On average performance and stability of economic model predictive control

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Abstract

Control performance and cost optimization can be conflicting goals in the management of industrial processes. Even when optimal or optimization-based control synthesis tools are applied, the economic cost associated with plant operation is often only optimized according to static criteria that pick, among all feasible *steady states*, those with minimal cost. In mathematical terms an economic cost functional differs from stage costs commonly adopted in MPC as it need not be minimal at its best equilibrium.

This note collects and illustrates some recent advances in receding horizon optimization of nonlinear systems that allow the control designer to simultaneously and dynamically optimize transient and steady-state economic performance.

In particular, we show that average performance of economic MPC is never worse than the optimal steady-state operation. We introduce a dissipation inequality and supply function that extend previous sufficient conditions for asymptotic stability of economic MPC. Dissipativity is also shown to be a sufficient condition for concluding that steady-state operation is optimal. We show how to modify an economic cost function so that steady-state operation is asymptotically stable when that feature is deemed desirable. Finally, for the case when steady-state operation is not optimal, we develop two modified MPC controllers that asymptotically guarantee (i) improved performance compared to optimal periodic control and (ii) satisfaction of constraints on average values of states and inputs.

1 Introduction

Standard practice in most industrial process control systems is to decompose the plant's management and optimization into two levels. The first level, usually referred to as real-time optimization (RTO) takes into account all sorts of

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different constraints (including production, safety or physical constraints) and essentially performs a static optimization. That is, it determines, among all feasible *steady-state* plant operating conditions (setpoints) those with minimal cost; see for instance [15, 5, 13, 12, 9, 4, 16]. The second level, instead, is responsible for deciding suitable dynamic control actions that steer the plant's operation to the desired steady-state operating condition within a reasonable amount of time. Since constraints are also of concern during transient operation, in many advanced industrial control systems, the dynamic operation is usually implemented with some kind of model predictive control (MPC) scheme.

Three considerations are worthwhile in this respect. First of all, computational complexity of hierarchical approaches is typically lower than that of alternative non-decoupled schemes. Having acknowledged that, however, hierarchical separation often means that the control law is designed disregarding the issue of transient costs. In fact, even though cost functionals are commonly employed in control design, they are usually shaped in order to yield control performance, that is induce quick asymptotic tracking of setpoints and need not bear any resemblance to the actual economic cost involved in plant operation, in particular to the one used in order to select the optimal steady state. On these grounds, a hierarchical approach is only meaningful provided a time-scale separation is assumed between constants characterizing system's dynamics and rates of variations of setpoints and constraints for the problem at hand. Only under such circumstances does the system spend most of its time close to equilibrium so that, in the long term, suboptimal transient profiles can be neglected. When this is not the case, however, transient costs could also be significant and therefore a hierarchical approach could be inappropriate.

Finally, plant's nonlinearities and nonconvex cost functionals can be responsible for somewhat counter-intuitive situations, in which the best operating regime for given plant and constraints could actually fail to be an equilibrium; periodic or even complex chaotic regimes might outperform the best possible steady states. This phenomenon has been widely recognized during the 1970s and 1980s, giving rise to intense research on periodic operation of chemical reactors, (see the survey [18]), and optimal periodic control (OPC), [6, 11].

From the practical point of view, there are two main stumbling blocks that are faced when trying to improve plant performance on periodic operation cycles:

- finding an optimal periodic cycle entails the solution of a nonconvex, infinite dimensional optimization problem;
- the solution found need not be asymptotically stable, therefore suitable feedback control is still needed once the OPC is solved.

This note presents some recent advances in MPC that allow the dynamic control layer to be used to achieve both transient and steady-state economic optimization simultaneously.

2 Standard MPC vs. Economic MPC

In the following we compare two approaches to model predictive control of nonlinear plants. The first one, referred to as Standard MPC, adopts a stage cost that need not be directly related to the economic cost incurred during plant operation. This cost is conveniently chosen to be minimal at the desired setpoint. In other words, given (x_s, u_s) is the best feasible pair of equilibrium state and associated control input, respectively, the following cost function is standard.

Assumption 1 *Standard MPC cost function*

$$0 = \ell(x_s, u_s) \leq \ell(x, u) \text{ for all admissible } (x, u) \tag{1}$$

in which $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ *denotes the stage cost,* $\mathbb{X}$ *is the state space and* $\mathbb{U}$ *the set of admissible input values.*

In economic MPC instead, the cost incurred for plant operation is used directly as a stage cost in the MPC optimization layer. Therefore, (1) cannot be generally assumed, and it may occur that $\ell(x_s, u_s) > 0$ or, even more fundamentally, that $\ell(x, u) < \ell(x_s, u_s)$ for some feasible pair (x, u) not corresponding to any equilibrium point. This unconventional formulation of MPC was originally proposed in [17] in the context of MPC in the presence of unreachable setpoints. A popular approach when the best setpoint from the economic point of view turns out to be infeasible, (for instance due to more stringent input or production constraints), is to replace it by the best feasible one and redesign the stage cost ℓ in order to be zero and minimal at this alternative setpoint. This approach is clearly compatible with the standard MPC paradigm; it may lead to suboptimal economic performance, however, when it comes to transient behavior. The alternative approach proposed in [17] was to leave unchanged the stage cost used to formulate the MPC algorithm, while only replacing its terminal constraint, namely by forcing the state x to reach the best *feasible steady state* at the end of the control horizon. It is clear that, in this alternative formulation, Assumption 1 need not hold. In order to compare the two approaches we take into account the fairly general set-up described below. We consider in particular finite-dimensional discrete-time nonlinear control systems

$$x^+ = f(x, u) \tag{2}$$

with state $x \in \mathbb{X} \subset \mathbb{R}^n$, input $u \in \mathbb{U} \subset \mathbb{R}^m$ and state-transition map $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$. The cost functional that we seek to optimize is given by:

$$\sum_k \ell(x(k), u(k)) \tag{3}$$

subject to the dynamic constraints provided by (2) as well as constant pointwise-in-time constraints:

$$(x(k), u(k)) \in \mathbb{Z} \qquad k \in \mathbb{I}_{\geq 0} \tag{4}$$

for some compact set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. Since there is no natural termination time for production of an industrial plant, ideally one would like to consider (3) over an infinite time horizon. This however introduces nontrivial computational and theoretical complications as optimization of (3) entails the solution of a nonconvex infinite dimensional problem; see for instance [7]. To overcome at least in part such difficulties, one may replace the cost functional in (3) by a similar one defined over a sufficiently long, but finite horizon:

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) \tag{5}$$

where $\mathbf{u} = [u(0), u(1), \ldots, u(N-1)]$ and $x^+ = f(x, u)$, $x(0) = x$. As is customary in MPC, (5) is repeatedly minimized in a receding horizon manner, that is applying only the first control sample of the optimal solution computed at any given time instant. The motivation is that, by doing so, the total resulting cost associated to the closed-loop behavior is not too far from that of the infinite horizon optimal solution. More in detail, with the notation adopted so far, the best feasible steady-state control-input pair fulfills:

$$\ell(x_s, u_s) = \min \left\{ \ell(x, u) \mid (x, u) \in \mathbb{Z},\, x = f(x, u) \right\} \tag{6}$$

A feedback control law $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ is implicitly defined by solving the following optimization problem:

$$\begin{aligned} &\min_{\mathbf{u}} \quad V_N(x, \mathbf{u}) \\ &\text{subject to} \begin{cases} x^+ = f(x, u) \\ (x(k), u(k)) \in \mathbb{Z} \quad k \in \mathbb{I}_{0:N-1} \\ x(N) = x_s, \quad x(0) = x \end{cases} \end{aligned} \tag{7}$$

An input sequence $\mathbf{u} = \{u(0), u(1), \ldots u(N-1)\}$ is termed *feasible* for initial state x if the input sequence and corresponding state sequence generated by the model $x^+ = f(x, u)$ with initial condition $x(0) = x$ together satisfy the constraints of the optimal control problem. We define the admissible set $\mathbb{Z}_N$ as this set of $(x, \mathbf{u})$ pairs

$$\begin{aligned} \mathbb{Z}_N = \{(x, \mathbf{u}) \mid \exists\, x(1), \ldots, x(N) : x^+ = f(x, u), \\ (x(k), u(k)) \in \mathbb{Z}, \forall k \in \mathbb{I}_{0:N-1}, \quad x(N) = x_s, \quad x(0) = x\} \end{aligned}$$

The set of admissible states $\mathcal{X}_N$ is then defined as the projection of $\mathbb{Z}_N$ onto $\mathbb{X}$

$$\mathcal{X}_N = \{x \in \mathbb{X} \mid \exists\, \mathbf{u} \text{ such that } (x, \mathbf{u}) \in \mathbb{Z}_N\}$$

The control law is then defined as

$$u = \kappa_N(x) = u^0(0; x) \qquad x \in \mathcal{X}_N \tag{8}$$

where $\mathbf{u}^0(x)$ denotes the optimal solution of (7) for initial state x, and $u^0(k;x)$ denotes the solution at time $k \in \mathbb{I}_{0:N-1}$. For the sake of simplicity we assume $\mathbf{u}^0(x)$ to be uniquely defined (the case of multivalued optima can be treated by arbitrarily assigning a constant *selection* map). The control algorithm defined by (7) and (8) is appropriate for describing both standard MPC and economic MPC. As previously remarked, the difference between the two approaches only comes with respect to Assumption 1, which holds for the former and does not for the latter. Finally, to ensure the existence of: (i) a solution to the optimal control problem (7) and (ii) an interesting admissible set, we make the following assumption.

