Dynamics of a viscous thread surrounded by another viscous fluid in a cylindrical tube under the action of a radial electric field: Breakup and touchdown singularities

Q. WANG\(^1\) AND D. T. PAPAGEORGIOU\(^2\)

\(^1\)Department of Mathematical Sciences, NJIT, NJ, 07102, USA
\(^2\)Department of Mathematics, Imperial College London, London SW7 2AZ, UK

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The nonlinear dynamics of a viscous filament surrounded by a second viscous fluid arranged in a core-annular configuration when a radial electric field acts in the annular region, are studied analytically and computationally using boundary element methods. The flow is characterized by the viscosity ratio, an electric Weber number measuring the strength of the electric field, a geometrical parameter measuring the thickness of the undisturbed annular region, as well as a computational parameter that fixes the wavenumber of the undulations. Axisymmetric solutions are computed by direct numerical simulations in the Stokes limit for general values of the parameters when the two fluids have equal viscosities, and an asymptotic theory is carried out to produce a novel evolution equation for thin film dynamics valid when the undisturbed annular thickness is small and the viscosity ratio is of order one. It is established (in agreement with previous computations in the absence of electric fields) that a sufficiently thick annulus enables thread breakup while a sufficiently thin one (approximately one fifth of the undisturbed thread radius for the case of equal viscosities, for instance) suppresses pinching and drives the interface to approach the tube wall asymptotically without actually touching it. The present simulations show that the electric field affects the dynamics drastically in several ways. First, it promotes interfacial wall touchdown in finite time and a comparison between direct simulations and the asymptotic solutions are in fair agreement. Second, the electric field acts to suppress pinching in the sense that solutions that lead to jet breakup due to a thick enough viscous annulus are driven to wall touchdown. When pinching takes place we find that the ultimate pinching solutions are self-similar and recover the non-electrified ones to leading order for the range of parameters studied.

1. Introduction

Viscous liquid thread or jet flows are ubiquitous in nature and applications and have been studied widely in the nonlinear regime in recent years. Recent and ongoing analytical studies concentrate on describing the pinching process asymptotically by utilizing the separation of radial and axial scales and mapping the dynamics to a class of self-similar solutions which are universal when inertia is present; notable studies include the work of Eggers (1993), Eggers & Dupont (1994), Papageorgiou (1995), Brenner et al. (1996) for jets surrounded by a passive medium; Craster et al. (2002), Craster et al. (2003), Craster et al. (2005), for surfactant-covered or compound jets; Conroy et al. (2010), for core-annular arrangements in the presence of electrokinetic effects. Significant computational work has also been carried out with the aim of simulating the phenomena and evaluating the asymptotic theories (the latter are considerably less demanding numerically) - see Newhouse & Pozrikidis (1992), Pozrikidis (1999), Lister & Stone (1998), Sierou & Lister (2003), who simulate Stokes flows using
boundary integral methods, and Ambravaneswaran et al. (2002), Chen et al. (2002), Notz et al. (2001), Notz & Basaran (2004), Collins et al. (2007), Hameed et al. (2008) who compute the flow at arbitrary Reynolds number and in some instances include the effects of surfactants and electric fields - the extensions and novel aspects of the present work are outlined later.

In this study we concentrate on flows characterised by small Reynolds numbers and hence governed by the Stokes equations. Such situations arise in small scale geometries and/or high viscosity fluids and numerous applications have been documented in recent years in the field of microfluidics, for example - see the reviews by Eggers (1997), Stone et al. (2004), Song et al. (2006), Eggers & Villermeaux (2008) and Craster & Matar (2009). The basic mechanism that drives a perfectly cylindrical thread (representing an exact solution of the Navier-Stokes equations) to breakup and drop formation is furnished by capillary instability so that axisymmetric linear disturbances that are longer than the undisturbed jet circumference are unstable - Plateau (1863), Rayleigh (1879). The situation is similar for viscous jets surrounded by another immiscible viscous fluid, in unbounded and bounded geometries as shown by Tomotika (1935) and Goren (1962), respectively.

When the jet is electrified by a concentric outer electrode, both the linear stability characteristics and nonlinear dynamics are altered and our objective lies in the theoretical description of such phenomena. A typical configuration has a perfectly conducting liquid jet surrounded by a dielectric (insulating) gas filling the region between the jet interface and the outer cylindrical electrode. A radial electric field is set up in the undisturbed configuration and several studies have carried out linear stability analyses and experiments of the system, starting with Basset (1894) who considered axisymmetric perturbations and included the effects of an ambient fluid (see also Taylor (1969)). Non-axisymmetric disturbances were considered by Huebner & Chu (1971), Schneider et al. (1967) and Saville (1971) for both inviscid and viscous jets, who conclude that non-axisymmetric modes are activated. Interestingly, for highly viscous jets surrounded by a gas, the non-axisymmetric modes dominate and can lead to the so-called whipping mode - a comprehensive linear study for highly viscous jets has also been carried out by Mestel (1996). The results of the linear theory are consistent with the experimental observations of Magarvey & Outhouse (1962) who show that in their configuration (liquid jets surrounded by a gas) the electric charge tends to reduce the size of the droplets after breakup.

The nonlinear dynamics and breakup of liquid jets stressed by radial electric fields have not been studied widely. One of the original inviscid studies is that of Setiawan & Heister (1997) who used a boundary-element method to follow the dynamics of a perfectly conducting axisymmetric jet surrounded by a dielectric gas. They report terminal states that are qualitatively similar to the non-electrified main and satellite-drop formation, but also ones where the jet does not pinch but evolves to an axisymmetric analogue of the Taylor cone (experimental evidence of such phenomena has been reported by Cloupeau & Prunet-Foch (1989) and Kelly (1994)). A more recent study which is relevant to the present work is that by Collins et al. (2007), who carry out direct numerical simulations of the dynamics of perfectly conducting axisymmetric jets of arbitrary viscosity surrounded by a dielectric gas and stressed by a radial electric field that is generated between the jet surface and an outer concentrically placed cylindrical electrode which is held at constant voltage potential. They also use a one-dimensional model introduced by López-Herrera et al. (1999) and compare results with the experiments of López-Herrera & Gañán-Calvo (2004) (performed for relatively small viscosity jets surrounded by a dielectric gas). A wide range of viscosities were investigated and several conclusions (in line with previous work where appropriate) were drawn. It was shown that in the Stokes limit satellites form at breakup even though the surrounding region is a gas - a phenomenon attributed to the electric field since a viscous external shear is required to produce satellites in the non-electrified case (see Lister & Stone (1998), Craster et al. (2003), Craster et al. (2005)). In addition, the size of the satellite drops was found to increase in the presence of the electric field (also implying that the main drop volume decreases as found by Huebner & Chu (1971) and Magarvey & Outhouse
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(1962)). After breakup, the drops are charged and can lose stability when the charge exceeds the Rayleigh limit (Rayleigh (1882)), possibly causing electro-spraying.

The present work is fundamentally different from that of Collins et al. (2007) in several ways - a viscous surrounding dielectric fluid is included rather than a hydrodynamically passive medium, and the outer bounding electrode is not placed too far so that phenomena other than pinching can be described, including interfacial wall touchdown. One motivation for this study is the theoretical understanding of electrified core-annular flows as could arise in multiphase flows in porous media, for example. In a recent study, Conroy et al. (2010), we considered the dynamics of an annular electrolyte film thus extending the study of Hammond (1983) to electrolytic multiphase systems, and showed that the film can either touch the wall in finite time (in contrast to the dynamics of the Hammond equation) or develop into non-uniform stationary states depending on the electric double layer interaction. In this article we simulate such flows numerically for arbitrary undisturbed annular layer thicknesses in order to establish the range of validity of the asymptotic models and provide results at arbitrary parameter values beyond this range. Also of interest are the boundary integral computations of Newhouse & Pozrikidis (1992) of the capillary driven dynamics of threads in core-annular arrangements in perfectly cylindrical tubes, who show that the dynamics are ultimately attracted to pinching or wall touchdown (strictly speaking the latter is precluded by the mathematical properties of the Hammond equation and some recent very large time computations studies of it - see Tseluiko & Papageorgiou (2007) and Lister et al. (2006), respectively), depending on whether the undisturbed annulus thickness is above or below a threshold value. Our numerical work accurately predicts this boundary in the solution space and we also quantify the effect of the electric field showing that relatively thick annuli that lead to pinching in the absence of a field, can be driven to wall touchdown in its presence, thus indicating a physical mechanism to suppress breakup into droplets. We also note that a one-dimensional Stokes model was derived and used by Wang et al. (2009); the model is valid for long waves and for a passive dielectric gas in the annulus. Analysis and computations of the model equations predict wall touchdown for undisturbed annular thicknesses below a parameter-dependent threshold, otherwise jet thinning and the formation of a quasi-static micro-thread. The latter phenomenon arises from the competition between the electric and capillary stresses which become comparable in the long wave - thin thread limit as quantified in Wang et al. (2009). Direct simulations showing such phenomena are provided here by allowing the thread to evolve to very small radii with pinching ultimately taking place.

The structure of the rest of the paper is as follows. Section 2 presents the governing equations and dimensionless physical parameters for a general electrified two-fluid viscous system when the core fluid is a perfect conductor. Section 3 derives the linear dispersion equation for axisymmetric perturbations while section 4 presents an asymptotic analysis valid for a thin annulus and derives and studies a novel evolution equation for the dynamics in this case. Boundary integral equations and their numerical implementation, and numerical results are presented in sections 5 and 6, respectively. Drop formation and local dynamics are discussed for a pinching thread as well as the transition to touchdown solutions. Comparison between boundary integral simulations and lubrication models are carried out at the end of Section 6. Section 7 is devoted to our conclusions.

