories and the corresponding representations and dimensions in the case of 4 flavours and one antiflavour

N_f	$d(0,3)$	$d(1,1)$	$d(2,2)$	$d(3,0)$	$d(4,1)$	$d(5,2)$	$d(6,0)$	deg $P(t,u)$	$dim(\mathcal{M})$
1	1	1	1	1	0	0	0	0	4
2	1	2	3	4	1	0	0	0	11
3	1	3	6	10	6	3	1	9,3	30
4	1	4	≥ 9 ≤ 10	≥ 16 ≤ 20	≥ 11 ≤ 20	≥ 6 ≤ 20	≥ 2 ≤ 10	?, ?	57
$N_f \geq 3$	1	N_f	?	?	?	?	?	?, ?	$27N_f - 51$

Table 5.17: Powers of $(1 - t^n u^m)$ in denominator of unrefined Hilbert series for E_6 SQCD theories with N_f flavours with $0 \leq N_f \leq 3$ and 1 antiflavour and upper and lower bounds for 4-flavour and 1-antiflavour case

only one type of flavour that we have been investigating so far, because these are still the same, though we do not as yet know how the notion of the dimension of the moduli space, i.e. the degree of the pole at $t = 1$, could be determinable from even partially refined series:

In this case, the generator of the chiral ring $[0, 0, 2, \ldots]$ is at order t^6 and the generator $[2, 0, 0, 1, \ldots]$ is at order t^6u^3. Since there are no generators of the second type in the complete-intersection case (3 flavours and 1 antiflavour), we guess for now that it does not occur in the denominator of the (partially) unrefined series for 4 flavours and 1 antiflavour.

We see that the unbroken gauge group depends only on the total number of flavours, not whether they are all flavours or some or all are antiflavours. We do not expect that this should always be the case, however. If we were to set $u = t$ in the partially (i.e. distinguishing flavours and antiflavours) unrefined series to fully unrefine them, we would not obtain the same Hilbert series as for the theory with the same number of total flavours but with all being of the same type. The powers of t and u in any term in either the Taylor series expansion of the Hilbert series or its expression as a rational function always differ by a multiple of 3.

The exponents of $(1-t^2u^2)$, $(1-t^3)$, $(1-t^4u)$, $(1-t^5u^2)$ and $(1-t^6)$ in the denominators of the unrefined Hilbert series of E_6 with N_f flavours and 1 antiflavour are the same as those of $(1-t^n)$, $2 \leq n \leq 6$ in those of the unrefined Hilbert series for F_4 with N_f flavours. That of $(1-tu)$ is N_f, and that of $(1-u^3)$ is 1, and the numerators in the $N_f = 3$ cases are $1+t^9u^3$ and $1+t^9$. Now, we recall that F_4 is the unbroken or residual gauge group when one flavour (or antiflavour) of E_6 is given a vacuum expectation value, each remaining (anti)flavour of E_6 decomposes into one fundamental of F_4 and one scalar giving N_f scalars in total, and there is only one (cubic) fully symmetric invariant of the VEVved (anti)fundamental of E_6. We revisit this in Section 5.6.

As we see later, the absence of the $(1-t^6u^3)$ term in the denominator for the (4,1), i.e. 4 flavours and 1 antiflavour, case could reduce the degree of the numerator of the refined F_4 series for 4 flavours to 216, or 54 in each flavour. We also will see that this upper bound also arises from the 5-flavour, no-antiflavour case for E_6.

We also, following [4], calculated invariants for the case of arbitrary flavour numbers, with no antiflavours, using the 'trial and error' approach noted earlier. We summarize our results as follows:

- We again found the same invariants as in [4] for each mass level up to 18, and with there being constraints at orders 15 and 18, we found the same constraints too.

- We found 28 more invariants at order 21, and 597 (!) constraints.

- The first order at which constraints occur is 15, which again is equal to the sum of the highest (12) and lowest (3) orders of generators in the case of the complete intersection (here 4 flavours). Again, constraints appear at order 15 at the next flavour up (here 5 flavours), and there are no invariants at this order with 5 flavours (indeed we do not find new ones here until we get to 8 flavours).

- In this case, the column of 27 boxes is an independent invariant, since there are no simpler invariants consisting of a single column. We only need to check this up to order 13.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[3]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0
[15]	0	0	0	0	0	0	0	1	3	1	0	0	0	0	0
[18]	0	0	0	0	0	0	0	0	0	3	5	1	0	0	0
[21]	0	0	0	0	1	1	1	0	0	0	0	6	14	4	1

Table 5.18: Invariants and even-order higher syzygies of E_6 SQCD theories with no antiflavours arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15]	0	0	0	0	2	2	1	0	0	0	0	0	0	0	0
[18]	0	0	0	0	3	13	24	17	7	0	1	0	0	0	0
[21]	0	0	0	0	6	35	99	154	158	103	39	2	1	0	0

Table 5.19: Relations and odd-order higher syzygies of E_6 SQCD theories with no antiflavours arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

Table 5.18 shows the number of invariants of E_6 (including second- and higher even-order syzygies) and Table 5.19 the number of relations (including higher odd-order syzygies) for a specific 'mass' level (i.e. number of fields) having a specific number of rows (i.e. the minimum number of flavours at which they appear) in their Young tableaux. The levels are specified in the first column of the table and the (minimum) number of flavours in the top row. (Only the results at order 21 are new, those at lower orders match those in [4].)

We see that the first second-order syzygies occur at order 21, with one each arising at 5, 6 and 7 flavours. As in the F_4 case, there is a diagonal pattern with the primitive invariants in that the minimum number of flavours at which new invariants occur for a given order increases with the order, but in this case there is not an 'end' to the primitive invariants and there does not seem to be one in sight. We expect to have to go to order 33 at least before we can confirm the end, though we know there is an invariant with one column of 27 boxes; the differences with the F_4 case are that invariants occur only at orders divisible by 3 and also that the only single-column invariant is the one of 27 boxes which is the dimension of the fundamental representation of E_6, while in the F_4 case we have single-column invariants with 9 and 17 boxes.

We did the same thing with the number of antiflavours fixed at 1, which was not done in [4].

Tables 5.20, 5.21, 5.22 and 5.23 show the number of invariants of E_6 (including second- and higher even-order syzygies) and Tables 5.24, 5.25, 5.26 and 5.27 show the number of relations (including higher odd-order syzygies) for a specific 'mass' level (i.e. number of fields) having a specific number of rows (i.e. the minimum number of flavours at which they appear; note the difference with the F_4 case, as here we can have 'antiflavours', or flavours of antifundamentals; in this section the number of antiflavours is fixed at 1) in their Young tableaux. The levels, with the number of fundamental fields first and the number of antifundamental fields second, are specified in the first column of the table and the minimum number of flavours (of fundamentals) in the top row. (There is also the $(1 - u^3)$ invariant solely in the antiflavour, which we cannot accommodate in these tables but we state here that it is present.)

There are diagonal patterns similar to in the F_4 case, but they are less clear and harder to visualize in this presentation.

Summing the number of invariants of each type for each number of flavour fields in the invariant over numbers of antiflavour fields, we show in Table 5.28 the numbers of invariants and even-order higher syzygies, and Table 5.29 the numbers of relations and odd-order higher syzygies, for each number of quark fields.

By inspection, with the exception of the invariant at order tu, the number of 'net' invariants (i.e. primary invariants and even-order higher syzygies minus primary relations and odd-order higher syzygies) for a given number of flavours and number of flavour fields in the invariant, summed over the number of antiflavour fields, is the same for the E_6 case with one antiflavour as for the F_4 case. We show this in Tables 5.30 and 5.31 for the case of invariants containing 21 flavour fields (assuming the 'invariants' to be second-order syzygies):

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[1,1]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[2,2]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3,0]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4,1]	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5,2]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,0]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,3]	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,1]	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,4]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,2]	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,5]	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,0]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,3]	0	0	1	0	1	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,6]	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,9]	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
[10,1]	0	0	0	1	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,4]	0	0	0	0	0	1	3	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,7]	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
[11,2]	0	0	0	0	0	3	4	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,5]	0	0	0	0	0	0	1	4	1	0	0	0	0	0	0	0	0	0	0	0

Table 5.20: Invariants and even-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[11,8]	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
[11,11]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,0]	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,3]	0	0	0	0	0	0	5	9	1	0	0	0	0	0	0	0	0	0	0	0
[12,6]	0	0	0	0	0	0	0	0	5	2	0	0	0	0	0	0	0	0	0	0
[12,9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,1]	0	0	0	0	0	0	3	5	1	0	0	0	0	0	0	0	0	0	0	0
[13,4]	0	0	0	0	0	0	0	6	14	2	0	0	0	0	0	0	0	0	0	0
[13,7]	0	0	0	0	0	0	0	0	0	3	2	0	0	0	0	0	0	0	0	0
[13,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,13]	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
[14,2]	0	0	0	0	0	0	0	10	15	2	0	0	0	0	0	0	0	0	0	0
[14,5]	0	0	0	0	0	0	0	0	1	14	4	0	0	0	0	0	0	0	0	0
[14,8]	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
[14,11]	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
[15,0]	0	0	0	0	0	0	0	1	3	1	0	0	0	0	0	0	0	0	0	0
[15,3]	0	0	0	0	0	0	0	0	16	25	4	0	0	0	0	0	0	0	0	0
[15,6]	0	0	0	0	2	1	0	0	0	0	14	7	1	0	0	0	0	0	0	0
[15,9]	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
[15,12]	0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0

Table 5.21: Invariants and even-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[15,15]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,1]	0	0	0	0	0	0	0	0	9	12	2	0	0	0	0	0	0	0	0	0
[16,4]	0	0	0	0	0	0	0	0	0	7	39	12	1	0	0	0	0	0	0	0
[16,7]	0	0	0	2	14	14	5	0	0	0	0	10	8	2	0	0	0	0	0	0
[16,10]	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
[16,13]	0	0	0	0	0	0	0	0	0	2	3	2	1	0	0	0	0	0	0	0
[17,2]	0	0	0	0	0	0	0	0	0	21	37	8	1	0	0	0	0	0	0	0
[17,5]	0	0	0	3	10	3	2	0	0	0	1	53	24	5	1	0	1	0	0	0
[17,8]	0	0	0	5	64	132	120	56	9	0	0	0	0	4	1	0	0	0	0	0
[17,11]	0	0	0	0	0	0	0	0	3	7	4	0	0	0	0	0	0	0	0	0
[17,14]	0	0	0	0	0	0	0	0	0	2	4	2	1	1	0	0	0	0	0	0
[17,17]	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
[18,0]	0	0	0	0	0	0	0	0	0	3	5	1	0	0	0	0	0	0	0	0
[18,3]	0	0	0	1	2	0	0	0	0	0	12	80	27	4	1	0	0	0	0	0
[18,6]	0	0	0	23	292	530	274	32	0	0	0	0	34	27	8	2	1	0	0	0
[18,9]	0	0	0	1	77	346	506	440	229	55	2	0	0	0	0	0	0	0	0	0
[18,12]	0	0	0	0	0	0	0	0	9	26	27	18	2	0	0	0	0	0	0	0
[18,15]	0	0	0	0	0	0	0	0	0	2	4	3	1	2	1	0	0	0	0	0
[18,18]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,1]	0	0	0	0	1	0	0	0	0	0	14	30	9	1	0	0	0	0	0	0

Table 5.22: Invariants and even-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[19,4]	0	0	0	12	128	128	23	1	0	0	1	1	117	54	13	3	1	1	0	0
[19,7]	0	0	0	46	932	3213	3997	2568	692	20	0	0	0	0	12	5	1	0	0	0
[19,10]	0	0	0	0	47	440	1109	1343	1134	695	236	10	0	0	0	0	0	0	0	0
[19,13]	0	0	0	0	0	0	0	0	20	65	89	76	50	23	5	0	0	0	0	0
[19,16]	0	0	0	0	0	0	0	0	0	2	4	4	3	1	0	0	0	0	0	0
[19,19]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,2]	0	0	0	1	7	7	2	0	0	0	1	11	95	38	7	1	0	0	0	0
[20,5]	0	0	0	67	1639	5722	6249	2439	101	0	0	0	0	75	66	24	7	2	0	0
[20,8]	0	0	0	50	1362	7238	14105	14589	9841	3997	474	0	0	0	0	0	0	0	0	0
[20,11]	0	0	0	0	17	371	1545	2695	3014	2688	1876	910	207	8	1	0	0	0	0	0
[20,14]	0	0	0	0	0	0	0	0	34	115	167	142	93	47	13	0	0	0	0	0
[20,17]	0	0	0	0	0	0	0	0	0	2	4	5	5	2	1	0	0	0	0	0
[20,20]	0	0	0	0	0	0	0	0	0	1	2	2	2	2	2	2	2	2	1	0
[21,0]	0	0	0	0	1	1	1	0	0	0	0	6	14	4	1	0	0	0	0	0
[21,3]	0	0	0	11	544	1967	1492	111	0	1	1	0	0	167	90	25	8	2	1	0
[21,6]	0	0	0	163	4928	25285	46283	42637	22907	4597	6	0	0	0	0	29	16	1	0	0
[21,9]	0	0	0	20	985	8934	26815	39401	37435	26053	12936	3240	55	2	0	0	0	0	1	0
[21,12]	0	0	0	0	3	227	1567	3953	5720	6216	5522	3914	2112	721	62	0	0	0	0	0
[21,15]	0	0	0	0	0	0	0	0	61	178	243	203	139	88	49	14	0	0	0	0
[21,18]	0	0	0	0	0	0	0	0	1	8	13	16	17	15	13	12	12	9	3	0
[21,21]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0

Table 5.23: Invariants and even-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 4)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[1,1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[2,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4,1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,4]	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,7]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,5]	0	0	0	2	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.24: Relations and odd-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[11,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,11]	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
[12,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,3]	0	0	0	5	7	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,6]	0	0	0	1	4	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,9]	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
[13,1]	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,4]	0	0	0	6	26	27	15	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,7]	0	0	0	0	3	9	13	11	4	0	0	0	0	0	0	0	0	0	0	0
[13,10]	0	0	0	0	0	0	0	0	1	2	2	0	0	0	0	0	0	0	0	0
[13,13]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,2]	0	0	0	3	19	21	7	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,5]	0	0	0	7	39	70	70	41	1	0	0	0	0	0	0	0	0	0	0	0
[14,8]	0	0	0	0	2	7	16	19	19	10	0	0	0	0	0	0	0	0	0	0
[14,11]	0	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0	0	0	0
[15,0]	0	0	0	0	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,3]	0	0	0	8	62	115	97	40	0	0	0	0	0	0	0	0	0	0	0	0
[15,6]	0	0	0	6	34	99	146	139	87	16	0	0	0	0	0	0	0	0	0	0
[15,9]	0	0	0	0	0	5	14	23	30	28	16	5	0	0	0	0	0	0	0	0
[15,12]	0	0	0	0	0	0	0	0	1	2	1	0	1	0	0	0	0	0	0	0

Table 5.25: Relations and odd-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[15,15]	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
[16,1]	0	0	0	1	14	34	29	8	0	0	0	0	0	0	0	0	0	0	0	0
[16,4]	0	0	0	17	119	308	390	313	140	2	0	0	0	0	0	0	0	0	0	0
[16,7]	0	0	0	0	5	63	159	235	231	155	45	0	0	0	0	0	0	0	0	0
[16,10]	0	0	0	0	0	3	11	24	35	42	35	24	8	0	0	0	0	0	0	0
[16,13]	0	0	0	0	0	0	0	0	1	2	2	1	1	1	0	0	0	0	0	0
[17,2]	0	0	0	5	67	203	265	192	70	1	1	0	0	0	0	0	0	0	0	0
[17,5]	0	0	0	8	78	381	782	924	739	357	44	0	0	0	0	0	0	0	0	0
[17,8]	0	0	0	0	0	10	91	200	308	337	247	98	12	0	0	0	0	0	0	0
[17,11]	0	0	0	0	0	0	7	21	37	44	37	29	24	8	0	0	0	0	0	0
[17,14]	0	0	0	0	0	0	0	0	1	2	3	2	1	1	1	0	0	0	0	0
[17,17]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,0]	0	0	0	0	3	13	24	17	7	0	1	0	0	0	0	0	0	0	0	0
[18,3]	0	0	0	10	150	610	1091	1166	806	323	4	0	0	0	0	0	0	0	0	0
[18,6]	0	0	1	0	1	64	484	1151	1602	1392	746	178	0	0	0	0	0	0	0	0
[18,9]	0	0	0	0	0	0	22	112	216	329	404	332	158	38	2	0	0	0	0	0
[18,12]	0	0	0	0	0	0	3	13	35	40	29	16	8	11	7	0	0	0	0	0
[18,15]	0	0	0	0	0	0	0	0	1	2	4	4	2	2	1	1	0	0	0	0
[18,18]	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0
[19,1]	0	0	0	1	31	154	295	321	215	76	1	0	0	0	0	0	0	0	0	0

Table 5.26: Relations and odd-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[19,4]	0	0	0	5	9	356	1743	3109	3361	2385	1039	124	0	0	0	0	0	0	0	0
[19,7]	0	0	0	0	0	0	56	452	1065	2024	2177	1332	444	41	1	0	0	1	0	0
[19,10]	0	0	0	0	0	0	0	29	105	185	226	301	314	168	42	2	0	0	0	0
[19,13]	0	0	0	0	0	0	0	7	25	38	28	13	8	9	10	7	0	0	0	0
[19,16]	0	0	0	0	0	0	0	0	2	4	6	7	5	3	2	2	3	1	0	0
[19,19]	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
[20,2]	0	0	0	5	65	496	1406	2090	2003	1313	498	19	0	0	0	0	0	0	0	0
[20,5]	0	0	0	0	0	6	312	1692	4796	6846	5403	2627	582	4	0	0	0	0	0	0
[20,8]	0	0	0	0	0	0	1	77	395	812	1708	2689	1937	765	140	12	2	2	0	0
[20,11]	0	0	0	0	0	0	0	0	35	118	144	101	80	158	128	37	0	0	0	0
[20,14]	0	0	0	0	0	0	0	4	19	38	35	17	5	5	4	14	12	2	0	0
[20,17]	0	0	0	0	0	0	0	0	1	4	6	7	6	3	3	2	2	3	1	0
[20,20]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[21,0]	0	0	0	0	6	35	99	154	158	103	39	2	1	0	0	0	0	0	0	0
[21,3]	0	0	0	2	0	33	902	3986	7438	7678	5171	2135	313	1	0	0	0	0	0	0
[21,6]	0	0	0	0	0	0	3	160	1216	3456	9405	9701	5371	1573	149	0	0	0	2	0
[21,9]	0	0	0	1	0	1	8	6	56	271	497	757	1978	2117	961	202	23	2	0	0
[21,12]	0	0	0	0	0	0	0	0	1	40	80	75	35	22	39	117	54	3	0	0
[21,15]	0	0	0	0	0	0	0	0	12	36	48	39	24	14	8	6	17	17	4	0
[21,18]	0	0	0	0	0	0	0	0	0	0	2	4	4	3	1	0	0	0	0	0
[21,21]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.27: Relations and odd-order higher syzygies of E_6 SQCD theories with one antiflavour arranged by total number of boxes in flavour Young tableau and single-row antiflavour tableau (down) and minimum number of flavours of fundamentals (across) (part 4)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[1]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[2]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	1	0	2	3	1	0	1	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	1	1	3	3	1	1	0	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	0	0	3	5	4	2	1	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	1	0	1	6	9	6	2	0	0	0	0	0	0	0	0	0	0
[13]	0	0	0	0	0	0	3	11	15	5	3	0	0	0	0	0	0	0	0	0
[14]	0	0	0	0	0	0	0	10	16	17	5	2	0	0	0	0	0	0	0	0
[15]	0	0	0	0	2	1	0	1	19	27	20	8	2	0	0	0	0	0	0	0
[16]	0	0	0	2	14	14	5	0	10	22	45	24	10	2	0	0	0	0	0	0
[17]	0	0	0	8	74	135	122	56	12	30	46	63	27	10	2	0	1	0	0	0
[18]	0	0	0	25	371	876	780	472	238	86	50	102	64	33	10	2	1	0	0	0
[19]	0	0	0	58	1108	3781	5129	3912	1846	782	344	121	179	79	30	8	2	1	0	0
[20]	0	0	0	118	3025	13338	21901	19723	12990	6803	2524	1070	402	172	90	27	9	4	1	0
[21]	0	0	0	194	6461	36414	76158	86102	66124	37053	18721	7379	2337	997	216	80	36	12	5	0

Table 5.28: Invariants and even-order higher syzygies of E_6 SQCD theories with one antiflavour summed over number of boxes in single-row antiflavour Young tableau arranged by total number of boxes in flavour tableau (down) and minimum number of flavours of fundamentals (across)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[1]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	2	3	3	0	0	0	1	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	6	11	11	6	0	1	1	0	0	0	0	0	0	0	0	0	0
[13]	0	0	0	7	31	36	28	11	5	2	2	0	0	0	0	0	0	0	0	0
[14]	0	0	0	10	60	98	93	60	21	12	2	1	0	0	0	0	0	0	0	0
[15]	0	0	0	14	98	221	258	202	118	46	17	6	1	0	0	0	0	0	0	0
[16]	0	0	0	18	138	408	589	580	407	201	82	25	9	1	0	0	0	0	0	0
[17]	0	0	0	13	145	594	1145	1337	1155	741	332	129	37	9	1	0	0	0	0	0
[18]	0	0	1	10	154	687	1624	2459	2668	2086	1189	530	169	51	11	1	1	0	0	0
[19]	0	0	0	6	40	510	2094	3918	4773	4712	3477	1777	771	222	55	11	3	2	0	0
[20]	0	0	0	5	65	502	1719	3863	7249	9131	7794	5460	2610	935	275	65	16	7	1	0
[21]	0	0	0	3	6	69	1012	4306	8881	11584	15242	12713	7726	3730	1158	325	94	22	6	0

Table 5.29: Relations and odd-order higher syzygies of E_6 SQCD theories with one antiflavour summed over number of boxes in single-row antiflavour Young tableau arranged by total number of boxes in flavour tableau (down) and minimum number of flavours of fundamentals (across)

	4	5	6	7	8	9	10
F_4 2nd-order	191	6455	36345	75147	81796	57695	26850
F_4 relations	0	0	0	1	0	452	1381
E_6 2nd-order	194	6461	36414	76158	86102	66124	37053
E_6 relations	3	6	69	1012	4306	8881	11584

Table 5.30: Comparison of numbers of net invariants at order 21 in flavours for E_6 with one antiflavour and F_4 (part 1)

	11	12	13	14	15	16	17	18	19	20
F_4 2nd-order	5681	2	0	0	0	0	0	0	0	0
F_4 relations	2202	5336	5389	2733	942	245	58	10	1	0
E_6 2nd-order	18721	7379	2337	997	216	80	36	12	5	0
E_6 relations	15242	12713	7726	3730	1158	325	94	22	6	0

Table 5.31: Comparison of numbers of net invariants at order 21 in flavours for E_6 with one antiflavour and F_4 (part 2)

Though the analysis is too long to present here, the actual invariants (including relations and higher syzygies), again excluding that at order tu, also match when those for E_6 are summed over the number of antiflavour fields in the invariant. This is more evidence that suggests the 'Higgsing' that we will discuss later in Section 5.6.

5.5.4 E_7 gauge group

Because the second Dynkin index of the fundamental is again even, this time 12, $\mathbb{Z}_2$ anomaly cancellation does not require the number of flavours to be even.

The refined and unrefined series for up to 3 flavours are known results, though they are not in [6].