Assumption 2 (Model, cost, and admissible set) *1. The model $f(\cdot)$ and stage cost $\ell(\cdot)$ are continuous. The admissible set $\mathcal{X}_N$ contains x_s in its interior.*

2. There exists γ of class $\mathcal{K}_\infty$ such that for each $x \in \mathcal{X}_N$ there exists a feasible $\mathbf{u}$*, with*

$$|\mathbf{u} - [u_s, \dots, u_s]'| \leq \gamma(|x - x_s|)$$

As in standard MPC it might be desirable to relax the terminal equality constraint and replace it by a terminal weighting function and possibly inequality constraints. This is investigated in [1].

3 Analysis of average asymptotic performance

A standard approach for closed-loop stability analysis of most variants of MPC algorithms is the definition of a cost-to-go function and its use as a Lyapunov function. In our setup we denote the optimal cost in problem (7) by $V_N^0(x)$. It is clear that Assumption 1 implies $0 = V_N^0(x_s) \leq V_N^0(x)$ for all $x \in \mathcal{X}_N$. Moreover, as is customary in MPC, along solutions of the closed-loop system the following inequality holds:

$$V_N^0(x^+) - V_N^0(x) \leq \ell(x_s, u_s) - \ell(x, u) \tag{9}$$

provided u is selected according to (8). In the case of standard MPC, (1) implies that $V_N^0(x^+) \leq V_N^0(x)$, that is monotonicity of the cost-to-go function evaluated along solutions of the closed-loop system; this provides a first important step towards the proof of asymptotic stability of MPC algorithms (usually completed under mild additional assumptions, such as $\ell(x_s, u_s)$ being a strict minimum of $\ell(x, u)$, by using standard Lyapunov-like analysis tools).

As remarked and exemplified in [16], if Assumption 1 does not hold, $V_N^0(x)$ need not be monotonically decreasing, even in the simplest case of a linear system with a convex stage cost and subject to convex constraints. Loss of $V_N^0(\cdot)$ as a Lyapunov function does not necessarily imply loss of stability, however. Indeed, for the case of strictly convex cost functionals and linear systems subject to convex constraints, x_s turns out to be asymptotically stable with a region of attraction $\mathcal{X}_N$. The original proof, contained in [17], is based on convexity arguments. Recently, a different proof based on Lyapunov arguments was developed [8].

For general nonlinear systems and/or nonconvex cost functionals, however, there is no reason why x_s should even be an equilibrium point of the closed-loop system (2)–(8) and therefore its stability cannot be expected in general. It is in fact conceivable that the optimal path from x_s at time 0 to x_s at time N (that is at the end of the control horizon N) be different from the constant solution $x(k) \equiv x_s$ for all $k \in \mathbb{I}_{0:N}$. While this can at first sight appear to be a dangerous drawback of economic MPC, it might be, for specific applications, one of its major strengths. Indeed, even though stability is not in general guaranteed, asymptotic performance of the controller is preserved, however, as was first investigated in [3]. For the sake of completeness we recall here the main result of [3] together with its proof.

Theorem 1 *Let $x(0) \in \mathcal{X}_N$ be a feasible initial condition such that for at least one admissible control sequence, the state is steered to x_s at time N without leaving $\mathcal{X}_N$. Then, system (2) in closed-loop with (8) has an average performance that is no worse than that of the best admissible steady state.*

Proof. Pick an arbitrary $x \in \mathcal{X}_N$ so $\mathcal{U}_N(x)$ is nonempty. It is easily seen that if $\mathbf{u} \in \mathcal{U}_N(x)$ is a feasible control sequence at time k, from initial state x and $u(0)$ gets applied, then $\{u(1), \ldots, u(N-1), u_s\}$ is also a feasible control sequence at time $k+1$, from initial state $f(x, u(0))$. Hence, by induction, feasibility of optimization problem (7) follows for all non-negative times for all $x \in \mathcal{X}_N$. In other words, solutions are globally defined for $k \geq 0$ and fulfill pointwise-in-time constraints.

In addition, along solutions of the closed-loop system

$$V_N^0(x^+) - V_N^0(x) \leq \ell(x_s, u_s) - \ell(x, u) \tag{10}$$

Taking averages in both sides of (10) yields:

$$\begin{aligned} & \liminf_{T \to +\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} \\ \leq \quad & \liminf_{T \to +\infty} \frac{\sum_{k=0}^{T} \ell(x_s, u_s) - \ell(x(k), u(k))}{T+1} \\ = \quad & \ell(x_s, u_s) - \limsup_{T \to +\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1} \end{aligned} \tag{11}$$

On the other hand, assuming without loss of generality $\ell(x, u) \geq 0$ for all $(x, u) \in$

$\mathbb{Z}$:

$$\begin{array}{rl} & \liminf_{T\to+\infty} \frac{\sum_{k=0}^{T} V_N^0(x(k+1)) - V_N^0(x(k))}{T+1} \\ = & \liminf_{T\to+\infty} \frac{V_N^0(x(T+1)) - V_N^0(x(0))}{T+1} \\ \geq & \liminf_{T\to+\infty} -\frac{V_N^0(x(0))}{T+1} = 0 \end{array} \tag{12}$$

Combining (11) and (12) we come to the conclusion:

$$\limsup_{T\to+\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1} \leq \ell(x_s, u_s)$$

which completes our proof. □

Remark 3.1 It is worth pointing out that only the asymptotic average cost is guaranteed to be not worse than that of the best steady-state. Transient averages, as defined for a given t by the following quantity:

$$\frac{\sum_{k=0}^{t} \ell(x(k), u(k))}{t+1},$$

are in fact allowed to take any value.

4 Stability and convergence analysis

Asymptotic stability of economic MPC was first proved in [17] under the assumption of linear plant dynamics and strictly convex cost functionals. The original proof heavily relied on convexity properties and had no apparent Lyapunov interpretation. Such interpretation is subsequently provided in [8], where it is shown how to define a standard MPC algorithm (on a "rotated" stage cost) yielding exactly the same closed-loop behavior of the original economic MPC scheme. Analysis of the standard MPC scheme can then proceed along the usual lines by choosing the rotated cost-to-go as a candidate Lyapunov function.

In the following we significantly relax the assumptions of [8] and highlight the role played by a suitable dissipativity inequality in proving stability of economic MPC.

Definition 4.1 A control system as in (2) is dissipative with respect to a supply rate $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ if there exists a function $\lambda : \mathbb{X} \to \mathbb{R}$ such that:

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) \tag{13}$$

for all $(x, u) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. If in addition $\rho : \mathbb{X} \to \mathbb{R}_{\geq 0}$ positive definite[1] exists such that:

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x) + s(x, u) \tag{14}$$

[1]A function is positive definite with respect to some point $x_s \in \mathbb{X}$ if it is continuous, $\rho(x_s) = 0$ and $\rho(x) > 0$ for all $x \neq x_s$.

then the system is said to be strictly dissipative. □

We now state the main result for this Section.