2. Mathematical formulation

We consider the dynamics of a viscous incompressible two-fluid system arranged in a core-annular configuration inside a cylindrical tube of constant circular cross section of radius $b$. Cylindrical polar coordinates $\mathbf{x} = r \mathbf{e}_r + \theta \mathbf{e}_\theta + z \mathbf{e}_z$ are used with the $z$-axis along the pipe axis. Fluid 1 of viscosity $\mu_1$ occupies the core region defined by $0 < r < S(z,t)$, while fluid 2 of viscosity $\mu_2$ occupies $S(z,t) < r < b$, where we have assumed axisymmetry (all variables are independent of the azimuthal angle $\theta$). In what follows the effect of gravity is neglected which is a good approximation when the ratio of gravitational to capillary forces are small - Hammond (1983). A radial electrostatic field is
imposed and in this study we take the core fluid to be a perfect conductor (without loss of generality the interface is a zero equipotential surface) and the tube wall to be an electrode at a prescribed constant non-zero potential. The annular fluid 2, is taken to be a perfect dielectric and supports an electric field that changes with the interfacial configuration. We define the electric field by \( E = -\nabla \phi \) where \( \phi(r,z,t) \) is the voltage potential in region 2 (this follows from the fact that \( \nabla \times E = 0 \) in the electrostatic limit). A schematic of the problem is provided in figure 1.

In the zero Reynolds number limit the flow in regions 1 and 2 is governed by the Stokes equations:

\[
-\nabla p_i + \mu_i \nabla^2 u_i = 0, \quad \nabla \cdot u_i = 0, \quad i = 1, 2,
\]

where \( u_i = u_i e_r + w_i e_z \) is the fluid velocity, \( p_i \) is the pressure and \( \mu_i \) is the fluid viscosity in each phase. The no-slip and no-penetration boundary condition at the wall requires

\[
u_2(b,z,t) = w_2(b,z,t) = 0.
\]

In region 2 where the voltage potential is non-zero, it satisfies Laplace’s equation (this follows from the fact that in the absence of charges in the bulk \( \nabla \cdot E = 0 \))

\[
\nabla^2 \phi = 0,
\]

with boundary conditions

\[
\phi(b,z,t) = V_0, \quad \phi(S(z,t), z, t) = 0.
\]

It remains to prescribe all other boundary conditions on the moving interface \( r = S(z,t) \). These are the usual kinematic condition

\[
u_i = S_t + w_i S_z, \quad \text{on} \quad r = S(z,t),
\]

and the tangential and normal stress balances

\[
[t \cdot \mathbf{T} \cdot \mathbf{n}]_1^2 = 0,
\]

\[
[n \cdot \mathbf{T} \cdot \mathbf{n}]_2^2 = -\gamma \kappa,
\]

where \( t = (S_z e_r + e_z)/(1 + S_z^2)^{1/2} \) and \( n = (e_r - S_z e_z)/(1 + S_z^2)^{1/2} \) are the unit tangent and normal (pointing into region 2) at any point on the interface \( r = S(z,t) \). The stress tensor \( \mathbf{T} \) has hydrodynamic and electrostatic (Maxwell stress) contributions which we write as \( \mathbf{T} = \sigma - \mathbf{M} \) where

\[
\sigma = -p I + \mu (\nabla u + \nabla u^T), \quad \mathbf{M} = \epsilon_p (E E - \frac{1}{2} I |E|^2),
\]
with \( I \) denoting the identity tensor and \( \epsilon_p \) the electrical permittivity of fluid 2 (the permittivity of free space is \( \epsilon_0 \approx 8.8542 \times 10^{-12} \text{ F/m} \)).

We nondimensionalize the problem using the undisturbed core radius \( a \) for lengths, \( \gamma/\mu_1 \) for velocities, \( \gamma/a \) for the pressure, \( \mu_1 a/\gamma \) for time and \( V_0 \) for the voltage potential (these dimensional characteristic quantities can also be found in figure 1). The Stokes equations become

\[
-\nabla p_i + \lambda_i \nabla^2 u_i = 0, \quad \nabla \cdot u_i = 0, \quad i = 1, 2, \tag{2.9}
\]

with \( \lambda_1 = 1, \lambda_2 = \lambda = \mu_2/\mu_1 \). The no-slip and no-penetration boundary condition at the wall requires \( u_2(d, z, t) = w_2(d, z, t) = 0 \), where \( d = b/a \). The Laplace equation (2.3) for \( \phi \) is unchanged and the boundary conditions become \( \phi(d, z, t) = 1 \) and \( \phi(S(z, t), z, t) = 0 \). The dimensionless kinematic condition and the tangential and normal stress balances at \( r = S(z, t) \) are

\[
S_t + w_i S_i = u_i, \tag{2.10}
\]

\[
\left[ \lambda_i \left( \{u_{iz} + w_{ir}\}(1 - S_z^2) + 2S_z(u_{ir} - w_{iz}) \right) \right]_2 = 0, \tag{2.11}
\]

\[
[-p_i (1 + S_z^2) + \lambda_i (2u_{ir} + 2S_z w_{iz} - 2S_z (u_{iz} + w_{ir}))]_2 \nonumber
\]

\[
-\frac{1}{2} (1 - S_z^2) (\phi_z^2 - \phi_r^2) - 2S_z \phi_r \phi_z = \frac{S_{zz} - (1 + S_z^2)/S}{\sqrt{1 + S_z^2}}. \tag{2.12}
\]

Three dimensionless parameters appear above; these are

\[
\lambda = \frac{\mu_2}{\mu_1}, \quad d = \frac{b}{a}, \quad E_b = \frac{\epsilon_p V_0^2}{\gamma a}, \tag{2.13}
\]

and represent the annulus to core fluid viscosity ratio, the ratio of the tube radius to the undisturbed core radius (the dimensionless undisturbed annulus thickness is \( d - 1 \)), and an electric Weber number measuring the ratio of electrical to capillary pressures.

Our main concern in this study is the nonlinear evolution of the system and we address this numerically by solving the boundary integral equations directly instead of performing a fully 2D simulation as in Collins et al. (2007), and also asymptotically in the physically relevant limit of thin annuli. In order to gain some insights into the dynamics and to obtain a basis to verify the numerical work, we begin by considering the linear stability characteristics of the system.

### 3. Linear Stability Analysis

It is well known that jet pinching is the result of capillary instability which is initiated from small amplitude interfacial perturbations whose wavelength is larger than the undisturbed jet circumference. The mechanism for such Rayleigh instability phenomena centers on the presence of surface tension and the associated pressure jump across the interface. It can be demonstrated that a local thinning of the jet by a sufficiently long perturbation increases the pressure inside the jet in the vicinity of the depression relative to the pressure far away, and thus causes a fluid motion away from the depression hence enhancing the instability. The argument is based on small perturbation estimates and is straightforward due to the explicit form of the pressure jump in terms of the local curvature given by the Young-Laplace equation. When an electric field is present the Young-Laplace equation contains Maxwell stresses involving the normal component of the field at the interface; these additional stresses are nonlocal in nature since Laplace’s equation for the voltage potential needs to be determined (technically, the Dirichlet to Neumann map of the solution is required in the simple case of perfectly conducting fluids). We have carried out such analysis (for brevity this is not included here) and we show that the excess pressure due to the field in the vicinity of a depression perturbation is positive and hence enhances the capillary instability. Analogous results hold in the vicinity of a local elevation the difference being that there is now a decrease in the local pressure.
due to capillary and electric stresses thus increasing the amplitude of the elevated interface. The mechanism can be simply understood as follows: an outward perturbation of the interface decreases the potential on it, and this in turn increases the magnitude of the radial electric field, it induces a more negative charge on the interface and produces an upward electrostatic force that enhances the instability. These mechanisms can be fully quantified using linear stability analysis and this is discussed next.

Linear dispersion relations pertaining to the stability of perfectly conducting viscous liquid threads surrounded by a hydrodynamically passive region (air) and stressed by a radial electric field, have been derived by Saville (1971) when the surrounding electrode is at infinity and Collins et al. (2007) when the electrode has a finite radius. Here we include a surrounding viscous fluid and note that the results of Collins et al. (2007) emerge in the zero viscosity limit of the surrounding fluid. We consider the stability of the quiescent perfectly cylindrical steady state (bars denote base-flow quantities)

\[ \mathbf{u}_j = 0, \quad p_1 - p_2 = 1 - \frac{E_0}{2(\ln d)^2}, \quad \mathbf{S} = 1, \quad \bar{\phi} = \frac{\ln r}{\ln d}. \]  

Perturbing about this state, linearising the boundary conditions and assuming normal mode solutions proportional to \( \exp(ikz + \omega t) \) provides an eigenvalue problem for \( \omega(k) \), with instability present whenever \( \text{Re} \omega > 0 \). The perturbed flow is expressed in terms of a streamfunction so that \( u_i = -\frac{1}{r} \partial \psi_i / \partial z \), \( w_j = \frac{1}{r} \partial \psi_i / \partial r \) and writing \( \psi_j(r,z,t) = \hat{\psi}(r) \exp(ikz + \omega t) + c.c \) provides the equation \( E_4 \hat{\psi}_1 = 0 \), where the operator \( E_4 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - k^2 \). The solutions are

\[ \hat{\psi}_j(r) = r^2 (E_{1j} I_1(kr) + F_{1j} K_1(kr) + E_{2j} r I_0(kr) + F_{2j} r K_0(kr)), \]

where \( E_{1j}, F_{1j}, E_{2j}, F_{2j} \) are unknown coefficients; regularity of solutions at the cylinder axis \( r = 0 \) implies \( F_{11} = F_{21} = 0 \). The solution for the linearized perturbation potential \( \hat{\phi}(r) \exp(ikz + \omega t) \) proceeds along similar lines and satisfies \( E_2 \hat{\phi} = 0 \). The solutions that satisfy the boundary conditions at the wall \( r = d \) and the interface given by \( r = 1 + \delta \hat{\eta} \exp(ikz + \omega t) \) (here \( \delta \ll 1 \) is the linearization parameter), are readily found to be

\[ \hat{\phi}(r) = \frac{\hat{\eta}}{\ln d} r^2 \left[ \frac{I_0(kr) K_0(kd) + K_0(kr) I_0(kd)}{I_0(kr) K_0(kd) - K_0(kr) I_0(kd)} \right]. \]

Four linear homogeneous algebraic equations involving the six coefficients \( E_{11}, E_{12}, E_{12}, E_{21}, E_{22}, F_{22}(k) \) are found from the no-slip conditions at \( r = d \), and continuity of velocities across the interface. Along with \( \hat{\eta} \) we have seven unknowns and the extra three equations required come from the linearized kinematic condition and the tangential and normal stress balances. Eliminating \( \hat{\eta} \) in favour of the other constants we arrive at a linear homogeneous system of six equations for the unknown vector \( \mathbf{w} = [E_{11}, E_{12}, E_{21}, F_{12}, F_{22}]^T \) written in matrix form as

\[ \mathbf{A} \mathbf{w} = 0. \]  

The coefficient matrix is given in the Appendix A (derivations in a similar context in the absence of electric fields can be found in Goren (1962) and Kwak & Pozrikidis (2001)). For a non-trivial solution of (3.4) we require \( \det(\mathbf{A}) = 0 \) and this provides the desired dispersion relation \( \omega(k) \).