The refined series are as follows:

$$\begin{aligned}
PL(g^{(1,E_7)}) &= t^4 \\
PL(g^{(2,E_7)}) &= [0]t^2 + [4]t^4 + [0]t^6 \\
PL(g^{(3,E_7)}) &= [0,1]t^2 + [4,0]t^4 + [0,3]t^6 + [2,0]t^8 + [0,0]t^{12} + [0,0]t^{18} - [0,0]t^{36}
\end{aligned}$$

N_f	$56N_f$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge group
1	56	1	0	55	78	E_6
2	112	7	0	105	28	D_4
3	168	36	1	133	0	$\emptyset$

Table 5.32: Numbers of invariants, relations, broken and unbroken generators and unbroken gauge groups for E_7 SQCD theories with up to 3 flavours

The unrefined series are as follows:

$$
\begin{aligned}
g^{(1,E_7)} &= \frac{1}{1-t^4} \\
g^{(2,E_7)} &= \frac{1}{(1-t^2)(1-t^4)^5(1-t^6)} \\
g^{(3,E_7)} &= \frac{1-t^{36}}{(1-t^2)^3(1-t^4)^{15}(1-t^6)^{10}(1-t^8)^6(1-t^{12})(1-t^{18})} \\
&= \frac{1+t^{18}}{(1-t^2)^3(1-t^4)^{15}(1-t^6)^{10}(1-t^8)^6(1-t^{12})}
\end{aligned}
$$

Again, the difference between the degree (as a polynomial) of the denominator and that of the numerator is equal to the number of degrees of freedom in the matter fields when the moduli space is not freely generated (here for 3 flavours): $3.2 + 15.4 + 10.6 + 6.8 + 1.12 = 18 + 3.56$ as required.

The numbers of invariants, relations and broken and unbroken generators of the gauge group and the unbroken gauge groups are listed in Table 5.32:

Because of memory constraints, we were again unable to calculate the unrefined Hilbert series for the 4-flavour case. However, we can again calculate lower and upper bounds for the degree of the numerator. As for F_4 (and E_6 with one more flavour), we have, again defining then $d(n)$ as in 5.35,

$$d(n)|_{N_f=4} \geq 2d(n)|_{N_f=3} - d(n)|_{N_f=2} \tag{5.40}$$

Again they must sum to the dimension of the moduli space, which is

$56N_f - 133 = 91$. The remaining 28 must be distributed across the various values of $d(n)$ in some way. They must also be bounded above by the number of primitive invariants at that order whose Young tableaux have

Order	Young tableau	$SU(4)$ representation	Dimension
2	[0,1,...]	[0,1,0]	6
4	[4,...]	[4,0,0]	35
6	[0,3,...]	[0,3,0]	50
8	[2,0,2,...]	[2,0,2]	84
12	[0,0,4,...]	[0,0,4]	35

Table 5.33: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of E_7 SQCD theories and the corresponding representations and dimensions in the case of 4 flavours

N_f	$d(2)$	$d(4)$	$d(6)$	$d(8)$	$d(12)$	deg $P(t)$	$dim(\mathcal{M})$
1	0	1	0	0	0	0	1
2	1	5	1	0	0	0	7
3	3	15	10	6	1	18	35
4	≥ 5 ≤ 6	≥ 25 ≤ 35	≥ 19 ≤ 50	≥ 12 ≤ 84	≥ 2 ≤ 50	?	91
$N_f \geq 3$	?	?	?	?	?	?	$56N_f - 133$

Table 5.34: Powers of $(1-t^n)$ in denominator of unrefined Hilbert series for E_7 SQCD theories with N_f flavours with $1 \leq N_f \leq 3$ and upper and lower bounds for 4-flavour case

N_f rows or fewer. Only orders divisible by 2 are relevant to the case of E_7; up to 3 flavours we have invariants at orders 2, 4, 6, 8 and 12, and we expect only these orders and these invariants to contribute to the denominator of both the unrefined and refined series; though there is an invariant of order 18 that is absorbed into the relation, and three of order 10, two others of order 12 and yet more of higher orders that arise in the 4-flavour case, we do not expect them to contribute (i.e. we expect no factors of $(1-t^{10})$), just as the $[2,0,0,1,\dots]$ invariant does not contribute in the case of gauge group F_4.

The relevant invariants are as shown in Table 5.33:

One sees that the upper bounds for $d(n)$ for n equal to 2, 4, 6, 8 and 12 must be 6, 35, 50, 84 and 35 respectively. Calculating the lower bounds from the values at 3 and 4 flavours, we summarize this information in Table 5.34:

We see by inspection that the case giving the numerator of lowest degree has $d(n)$ at 6, 35, 36, 12 and 2, giving a numerator of degree 2.6+4.35+6.36+

$8.12 + 12.2 - 4.56 = 264$, this is the lower bound. The refined series would have a numerator of degree $2.6 + 4.35 + 6.50 + 8.84 + 12.35 - 4.56 = 1320$, or 330 in each flavour fugacity!

We also, following [4], calculated invariants for the case of arbitrary flavour numbers using the 'trial and error' approach noted earlier. We summarize our results as follows:

- We found the same invariants as in [4] for each order up to 18.
- We found 2686 (!) more invariants at order 20, two of which have a column of 11 boxes, and 15 constraints.
- The first constraints occur at order 20, which is the sum of the highest (18) and lowest (2) orders of generators in the 3-flavour case which is the complete intersection case. 18 is the dual Coxeter number of E_7. (The same pattern occurs for G_2 (2+4=6), F_4 (2+9=11) and E_6 with no antiflavours (3+12=15).)
- At 4 flavours, only constraints (9 of them) occur at order 20. New invariants occur at that order at 5 flavours, 75 of them (!), along with 5 more constraints. The final constraint at order 20 emerges at 6 flavours.
- Invariants can only have columns of up to 56 boxes, since this is the dimension of the fundamental representation of E_7. However, relations and higher syzygies can have columns with more than 56 boxes. It would be an extremely time-consuming task to even approach obtaining these, however.
- Unlike in the E_6 case, since there is a completely antisymmetric invariant with 2 boxes, the column of 56 boxes is not an independent invariant but rather the 28th power of the former.

Table 5.35 shows the number of invariants of E_7 (including second- and higher even-order syzygies) and Table 5.36 the number of relations (including higher odd-order syzygies) for a specific 'mass' level (i.e. number of fields) having a specific number of rows (i.e. the minimum number of flavours at which they appear) in their Young tableaux. The levels are specified in the first column of the table and the (minimum) number of

	1	2	3	4	5	6	7	8	9	10	11	12
[2]	0	1	0	0	0	0	0	0	0	0	0	0
[4]	1	0	0	0	0	0	0	0	0	0	0	0
[6]	0	1	0	0	0	0	0	0	0	0	0	0
[8]	0	0	1	0	0	0	0	0	0	0	0	0
[10]	0	0	0	3	0	0	0	0	0	0	0	0
[12]	0	0	1	2	5	1	0	0	0	0	0	0
[14]	0	0	0	4	10	15	1	0	0	0	0	0
[16]	0	0	0	5	30	49	37	4	0	0	0	0
[18]	0	0	1	5	60	178	195	114	15	1	0	0
[20]	0	0	0	0	75	482	879	792	389	67	2	0

Table 5.35: Invariants and even-order higher syzygies of E_7 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

flavours in the top row. (Only the results at order 20 are new, those at lower orders match those in [4].)

5.6 Higgsing

In 'normal' (i.e. non-supersymmetric) gauge theories, the Higgs mechanism is the breaking of the symmetry group that occurs in the vacuum when the potential has a minimum at a non-zero value of the matter field(s). It is often associated with spontaneous symmetry breaking in ϕ^4 scalar field theories where the mass squared takes on the appearance of being negative ('tachyonic') but actually gives a non-zero vacuum expectation value (VEV) for the scalar(s).

Before fixing the gauge, the choice of vacuum leads to the appearance of a number of massless scalar modes called 'Goldstone bosons'. There is one for every generator of the gauge group that does not leave the vacuum state invariant, i.e. every generator that is 'broken' by the choice of vacuum. (The unbroken subgroup is called the 'stability subgroup' or 'little group', similar to the case in string theory where it is the part of the Lorentz symmetry that commutes with the momentum; when the unbroken gauge group is $SU(N)$, $SO(N)$ or $Sp(N)$ and the matter is in the (anti)fundamental representation, the little group is the same but with $N-1$, when it is in a non-fundamental representation or the group is exceptional the patterns must be learned, or

	1	2	3	4	5	6	7	8	9	10	11	12
[2]	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	0	0	0	0	0	0	0	0	0
[12]	0	0	0	0	0	0	0	0	0	0	0	0
[14]	0	0	0	0	0	0	0	0	0	0	0	0
[16]	0	0	0	0	0	0	0	0	0	0	0	0
[18]	0	0	0	0	0	0	0	0	0	0	0	0
[20]	0	0	0	9	5	1	0	0	0	0	0	0

Table 5.36: Relations and odd-order higher syzygies of E_7 SQCD theories arranged by total number of boxes in Young tableau (down) and minimum number of flavours (across)

'guessed' from the number of unbroken generators left over.) One can then go to 'unitary gauge', following the procedure in [41], where the Goldstone bosons are 'eaten' by the massless gauge fields making them massive and breaking the gauge group to the corresponding little group. There is one broken generator for every Goldstone boson. Each eaten scalar has 1 degree of freedom, the massless gauge field $D-2$ in D dimensions, and the massive vector field $D-1$, so there is no mismatch.

In supersymmetric gauge theories, the 'eating' of a Goldstone boson by a massless gauge field to give a massive vector field is extended to the eating of a massless chiral multiplet (in $\mathcal{N}=1$) or hypermultiplet ($\mathcal{N}=2$) by a massless vector multiplet to give a massive vector multiplet. In both cases the number of (on-shell) degrees of freedom in the two massless multiplets is the same (respectively 4 and 8 including CPT conjugates) and the number in the massive vector multiplet is twice this (including CPT conjugates in the $\mathcal{N}=1$ case but not the $\mathcal{N}=2$ case).

In the standard model, the Higgs mechanism also gives rise to fermion masses. Because neutrinos were only observed in their left-handed form (and antineutrinos right-handed), the right-handed electron, muon and tau lepton were assigned to singlets of the 'weak isospin' $SU(2)$ part of the gauge group, while their left-handed equivalents combined with their respective neutrinos to form doublets. The same procedure was applied to quarks, even though in this case both members of every isospin (no 'weak' here) doublet were known to be massive. Left-handed quarks and leptons form doublets of the $SU(2)$, and the right-handed ones form singlets. Fermion mass terms can only occur by the interaction of left- and right-handed fermions, which as they transform differently under $SU(2)$ means explicit mass terms cannot occur in the Lagrangian; instead, a new scalar (Higgs) field transforming as a doublet of $SU(2)$ must be introduced to give a gauge-invariant Yukawa-like term which gives rise to fermion masses when the Higgs acquires its VEV.

(The true 'super-Higgs mechanism' is not merely the supersymmetric extension of the non-supersymmetric Higgs mechanism, which breaks no supersymmetry, only the gauge symmetry, but rather spontaneous supersymmetry breaking by gravitinos eating 'goldstinos', which are not the superpartners of Goldstone bosons but arise independently and by a different process, in supergravity rather than gauge theories.)

Since the scalar potential in a supersymmetric gauge theory can come from both the F- and D-term constraints, both can give rise to Higgsing of the gauge group. In SQCD, the (classical) superpotential is zero so only D-term constraints contribute. The procedure for $SU(N)$ gauge groups is described in [6].

In [1], the Higgsing is used to 'derive' the form of the Hilbert series for $N_f < N_c - 1$. At a generic point in the moduli space, the VEVs for the quark and antiquark fields can be rotated to make the top left $N_f \times N_f$ submatrix diagonal and the rest zero; the gauge group $SU(N_c)$ is broken to $SU(N_c - N_f)$, its subgroup which commutes with the VEVs, and the number of broken generators is given by

$$(N_c^2 - 1) - ((N_c - N_f)^2 - 1) = 2N_f N_c - N_f^2 \tag{5.41}$$

Since the number of fundamental fields is $2N_f N_c$, the number that remain massless after Higgsing is N_f^2. These can be parametrized in terms of the original fields as

$$M_b^a = Q_i^a \tilde{Q}_b^i \tag{5.42}$$

These are not the same 'type' of the field as the original fields in that they are products of more than one of them; this is not inconsistent with the construction as in [41].

The same procedure is followed in [2] to give the Hilbert series for $N_f < N_c$

in $SO(N_c)$ theories and $N_f \leq N_c$ (or $N_f < N_c + 1$) in $Sp(N_c)$ theories and gives gauge-invariant fields of the same type as for $SU(N_c)$ (mesonic, i.e. constructed with traces, although the symplectic trace is antisymmetric). For higher values of N_f, the gauge group is completely broken leaving a moduli space of dimension $2N_fN_c - N_c^2 + 1$, but the number of invariants, which now includes baryons (i.e. constructed with determinants), exceeds the number of generators and so there must be relations between them too, and when we compute the form of the relations we see they over-cancel the extra generators so there must be higher syzygies between the relations and the primitive invariants, and so on...

In [44], the Higgsing of E_6 progressively to F_4 and $D_4 = SO(8)$ (though not $A_2 = SU(3)$) is demonstrated explicitly by expressing it and its representations in terms of its maximal subgroups $SU(3) \times SU(3) \times SU(3)$ and $SU(6) \times SU(2)$ and their representations. The (anti)fundamental representation, of dimension 27, can be written as a triplet of three 3x3 matrices, or as a 6x6 antisymmetric matrix and a general 2x6 matrix, with the elements of the two being identified appropriately. The VEV can be rotated so that it takes the form of one of the 3x3 matrices being equal to the identity with the other two remaining zero, and consequently the 6x6 matrix takes the form $\sigma_2 \times \mathbb{1}_3$ with the 2x6 matrix remaining zero. This breaks the two $SU(3) \times SU(3)$ corresponding to the un-VEVved 3x3 matrix to their diagonal subgroup $SU(3)$ and the $SU(6)$ to $Sp(6)$ (or $Sp(3)$ if one prefers, in any case it is C_3); one sees that F_4 has $SU(3) \times SU(3)$ and $SU(2) \times C_3$ as subgroups, demonstrating the Higgsing. Similarly giving a VEV to a second flavour is shown to break one $SU(3)$ to its Cartan subgroup $U(1) \times U(1)$ and C_3 to the product of three $SU(2)$; these are the subgroups of D_4 'normal' to $SU(3)$ and $SU(2)$ respectively, showing the second Higgsing.

(We can see by inspection that giving a VEV to one single antisymmetric second-rank tensor of $SU(2N)$ does result in a residual gauge group of $Sp(2N)$, $Sp(N)$ or C_N depending on notation (we will use $Sp(N)$); the $SU(2N)$ gauge group has dimension $4N^2 - 1$, there are $N(2N - 1)$ fundamental fields and there is one invariant at order N, giving $N(2N + 1)$ unbroken generators, which is the number of generators of $Sp(N)$. (Note the second Dynkin index of this representation is $2N - 2$ and therefore even so we can have theories with only one 'flavour' of them.) A second antisymmetric second-rank tensor, in either the $[0, 1, 0, \ldots, 0]$ or $[0, \ldots, 0, 1, 0]$ representation (assume WLOG the first one was in the first of these), decomposes to a symplectic traceless second-rank antisymmetric tensor and symplectic trace and the first of these further Higgses $Sp(N)$ down to $SU(2)^N$, as in the example above.)

In general, however, Higgsing of exceptional groups, and of classical groups by matter in non-(anti)fundamental representations, is not demonstrated explicitly and is usually assumed simply from finding a subgroup of the original group with the required number of generators (i.e. the unbroken ones of the original group). We follow this procedure here too, determining the Higgsing from the invariants (i.e. the Hilbert series).

Higgsing, in principle, involves 'integrating out' the broken (i.e. made massive) generators of the gauge group by flowing to a mass scale much lower than the mass of the broken generators, which is that of the VEVs of the fundamental fields of the original theory to which non-zero VEVs were given. Other than its application in SQCD to determine the number and form of the gauge invariant and hence still massless new fields, it is also used in brane tiling theories, where the Higgsing is implemented by removing one edge from the fundamental domain (as the corresponding field is now massive) and coalescing the gauge groups corresponding to faces on opposite sides of the edge. The Higgsing procedure gives rise to a toric diagram with one fewer external point; however rather than using a 'trial and error' method of choosing a point to remove and using the 'inverse algorithm' to go from toric diagram to brane tiling (as opposed to the 'forward algorithm' going from brane tiling to toric diagram and Hilbert series), the new perfect matchings, Chern-Simons levels (in M2-brane theories) and other properties, and hence the new Hilbert series, are calculated from the new tiling. (Since there is no inverse algorithm yet formulated for M2-brane theories, this second method is the only one that can be used for them.)

In these theories the reverse process of un-Higgsing can be easily implemented by adding an edge to the fundamental domain and splitting the gauge group corresponding to the face into two new ones. (Integrating out massive fields is also the rationale behind only considering 'irreducible' brane tilings, i.e. those in which every node is connected to at least three other nodes of the opposite colour, because a node connected to only two other nodes corresponds to a term of order 2 in the fundamental fields and this would give rise to a mass for those fields, this is described in [39] and

[29].)

Higgsing is also seen in string theory, in particular in the closed bosonic string when, in the simplest example, the string is compactified on one circular dimension of radius R and we have extra massless vector modes and therefore an enhanced $SU(2)\times SU(2)$ symmetry when $R=\alpha'^{1/2}$ where α' is the string scale. When R moves away from the critical value, these extra massless modes gain a mass proportional to $(R-\alpha'^{1/2})$ and the gauge group is broken to $U(1)\times U(1)$. This is explained in more detail in [53]. Enhanced gauge groups, and thus Higgsing, also occur for the heterotic string (of either type), but not for type IIA, IIB or I strings. Yet another example occurs with gauge theories on D-branes; when initially there are N branes coincident the gauge group is $U(N)$, but when they are separated the strings between them gain a mass proportional to the separation of the two branes on which they end and the gauge group is broken to a product of $U(N_i)$ where i counts the number of different positions of the separated branes and N_i is the number of branes at position x_i, with the N_i summing to N.

The smallest classical groups that contain F_4, E_6 and E_7 are $SO(26)$, $SU(27)$ and $Sp(28)$ respectively and cannot be Higgsed to the corresponding exceptional groups anyway because the smallest representations that contain the adjoints of the exceptional groups are the adjoints of the larger groups and the presence of an adjoint field necessarily Higgses the classical group down to its maximal torus $U(1)^r$ where r is 13, 26 and 28 respectively, and this does not contain the exceptional group.

In this section, we do not derive the number or form of the gauge invariant quantities by counting the number of generators of the gauge group broken by the Higgsing, but rather, inspired by the relations between the Hilbert series for F_4 theories with a specified number of flavours and those for E_6 with the same number of flavours and one antiflavour, we compare Hilbert series for other gauge groups related by Higgsing on only some of their fundamental fields, i.e. the one antiflavour in these E_6 theories.

As opposed to E_6 and F_4, however, we will start with the simpler case of $B_3=SO(7)$ and $D_4=SO(8)$ being Higgsed to G_2, and G_2 being Higgsed to $A_2=SU(3)$.

We will also demonstrate cases where the Higgsed flavours are not distinguished in the original Hilbert series from those remaining in the child theory, in which case the original series will have to be refined.

The comparisons of Hilbert series for gauge groups related by Higgsing, along with the classical-group cases where N_c is simply reduced to $N_c - N_f$ (with N_f being the number of Higgsed flavours, not the total number), that we will investigate in this thesis are as follows. We show how the adjoint and the field(s) being Higgsed on decompose under the branching to the residual gauge group, giving rise to a number of invariants in the Higgsed fields equal to the number of scalars in the decomposition of the matter fields but not the adjoint, and in brackets how any other potentially relevant fundamental representations of the original group decompose in the residual group. (For $D_5 = SO(10)$ the two types of spinors are conjugate to each other; for $D_4 = SO(8)$ they are both self-conjugate, but we will still call them spinors and conjugate spinors.)

- B_3 to G_2 on one spinor
  - Adjoint: $\mathbf{21} \to \mathbf{14} + \mathbf{7}$
  - Spinor: $\mathbf{8} \to \mathbf{7} + \mathbf{1}$
  - (Vector: $\mathbf{7} \to \mathbf{7}$)
- D_4 to B_3 on one vector, spinor or conjugate spinor (WLOG vector)
  - Adjoint: $\mathbf{28} \to \mathbf{21} + \mathbf{7}$
  - Vector: $\mathbf{8}_v \to \mathbf{7} + \mathbf{1}$
  - (Spinor, conjugate spinor: $\mathbf{8}_s, \mathbf{8}_c \to \mathbf{8}$)
- D_4 to G_2 on one each of two types of field
  - Adjoint: $\mathbf{28} \to \mathbf{14} + 2.\mathbf{7}$
  - Vector, spinor, conjugate spinor: $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c \to \mathbf{7} + \mathbf{1}$
- D_5 to G_2 on two spinors or conjugate spinors (it is not Higgsed at all by one spinor and goes to A_3 on one of each type)
  - Adjoint: $\mathbf{45} \to \mathbf{14} + 4.\mathbf{7} + 3.\mathbf{1}$
  - Spinor, conjugate spinor: $\mathbf{16}, \bar{\mathbf{16}} \to 2.\mathbf{7} + 2.\mathbf{1}$
  - (Vector: $\mathbf{10} \to \mathbf{7} + 3.\mathbf{1}$)
- E_6 to F_4 on one (anti)fundamental

  - Adjoint: $\mathbf{78} \to \mathbf{52} + \mathbf{26}$
  - (Anti)fundamental: $\mathbf{27}, \bar{\mathbf{27}} \to \mathbf{26} + \mathbf{1}$

- E_6 to D_4 on any combination of two fundamentals or antifundamentals
  - Adjoint: $\mathbf{78} \to \mathbf{28} + 2.\mathbf{8}_v + 2.\mathbf{8}_s + 2.\mathbf{8}_c + 2.\mathbf{1}$
  - (Anti)fundamental: $\mathbf{27}, \bar{\mathbf{27}} \to \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c + 3.\mathbf{1}$

- E_7 to E_6 on one fundamental
  - Adjoint: $\mathbf{133} \to \mathbf{78} + \mathbf{27} + \bar{\mathbf{27}} + \mathbf{1}$
  - Fundamental: $\mathbf{56} \to \mathbf{27} + \bar{\mathbf{27}} + 2.\mathbf{1}$

- E_7 to D_4 on two fundamentals
  - Adjoint: $\mathbf{133} \to \mathbf{28} + 4.\mathbf{8}_v + 4.\mathbf{8}_s + 4.\mathbf{8}_c + 9.\mathbf{1}$
  - Fundamental: $\mathbf{56} \to 2.\mathbf{8}_v + 2.\mathbf{8}_s + 2.\mathbf{8}_c + 8.\mathbf{1}$

- F_4 to D_4 on one fundamental
  - Adjoint: $\mathbf{52} \to \mathbf{28} + \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c$
  - Fundamental: $\mathbf{26} \to \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c + 2.\mathbf{1}$

- G_2 to A_2 on one fundamental.
  - Adjoint: $\mathbf{14} \to \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}}$
  - Fundamental: $\mathbf{8} \to \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}$

When the gauge group is Higgsed to its residual subgroup by all or some of the matter fields, the non-singlets in the decomposition of the matter fields cancel out that of the adjoint of the gauge group leaving behind the adjoint of the residual group and possibly some singlets, and the number of invariants is the number of singlets in the decomposition of the matter fields minus the number in the decomposition of the adjoint of the gauge group.