Theorem 2 *Suppose Assumption 2 holds and consider a nonlinear control system as in* (2) *and the MPC control scheme defined by* (7) *and* (8)*, where* (x_s, u_s) *is a best feasible equilibrium-control pair as defined in* (6)*. If system* (2) *is strictly dissipative with respect to the supply rate:*

$$s(x,u) = \ell(x,u) - \ell(x_s,u_s) \tag{15}$$

then x_s *is an asymptotically stable equilibrium point of the closed-loop system with region of attraction* $\mathcal{X}_N$*.*

Remark 4.2 It is worth pointing out that the assumption of dissipativity can equivalently be stated as follows: there exists a function $\lambda(\cdot)$ such that

$$\min_{(x,u)\in\mathbb{Z}} \ell(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \ell(x_s,u_s) \tag{16}$$

In the case of linear Lyapunov-like functions $\lambda(x) = \bar{\lambda}'x$ for some $\bar{\lambda} \in \mathbb{R}^n$, condition (16) reads

$$\min_{(x,u)\in\mathbb{Z}} \ell(x,u) + \bar{\lambda}'(x - f(x,u)) \geq \ell(x_s,u_s) \tag{17}$$

This is a classic condition in the context of infinite horizon optimal control, see Assumption 4.2 (ii) in [7], usually referred to as *strong duality* (see for instance [8]). Indeed, defining the *rotated* stage-cost

$$L(x,u) = \ell(x,u) + \lambda(x) - \lambda(f(x,u)) \tag{18}$$

the following holds for all $\lambda(\cdot)$:

$$\begin{aligned} \min_{(x,u)\in\mathbb{Z}} L(x,u) &\leq \min_{(x,u)\in\mathbb{Z}, x=f(x,u)} L(x,u) \\ &= \min_{(x,u)\in\mathbb{Z}, x=f(x,u)} \ell(x,u) = \ell(x_s,u_s) \end{aligned}$$

so that condition (17) can be interpreted as the absence of a duality gap:

$$\begin{aligned} &\max_{\bar{\lambda}\in\mathbb{R}^n} \min_{(x,u)\in\mathbb{Z}} \ell(x,u) + \bar{\lambda}'(x - f(x,u)) \\ = &\min_{(x,u)\in\mathbb{Z}, x=f(x,u)} \ell(x,u) \end{aligned}$$

As remarked in [8] this is always fulfilled for linear control systems with a strictly convex stage cost satisfying a Slater condition. □

We show next, by means of an example, that the dissipativity definition 4.1 is an effective way of relaxing strong duality.

Example 4.3 Consider the following scalar linear system:

$$x^+ = \alpha x + (1-\alpha)u \tag{19}$$

where $\alpha \in [0,1)$ is a parameter to be discussed later, along with the non-convex cost functional:

$$\ell(x,u) = (x + u/3)(2u - x) + (x-u)^4. \tag{20}$$

Notice that, regardless of α, for each input u, there exists a unique equilibrium $x_e = u$. Moreover:

$$\ell(x,u)|_{x=u} = \frac{4}{3}u^2 \tag{21}$$

so that $(x_s, u_s) = (0,0)$ is the best steady-state and $\ell(x_s,u_s) = 0$. The point $(0,0)$ is not, however, the global minimum of $\ell(x,u)$, which in fact has 2 global minima for $(x,u) = \pm(21\sqrt{6}/64, 7\sqrt{6}/192)$. In fact $(0,0)$ is a saddle-point of $\ell(x,u)$ and the level-set $\mathcal{L}_0 = \{(x,u) : \ell(x,u) = 0\}$ is in $(0,0)$ tangent to the lines of equation $u = -3x$ and $u = x/2$.

Notice that zero-average period-2 solutions of (19) are possible. These correspond to input sequences of alternating signs: $+u_0, -u_0, +u_0, -u_0, \ldots$, with the resulting periodic state sequence:

$$-(1+\alpha)/(1-\alpha)u_0, (1+\alpha)/(1-\alpha)u_0, -(1+\alpha)/(1-\alpha)u_0, \ldots$$

Non-zero average period-2 solutions are also possible, but for the sake of simplicity and for the purpose of this analysis we will not take them into account.

Choosing $\alpha = 0$ or sufficiently small, yields period 2 solutions which, suitably tuning the input amplitude u_0, belong to the sublevel set $\mathcal{L}_{\leq 0} = \{(x,u) : \ell(x,u) \leq 0\}$, thus outperforming the best steady state. Under such circumstances, one cannot expect dissipativity to hold. For larger values of α, however, the period 2 solution leave $\mathcal{L}_{\leq 0}$ and get closer to the u-axis. One may therefore wonder for which values of α (if any) the system fulfills *dissipativity* or *strong duality*.

In order to prove strong duality we look for λ such that:

$$\lambda x^+ - \lambda x \leq \ell(x,u) \tag{22}$$

Notice that the left-hand side of (22) defines, regardless of λ and α, a linear function in (x,u). Since $\ell(x,u)$ has a saddle point in $(0,0)$, there is no choice of λ and α which can fulfill (22).

Next we look for a candidate quadratic storage function $\lambda(x) = kx^2$. Dissipativity holds for all $\alpha \in [0,1)$ for which there exists k and $\epsilon > 0$ so that:

$$k(x^+)^2 - kx^2 \leq -\varepsilon x^2 + \ell(x,u) \tag{23}$$

holds for all $(x,u) \in \mathbb{R}^2$. For (23) to hold, it is enough that:

$$k(\alpha x + (1-\alpha)u)^2 - kx^2 \leq -\varepsilon x^2 + (x + u/3)(2u - x). \tag{24}$$

All terms in (24) are quadratic functions in x and u, therefore, inequality (24) is fulfilled if the following matrix is positive definite:

$$Q = \begin{bmatrix} k(1-\alpha^2)-1 & \frac{5}{6}-k\alpha(1-\alpha) \\ \frac{5}{6}-k\alpha(1-\alpha) & \frac{2}{3}-k(1-\alpha)^2 \end{bmatrix}. \tag{25}$$

Requiring that diagonal entries of Q be positive we find the conditions:

$$\frac{1}{1-\alpha^2} < k < \frac{2}{3(1-\alpha)^2} \tag{26}$$

These are fulfilled for some values of k provided $\alpha \in (1/5, 1)$. Finally, requiring $\det(Q) > 0$ yields:

$$-k^2(1-\alpha)^2 + k(1-\alpha)(4\alpha/3 + 5/3) - 49/36 > 0. \tag{27}$$

Letting $Z = k(1-\alpha)$ we can rewrite condition (27) as:

$$-Z^2 + Z(4\alpha/3 + 5/3) - 49/36 > 0, \tag{28}$$

which is fulfilled for Z between

$$\frac{4\alpha + 5 \pm \sqrt{16\alpha^2 + 40\alpha - 24}}{6}, \tag{29}$$

provided $\alpha > 1/2$. Condition (29) implies that k should be chosen between

$$\frac{4\alpha + 5 \pm \sqrt{16\alpha^2 + 40\alpha - 24}}{6(1-\alpha)}. \tag{30}$$

In fact it can be shown that

$$k = \frac{4\alpha + 5}{6(1-\alpha)}$$

is a feasible choice for all $\alpha \in (1/2, 1)$. Hence, for all such αs condition (23) is fulfilled and economic MPC yields convergence to the best equilibrium. Notice that $\alpha = 1/2$ is exactly the value of α for which the zero average period 2 solutions exit the sublevel set $\mathcal{L}_{\leq 0}$; this shows that dissipativity is in this case a tight condition (see Fig. 1). □

Definition of the *rotated* stage-cost $L(x, u)$ also allows to state an *auxiliary* optimization problem by replacing in (5) ℓ by L:

$$\tilde{V}_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} L(x(k), u(k)) \tag{31}$$

and considering this new cost functional for optimization:

$$\begin{aligned} &\min_{\mathbf{u}} \quad \tilde{V}_N(x, \mathbf{u}) \\ &\text{subject to} \begin{cases} x^+ = f(x, u) \\ (x(k), u(k)) \in \mathbb{Z} \quad k \in \mathbb{I}_{0:N-1} \\ x(N) = x_s, \quad x(0) = x \end{cases} \end{aligned} \tag{32}$$

The following Lemma (which closely follows the steps of a related one in [8]) is crucial to the proof of Theorem 2).

