Dynamic Behaviour of Blast Loaded Hybrid Structural Systems

By

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Declarations

Declaration of Originality

I carried out the work presented in this thesis in the department of Civil and Environmental Engineering at Imperial College London. This thesis is a result of my own original work and quotations and information from other works, published or unpublished, have been acknowledged in the thesis and referenced. The thesis is not the same as any work that has been submitted in any institution for the award of any degree.

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Ebuka Nwankwo
Abstract

A hybrid system consists of two or more different constituent materials combined to form a single system to achieve increased mechanical properties and structural performances. The combinations of constituent materials are on a macroscopic level. The improved performances achieved in hybrid systems are in fatigue, impact, corrosion resistance, weight savings, and improved strength to weight performances.

The increasing demand for high-performance and lightweight structures forms the motivation for this thesis. In the light of these, three different hybrid systems under different blast scenarios have been studied and the reason for their high-performance over monolithic systems discussed. The possibility of debonding of the strengthening composite patch from the stainless steel panel in a hybrid system of strengthened blast wall leads to the study of fibre metal laminates (FMLs) and lap joints. Since composites form a significant part of these hybrid systems, simplified damage models for composites are developed and applied to the various hybrid systems studied in order to investigate their overall response.

First, this thesis presents a hybrid system of a stainless steel blast wall with retrofitting composite patches. An analytical model, which allows for multiple deformation modes, is developed to study the hybrid system of strengthened blast wall. Maximum displacements predicted by the analytical models correlated well with maximum displacements predicted by the numerical models of the proposed hybrid system in Abaqus. It is observed that fibre fracture, which is a more detrimental failure mode, did not occur in the composite patch in the numerical model. The hybrid system of composite strengthened blast walls allows for increased energy absorption by the development of four plastic hinges compared to the development of three plastic hinges of the monolithic system. This behaviour renders it superior to a monolithic system in a gas explosion scenario.

In order to simplify the system presented in Chapter 3, an analytical solution for evaluating the maximum displacement of a continuous system with semi-rigid supports subjected to pulse loads is presented. The maximum elastic displacement presented by the numerical models in Chapter 3 is compared with the maximum displacement presented by the simplified model. The limitation of the simplified model is subsequently discussed. Using the simplified model, an elastic pressure-impulse diagram for the blast wall studied in Chapter 3 is presented under typical hydrocarbon explosions. In addition, unique pulse-shape independent pressure-impulse diagrams

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1 The three blast scenarios considered are vapour cloud (hydrocarbon) explosions with overpressures with finite rise time; high explosive scenarios with very short or negligible rise times and localised blast overpressure from small charges (bombs)
for elastic and elastic-plastic responses are developed using dimensionless parameters for typical high explosive events. However, the major limitations of this model are its inability to account for membrane effect, travelling plastic hinge, support shear hinge and connection pull-in.

Secondly, the response of an FML is studied in order to obtain an insight into debonding between composite and metal, which was assumed to be prevented in Chapter 3. An FML was chosen because of the availability of experimental data on the blast response for this kind of hybrid system in the open literature. In addition, other researchers have proven that FMLs performed better the monolithic aluminium with similar areal density. A modified Hashin model is used to model damage in the composite layers of fibre metal laminates (FMLs) under blast loads. The FML studied comprises 2024-O aluminium alloys (O represents the temper of the aluminium alloy-i.e.no heat treatment) and woven glass-fibre/polypropylene composites. Thus, this work presents an improved and simplified model to analyse the damage initiation, damage progression, and failure of the aluminium layers and the three-dimensional woven composite layers.

In order to gain an insight on how bonded substrates influence the stress in adhesive layers and because interfacial stresses cannot be obtained directly from cohesive elements in Abaqus (i.e. adhesive layers in the studied FML), an analytical model to predict the maximum peel and shear stresses in an elastic adhesive in a single lap joint (metal-metal adherends) subjected to transverse pulse loads is presented. The analytical model for a metal-adhesive-metal system, which was validated with numerical models in Abaqus, gave an insight into the relationship of interfacial stresses in adhesive layers with bonded layers. Inference drawn from this model supports the assumption that bonded materials with similar in-plane stiffness would result in minimal interfacial stresses under blast scenarios as originally assumed in Chapter 3. Finally, a lap joint with similar adherends under in-plane blast load is compared with a hybrid system of metal and composite lap joint. The interfacial stresses produced by the hybrid system showed some reduction and fibre failure was not observed in the composite. This reinforces the improved performance of hybrid systems.
Acknowledgements

I wish to express my heartfelt appreciation to all those who were instrumental to the successful completion of this research. I must say that without them it would have been impossible to carry out this research in the first place.