It is also possible for a group to be partially Higgsed only on one or some rather than all of the invariants. For example, F_4, E_6 and E_7 with one flavour, one flavour and one antiflavour and two flavours respectively can be Higgsed on only the 2nd-order symmetric invariant (at order t^2), the delta invariant between a flavour and an antiflavour (at tu) and the 2nd-order antisymmetric invariant (at t^2) respectively to give B_4, D_5 and B_5

[34]. There are 1, 2 and 3 vectors left over from the decomposition of the matter fields that can then further Higgs these gauge groups down to D_4 as is the case for full Higgsing in the first place. We do not consider this partial Higgsing in this thesis, but we summarize it here:

- E_6 to D_5 on the delta invariant of one fundamental and one antifundamental
  - Adjoint: $\mathbf{78} \to \mathbf{45} + \mathbf{16} + \bar{\mathbf{16}} + \mathbf{1}$
  - Fundamental: $\mathbf{27} \to \mathbf{16} + \mathbf{10} + \mathbf{1}$
  - Antifundamental: $\bar{\mathbf{27}} \to \bar{\mathbf{16}} + \mathbf{10} + \mathbf{1}$
- E_7 to B_5 on the 2nd-rank antisymmetric invariant of two fundamentals
  - Adjoint: $\mathbf{133} \to \mathbf{55} + 2.\mathbf{32} + \mathbf{11} + 3.\mathbf{1}$
  - Fundamental: $\mathbf{56} \to \mathbf{32} + 2.\mathbf{11} + 2.\mathbf{1}$
- F_4 to B_4 on the 2nd-rank symmetric invariant of one fundamental
  - Adjoint: $\mathbf{52} \to \mathbf{36} + \mathbf{16}$
  - Fundamental: $\mathbf{26} \to \mathbf{16} + \mathbf{9} + \mathbf{1}$

The decomposition of E_7 to B_5 involves an intermediate step to $D_6 \times A_1$, where the adjoint decomposes as $\mathbf{133} \to (\mathbf{66}, \mathbf{1}) + (\bar{\mathbf{32}}, \mathbf{2}) + (\mathbf{1}, \mathbf{3})$ and the fundamental as $\mathbf{56} \to (\mathbf{32}, \mathbf{1}) + (\mathbf{12}, \mathbf{2})$. The two spinors of D_6 map to the unique spinor of B_5, the adjoint to the adjoint plus a vector and the vector to a vector plus a scalar.

We also list below the Higgsings of higher special orthogonal groups by matter in spinor representations, where there is no general rule as there is for Higgsing by vector matter and the residual gauge groups have to be 'guessed':

- B_4 to B_3 on one spinor (invariant at order 2)
  - Adjoint: $\mathbf{36} \to \mathbf{21} + \mathbf{8} + \mathbf{7}$
  - Spinor: $\mathbf{16} \to \mathbf{8} + \mathbf{7} + \mathbf{1}$
  - (Vector: $\mathbf{9} \to \mathbf{8} + \mathbf{1}$)
- B_5 to A_4 on one spinor (invariant at order 4)
  - Adjoint: $\mathbf{55} \to \mathbf{24} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{1}$

- Spinor: $\mathbf{32} \to \mathbf{10} + \bar{\mathbf{10}} + \mathbf{5} + \bar{\mathbf{5}} + 2.\mathbf{1}$
- (Vector: $\mathbf{11} \to \mathbf{5} + \bar{\mathbf{5}} + \mathbf{1}$)

- D_6 to A_5 on one spinor (invariant at order 4)
  - Adjoint: $\mathbf{66} \to \mathbf{35} + \mathbf{15} + \bar{\mathbf{15}} + \mathbf{1}$
  - Spinor: $\mathbf{32} \to \mathbf{15} + \bar{\mathbf{15}} + 2.\mathbf{1}$
  - (Conjugate spinor: $\bar{\mathbf{32}} \to \mathbf{20} + \mathbf{6} + \bar{\mathbf{6}}$)
  - (Vector: $\mathbf{12} \to \mathbf{6} + \bar{\mathbf{6}}$)

- D_7 to $G_2 \times G_2$ on one spinor (invariant at order 8)
  - Adjoint: $\mathbf{91} \to (\mathbf{14}, \mathbf{1}) + (\mathbf{1}, \mathbf{14}) + (\mathbf{7}, \mathbf{7}) + (\mathbf{7}, \mathbf{1}) + (\mathbf{1}, \mathbf{7})$
  - Spinor, conjugate spinor: $\mathbf{64}, \bar{\mathbf{64}} \to (\mathbf{7}, \mathbf{7}) + (\mathbf{7}, \mathbf{1}) + (\mathbf{1}, \mathbf{7}) + (\mathbf{1}, \mathbf{1})$
  - (Vector: $\mathbf{14} \to (\mathbf{7}, \mathbf{1}) + (\mathbf{1}, \mathbf{7})$)

$D_5 = SO(10)$ has no single-row invariants in either the spinor or the conjugate spinor so it is not Higgsed by one flavour of spinor matter of either type.

The Higgsing of B_4 to B_3 on one spinor must also be broken into two steps, the first being the decomposition into D_4, where the B_4 adjoint becomes $\mathbf{28} + \mathbf{8}_v$, the spinor $\mathbf{8}_s + \mathbf{8}_c$ and the vector $\mathbf{8}_v + \mathbf{1}$. The difference is that here the second decomposition into B_3 does not take the vector of D_4 into a vector plus scalar of B_3, but rather one of the two spinor representations of D_4 decomposes as such, with the vector and the other spinor of D_4 becoming B_3 spinors. Another way to see this is that Higgsing B_4 on a vector takes it to D_4, under which the B_4 spinor breaks up into one each of the two D_4 spinors and they Higgs D_4 down to G_2, while Higgsing B_4 on the spinor first takes it to B_3; this is further Higgsed to G_2 by one spinor but to A_3 by a vector, and since progressive Higgsing must be independent of the order, the B_4 vector must decompose to a B_3 spinor (plus scalar).

The Higgsing of D_7 to $G_2 \times G_2$ is discussed in [4]. Higgsing D_7 on one vector breaks it to B_6, while the spinor (either one) remains unchanged, while Higgsing G_2 on one fundamental breaks it to A_2; we deduce therefore that one spinor of B_6 Higgses it to $A_2 \times A_2$. (There are two fully symmetric invariants, at orders 4 and 8, so the dimensions match; $78 - 64 + 2 = 16$.)

5.6.1 B_3 gauge group

B_3, or $SO(7)$, is not an exceptional group, but it does have G_2 as a subgroup. This is important phenomenologically because when M-theory is compactified down to 4 dimensions on a 7-manifold, the holonomy group of the 7-manifold is a subgroup of $SO(7)$, and G_2 is the subgroup that breaks the spinor of $SO(7)$, the **8**, down to a **7** and a singlet. The **32** supercharge of the $SO(1,10)$ Lorentz group of M-theory decomposes into a $\mathbf{4} \times \mathbf{8}$ of $SO(1,3) \times SO(7)$, and we require $\mathcal{N} = 1$ for a phenomenologically consistent theory. Imposing G_2 holonomy on the 7-manifold achieves this, not that the actual construction of a (compact) such manifold is trivial.

The double cover of $SO(7)$, called $Spin(7)$, is the other important 'special' holonomy subgroup according to Berger's classification of reduced holonomy groups, though less important than G_2. The **32** supercharge of the $SO(1,10)$ Lorentz group of M-theory decomposes into a $\mathbf{2} \times (\mathbf{8}_s + \mathbf{8}_c)$ of $SO(1,2) \times SO(8)$; $Spin(7)$ can be embedded into $Spin(8)$, the double cover of $D_4 = SO(8)$, in three ways, each choosing one of the three 8-dimensional representations to decompose to a vector plus scalar of $Spin(7)$ and the other two become spinors. In this context we choose one of the spinors to decompose into the vector plus scalar, while the other remains a spinor; this gives one singlet out of 16 total $Spin(8)$ spinor degrees of freedom, preserving only 1/16 of the supersymmetry, i.e. we have $\mathcal{N} = 1$ in 3D. (Similarly, when D_4 decomposes to $A_3 = SU(4)$, i.e. we compactify on a Calabi-Yau 4-fold, one spinor becomes the 'vector' **6** plus two scalars and the other $\mathbf{4} + \bar{\mathbf{4}}$. There are only two singlets out of 16 total degrees of freedom, so 1/8 of the supersymmetry is preserved and we have $\mathcal{N} = 2$ in 3D.)

The spinor of B_3, which is $[0,0,1]$ in Dynkin notation, has 8 degrees of freedom. By either taking the plethystic exponential of its character, weighted by a counting fugacity t, and performing Molien-Weyl integration, or simply by inspection of the singlets in successive symmetrizations, one sees that there is only one primitive totally symmetric invariant of the spinor at order 2. The explicit symmetrization is as follows:

$$Sym^k[0,0,1]_{B_3} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} [0,0,k-2m]_{B_3} \tag{5.43}$$

We see that there is one singlet at every even order, which interestingly is the same as for the vector! Converting this to a Hilbert series, we get

$$g^{(1,0,B_3)}(t,u) = \frac{1}{1-t^2} \tag{5.44}$$

The refined series, with the number of spinors, counted by t, listed first followed by the number of vectors counted by u, are as follows for total number of matter fields up to 5:

$$\begin{aligned}
PL(g^{(1,0,B_3)}(t,u)) &= t^2 \\
PL(g^{(2,0,B_3)}(t,u)) &= [2]t^2 \\
PL(g^{(3,0,B_3)}(t,u)) &= [2,0]t^2 \\
PL(g^{(4,0,B_3)}(t,u)) &= [2,0,0]t^2 + [0,0,0]t^4 \\
PL(g^{(5,0,B_3)}(t,u)) &= [2,0,0,0]t^2 + [0,0,0,1]t^4 - [0,0,0,0]t^{10} \\
PL(g^{(2,1,B_3)}(t,u)) &= [2]t^2 + [0]t^2u + [0]u^2 \\
PL(g^{(3,1,B_3)}(t,u)) &= [2,0]t^2 + [0,1]t^2u + [0,0]u^2 \\
PL(g^{(4,1,B_3)}(t,u)) &= [2,0,0]t^2 + [0,0,0]t^4 + [0,1,0]t^2u + [0,0,0]t^4u + [0,0,0]u^2 \\
&\quad -[0,0,0]t^8u^2 \\
PL(g^{(1,2,B_3)}(t,u)) &= [0]t^2 + [2]u^2 \\
PL(g^{(1,3,B_3)}(t,u)) &= [0,0]t^2 + [0,0]t^2u^3 + [2,0]u^2 \\
PL(g^{(1,4,B_3)}(t,u)) &= [0,0,0]t^2 + [0,0,1]t^2u^3 + [0,0,0]t^2u^4 + [2,0,0]u^2 - [0,0,0]t^4u^8 \\
PL(g^{(2,2,B_3)}(t,u)) &= [2;0]t^2 + [0;1]t^2u + [0;0]t^2u^2 + [0;2]u^2 \\
PL(g^{(3,2,B_3)}(t,u)) &= [2,0;0]t^2 + [0,1;1]t^2u + [0,1;0]t^2u^2 + [0,0;2]u^2 - [0,0;0]t^6u^4 \\
PL(g^{(2,3,B_3)}(t,u)) &= [2;0,0]t^2 + [0;1,0]t^2u + [0;0,1]t^2u^2 + [2;0,0]t^2u^3 + [0;2,0]u^2 \\
&\quad -[0;0,0]t^4u^3 - [0;0,0]t^4u^6
\end{aligned}$$

The (partially, keeping t and u separate) unrefined series, including the cases for 6 and 7 spinors and no vectors which are not complete intersections, which were calculated as for those for G_2 with 5 to 8 flavours by unrefining the character expansion and knowing the arithmetic progression of the exponents of $(1-t^2)$ and $(1-t^4)$, are as follows:

$$\begin{aligned}
g^{(1,0,B_3)}(t,u) &= \frac{1}{1-t^2}\\
g^{(2,0,B_3)}(t,u) &= \frac{1}{(1-t^2)^3}\\
g^{(3,0,B_3)}(t,u) &= \frac{1}{(1-t^2)^6}\\
g^{(4,0,B_3)}(t,u) &= \frac{1}{(1-t^2)^{10}(1-t^4)}\\
g^{(5,0,B_3)}(t,u) &= \frac{1-t^{10}}{(1-t^2)^{15}(1-t^4)^5} = \frac{1+t^2+t^4+t^6+t^8}{(1-t^2)^{14}(1-t^4)^5}\\
g^{(6,0,B_3)}(t,u) &= ((1-t^2)^{18}(1-t^4)^9)^{-1}\times\\
&\quad (1+3t^2+12t^4+28t^6+57t^8+78t^{10}+92t^{12}+78t^{14}+\\
&\quad 57t^{16}+28t^{18}+12t^{20}+3t^{22}+t^{24})\\
g^{(7,0,B_3)}(t,u) &= ((1-t^2)^{22}(1-t^4)^{13})^{-1}\times\\
&\quad (1+6t^2+43t^4+188t^6+701t^8+1966t^{10}+4621t^{12}+8708t^{14}+13818t^{16}+\\
&\quad 17976t^{18}+19782t^{20}+17976t^{22}+13818t^{24}+8708t^{26}+\\
&\quad 4621t^{28}+1966t^{30}+701t^{32}+188t^{34}+43t^{36}+6t^{38}+t^{40})\\
g^{(2,1,B_3)}(t,u) &= \frac{1}{(1-t^2)^3(1-t^2u)(1-u^2)}\\
g^{(3,1,B_3)}(t,u) &= \frac{1}{(1-t^2)^6(1-t^2u)^3(1-u^2)}\\
g^{(4,1,B_3)}(t,u) &= \frac{1-t^8u^2}{(1-t^2)^{10}(1-t^4)(1-t^2u)^6(1-t^4u)(1-u^2)}\\
g^{(1,2,B_3)}(t,u) &= \frac{1}{(1-t^2)(1-u^2)^3}\\
g^{(1,3,B_3)}(t,u) &= \frac{1}{(1-t^2)(1-t^2u^3)(1-u^2)^6}\\
g^{(1,4,B_3)}(t,u) &= \frac{1-t^4u^8}{(1-t^2)(1-t^2u^3)^4(1-t^2u^4)(1-u^2)^{10}}\\
g^{(2,2,B_3)}(t,u) &= \frac{1}{(1-t^2)^3(1-t^2u)^2(1-t^2u^2)(1-u^2)^3}\\
g^{(3,2,B_3)}(t,u) &= \frac{1-t^6u^4}{(1-t^2)^6(1-t^2u)^6(1-t^2u^2)^3(1-u^2)^3}\\
g^{(2,3,B_3)}(t,u) &= \frac{(1-t^4u^3)(1-t^4u^6)}{(1-t^2)^3(1-t^2u)^3(1-t^2u^2)^3(1-t^2u^3)^3(1-u^2)^6}
\end{aligned}$$

Both the spinor and the vector have second Dynkin index 2 (it is always 2 for a vector of an $SO(N)$ group, and for a spinor it is the dimension divided by 4), and the adjoint has Dynkin index 10, which is equal to twice the dual Coxeter number of $SO(7)$. By the formula, the complete intersection should occur at 5 total spinors and vectors. As we know from [2], this does not occur for 5 vectors (but rather at 7), but it does when at least one fundamental field is a spinor. The first relation should occur at (total) order 10; this is the case for 5 spinors, for 4 spinors and 1 vector and for 3 and 2 respectively but not for 2 and 3 (where it has order 7, though here two relations appear and the second is at order 10) or for 1 and 4, where it appears at order 12 (recall there is no relation for 5 vectors, the relation for 7 vectors has order 14).

In the cases where there is exactly 1 spinor field, with N_v vectors (i.e. the $(1, N_v)$ cases), we see that removing the $(1-t^2)$ term from the denominator, setting t to 1 and then relabelling u as t gives the Hilbert series for G_2 with N_v flavours.

Refining them further by setting $u = t$, one can determine the dimension of the moduli space, which is the order of the pole at $t = 1$. This is equal to the number of degrees of freedom in the fundamental fields (8 per spinor, 7 per vector) minus the number of broken generators of the gauge group. We see that the residual gauge group depends only on the total number of flavours, not how many are spinors and how many vectors, as long as at least one is a spinor; this follows from the fact the Higgsing by one spinor gives G_2 and both spinors and vectors decompose to fundamentals, although spinors give an extra scalar which accounts for the extra degree(s) of freedom and the extra dimensions of the moduli space. The fully refined series for the complete intersections (5 total spinors and vectors) are as follows:

$$
\begin{aligned}
g^{(5,0,B_3)}(t,u) &= \frac{1+t^2+t^4+t^6+t^8}{(1-t^2)^{14}(1-t^4)^5} \\
g^{(4,1,B_3)}(t,u) &= \frac{1+t^5}{(1-t^2)^{11}(1-t^3)^6(1-t^4)} \\
g^{(3,2,B_3)}(t,u) &= \frac{1+t^2+t^4+t^6+t^8}{(1-t^2)^8(1-t^3)^6(1-t^4)^3} \\
g^{(2,3,B_3)}(t,u) &= \frac{1+t^2+t^4+t^5+t^6+t^8+t^{10}}{(1-t^2)^8(1-t^3)^3(1-t^4)^3(1-t^5)^2} \\
g^{(1,4,B_3)}(t,u) &= \frac{1+t^6}{(1-t^2)^{11}(1-t^5)^4}
\end{aligned}
$$

Returning to the partially (un)refined series, in all the cases which are either freely generated or complete intersections, we can easily combine the $U(1)$ counting and $SU(N)$ fugacities into $U(N)$ fugacities t_i (or u_i etc), $1 \leq i \leq N$, with the character of the fundamental being simply the sum of all the t_i. With the usual Dynkin weights used for $SU(N)$ weights, recall that the formula for conversion from $U(1) \times SU(N)$ to $U(N)$ is as follows:

$$
\begin{aligned}
tz_1 &\rightarrow t_1 \\
t\frac{z_i}{z_{i-1}} &\rightarrow t_i, 1 < i < N \\
t/z_{N-1} &\rightarrow t_N
\end{aligned}
\tag{5.45}
$$

Both methods are explored in [1] and [2], the first for the character expansion, the second for the initial calculation of the refined series.

Writing the refined Hilbert series in this new form, we obtain

$$
\begin{aligned}
g^{(2,0,B_3)}(t,u) &= \frac{1}{(1-t_1^2)(1-t_1t_2)(1-t_2^2)} \\
g^{(3,0,B_3)}(t,u) &= \frac{1}{\prod_{1\le i\le j\le 3}(1-t_it_j)} \\
g^{(4,0,B_3)}(t,u) &= \frac{1}{(\prod_{1\le i\le j\le 4}(1-t_it_j))(1-t_1t_2t_3t_4)} \\
g^{(5,0,B_3)}(t,u) &= \frac{1-\prod_{i=1}^{5}t_i^2}{(\prod_{1\le i\le j\le 5}(1-t_it_j))(\prod_{1\le i<j<k<l\le 5}(1-t_it_jt_kt_l))} \\
g^{(2,1,B_3)}(t,u) &= \frac{1}{(1-t_1^2)(1-t_1t_2)(1-t_2^2)(1-t_1t_2u)(1-u^2)} \\
g^{(3,1,B_3)}(t,u) &= \frac{1}{(\prod_{1\le i\le j\le 3}(1-t_it_j))(1-t_1t_2u)(1-t_1t_3u)(1-t_2t_3u)(1-u^2)} \\
g^{(4,1,B_3)}(t,u) &= (1-(\prod_{i=1}^{4}t_i^2)u^2)(\prod_{1\le i\le j\le 3}(1-t_it_j)^{-1})(1-t_1t_2t_3t_4)^{-1} \\
&\times (\prod_{1\le i<j\le 4}(1-t_it_ju)^{-1})(1-t_1t_2t_3t_4u)^{-1}(1-u^2)^{-1} \\
g^{(1,2,B_3)}(t,u) &= \frac{1}{(1-t^2)(1-u_1^2)(1-u_1u_2)(1-u_2^2)} \\
g^{(1,3,B_3)}(t,u) &= \frac{1}{(1-t^2)(1-t^2u_1u_2u_3)(\prod_{1\le i\le j\le 3}(1-u_iu_j))} \\
g^{(1,4,B_3)}(t,u) &= (1-t^4\prod_{i=1}^{4}u_i^2)(1-t^2)^{-1}(\prod_{1\le i<j<k\le 4}(1-t^2u_iu_ju_k)^{-1}) \\
&\times (1-t^2u_1u_2u_3u_4)^{-1}(\prod_{1\le i\le j\le 4}(1-u_iu_j)^{-1}) \\
g^{(2,2,B_3)}(t,u) &= (1-t_1^2)^{-1}(1-t_1t_2)^{-1}(1-t_2^2)^{-1}(1-t_1t_2u_1)^{-1}(1-t_1t_2u_2)^{-1} \\
&\times (1-t_1t_2u_1u_2)^{-1}(1-u_1^2)^{-1}(1-u_1u_2)^{-1}(1-u_2^2)^{-1} \\
g^{(3,2,B_3)}(t,u) &= (1-t_1^2t_2^2t_3^2u_1^2u_2^2)(\prod_{1\le i\le j\le 3}(1-t_it_j)^{-1})(\prod_{1\le i\le j\le 3}(1-t_it_ju_1)(1-t_it_ju_2)^{-1}) \\
&\times (\prod_{1\le i<j\le 3}(1-t_it_ju_1u_2)^{-1})(1-u_1^2)^{-1}(1-u_1u_2)^{-1}(1-u_2^2)^{-1} \\
g^{(2,3,B_3)}(t,u) &= (1-t_1^2t_2^2u_1u_2u_3)(1-t_1^2t_2^2u_1^2u_2^2u_3^2)(1-t_1^2)^{-1}(1-t_1t_2)^{-1}(1-t_2^2)^{-1} \\
&\times (\prod_{i=1}^{3}(1-t_1t_2u_i)^{-1})(\prod_{1\le i<j\le 3}(1-t_1t_2u_iu_j)^{-1})(\prod_{1\le i\le j\le 2}(1-t_it_ju_1u_2u_3)^{-1}) \\
&\times (\prod_{1\le i\le j\le 3}(1-u_iu_j)^{-1})
\end{aligned}
$$

We see that when we choose one of the t_i fugacities (usually the highest numbered one, i.e. t_{N_s} denoting the number of spinors by N_s), remove $(1 - t_i^2)$ (corresponding to the invariant in the Higgsed spinor) from the denominator and then set the fugacity to 1, we get $N_s - 1$ factors in $(1 - t_j)$ for $j < N_s$; these correspond to the scalars in the decomposition of the remaining spinors, so removing these and relabelling the u_i fugacities $u_i \to t_{i+N_s-1}$ for $1 \leq i \leq N_v$, we get the G_2 Hilbert series for $(N_s + N_v - 1)$ flavours.

We will now consider the case with 1 spinor and 5 vectors, for which we calculated the Hilbert series using Mathematica. We still list the number of spinors (here fixed at 1) first, but switch the order of the fugacities, so u counts the number of spinor fields and t the number of vector fields. This is to facilitate comparison with the G_2 series, where with only one type of matter field we used t to count them.

$$g^{(B_3,1,5)}(u,t) = (1 + 3u^2t^3 + 5u^2t^4 + u^4t^6 + 5u^4t^7 - 5u^4t^9 - u^4t^{10} - 5u^6t^{12} - 3u^6t^{13} - u^8t^{16})/((1-t^2)^{15}(1-u^2)(1-t^3u^2)^7)$$

The $(1 - u^2)$ term again corresponds to the symmetric invariant of order 2 of the spinor, and there are no $(1 - t)$ terms in the denominator because there are no further spinors that decompose under G_2 into a **7** and a scalar. Vectors of $SO(7)$ do not decompose under G_2, while the adjoint gives an adjoint and a fundamental.

We see that the power of $(1 - t^2)$ in the denominator is not 14, as in the G_2 series, but rather 15, which is the number of invariants of order 2; these transform as a second-rank symmetric tensor of the global $SU(5)$ with dimension 15. As we will see later, this will cause issues when we consider reversing the Higgsing process ('un-Higgsing') and trying to construct Hilbert series of 'parent' gauge groups, in this case B_3, in terms of those of 'child' groups, here G_2.