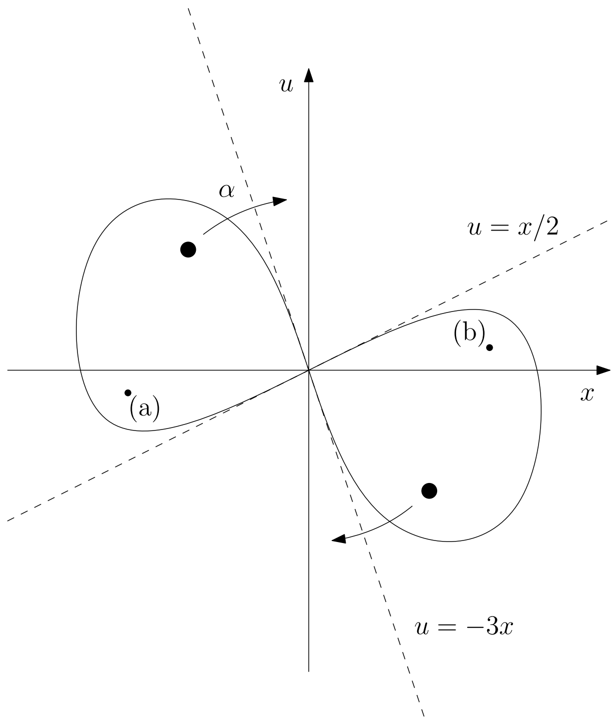

Figure 1: Qualitative picture of $\mathcal{L}_0$: global minima (a),(b); α-dependent period-2 solution (black dots)

Lemma 4.4 The set of points $x \in \mathbb{X}$ for which optimization problems (7) and (32) are feasible coincide. Moreover, any optimal trajectory $\mathbf{u}^0$ of (32) is also optimal for (7) (and vice versa). □

Proof. Notice that problems (7) and (32) share the same set of constraints and only differ as far as cost functionals are concerned. Hence the first claim of the Lemma trivially follows. Take any feasible pair $(\mathbf{x}, \mathbf{u})$. Straightforward computations show that:

$$\begin{aligned}\tilde{V}_N(x, \mathbf{u}) &= \sum_{k=0}^{N-1} \ell(x(k), u(k)) + \lambda(x(k)) - \lambda(f(x(k), u(k))) \\ &= \sum_{k=0}^{N-1} \ell(x(k), u(k)) + \lambda(x(k)) - \lambda(x(k+1)) \\ &= \lambda(x(0)) - \lambda(x(N)) + \sum_{k=0}^{N-1} \ell(x(k), u(k)) \\ &= \lambda(x(0)) - \lambda(x_s) + V_N(x, \mathbf{u})\end{aligned}$$

Hence, on any feasible solution the objective functionals (31) and (5) only differ by a constant quantity (that does not depend upon optimization variables). This completes the proof of the Lemma.

□

We are now ready to prove the main result for this Section, that is Theorem 2.

Proof. By virtue of Lemma 4.4 we may analyze stability of (7) by considering the MPC closed-loop system induced by the auxiliary optimization problem (32). Notice that, thanks to dissipativity:

$$L(x_s, u_s) = \ell(x_s, u_s) \leq \min_{(x,u) \in \mathbb{Z}} L(x, u) \tag{33}$$

which shows, assuming without loss of generality $L(x_s, u_s) = 0$, how the newly formulated MPC scheme fulfills conditions (1) and can therefore be analyzed along the lines of a standard MPC scheme. To this end, define the rotated cost-to-go as the optimal cost in problem (32), and denote it by $\tilde{V}_N^0(x)$ We claim that $\tilde{V}_N^0(x)$ can serve as a Lyapunov function in order to assess stability of the closed-loop system. Indeed, $\tilde{V}_N^0(x)$ is continuous for $x = x_s$ thanks to the assumption 2 and positive-definite with respect to x_s (assuming without loss of generality $L(x_s, u_s) = 0$). Moreover, exploiting strict dissipativity the following holds along solutions of the closed-loop system:

$$\begin{aligned}\tilde{V}_N^0(x^+) &\leq \tilde{V}_N^0(x) + L(x_s, u_s) - L(x, u) \\ &\leq \tilde{V}_N^0(x) - \rho(x).\end{aligned} \tag{34}$$

Hence, $\tilde{V}_N^0$ is strictly decreasing and therefore x_s is an asymptotically stable equilibrium with region of attraction $\mathcal{X}_N$. □

When using purely economic cost functions, we have seen that the optimal steady state may not be stable. If steady operation is deemed desirable, we next show one method to modify the cost function to achieve this goal. We consider the following modified stage cost in which we shall determine the function $\alpha : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$

$$\bar{\ell}(x,u) = \ell(x,u) + \alpha(x,u) \tag{35}$$

in which $\alpha(\cdot)$ is chosen positive definite with respect to (x_s, u_s). Hence, $\bar{\ell}(\cdot)$ and $\ell(\cdot)$ share the same optimal steady state, (x_s, u_s). To achieve strict dissipativity, it is sufficient to satisfy the following inequality for some $\bar{\lambda} \in \mathbb{R}^n$ and all $(x,u) \in \mathbb{Z}$

$$\bar{\lambda}'(x - f(x,u)) \leq -\rho(x) + \ell(x,u) - \ell(x_s,u_s) + \alpha(x,u)$$

Rearranging, we must satisfy for some $\bar{\lambda}$ and all $(x,u) \in \mathbb{Z}$

$$\alpha(x,u) \geq h(x,u,\bar{\lambda})$$

in which $h(x,u,\bar{\lambda}) := \bar{\lambda}'(x - f(x,u)) + \rho(x) - \ell(x,u) + \ell(x_s,u_s)$. The function $h(x,u,\bar{\lambda})$ is continuous in (x,u) for all $\bar{\lambda}$, so define for $r \in \mathbb{R}_{\geq 0}$

$$\bar{h}(r,\bar{\lambda}) = \max_{\substack{(x,u)\in\mathbb{Z} \\ |(x,u)-(x_s,u_s)|\leq r}} h(x,u,\bar{\lambda})$$

in which the maximum exists for all $r \in \mathbb{R}_{\geq 0}$ by the Weierstrass theorem. Using $\alpha(x,u) = \bar{h}(|(x,u) - (x_s,u_s)|, \bar{\lambda})$ for any $\bar{\lambda} \in \mathbb{R}^n$ is positive definite with respect to (x_s,u_s) and suffices for strict dissipativity. If desired, we can select the *weakest* modification to the purely economic problem by searching[2] over $\bar{\lambda}$. We then have the following stability result.

Theorem 3 *Consider a nonlinear control system as in* (2) *and the MPC control scheme defined by* (7) *and* (8)*, where* (x_s,u_s) *is a best feasible equilibrium-control pair as defined in* (6)*. If the stage cost is chosen according to* (35) *with* $\alpha(x,u) = \bar{h}(|(x,u) - (x_s,u_s)|, \bar{\lambda})$ *for any* $\bar{\lambda} \in \mathbb{R}^n$*, then* x_s *is an asymptotically stable equilibrium point of the closed-loop system with region of attraction* $\mathcal{X}_N$*.*

Proof. By construction of $\alpha(\cdot)$, strict dissipativity is satisfied and Theorem 2 applies, giving asymptotic stability of x_s. □

Theorem 3 essentially says that, for any continuous nonlinear system dynamics $f(x,u)$, we can turn up the convexity in the stage cost $\bar{\ell}(x,u)$ until we stabilize the optimal steady state. From this perspective, the pure economic problem can be viewed as one extreme in which $\ell(x,u)$ represents the economic cost, $\alpha(\cdot)$ is chosen to be zero, and the optimal steady state may not be asymptotically stable. The standard tracking problem can be viewed as the other extreme in which the economics are completely ignored, $\ell(\cdot)$ is set to zero, and $\alpha(x,u) = (1/2)(|x - x_s|_Q^2 + |u - u_s|_R^2)$ is the usual tracking objective, and the optimal steady state is asymptotically stable by design. The function $\alpha(\cdot)$ allows us to capture a range of behaviors between these two extremes. We illustrate this feature in the later example.

[2] Note that to fully exploit the generality of dissipativity, one could extend this result and search over *continuous functions* $\lambda(\cdot)$ rather than scalar $\bar{\lambda}$.