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Finally, I would like to thank my wife, Mrs Linda Nwankwo; my parents; my cousin, Mrs Nonye Soludo; my siblings and my mother in-law, Mrs Stella Okafor for the spiritual support they give me during the course of my PhD study here in London.
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## Nomenclature

### Latin

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Cross section of steel panel</td>
</tr>
<tr>
<td>$A_c$</td>
<td>Cross sectional area of adhesive</td>
</tr>
<tr>
<td>$C_t(t)$</td>
<td>Generalised displacement</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s Modulus</td>
</tr>
<tr>
<td>$E_c$</td>
<td>Elastic modulus of the adhesive</td>
</tr>
<tr>
<td>$E_h$</td>
<td>Hardening Modulus</td>
</tr>
<tr>
<td>$F$</td>
<td>Langrangian function</td>
</tr>
<tr>
<td>$F_X$</td>
<td>Axial force</td>
</tr>
<tr>
<td>$F_m$</td>
<td>Maximum amplitude of pulse</td>
</tr>
<tr>
<td>$G_c$</td>
<td>Shear modulus of the adhesive</td>
</tr>
<tr>
<td>$G_{m,g_{m}}$</td>
<td>Specific fracture energy corresponding to each damage direction</td>
</tr>
<tr>
<td>$h_c$</td>
<td>Thickness of adhesive</td>
</tr>
<tr>
<td>$\dot{H} = \frac{\partial H}{\partial t}, H' = \frac{\partial H}{\partial x}$</td>
<td>Temporal and spatial derivatives of a generic function $H(x,t)$</td>
</tr>
<tr>
<td>$i$</td>
<td>Mode shape index</td>
</tr>
<tr>
<td>$I$</td>
<td>Second moment of area (moment of inertia)</td>
</tr>
<tr>
<td>$K_{xe}$</td>
<td>Elastic stiffness of horizontal translational spring</td>
</tr>
<tr>
<td>$K_{xp}$</td>
<td>Plastic stiffness of horizontal translational spring</td>
</tr>
<tr>
<td>$K_y$</td>
<td>Stiffness of vertical translational spring</td>
</tr>
<tr>
<td>$K_{oe}$</td>
<td>Elastic stiffness of support rotational spring</td>
</tr>
<tr>
<td>$K_{op}$</td>
<td>Plastic stiffness of support rotational spring</td>
</tr>
</tbody>
</table>
\( K_\theta \) Plastic stiffness of rotational spring in beam

\( L_h \) Plastic hinge length used in the analytical model

\( L_1 \) and \( L_3 \) Lengths of unbonded regions of lap joint

\( L_2 \) Length of bounded region of lap joint

\( L_b \) Length of the unstrengthened area

\( M \) Bending moment

\( M_\theta \) Full plastic moment capacity of the beam

\( M_\theta^a \) Full plastic moment capacity of support springs

\( M_{ul} \) Moment at point of formation of plastic hinge in the beam

\( M_{sup} \) Moment at support

\( M_{11}(x,t), S_{11}(x,t), N_{11}(x,t) \) Internal moment, shear and normal forces in unbonded region \( L_1 \)

\( M_{21}(x,t), S_{21}(x,t), N_{21}(x,t) \) Internal moment, shear and normal forces in unbonded region \( L_2 \)

\( M_{22}(x,t), S_{22}(x,t), N_{22}(x,t) \) Internal moment, shear and normal forces in unbonded region \( L_3 \)

PE Potential energy loss due to loading

\( P(x, t) \) Blast pressure profile

\( q_i(x) \) The \( i \)-th shape function

\( R_{mx} \) Maximum resistance of horizontal translational spring

\( t \) Time

\( t_{eq} \) Thickness of the equivalent rectangular beam section

\( t_1 \) Time at the end of stage I

\( t_2 \) Time at end of stage IIa or IIb

\( t_3 \) Time at end of stage III
\( t_d \)  \quad \text{Duration of pulse}

\( T \)  \quad \text{Kinetic energy}

\( U \)  \quad \text{Strain energy}

\( U_b \)  \quad \text{Flexural strain energy}

\( U_{se} \)  \quad \text{Elastic strain energy in rotational spring at the support}