Nevertheless, removing the $(1-u^2)$ term from the denominator and setting u to 1 (there are no $(1-t)$ terms) does still result in the G_2 Hilbert series for 5 flavours, factoring out a $(1 - t^2)$ from both numerator and denominator. The power of $(1-t^3u^2)$ in the denominator, 7, is the same as that of $(1-t^3)$ for the G_2 case. (This invariant arises because the 3rd antisymmetric power of the [1,0,0] vector representation and the 2nd symmetric power of the [0,0,1] spinor both contain the 35-dimensional [0,0,2]. It transforms in the $[0,0,1,\ldots]$ representation of the global $SU(N_v)$, as for the G_2 case.)

Fully unrefining this by setting $u = t$ and thereby identifying vector and spinor fields, we obtain the following Hilbert series:

$$\begin{aligned} g^{(B_3,1,5)}(t,t) = (1 + t^2 + t^4 + 3t^5 + 6t^6 + 3t^7 + 6t^8 + 3t^9 + 7t^{10} + \\ 8t^{11} + 7t^{12} + 3t^{13} + 6t^{14} + 3t^{15} + 6t^{16} + \\ 3t^{17} + t^{18} + t^{20} + t^{22})/((1 - t^2)^{15}(1 - t^5)^7) \end{aligned}$$

The dimension of the moduli space is the degree of the pole at $t = 1$, which is 22. This is equal to the number of degrees of freedom in the fundamental fields, 1 spinor and 5 vectors giving $1.8 + 5.7 = 43$, minus the dimension of the gauge group, 21, because the gauge group is completely broken.

The series for 1 spinor and 6 and 7 vectors are given below. Note that in the 7-vector case, the power of $(1-t)$ is 28, but that of $(1+t)$ is only 27. We present two forms of the series, the first in lowest terms and the second with both numerator and denominator multiplied by $(1 + t)$ to remove terms of order t from both. The power of $(1 - t^3u^2)$ does follow the same arithmetic progression as that of $(1 - t^3)$ in the G_2 case. In all cases, removing the $(1 - u^2)$ term, setting u to 1 and putting the resulting fraction into lowest terms results in the same Hilbert series as for G_2 with N_v flavours.

We were unable to obtain the series for 1 spinor and 8 vectors because of memory constraints; we expect however that the power of $(1 + t)$ follows an arithmetic progression from 5 vectors on, but that of $(1 - t)$ continues to be the dimension of the second-rank symmetric tensor of $SU(N_v)$, or the number of invariants of order 2.

The series for 1 spinor and 6 vectors is as follows:

$$g^{(B_3,1,6)}(u,t) = (1 + 10t^3u^2 + 15t^4u^2 + 20t^6u^4 + 60t^7u^4 - 70t^9u^4 - 21t^{10}u^4 + 10t^9u^6 + 45t^{10}u^6 - 196t^{12}u^6 - 126t^{13}u^6 + 105t^{14}u^6 + 70t^{15}u^6 - 6t^{17}u^6 - t^{18}u^6 + t^{12}u^8 + 6t^{13}u^8 - 70t^{15}u^8 - 105t^{16}u^8 + 126t^{17}u^8 + 196t^{18}u^8 - 45t^{20}u^8 - 10t^{21}u^8 + 21t^{20}u^{10} + 70t^{21}u^{10} - 60t^{23}u^{10} - 20t^{24}u^{10} - 15t^{26}u^{12} - 10t^{27}u^{12} - t^{30}u^{14})/((1-t^2)^{21}(1-u^2)(1-t^3u^2)^{10})$$

Here, not only does the power of $(1-t^2)$ in the denominator (21) not match that in the G_2 series with 6 flavours (18), but we have instances, the first occurring at t^9, of there being two terms in the numerator with the same power of t but different powers of u and therefore there not being a direct mapping of terms from the G_2 series to this series. This makes the process of reversing the Higgsing, or 'un-Higgsing', difficult if not impossible.

$$1 + 5t^4u^2 - 5t^6u^4 - 10t^7u^4 - 5t^9u^4 - t^{10}u^4 + 5t^9u^6 + 10t^{12}u^6 + 10t^{13}u^8 + 5t^{16}u^8 - t^{15}u^{10} - 5t^{16}u^{10} - 10t^{18}u^{10} - 5t^{19}u^{10} + 5t^{21}u^{12} + t^{25}u^{14}$$

$$1 + 5t^4 - 5t^6 - 10t^7 - t^{10} + 10t^{12} + 10t^{13} - t^{15} - 10t^{18} - 5t^{19} + 5t^{21} + t^{25}$$

The series for 1 spinor and 7 vectors is as follows, presented in the two ways described earlier:

$$
\begin{aligned}
g^{(B_3,1,7)}(u,t) = (1 - t + t^2 - t^3 + t^4 - t^5 + t^6 + 22t^3u^2 + 13t^4u^2 - \\
13t^5u^2 + 13t^6u^2 - 13t^7u^2 + 13t^8u^2 - 13t^9u^2 + \\
113t^6u^4 + 216t^7u^4 - 216t^8u^4 - 274t^9u^4 + \\
78t^{10}u^4 - 78t^{11}u^4 + 78t^{12}u^4 + 190t^9u^6 + \\
580t^{10}u^6 - 580t^{11}u^6 - 2458t^{12}u^6 + 302t^{13}u^6 + \\
2344t^{14}u^6 + 8t^{15}u^6 - 294t^{16}u^6 - 84t^{17}u^6 + \\
113t^{12}u^8 + 461t^{13}u^8 - 461t^{14}u^8 - 3655t^{15}u^8 - \\
881t^{16}u^8 + 8819t^{17}u^8 + 3613t^{18}u^8 - 7014t^{19}u^8 - \\
2814t^{20}u^8 + 1506t^{21}u^8 + 846t^{22}u^8 - 90t^{23}u^8 - \\
106t^{24}u^8 - 6t^{25}u^8 + 6t^{26}u^8 + t^{27}u^8 + 22t^{15}u^{10} + \\
111t^{16}u^{10} - 111t^{17}u^{10} - 1373t^{18}u^{10} - 783t^{19}u^{10} + \\
6075t^{20}u^{10} + 6357t^{21}u^{10} - 10584t^{22}u^{10} - \\
10584t^{23}u^{10} + 6357t^{24}u^{10} + 6075t^{25}u^{10} - \\
783t^{26}u^{10} - 1373t^{27}u^{10} - 111t^{28}u^{10} + 111t^{29}u^{10} + \\
22t^{30}u^{10} + t^{18}u^{12} + 6t^{19}u^{12} - 6t^{20}u^{12} - \\
106t^{21}u^{12} - 90t^{22}u^{12} + 846t^{23}u^{12} + 1506t^{24}u^{12} - \\
2814t^{25}u^{12} - 7014t^{26}u^{12} + 3613t^{27}u^{12} + \\
8819t^{28}u^{12} - 881t^{29}u^{12} - 3655t^{30}u^{12} - 461t^{31}u^{12} + \\
461t^{32}u^{12} + 113t^{33}u^{12} - 84t^{28}u^{14} - 294t^{29}u^{14} + \\
8t^{30}u^{14} + 2344t^{31}u^{14} + 302t^{32}u^{14} - 2458t^{33}u^{14} - \\
580t^{34}u^{14} + 580t^{35}u^{14} + 190t^{36}u^{14} + 78t^{33}u^{16} - \\
78t^{34}u^{16} + 78t^{35}u^{16} - 274t^{36}u^{16} - 216t^{37}u^{16} + \\
216t^{38}u^{16} + 113t^{39}u^{16} - 13t^{36}u^{18} + 13t^{37}u^{18} - \\
13t^{38}u^{18} + 13t^{39}u^{18} - 13t^{40}u^{18} + 13t^{41}u^{18} + \\
22t^{42}u^{18} + t^{39}u^{20} - t^{40}u^{20} + t^{41}u^{20} - t^{42}u^{20} + \\
t^{43}u^{20} - t^{44}u^{20} + t^{45}u^{20})/((1-t^2)^{27}(1-t)(1-u^2)(1-t^3u^2)^{13})
\end{aligned}
$$

$$g^{(B_3,1,7)}(u,t) = (1 + t^7 + 22t^3u^2 + 35t^4u^2 - 13t^{10}u^2 + 113t^6u^4 + 329t^7u^4 - 490t^9u^4 - 196t^{10}u^4 + 78t^{13}u^4 + 190t^9u^6 + 770t^{10}u^6 - 3038t^{12}u^6 - 2156t^{13}u^6 + 2646t^{14}u^6 + 2352t^{15}u^6 - 286t^{16}u^6 - 378t^{17}u^6 - 84t^{18}u^6 + 113t^{12}u^8 + 574t^{13}u^8 - 4116t^{15}u^8 - 4536t^{16}u^8 + 7938t^{17}u^8 + 12432t^{18}u^8 - 3401t^{19}u^8 - 9828t^{20}u^8 - 1308t^{21}u^8 + 2352t^{22}u^8 + 756t^{23}u^8 - 196t^{24}u^8 - 112t^{25}u^8 + 7t^{27}u^8 + t^{28}u^8 + 22t^{15}u^{10} + 133t^{16}u^{10} - 1484t^{18}u^{10} - 2156t^{19}u^{10} + 5292t^{20}u^{10} + 12432t^{21}u^{10} - 4227t^{22}u^{10} - 21168t^{23}u^{10} - 4227t^{24}u^{10} + 12432t^{25}u^{10} + 5292t^{26}u^{10} - 2156t^{27}u^{10} - 1484t^{28}u^{10} + 133t^{30}u^{10} + 22t^{31}u^{10} + t^{18}u^{12} + 7t^{19}u^{12} - 112t^{21}u^{12} - 196t^{22}u^{12} + 756t^{23}u^{12} + 2352t^{24}u^{12} - 1308t^{25}u^{12} - 9828t^{26}u^{12} - 3401t^{27}u^{12} + 12432t^{28}u^{12} + 7938t^{29}u^{12} - 4536t^{30}u^{12} - 4116t^{31}u^{12} + 574t^{33}u^{12} + 113t^{34}u^{12} - 84t^{28}u^{14} - 378t^{29}u^{14} - 286t^{30}u^{14} + 2352t^{31}u^{14} + 2646t^{32}u^{14} - 2156t^{33}u^{14} - 3038t^{34}u^{14} + 770t^{36}u^{14} + 190t^{37}u^{14} + 78t^{33}u^{16} - 196t^{36}u^{16} - 490t^{37}u^{16} + 329t^{39}u^{16} + 113t^{40}u^{16} - 13t^{36}u^{18} + 35t^{42}u^{18} + 22t^{43}u^{18} + t^{39}u^{20} + t^{46}u^{20})/((1-t^2)^{28}(1-u^2)(1-t^3u^2)^{13})$$

Fully unrefining, we obtain the following Hilbert series, in each case presented first in lowest terms with $(1+t)$ factors in the denominator and secondly with all factors in the denominator being of the form $(1-t^n)$ with

$n > 1$:

$$g^{(B_3,1,6)}(t,t) = (1 + 2t + 4t^2 + 6t^3 + 9t^4 + 22t^5 + 51t^6 + 90t^7 + 145t^8 + 210t^9 + 311t^{10} + 482t^{11} + 689t^{12} + 896t^{13} + 1118t^{14} + 1350t^{15} + 1642t^{16} + 1944t^{17} + 2110t^{18} + 2160t^{19} + 2180t^{20} + 2160t^{21} + 2110t^{22} + 1944t^{23} + 1642t^{24} + 1350t^{25} + 1118t^{26} + 896t^{27} + 689t^{28} + 482t^{29} + 311t^{30} + 210t^{31} + 145t^{32} + 90t^{33} + 51t^{34} + 22t^{35} + 9t^{36} + 6t^{37} + 4t^{38} + 2t^{39} + t^{40})/((1 - t^2)^{19}(1 + t)^2(1 - t^5)^{10})$$

$$g^{(B_3,1,6)}(t,t) = (1 + t^2 + t^4 + 10t^5 + 16t^6 + 10t^7 + 16t^8 + 10t^9 + 36t^{10} + 70t^{11} + 36t^{12} + 15t^{14} + 10t^{15} + 60t^{16} + 10t^{17} - 136t^{18} - 116t^{19} - 30t^{20} - 40t^{21} - 30t^{22} - 116t^{23} - 136t^{24} + 10t^{25} + 60t^{26} + 10t^{27} + 15t^{28} + 36t^{30} + 70t^{31} + 36t^{32} + 10t^{33} + 16t^{34} + 10t^{35} + 16t^{36} + 10t^{37} + t^{38} + t^{40} + t^{42})/((1 - t^2)^{21}(1 - t^5)^{10})$$

$$g^{(B_3,1,7)}(t,t) = (1 + 4t + 12t^2 + 28t^3 + 58t^4 + 130t^5 + 311t^6 + 713t^7 + 1523t^8 + 2996t^9 + 5611t^{10} + 10267t^{11} + 18223t^{12} + 31020t^{13} + 50499t^{14} + 78835t^{15} + 119213t^{16} + 175282t^{17} + 249373t^{18} + 342255t^{19} + 453810t^{20} + 583258t^{21} + 729077t^{22} + 885489t^{23} + 1041499t^{24} + 1186222t^{25} + 1311646t^{26} + 1410834t^{27} + 1476500t^{28} + 1499934t^{29} + 1476500t^{30} + 1410834t^{31} + 1311646t^{32} + 1186222t^{33} + 1041499t^{34} + 885489t^{35} + 729077t^{36} + 583258t^{37} + 453810t^{38} + 342255t^{39} + 249373t^{40} + 175282t^{41} + 119213t^{42} + 78835t^{43} + 50499t^{44} + 31020t^{45} + 18223t^{46} + 10267t^{47} + 5611t^{48} + 2996t^{49} + 1523t^{50} + 713t^{51} + 311t^{52} + 130t^{53} + 58t^{54} + 28t^{55} + 12t^{56} + 4t^{57} + t^{58})/((1-t^2)^{23}(1+t)^4(1-t^5)^{13})$$

$$g^{(B_3,1,7)}(t,t) = (1 + 2t^2 + 3t^4 + 22t^5 + 39t^6 + 45t^7 + 75t^8 + 68t^9 + 224t^{10} + 420t^{11} + 360t^{12} + 282t^{13} + 300t^{14} + 334t^{15} + 1010t^{16} + 464t^{17} - 1318t^{18} - 1562t^{19} - 887t^{20} - 662t^{21} - 742t^{22} - 4256t^{23} - 5217t^{24} + 110t^{25} + 2873t^{26} + 1075t^{27} - 349t^{28} - 1424t^{29} + 4074t^{30} + 9272t^{31} + 4074t^{32} - 1424t^{33} - 349t^{34} + 1075t^{35} + 2873t^{36} + 110t^{37} - 5217t^{38} - 4256t^{39} - 742t^{40} - 662t^{41} - 887t^{42} - 1562t^{43} - 1318t^{44} + 464t^{45} + 1010t^{46} + 334t^{47} + 300t^{48} + 282t^{49} + 360t^{50} + 420t^{51} + 224t^{52} + 68t^{53} + 75t^{54} + 45t^{55} + 39t^{56} + 22t^{57} + 3t^{58} + 2t^{60} + t^{62})/((1-t^2)^{27}(1-t^5)^{13})$$

As with the only partially unrefined series in the 7-vector case, the powers of $(1-t)$ and $(1+t)$ in the denominator are different in both cases, though this time there are extra powers of $(1+t)$. The dimensions of the moduli spaces are again given by the dimension of the pole at $t=1$, which are 29 and 36; they are again equal to the number of degrees of freedom in the fundamental fields (50 and 57 respectively) minus the number of broken generators, which is again all 21.

For $N_v \geq 5$, the fully unrefined Hilbert series can be written in the following form:

$$g^{(B_3,1,N_v)}(t,t) = \frac{P_{18N_v-68}(t)}{(1-t^2)^{4N_v-5}(1+t)^{2N_v-10}(1-t^5)^{3N_v-8}} \qquad (5.46)$$

This gives a moduli space of dimension $7N_v - 13 = 7N_v + 8 - 21$, and a (degree of denominator)-(degree of numerator) of $7N_v + 8$ as required.

As we did for E_6 with one antiflavour, we will now calculate invariants for the case of arbitrary vector flavour numbers, with one spinor flavour, using the 'trial and error' approach noted earlier. We summarize our results as follows:

Tables 5.37, 5.38, 5.39 and 5.40 show the number of invariants of B_3 (including second- and higher even-order syzygies) and Tables 5.41, 5.42, 5.43 and 5.44 show the number ofrelations (including higher odd-order syzygies) for a specific 'mass' level (i.e. number of fields) having a specific number of rows (i.e. the minimum number of flavours of vectors at which they appear, with the number of spinor flavours fixed at 1) in their Young tableaux. The levels, with the number of vector fields first and the number of spinor fields second, are specified in the first column of the table and the minimum number of flavours (of vectors) in the top row. As in the E_6 with one antiflavour case, we cannot accommodate the $(1-u^2)$ invariant solely in the spinors in these tables, but we understand it is present.

Summing the number of invariants of each type for each number of vector fields in the invariant over numbers of spinor fields, we show the number of invariants (including even-order higher syzygies) at each order in the vector-counting fugacity in Table 5.45 and the number of relations (including odd-order higher syzygies) in Table 5.46:

By inspection, the number of 'net' invariants (i.e. primary invariants and even-order higher syzygies minus primary relations and odd-order higher syzygies) for a given number of vectors and number of vector fields in the invariant, summed over the number of spinor fields, is the same for the B_3 case with one spinor as for the G_2 case. We show this in Tables 5.47 and 5.48 for the case of invariants containing 21 vector fields (assuming the

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2,0]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3,2]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4,2]	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,0]	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,6]	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
[10,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,6]	0	0	0	0	2	4	4	4	2	1	0	0	0	0	0	0	0	0	0	0
[11,0]	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
[11,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,6]	0	0	0	0	2	5	6	5	4	2	1	0	0	0	0	0	0	0	0	0
[12,2]	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0	0	0	0	0
[12,6]	0	0	0	0	2	4	5	4	2	1	1	0	0	0	0	0	0	0	0	0
[12,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.37: Invariants and even-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[13,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,2]	0	0	0	0	0	0	0	1	2	2	2	1	0	0	0	0	0	0	0	0
[13,4]	0	0	0	0	0	0	3	5	6	5	4	2	1	0	0	0	0	0	0	0
[13,6]	0	0	0	0	2	3	3	3	1	0	0	0	0	0	0	0	0	0	0	0
[13,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,4]	0	0	0	0	0	0	7	12	14	14	11	8	4	2	0	0	0	0	0	0
[14,6]	0	0	0	0	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0
[14,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,0]	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
[15,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,4]	0	0	0	0	0	0	4	6	3	2	3	1	1	1	1	0	0	0	0	0
[15,6]	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,10]	0	0	0	0	1	11	27	36	36	27	17	8	3	1	0	0	0	0	0	0
[16,0]	0	0	0	0	0	0	0	1	2	2	2	2	2	2	1	0	0	0	0	0
[16,2]	0	0	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0	0	0
[16,4]	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.38: Invariants and even-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[16,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,10]	0	0	0	0	5	57	164	253	265	226	154	91	40	16	4	1	0	0	0	0
[17,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,2]	0	0	0	0	0	0	1	4	8	11	12	11	10	7	4	2	1	0	0	0
[17,4]	0	0	0	0	0	0	1	0	2	3	4	5	3	1	0	0	0	0	0	0
[17,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,10]	0	0	0	0	12	137	454	762	870	759	567	351	193	82	32	8	2	0	0	0
[18,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,2]	0	0	0	0	0	0	1	4	9	13	14	13	11	9	6	4	2	1	0	0
[18,4]	0	0	0	0	0	0	0	0	9	23	30	32	28	18	8	2	0	0	0	0
[18,6]	0	0	0	0	0	0	0	6	10	11	10	7	3	1	0	0	0	0	0	0
[18,8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,10]	0	0	0	0	20	235	802	1435	1678	1513	1115	728	405	206	84	32	8	2	0	0
[18,12]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,0]	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
[19,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,4]	0	0	0	0	0	0	0	0	7	16	17	16	15	10	6	3	1	0	0	0

Table 5.39: Invariants and even-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[19,6]	0	0	0	0	0	0	0	36	119	208	261	273	234	157	87	35	9	0	0	0
[19,8]	0	0	0	0	0	0	49	216	404	500	496	404	292	178	97	41	16	4	1	0
[19,10]	0	0	0	0	24	299	1074	1942	2336	2062	1500	919	507	245	112	43	16	4	1	0
[19,12]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,0]	0	0	0	0	0	0	0	0	3	7	8	8	8	8	8	7	3	0	0	0
[20,2]	0	0	0	0	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0
[20,4]	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
[20,6]	0	0	0	0	0	0	0	78	295	491	570	548	447	321	197	110	48	17	3	0
[20,8]	0	0	0	0	0	1	251	1112	2255	3013	3089	2690	2006	1348	775	404	167	64	16	4
[20,10]	0	0	0	0	22	301	1103	2073	2496	2215	1531	894	438	189	72	26	9	3	1	0
[20,12]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[21,0]	0	0	0	0	0	0	0	1	2	4	6	7	7	7	5	4	3	2	1	1
[21,2]	0	0	0	0	0	0	0	3	33	83	110	119	119	110	93	66	38	16	4	0
[21,4]	0	0	0	0	0	0	0	0	1	0	2	3	4	5	3	1	0	0	0	0
[21,6]	0	0	0	0	0	0	0	79	201	236	221	165	127	81	44	22	10	6	3	1
[21,8]	0	0	0	0	0	2	462	2090	4425	6120	6533	5721	4419	2995	1877	1034	532	223	88	23
[21,10]	0	0	0	0	16	232	878	1678	2078	1822	1249	688	319	122	38	9	1	0	0	0
[21,12]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[21,14]	0	0	0	0	3	127	860	2341	3705	4103	3627	2684	1735	961	471	193	70	20	5	1

Table 5.40: Invariants and even-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 4)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,4]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7,4]	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,4]	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[9,0]	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[9,4]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,2]	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
[10,4]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[11,2]	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0
[11,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[12,8]	0	0	0	0	1	3	6	5	4	2	1	0	0	0	0	0	0	0	0	0

Table 5.41: Relations and odd-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[13,0]	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
[13,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,8]	0	0	0	0	3	14	26	29	26	16	9	3	1	0	0	0	0	0	0	0
[14,0]	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0
[14,2]	0	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0	0	0	0
[14,4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,8]	0	0	0	0	5	28	52	67	57	46	25	14	4	2	0	0	0	0	0	0
[15,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,2]	0	0	0	0	0	0	0	0	1	2	2	2	1	0	0	0	0	0	0	0
[15,4]	0	0	0	0	0	0	0	2	3	4	5	3	1	0	0	0	0	0	0	0
[15,6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[15,8]	0	0	0	0	7	36	72	86	78	55	35	18	9	3	1	0	0	0	0	0
[15,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,4]	0	0	0	0	0	0	0	9	22	29	28	24	16	8	2	0	0	0	0	0

Table 5.42: Relations and odd-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 2)