Before presenting general linear stability results we note the long-wave limit \( k \ll 1 \) result,

\[ \omega = k^2 \left( 1 - \frac{\ln d - 1}{(\ln d)^2} E_0 \right) \frac{1}{16\lambda} \left[ (1 - d^2)^2 \left( 1 - \frac{(1 - d^2)^2}{(d^4 + \lambda - 1)} \right) - 4 \ln(1/d) - 2d^2 + 2 \right] + \ldots, \]  

which is valid for \( \lambda \) of order one, and is consistent with Georgiou et al. (1991) for the uncharged case. The single fluid case \( \lambda = 0 \), studied by Wang et al. (2009) also has very similar long-wave stability characteristics. The expression (3.5) indicates that the electric field can be used to enhance
or reduce long-wave instability and the nonlinear manifestation of such mechanisms are considered later in our computations.

To discuss the general linear stability properties we note that in the absence of electric fields, the core thread is subject to Rayleigh instability when the wavenumber of the perturbation is below the critical value $k = 1$. The presence of the annular fluid provides a long wavelength cut-off with the growth rate becoming zero at $k = 0$ as opposed to the constant value predicted by Stokes flow (see Timmermans & Lister (2002) for a discussion of this and several other limits in the linear regime). Such effects are shown in figure 2 when an electric field is also present; the dotted lines in panels (a) and (b) represent the results for a single fluid Stokes jet and agree with Wang et al. (2009). Panels (a) and (b) have $d = 2.5$ and $d = 3$, respectively, and these values are motivated by the results of Wang et al. (2009) (or (3.5)) which predict long-wave enhanced instability or stability for $d < e \approx 2.7183$ or $d > e$, respectively, as $E_b$ increases. Increasing $E_b$, therefore, enhances instability for both $\lambda = 0$ and $\lambda = 0.001$ as observed in panel (a) of figure 2, while panel (b) shows that long waves are stabilized with increasing $E_b$; at the same time, however, short waves are destabilized by the electric field as indicated by the arrows.

Figure 3 investigates the effect of the different parameters ($\lambda$, $d$ and $E_b$) on the growth rate. In
panel (a) the solid lines represent the growth rate for \( d = 2.5 \) and \( E_b = 0 \) with varying \( \lambda \), and shows the known result that the presence of a more viscous annular fluid reduces the maximum growth rate significantly. For the case \( \lambda = 1 \) presented in panel (a), an increase of \( E_b \) increases the maximum growth rate as well as the band of unstable wavenumbers - dashed and dotted lines correspond to \( E_b = 0.25 \) and \( E_b = 0.5 \) respectively. Figure 3(b) shows the dependence of the growth rate on \( d \) for fixed \( E_b = 0.5 \) and \( \lambda = 1 \); \( d = 1.75 \) gives a wide range of unstable modes the reason being that the tube wall electrode is relatively close to the fluid interface in this case and the Maxwell stresses are increased. Physically, a small annulus supports an intense field that induces a large electric force on the interface. As an additional check, long wave results obtained from (3.5) are superimposed in panel 3(b) and shown as dotted lines with the same parameters; agreement is excellent.

### 4. Thin Annulus Limit

In this Section we analyse the problem when the undisturbed annular thickness is small relative to the tube radius. The analysis is an electrical modification of Hammond (1983) and an electrostatic version of the electrokinetic study of Conroy et al. (2010). We introduce the small parameter \( \epsilon = d^{-1} \) and assume that the viscosity ratio is of order one, so that in particular \( \epsilon \lambda \ll 1 \). The annulus is described by a stretched variable \( y = (d - r)/\epsilon \) so that the interface \( r = S(z, t) \) can be written as \( S(z, t) = d - \epsilon h(z, t) \) with \( h(z, t) = O(1) \) (thus, the interfacial deflections scale with the annulus thickness \( \epsilon \) and the model is capable of predicting wall touchdown). The flow in the film is driven by the capillary pressure and a familiar lubrication approximation can be invoked as detailed by Hammond (1983). Briefly, the normal stress balance (2.12) sets the perturbation pressure in the film to be of size \( \epsilon \) which in turn drives axial and radial flows in the film of sizes \( \epsilon^3 \) and \( \epsilon^4 \), respectively. Continuity of velocities across the interface imply that the perturbation velocity and pressure fields in the core region (here axial and radial length scales are both of order one) are of order \( \epsilon^3 \). Thus, we look for asymptotic solutions in powers of \( \epsilon \) in the form

\[
\begin{align*}
  u_2(y, z, t) &= \epsilon^3 u_2^0 + \ldots, \\
  w_2(y, z, t) &= \epsilon^3 w_2^0 + \ldots, \\
  u_1(r, z, t) &= \epsilon^4 u_1^0 + \ldots, \\
  w_1(r, z, t) &= \epsilon^4 w_1^0 + \ldots, \\
  p_2(y, z, t) &= -1 + \epsilon p_2^0 + \ldots, \\
  p_1(r, z, t) &= \epsilon^5 p_1^0 + \ldots.
\end{align*}
\]

The leading order Stokes film problem gives the following equations for the radial momentum, axial momentum and continuity equation, respectively

\[
\begin{align*}
  \rho_2^0 \frac{\partial}{\partial y} u_2^0 &= 0, \\
  -p_2^0 + \lambda w_2^0 \frac{\partial}{\partial y} u_2^0 &= 0, \\
  -u_2^0 + w_2^0 &= 0. \\
\end{align*}
\]  

Solving gives \( p_2^0 \equiv p_2^0(z, t) \), and using the no slip conditions at the wall \( (y = 0) \) yields

\[
\begin{align*}
  u_2^0 &= \frac{\rho_2^0}{2\lambda} y^2 + \frac{1}{\lambda} y A(z, t), \\
  u_2^0 &= \frac{y^3}{6\lambda} p_2^0 + \frac{y^2}{2\lambda} A_z(z, t), \\
\end{align*}
\]  

with \( p_2^0(z, t) \) and \( A(z, t) \) to be found. For order one viscosity ratios \( \lambda \) the problem in the film closes in the sense that there is no coupling with the hydrodynamics in the core. This can be established from the leading order tangential stress balance (2.11) which becomes \( u_{2y}^0 = 0 \) at \( y = h(z, t) \), and in turn yields \( A(z, t) = -p_2^0 h(z, t) \), hence

\[
\begin{align*}
  w_2^0 &= \frac{\rho_2^0}{\lambda} \left[ \frac{y^2}{2} - yh \right], \\
  u_2^0 &= \frac{\rho_2^0}{\lambda} \left[ \frac{y^3}{6} - \frac{y^2}{2} h \right] - \frac{y^2}{2} p_2^0 h_z.
\end{align*}
\]

For completeness we note that even though the film problem is closed (it is seen below that the normal stress balance involves film variables alone to leading order) its solution drives the hydrodynamics in the core through the continuity of velocities condition at the interface which are, to leading order,

\[
\begin{align*}
  w_1^0(1, z, t) &= u_1^0(h(z, t), z, t), \\
  u_1^0(1, z, t) &= 0.
\end{align*}
\]
Before proceeding to the normal stress boundary condition to find \( p_2^0(z,t) \), we analyze the electric field problem in the annulus in order to determine the required Maxwell stresses. Writing \( \phi(y,z,t) = \phi_0(y,z,t) + c \phi_1(y,z,t) + \ldots \) and substituting into (2.3) gives \( \phi_{yy} = 0 \), to leading order, whose solution, satisfying the boundary conditions \( \phi_0(0,z,t) = 1 \) and \( \phi_0(h(z,t),z,t) = 0 \) (these are the dimensionless leading order analogues of (2.4)), is

\[
\phi = \frac{h(z,t) - y}{h(z,t)}.
\]

Hence the leading order electric field contribution in the normal stress balance (2.12) has size \( E_b/\epsilon^3 \) and for a balance between capillary forces with electrostatic ones we require the canonical limit

\[
E_b = \epsilon^3 \beta, \quad 0 \leq \beta = O(1).
\]

The perturbation pressure \( p_0^0 \) in the film now follows from the normal stress balance and reads

\[
p_0^0(z,t) = -(h + h_{zzz}) + \frac{\beta}{2h^2},
\]

enabling the leading order velocity field (4.3) to be expressed in terms of \( h(z,t) \) and its derivatives. The desired evolution equation now follows from the leading order terms (order \( \epsilon^4 \) in fact) of the kinematic condition after the introduction of a slow time-scale \( t \to (\epsilon^3/\lambda)(\partial/\partial t) \), and substitution of the leading order velocities (4.3) evaluated at \( y = h(z,t) \). The result is

\[
h_t + \frac{1}{3} [h^3(h_z + h_{zzz}) + \beta h_z]_z = 0.
\]

Equation (4.8) is an electrostatically modified thin film equation and the Hammond equation arises when \( \beta = 0 \). The new equation is interesting both physically and mathematically because when \( \beta \neq 0 \) solutions are found to terminate (under fairly general initial conditions) in touchdown singularities in finite time, as opposed to the absence of touchdown in the Hammond equation which supports intricate multi-scale dynamics over extremely large time-scales (for rigorous results see Tseluiko & Papageorgiou (2007) and for extensive numerical work see Lister et al. (2006)). We also note that (4.8) emerges in the zero electrokinetic limit of the more general electrolytic fluids version derived by Conroy et al. (2010); in addition, it provides the structures near touchdown of the electrokinetic model, if the latter terminates in a singularity.