\( U_{op} \)  \quad \text{Plastic strain energy in rotational spring at the support}

\( U_{\theta} \)  \quad \text{Plastic strain energy in the internal rotational spring}

\( U_{xe} \)  \quad \text{Elastic strain energy in horizontal spring at the support}

\( U_{xp} \)  \quad \text{Plastic strain energy in horizontal spring at the support}

\( U_{ye} \)  \quad \text{Elastic strain energy in vertical spring at the support}

\( U_{me} \)  \quad \text{Elastic membrane strain}

\( V \)  \quad \text{Total potential energy}

\( w(x, t) \)  \quad \text{Transverse displacement function}

\( \dot{w}(x, t) \)  \quad \text{Transverse velocity field}

\( w_I(x, t) \)  \quad \text{Transverse displacement function in stage I}

\( w_{IIA}(x, t) \)  \quad \text{Transverse displacement function in stage IIa}

\( w_{IIB}(x, t) \)  \quad \text{Transverse displacement function in stage IIb}

\( w_{III}(x, t) \)  \quad \text{Transverse displacement function in stage III}

\( w_{11}(x, t), u_{11}(x, t) \)  \quad \text{Transverse and longitudinal displacements in } L_1

\( w_I(x, t), u_I(x, t) \)  \quad \text{Transverse and longitudinal displacements in } L_2

\( w_{22}(x, t), u_{22}(x, t) \)  \quad \text{Transverse and longitudinal displacements in } L_3
Greek

\( \alpha \)  
Dimensionless CFRP patch length

\( \gamma_c \)  
Shear strain in adhesive

\( \delta W \)  
Virtual work done by non-conservative forces and external forces

\( \varepsilon \)  
Strain

\( \varepsilon_c \)  
Longitudinal strain in adhesive

\( \Upsilon \)  
Non-dimensional parameter

\( \eta \)  
Non-dimensional parameter

\( \kappa \)  
Curvature

\( \lambda \)  
Natural frequency of system

\( \nu \)  
Poison’s ration

\( \rho \)  
Density of adherend

\( \rho_c \)  
Density of adhesive

\( \tau \)  
Non-dimensional values of time

\( \sigma \)  
Stress

\( \varphi(x) \)  
Shape function

\( \Omega \)  
Relevant domain of analysis

\( \eta \) and \( \Upsilon \)  
Nondimensional values for load evaluation
Chapter 1

Introduction

1.1. Background

During the past decade, increasing demand for lightweight high performance structures has resulted in a number of research works on how to optimise the weight of engineering structures without compromising on its mechanical performance. One viable option that has been adopted is the use of composites (fibre-reinforced polymers) as constituent parts of structures. The desire for overall lightweight construction has seen the use of fibre reinforced composites in retrofitting steel and concrete structures. This milestone achieved in the science of hybridised lightweight structures has been because of improved technology in the manufacture of high strength composites [1].

Lightweight structures are very desirable in blast and ballistic protection scenarios. For instance, the convenience of wearing lightweight armour by military personnel has led to numerous research on laminated (hybrid) and sole composite armour steel [2].

In the light of this, it is deemed necessary to study the behaviour of hybrid systems in order to have an insight into their behaviour. There are different configurations that can be achieved by hybridization. Strictly speaking, each specific hybrid system requires individual investigation of its behaviour. This thesis investigates the behaviour of three different hybrid configurations (i.e. strengthened blast walls, fibre metal laminates (FMLs) and lap joints) with a view of drawing a common behaviour between them.

This chapter looks at the fundamental issues relating to the family of hybrid systems studied in this research. Even though hybrid systems in the context of this research refer to a combination of constituent materials (composite plastics, adhesives, and metals) on a macroscopic level, this chapter also discusses the broader meaning of hybrid systems and its evolution. First, two fundamental issues are addressed, i.e., (1), what does a hybrid system mean in general and in the context of this research? (2), why is there an increasing demand for hybrid systems in engineering applications? Some practical applications of the particular classes of hybrid systems studied in this research are discussed and some of the reasons that gave rise to their birth highlighted. Finally, this chapter discusses the objectives, motivation, and scope of this research.
1.2. What is a Hybrid System?