[16,6]	0	0	0	0	0	0	13	33	44	46	40	30	17	9	3	1	0	0	0	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[16,8]	0	0	0	0	6	37	73	86	74	47	26	11	5	2	1	0	0	0	0	0
[16,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,0]	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
[17,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,4]	0	0	0	0	0	0	0	7	15	14	13	11	8	5	3	1	0	0	0	0
[17,6]	0	0	0	0	0	1	39	116	180	195	181	140	97	53	27	9	3	0	0	0
[17,8]	0	0	0	0	5	27	56	67	57	34	16	6	1	0	0	0	0	0	0	0
[17,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,0]	0	0	0	0	0	0	0	1	3	5	4	5	4	5	4	2	0	0	0	0
[18,2]	0	0	0	0	0	0	0	0	0	0	1	2	2	1	0	0	0	0	0	0
[18,4]	0	0	0	0	0	0	0	3	4	2	0	1	1	0	0	0	0	0	0	0
[18,6]	0	0	0	0	0	1	46	132	199	219	191	147	99	66	37	21	8	3	0	0
[18,8]	0	0	0	0	3	16	28	36	31	17	7	3	1	0	0	0	0	0	0	0
[18,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,12]	0	0	0	0	2	36	149	281	345	315	242	154	83	37	13	4	1	0	0	0
[19,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,2]	0	0	0	0	0	0	0	7	26	41	47	48	45	38	28	16	8	2	0	0
[19,4]	0	0	0	0	0	0	0	2	2	2	3	4	6	3	1	0	0	0	0	0

Table 5.43: Relations and odd-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[19,6]	0	0	0	0	0	0	31	48	20	6	2	0	0	0	0	0	0	0	1	0
[19,8]	0	0	0	0	1	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,12]	0	0	0	0	10	223	1081	2336	3158	3126	2556	1752	1055	528	234	80	25	5	1	0
[20,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,2]	0	0	0	0	0	0	0	7	28	53	67	69	63	54	41	28	16	8	2	0
[20,4]	0	0	0	0	0	0	5	33	89	148	192	194	178	143	101	59	30	13	5	1
[20,6]	0	0	0	0	0	0	17	10	6	10	11	10	7	3	1	0	0	0	0	0
[20,8]	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,12]	0	0	0	0	28	691	3692	8898	12903	13699	11585	8479	5308	2982	1411	607	202	63	12	3
[21,0]	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
[21,2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[21,4]	0	0	0	0	0	0	11	76	218	388	506	519	473	383	284	185	113	57	27	9
[21,6]	0	0	0	0	0	0	7	1	38	106	173	215	249	227	154	80	27	6	0	0
[21,8]	0	0	0	0	0	0	0	18	40	50	48	39	25	13	4	1	0	0	0	0
[21,10]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[21,12]	0	0	0	0	52	1407	8253	21085	32602	35775	31451	23225	15185	8691	4520	2029	832	273	83	17
[21,14]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.44: Relations and odd-order higher syzygies of B_3 SQCD theories with one spinor arranged by total number of boxes in vector Young tableau and single-row spinor tableau (down) and minimum number of flavours of vectors (across) (part 4)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2]	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	0	2	4	4	4	2	1	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	0	2	5	6	5	5	2	1	0	0	0	0	0	0	0	0	0
[12]	0	0	0	0	2	4	5	5	4	3	2	0	0	0	0	0	0	0	0	0
[13]	0	0	0	0	2	3	6	9	9	7	6	3	1	0	0	0	0	0	0	0
[14]	0	0	0	0	1	2	8	13	15	14	11	8	4	2	0	0	0	0	0	0
[15]	0	0	0	0	1	12	31	42	39	29	21	9	4	2	1	0	0	0	0	0
[16]	0	0	0	0	5	57	165	255	267	229	158	95	43	18	5	1	0	0	0	0
[17]	0	0	0	0	12	137	456	766	880	773	583	367	206	90	36	10	3	0	0	0
[18]	0	0	0	0	20	235	803	1445	1706	1560	1169	780	447	234	98	38	10	3	0	0
[19]	0	0	0	0	24	299	1123	2194	2866	2786	2274	1612	1049	590	302	122	42	8	2	0
[20]	0	0	0	0	22	302	1354	3263	5050	5727	5198	4141	2901	1868	1053	547	227	84	20	4
[21]	0	0	0	0	19	361	2200	6192	10445	12368	11748	9387	6730	4281	2531	1329	654	267	101	26

Table 5.45: Invariants and even-order higher syzygies of B_3 SQCD theories with one spinor summed over number of boxes in single-row spinor Young tableau arranged by total number of boxes in vector tableau (down) and minimum number of flavours of vectors (across)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[4]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6]	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[7]	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[8]	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[9]	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[10]	0	0	0	0	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
[11]	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0
[12]	0	0	0	0	1	3	6	5	4	2	1	0	0	0	0	0	0	0	0	0
[13]	0	0	0	0	3	14	26	29	26	17	9	3	1	0	0	0	0	0	0	0
[14]	0	0	0	0	5	28	53	67	59	48	28	15	5	2	0	0	0	0	0	0
[15]	0	0	0	0	7	36	72	88	82	61	42	23	11	3	1	0	0	0	0	0
[16]	0	0	0	0	6	37	86	128	140	122	94	65	38	19	6	1	0	0	0	0
[17]	0	0	0	0	5	28	95	190	252	243	210	158	106	58	30	10	3	0	0	0
[18]	0	0	0	0	5	53	223	453	582	558	445	312	190	109	54	27	9	3	0	0
[19]	0	0	0	0	11	230	1112	2393	3206	3175	2608	1804	1106	569	263	96	33	7	2	0
[20]	0	0	0	0	28	692	3714	8948	13026	13910	11855	8752	5556	3182	1554	694	248	84	19	4
[21]	0	0	0	0	52	1407	8271	21180	32898	36319	32178	23998	15932	9315	4962	2295	972	336	110	26

Table 5.46: Relations and odd-order higher syzygies of B_3 SQCD theories with one spinor summed over number of boxes in single-row spinor Young tableau arranged by total number of boxes in vector tableau (down) and minimum number of flavours of vectors (across)

	5	6	7	8	9	10	11
G_2 6th-order	0	3	15	28	32	29	23
G_2 5th-order	33	1049	6086	15016	22485	23980	20453
B_3 6th-order	19	361	2200	6192	10445	12368	11748
B_3 5th-order	52	1407	8271	21180	32898	36319	32178

Table 5.47: Comparison of numbers of net invariants at order 21 in vectors for B_3 with one spinor and G_2 (part 1)

	12	13	14	15	16	17	18	19	20
G_2 6th-order	15	8	4	2	0	0	0	0	0
G_2 5th-order	14626	9210	5038	2433	966	318	69	9	0
B_3 6th-order	9387	6730	4281	2531	1329	654	267	101	26
B_3 5th-order	23998	15932	9315	4962	2295	972	336	110	26

Table 5.48: Comparison of numbers of net invariants at order 21 in vectors for B_3 with one spinor and G_2 (part 2)

'invariants' to be second-order syzygies):

Though the analysis is too long to present here, the actual invariants (including relations and higher syzygies) also match when those for B_3 are summed over the number of spinor fields in the invariant. This echoes the parallels between E_6 with one antiflavour and F_4 which we discovered earlier.

5.6.2 D_4 gauge group

D_4 is again not an exceptional group, but we recall that it is Higgsed to B_3 by one vector, and by triality, also by one spinor or one conjugate spinor. It can be decomposed to B_3 in three ways, each choosing one of the three fundamental representations to decompose into a vector and scalar of B_3, with the other two becoming spinors. If there are more fields of the same type, they become vectors of B_3, with an additional scalar; fields of the other two types become spinors of B_3:

$$D_4(N_v, N_s, N_c) \to B_3(N_s + N_c, N_v - 1)$$

with the number of fields of each type being permuted if the field being Higgsed on is actually a spinor or conjugate spinor.

The second Dynkin index of all three fundamental representations is 2, for the vector it is defined to be so and for the spinors one can show it either by triality or by the fact that it is the dimension, which is here 8, divided by 4. That of the adjoint is 12, which is equal to the (dual) Coxeter number. One expects therefore that the complete intersection would occur at 6 total flavours and with the relation at total order 12, again except for the case where all fundamental fields are the same, here WLOG vectors, where we know it to occur at 8 flavours with the relation at order 16.

The refined series for up to 6 total flavours, or 8 if they are all the same type (WLOG vectors) are as follows: (note the number of vector fields comes first, then spinors, then conjugate spinors; though when there are two types of fields present they are usually considered to be spinors and conjugate spinors, though the fugacities t and u are used here rather than u and v)

$$
\begin{aligned}
PL(g^{(N_f,0,0,D_4)}) &= [2,\ldots]t^2, 1 \leq N_f \leq 7 \\
PL(g^{(8,0,0,D_4)}(t,u,v)) &= [2,0,0,0,0,0,0]t^2 + [0,0,0,0,0,0,0]t^8 - [0,0,0,0,0,0,0]t^{16} \\
PL(g^{(1,1,0,D_4)}(t,u,v)) &= t^2 + u^2 \\
PL(g^{(2,1,0,D_4)}(t,u,v)) &= [2]t^2 + u^2 \\
PL(g^{(3,1,0,D_4)}(t,u,v)) &= [2,0]t^2 + u^2 \\
PL(g^{(4,1,0,D_4)}(t,u,v)) &= [2,0,0]t^2 + [0,0,0]t^4u^2 + u^2 \\
PL(g^{(5,1,0,D_4)}(t,u,v)) &= [2,0,0,0]t^2 + [0,0,0,1]t^4u^2 + u^2 - [0,0,0,0]t^{10}u^4 \\
PL(g^{(2,2,0,D_4)}(t,u,v)) &= [2;0]t^2 + [0;2]u^2 + [0;0]t^2u^2 \\
PL(g^{(3,2,0,D_4)}(t,u,v)) &= [2,0;0]t^2 + [0,0;2]u^2 + [0,1;0]t^2u^2 \\
PL(g^{(4,2,0,D_4)}(t,u,v)) &= [2,0,0;0]t^2 + [0,0,0;2]u^2 + [0,1,0;0]t^2u^2 + [0,0,0;2]t^4u^2 \\
&\quad -[0,0,0;0]t^4u^4 - [0,0,0;0]t^8u^4 \\
PL(g^{(3,3,0,D_4)}(t,u,v)) &= [2,0;0,0]t^2 + [0,0;2,0]u^2 + [0,1;0,1]t^2u^2 - [0,0;0,0]t^6u^6 \\
PL(g^{(1,1,1,D_4)}(t,u,v)) &= t^2 + u^2 + v^2 + tuv \\
PL(g^{(2,1,1,D_4)}(t,u,v)) &= [2]t^2 + [0]u^2 + [0]v^2 + [1]tuv \\
PL(g^{(3,1,1,D_4)}(t,u,v)) &= [2,0]t^2 + [0,0]u^2 + [0,0]v^2 + [1,0]tuv + [0,0]t^3uv \\
PL(g^{(4,1,1,D_4)}(t,u,v)) &= [2,0,0]t^2 + [0,0,0]u^2 + [0,0,0]v^2 + [1,0,0]tuv + [0,0,1]t^3uv \\
&\quad +[0,0,0]t^4u^2 + [0,0,0]t^4v^2 - [0,0,0]t^4u^2v^2 - [0,0,0]t^8u^4v^4 \\
PL(g^{(2,2,1,D_4)}(t,u,v)) &= [2;0]t^2 + [0;2]u^2 + [0;0]v^2 + [1;1]tuv + [0;0]t^2u^2 \\
PL(g^{(3,2,1,D_4)}(t,u,v)) &= [2,0;0]t^2 + [0,0;2]u^2 + [0,0;0]v^2 + [1,0;1]tuv + [0,1;0]t^2u^2 \\
&\quad +[0,0;1]t^3uv - [0,0;0]t^6u^4v^2 \\
PL(g^{(2,2,2,D_4)}(t,u,v)) &= [2;0;0]t^2 + [0;2;0]u^2 + [0;0;2]v^2 + [1;1;1]tuv + [0;0;0]t^2u^2 \\
&\quad +[0;0;0]t^2v^2 + [0;0;0]u^2v^2 + [0;0;0]t^2u^2v^2 - [0;0;0]t^4u^4v^4
\end{aligned}
$$

The unrefined series, including the non-complete intersection cases for 9, 10, 11 and 12 vectors but no spinors or conjugate spinors, are as follows:

$$
\begin{aligned}
g^{(N_v,0,0,D_4)} &= \frac{1}{(1-t^2)^{\frac{1}{2}N_v(N_v+1)}}, 1 \le N_v \le 7 \\
g^{(8,0,0,D_4)}(t,u,v) &= \frac{1-t^{16}}{(1-t^2)^{36}(1-t^8)} = \frac{1+t^8}{(1-t^2)^{36}} \\
g^{(9,0,0,D_4)} &= \frac{1+t^2+t^4+t^6+10t^8+t^{10}+t^{12}+t^{14}+t^{16}}{(1-t^2)^{44}} \\
g^{(10,0,0,D_4)} &= (1-t^2)^{-52}(1+3t^2+6t^4+10t^6+60t^8+57t^{10}+56t^{12}+57t^{14}+ \\
&\quad 60t^{16}+10t^{18}+6t^{20}+3t^{22}+t^{24}) \\
g^{(11,0,0,D_4)} &= (1-t^2)^{-60}(1+6t^2+21t^4+56t^6+291t^8+648t^{10}+ \\
&\quad 1078t^{12}+1562t^{14}+2112t^{16}+1562t^{18}+1078t^{20}+ \\
&\quad 648t^{22}+291t^{24}+56t^{26}+21t^{28}+6t^{30}+t^{32}) \\
g^{(12,0,0,D_4)} &= (1-t^2)^{-68}(1+10t^2+55t^4+220t^6+1210t^8+ \\
&\quad 4378t^{10}+11495t^{12}+24530t^{14}+45695t^{16}+62270t^{18}+ \\
&\quad 68354t^{20}+62270t^{22}+45695t^{24}+24530t^{26}+11495t^{28}+ \\
&\quad 4378t^{30}+1210t^{32}+220t^{34}+55t^{36}+10t^{38}+t^{40})
\end{aligned}
$$

$$
\begin{aligned}
g^{(1,1,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)(1-u^2)} \\
g^{(2,1,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^3(1-u^2)} \\
g^{(3,1,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^6(1-u^2)} \\
g^{(4,1,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^{10}(1-t^4u^2)(1-u^2)} \\
g^{(5,1,0,D_4)}(t,u,v) &= \frac{1-t^{10}u^4}{(1-t^2)^{15}(1-t^4u^2)^5(1-u^2)} \\
g^{(2,2,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^3(1-u^2)^3(1-t^2u^2)} \\
g^{(3,2,0,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^6(1-u^2)^3(1-t^2u^2)^3} \\
g^{(4,2,0,D_4)}(t,u,v) &= \frac{(1-t^4u^4)(1-t^8u^4)}{(1-t^2)^{10}(1-u^2)^3(1-t^2u^2)^6(1-t^4u^2)^3} \\
g^{(3,3,0,D_4)}(t,u,v) &= \frac{1-t^6u^6}{(1-t^2)^6(1-u^2)^6(1-t^2u^2)^9} \\
g^{(1,1,1,D_4)}(t,u,v) &= \frac{1}{(1-t^2)(1-u^2)(1-v^2)(1-tuv)} \\
g^{(2,1,1,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^3(1-u^2)(1-v^2)(1-tuv)^2} \\
g^{(3,1,1,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^6(1-u^2)(1-v^2)(1-tuv)^3(1-t^3uv)} \\
g^{(4,1,1,D_4)}(t,u,v) &= \frac{(1-t^4u^2v^2)(1-t^8u^2v^2)}{(1-t^2)^{10}(1-u^2)(1-v^2)(1-tuv)^4(1-t^3uv)^4(1-t^4u^2)(1-t^4v^2)} \\
g^{(2,2,1,D_4)}(t,u,v) &= \frac{1}{(1-t^2)^3(1-u^2)^3(1-v^2)(1-tuv)^4(1-t^2u^2)} \\
g^{(3,2,1,D_4)}(t,u,v) &= \frac{1-t^6u^4v^2}{(1-t^2)^6(1-u^2)^3(1-v^2)(1-tuv)^6(1-t^2u^2)^3(1-t^3uv)^2}
\end{aligned}
$$

$$
g^{(2,2,2,D_4)}(t,u,v)
$$
$$
= \frac{1-t^4u^4v^4}{(1-t^2)^3(1-u^2)^3(1-v^2)^3(1-tuv)^8(1-t^2u^2)(1-t^2v^2)(1-u^2v^2)(1-t^2u^2v^2)}
$$

As with the B_3 case, the residual gauge group is the same for any given total number of fundamental fields, as long as they are of at least 2 different types; the reason is that Higgsing on one field reduces the gauge group to B_3, with all other fields of the same type becoming vectors (plus a scalar) and fields of other types becoming spinors, and further Higgsing on one spinor reduces the gauge group to G_2, and now all fundamental fields of D_4 decompose to fundamentals of G_2 plus scalars.

Recalling that the number of invariants for E_6 with 2, 3 or 4 total flavours was the same however many were fundamentals and however many antifundamentals, we see that this is not the case with 2 total spinors here; there are three invariants, at order t^2, for 2 spinors of the same type, but only 2, at orders t^2 and u^2, for one of each type.

The moduli space is the dimension of the pole at $t = 1$ in the fully unrefined series when all fields are identified and counted by the same fugacity t; it is equal to the number of degrees of freedom in the fundamental fields, which here is 8 times the total number of fields, minus the number of broken generators of the gauge group. This is easy to see by inspection for the freely generated cases and the cases with only one type of field (WLOG only vectors); for the complete intersections with more than one type of field the fully refined Hilbert series are as follows:

$$
\begin{aligned}
g^{(5,1,0,D_4)}(t,t,t) &= \frac{1+t^2+t^4+t^6+t^8+t^{10}+t^{12}}{(1-t^2)^{15}(1-t^6)^5} \\
g^{(4,2,0,D_4)}(t,t,t) &= \frac{1+t^4+t^6+t^{10}}{(1-t^2)^{13}(1-t^4)^5(1-t^6)^2} \\
g^{(3,3,0,D_4)}(t,t,t) &= \frac{1+t^4+t^8}{(1-t^2)^{12}(1-t^4)^8} \\
g^{(4,1,1,D_4)}(t,t,t) &= \frac{1+t^4+t^6+t^{10}}{(1-t^2)^{12}(1-t^3)^3(1-t^5)^4(1-t^6)} \\
g^{(3,2,1,D_4)}(t,t,t) &= \frac{1+t^4+t^8}{(1-t^2)^{10}(1-t^3)^6(1-t^4)^2(1-t^5)^2} \\
g^{(2,2,2,D_4)}(t,t,t) &= \frac{1+t^6}{(1-t^2)^9(1-t^3)^8(1-t^4)^3}
\end{aligned}
$$

In all cases the moduli space has dimension 20, which is the number of fundamental fields (48) minus the dimension of the group (28).

Recall that Higgsing of E_6 on two total flavours of fundamentals or antifundamentals, or F_4 on one fundamental, gives D_4 as the residual gauge group, and the (anti)fundamental of E_6 and fundamental of F_4 both decompose under D_4 to give a vector, a spinor and a conjugate spinor, plus scalars

N_f	$24N_f$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge group
1	24	4	0	20	8	A_2
2	48	21	1	28	0	$\emptyset$

Table 5.49: Numbers of invariants, relations, broken and unbroken generators and unbroken gauge groups for D_4 SQCD theories with up to 2 flavours each consisting of a vector, spinor and conjugate spinor

to fill out the representation of the parent group. From now on we will only consider these cases, where the number of vectors, spinors and conjugate spinors are equal and identified by the same fugacities t or t_i. Identifying t, u and v in the (1,1,1) and (2,2,2) cases, we have the following refined series:

$$\begin{aligned} PL(g^{(1,1,1,D_4)}(t,t,t)) &= 3t^2 + t^3 \\ PL(g^{(2,2,2,D_4)}(t,t,t)) &= 3[2]t^2 + ([3]+2[1])t^3 + 3[0]t^4 + [0]t^6 - [0]t^{12} \end{aligned}$$

The unrefined series are as follows:

$$\begin{aligned} g^{(1,1,1,D_4)}(t,t,t) &= \frac{1}{(1-t^2)^3(1-t^3)} \\ g^{(2,2,2,D_4)}(t,t,t) &= \frac{1-t^{12}}{(1-t^2)^9(1-t^3)^8(1-t^4)^3(1-t^6)} = \frac{1+t^6}{(1-t^2)^9(1-t^3)^8(1-t^4)^3} \end{aligned}$$

The numbers of invariants, relations and broken and unbroken generators of the gauge group and the unbroken gauge groups are listed in Table 5.49:

The invariants and their form in the 3-flavour case are as shown in Table 5.50:

One sees that the upper bounds for $d(n)$ for n equal to 2, 3 and 4 must be 18, 27 and 27 (the latter two 26 and 18 excluding the invariants that do not occur in the 2-flavour case) respectively. Calculating the lower bounds from the values at 1 and 2 flavours, we summarize this information in Table 5.51:

We were unable to calculate either the refined or unrefined Hilbert series in the 3-flavour case; from these bounds we know that the minimum degree of the numerator of the unrefined series, assigning each of the 8 'remaining'

Order	Young tableau	$SU(3)$ representation	Dimension
2	3[2,...]	3[2,0]	18
3	[3,...]	[3,0]	10
3	2[1,1,...]	2[1,1]	16
3	[0,0,1,...]	[0,0]	1
4	3[0,2,...]	3[0,2]	18
4	3[1,0,1,...]	3[1,0]	9

Table 5.50: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of D_4 SQCD theories and the corresponding representations and dimensions in the case of 3 flavours

N_f	$d(2)$	$d(3)$	$d(4)$	deg $P(t)$	$dim(\mathcal{M})$
1	3	1	0	0	4
2	9	8	3	6	20
3	≥ 15 ≤ 18	≥ 15 ≤ 26	≥ 6 ≤ 18	?	44
$N_f \geq 2$	?	?	?	?	$24N_f - 28$

Table 5.51: Powers of $(1-t^n)$ in denominator of unrefined Hilbert series for D_4 SQCD theories with N_f flavours with $1 \leq N_f \leq 2$ and upper and lower bounds for 3-flavour case

poles to the lowest possible order maintaining the bounds, is $2.18 + 3.20 + 4.6 - 3.24 = 48$, and the highest is $2.15 + 3.15 + 4.14 - 3.24 = 59$. If we include the [0,0,1] and 3[1,0,1] invariants, the refined series has order $2.18 + 3.27 + 4.27 - 3.24 = 153$ (51 in each flavour), and if we do not, it has order $2.18 + 3.26 + 4.18 - 3.24 = 114$ (38 in each flavour).

We could try calculating the (refined) Hilbert series through un-Higgsing of the Hilbert series for A_2 with 6 flavours of both quarks and antiquarks, identifying fugacities $t_i = u_i$ and then the t_i in groups of 3. Recall that one flavour of V+S+C in D_4 decomposes to give three each of the fundamental and antifundamental plus six scalars in A_2, and the adjoint gives the adjoint of A_2 plus three quark-antiquark pairs and two scalars, so the 3-flavour (V+S+C) case of D_4 corresponds to the 6-flavour case of A_2 with 12 added scalars.

5.6.3 D_5 gauge group

While D_5 is not an exceptional group, it follows the 'sequence' of exceptional groups down from E_8 by progressively removing the rightmost node in the Dynkin diagram, drawing them so that the 'off-line' node is attached to the third node from the left. As with the E_n groups, it is the group that arises from Kaluza-Klein toroidal compactification of $n = 5$ dimensions of M-theory (or $n - 1 = 4$ of either type II string theory), though the actual U-duality group is only its discrete subgroup $D_{5(5)}(\mathbb{Z})$ (or $E_{n(n)}(\mathbb{Z})$).

The second Dynkin index of both spinor representations is 4, because the normalization of the vector to have Dynkin index 2, for any special orthogonal group, means that the spinors have a value of the dimension of the representation, which is here 16, divided by 4. That of the adjoint is 16, which is equal to the (dual) Coxeter number. One expects therefore that the complete intersection would occur at 4 total flavours of spinors, with the relation at total order 16.