Equation (4.8) was solved numerically on spatially periodic domains, and a typical run is provided in figure 4. Panel (a) shows the evolution of the thin film with an initial condition \( h(z,0) = 1 + 0.5 \cos(\pi z/5) \). Recall that \( h = 1 \) corresponds to the case when the interface is undisturbed, while when touchdown occurs \( h \) vanishes at some position(s) \( z_s \) - for the symmetric initial conditions used here (symmetry is not enforced in the code) the touchdown occurs in the middle of the domain, \( z_s = 5 \). The results show clearly that there is finite-time touchdown and in fact the local shape appears to be cusp-like.

The dynamics in the vicinity of the singular event \( t = t_s \), can be analyzed by searching for self-similar solutions as \( h \to 0 \). Writing \( h(z,t) = (t_s - t)^\alpha f\left( \frac{z - z_s}{(t_s - t)^\gamma} \right) \) where \( \alpha > 0, \gamma > 0 \) allows us to estimate each term in (4.8) as \( t \to t_s^- \) as follows

\[
h_t \sim (t_s - t)^{\alpha - 1}, \quad (h^3h_z)_z \sim (t_s - t)^{4\alpha - 2\gamma}, \quad (h^3h_{zzz})_z \sim (t_s - t)^{4\alpha - 4\gamma}, \quad h_{zzz} \sim (t_s - t)^{\alpha - 2\gamma}.
\]

Since \( \alpha > 0 \) and \( (t_s - t) \ll 1 \), we have \( (h^3h_z)_z \ll h_{zz} \) and so balancing the terms \( h_t \sim (h^3h_{zzz})_z \sim h_{zz} \) gives two equations for \( \alpha \) and \( \gamma \), namely \( \alpha - 1 = 4(\alpha - \gamma) = \alpha - 2\gamma \), that is

\[
\alpha = \frac{1}{3}, \quad \gamma = \frac{1}{2}.
\]
Figure 4. Panel (a) shows the evolution of thin film under electric field ($\beta = 1$, $L = 10$). Panel (b) shows the rescaled second derivative $h^2_{\min}h_{zz}$ versus the rescaled axial distance $(z - z_s)/h^3_{\min}$, where $z_s = 5$ here. Solid curves correspond to profiles at $h_{\min} = 0.3221, 0.1570, 0.0856, 0.0408, 0.0177$ and $0.0098$. Circle corresponds to the profile at $h_{\min} = 0.00497$. Rapid convergence to a self-similar profile is observed by $h_{\min} \approx 0.02$.

A similarity solution emerges asymptotically and has the form

$$h(z, t) = (t_s - t)^{1/3} f \left( \frac{z - z_s}{(t_s - t)^{1/2}} \right).$$

(4.11)

Defining $\xi = z - z_s/(t_s - t)^{1/2}$, substituting (4.11) into (4.8) and keeping the leading terms as $t \rightarrow t_s$, gives the following nonlinear problem for the scaling function $f(\xi)$ (primes denote $\xi$-derivatives):

$$-\frac{1}{3} f + \frac{1}{2} \xi f' + \frac{1}{3} \left( f^3 f'' + \beta f' \right)' = 0, \quad -\infty < \xi < \infty.$$  (4.12)

To verify the self-similar behavior close to breakup, panel (b) illustrates that the rescaled second derivative $h^2_{\min}h_{zz}$ and the rescaled axial distance $(z - z_s)/h^3_{\min}$ follow the self-similar scalings and are order one quantities which collapse to a single curve as touchdown is approached. Convergence to the self-similar scaling function given by (4.12) has also been verified but is not included for brevity. One of our main objectives is the validity of the asymptotic equations derived here and we turn next to the direct simulation of the flow for arbitrary annulus thicknesses. This is achieved by an accurate boundary integral method which is described next.

5. Boundary integral simulations

The boundary integral method has been used widely to study fluid mechanics problems - for example see Acrivos & Rallison (1978), Newhouse & Pozrikidis (1992), and Dubash & Mestel (2007) for an electrohydrodynamic application, and the derivation and theory regarding the method for Stokes flow are well documented in Pozrikidis (1992). Here we start with the equation in axisymmetric form

$$u_\alpha(x_0) + \frac{\lambda - 1}{4\pi(\lambda + 1)} \int_I Q_{\alpha\beta\gamma}(x, x_0)u_{\beta}(x)n_{\gamma}(x)dl(x)$$

$$= -\frac{1}{4\pi(\lambda + 1)} \int_I M_{\alpha\beta}(x, x_0)\Delta f_{\beta}(x)dl(x),$$

(5.1)

where the subscripts run over cylindrical coordinates $z$ or $r$, $Q$ denotes the principal value of the double-layer integral, $I$ is the domain of integration comprised of the interface over one period, $x_0$ lies on the interface and $n$ is the outward point normal to the interface. The kernels $M$ and $Q$ are
the periodic Green’s functions of axisymmetric Stokes flow for the velocity and stress inside the tube. $\Delta f$ is the force jump across the thread surface and is given by

$$\Delta f = F_n, \quad F = \left[ \kappa - \frac{E_b}{2} \left( \frac{\partial \phi}{\partial n} \right)^2 \right], \quad (5.2)$$

where $\kappa$ is the curvature and $E_b$ is the electric parameter defined in section 2; $\phi_n$ will be calculated from the following integral equation

$$\phi(x_0) = 1 - \int P(x, x_0) \phi_n(x)r(x)dl(x), \quad (5.3)$$

where $P$ is the Green’s function vanishing on the wall, and $\phi = 1$ is satisfied at the wall. A general derivation of (5.3) is included in Appendix B. Due to the assumption that solutions are $L$-periodic, we impose no flux conditions at $z = 0$ and $z = L$, i.e. $\phi_n(z = 0, r) = \phi_n(z = L, r) = 0$. Hence, axial images are chosen so that $P_n(z = 0, r) = P_n(z = L, r) = 0$ is satisfied. The derivation of $P$ is similar to Stokeslets inside a cylindrical tube and the details can be found in Wang (2010).

The algorithm used to implement the boundary-integral method is as follows. The computation of the velocity field (5.1) requires the electric field component $\phi_n$ through the force jump (5.2). Given an interfacial position, $\phi_n$ is computed independently of the flow and used in (5.1) to obtain the velocities. The interface is then advanced by integrating the kinematic condition using an Euler method. The weak singularities in the kernels in the single-layer potential terms can be handled by Gauss-log quadratures and the regular integrals computed by standard Gauss-Legendre quadratures. Another way to tame the singularity in (5.1) is to take advantage of the result that if $x_0$ is a point on the interface $I$, then

$$\int_I M_{\alpha\beta}(x, x_0)n_\beta(x)(F(x) - F(x_0))dl(x) = 0. \quad (5.4)$$

Hence the single-layer term in (5.1) can be rewritten as

$$\int_I M_{\alpha\beta}(x, x_0)n_\beta(x)(F(x) - F(x_0))dl(x), \quad (5.5)$$

whose kernel now is regular, hence Gauss-Legendre quadrature can be applied directly. Both of these methods were used and compared - in most of the results reported here we use Gauss-Legendre quadratures on equation (5.5). The curvature of the interface is calculated by using cubic-splines, which are also used to redistribute the nodal points along the interface at each time step in order to maintain good resolution. The linear system obtained after discretization of the integral equations, is solved using the fortran package LAPACK. A for-aft symmetry is also enforced, hence only half the period of the thread needs to be computed. In what follows we report results for the simpler case $\lambda = 1$ due to the computational cost in calculating the double layer potential terms.

6. Numerical Results

In all direct simulations that follow, the initial condition for the interface is taken to be

$$S(z, 0) = a + a_1 \cos(kz), \quad (6.1)$$

where $a > 0$ denotes the dimensionless unperturbed thread radius and $|a_1| < a$. Results for $a \neq 1$ can be re-scaled to those having $a = 1$, and we have chosen to perform computations with $a \neq 1$ in order to facilitate comparisons with simulations by other authors (e.g. Pozrikidis (1999)). Before presenting results for electrified threads we test the algorithm against the linear theory of section 3. Figure 5 shows the evolving amplitude from a nonlinear computation for a set of parameters, $d = 2.5$, $E_b = 0.47$, $\lambda = 1$, dimensionless wavenumber $ka = 2/3$ and an initial perturbation of
Figure 5. Comparison between linear theory (dashed line) and simulation (solid line): Evolution of $A/a_1$ where $A$ is half the difference between the maximum and minimum value of the interfacial shape. Here $\lambda = 1$, $E_b = 0.47$, $d = 2.5$.

Figure 6. Interfacial evolution in the presence of a radial electric field (profiles shown over half a period) with $\lambda = 1$. Panel (a) $ka = 0.3$, $d = 2$ and $E_b = 0.1047$. Panel (c) $ka = 0.5$, $d = 5$ and $E_b = 2.513$. The corresponding evolution of $\phi_n^2$ measuring the electric pressure, is plotted in panels (b) and (d), respectively.

amplitude $a_1 = 0.01a$. It can be seen that the solid line corresponding to the nonlinear computation is in excellent agreement with the dashed line that is predicted by the linear theory of section 3; deviations take place at $t \approx 140$ when nonlinear effects enter. We also tested the code with the calculations of Newhouse & Pozrikidis (1992) for different annulus thicknesses in the non-electrified case, where a sufficiently thin annulus leads to near touchdown as observed in Hammond (1983) through a lubrication model. The results are in good agreement and the boundary between the pinching and touching regimes is determined numerically and is given by $d \approx 1.2$. In Section 6.4 we extend such phase behavior results to non-zero electric fields.
Electrified threads

Figure 7. Final computed electrified pinching solutions with $\lambda = 1$. Panel (a) $ka = 2/3$, $d = 2.5$, $E_b = 0.0$, $0.4712$, $0.9425$ correspond to dotted, long-short dashed and solid lines respectively. Panel (b) $ka = 0.5$, $d = 5$, $E_b = 0.0$, $2.5133$, $5.0625$ correspond to dotted, long-short dashed and solid lines respectively. The upper and lower boundaries of the figures define the position of the tube wall electrode.