A hybrid system consists of two or more materials combined on a macroscopic or microscopic level to form a unit with improved mechanical behaviour. The generic definition of ‘composite’ materials, however, falls within the definition given. The traditional ‘composite’ material refer to cement, concrete, plywood, fibre-reinforced polymer, metal composites, ceramic composites, and other possible homogenised materials. In this research, composites refer to laminated fibre-reinforced polymers such as Glass Fibre Reinforced Plastics (GFRP), Carbon Fibre Reinforced Plastics (CFRP) etc.

Scientists noticed the improved mechanical properties achieved by combining materials on macroscopic or microscopic levels many centuries ago. For example, the Egyptians documented the ancient brick making process (straw and mud combined to form brick). Other ‘composites’ such as plywood, cartonnage layers of linen or papyrus soaked in plaster, concrete, papier-mâché have long histories. Bakelite, the first recorded Fibre Reinforced Plastic (FRP) dates back to 1907 [3]. Over the years, the use of FRP has broadened in civil engineering. FRP is used for (1) rehabilitation (which includes strengthening and retrofit of structures) (2) new construction with all FRP solutions [4]. For example, unconventional materials are used in the repair and retrofitting of seismically loaded bridge columns. These materials are fibreglass, carbon, and hybrid composite jackets. These retrofitting materials are as effective as conventional steel jackets. The success achieved with advanced composites in strengthening has led to the development of new lightweight structural concepts utilizing FRP shells and tubes to form new structural systems [4]. Solent Composite Systems\textsuperscript{2} has utilised composite in the blades of wind turbines, various industrial application and in the oil and gas industry (in blast walls and firewalls).

As mentioned in the introduction, hybrid systems in the context of this research refer to a combination of materials on a macroscopic level.

1.2.1. Why Use a Hybrid System?

The improved mechanical performance of hybrid systems and composites has made it very attractive in engineering. The set of hybrid systems studied in this research are Carbon Fibre Reinforced Composite (CFRP) strengthened blast walls, Fibre Metal Laminates (FML) and Lap

\textsuperscript{2} Solent Composite Systems designs and manufactures bespoke composite solutions for high performance asset protection in the offshore Oil and Gas industry.
Joints (metal-adhesive-metal). The improved performance seen in engineering composites are also shared by these set of hybrids studied in this research. These characteristics are:

(a) High strength-to-weight ratio

Composite materials are traditionally known to have a high strength-to-weight ratio, thus, its inclusion in engineering systems results in lesser overall weight of the system without jeopardising strength criteria.

(b) Corrosion resistance

Inclusion of composite materials in engineering systems reduces the vulnerability of the system to corrosion. Composites have excellent corrosion resistance.

(c) Low life cycle cost

Due to the reduced resultant weight of systems that have components made of composites, the fuel consumption cost of such systems is at a minimum. Thus, despite the relatively high construction cost (CAPEX- capital expenditure), the operating cost (OPEX) is relatively low.

(d) Damage Tolerance

Systems with components made of composite can have improved acoustic damping properties, and improved impact and resistance properties [5].

(e) Benefits associated with adhesive bonding

The hybrid systems studied in this thesis utilize adhesive bond in joining its component parts. The benefits accruing from this joining method, which is beneficial by the whole system, are:

- Allowance for great flexibility in design
- Reduction in product weight (bolts and rivets are heavy components)
- The damping ability of adhesives reduces noise and vibration
- Sealing function and protection against corrosion
- Elimination of corrosion associated with joining dissimilar metals with different galvanic potential (i.e. steel and aluminium)
- Adhesive bond does not produce any deformation in the components of the systems thus improving aesthetics and eliminating susceptibility to fatigue crack initiation
- Ease of manufacturing systems with components having different geometries, sizes and composition [5].
1.3. Motivation for Research and Common Feature of Thesis

The duration of construction of a structure can be greatly reduced by reducing the weight of its component parts. In the case of a hybrid system, comprising two different materials (i.e. metal and composites) combined to form a single unit, achieving a weight reduction when compared with a monolithic metal is a very desirable characteristic. Composites (because of their high strength-to-weight ratio) are materials that offer significant weight reduction.