The refined series, for up to 4 total spinors, are as follows:

$$\begin{aligned}
PL(g^{(2,0,D_5)}(t,u)) &= [0]t^4 \\
PL(g^{(3,0,D_5)}(t,u)) &= [0,2]t^4 \\
PL(g^{(4,0,D_5)}(t,u)) &= [0,2,0]t^4 - [0,0,0]t^{16} \\
PL(g^{(1,1,D_5)}(t,u)) &= tu + t^2u^2 \\
PL(g^{(2,1,D_5)}(t,u)) &= [1]tu + [2]t^2u^2 + [0]t^4 \\
PL(g^{(3,1,D_5)}(t,u)) &= [1,0]tu + [2,0]t^2u^2 + [0,2]t^4 + [2,0]t^5u \\
&\quad -[0,0]t^9u - [0,0]t^{12}u^4 \\
PL(g^{(2,2,D_5)}(t,u)) &= [1;1]tu + [2;2]t^2u^2 + [1;1]t^3u^3 + [0;0]t^4 + [0;0]u^4 \\
&\quad +[0;0]t^6u^2 + [0;0]t^2u^6 - [0;0]t^6u^6 - [0;0]t^8u^8
\end{aligned}$$

We see that the relation in the (4,0) case, and the higher of the two relations in the (3,1) and (2,2) cases, indeed is at total order 16.

The partially unrefined series are as follows:

$$
\begin{aligned}
g^{(2,0,D_5)}(t,u) &= \frac{1}{1-t^4} \\
g^{(3,0,D_5)}(t,u) &= \frac{1}{(1-t^4)^6} \\
g^{(4,0,D_5)}(t,u) &= \frac{1-t^{16}}{(1-t^4)^{20}} = \frac{1+t^4+t^8t^{12}}{(1-t^4)^{19}} \\
g^{(1,1,D_5)}(t,u) &= \frac{1}{(1-tu)(1-t^2u^2)} \\
g^{(2,1,D_5)}(t,u) &= \frac{1}{(1-tu)^2(1-t^2u^2)^3(1-t^4)} \\
g^{(3,1,D_5)}(t,u) &= \frac{(1-t^9u)(1-t^{12}u^4)}{(1-tu)^3(1-t^2u^2)^6(1-t^4)^6(1-t^5u)^6} \\
g^{(2,2,D_5)}(t,u) &= \frac{(1-t^6u^6)(1-t^8u^8)}{(1-tu)^4(1-t^2u^2)^9(1-t^3u^3)^4(1-t^4)(1-u^4)(1-t^6u^2)(1-t^2u^6)} \\
&= \frac{(1+t^3u^3)(1+t^2u^2+t^4u^4+t^6u^6)}{(1-tu)^4(1-t^2u^2)^8(1-t^3u^3)^3(1-t^4)(1-u^4)(1-t^6u^2)(1-t^2u^6)}
\end{aligned}
$$

Recalling that the number of invariants for E_6 with 2, 3 or 4 total flavours was the same however many were fundamentals and however many antifundamentals, we see that this is not the case with 2 total spinors here; there is only one invariant, at order t^4, for 2 spinors of the same type, but there are 2, at orders tu and t^2u^2, for one of each type. There are no invariants at all when there is only one spinor (of either type); this can be seen from the general formula for the symmetric power of the spinor, which is as follows:

$$
Sym^k[0,0,0,0,1]_{D_5} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} [m,0,0,0,k-2m]_{D_5}
$$

As we see, the singlet only occurs for $k = 0$.

The dimension of the moduli space is given by the order of the pole at $t = 1$ in the fully unrefined (setting u to t) series, and is given by the number of degrees of freedom in the fundamental fields minus the number of broken generators of the gauge group. It can be seen by inspection of the partially refined series, except in the (3,1) case where the relation does not factorize into terms in the denominator and we must therefore write the series in its

(N_f, N_a)	$16(N_f + N_a)$	No. invariants	No. relations	No. broken gens	No. unbroken gens	Unbroken gauge group
(1,1)	32	2	0	30	15	A_3
(2,0)	32	1	0	31	14	G_2
(3,0) or (2,1)	48	6	0	42	3	A_1
(4,0)	64	20	1	45	0	$\emptyset$
(3,1) or (2,2)	64	21	2	45	0	$\emptyset$

Table 5.52: Numbers of invariants, relations, broken and unbroken generators and unbroken gauge groups for D_5 SQCD theories with up to 4 total flavours of spinors

fully unrefined form to show the dimension:

$$g^{(3,1,D_5)}(t,t) = \frac{(1+t^2+t^4+t^6+t^8)(1+t^4+t^8+t^{12})}{(1-t^2)^2(1-t^4)^{11}(1-t^6)^6}$$

We see the dimension is 19, which is equal to 64 (the number of matter degrees of freedom) minus 45 (the dimension of D_5 which is completely broken by the Higgsing). This is the same as in the (4,0) and (2,2) cases.

The numbers of invariants, relations and broken and unbroken generators of the gauge group and the unbroken gauge groups are listed in Table 5.52:

We see that although the Higgsing on two spinors of the same type gives a different residual gauge group (G_2) to doing it on one spinor of each type ($A_3 = SU(4)$), subsequent Higgsing on another flavour of either type of spinor is the same in both cases ($A_1 = SU(2)$). This occurs because either spinor of D_5 decomposes to two G_2 fundamentals (and two scalars), and these Higgs G_2 to A_1, while it would decompose to two flavours (quark plus antiquark) of A_3 and Higgsing on these would also give A_1 as the final gauge group.

We will demonstrate the Higgsing of the (4,0) case on 2 flavours using the $SU(4) \times U(1)$ notation as follows. Higgsing on the fourth flavour, we obtain using the method outlined in the B_3 (with only spinors) case earlier:

- $[0,2,0;4] \to [2,0;2] + [1,1;3] + [0,2;4]$ - so no scalars or invariants, as expected from the fact that D_5 is not Higgsed by just one flavour
- $[0,0,0;16] \to [0,0;12]$ (relation)

Performing the second Higgsing on the third flavour, we obtain:

- $[2,0;2] \to [0;0]+[1;1]+[2;2]$ - one invariant from Higgsed two flavours, two scalars from decomposition of one remaining spinor of D_5 to two fundamentals of G_2 and two scalars

- $[1,1;3] \to [1;1]+[2;2]+[0;2]+[1;3]$ - two scalars from decomposition of the other remaining spinor

- $[0,2;4] \to [2;2]+[1;3]+[0;4]$

- $[0,0;12] \to [0;8]$ (relation)

We see that we have remaining $(3[2]+[0])t^2 + 2[1]t^3 + t^4 - t^8$. Identifying the fugacities t_1 with t_3 and t_2 with t_4 in the refined series for G_2 with 4 flavours, which corresponds to the mapping of Dynkin labels from $SU(4)$ to $SU(2)$ as $[n_1, n_2, n_3] \to [n_1 + n_3]$ (simply discarding n_2!), we find that the $[2,0,0]t^2 + [0,0,1]t^3 + t^4 - t^8$ indeed map to this.

Higgsing the (3,1) case on two of the three flavours, we have, doing it on the third flavour:

- $[1,0;1;1] \to [0;0;1]+[1;1;1]$ - one scalar from decomposition of remaining flavour, discard this in second step

- $[2,0;2;2] \to [0;0;2]+[1;1;2]+[2;2;2]$

- $[0,2;4;0] \to [2;2;0]+[1;3;0]+[0;4;0]$

- $[2,0;5;1] \to [0;2;1]+[1;3;1]+[2;4;1]$

- $[0,0;9;1] \to [0;6;1]$ (relation)

- $[0,0;12;4] \to [0;8;4]$ (relation)

Higgsing again on the second flavour:

- $[1;1;1] \to [0;1]+[1;1]$ - second scalar from decomposition of remaining flavour

- $[0;0;2] \to [0;2]$

- $[1;1;2] \to [0;2]+[1;2]$

- $[2;2;2] \to [0;2]+[1;2]+[2;2]$

- $[2;2;0] \rightarrow [0;0] + [1;0] + [2;0]$ - one scalar from decomposition of antiflavour plus one invariant in Higgsed flavours
- $[1;3;0] \rightarrow [1;0] + [2;0]$ - second scalar from decomposition of antiflavour
- $[0;4;0] \rightarrow [2;0]$
- $[0;2;1] \rightarrow [1;1]$
- $[1;3;1] \rightarrow [1;1] + [2;1]$
- $[2;4;1] \rightarrow [1;1] + [2;1] + [3;1]$
- $[0;6;1] \rightarrow [3;1]$ - relation cancels out invariant in last step
- $[0;8;4] \rightarrow [4;4]$ - relation

We can, after discarding the invariants and scalars, reassemble these latter into $SU(2)$ representations (taking $t \rightarrow tz, u \rightarrow t/z$):

- $3([0;2] + [1;1] + [2;0]) \rightarrow 3[2]t^2$
- $2([1;2] + [2;1]) \rightarrow 2[1]t^3$
- $[1;1] \rightarrow [0]t^2$
- $[2;2] \rightarrow [0]t^4$
- $[4;4] \rightarrow [0]t^8$ (relation)

We see that this matches the Higgsing of the (4,0) case on two flavours, and therefore also matches the G_2 Hilbert series for 4 flavours (with the mapping from $SU(4)$ to $SU(2)$ understood).

Higgsing the (2,2) case on the two antiflavours is simple, just set u to 1, one obtains one invariant in the Higgsed antiflavours from $[0;0]u^4$, four scalars from the decomposition of the flavours from $[1;1]tu$ and the same Hilbert series as for the other two Higgsings.

The process of un-Higgsing would seem to be particularly hard in this case, in the case of Higgsing on one flavour and one antiflavour or two antiflavours there is the difficulty with the antiflavour(s) as we have seen with B_3, in the case of Higgsing on two flavours there are the issues that

Order(t,u)	Young tableau	$SU(3) \times SU(2)$ representation	Dimension
(2,2)	[2,. . .;2]	[2,0;2]	18
(2,6)	[0,1,. . .;0]	[0,1;0]	3
(3,3)	[1,1,. . .;1]	[1,1;1]	16
(4,0)	[0,2,. . .;0]	[0,2;0]	6
(6,2)	[0,3,. . .;0]	[0,3;0]	10

Table 5.53: Young tableaux (in $SU(N)$ representation form) corresponding to primitive invariants of D_5 SQCD theories and the corresponding representations and dimensions in the case of 3 flavours and 2 antiflavours

N_f	$d(0,4)$	$d(1,1)$	$d(2,2)$	$d(2,6)$	$d(3,3)$	$d(4,0)$	$d(6,2)$	deg $P(t,u)$	$dim(\mathcal{M})$
0	1	0	0	0	0	0	0	0	1
1	1	2	3	0	0	0	0	0	6
2	1	4	9	1	4	1	1	14,14	19
3	1	6	≥ 15 ≤ 18	≥ 2 ≤ 3	≥ 8 ≤ 16	≥ 2 ≤ 6	≥ 2 ≤ 10	?, ?	35
$N_f \geq 2$	1	$2N_f$	?	?	?	?	?	?, ?	$16N_f - 13$

Table 5.54: Powers of $(1 - t^n u^m)$ in denominator of unrefined Hilbert series for D_5 SQCD theories with N_f flavours with $0 \leq N_f \leq 3$ and 2 antiflavours and upper and lower bounds for 3-flavour and 2-antiflavour case

the Higgsing must be done twice and also that the $SU(2)$ representations must be 'expanded' into $SU(4)$ ones in some way. We will not discuss this further here.

Returning to the calculation of Hilbert series with 5 total spinors, we were again unable to do so using Mathematica because of memory constraints. Once again, however, we provide in Table 5.53 the analysis of the lower and upper bounds based on the lower cases, with the number of antiflavours (conjugate spinors) fixed at 2 in the first case and 1 in the second case:

The lower and upper bounds for the powers of $(1 - t^n u^m)$ in the denominator in the $(N_f, 2)$ case are shown in Table 5.54:

Setting $u = t$, the coefficients of $(1 - t^2)$ in the 3-spinor case is fixed at $2N_f = 6$, that of $(1-t^4)$ is between 18 and 29, that of $(1-t^6)$ between 8 and 16 (or 20 if the [0,0,1;3] invariant is considered) and that of $(1-t^8)$ between 2 and 10. Knowing the moduli space must have dimension $35 = 5.16 - 45$, there must be 6, 18, 8 and 2 at each order with one extra at order 4, 6 or 8; the degree of the numerator must range between $6.2 + 18.4 + 8.6 + 2.8 - 5.16 + 4 = 72$ and 76. (That for 5 spinors and no conjugate spinors

Order(t,u)	Young tableau	$SU(4)$ representation	Dimension
(2,2)	[2,...]	[2,0,0]	10
(4,0)	[0,2,...]	[0,2,0]	20
(5,1)	[2,0,1,...]	[2,0,1]	36

Table 5.55: Young tableaux (in $SU(N) \times SU(N)$ representation form) corresponding to primitive invariants of D_5 SQCD theories and the corresponding representations and dimensions in the case of 4 flavours and one antiflavour

N_f	$d(1,1)$	$d(2,2)$	$d(4,0)$	$d(5,1)$	deg $P(t,u)$	$dim(\mathcal{M})$
1	1	0	0	0	0	2
2	2	3	1	0	0	6
3	3	6	6	6	21,5	19
4	4	≥ 9 ≤ 10	≥ 11 ≤ 20	≥ 12 ≤ 36	$?,?$	35
$N_f \geq 3$	N_f	?	?	?	$?,?$	$16N_f - 29$

Table 5.56: Powers of $(1 - t^n u^m)$ in denominator of unrefined Hilbert series for D_5 SQCD theories with N_f flavours with $1 \leq N_f \leq 3$ and 1 antiflavour and upper and lower bounds for 4-flavour and 1-antiflavour case

necessarily has order $35.4 - 5.16 = 60$).

For the case with one antiflavour, we have the invariants as in Table 5.55:

And in Table 5.56 are the (bounds for the) powers in the denominator:

Similarly, in the fully unrefined case we have powers of $(1 - t^2)$ fixed at $N_f = 4$, of $(1 - t^4)$ between 20 and 30 and of $(1 - t^6)$ between 12 and 36 respectively. With these necessarily adding up to 36, one of the powers must be reduced by one; supposing it is the power of $(1 - t^2)$, the numerator is of degree $3.2 + 20.4 + 12.6 - 5.16 = 78$.

5.6.4 E_6 gauge group

We have already considered Higgsing of E_6 with N_f flavours and one antiflavour on the antiflavour, where we remove the $(1 - u^3)$ factor from the denominator, set the antiflavour fugacity u to 1 and remove factors of $(1-t)$ (in the partially unrefined case) or $(1-t_i)$ (fully refined) to get the F_4 series with N_f flavours. In this section we will consider Higgsing in more general theories with either no antiflavours or 2 or more of them, and also Higgsing on more than one (anti)flavour, although we have already dealt with the case of exactly 2 antiflavours in the D_4 section, E_6 being Higgsed to D_4 on two total flavours.

We have already described the explicit Higgsing in terms of explicit representations of E_6 fundamentals as three 3×3 matrices, corresponding to its decomposition as its maximal subgroup $SU(3)^3$, and also as a 6×6 antisymmetric matrix and a 2×6 matrix, corresponding to that as $SU(6) \times SU(2)$, which is also a maximal subgroup, and how the interaction of the breakings of the subgroups results in the breaking of E_6 to F_4 and D_4 progressively; this is described in more detail in [44].

The k-th symmetric product of the fundamental of E_6 is given by the following formula:

$$Sym^k[1,0,0,0,0,0]_{E_6} = \sum_{m=0}^{\lfloor \frac{k}{3} \rfloor} \sum_{n=0}^{\lfloor \frac{k-3m}{2} \rfloor} [m,0,0,0,k-2m,0]_{E_6}$$

This expression shows that we do have symmetric invariants solely in flavours or antiflavours, the primitive one being of order 3.

This can be expressed in PE form as follows:

$$PE(t[1,0,0,0,0,0]_{E_6}) = \frac{1}{1-t^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [m,0,0,0,n,0] t^{m+2n}$$

For the Hilbert series, we consider first the case with up to 4 flavours but no antiflavours. As before, we rewrite the refined series in terms of fugacities t_i for $1 \leq i \leq N_f$:

$$\begin{aligned}
g^{(1,0,E_6)}) &= \frac{1}{1-t^3} \\
g^{(2,0,E_6)}) &= \frac{1}{(1-t_1^3)(1-t_1^2 t_2)(1-t_1 t_2^2)(1-t_2^2)} \\
g^{(3,0,E_6)}) &= \frac{1}{(\prod_{1 \leq i \leq j \leq k \leq 3}(1-t_i t_j t_k))(1-t_1^2 t_2^2 t_3^2)}
\end{aligned}$$

The invariants at order 3 are as for 3 flavours but with i, j and k ranging from 1 to 4 instead of 3; that at order 12 is $\prod_{i=1}^4 t_i^3$ and the relation at order 24 is $\prod_{i=1}^4 t_i^6$, which reduces to a $(1 + \prod_{i=1}^4 t_i^3)$ term in the numerator.

Because the invariants at order 6 transform in the $[0, 0, 2, \ldots]$ representation ([0,0,2] of $SU(4)$ specifically), we cannot list them easily in a product form similarly to above. (This is not an issue for 3 flavours because it reduces to [0,0], i.e. the singlet.) They are given by the second symmetric product of the [0,0,1] representation, i.e. $Sym^2(t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)$.

When the theory is Higgsed on one flavour, the factor $(1-t_{N_f}^3)$ is removed from the denominator, then t_{N_f} is set to 1 and terms in $(1 - t_i)$ for $1 \leq i \leq N_f - 1$, corresponding to the scalars arising from the reduction of the remaining flavours to F_4, are removed too.

When $N_f = 2$ in the original theory, after removing the $(1 - t_2^3)$ term and the one scalar, we are left with one invariant t_1^2 and one t_1^3, which is the same as in the F_4 theory with one flavour. When $N_f = 3$, there are two scalars and we are left with 3, 4 and 1 invariant at orders 2, 3 and 4 respectively, again as for F_4 with one less flavour. In the $N_f = 4$ case, the Higgsing results in the following invariants, after removing the $(1-t_4^3)$ term and the three scalars from $t_it_4^2$ (here $i, j, \ldots$ take values 1 to 3):

- t_it_j, $i \leq j$ (from $t_it_jt_4$) in the [2,0] representation of $SU(3)$, there are 6 of them
- $t_it_jt_k$, $i \leq j \leq k$ (unchanged) in [3,0] (10)
- $t_i^2t_j^2$, $i < j$ and $t_i^2t_jt_k$, $i \neq j \neq k$ (cancelling out t_4^2) in [0,2] (6) (the second symmetric power of $t_1t_2 + t_1t_3 + t_2t_3$)
- $t_i^2t_j^2t_k$, $i \neq j \neq k$ (cancelling out t_4) in [0,1] (3)
- $t_1^2t_2^2t_3^2$ (unchanged) in [0,0] (1)
- $t_1^3t_2^3t_3^3$ (cancelling out t_4^3) in [0,0] (1)
- $t_1^6t_2^6t_3^6$ (cancelling out t_4^6) (relation) in [0,0] (1)

In $U(4) \to U(3)$, or $SU(4) \times U(1) \to SU(3) \times U(1)$ notation, with the $U(1)$ charge being the order of the invariant, we have

- $[3, 0, 0; 2] \to [0, 0; 0] + [1, 0; 1] + [2, 0; 2] + [3, 0; 3]$; first term is invariant of t_4, second is scalars
- $[0, 0, 2; 6] \to [0, 2; 4] + [0, 1; 5] + [0, 0; 6]$

- $[0,0,0;4n] \to [0,0;3n]$, here for $n=3$ (invariant) and $n=6$ (relation)

We see that, in the freely generated and complete intersection cases, Higgsing of E_6 theories with N_f flavours and no antiflavours on one flavour results in the F_4 theory with $N_f - 1$ flavours.

We will now consider theories with both flavours and antiflavours (still no more than 4 total). We have already seen the Higgsing on one (anti)flavour when there is only one of this type of field, so we will not consider this case further. This time we will write the plethystic logarithm of the series in t_i form:

$$\begin{aligned} PL(g^{(1,1,E_6)}(t,u)) &= tu + t^3 + u^3 + t^2u^2 \\ PL(g^{(2,1,E_6)}(t,u)) &= (t_1+t_2)u + t_1^3 + t_1^2t_2 + t_1t_2^2 + t_2^3 + u^3 + (t_1^2 + t_1t_2 + t_2^2)u^2 \\ &\quad + t_1^2t_2^2u \end{aligned}$$

The first case has already been covered, whether we Higgs on t or u; Higgsing the second on t_2 gives rise to the one (cubic) invariant of the Higgsed flavour, two scalars from the decomposition of the remaining flavour and the antiflavour, three invariants at order 2, four at order 3 and one at order 4, as required for the F_4 theory with 2 flavours.

For simplicity we will stick with $SU(N_f)$ (and $SU(N_a)$) notation for the (3,1) and (2,2) cases:

$$\begin{aligned} PL(g^{(3,1,E_6)}(t,u)) &= [1,0]tu + [3,0]t^3 + [0,0]u^3 + [2,0]t^2u^2 + [0,2]t^4u \\ &\quad + [0,1]t^5u^2 + [0,0]t^6 + [0,0]t^9u^3 - [0,0]t^{18}u^6 \\ PL(g^{(2,2,E_6)}(t,u)) &= [1;1]tu + [3;0]t^3 + [0;3]u^3 + [2;2]t^2u^2 + [0;1]t^4u \\ &\quad + [1;0]tu^4 + [1;1]t^3u^3 + [0;0]t^4u^4 + [0;0]t^6u^6 - [0;0]t^{12}u^{12} \end{aligned}$$

In the (3,1) case, Higgsing on one of the 3 flavours decomposes the $SU(3) \times U(1)(\times U(1))$ global symmetry group to $SU(2) \times U(1)(\times U(1))$:

- $[1,0;1;1] \to [1;1;1] + [0;0;1]$; second term corresponds to scalar in decomposition of antiflavour
- $[3,0;3;0] \to [3;3;0] + [2;2;0] + [1;1;0] + [0;0;0]$; last term is invariant in Higgsed flavour, previous one is scalars in decomposition of other two flavours

- $[0,0;0;3] \to [0;0;3]$
- $[2,0;2;2] \to [2;2;2] + [1;1;2] + [0;0;2]$
- $[0,2;4;1] \to [0;4;1] + [1;3;1] + [2;2;1]$
- $[0,1;5;2] \to [0;4;2] + [1;3;2]$
- $[0,0;6;0] \to [0;4;0]$
- $[0,0;9;3] \to [0;6;3]$
- $[0,0;18;6] \to [0;12;6]$ (relation)

The first $U(1)$ (the second term) gives the overall power of the t_i fugacities, and the $SU(2)$ (the first term) their breakdown into t_1 and t_2 powers; the second $U(1)$ (the last term) gives the power of u. F_4 only has one type of fundamental, so we must incorporate the u fugacity into the t_i; we relabel u to t_{N_f} (here t_3). Summing terms with the same sum of the two $U(1)$ charges into $SU(3) \times U(1)$ representations, one obtains:

- $[1;1;1] + [2;2;0] + [0;0;2] \to [2,0;2]$
- $[3;3;0] + [0;0;3] + [1;1;2] + [2;2;1] \to [3,0;3]$
- $[2;2;2] + [1;3;1] + [0;4;0] \to [0,2;4]$
- $[0;4;1] + [1;3;2] \to [0,1;5]$
- $[0;4;2] \to [0,0;6]$
- $[0;6;3] \to [0;0;9]$
- $[0;12;6] \to [0;0;18]$ (relation)

These are the same invariants and relations that occur in the F_4 theory with 3 flavours, as expected.