6.1. Pinching, drop formation and onset of electrohydrodynamic atomization

Next we turn to simulations that incorporate electric field effects. Figure 6 shows the evolution of the thread (left) and its corresponding electric force term $\phi_n^2$ (right) that appears in (5.2). Panels (a) and (b) have perturbation wavenumbers $ka = 3/10$ and a tube radius $d = 2$, while panels (c) and (d) correspond to $ka = 1/2$ and $d = 5$. We note that the dimensionless local charge distribution on the interface is given by Gauss’s law once the electric field problem has been solved and is equal to $q = \phi_n$ (the permittivity of the annular fluid is used in the non-dimensionalization). As the interface deforms, surface charge tends to accumulate on the satellite drop regions and the Maxwell stresses contribute an additional driving force to break the thread up. A large value of $\phi_n^2$ is observed in the vicinity of the breakup point but not at the breakup point itself (we will discuss this later in Section 6.3). Figure 7 compares the final pinching solutions for varying electric field strengths with $d = 2.5$, $ka = 2/3$ in panel (a) and $d = 5$, $ka = 1/2$ in panel (b). It can be seen that the effect of the electric field is to deform the main drops radially in the direction of the electric field and change their shapes significantly. Furthermore the volume of the satellite drops is increased due to the accumulation of charge at the interface. This can be understood physically by considering the normal stress balance (2.12) which shows that the electric field contribution to the pressure jump $p_1 - p_2$ is $-E_b\phi_n^2$; the large values of $\phi_n^2$ found in the computations (see figure 6) show that there is a relative reduction in the local pressure inside the incipient satellite region and hence more fluid enters to increase the satellite volume. Such increased satellite volumes have also been observed in the related studies of Setiawan & Heister (1997), López-Herrera et al. (1999) and Collins et al. (2007). Following López-Herrera et al. (1999), we can assess the stability of the main and satellite drops by calculating the charge $Q$ that they carry at pinch-off (and assume that this is retained beyond the topological transition) and compare with the Rayleigh charge limit (denoted by $Q_R$), which provides an upper limit on the amount of charge that a spherical drop can bear without entering an electrospay mode (see Rayleigh (1882)). In terms of the dimensionless groups used in
Table 1. Some characteristics of the main and satellite drops after pinch-off for the case \( d = 2.5 \) and \( ka = 2/3 \)

<table>
<thead>
<tr>
<th>( E_b )</th>
<th>( R_s/R_{s0} )</th>
<th>( R_m/R_{m0} )</th>
<th>( Q_{Rs} )</th>
<th>( Q_s/Q_{Rs} )</th>
<th>( Q_{Rm} )</th>
<th>( Q_m/Q_{Rm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>1.0214</td>
<td>0.9999</td>
<td>1.0057</td>
<td>4.6604</td>
<td>15.3487</td>
<td>4.6084</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0429</td>
<td>0.9999</td>
<td>1.4667</td>
<td>3.2181</td>
<td>21.7063</td>
<td>3.2610</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2403</td>
<td>0.9991</td>
<td>4.2554</td>
<td>1.1577</td>
<td>48.4754</td>
<td>2.0429</td>
</tr>
<tr>
<td>0.1</td>
<td>1.5613</td>
<td>0.9965</td>
<td>8.4990</td>
<td>1.1663</td>
<td>68.2937</td>
<td>1.4934</td>
</tr>
</tbody>
</table>

Table 2. Some characteristics of the main and satellite drops after pinch-off for the case \( d = 5 \) and \( ka = 0.5 \)

<table>
<thead>
<tr>
<th>( E_b )</th>
<th>( R_s/R_{s0} )</th>
<th>( R_m/R_{m0} )</th>
<th>( Q_{Rs} )</th>
<th>( Q_s/Q_{Rs} )</th>
<th>( Q_{Rm} )</th>
<th>( Q_m/Q_{Rm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0726</td>
<td>0.9985</td>
<td>13.88</td>
<td>0.7167</td>
<td>90.75</td>
<td>0.4423</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1634</td>
<td>0.9955</td>
<td>22.17</td>
<td>0.4991</td>
<td>127.83</td>
<td>0.311</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4631</td>
<td>0.9862</td>
<td>44.22</td>
<td>0.2217</td>
<td>178.15</td>
<td>0.3716</td>
</tr>
<tr>
<td>0.43</td>
<td>1.5347</td>
<td>0.9831</td>
<td>49.25</td>
<td>0.2097</td>
<td>183.83</td>
<td>0.4344</td>
</tr>
</tbody>
</table>

the present work, we have

\[
Q_R^2 = 48\pi E_b V, \tag{6.2}
\]

where \( V \) is the dimensionless volume of a drop. Due to the highly deformed drop shapes after breakup, the inequality \( Q/Q_R < 1 \) does not guarantee stability of the drops after pinch-off. On the other hand \( Q/Q_R > 1 \) usually implies instability and can be used as a phenomenological criterion for electrohydrodynamic atomization. Table 1 summarizes our results and atomization predictions for the case \( d = 2.5 \) and \( ka = 2/3 \), while Table 2 provides corresponding results for \( d = 5 \) and \( ka = 1/2 \). For all quantities appearing in the tables, subscripts \( s \) and \( m \) represent the satellite and main drops, respectively. The effective drop radius is defined as

\[
R = \left( \frac{3V}{4\pi} \right)^{1/3},
\]

and \( R_0 \) corresponds to the effective radius in the non-electrified case, while \( R_{s0} \) and \( R_{m0} \) refer to the effective satellite and main drop radii in the non-electrified case and are used as benchmarks. The computed total charge (found by integrating Gauss’s law over the surface of main and satellite drops just prior to pinch-off - note that the charge at the pinch point is found to vanish asymptotically so that this calculation is asymptotically correct) on the main and satellite drops is denoted by \( Q_m \) and \( Q_s \), respectively, while the corresponding Rayleigh limits given by (6.2) are denoted by \( Q_{Rm} \) and \( Q_{Rs} \). Table 1 shows that the effective radius (and hence the volume) of satellite drops increases with \( E_b \) as found already in the results of figure 7, for instance. At the same time the size of the main drop decreases as expected by mass conservation. It is interesting to note that in the case \( d = 2.5 < e \) both the ratios \( Q_s/Q_{Rs} \) and \( Q_m/Q_{Rm} \) are always greater than one so that further breakup and atomization is expected to follow. In addition, for the range of values of \( E_b \) used in our numerical experiments we observe that \( Q_s/Q_{Rs} \) and \( Q_m/Q_{Rm} \) decrease as \( E_b \) increases and this indicates a certain stabilizing effect of the electric field. The change of the charge ratio is not monotonic, however, and this finding differs
from the results shown for most cases in Collins et al. (2007). For example, $Q_s/Q_{Rs}$ for $E_b a = 0.1$ is slightly larger than the value for $E_b a = 0.05$ and something similar happens for the case $d = 5$ with a small increase in $Q_m/Q_{R_m}$ as $E_b a$ increases from 0.4 to 0.43. As will be shown later this is due to a transition between different breakup scenarios as well as a relatively small $d$ in the present paper. For the larger tube radius $d = 5$ we find once again larger satellite and smaller main drops as compared to the non-electrified case. More importantly, the charge ratios $Q_s/Q_{Rs}$ and $Q_m/Q_{R_m}$ are all below unity for the range of $E_b$ tested and this indicates that if the outer cylindrical electrode is located sufficiently far then the post pinching drops are relatively stable and are not expected to undergo electrohydrodynamic atomization. The atomization possible for $d = 2.5$, however, appears to be more desirable because both main and satellite drops have charge ratios considerably above unity.

Finally we mention some numerical findings for long perturbation wavelengths $L/a = 10\pi$ (i.e. $ka = 1/5$) and a tube radius $d = 5$ (for brevity, graphs are not included). We find that in addition to the larger and longer satellites formed due to the presence of the field (see earlier discussion), more and relatively larger amplitude interfacial oscillations are induced over the satellite region as compared to the non-electrified case, indicating the possibility of secondary pinchoff and sub-satellite drop formation. Similar secondary instabilities but arising by a different mechanism, namely shear, have been documented in the calculations of Tjahjadi et al. (1992) for the uncharged problem. Note also that undulations along the electrified jet surface were also observed in the calculations of Setiawan & Heister (1997) for inviscid flows subject to long wave length perturbation ($ka = 2/5$ in terms of our notation, i.e. $L/a = 5\pi$).

6.2. Suppression of pinching: Transition to interfacial singularities and wall touchdown

The results in Section 6.1 studied the effect of a radial electric field on flows that lead to pinching and drop formation in the absence of a field (pinching takes place because the dimensionless tube radii are sufficiently large). In what follows we consider the effect of significantly larger electric fields on such flows and in particular evaluate numerically the ultimate dynamics in such instances. It has been established numerically that a sufficiently strong electric field is capable of suppressing pinchoff and our results indicate that the interface gets attracted to the wall electrode and touches it in finite time. A map of the dynamics as $E_b$ varies is depicted in figure 8. The figure considers two typical cases, the first having a relatively small tube radius $d = 2$ and the second having a relatively large one with $d = 5$. The perturbation wavenumbers for the two cases are $ka = 3/10$ and $ka = 1/2$, respectively. The results are characterized in terms of the ultimate solutions just prior to a singular event (either pinching or wall touchdown) and the figure tracks the singular time $t_b$ with $E_b$ (in all cases $t_b$ represents the time where a mathematical singularity of the system being solved is expected and the computations are halted just before such times). Sample final shapes are also presented as insets in order to emphasize and quantify the separate behaviors of pinching and wall touchdown.