In the light of this, there is need to investigate the overall behaviour of hybrid systems which are materials with significant weight reduction. A hybrid system of composite strengthened blast wall is first investigated. Subsequently, in order to gain an insight into debonding failure modes, FMLs and lap joints were investigated. In the course of this, various conclusions are drawn from the investigation of the individual systems and a general conclusion as well.

As would be discussed in Chapter 2, blast loads have different configurations (i.e. spatially and temporally). Vapour cloud explosions, high detonation explosions, and close-in blasts due to proximal bombs have different spatial and temporal configurations. Thus, it is important to investigate the response of individual hybrids under the kind of possible loads they might experience. Thus, it makes sense to investigate the performance of blast walls under vapour cloud explosions, FMLs under localised blast – typical of a bomb explosion in the luggage container of an aircraft, and hybrid joints under lateral and in-plane loads representative of distal and proximal charges.

1.4 Studied Hybrid Systems

As discussed, in section 1.3, the secondary motivation for studying the three systems in this research is to improve the understanding of response of hybrid systems to blast. Sections 1.4.1 to 1.4.3 give an overview of the applications of the specific systems studied. However, it must be emphasized that the analytical models developed in this thesis can be applied to a wide range of structures outside the purview of the systems discussed in Sections 1.4.1 to 1.4.3. For example, the analytical beam model used to describe a blast wall can be applied to any structure that can be idealised as a beam with semi-rigid (i.e. flexible) connections.
1.4.1. Strengthened Blast Wall

On the 6th July 1988 events in the Piper Alpha accident (which claimed 167 lives), known to be one of the world’s most devastating explosion, shows that blast loads could be of orders higher than designed load. The first blast in the offshore platform devasted the control room, buckled bulkheads, blew doors off their hinges and men from their bunks, and started a major fire in the oil-separation module fed by two other platforms, Tartan and Claymore, which Piper Alpha was connected.

The accommodation blocks sank under the waves taking with them many workers who had gathered in the galley after the first alarms. The immediate cause of the disaster from the public inquiry lead by Lord Cullen reported that an unsafe permit-to-work (PTW) system and other safety failures were responsible for the disaster [6].

In the light of this there is need to investigate means of strengthening existing platforms. The introduction of composite patches, thus, forming a hybrid solution proves to be a very viable option. Figure 1.1 shows a snap short of the Piper Alpha accident. Figure 1.2 shows installed blast walls on an offshore module.

Figure 1. 1: Piper Alpha accident [7]

Figure 1. 2: Living quarter module of an offshore platform showing installed blast walls (Picture courtesy of Van Dam B.V., Nederland)
1.4.2. Fibre Metal Laminates (FMLs)

An FML comprises several thin layers of metal (usually aluminium) interspaced with layers of glass-fibre ‘pre-preg’ bonded together with a matrix such as epoxy [8]. FMLs are widely used in the aviation industry because of its (a) better damage tolerance behaviour i.e. especially impact and metal fatigue, (b) better corrosion resistance, (c) better fire resistance, (d) lower specific weight [8].

Some major events in history have resulted in the blast investigation of FMLs used in the aviation industry. First, on December 21, 1988 a Pan American Airways (Pan AM) from London to New York exploded and crashed over Lockerbie, Scotland. An improvised explosive planted in the luggage compartment of the aircraft caused this disaster. On June 23, 1985, Air India aircraft crashed into the sea as result of an explosion in the forward cargo killing 329 persons on board. On November 29, 1987 Korean Air Flight 858 exploded in flight. An explosive device located in an overhead bin in the aircraft caused this accident. One hundred and fifteen persons lost their lives in this incidence. On September 19, 1989 a Union de Transports Aeriens (UTA) flight was destroyed over the Sahara Desert from an explosion on the forward cargo [9].

These major accidents have increased the interest in research from Governments, universities and various research institutes on ways of mitigating these disastrous incidents. For example Fleischer tested a lightweight container based on GLARE™ (a glass fibre reinforced epoxy/aluminium FML) and reported that it was capable of withstanding a bomb blast greater than that in the Lockerbie air disaster [10]. Figure 1.3a and 1.3b show the newly developed blast resistant air cargo container and the wreckage of Pan AM 103 respectively.