5 Extensions of economic MPC

The observation that economic MPC need not converge to the best feasible equilibrium introduces the possibility for further performance improvements as well as the introduction of additional features to the control algorithm. We briefly discuss below two variants of economic MPC schemes developed in order to deal with the following issues:

- outperforming Optimal Periodic solutions
- dealing with average constraints

5.1 Periodic terminal constraint

For a plant that is not optimally operated at steady state, it is meaningful to aim at an average asymptotic cost that is strictly less than that of the best feasible equilibrium. In [3] this is achieved by considering the situation in which a Q-periodic solution $x^*(k), k \in \mathbb{I}_{0:Q-1}$ that outperforms the best feasible steady state has been precomputed. This can be done by solving the following optimization problem:

$$\min_{x(0),\mathbf{u}} V_Q(x(0),\mathbf{u}) = \sum_{k=0}^{Q-1} \ell(x(k),u(k))$$
$$\text{subject to} \begin{cases} x^+ = f(x,u) \\ (x(k),u(k)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{0:Q-1} \\ x(Q) = x(0) \end{cases} \tag{36}$$

with $\mathbf{u} = \{u(0), u(1), \ldots, u(Q-1)\}$. Denote the optimal state and input sequence for this problem as $(x^*(k), u^*(k)), k \in \mathbb{I}_{0:Q-1}$.

Next we obtain a time-varying state feedback law by solving online the following optimization problem over the set of terminal constraints indexed by integer $q \in \mathbb{I}_{0:Q-1}$

$$\min_{\mathbf{v}} V_N(x,\mathbf{v},q) \quad \text{subject to} \begin{cases} z^+ = f(z,v) \\ (z(k),v(k)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{0:N-1} \\ z(N) = x^*(q) \\ z(0) = x \end{cases} \tag{37}$$

Let $(\mathbf{z}^0(x,q), \mathbf{v}^0(x,q))$ denote the optimal state and input of (37) (assumed unique) for initial state x using the qth element of the periodic terminal constraint. Next define the function $\kappa_N(\cdot)$ to be the first element of the optimal input sequence using the qth constraint

$$\kappa_N(x,q) = v^0(0;x,q) \tag{38}$$

and system (2) is controlled by selecting inputs according to the time-varying feedback control law

$$u(x,t) = \kappa_N\big(x, t \bmod Q\big) \qquad t \in \mathbb{I}_{\geq 0} \tag{39}$$

This law is defined on the set of x for which problem (37) is feasible. Notice that, due to the periodic terminal constraint, the resulting closed-loop system is also a Q-periodic nonlinear system. It is shown in [3] that the feedback law (39) induces an asymptotic average cost that is not worse than that of the optimal periodic solution. As in the case of optimal equilibria, asymptotic convergence to the periodic solution is not generally to be expected and can only be ensured provided suitable "dissipativity" assumptions are in place. For the sake of completeness we state here the main result concerning performance of periodic economic MPC and defer its proof to the appendix.

Theorem 4 *Consider a nonlinear control system as in* (2) *under periodic state-feedback as defined by* (38) *and* (39). *Then, the average asymptotic performance of the closed-loop system fulfills:*

$$\limsup_{T\to+\infty} \frac{\sum_{k=0}^{T} \ell(x(k),u(k))}{T+1} \leq \frac{\sum_{k=0}^{Q-1} \ell(x^*(k),u^*(k))}{Q} \tag{40}$$

5.2 Average constraints

Shifting the focus from convergence to average performance leads naturally to the consideration of constraints on average values of variables (typically inputs and states), besides pointwise in time hard bounds as discussed in the previous Sections and customary in MPC. As standard MPC guarantees convergence to equilibrium, asymptotic averages of any variable are in fact determined by the value of such quantity at equilibrium; therefore average constraints do not deserve special attention as they can be taken into account as static constraints in the RTO layer. We present in the following an adaptation of the control scheme (7) (also discussed in a preliminary version in [3]) that, together with a guaranteed average cost, also ensures satisfaction of asymptotic constraints on average quantities. To this end, for any given vector valued bounded signal $v : \mathbb{I}_{\geq 0} \to \mathbb{R}^{n_v}$, we define the set of asymptotic averages:

$$\mathrm{Av}[v] = \left\{ \bar{v} \in \mathbb{R}^{n_v} \mid \exists\, t_n \to +\infty : \lim_{n\to+\infty} \frac{\sum_{k=0}^{t_n} v(k)}{t_n+1} = \bar{v} \right\}$$

Notice that, $\mathrm{Av}[v]$ is always nonempty (because bounded signals have limit points). It need not be a singleton, though, as there may be more than one asymptotic average for each given signal. As an example take the sequence v as:

$$0,1,0,0,1,1,0,0,0,0,1,1,1,1,\ldots$$

in which the number of consecutive 0 and 1s doubles at each time. The highest possible asymptotic average is achieved by sampling averages at the end of each string of 1s, in particular for $t_n = 2(2^n-1)-1$. For such a choice of t_n we have $\sum_{k=0}^{t_n} v(k) = (2^n-1)$, so that:

$$\frac{\sum_{k=0}^{t_n} v(k)}{t_n+1} = \frac{1}{2} \qquad \forall\, n.$$

The lowest possible asymptotic average is achieved instead by sampling averages at the end of each string of 0s, in particular for $t_n = (2^n + 2^{n-1} - 3)$. It holds:

$$\frac{\sum_{k=0}^{t_n} v(k)}{t_n + 1} = \frac{2^{n-1} - 1}{2^n + 2^{n-1} - 2} \to \frac{1}{3} \quad \text{as } n \to +\infty$$

Hence $\text{Av}[v]$ contains at least two distinct points. In fact it is possible to show that $\text{Av}[v] = [1/3, 1/2]$.

Also, it is straightforward to verify that, whenever $w(k) = v(k + P)$ for some finite $P \in \mathbb{I}_{\geq 0}$, we have: $\text{Av}[(w, v)] \subseteq \{[v_1, v_2] \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_v} \mid v_1 = v_2\}$. In particular then, $\text{Av}[v] = \text{Av}[w]$. The notation adopted, which does not emphasize time dependence, is indeed consistent with the above shift-invariance and does not create misunderstandings. It is worth pointing out that the above construction leads to tighter asymptotic averages than those obtained by taking component wise averages of vector signals and, for technical reasons, it appears more natural in our context.

Let $\mathbb{Y} \subseteq \mathbb{R}^p$ be a closed and *convex* set and y an auxiliary output variable defined according to:

$$y = h(x, u) \tag{41}$$

for some continuous map $h : \mathbb{Z} \to \mathbb{R}^p$. The following nestedness condition is assumed:

$$h(x_s, u_s) \in \mathbb{Y} \tag{42}$$

Our goal is to design a receding horizon control strategy that ensures the following set of constraints:

$$\begin{array}{rcl} \text{Av}[\ell(x, u)] & \subseteq & (-\infty, \ell(x_s, u_s)] \\ (x(k), u(k)) & \in & \mathbb{Z} \quad k \in \mathbb{I}_{\geq 0} \\ \text{Av}[y] & \subseteq & \mathbb{Y} \end{array} \tag{43}$$

At each time t we solve the following optimization problem:

$$\min_{\mathbf{v}} \sum_{k=0}^{N-1} \ell(z(k), v(k)) \tag{44}$$

subject to the following constraints

$$\begin{array}{c} z^+ = f(z, v) \\ (z(k), v(k)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{0:N-1} \\ z(N) = x_s, \quad z(0) = x \\ \sum_{k=0}^{N-1} h(z(k), v(k)) \in \mathbb{Y}_t \end{array} \tag{45}$$

The time-varying output constraint set is the new feature of this problem. To enforce the average constraints, we define the constraint sets recursively

$$\mathbb{Y}_{i+1} = \mathbb{Y}_i \oplus \mathbb{Y} \ominus h(x(i), u(i)) \qquad \text{for } i \in \mathbb{I}_{\geq 0} \tag{46}$$

in which the symbols $\oplus$ and $\ominus$ denote standard set addition, and subtraction, respectively. We initialize the recursion using

$$\mathbb{Y}_0 = N\mathbb{Y} + \mathbb{Y}_{00} \tag{47}$$

in which the set $\mathbb{Y}_{00} \subset \mathbb{R}^p$ is an arbitrary compact set containing the origin. By adjusting the output constraint sets with the closed-loop behavior, we force the average constraints to be satisfied asymptotically. The main result for this Section is stated below (see the Appendix for a proof).

Theorem 5 *Let $\mathbf{u}^0(x,t)$ be any minimizer of (44) subject to (45) with initial state x at time t. Consider the closed-loop system obtained by letting $u(x,t) = u^0(0;x,t)$ at each time $t \in \mathbb{I}_{\geq 0}$*

$$x(t+1) = f(x(t), u^0(0;x,t)) \tag{48}$$

Then, provided $x(0)$ is a feasible initial condition, feasibility is ensured for all subsequent times and (43) holds for the closed-loop signals $x(t)$, $u(t)$ and $y(t) = h(x(t), u(t))$.