In the (2,2) case, we Higgs on one of the antiflavours and decompose the second $SU(2) \times U(1)$ to $U(1)$:

- $[1;1;1;1] \to [1;1;1] + [1;1;0]$; second term is scalar in decomposition of remaining antiflavour
- $[3;3;0;0] \to [3;3;0]$

- $[0;0;3;3] \to [0;0;3] + [0;0;2] + [0;0;1] + [0;0;0]$; last term is invariant in Higgsed antiflavour, previous one is scalars in decomposition of flavours
- $[2;2;2;2] \to [2;2;2] + [2;2;1] + [2;2;0]$
- $[0;4;1;1] \to [0;4;1] + [0;4;0]$
- $[1;1;0;4] \to [1;1;2]$
- $[1;3;1;3] \to [1;3;2] + [1;3;1]$
- $[0;4;0;4] \to [0;4;2]$
- $[0;6;0;6] \to [0;6;3]$
- $[0;12;0;12] \to [0;12;6]$ (relation)

Since these are the same as in the Higgsing of the $(3,1)$ case on one of the 3 flavours, they again recombine to form the same invariants and relations as in the F_4 theory with 3 flavours, as required.

When considering higher (total) numbers of flavours, unfortunately because of memory and processor constraints we were unable to calculate Hilbert series, even the unrefined cases, for any of the cases with 5 total flavours or more, i.e. any non-complete intersections, unlike in the case of B_3 with one spinor and up to 7 vectors. We do have the unrefined series for F_4 with 4 flavours, but not the refined series.

We will now move on to the case of Higgsing on more than one (anti)flavour. Recall that the residual gauge group is D_4. The refined Hilbert series for the case with no antiflavours are as follows:

$$\begin{aligned} PL(g^{(3,0,E_6)}(t,u)) &= [3,0]t^3 + [0,0]t^6 \\ PL(g^{(4,0,E_6)}(t,u)) &= [3,0,0]t^3 + [0,0,2]t^6 + [0,0,0]t^{12} - [0,0,0]t^{24} \end{aligned}$$

and when there are antiflavours:

$$
\begin{aligned}
PL(g^{(2,1,E_6)}(t,u)) &= [1]tu + [3]t^3 + [0]u^3 + [2]t^2u^2 + [0]t^4u \\
PL(g^{(3,1,E_6)}(t,u)) &= [1,0]tu + [3,0]t^3 + [0,0]u^3 + [2,0]t^2u^2 + [0,2]t^4u \\
&\quad +[0,1]t^5u^2 + [0,0]t^6 + [0,0]t^9u^3 - [0,0]t^{18}u^6 \\
PL(g^{(2,2,E_6)}(t,u)) &= [1;1]tu + [3;0]t^3 + [0;3]u^3 + [2;2]t^2u^2 + [0;1]t^4u \\
&\quad +[1;0]tu^4 + [1;1]t^3u^3 + [0;0]t^4u^4 + [0;0]t^6u^6 - [0;0]t^{12}u^{12}
\end{aligned}
$$

Higgsing the (3,0) case on two flavours, the invariants at order 3 give the 4 invariants of the Higgsed flavours, 3 scalars corresponding to the reduction of the remaining flavour to a vector, a spinor, a conjugate spinor and three scalars, two invariants at order 2 ($t_1^2t_2$ and $t_1^2t_3$ in the original theory) and one at order 3 (t_1^3 which remains unchanged). That at order 6, $t_1^2t_2^2t_3^2$, reduces to t_1^2, another invariant at order 2, giving 3 in total, along with 1 at order 3. This is as in the D_4 theory with one flavour of (vector+spinor+conjugate spinor).

Higgsing the (2,1) case on the two flavours, we get the 4 invariants of the Higgsed flavours from the $[3]t^3$ term, two scalars from the $[1]tu$ term (remember u is the fugacity we are keeping, though the $SU(2)$ Dynkin label relates to t) and one from $[0]t^4u$ giving 3 in total, 3 invariants at order 2 from $[2]t^2u^2$ and one at order 3 from $[0]u^3$, again as in the D_4 theory with one flavour of V+S+C.

Higgsing the (2,1) case on one flavour and one antiflavour, we get the four invariants of the Higgsed flavours one each from the first four terms, three scalars from the first, second and fourth terms, three invariants of order 2 from the second, fourth and fifth terms and one invariant of order 3 from the second term, again as in the D_4 theory with one flavour of V+S+C.

We will not show the various Higgsings of the three E_6 theories with 4 total flavours, but they all give the 4 invariants of the two Higgsed (anti)flavours and the 6 scalars from the decomposition of the two remaining (anti)flavours, and the 9 invariants at order 2, 8 at order 3, 3 at order 4 and one at order 6 and the relation at order 12, which are the same as in the D_4 theory with two flavours of V+S+C, as required.

We will now consider reversing the process and un-Higgsing on F_4 series. Reverting temporarily back to the case with exactly one antiflavour, we recall that in the non-complete intersection case of B_3 with one spinor and $N_v \geq 5$ vectors, the power of $(1-t^2)$ (where t is the vector fugacity) is not the same as that in the G_2 series with N_v fundamentals, but rather that of $(1-t)$ goes as the dimension of the quadratic invariant $[2,\ldots]$ in $SU(N_v)$ and that of $(1+t)$ goes as $6N_v - 15$ for $N_v \geq 5$ (as far as we know, i.e. up to 7 vectors). Therefore by analogy we do not expect the powers, of $(1-t^2u^2)$, $(1-t^3)$, $(1-t^4u)$, $(1-t^5u^2)$ and $1-t^6$, to be the same in the E_6 series with one antiflavour (although that of $(1-t^3u^2)$ in the B_3 series is the same as that of $(1-t^3)$ in the G_2 series).

We know, however, that they are at least as high, because after removal of the $(1-u^3)$ and $(1-tu)^{N_f}$ factors corresponding to the invariant in the Higgsed antiflavour and the scalars resulting from the decomposition of the N_f fundamentals under F_4, and setting u to 1, the Hilbert series are the same (in their lowest terms).

Assuming that they are the same, as our lower bound, we see therefore that the degree of the numerator in t, the flavour fugacity, of the E_6 unrefined Hilbert series with 4 flavours and 1 antiflavour is at least 84, which is the degree of the numerator of the unrefined series for F_4 flavours; the four extra powers coming from the $(1-tu)^4$ in the denominator are cancelled out by the four extra degrees of freedom which become scalars under the decomposition. As for the degree in u, we have, again as a lower bound, $1.3+4.1+10.2+16.0+14.1+8.2+4.0-1.27 = 30$. As for the B_3 case with one spinor, however, we do not expect to be able to do the un-Higgsing.

We will now return to un-Higgsing of F_4 theories to E_6 theories with only one type of fundamental field, WLOG flavours. Recall that for Higgsing E_6 with N_f flavours on one flavour, there is one cubic invariant from the Higgsed flavour and one scalar from the decomposition of each of the remaining flavours, and the refined series for F_4 are as follows:

$$
\begin{aligned}
PL(g^{(1,F_4)}(t)) &= t^2 + t^3 \\
PL(g^{(2,F_4)}(t)) &= [2]t^2 + [3]t^3 + [0]t^4 \\
PL(g^{(3,F_4)}(t)) &= [2,0]t^2 + [3,0]t^3 + [0,2]t^4 + [0,1]t^5 + [0,0]t^6 + [0,0]t^9 - [0,0]t^{18}
\end{aligned}
$$

In the 1-flavour (of F_4) case, the invariant from the Higgsed flavour, the scalar and the two invariants assemble into a [3] of $SU(2)$ at order 3, as for E_6 with 2 flavours; in the 2-flavour case, the invariant from the Higgsed flavour is in the [0;0] representation of $SU(2)\times U(1)$ and the scalars are in [1;1] and these along with the quadratic assemble into a [3,0;3] of $SU(3)\times U(1)$, with the [0;4] becoming a [0,0;6], as for E_6 with 3 flavours. In the 3-flavour case, we have the invariant from the Higgsed flavour in [0,0;0] and the scalars in [1,0;1]; assembling them into $SU(4) \times U(1)$ representations, we have

- $[0,0;0] + [1,0;1] + [2,0;2] + [3,0;3] \to [3,0,0;3]$
- $[0,2;4] + [0,1;5] + [0,0;6] \to [0,0,2;6]$
- $[0,0;9] \to [0,0,0;12]$
- $[0,0;18] \to [0,0,0;24]$ (relation)

These are as in the E_6 theory with 4 flavours, as required.

For us to be able to un-Higgs the 4-flavour F_4 case to E_6 with 5 flavours, we would need to know the 4-flavour F_4 Hilbert series in its refined form, which we are not close to at the moment (the numerator has degree 216, or 54 in each flavour fugacity). We could try to derive this latter series by un-Higgsing the D_4 series with 3 flavours of V+S+C, which as things stand we would have to obtain itself by un-Higgsing the A_2 series with 6 flavours, identifying quark and antiquark fugacities and then the flavours in groups of 3.

Un-higgsing to the case of E_6 with $N_f \geq 3$ or more flavours and 2 antiflavours would have to start from D_4 with N_f flavours of V+S+C and would face the same problems as with obtaining E_6 Hilbert series with N_f flavours and one antiflavour from those of F_4 with N_f flavours.

5.6.5 E_7 gauge group

We recall that E_7 is Higgsed by one fundamental to E_6, with any remaining fundamentals being decomposed to a fundamental **27**, an antifundamental $\bar{\mathbf{27}}$ and two scalars. Recall the E_7 refined series:

$$
\begin{aligned}
PL(g^{(1,E_7)}) &= t^4 \\
PL(g^{(2,E_7)}) &= [0]t^2 + [4]t^4 + [0]t^6 \\
PL(g^{(3,E_7)}) &= [0,1]t^2 + [4,0]t^4 + [0,3]t^6 + [2,0]t^8 + [0,0]t^{12} + [0,0]t^{18} - [0,0]t^{36}
\end{aligned}
$$

We can write the PL in the 2-flavour case in terms of distinct fugacities t_1, t_2 as follows:

$$PL(g^{(2,E_7)}) = t_1t_2 + t_1^4 + t_1^3t_2 + t_1^2t_2^2 + t_1t_2^3 + t_2^4 + t_1^3t_2^3$$

Higgsing this on t_2 gives the invariant in the Higgsed flavour, two scalars resulting from the decomposition of the remaining flavour and one invariant at order 2, two at order 3 and one at order 4.

We will again use the $SU(3) \times U(1)$ notation for the 3-flavour case:

- $[0, 1; 2] \to [0; 2] + [1; 1]$; second term is one scalar for each of flavours 1 and 2
- $[4, 0; 4] \to [4; 4] + [3; 3] + [2; 2] + [1; 1] + [0; 0]$; last term is invariant in Higgsed flavour 3, previous term is other scalar for each of flavours 1 and 2
- $[0, 3; 6] \to [0; 6] + [1; 5] + [2; 4] + [3; 3]$
- $[2, 0; 8] \to [2; 6] + [1; 5] + [0; 4]$
- $[0, 0; 12] \to [0; 8]$
- $[0, 0; 18] \to [0; 12]$
- $[0, 0; 36] \to [0; 24]$ (relation)

Setting $u = t$ in the E_6 series with $N_f = N_a$, we get the following series:

$$\begin{aligned} PL(g^{(1,1,E_6)}(t,t)) &= t^2 + 2t^3 + t^4 \\ PL(g^{(2,2,E_6)}(t,t)) &= ([2] + [0])t^2 + 2[3]t^3 + ([4] + [2] + [0])t^4 + 2[1]t^5 \\ &\quad +([2] + [0])t^6 + [0]t^8 + [0]t^{12} - [0]t^{24} \end{aligned}$$

We see that the series resulting from Higgsing of E_7 series with 2 and 3 flavours on one flavour are the same as these, as required.

We will again Higgs the series for 3 flavours on 2 of those flavours, where the residual gauge group is D_4:

- The [0,1] term gives rise to one invariant of the Higgsed flavours (from t_2t_3) and two scalars (from t_1t_2 and t_1t_3).

- The [4,0] term gives five more invariants of the Higgsed flavours, four more scalars, three invariants at order 2 in t_1, two at order 3 and one at order 4.
- The [0,3] term, which is the 3rd symmetric power of $(t_1t_2 + t_1t_3 + t_2t_3)$, gives rise to one invariant of the Higgsed flavours, two further scalars, three invariants at order 2 in t_1 and four invariants at order 3. This completes the 7 invariants on the two Higgsed flavours and the 8 scalars from the decomposition of the remaining **56** of E_7 into two (vector+scalar+conjugate spinor) flavours of D_4 and eight scalars.
- The [2,0] term gives rise to 3 invariants at order 2, 2 at order 3 and one at order 4.
- The [0,0] term at order 12 gives rise to one invariant at order 4, that at order 18 gives one at order 6 and the relation at order 36 gives a relation at order 12. In total at orders 2, 3 and 4 we get 9, 8 and 3 invariants, as in the D_4 theory with two flavours of V+S+C, as required.

Considering un-Higgsing on one flavour from E_6 with (N_f, N_f) (they must be equal) to E_7 with $N_f + 1$ flavours, there is one (quartic) invariant in the Higgsed flavour and $2N_f$ scalars from the decomposition of the remaining flavours under E_6. In the (1,1) case, $1+t+t^2+t^3+t^4$ from the PL augmented by the invariant and scalars assemble to give a [4] at order t^4, and the extra terms $t + t^3$ give [0] at orders t^2 and t^6 as required for E_7 with 2 flavours. In the (2,2) case, the invariant forms a [0] at order 0 and the four scalars two [1]s at order 1, so we have:

- $[0;0] + [1;1] + [2;2] + [3;3] + [4;4] \to [4,0;4]$
- $[1;1] + [0;2] \to [0,1;2]$
- $[3;3] + [2;4] + [1;5] + [0;6] \to [0,3;6]$
- $[0;4] + [1;5] + [2;6] \to [2,0;8]$
- $[0;8] \to [0;12]$
- $[0;12] \to [0;18]$

- $[0;24] \to [0;36]$ (relation)

These are as in the E_7 theory with 3 flavours, as required.

When going to higher number of flavours, for which we know the degree of the numerator is at least 264 even in the unrefined series (and 1320, 330 in each flavour, in the refined one!), there is also the issue of obtaining the E_6 series for (N_f, N_f) specifically, rather than some other combination of $2N_f$ total flavours, for $N_f \geq 3$, since the Higgsing process on E_6 with arbitrary (N_f, N_a) leaves no trace of how many of the original fields were fundamentals and how many antifundamentals, since they both decompose to the same F_4 fundamental under Higgsing on one (anti)flavour, and so on etc.

We will not discuss the un-Higgsing on two flavours from D_4 up to E_7 here.

5.6.6 F_4 gauge group

We recall that F_4 is Higgsed by one fundamental to D_4, with any remaining fundamentals being decomposed to a vector, a spinor, a conjugate spinor and two scalars. Recall the F_4 refined series:

$$\begin{aligned}
PL(g^{(1,F_4)}(t)) &= t^2 + t^3 \\
PL(g^{(2,F_4)}(t)) &= [2]t^2 + [3]t^3 + [0]t^4 \\
PL(g^{(3,F_4)}(t)) &= [2,0]t^2 + [3,0]t^3 + [0,2]t^4 + [0,1]t^5 + [0,0]t^6 + [0,0]t^9 - [0,0]t^{18}
\end{aligned}$$

We can write the PL in the 2-flavour case in terms of distinct fugacities t_1, t_2 as follows:

$$PL(g^{(2,F_4)}) = t_1^2 + t_1 t_2 + t_2^2 + t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3 + t_1^2 t_2^2$$

Higgsing this on t_2 gives the invariant in the Higgsed flavour, two scalars resulting from the decomposition of the remaining flavour and three invariants at order 2 and one at order 3, as in the D_4 theory with one flavour of V+S+C as required.

As when Higgsing the (3,0) and (3,1) cases for E_6 gauge group and the 3-flavour case for E_7, we will use the $SU(3) \times U(1)$ notation:

- $[2,0;2] \to [2;2] + [1;1] + [0;0]$; last term is quadratic invariant in

Higgsed flavour 3, second term is one scalar for each of flavours 1 and 2

- $[3,0;3] \to [3;3] + [2;2] + [1;1] + [0;0]$; last term is cubic invariant in Higgsed flavour 3, previous term is the other scalar for each of flavours 1 and 2
- $[0,2;4] \to [0;4] + [1;3] + [2;2]$
- $[0,1;5] \to [0;4] + [1;3]$
- $[0,0;6] \to [0;4]$
- $[0,0;9] \to [0;6]$
- $[0,0;18] \to [0;12]$ (relation)

We obtain the two invariants of the Higgsed flavour, two scalars for each of the remaining flavours, nine invariants at order 2 transforming in three [2] representations of $SU(2)$, eight at order 3 in one [3] and two [1]s and three at order 4 and one at order 6 all transforming in [0]s, plus one relation at order 12 also transforming in a [0] of $SU(2)$, as required.

Higgsing on two flavours, the reduced gauge group is $A_2 = SU(3)$. We get the 8 invariants of the two Higgsed flavours (3 quadratic, 4 cubic, one quartic), 8 scalars from the decomposition of the remaining flavour to three $(\mathbf{3}+\bar{\mathbf{3}})$ pairs and eight scalars, and the nine quadratic and two cubic (identifying quark and antiquark fugacities t_i and u_i) invariants and one relation at order 6, as required.

We cannot demonstrate Higgsing of F_4 theories with higher numbers of flavours, because we have not been able to compute the refined series, which has a numerator of degree 216 (54 in each flavour) in the 4-flavour case.

We will now consider un-Higgsing on one flavour from D_4 with N_f flavours of V+S+C up to F_4 with N_f+1 flavours. Recall the refined series for D_4:

$$
\begin{aligned}
PL(g^{(1,1,1,D_4)}(t,t,t)) &= 3t^2 + t^3 \\
PL(g^{(2,2,2,D_4)}(t,t,t)) &= 3[2]t^2 + ([3]+2[1])t^3 + 3[0]t^4 + [0]t^6 - [0]t^{12}
\end{aligned}
$$

There are two invariants from the Higgsed flavour of F_4 and two scalars from each of the N_f remaining flavours. For $N_f = 1$, we have $2 + 2t + 3t^2 + t^3$;

$1+t+t^2$ gives $[2]t^2$, $1+t+t^2+t^3$ gives $[3]t^3$, and t^2 gives $[0]t^4$, as required for F_4 with 2 flavours. For $N_f = 2$ we have two scalars in the [0] of $SU(2)$ and four scalars forming 2[1], so we have:

- $[0;0] + [1;1] + [2;2] \to [2,0;2]$
- $[0;0] + [1;1] + [2;2] + [3;3] \to [3,0;3]$
- $[2;2] + [1;3] + [0;4] \to [0,2;4]$
- $[1;3] + [0;4] \to [0,1;5]$
- $[0;4] \to [0,0;6]$
- $[0;6] \to [0,0;9]$
- $[0;12] \to [0,0;18]$ (relation)

This is as for F_4 with 3 flavours, as required.

Again, un-Higgsing D_4 with $N_f \geq 3$ flavours of V+S+C to F_4 with N_f+1 flavours requires knowledge of the refined series in the D_4 case. This should be obtainable by un-Higgsing A_2 with $3(N_f-1)$ flavours, indeed the $N_f = 3$ case should be obtainable via Mathematica since it is the result of Higgsing the F_4 theory with 4 flavours and is therefore necessarily less complex, but has not been found so far.

5.6.7 G_2 gauge group

When the theory with N_f flavours is Higgsed on one of them, the gauge group is broken to $A_2 = SU(3)$ and the remaining $N_f - 1$ flavours each decompose into one fundamental of A_2, one antifundamental and one scalar.

We begin by writing the refined series for up to 4 flavours in terms of $U(N_f)$ fugacities t_i:

$$\begin{aligned}
PL(g^{(1,G_2)}(t)) &= t^2 \\
PL(g^{(2,G_2)}(t)) &= t_1^2 + t_1t_2 + t_2^2 \\
PL(g^{(3,G_2)}(t)) &= t_1^2 + t_1t_2 + t_1t_3 + t_2^2 + t_2t_3 + t_3^2 + t_1t_2t_3 \\
PL(g^{(4,G_2)}(t)) &= (\sum_{1\leq i\leq j\leq 4} t_it_j) + t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 + t_1t_2t_3t_4 \\
&\quad -t_1^2t_2^2t_3^2t_4^2
\end{aligned}$$

By inspection, Higgsing on t_{N_f} gives in each case one invariant in the Higgsed flavour, $(N_f - 1)$ scalars resulting from the decomposition of the other flavours and $(N_f - 1)^2$ invariants at order 2, and in the case $N_f = 4$, $2 = 2(\frac{N_f-1}{3})$ invariants at order 3 and one relation at order 6, as in accordance with the A_2 series with $N_f - 1$ flavours (where each flavour consists of a quark and an antiquark) identifying each quark fugacity t_i with the corresponding antiquark fugacity u_i.

5.7 Adjoint SQCD

We will just provide a brief summary of Hilbert series of exceptional gauge groups with adjoint matter, and $SO(N)$ groups (for $N \geq 7$) with spinor matter. We were unable to obtain any series through Molien-Weyl integration in Mathematica.

As we know, the Hilbert series of any group with one adjoint and no other matter is given by $\prod_{i=1}^{rank(G)}(1-s^{d_i})^{-1}$, where d_i is the dimension of the i-th Casimir invariant of the group. Adjoint matter therefore always Higgses a group down to its maximal torus $U(1)^{rank(G)}$, generated by the Cartan subalgebra of its corresponding Lie algebra. (This is similar to the effect that happens in non-SQCD SUSY gauge theories where the scalars in the vector multiplets, rather than the hypermultiplets or chiral multiplets, are given VEVs; the group is broken to $U(1)^{rank(G)}$ there too, and this is the origin of the term 'Coulomb branch' for this branch.)

B_3 with one spinor and one adjoint gives a complete intersection:

$$\frac{1 - t^8 s^{18}}{(1-t^2)(1-s^2)(1-s^4)(1-s^6)(1-t^2s^2)(1-t^2s^4)(1-t^2s^6)(1-t^4s^6)(1-t^4s^9)}$$

We can use the Higgsing: the spinor Higgses B_3 to G_2, under which the adjoint of B_3 breaks down into an adjoint and a fundamental of G_2. The Hilbert series of G_2 with one fundamental and one adjoint is a complete intersection [3]:

$$\frac{1 - s^{12}t^6}{(1-s^2)(1-s^6)(1-s^3t)(1-t^2)(1-s^2t^2)(1-s^4t^2)(1-s^3t^3)(1-s^6t^3)}$$

We see that, removing the $(1 - t^2)$ term from the denominator of the B_3 series and setting t to 1, we get the same series as if we set $t = s$ in the G_2 series. We must identify t and s because the adjoint of B_3 maps to the adjoint, counted by s, and the fundamental, counted by t, of G_2.

By triality, the Hilbert series of D_4 with one spinor (of either type) and one adjoint is freely generated, and that of two spinors of the same type and one adjoint is a complete intersection. However, that of one spinor of each type and one adjoint is a non-complete intersection. We can see this by Higgsing on, say, the spinor; the adjoint decomposes to the adjoint and vector of B_3 and the conjugate spinor to a spinor of B_3. Higgsing on this second spinor breaks the adjoint to the adjoint and fundamental of G_2 and the vector becomes a fundamental; this leads to the Hilbert series for G_2 with two fundamentals and one adjoint, which we know to be a non-complete intersection.

All other cases of Hilbert series of exceptional groups with one adjoint and some other matter, or of $SO(N)$ groups with spinor matter, are non-complete intersections. We show this again by Higgsing for B_4 with one adjoint and one spinor; the spinor Higgses B_4 to B_3 and the adjoint breaks up into an adjoint, a vector and a spinor of B_3, and we saw earlier that this gives a non-complete intersection.

5.8 Conclusions

In this section of this thesis we have achieved and shown the following:

- G_2 theories with 1, 2 and 3 flavours have a moduli space which is freely generated, for 4 flavours it is a complete intersection, and for 5 or more flavours it is a non-complete intersection. The invariants, relations and higher syzygies agree with those found in [4] up to order 11 but not at orders 12 or 13, which is the highest order reached in that paper.