The further understanding of the blast response of FMLs forms a major motivation for this part of the research. Figure 1.4 illustrates the configuration of an FML.
Figure 1. 3: (a) Modern blast resistant air cargo container (picture courtesy of DIAB Group) (b) Wreckage of Pan AM Flight 103 (courtesy of AFP)

Figure 1. 4: Section through fibre metal laminate (Glare) [11]

1.4.3. Lap Joints

As mentioned in the case of strengthening schemes, debonding plays a very important role in the response of hybrid systems. The strength and fracture mechanics approach present procedures used in evaluating the initiation of debonding. Thus, in order to study debonding, this research investigates the interfacial stresses in adhesives of lap joints. Lap joints have become very attractive in various engineering applications.

In the aviation industry, adhesive and rivets connect adherends in lap-jointed configurations. Such connection is seen on aircraft wing, fuselage, and furnishing components. Figure 1.5a shows lap joints on the fuselage of a commercial aircraft.

However, problems associated with rivets have increased the attractiveness of full adhesive bonded lap jointed solutions. For instance, one of the numerous accidents caused by structural failure of fuselage is the 1988 Aloha’s Boeing 937 aircraft accident. The upper part of the fuselage of this aircraft was lost in full flight at 24000 ft (7315.2m). Multiple fatigue cracks
occurred in the remaining aircraft structure. Riveting exacerbates these kinds of cracks. Figures 1.5b and 1.5c show the damage in the aircraft’s fuselage [12].

Figure 1. 5: (a) Lap joints on the fuselage of a commercial aircraft (b) Belly of Aloha’s Boeing 737 showing sunken rivets (c) Illustration of cracks on the fuselage of Aloha’s aircraft [12]

1.5. Contribution to the Study Blast Engineering
The primary aim of studying FMLs and lap joints was to have a further insight into debonding which is a possible failure mode in a hybrid system of strengthened blast wall. However, considering the vast applications of the studied hybrid systems, this research also aims at improving the understanding of the behaviour and response of these systems under blast loads. A better understanding of their responses under blast loads would increase their use (i.e. hybrid systems studied in this research) in engineering applications.
An improved understanding of the transient response of these systems would also increase the confidence of scientists and engineers involved in the design of these systems for defence, oil and gas, marine and aeronautical applications.

Specific and novel contributions of this research to the science of the dynamic response of hybrid structural systems are:

1. Development of a novel Pressure-Impulse \((P-I)\) diagram (for pulse shapes with zero rise time) that can be applied to offshore structures, onshore structures and other engineering systems that can be conceptualised as continuous systems
2. Improving the understanding of the dynamic response of hybrid systems formed by partially strengthening structures with composite (with offshore blast wall as a case study) and developing a technique or technology for efficient strengthening of blast loaded blast walls
3. Improving the understanding of the dynamic response of fibre metal laminates (FMLs) and subsequently developing tools that would increase the confidence of analysts in designing these systems
4. Improving the understanding of adhesively bonded lap joints under blast loads

1.6. Scope and Outline of the Thesis

This thesis comprises of seven chapters. It starts with an introduction, which states the motivation for studying the classes of hybrid systems presented, followed by a background literature review. The other part of the thesis comprises of four chapters, which investigate the blast response behaviours of the types of hybrid system (hybrid system of a strengthened blast wall, fibre metal laminates, and metal-metal lap joints) studied in this research, and the last chapter concludes the research. The concluding chapter provides a general conclusion (drawn from the individual systems studied) on the response of the hybrid systems. The conclusion also discusses possible future area of research on the systems studied.

Chapter 2 presents an overview of the relevant literature, which provides insight into the dynamic response of hybrid systems. This chapter discusses the science of blast and its estimation. In addition, Chapter 2 discusses the fundamental principle of linear and nonlinear structural dynamics, which forms the backbone of the analysis in this thesis. This chapter presents an overview of the mechanics of composites, which is an important component of the first two families of hybrids, studied. It concludes with a literature on blast walls, FMLs, and lap joints.
Chapter 3 presents a model for a hybrid system of carbon-fibre-reinforced plastic (CFRP) strengthened blast wall. This chapter proposes an analytical model for the response of a CFRP strengthened blast wall. The results of the analytical model correlate well with results from Finite Element (FE) model in Abaqus. This chapter compares the response of the hybrid to the response of a monolithic blast wall in order to demonstrate the benefits of hybridisation. Furthermore, this chapter discusses the merits and philosophy of the proposed strengthening scheme and the overall response drawn from this structure on hybrid systems in general.