6 On optimality of steady-state operation

As remarked in previous Sections, nonlinearity of plant dynamics and non-convexity of cost functionals may be responsible for the existence of complex operation regimes outperforming the best feasible equilibria. In order to classify in system-theoretic terms such possibilities, the following notions were introduced in [3].

Definition 6.1 We say that a control system $x^+ = f(x,u)$ is *optimally operated at steady state* with respect to the cost functional $\ell(x,u)$, if for any solution such that $(x(k), u(k)) \in \mathbb{Z}$ for all $k \in \mathbb{I}_{\geq 0}$, it holds:

$$\text{Av}[\ell(x,u)] \subseteq [\ell(x_s, u_s), +\infty)$$

where x_s is the best admissible steady state defined in (6). If, in addition, at least one of the conditions below holds:

1. $\text{Av}[\ell(x,u)] \subseteq (\ell(x_s, u_s), +\infty)$
2. $\liminf_{k\to\infty} |x(k) - x_s| = 0$

we say that the system is *suboptimally operated off steady state.* □

Similarly, if average constraints are considered, we may define:

Definition 6.2 We say that a control system $x^+ = f(x,u)$ is *optimally operated at steady state* with respect to the cost functional $\ell(x,u)$ and average constraints, if for any solution satisfying $(x(k), u(k)) \in \mathbb{Z}$ for all $t \in \mathbb{I}_{\geq 0}$ and $\text{Av}[h(x,u)] \subseteq \mathbb{Y}$, the following holds

$$\text{Av}[\ell(x,u)] \subseteq [\ell(x_s, u_s), +\infty)$$

where x_s is the best admissible steady state defined in (6). If, in addition, at least one of the conditions below holds:

1. $\text{Av}[\ell(x,u)] \subseteq (\ell(x_s,u_s),+\infty)$
2. $\liminf_{k\to\infty} |x(k)-x_s| = 0$

we say that the system is *suboptimally operated off steady state.* □

Note that suboptimal operation off steady state implies only that the steady state is a limit point of solutions of the closed-loop system, not that the closed-loop system converges to the steady state (in general multiple limit points may exist). We show next that dissipativity and strict dissipativity are sufficient conditions for optimal operation at steady state and suboptimal operation off steady state, respectively. In order to treat the case of systems with average constraints we take into account the case in which $\mathbb{Y}$ is a polyhedron defined as follows:

$$\mathbb{Y} = \{y \in \mathbb{R}^p \mid Ay \le b\} \tag{49}$$

for some matrix $A \in \mathbb{R}^{p\times n_c}$ and some $b \in \mathbb{R}^{n_c}$. In this case we may define:

Definition 6.3 A system is dissipative with respect to the supply function $s(x,u)$ on averagely constrained solutions if there exists a function $\lambda : \mathbb{X} \to \mathbb{R}$ and a multiplier $\bar{\lambda} \in [0,+\infty)^{n_c}$ such that:

$$\lambda(f(x,u)) \le \lambda(x) + s(x,u) + \bar{\lambda}'[Ah(x,u)-b] \tag{50}$$

for all $(x,u) \in \mathbb{Z}$. If in addition, for some positive definite $\rho(\cdot)$ we have:

$$\lambda(f(x,u)) \le \lambda(x) - \rho(x) + s(x,u) + \bar{\lambda}'[Ah(x,u)-b] \tag{51}$$

then the system is strictly dissipative on averagely constrained solutions. □

We are now ready to state the main result for this Section.

Proposition 6.4 Assume that the control system (2) is dissipative (strictly dissipative) on averagely constrained solutions with respect to the supply function $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$, then the system is optimally operated at steady state on averagely constrained solutions (suboptimally operated off steady state). □

Proof. Through the following simple manipulations we come to the desired estimate:

$$\begin{aligned} 0 &= \lim_{T\to+\infty} \frac{\lambda(x(T)) - \lambda(x(0))}{T} \\ &= \lim_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \lambda(x(k+1)) - \lambda(x(k))}{T} \\ &\leq \lim_{T\to+\infty} \frac{\sum_{k=0}^{T-1} s(x(k),u(k)) + \bar{\lambda}'[Ah(x(k),u(k)) - b]}{T} \\ &\leq \liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} - \ell(x_s,u_s) \\ &\quad + \limsup_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \bar{\lambda}'[Ah(x(k),u(k)) - b]}{T} \\ &\leq \liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} - \ell(x_s,u_s) \\ &\quad + \max_{y\in\mathbb{Y}} \bar{\lambda}'[Ay - b] \\ &\leq \liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} - \ell(x_s,u_s) \end{aligned} \tag{52}$$

Notice that the first inequality follows by dissipativity, while the third is a consequence of average constraints and non-negativity of $\bar{\lambda}$. This concludes the proof of the first part of the claim.

Similar manipulations for the case of strictly dissipative systems allow to derive the following inequality:

$$0 \leq \liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k)) - \rho(x(k))}{T} - \ell(x_s,u_s). \tag{53}$$

Hence:

$$\begin{aligned} \liminf_{T\to+\infty} \tfrac{\sum_{k=0}^{T-1} \rho(x(k))}{T} &\leq \\ \leq \liminf_{T\to+\infty} \tfrac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} &- \ell(x_s,u_s) \end{aligned} \tag{54}$$

The following two cases are possible:

1. $\liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} > \ell(x_s,u_s)$, that is $\mathrm{Av}(\ell(x,u)) \subseteq (\ell(x_s,u_s),+\infty)$;
2. alternatively $\liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} = \ell(x_s,u_s)$; by virtue of (54) then:

   $$\liminf_{T\to+\infty} \frac{\sum_{k=0}^{T-1} \rho(x(k))}{T} = 0$$

   and this in turn implies, by positive definiteness of $\rho(\cdot)$,

   $$\liminf_{k\to+\infty} |x(k) - x_s| = 0 \tag{55}$$

   This concludes the proof of suboptimal operation off steady state.

□

Remark 6.5 It is worth pointing out that Proposition 6.4 can be used to perform some convergence analysis of economic MPC subject to average constraints for strictly dissipative systems as in (51). A Lyapunov analysis of such an MPC scheme is not currently available, nevertheless, combining Proposition 6.4 and Theorem 5 allows to conclude convergence of $x(k)$ to x_s at least in a weak sense, that is as derived in equation (55). □

7 An example: consecutive-competitive reactions

We consider next the control of a nonlinear isothermal chemical reactor with consecutive-competitive reactions [14]. Such networks arise in many chemical and biological applications such as polymerizations, and are characterized by a set of reactions of the following form:

$$\begin{array}{rcl} P_{i-1} + B & \to & P_i \\ i & \in & \{1, 2, \dots, R\}. \end{array} \tag{56}$$

Typically a desirable distribution of products in the effluent is a primary objective in the reactor design for these processes. For simplicity we consider the case of two reactions:

$$\begin{aligned} P_0 + B &\longrightarrow P_1 \\ P_1 + B &\longrightarrow P_2 \end{aligned}$$

The dimensionless mass balances for this problem are:

$$\begin{aligned} \dot{x}_1 &= u_1 - x_1 - \sigma_1 x_1 x_2 \\ \dot{x}_2 &= u_2 - x_2 - \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_3 &= -x_3 + \sigma_1 x_1 x_2 - \sigma_2 x_2 x_3 \\ \dot{x}_4 &= -x_4 + \sigma_2 x_2 x_3 \end{aligned}$$

where x_1, x_2, x_3 and x_4 are the concentrations of P_0, B, P_1, and P_2 respectively, while u_1 and u_2 are inflow rates of P_0 and B and are the manipulated variables. The parameters σ_1 and σ_2 have values 1 and 0.4 respectively. The time average value of u_1 is constrained to lie between 0 and 1.

$$Av[u_1] \subseteq [0, 1].$$

The primary objective for this system is to maximize the average amount of P_1 in the effluent flow ($\ell(x, u) = -x_3$). Previous analysis has clearly highlighted that periodic operation can outperform steady-state operation [14]. The steady-state problem has a solution $x_s = \begin{bmatrix} 0.3874 & 1.5811 & 0.3752 & 0.2373 \end{bmatrix}'$ with the optimal input $u_s = \begin{bmatrix} 1 & 2.4310 \end{bmatrix}'$.