- F_4 theories with 1 and 2 flavours have a moduli space which is freely generated, for 3 flavours it is a complete intersection, and for 4 or more flavours it is a non-complete intersection.

- E_6 theories with 1, 2 and 3 total flavours (flavours plus antiflavours) have a moduli space which is freely generated, for 4 total flavours it is a complete intersection, and for 5 or more total flavours it is a non-complete intersection. This time, the invariants, relations and higher syzygies found in the cases with no antiflavours agree with those found in [4] all the way up to order 18, the highest found in that paper.

- E_7 theories with 1 and 2 flavours have a moduli space which is freely generated, for 3 flavours it is a complete intersection and for 4 or more flavours it is a non-complete intersection. Again the invariants, relations and higher syzygies agree with those found in [4] up to order 18. (Note: these results are known except for F_4 with 4 flavours, some of the higher flavour numbers of G_2 and the invariants not mentioned as being in [4].)

- These results agree with the formula for the 'critical' number of (total) flavours at which the moduli space is a complete intersection (and freely generated for fewer flavours and a non-complete intersection for more), which is, except for the cases of $SO(N)$ gauge groups with matter in the vector representation (where it is N), given by

$$N_f^{crit} = \frac{I^2(Ad)}{I^2(R_{mat})}$$

  where $I^2(R)$ is the second Dynkin index of a specified representation R, Ad is the adjoint representation of the gauge group and R_{mat} is the representation in which the matter flavours transform. The second Dynkin index of the adjoint representation is equal to twice the dual Coxeter number of the group.

- The invariants found for F_4 with N_f flavours agree with those found for E_6 with N_f flavours and one antiflavour, excluding the one at order tu, summed over the number of antiflavour fields (i.e. setting the antiflavour fugacity u to 1).

- The same relationship exists between G_2 and B_3 with one spinor. We associate this relationship, in both cases, to the fact that the antifundamental of E_6, and the spinor of B_3, Higgs the group down to F_4 and G_2 respectively.

- However, while the invariants (summed over the number of antiflavour or spinor fields) are the same, we cannot use this property to un-Higgs the simpler theory to the more complex one, because the powers of $(1-t^2)$ in the denominator are not the same in the B_3 and G_2 cases and we expect similar discrepancies between the E_6 and F_4 series.

- We found the same relationship between D_4 with N_f flavours consisting of a vector, a spinor and a conjugate spinor, and E_6 Higgsed on N_f flavours and 2 antiflavours.

- We believe that we found all the primitive invariants of F_4; the highest invariant that seems to be primitive is the 18-box invariant $[2,\ldots,1_{16},\ldots]$ in the notation of Section 5.4.1.

- We also went to order 24 (from 13 in [4]) for G_2, 21 (from 18) for E_6 (and the same for the case with antiflavours) and 20 (from 18) for E_7, and did up to 21 for B_3 with one spinor and any number of vectors, 19 for D_4 with V+S+C flavours and 21 for E_6 with 2 antiflavours.

The formulae for the Higgsing relations between (partially (un)refined) Hilbert series are as follows:

$$
\begin{aligned}
g^{(G_2,N_f)}(t) &= \lim_{u\to 1}(1-u^2)g^{(B_3,N_f,1)}(t,u) \\
&= \lim_{u,v\to 1}(1-u^2)(1-v^2)(1-tuv)^{N_f}g^{(D_4,N_f,1,1)}(t,u,v) \\
g^{(B_3,N_s,0)}(t) &= \lim_{u\to 1}(1-u^2)g^{(D_4,N_f,1,0)}(t,u) \\
g^{(F_4,N_f)}(t) &= \lim_{u\to 1}(1-u^3)(1-tu)^{N_f}g^{(E_6,N_f,1)}(t,u) \\
g^{(D_4,N_f,N_f,N_f)}(t,t,t) &= \lim_{u\to 1}(1-u^3)^4(1-tu)^{2N_f}(1-tu^4)^{N_f}g^{(E_6,N_f,2)}(t,u)
\end{aligned}
$$

5.9 Discussion and outlook

Because of memory constraints, we were unable to compute the Hilbert series, even unrefined, for E_6 with more than 4 total flavours, E_7 with more than 3 flavours, F_4 with more than 4 flavours (we were lucky to be able to calculate it for 4 flavours!), B_3 with 1 spinor and 8 or more vectors, D_4 with more than 7 total flavours of matter not all of the same type (vector, spinor and conjugate spinor) and D_5 with more than 5 total flavours of spinor matter.

There are two methods we could try to overcome these problems. One way, as we have seen, is to obtain (refined) Hilbert series by un-Higgsing, as we have done for the freely generated and complete intersection cases. However, this method has limitations. Firstly, it requires the refined series of the 'child' theory, which is often itself difficult to obtain. Secondly, in cases where the parent theory has more than one type of basic field that maps to the same field in the child theory, added complications arise, especially when there is only one of one particular type of field in the parent theory, such as 1 spinor of B_3 or 1 antiflavour of E_6.

An alternative method is to try to find an alternative picture of the theory with the same global symmetry group but a different gauge symmetry, a phenomenon known as duality. Duality occurs in many other areas of physics, such as T-duality in string theory relating two circular or toroidal compactifications where the radii satisfy $R' = \frac{\alpha'}{R}$, and also in string theory S-duality between strong and weak coupling and U-duality which is the combination of these two, and the general gauge-gravity duality, of which the AdS/CFT correspondence is the most important case, of a gauge theory in d dimensions giving another, usually simpler, picture of a gravity theory in $d+1$ dimensions. The dualities considered here, though, are between two gauge theories, usually considered as an electric-magnetic duality, like that in Maxwell's theory.

Though it has not been relevant to much of the discussion in this paper, fields in a supersymmetric gauge theory have a charge under the so-called R-symmetry, or the R-charge. The R-symmetry group is the subgroup of the internal symmetry group that does not commute with the supercharges, i.e. for $U(1)$ R-symmetry group, $[R, Q_\alpha] = Q_\alpha$, $[R, \tilde{Q}_{\dot{\alpha}}] = -\tilde{Q}_{\dot{\alpha}}$. The R-symmetry group is determined by $\mathcal{N}$ and by the number of dimensions, e.g. for 4 dimensions it is $(S)U(\mathcal{N})$ (special for $\mathcal{N} = 4$) but for 6 dimensions it is $Sp(\mathcal{N}_L) \times Sp(\mathcal{N}_R)$, because there are two types of (symplectic Weyl-Majorana) supercharge. The R-charge is usually used to refer to one specific $U(1)$ subgroup of the whole R-symmetry.

In simple SQCD theories with only one type of field (or two types which are conjugate to each other), the R-charge of a matter field is given by, where R_i is the representation in which the i-th matter field transforms:

$$R(Q) = \frac{(\sum_i I^2(R_i)) - I^2(Ad)}{\sum_i I^2(R_i)}$$

In the case of one (or two conjugate) types of field, all the R_i are the same; however if the fields have the same invariants, e.g. when both the vector and (conjugate) spinor have a quadratic invariant, this formula is used for more than one type of field, as in [4] for $D_8 = SO(16)$. (In brane tiling theories there may be more than one $U(1)$ subgroup of the R-symmetry group, so determining which combination is the R-charge may involve a complicated minimization procedure. In the theories discussed in [42], a toric variety of dimension 3 always has 3 $U(1)$ charges from the metric, one is the R-charge and the other two are the other mesonic charges.)

If there is a superpotential, it must have R-charge 2, because the R-charge of $d\theta$ is -1.

If there exists a gauge-invariant quantity with R-charge less than 2/3, the theory must have a dual [4].

Seiberg's original formulation of his duality, as in [77], relates SQCD with N_f flavours of matter in the fundamental Q and antifundamental $\tilde{Q}$ of $SU(N_c)$, with the fundamentals transforming in the fundamental of one $SU(N_f)$ and as a singlet of another and the antifundamentals as a singlet of the first and the antifundamental of the second, for $\frac{3}{2}N_c \leq N_f \leq 3N_c$, to a similar theory with N_f flavours of matter in the fundamental q and antifundamental $\tilde{q}$ of $SU(N_f - N_c)$, a meson as a basic field transforming in the antifundamental of the first $SU(N_f)$ and the fundamental of the second (rather than as a gauge-invariant combination which is schematically $q\tilde{q}$), and a superpotential $W \sim Mq\tilde{q}$.

Seiberg-like dualities for exceptional gauge groups are outlined in [34]. Some of the dualities seem incredible, because they are between non-chiral $SO(N)$ theories (for $7 \leq N \leq 10$) and chiral $SU(N-5)$ ones, but the resemblances are demonstrated, although as with most dualities they are not always rigorously defined. Their relevance to exceptional group theories is that exceptional groups can be Higgsed, including by the partial Higgsing on only one of their invariants, to $SO(N)$ groups with N between 7 and 11, and $SO(7)$ is also Higgsed by a spinor to G_2.

In brane-tiling theories, Seiberg duality is the same as toric duality, where one brane tiling is 'dualized' by moving the nodes of another face and creating new lines and faces to give another tiling that gives the same toric diagram when the forward algorithm is applied. As with (un)Higgsing, though, non-brane-tiling cases are considerably more complicated both to formulate and to understand.

So far, the duality between two theories related by Seiberg or other duality is not manifest in the Hilbert series, at least not when calculated classically; it is stated in [25] that the duality exchanges the classical and quantum branches, so unless there is a way of calculating quantum moduli spaces for the dual theory, it is difficult to use. Such a method does exist for instanton moduli spaces, where the Coulomb branch of 3d $\mathcal{N} = 2$ gauge theories on quiver gauge theories specified by quivers in the shape of the extended Dynkin diagram for a given gauge group, not necessarily simply laced, is the same as the instanton moduli space for that group, at least for one instanton [60, 61, 62, 63].

We hope to develop these methods further and use them to calculate Hilbert series, invariants and other properties that we have not yet been able to do using the methods discussed in this thesis.

5.10 A sample LiE program

A sample LiE [5] program used to compute the refined Hilbert series is as follows. This program computes the refined Hilbert series for E_6 with 5 flavours and one antiflavour, using t as the fugacity for the flavours and u for the antiflavour. (In this particular example, as in all examples with exactly one antiflavour, there is a missing generator u^3 corresponding to a term $(1 - u^3)$ in the denominator, which can be manually added in.)

```
on monitor
maxobjects 99999999
maxlev=21;
nf=5;
for i = 0 to maxlev do
for part row partitions(i) do
part1=null(nf);
valid=1;
```

```
if (i<=nf) then
for j = 1 to i do part1[j]=part[j] od;
else if (part[nf+1]==0) then
for j = 1 to nf do part1[j]=part[j] od;
else valid=0;
fi;
fi;
if (valid) then
part2=null(nf-1);
for j=1 to nf-1 do part2[j]=part1[j]-part1[j+1] od;
symm=plethysm(part1,[1,0,0,0,0,0],E6);
uind=null(maxlev+1);
for k = 1 to length(symm) do
repwt=expon(symm,k);
if (repwt[2]==0) && (repwt[3]==0) && (repwt[4]==0) && (repwt[5]==0) then
uind[repwt[1]+2*repwt[6]+1]+=coef(symm,k);
fi;
od;
for j=1 to maxlev+1 do
if (uind[j]) then
print(uind[j]+" "+part2+" t^"+i+" u^"+(j-1));
fi;
od;
fi;
gcol;
fi;
od;
od;
```

Bibliography

[1] J. Gray, A. Hanany, Y. H. He, V. Jejjala and N. Mekareeya, "SQCD: A Geometric Apercu," JHEP **0805** (2008) 099 [arXiv:0803.4257 [hep-th]].

[2] A. Hanany and N. Mekareeya, "Counting Gauge Invariant Operators in SQCD with Classical Gauge Groups," JHEP **0810** (2008) 012 [arXiv:0805.3728 [hep-th]].

[3] A. Hanany, N. Mekareeya and G. Torri, "The Hilbert Series of Adjoint SQCD," Nucl. Phys. B **825** (2010) 52 [arXiv:0812.2315 [hep-th]].

[4] P. Pouliot, "Spectroscopy of gauge theories based on exceptional Lie groups," J. Phys. A **34** (2001) 8631 [hep-th/0107151].

[5] M. van Leeuwen, A. Cohen and B. Lisser, LiE Manual, version 2.1, Amsterdam, The Netherlands, 1996. http://wwwmathlabo.univ-poitiers.fr/ maavl/LiE/

[6] Cox, A. The Hilbert Series for the Moduli Space of Supersymmetric Gauge Theories, Masters thesis (unpublished work), Imperial College London and University of Milan, 2012.

[7] D. Lust, N. Mekareeya, O. Schlotterer and A. Thomson, "Refined Partition Functions for Open Superstrings with 4, 8 and 16 Supercharges," Nucl. Phys. B **876** (2013) 55 [arXiv:1211.1018 [hep-th]].

[8] A. Hanany, D. Forcella and J. Troost, "The Covariant perturbative string spectrum," Nucl. Phys. B **846** (2011) 212 [arXiv:1007.2622 [hep-th]].

[9] W. Fulton and J. Harris, Representation Theory, A First Course, Springer-Verlag, 1991.

[10] T. L. Curtright and C. B. Thorn, "Symmetry Patterns in the Mass Spectra of Dual String Models," Nucl. Phys. B **274** (1986) 520.

[11] T. Eguchi and A. Taormina, "On the Unitary Representations of $N = 2$ and $N = 4$ Superconformal Algebras," Phys. Lett. B **210** (1988) 125.

[12] S. Odake, "Character Formulas of an Extended Superconformal Algebra Relevant to String Compactification," Int. J. Mod. Phys. A **5** (1990) 897.

[13] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions," J. Phys. A **42** (2009) 304010 [arXiv:0812.1151 [math-ph]].

[14] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface," Commun. Num. Theor. Phys. **3** (2009) 531 [arXiv:0904.0911 [math-ph]].

[15] W. Z. Feng, D. Lust and O. Schlotterer, "Massive Supermultiplets in Four-Dimensional Superstring Theory," Nucl. Phys. B **861** (2012) 175 [arXiv:1202.4466 [hep-th]].

[16] S. Zwegers, "Mock Theta Functions," arXiv:0807.4834 [math.NT].

[17] S. Gerigk, "String States on $AdS_3 x S^3$ from the Supergroup," JHEP **1210** (2012) 084 [arXiv:1208.0345 [hep-th]].

[18] L. Mazzucato, "Superstrings in AdS," Phys. Rept. **521** (2012) 1 [arXiv:1104.2604 [hep-th]].

[19] B. Feng, A. Hanany and Y. H. He, "Counting gauge invariants: The Plethystic program," JHEP **0703** (2007) 090 [hep-th/0701063].

[20] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, "The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories," Adv. Theor. Math. Phys. **8** (2004) 603 [hep-th/0310285].

[21] P. Pouliot, "Molien function for duality," JHEP **9901** (1999) 021 [hep-th/9812015].

[22] S. Benvenuti, B. Feng, A. Hanany and Y. H. He, "Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics," JHEP **0711** (2007) 050 [hep-th/0608050].

[23] Zhou, J. Introductions to Symmetric Polynomials and Symmetric Functions, http://faculty.math.tsinghua.edu.cn/~jzhou/SymmetricF.pdf

[24] R. Slansky, "Group Theory for Unified Model Building," Phys. Rept. **79** (1981) 1.

[25] J. Terning, "Modern supersymmetry: Dynamics and duality," (International series of monographs on physics. 132)

[26] J. Davey, A. Hanany, N. Mekareeya and G. Torri, "M2-Branes and Fano 3-folds," J. Phys. A **44** (2011) 405401 [arXiv:1103.0553 [hep-th]].

[27] J. Davey, A. Hanany and R. K. Seong, "Counting Orbifolds," JHEP **1006** (2010) 010 [arXiv:1002.3609 [hep-th]].

[28] J. Davey, A. Hanany, N. Mekareeya and G. Torri, "Brane Tilings, M2-branes and Chern-Simons Theories," arXiv:0910.4962 [hep-th].

[29] J. Davey, A. Hanany and J. Pasukonis, "On the Classification of Brane Tilings," JHEP **1001** (2010) 078 [arXiv:0909.2868 [hep-th]].

[30] J. Davey, A. Hanany, N. Mekareeya and G. Torri, "Higgsing M2-brane Theories," JHEP **0911** (2009) 028 [arXiv:0908.4033 [hep-th]].

[31] J. Davey, A. Hanany, N. Mekareeya and G. Torri, "Phases of M2-brane Theories," JHEP **0906** (2009) 025 [arXiv:0903.3234 [hep-th]].

[32] A. Hanany, V. Jejjala, S. Ramgoolam and R. K. Seong, "Calabi-Yau Orbifolds and Torus Coverings," JHEP **1109** (2011) 116 [arXiv:1105.3471 [hep-th]].

[33] A. Hanany and R. K. Seong, "Symmetries of Abelian Orbifolds," JHEP **1101** (2011) 027 [arXiv:1009.3017 [hep-th]].

[34] J. Distler and A. Karch, "N=1 dualities for exceptional gauge groups and quantum global symmetries," Fortsch. Phys. **45** (1997) 517 [hep-th/9611088].

[35] L. E. Ibanez and A. M. Uranga, "String theory and particle physics: An introduction to string phenomenology," Cambridge, UK: Univ. Pr. (2012) 673 p

[36] A. Hanany and R. Kalveks, "Highest Weight Generating Functions for Hilbert Series," JHEP **1410** (2014) 152 [arXiv:1408.4690 [hep-th]].

[37] Sypsas, S. Massive Spectrum and Symmetric Functions, unpublished work, Kings College London, 2011.

[38] Torri, G., Counting gauge invariant operators in supersymmetric theories using Hilbert series, PhD thesis, Imperial College London, 2012. http://spiral.imperial.ac.uk/handle/10044/1/9989

[39] J. P. Davey, "Brane Tilings, M2-branes and Orbifolds," arXiv:1110.6658 [hep-th].

[40] F. Quevedo, S. Krippendorf and O. Schlotterer, "Cambridge Lectures on Supersymmetry and Extra Dimensions," arXiv:1011.1491 [hep-th].

[41] Stelle, K., Unification lecture notes, Imperial College London, 2012. https://workspace.imperial.ac.uk/theoreticalphysics/Public/Forms/AllItems.aspx?RootFolder=

[42] S. Benvenuti, A. Hanany and N. Mekareeya, "The Hilbert Series of the One Instanton Moduli Space," JHEP **1006** (2010) 100 [arXiv:1005.3026 [hep-th]].

[43] Y. Chen and N. Mekareeya, "The Hilbert series of U/SU SQCD and Toeplitz Determinants," Nucl. Phys. B **850** (2011) 553 [arXiv:1104.2045 [hep-th]].

[44] P. L. Cho, "Moduli in exceptional SUSY gauge theories," Phys. Rev. D **57** (1998) 5214 [hep-th/9712116].

[45] I. Macdonald, Symmetric Polynomials and Hall Polynomials (second edition), Oxford University Press, 1998.

[46] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "An octonionic formulation of the M-theory algebra," JHEP **1411** (2014) 022 [arXiv:1402.4649 [hep-th]].

[47] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "A magic pyramid of supergravities," JHEP **1404** (2014) 178 [arXiv:1312.6523 [hep-th]].

[48] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "Super Yang-Mills, division algebras and triality," JHEP **1408** (2014) 080 [arXiv:1309.0546 [hep-th]].

[49] L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "Magic Square from Yang-Mills Squared," Phys. Rev. Lett. **112** (2014) 13, 131601 [arXiv:1301.4176 [hep-th]].

[50] A. Hanany, N. Mekareeya and A. Zaffaroni, "Partition Functions for Membrane Theories," JHEP **0809** (2008) 090 [arXiv:0806.4212 [hep-th]].

[51] D. Forcella, A. Hanany, Y. H. He and A. Zaffaroni, "The Master Space of N=1 Gauge Theories," JHEP **0808** (2008) 012 [arXiv:0801.1585 [hep-th]].

[52] D. Forcella, A. Hanany, Y. H. He and A. Zaffaroni, "Mastering the Master Space," Lett. Math. Phys. **85** (2008) 163 [arXiv:0801.3477 [hep-th]].

[53] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge, UK: Univ. Pr. (1998) 402 p

[54] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge, UK: Univ. Pr. (1998) 531 p

[55] Pope, C. Kaluza-Klein Theory, lecture notes, Texas A&M University, 2000. http://people.physics.tamu.edu/pope/ihplec.pdf

[56] Coman, I. and Hanany, A. Generating functions of the SO(D-1) covariant perturbative spectra of bosonic string theory on $\mathbb{R}^{1;24} \times S^1$ and Type IIB string theory on $\mathbb{R}^{1;8} \times S^1$, Masters thesis, Imperial College London, 2011.

[57] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," Submitted to: Cambridge Monogr.Math.Phys..

[58] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[59] Dougherty, J.P. Methods of Mathematical Physics, Cambridge lecture notes, 1997.

[60] S. Cremonesi, A. Hanany and A. Zaffaroni, "Monopole operators and Hilbert series of Coulomb branches of $3d\,\mathcal{N} = 4$ gauge theories," JHEP **1401** (2014) 005 [arXiv:1309.2657 [hep-th]].

[61] S. Cremonesi, G. Ferlito, A. Hanany and N. Mekareeya, "Coulomb Branch and The Moduli Space of Instantons," JHEP **1412** (2014) 103 [arXiv:1408.6835 [hep-th]].

[62] Marin Amat, E., N=4 Gauge Theories in 3D, Hilbert Series, Mirror Symmetry and Enhanced Global Symmetries, Masters thesis, Imperial College London, 2013. https://workspace.imperial.ac.uk/theoreticalphysics/Public/MSc/Dissertations/2013/Eloi

[63] Ferlito, G., Mirror Symmetry in 3D Supersymmetric Gauge Theories, Masters thesis, Imperial College London, 2013. https://workspace.imperial.ac.uk/theoreticalphysics/Public/MSc/Dissertations/2013/Giulia

[64] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[65] M. C. N. Cheng, "K3 Surfaces, N=4 Dyons, and the Mathieu Group M24," Commun. Num. Theor. Phys. **4** (2010) 623 [arXiv:1005.5415 [hep-th]].

[66] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu twining characters for K3," JHEP **1009** (2010) 058 [arXiv:1006.0221 [hep-th]].

[67] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," JHEP **1010** (2010) 062 [arXiv:1008.3778 [hep-th]].

[68] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," Phys. Lett. B **694** (2011) 446 [arXiv:1008.4924 [hep-th]].

[69] S. Govindarajan, "Unravelling Mathieu Moonshine," Nucl. Phys. B **864** (2012) 823 [arXiv:1106.5715 [hep-th]].

[70] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, "Umbral Moonshine," Commun. Num. TheorPhys. **08** (2014) 101 [arXiv:1204.2779 [math.RT]].

[71] A. Hanany, N. Mekareeya and S. S. Razamat, "Hilbert Series for Moduli Spaces of Two Instantons," JHEP **1301** (2013) 070 [arXiv:1205.4741 [hep-th]].

[72] R. Feger and T. W. Kephart, "LieART - A Mathematica Application for Lie Algebras and Representation Theory," arXiv:1206.6379 [math-ph].

[73] E. Kiritsis, "Introduction to superstring theory," (Leuven notes in mathematical and theoretical physics. B9) [hep-th/9709062].

[74] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: A modern introduction," Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p

[75] M. Bianchi, J. F. Morales and H. Samtleben, "On stringy AdS(5) x S**5 and higher spin holography," JHEP **0307** (2003) 062 [hep-th/0305052].

[76] H.Derksen and G.Kemper, Computational Invariant Theory, New York: Springer (2002).

[77] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," Nucl. Phys. B **435** (1995) 129 [hep-th/9411149].