Chapter 4 presents a simplified model for the analysis of an unstrengthened blast wall presented in Chapter 3. The displacement predicted by the simplified analytical scheme in this chapter is compared with the displacements predicted by the numerical and analytical models in Chapter 3. A good correction is achieved in the elastic response regime and subsequently the limitations of this model are discussed. Due to the inability to model membrane effects, the simplified model is used to develop an elastic pressure-impulse diagram ($P$-$I$) for the unstrengthened blast wall studied in Chapter 3. Subsequently, a fundamental pulse shape independent $P$-$I$ diagram (for elastic and elastic-plastic response) is presented based on dimensionless parameters for pulses with zero rise time.

Since debonding was prevented in Chapter 3, Chapter 5 uses cohesive elements to model the adhesive in a hybrid system in order to give an insight into the development of debonding and how it affects hybrid systems in a blast scenario. In addition, Chapter 5 presents an improved numerical model for prediction of the blast response of FMLs. It presents a simplified and improved model to analyse the damage initiation, progression, and failure of the three-dimensional solid woven glass-fibre polypropylene composite (GFPP), which is a constituent of FMLs. The developed numerical model is able to capture the state of the aluminium and composite and the maximum displacement of the faces of the FML studied.

In order to understand the stress (i.e. the relationship between the bonded components and the stress in the adhesive) in the adhesive layer of a hybrid system, Chapter 6 investigates the peel stresses and shear stresses in adhesive bonded lap joints. Debonding initiates when some stress or strain based initiation criteria have been reached in the adhesive. In other words, this chapter investigates the blast response of a system of metal-adhesive-metal system under transverse hydrocarbon type explosion with the sole intention of understanding the behaviour of adhesives which bond structural components in hybrid systems. It presents an analytical procedure for investigating the maximum peel and shear stresses developed in a transversely loaded balanced lap joint system. The maximum peel and shear stresses predicted by the analytical model show a good correlation with the FE model of the system in Abaqus. The equations of motion of the
system are derived in space of relevant displacement components and by the application of the principle of least action – Hamilton’s principle. The resulting partial differential equations (PDEs) are reduced to ordinary differential equations (ODEs) using the Galerkin’s weighted residual method. The final equations of motion are solved using the Central Difference Method. In addition, Chapter 6 investigates the stress distribution in the adhesive layer of a metal-adhesive-composite hybrid system. The advantages of the metal-composite adherends are discussed.

Chapter 7 concludes the research and discusses the general conclusion on the performance of hybrid systems drawn from the systems studied. Chapter 7 concludes with a discussion on possible areas for future research.
1.5 Cited References


Chapter 2

Literature Review

2.1 Introduction

This chapter presents an introduction to blast loading and its estimation. Subsequently, various methods of numerical integration are discussed. It presents a review of the mechanics of composites, which is a major component of the two types of hybrid system investigated. In addition, it presents a review of strengthened blast walls, fibre metal laminates (FMLs), and lap joints. This includes a review on blast walls and recent works that have been done on strengthening blast walls; analysis of fibre metal laminates (FMLs) and recent research on the dynamic response of FMLs and the analysis of dynamically loaded lap joints.

2.2 Blast Loads

Analysing blast effects on structures requires a proper understanding of blast load modelling. From an analysis point of view, the general term blast refers to both fluctuation of air pressure due to man-made explosions and to vibrations induced by ground. Air blast loads may act internally or externally depending on the position of the explosive relative to the structure. Internal blast loads may be because of detonation of high explosives, usually accidentally or intentionally placed or from detonation of chemical ammunitions or also from the deflagration of low explosives (usually accumulations of flammable gas/air mixtures). While external blast loadings may be from one or other of these, there is usually an added possibility of load action over the entire structure as a result of distant atomic explosions [1].