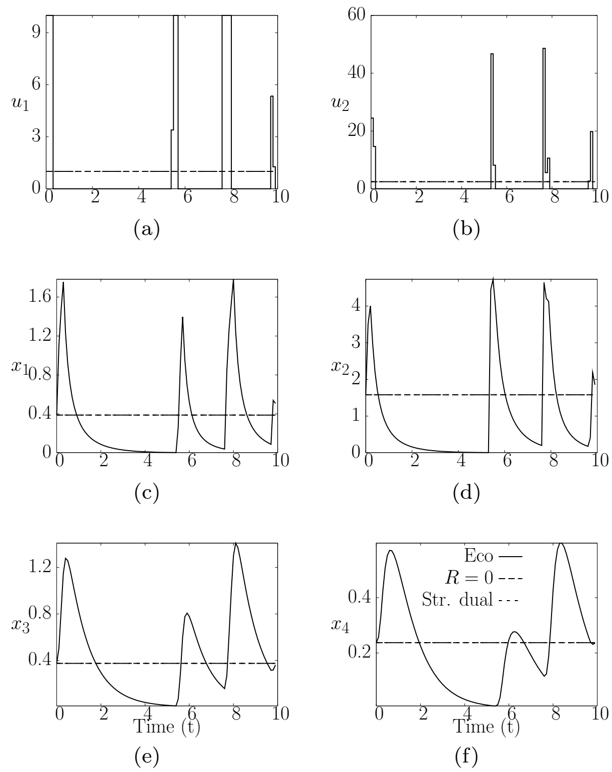

Figure 2: Open-loop input (a), (b) and state (c), (d), (e), (f) profiles with different cost functions and initial steady states.

We solve the dynamic regulation problem using the simultaneous approach [10]. The state space is divided into a fixed number of finite elements. The input is parameterized according to zero order hold with the input value constant across a finite element. An additional upper bound of 10 is imposed on $u_1(t)$. A terminal state constraint is used in all the simulations.

The system is first initialized at the steady state to check suboptimality of steady-state operation. A horizon of 100 is chosen with a sample time $T_s = 0.1$. The steady-state solution is used as the initial guess for the nonlinear solver. The solution of the dynamic problem is seen to be unstable (Figure 2). The solution returned by the optimizer shows the inputs jumping between the upper and lower bounds. Different initial guesses gave different locations of these jumps suggesting that these solutions are local optima, with a negligible cost difference.

In order to stabilize the system a convex term is added to the objective and

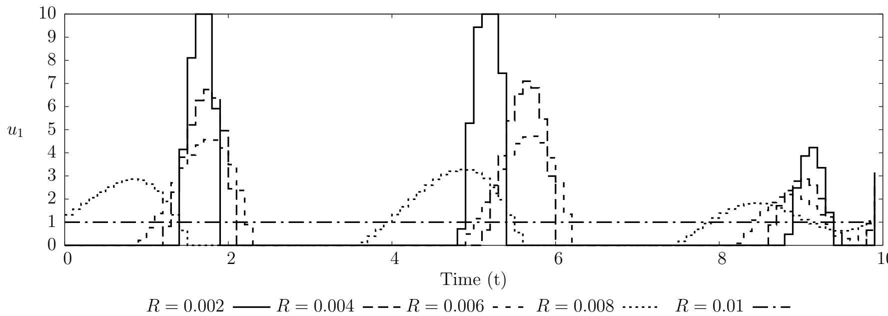

Figure 3: Stabilization via variation in u penalty.

R	avg profit
0	0.4472
0.002	0.4397
0.004	0.4233
0.006	0.4076
0.008	0.3938
0.01	0.3752

Table 1: Average economic profit for open-loop profiles with varying R penalty.

the penalties are varied.

$$\ell(x,u) = -x_3 + (1/2)\big(\,|x - x_s|_Q^2 + |u - u_s|_R^2\,\big)$$

Next the R penalty is tuned in fine amounts (keeping $Q = 0$) to see the effect of adding the convex term. As R is increased from 0.004 to 0.04, the optimal solution from the previous case is used as the initial guess for the next case. Economic profit (time average of $P_1(x_3)$) is computed for all these cases and is compared in Table 1. It is seen that increasing the convex term by increasing the R penalty, starts dampening the system (Figure 3). Consequently a loss in profit is observed proving that steady-state operation is suboptimal in the sense of economic profit.

Just increasing the R penalty to 0.02 does not make the steady-state problem strongly dual [8]. So we increase also the Q penalty until the steady-state problem is strongly dual, which is achieved for $Q = 0.36I_4$ and $R = 0.002I_2$. Strong duality is checked by numerically solving the dual problem [8] and checking its solution. Figure 2 shows that this case yields the steady-state solution.

Next, the system is initialized at a random state. We compare the performance of the three cases along with a purely tracking MPC controller with same Q and R penalties as the strongly dual case. The tracking controller cost is:

$$\ell(x,u) = (1/2)\big(\,|x - x_s|_Q^2 + |u - u_s|_R^2\,\big)$$

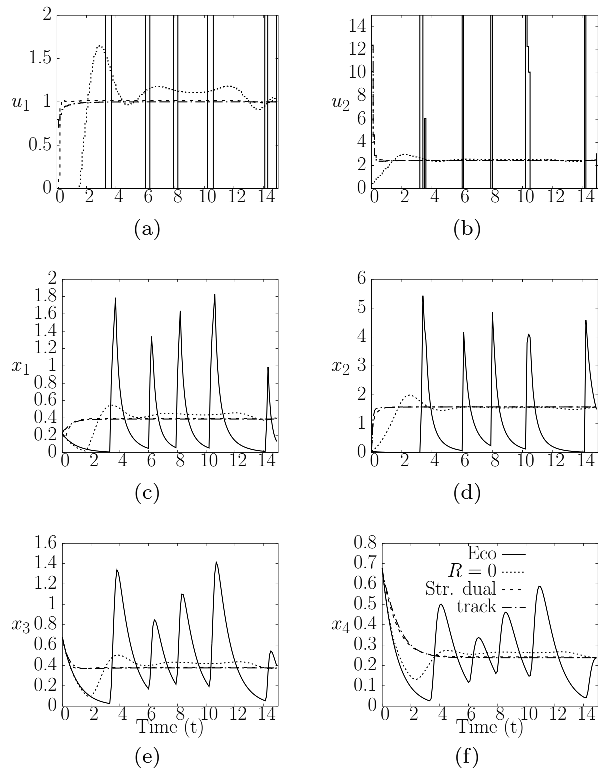

Figure 4: Open-loop input (a), (b) and state (c), (d), (e), (f) profiles with different cost functions and a random initial state.

Figure 4 shows the open-loop input and state profiles for these four cases. The economic controller gives an unstable solution. In order to compare the behavior of closed-loop systems over infinite time-horizons two different kinds of performance measures are of interest. One is average profit, that is $Av[x_3]$, the other is the transient profit, namely:

$$\sum_{k=0}^{\infty} \ell(x(k), u(k)) - \ell(x_s, u_s)$$

Notice that transient profit need not be defined for pure economic MPC. On the other hand, all MPC controllers that guarantee asymptotic tracking yield the same average profit (corresponding to x_{3s}), but possibly different transient profits. The average and transient profits are compared in Table 2. We see that the tracking controller yields the least amount of average P_1.

Case	avg profit	trans profit
Economic	0.4648	∞
$R = 0$	0.3916	2.6201
Strongly dual	0.3848	1.6034
Tracking	0.3812	1.0587

Table 2: Average profit and transient profit for open-loop profiles with varying R penalty.

8 Conclusions

The paper adresses questions of stability and performance of economic MPC control schemes by means of suitable Lyapunov analysis techniques. The contribution of the paper is to summarize several results previously scattered in conference Proceedings and to refine them in a self-contained unified treatment. In particular

1. definition of Economic MPC and variants for periodic terminal constraint and asymptotic average constraints (partial results previously published in [3])

2. performance analysis for standard, periodic and constrained in average, Economic MPC (partial results previously published in [3])

3. definition of the dissipativity notion to extend the concept of strong duality (partial results previously published in [2])

4. Lyapunov-based stability analysis of Economic MPC subject to dissipativity of the underlying dynamics (new results, generalizing the proof in [8])

5. Notions of optimality for steady-state operation and sufficient Lyapunov-based conditions for their test (partial results appeared in [2])

6. Example of application of Economic MPC to a chemical reactor.

The results provide a rigorous self-contained basis for anyone wishing to work on the area of Economic Model Predictive Control.