There are several fundamental physical features that emerge from the simulations summarized in figure 8. First, the parameters have been chosen so that the thread pinches in finite time when the field is absent - the time to breakup is approximately twice as large for the smaller $d$ and this can be attributed to the viscous resistance due to the proximity of the bounding tube wall. Switching on the electric field initially causes a slow-down of the pinching event over a range of $E_b$; a monotonic increase in $t_b$ is observed in both cases with a behavior that appears to be linear with $E_b$. Sample final computed profiles are shown which re-iterate our previous findings of the reduction/increase in volume for the main/satellite drops, as $E_b$ increases. Both cases $d = 2$ and $d = 5$ show clear maxima of $t_b$ at a certain value of $E_b$. In the former case (top panel) a transition in solution type takes place near this maximum with the interface being driven by the electric field to the upper wall and touching it in finite time, rather than pinching to form drops. This can be seen by the inset in the top panel showing the final computed interface at $E_b = 0.2513$; the previous inset at $E_b = 0.1676$ shows a pinching solution but with the main drop coming to close proximity with the
Figure 8. Influence of the electric field parameter $E_b$ on the breakup time for the case $ka = 0.3$, $d = 2$ (top panel), and $ka = 0.5$, $d = 5$ (bottom panel). In both cases $\lambda = 1$. Different breakup scenarios are depicted as insets at different values of $E_b$.

wall. This competition between pinching and touchdown for the case $d = 2$ gives way to the latter as $E_b$ increases between 0.18 and 0.25, approximately. After this transition, the time to a singular event decreases monotonically and for the largest computed value of $E_b = 0.5027$ a touchdown takes place at $t_b = 26.85$, a reduction by more than a factor of two of the time to pinching obtained when $E_b = 0$. The final shape of the thread shown in the inset is characterized by a short-scaled feature near touchdown while at the same time there is no significant thread thinning attained due to the suppression of the capillary instability by the relatively large values of $E_b$.

The lower panel of figure 8 has a much larger dimensionless tube radius $d = 5$. The initial trends of a linear slow-down of the time to pinching with increasing $E_b$ (found for the smaller $d = 2$ case) persist and are generic. Much larger values of $E_b$ are required, however, and this is expected since the outer electrode is now placed further away while being maintained at the same voltage. A local maximum in $t_b$ is found once more, but the transition from pinching to touchdown solutions now happens at even larger $E_b$. The insets at $E_b = 2.513$ and close to the local maximum position $E_b \approx 5.404$ show pinching solutions with the familiar volume reduction of main drops and increase of satellite drops as $E_b$ increases. As the value of $E_b$ is increased further a transition from pinching to wall-touching solutions takes place and the singular time $t_b$ decreases monotonically as in the results for $d = 2$. The main difference is that due to the larger distance to the electrode, a larger body of fluid is drawn towards the electrode as the interface is accelerated towards the wall by
Electrified threads

The electric field. This can be seen particularly well in the panel corresponding to $E_b = 6.283$ that depicts the final computed solution just prior to touchdown - an outer corner like region emerges but the interface is smooth and has a well defined maximum at $z = 0$ albeit with a large curvature. Such computations become increasingly challenging near such events and require adaptive meshing. There is a competition between a singular wall touchdown event and a pinching event in regions where the jet is thinning and it is crucial to have sufficient resolution. If the touchdown region is under-resolved, for example, then apparently different solutions are found that include interface turning and tip formation away from the wall. We have shown that all such solutions disappear with grid resolution but emphasize that they have been observed (using highly accurate adaptive mesh computations) by Collins et al. (2007) in the case when the outer electrode is even further away. Such competing phenomena are more important at the intermediate values of $E_b$ such as the case $E_b = 6.283$ reported, and less so for larger values, e.g. $E_b = 10.05$ also shown as an inset. In the latter case jet thinning is not pronounced enough by the time the interface approaches the wall in finite time. A heuristic argument supporting touchdown events can be made using some of the analytical findings of Section 4 and the long wave fully nonlinear theory of Wang et al. (2009) when the annular region is hydrodynamically passive. Both theories indicate that the minimum distance of the interface from the wall behaves like $(t_{sw} - t)^{1/3}$ as the singular time $t_{sw}$ is approached; pinching, on the other hand, involves vanishing jet radii scaling with $(t_{sp} - t)$ as the singular time $t_{sp}$ is approached. Hence, the acceleration involved in wall touchdown is asymptotically larger than that in pinching, and if the value of $d$ is not very large we can expect wall touchdown to dominate as found in the results reported here. The dynamics is expected to be quite intricate, therefore, and as an example we refer to computations by Collins et al. (2007) that report a solution at a larger value of $d = 10$ that appears to tend towards a tip formation scenario but which is ultimately found to pinch, i.e. a fluid ejection phenomenon is almost achieved but is then overtaken by pinching dynamics. The case where the outer electrode is removed to infinity would be of particular interest and is the subject of ongoing work.

6.3. Local Dynamics of Pinching Solutions

In the absence of electric fields the computations of Lister & Stone (1998) show self-similar behavior near pinching for a viscous thread surrounded by a second viscous fluid. All length scales near pinching scale proportionally to $\tau = t_s - t$ where $t_s$ is the time to the singularity. The dominant balance is between the capillary pressure and the viscous stresses due to the presence of the external fluid. Such findings have also been established in the studies of Cohen et al. (1999) and Cohen & Nagel (2001); a more complete study of the self-similar equations and their solutions covering a much larger range of viscosity ratios, can be found in Sierou & Lister (2003) where a stability analysis of the constructed solutions is also carried out indicating oscillations when the viscosity of the outer fluid becomes sufficiently small (approximately when the inner fluid is 32 times more viscous than the outer one). It is important to note that the presence of an outer viscous fluid alters possible self-similar structures near pinching in a fundamental manner since the dominant axial and radial scales are now of the same order as opposed to the case of a passive outer fluid where long wave dynamics are in control (see Eggers (1993), Eggers & Dupont (1994), Papageorgiou (1995)). This difference is crucial in the electrified problem as we will show below, because for a hydrodynamically passive surrounding phase it has been shown from the extensive numerical simulations of Collins et al. (2007) that near pinching for Stokes flows the electric forces are of the same order as the capillary ones (the simulations of Collins et al. (2007) consider arbitrary Ohnesorge numbers $N_{Oh}$ with the Stokes regime emerging when $N_{Oh} \gg 1$). In contrast, our Stokes flow simulations of the two-phase problem predict that the electrostatic forces are asymptotically smaller near pinching and the physical reason for this can be attributed to the electrostatic shielding of the pinch region that is possible in this case due to the balance of the axial and radial length scales - for long wave pinching dynamics such shielding is not in operation as discovered first by Collins et al. (2007).
We have carried out extensive computations to evaluate the effect of the electric field on both the global dynamics (i.e. main/satellite formation and volumes) and local structures near pinch-off. In what follows we present numerical evidence that the electric field does not influence the dynamics in the direct vicinity of the pinch point. Figure 9(a) tracks the ratio of electric to capillary forces defined by $f_e/f_s = \frac{1}{2}(E_b \phi^2_\infty/\kappa)$ as a function of the minimum thread radius $S_{min}$ - the latter tends to zero as pinching is approached, so time in the figure runs from right to left. Results are given for a range of dimensionless tube radii $d$, and electric field parameters $E_b$ as noted in the figure. It is clear from these results that as pinching is approached the force ratio tends to zero establishing that the local dynamics are governed by capillary forces. Panel (b) in figure 9 depicts the evolution of $S_{min}$ with time for a range of values of $d$ and $E_b$. The non-electrified cases with $d=\infty$, 2, 5 (recall that $\lambda = 1$ throughout) are also included for comparison along with the theoretical result of Cohen et al. (1999) ($d=\infty$) which predicts a linear thinning law of the form $-0.0335(t_s - t)$; the latter is represented by the dashed straight line of slope $-0.0335$. The results for non-zero $E_b$ and finite $d$ indicate a similar behavior with the same linear thinning law. In addition, the pinching time is delayed by the field as well as the closer proximity of the wall, i.e. smaller $d$ implies a larger singular time. Additional numerical evidence is collected in panels (c) and (d) of figure 9 which show the rescaled self-similar profiles (following the construction of Lister & Stone (1998), $H(\xi; t) = S(z, t)/S_{min}(t)$ where $\xi = (z - z_{min}(t))/S_{min}(t)$ as $t \to t_s$), and the logarithmic growth in the axial velocity in the necking region, respectively, for various $d$ and $E_b$ as indicated in figure. The dashed line in panel (d) is the theoretical result for the uncharged case (see Lister & Stone (1998)). We conclude, therefore, that the dominant local dynamics near pinching are insensitive to the presence of the electric field and bounding wall electrode - their effect is to delay the pinching event if the latter takes place. None, however, that the theory is local and does not contain large
scale information such as main/satellite drop volumes and positions which are affected by both the electric field and the radius of the tube.

We conclude this section with an asymptotic analysis that supports the numerical finding of the vanishing ratio of $\phi_n^2/\kappa$ at the pinch point. We address this issue using a double cone intermediate geometry similar to Lister & Stone (1998) for non-electrified flows; the intermediate region connects the curved pinching region whose dimensions become asymptotically small as the singular time is approached, with the far-field regions which are quasi-static as far as pinching is concerned. The analysis provides an estimate for the electric field shielding effect on the pinching dynamics. Thus, a perfect cone with extent $0 < y < R$ of half angle $\alpha$ is considered, so that the outward normal is $n = (\cos \alpha, -\sin \alpha)$ and the surface area element $dS = 2\pi y \sin \alpha \, dy$, where $y$ is the radial distance from the apex. Then (5.3) on the interface becomes

$$0 = \phi \sim 1 - \frac{1}{4\pi} \int_0^R \phi_n \frac{2\pi y \sin \alpha \, dy}{(y^2 - 2yz \cos \alpha + z^2)^{1/2}} + \cdots. \tag{6.3}$$

Assuming pinching occurs at the origin, we consider the leading order contribution for $\phi_n$ near there, so that

$$\frac{2}{\sin \alpha} \sim \phi_n(0, z) \int_0^R \frac{y \, dy}{(y^2 - 2yz \cos \alpha + z^2)^{1/2}} + \cdots = \phi_n(0, z) I(\alpha, z/R) + \cdots, \tag{6.4}$$

where $\alpha$ is assumed to be non-vanishing and finite and the integral $I$ can be written as

$$I(\alpha, z/R) = \int_0^R \frac{(y - z \cos \alpha) \, dy}{(y^2 - 2yz \cos \alpha + z^2)^{1/2}} + z \cos \alpha \int_0^R \frac{dy}{(y^2 - 2yz \cos \alpha + z^2)^{1/2}}.$$

The two parts can be integrated analytically,

$$I_1 = z \sin \alpha \left[ 1 + \left( \frac{y - z \cos \alpha}{z \sin \alpha} \right)^2 \right]^{1/2}, \quad I_2 = z \cos \alpha \left[ \sinh^{-1} \left( \frac{y - z \cos \alpha}{z \sin \alpha} \right) \right]_0^R. \tag{6.6}$$