9 Acknowledgment

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Proof of Theorem 5

The proof can be divided into 3 steps.

1. *Feasibility*: The proof is by induction, showing that feasibility at time t implies feasibility at time $t+1$. At time t, state $x(t)$, let $\{z(0), z(1), \dots z(N)\}$ and $\{v(0), v(1), \dots v(N-1)\}$ denote the optimal solution to (44) subject to (45). We claim that, at time $t+1$, state $x(t+1) = f(x(t), v(0))$, the optimization problem is again feasible. Construct the usual candidate sequences at time $t+1$, state $x(t+1)$: $\tilde{\mathbf{z}} = \{z(1), z(2), \dots, z(N-1), z(N), x_s\}$, $\tilde{\mathbf{v}} = \{v(1), z(2), \dots, v(N-1), u_s\}$. By construction, these satisfy the model, initial condition, $\tilde{z}(0) = x(t+1)$, terminal condition $\tilde{z}(N) = x_s$, and pointwise-in-time constraints $(\tilde{z}(k), \tilde{v}(k)) \in \mathbb{Z}$ for $k \in \mathbb{I}_{0:N-1}$. We also have that

$$\begin{aligned}\sum_{k=0}^{N-1} h(\tilde{z}(k), \tilde{v}(k)) &\in \mathbb{Y}_t \ominus h(x(t), u(t)) \oplus h(x_s, u_s) \\ &\subseteq \mathbb{Y}_t \oplus \mathbb{Y} \ominus h(x(t), u(t)) \\ &= \mathbb{Y}_{t+1}\end{aligned}$$

where the inclusion follows because $h(x_s, u_s) \in \mathbb{Y}$. Therefore, the time-varying output constraint is also satisfied and the candidate sequences satisfy (45) at time $t+1$, state $x(t+1)$.

2. *Average performance*: The proof of the inclusion $\mathrm{Av}[\ell(x,u)] \subseteq (-\infty, \ell(x_s,u_s)]$ can be performed exactly along the lines of Section 3, given the feasibility of the candidate control sequence $\tilde{\mathbf{v}}$ defined above.

3. *Average constraints*: We show next that $\mathrm{Av}[y] \subseteq \mathbb{Y}$. Solving the recursion (46) with initial condition (47) gives

$$\mathbb{Y}_t = \mathbb{Y}_{00} \oplus (t+N)\mathbb{Y} \ominus \sum_{k=0}^{t-1} y(k)$$

in which $y(k) = h(x(k), u(k))$. The average constraint in (45) then gives for all $t \in \mathbb{I}_{\geq 0}$.

$$\sum_{k=0}^{N-1} h(z(k), v(k)) + \sum_{k=0}^{t-1} y(k) \in \mathbb{Y}_{00} \oplus (t+N)\mathbb{Y}$$

for solution $\mathbf{z}, \mathbf{v}$ to (44) at time t. Notice that the first sum on the left-hand side of the previous equation involves only N terms, irrespective of t, and each one of them can be bounded by a quantity independent of t thanks to compactness of $\mathbb{Z}$ and continuity of $h(\cdot)$. Hence, by letting t grow to infinity along any subsequence t_n such that $\sum_{k=0}^{t_n} y(k)/(t_n+1)$ admits a limit we obtain

$$\begin{aligned} \lim_{n\to+\infty} \frac{\sum_{k=0}^{t_n} y(k)}{t_n+1} &\in \lim_{n\to+\infty} \frac{\mathbb{Y}_{00} \oplus (t_n+1+N)\mathbb{Y}}{t_n+1} \\ &= \mathbb{Y} \end{aligned}$$

where the set-limit holds in the Hausdorff topology sense. This shows indeed that $\mathrm{Av}[y] \subseteq \mathbb{Y}$ and concludes the proof of Theorem 5.

Proof of Theorem 4

Let $V_N^0(x,q)$ denote the optimal cost relative to the q-th terminal constraint:

$$\begin{aligned} V_N^0(x,q) \quad &= \quad \min_{\mathbf{v}} V_N(x,\mathbf{v},q) \\ &\qquad \text{subject to} \begin{cases} z^+ = f(z,v) \\ (z(k),v(k)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{0:N-1} \\ z(N) = x^*(q) \\ z(0) = x \end{cases} \end{aligned} \tag{57}$$

We may define a periodic Lyapunov-like function by letting $V(t,x) := V_N^0(x, t \bmod Q)$. Let $x(t)$ be a feasible initial condition for problem (57) at time t subject to the terminal constraint $z(N) = x^*(t \bmod Q)$ and let $\{z(0),\ldots,z(N)\}$, $\{v(0),\ldots,v(N-1)\}$ be the optimal state and control sequences. Then $\{z(1),\ldots,z(N),x^*((t+$

1) mod $Q)\}$, $\{v(1), \ldots, v(N-1), u^*(t \bmod Q)\}$ are a feasible $(\mathbf{z}, \mathbf{v})$ pair at time $t+1$ from initial state

$$x(t+1) = f(x(t), \kappa_N(x(t), t \bmod Q))$$

Hence, by induction, and provided $x(0)$ is a feasible initial condition for (57) subject to the terminal constraint $z(N) = x^*(0)$, closed-loop solutions are well defined for all subsequent times and fulfill the constraints.

Furthermore, evaluating increments of $V(t, x)$ along solutions of the closed-loop system by taking into account suboptimality of $\{z(1), \ldots, z(N), x^*((t+1) \bmod Q)\}$, $\{v(1), \ldots, v(N-1), u^*(t \bmod Q)\}$ at time $t+1$ yields

$$\begin{aligned} & V(t+1, f(x(t), u(t))) - V(t, x(t)) \\ \leq \quad & \ell\Big(x^*(t \bmod Q), u^*(t \bmod Q)\Big) - \ell\Big(x(t), u(t)\Big) \end{aligned} \tag{58}$$

for all $t \in \mathbb{I}_{\geq 0}$. Taking asymptotic averages of (58) and denoting by $\tau(k) := k \bmod Q$ yields:

$$\begin{aligned} 0 &= \liminf_{T \to +\infty} \frac{V(T+1, x(T+1)) - V(0, x(0))}{T+1} \\ &= \liminf_{T \to +\infty} \frac{\sum_{k=0}^{T} V(k+1, f(x(k), u(k))) - V(k, x(k))}{T+1} \\ &\leq \liminf_{T \to +\infty} \frac{\sum_{k=0}^{T} \ell(x^*(\tau(k)), u^*(\tau(k))) - \ell(x(k), u(k))}{T+1} \\ &= \frac{\sum_{q=0}^{Q-1} \ell(x^*(q), u^*(q))}{Q} - \limsup_{T \to +\infty} \frac{\sum_{k=0}^{T} \ell(x(k), u(k))}{T+1} \end{aligned}$$

where the last equality follows by taking the elementary steps detailed below:

$$\begin{aligned} & \lim_{T \to +\infty} \frac{\sum_{k=0}^{T} \ell(x^*(\tau(k)), u^*(\tau(k)))}{T+1} \\ &= \lim_{T \to +\infty} \frac{\Big(\sum_{k=1}^{\lfloor T/Q \rfloor} \sum_{\tau=0}^{Q-1} \ell(x^*(\tau), u^*(\tau))\Big)}{T+1} \\ & \qquad + \frac{\sum_{\theta=0}^{T \bmod Q} \ell(x^*(\theta), u^*(\theta))}{T+1} \\ &= \lim_{T \to +\infty} \frac{\lfloor T/Q \rfloor \ \sum_{\tau=0}^{Q-1} \ell(x^*(\tau), u^*(\tau))}{T+1} \\ & \qquad + \frac{\sum_{\theta=0}^{T \bmod Q} \ell(x^*(\theta), u^*(\theta))}{T+1} \\ &= \lim_{T \to +\infty} \frac{(T - T \bmod Q) \ \sum_{\tau=0}^{Q-1} \ell(x^*(\tau), u^*(\tau))}{Q(T+1)} \\ &= \frac{\sum_{\tau=0}^{Q-1} \ell(x^*(\tau), u^*(\tau))}{Q} \end{aligned}$$

This concludes the proof of Theorem 4.