In the limit $z/R \to 0$ (we assume $R$ is finite), $I_2 = O(z \ln(1/z))$ and so vanishes asymptotically, while $I_1 \sim R$ which is of order one. Therefore (6.4) at leading order becomes

$$\phi_n(0, z) \sim \frac{2}{R \sin \alpha}. \tag{6.7}$$

implying that the electric force $\phi_n^2$ is finite near the pinch point, and thus small compared to the curvature term. On the other hand, (6.7) becomes large when the cone angle is small and this may explain the results seen by Collins et al. (2007) in the moderate to large Ohnesorge number $N_{Oh}$ cases reported. The results in figure 6 also indicate that $\partial \phi_n/\partial z$ is large in the vicinity of the pinch point. Evidence for this can be obtained by differentiating (6.3) with respect to $z$ and estimating $\partial \phi_n/\partial z$ as above. We find

$$\phi_z \sim \frac{1}{2} \phi_n \frac{y \sin(\alpha)(z - y \cos(\alpha)) \, dy}{(y^2 - 2 \cos(\alpha)yz + z^2)^{3/2}} - \frac{1}{2} \int_0^R \frac{\partial \phi_n}{\partial z} \frac{y \sin(\alpha) \, dy}{(y^2 - 2 \cos(\alpha)yz + z^2)^{1/2}} + \cdots, \tag{6.8}$$

where $\phi_n \sim O(1)$ and $\phi_z$ is linearly related to $\phi_n$ hence is also of order one. Using the asymptotic forms of the integrals $I_1$ and $I_2$ for small $z$, yields the estimate

$$\partial \phi_n/\partial z(0, z) \sim \ln(1/z) \gg 1, \quad \frac{z}{R} \to 0, \tag{6.9}$$

which is consistent with the simulations.
In this Section we concentrate on singular solutions involving interfacial wall touchdown in finite time. As described earlier, a sufficiently thin annulus can suppress pinching even without an electric field and direct numerical simulations of the Stokes equations show that this happens whenever $d \lesssim 1.2$, approximately, for the case $\lambda = 1$ and the present simulations. Computations in the thin annular limit become challenging particularly when $\lambda \neq 1$, and considerable analytical insights can be obtained by considering asymptotic models that take into account the multi-scale nature of the phenomena. In the absence of electric fields such a lubrication model was derived and solved by Hammond (1983) where numerical evidence was given that precludes a finite time touchdown at least in the absence of attractive van der Waals forces, establishing instead a slow drainage for all time. The capillary drainage and details of the long-time behavior of the thin film were recently revisited and explored in detail by Lister et al. (2006), who show that the lobe and collar formation depends on the axial length of the film, and at large times they may interact and merge to minimize the surface energy.

We have carried out analogous computations of the full Stokes equations (see figure 10 in the absence of electric fields) in order to evaluate the long wave theory and computations of Lister et al. (2006) - as far as we know a direct comparison at long times has not been carried out. In making a comparison between asymptotic solutions and simulations, we took $d = 1.14$ (i.e. $\epsilon = d - 1 = 0.14$) and a perturbation wavenumber $ka = \pi/10$ corresponding to a relatively long wavelength $2L/a = 20$. Recalling the analysis of Section 4, given a value of $\epsilon$ predicts a slow unscaled time $t = \epsilon^3$ (note that $t$ here corresponds to the scaled $t$ appearing in the thin film equation (4.8)). For $\epsilon = 0.14$, therefore, the boundary integral simulations need to be computed to times of order $364$ where $t$ is the computational time of the thin film equation, making parametric studies involving large time computations prohibitive, and emphasizing the importance of the reduced model and its evaluation with direct simulations. Collective results are presented in figure 10; the upper panel contains three curves all scaled axially and radially with $L$. The solid curve is the computed boundary integral solution at $t = 9666.1$ (i.e. $\hat{t} = 26.52$), while the dotted and dashed lines are thin film solutions at $\hat{t} = 26.5$ (so that $t \approx 9657$) and $\hat{t} \approx 200$ ($t \approx 7.289 \times 10^4$), respectively. The appropriate comparison between asymptotic solutions and simulations is between the solid and dotted curves which correspond to almost the same time (the solution given by the dashed curve at a larger time is included to provide the ultimate trends which are beyond our computational capabilities using the boundary integral method since we would need to integrate to times as large as $7 \times 10^4$). The conclusion is that the thin film model does quite well in predicting both the film thickness minima and their shapes, as well as the global characteristics especially in the middle of the domain. We note that the computational times required for the thin film equation are orders of magnitude smaller than those needed for the direct simulations. We also note in passing, that when $d > 1.2$ a pinching regime can emerge from the direct simulations (with an array of main and satellite drops along the thread axis suspended in the carrier fluid) if the initial annular thickness is sufficiently large. This implies, therefore, that both attractors (pinching or near touchdown) are present and the initial state determines which one wins at large times.

It has already been shown in Section 4 that the electrically modified Hammond equation (4.8) generically terminates in touchdown singularities in finite time (also see Conroy et al. (2010)). We can expect, therefore, that simulations of the full equations will support analogous behavior and some results are presented in the lower panel of figure 10, which shows a comparison between boundary integral simulations depicted with a solid line along with the numerical solution of (4.8) showed with a dashed line; parameters are chosen as indicated on the figure so that finite time rupture occurs. Agreement is fair in the sense that the wavy interfacial structures are approximately obtained. Our boundary integral calculation is prohibited from going further due to the rapid dynamics of the sharp tip; an adaptive code is required to fully resolve the tip region and to investigate the local dynamics.
Figure 10. Upper panel: Formation of lobes and collars in the thin annulus limit for $\lambda = 1$, $E_b = 0$, $ka = \pi/10$ and $d = 1.14$ at $t = 9666.1$ - solid curve, boundary integral simulation; dotted and dashed lines, numerical solutions of (4.8) with $\beta = 0$ at time $t = 26.5$ ($t = 9666.1$) and $t \approx 200$, respectively. Lower panel: Solid line, direct simulation towards finite time touchdown at $t = 1.46$ with $E_b = 0.1871$, $ka = 0.45$ and $d = 1.25$; comparison with (4.8) for $\beta = 11.97 = E_b/(0.25^3)$ at $t = 0.135$ (dotted line, $t = 8.64$) and the final computed solution at $t = 0.149$ (dashed line, $t = 9.536$). The discrepancy of computed time may be due to the relatively large annular thickness, $d - 1 = 0.25$.

- this is left for future work. In addition, a sufficiently strong electric field can drive the interface to touchdown (for a particular value of $d$) in cases where a smaller electric field cannot overcome the capillary instability and pinching is attained. These physical predictions are fully confirmed by our simulations presented in figure 11. The figure presents three typical cases with increasingly larger undisturbed annular thicknesses characterized by $d = 1.3$ (top), $d = 1.5$ (middle) and $d = 2.0$ (bottom), with corresponding perturbation wavenumbers $ka = 0.5, 0.5$ and 0.2, respectively. We note that for all these cases (and same initial conditions) the thread is driven to pinching in finite time when the field is switched off, $E_b = 0$ for the dashed lines. The results show, however, that values of $E_b = 0.1005$, $E_b = 0.1257$ or $E_b = 1.5708$ are sufficient to alter the ultimate dynamics and drive the flow to touchdown. As expected, the threshold value of $E_b$ required to achieve a transition from pinching to touchdown singularities, increases as $d$ increases.

A bifurcation diagram has been constructed numerically in order to separate pinching and touchdown regions as $\epsilon$ and $E_b$ vary (recall that $d = 1 + \epsilon$). Clearly, the boundary is not sharp in the sense that touching solutions in its vicinity can be driven to pinching ones if the initial condition is favourable. In order to fix matters we consider initial conditions with moderate amplitudes and the same wavenumber $ka = 0.5$ (also $\lambda = 1$) and compute solutions as $E_b$ increases until we get a switchover from pinching to touching ultimate states. The range of values explored is $1 < d \lesssim 5$ (i.e. $0 < \epsilon < 4$) and the computed boundary is plotted in figure 12 with $E_b^{1/2}$ on the ordinate and $\ln(d/d_0)$ on the abscissa, where $d_0$ is the largest value of $d$ below which the thread pinches in the absence of an electric field. Our computations find $d_0 \approx 1.2$ and this is used in figure 12. The results indicate that if $E_b$ is not too large, there is a linear dependence of the form

$$E_b^{1/2} = K \ln(d/d_0), \quad (6.10)$$

where $K$ is a constant that depends on $ka$ and $\lambda$. The law (6.10) can be derived physically if we assume that pinching will give way to touching if the electric field at the unperturbed interface is
sufficiently large. The voltage potential, $\phi_0$ say, in the annulus $a \leq r \leq b$ is given by (refer to figure 1 in the paper for the dimensional geometry) $\phi_0 = V_0 \ln(r/a) \ln(b/a)$ so that the magnitude of the radial electric field at the interface becomes

$$E_0 = \frac{V_0}{a \ln(b/a)}.$$  

(6.11)
Writing (6.11) in terms of $E_b$ (recall from (2.13) that $E_b = \frac{\epsilon_0 V^2}{\gamma a}$) we obtain $E_0 = \left(\frac{\gamma a}{\frac{8V^2}{\epsilon_0}}\right)^{1/2} E_b^{1/2}$ which can be scaled to provide the dimensionless value $\overline{E}_0$, say,

$$\overline{E}_0 = \frac{E_0^{1/2}}{\ln(b/a)} = \frac{E_b^{1/2}}{\ln(d)}.$$  

(6.12)

A linear law based on (6.12) explains the numerical results; physically, the electric field at the interface must exceed a threshold whose boundary is postulated to be

$$E_b^{1/2} = K \ln(d) + K_1,$$  

(6.13)

where $K$ and $K_1$ are constants. In the absence of a field we have $d = d_0$ (see above; $d_0 \approx 1.2$ in these runs), hence $K_1 = -K \ln(d_0)$ and

$$E_b^{1/2} = K \ln(d/d_0).$$  

(6.14)

The results in figure 12 follow the law (6.14) extremely well for values of $d$ less than 2, approximately, for which the electric field is relatively small, $0 \leq E_b \leq 0.275$. For larger $d$ higher values of $E_b$ are required to switch the solutions from pinching to wall-touching, and the simple physically derived law (6.14) underestimates the required value of $E_b$, for a given $d$.

7. Conclusions

This study considered the nonlinear instability of electrified viscous liquid threads surrounded by another viscous liquid. The Stokes regime was analysed and direct numerical simulations based on accurate boundary integral methods have been developed and implemented. The model is quite general (see (B 7) and (B 8) for leaky-dielectric core annular flows) and is characterised by a number of dimensionless parameters including $\lambda$, the outer to inner fluid viscosity ratio; $d$, the ratio of the tube radius to the undisturbed core radius; $E_b$, an electric Weber number measuring the ratio of electrical to capillary pressures. The initial condition for the thread is given by a single wavenumber which is another parameter affecting the dynamics. We have computed solutions for a wide range of these parameters when $\lambda = 1$, including the limiting case of a thin annular layer characterised by $d - 1 \ll 1$ (and arbitrary viscosity ratio $\lambda = O(1)$).

The thin annular limit was studied asymptotically using lubrication theory to derive the single scaled evolution equation (4.8). This extends the work of of Hammond (1983) by incorporating electric field effects in the annulus and is also a special case of the more general equation of Conroy et al. (2010) that includes electrokinetic effects. The fundamental mathematical difference between (4.8) and Hammond’s equation, is that the electric field generically induces interfacial wall touchdown in finite time whereas in its absence no such touchdown is possible. A comparison between solutions of (4.8) and a direct numerical simulation at a finite but small value of $\epsilon = d - 1 = 0.14$ shows fair agreement in the nonelectrified case as well as a finite time touchdown solution for $d - 1 = 0.25$.

Extensive numerical simulations also show that the electric field can be used to control the nonlinear evolution and ultimate interfacial configuration as follows: In the absence of a field a sufficiently thick annulus (in fact numerics shows that for $\lambda = 1$ this must exceed 0.2 approximately, i.e. $d \geq 1.2$ in terms of our non-dimensionalization), the filament pinches and forms drops, whereas if $d < 1.2$ the interface approaches the wall asymptotically. Introducing the field has a suppressive effect on the pinching and promotes wall touchdown in finite time. A phase diagram in the $E_b$ versus $d$ plane, that delineates regions supporting pinching or wall touchdown is provided in figure 12. The conclusion is that an unperturbed configuration that has a relatively thick annulus so that pinching emerges generically in the absence of a field, can be driven to wall touchdown instead, if the electric field is strong enough. Further more, the electric field initially induces larger jet breakup times relative
to the non-electrified case, but sufficiently large fields lead to touchdown events at comparatively shorter times — see figure 8.

At order one values of \( d - 1 \) direct numerical simulations are required to predict the dynamics as already described above. Two ultimate behaviors have been identified for the range of parameters investigated: (i) Breakup and drop formation, and (ii) wall touchdown (see above). For case (i) we find (in agreement with previous authors who study \( \lambda = 0 \)) that the electric field increases the volumes of formed satellites and in particular induces charges on them which are beyond the Rayleigh limit so that droplet disintegration (atomization) is expected. In addition, our simulations show that the pinching dynamics are equivalent to those of the non-electrified case and the same self-similar structures emerge, to leading order near the pinching event. Case (ii) emerges (i.e. touchdown dominates pinching) if the annulus is sufficiently thin or the electric field is sufficiently strong. The phase-space of terminal solutions characterized by the variation of the breakdown time \( t_b \) versus the electric field parameter \( E_b \) is given in figure 8 for \( d = 2 \) and 5. All other things being equal, it is found that \( t_b \) initially increases monotonically with \( E_b \) (and gives larger satellites), whereas a further increase in \( E_b \) causes a drastic decrease of \( t_b \) and attraction to touchdown solutions. Finally, we note that all pinching solutions computed here are driven by capillary forces and indeed the electric to capillary pressure ratio \( f_e / f_c \) is shown to become asymptotically small at pinching (see Section 6.3). As a result, self-similar structures emerge locally to leading order which are in complete agreement with the results of Lister & Stone (1998).

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Appendix A. The Matrix for the linear stability problem

In this Appendix the elements of matrix \( A \) defined in equation (3.4) are given. We find

\[
A = \begin{bmatrix}
I_1(k) & I_0(k) & -I_1(k) & -I_0(k) & -K_1(k) & -K_0(k) \\
kI_0(k) & L_1 & -kI_0(k) & L_2 & kK_0(k) & L_3 \\
0 & 0 & I_1(kd) & dI_0(kd) & K_1(kd) & dK_0(kd) \\
0 & 0 & kI_0(kd) & L_4 & -kK_0(kd) & L_5 \\
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
T_1 & T_2 & T_3 & T_4 & T_5 & T_6 
\end{bmatrix}
\]

(A 1)

where

\[
L_1 = 2I_0(k) + kI_1(k), \quad L_2 = -2I_0(k) - kI_1(k), \\
L_3 = -2K_0(k) + kK_1(k), \quad L_4 = 2I_0(kd) + kdI_1(kd), \\
L_5 = 2K_0(kd) - kdK_1(kd), \\
S_1 = 2kI_1(k), \quad S_2 = 2(I_1(k) + kI_0(k)), \\
S_3 = -2\lambda kI_1(k), \quad S_4 = -2\lambda(I_1(k) + kI_0(k)), \\
S_5 = -2\lambda kK_1(k), \quad S_6 = -2\lambda(kK_0(k) - K_1(k)), \\
T_1 = 2k(kI_0(k) - I_1(k)) - \frac{kI_1(k)}{\omega} \left(1 - k^2 - \frac{E_b}{\ln^2(d)} \left(1 + \frac{kK_0(kd)I_1(k) + I_0(kd)K_1(k)}{I_0(k)K_0(kd) - I_0(kd)K_0(k)}\right)\right), \\
T_2 = 2kI_1(k) - \frac{kI_0(k)}{\omega} \left(1 - k^2 - \frac{E_b}{\ln^2(d)} \left(1 + \frac{kK_0(kd)I_1(k) + I_0(kd)K_1(k)}{I_0(k)K_0(kd) - I_0(kd)K_0(k)}\right)\right), \\
T_3 = -2\lambda k(kI_0(k) - I_1(k)), \quad T_4 = -2\lambda k^2 I_1(k), \\
T_5 = 2\lambda k(kK_0(k) + K_1(k)), \quad T_6 = 2\lambda kK_1(k). 
\]
Appendix B. Boundary integral equations for the electric field problem

Here we present boundary integral equations for the electrostatic problem in a more general case that involves two leaky dielectric fluids in regions $i = 1, 2$ as defined in figure 1; subscripts $i$ denote variables in region $i$. The complementary part of a ring of point forces (Green’s function), as well as its $z$- and $r$- derivatives, in a cylindrical tube satisfying $G + G^C|_{r=d} = 0$, are given by

$$G^C(r, z, r_0, z_0) = -\frac{1}{\pi} \int_0^\infty \frac{K_0(kd)}{I_0(kd)} I_0(kr_0) I_0(kr_0) \cos(k(z - z_0)) dk,$$

$$\frac{\partial G^C}{\partial z}(r, z, r_0, z_0) = \frac{1}{\pi} \int_0^\infty k \frac{K_0(kd)}{I_0(kd)} I_0(kr_0) I_0(kr_0) \sin(k(z - z_0)) dk,$$

$$\frac{\partial G^C}{\partial r}(r, z, r_0, z_0) = -\frac{1}{\pi} \int_0^\infty k \frac{K_0(kd)}{I_0(kd)} I_1(kr_0) I_0(kr_0) \cos(k(z - z_0)) dk.$$  \tag{B 1-3}

Additionally, since $\phi_{iz}(z = 0) = \phi_{iz}(z = L) = 0$, axial images are chosen so that Neumann boundary conditions are satisfied at two ends of the thread (see Wang (2010)). Finally, the periodic version of the Green’s function for the potential problem is obtained in a similar fashion to Pozrikidis (1992); formally it can be written as

$$P(r, z, r_0, z_0) = \sum_{n = -\infty}^{\infty} \left( P^R(r, z, r_0, z_0 + 2nL) + P^R(r, z, r_0, z_0 + 2nL) \right),$$

where $P^R = G + G^C$. In the interest of generality, the integral equations for a leaky-dielectric model under a radial electric field are considered. The electric fields $\phi_1$ and $\phi_2$ satisfy the integral equations

$$\frac{1}{2} \phi_1(x_0) + \int_I \phi_1(x) P_n(x, x_0) r(x) dl = \int_I \phi_{1n}(x) P(x, x_0) r(x) dl,$$

$$\frac{1}{2} \phi_2(x_0) + \int_I \phi_2(x) P_n(x, x_0) r(x) dl + 1 = \int_I \phi_{2n}(x) P(x, x_0) r(x) dl,$$  \tag{B 5-6}

where the term 1 in (B 6) is the contribution of the tube wall. By introducing dimensionless parameters, $Q = \varepsilon_2 / \varepsilon_1$, $R = \sigma_2 / \sigma_1$ representing the ratios of electric permittivity and conductivity, respectively, and using the interfacial boundary conditions, $\phi_1 = \phi_2$ and $\phi_{1n} = R \phi_{2n}$ (in the latter we have assumed a fast surface charge relaxation limit) we can combine (B 5) and (B 6) as follows

$$1 + \frac{R}{2} \phi_1(x_0) + (1 - R) \int_I \phi_1(x) P_n(x, x_0) r(x) dl = R.$$  \tag{B 7}

Subtracting (B 5) from (B 6) leads to an equation for $\phi_{2n}$

$$\phi_2(x_0) = 1 + (R - 1) \int_I \phi_2(x) P(x, x_0) r(x) dl.$$  \tag{B 8}

For the perfect conducting core fluid case, $\phi_1 \equiv 0$ and $R \to 0$, which recovers equation (5.3) used in present paper as expected. The integral equations for a finite drop in an axial electric field can be found in Lac & Homsy (2007), for example.

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