In blast load analyses, the excitation load is usually defined as a given load-time history. The objective of this section is to highlight the modelling of air blast loading in structures due to unconfined explosions in a mathematical form that are applied in structural response computations. A global look is given to explosions from sources such as nuclear, conventional weapons and from unconfined gas or dust cloud explosion (i.e. from oil and gas facilities).
2.2.1 Explosion Process

Explosion is a process resulting from rapid and sudden release of a large amount of energy [2]. Traditional explosives such as TNT (Trinitrotoluene) depend on a rearrangement of their atoms for the release of energy, while nuclear explosions depend on the release of energy building protons and neutrons from the atomic nuclei. In the case of flammable materials, energy is mainly derived from the chemical reaction. In many cases, only a fraction of the total mass of the explosive is involved in the explosion process. The rest of the mass is usually consumed by deflagration resulting in a large amount of the material’s chemical energy being dissipated as thermal energy which subsequently may cause fires [1].

When hot gases produced by an explosion forcibly push the atmosphere surrounding a scene of an explosion back, shock waves or blast waves are generated. Figure 2.1 shows the propagation of these shock waves. The waves move outward from the explosive source. The wave front (i.e. front of the wave) is similar to a wall of highly compressed air and has an overpressure much greater than the region behind it. As the wave front propagates the pressure behind the front drops, below the ambient pressure, this negative phase creates a partial vacuum and air is sucked in [1].

![Shock wave front propagation](image)

**Figure 2.1: Propagation of blast wave front**

After leaving the source of explosion, the blast wave travels as an incident until it strikes an object of a higher density than the atmosphere. After striking an object, a reflected wave is
generated which travels back towards the source of explosion. The reflected wave might have a pressure much greater than the incident wave. Thus, in some cases this results in the reflected waves catching up with the incident wave at some point between the explosion and the object. This results in a vertical wave front called a Mach system, which propagates horizontally along the surface of the ground. The implication of this is that structures below this path will experience a single shock wave while structures above this path will experience two shock waves (reflected and incident).

2.2.2 Blast Wave Configuration

The blast wave configurations discussed in this section are used in this study. Sections 2.2.2.1 and 2.2.2.2 describe the configurations of the types of global blast loads encountered in blast engineering.

2.2.2.1 High Explosives

The combination of high explosive materials with an oxidizer (oxygen or any dedicated material) results in a rapid exothermic reaction. This high rate reaction is known as detonation (a supersonic propagation of exothermic oxidation reaction front through the quiescent portion of explosive mass). As detonation progresses through the explosive mass, combustion products are produced and rapidly expand. This expansion process results in large pressure gradients in the surrounding medium resulting in the propagation of shock wave. Figure 2.2 illustrates the characteristic of this peculiar shock wave as it passes undisturbed through the surrounding medium. In order to facilitate calculation, $t_r$, which represents the finite rise time, is taken as zero in cases of high explosive shocks. The negative overpressure phase, which has a low peak magnitude, is often ignored in analysis. Thus, it is the peak overpressure $P_0$ and phase durations ($t_{dp}$ and $t_{dn}$) that define the impulse of the shock wave [3].
2.2.2.2 Vapour Clouds

The mixture of air and flammable gases, or accumulation atomised liquids results in vapour clouds [3]. This usually occurs in petrochemical plants as a result of leakages from pressurised systems. When vapour clouds occur, fire might subsequently occur if there is a source of ignition or the vapour cloud is within flammability limits. Ignition might occur immediately, or might be delayed, which results in deflagration. It is worth mentioning that under certain conditions, the deflagration process (which is a subsonic propagation of the flame front), may transition to detonation, with subsequent supersonic propagation resulting in significantly more energetic explosions.

The characteristic pressure time history of vapour cloud explosions differ from high explosions. Figure 2.3 shows the typical overpressure of a vapour cloud with relatively longer rise time, $t_r$ [3].
2.2.3 TNT Equivalence and Scaling

In practice, test data on blast is related to equivalent charge of TNT (Trinitrotoluene) explosive. The effective charge weights of explosives are related to an equivalent weight of TNT. The energy output of an explosive can be written as a function of heat of detonation as in Equation (2.1).

$$W_{TNT} = \frac{H_{exp}}{H_{TNT}} w_{exp}$$  \hspace{1cm} (2.1)

Here, $W_{TNT}$ is the equivalent TNT charge weight, $w_{exp}$ is the weight of the explosive investigated, $H_{TNT}$ is the heat of the detonation of TNT, and $H_{exp}$ is the heat of detonation of explosive being investigated.

The features of blast waves released depend on explosive energy release source and the medium in which it propagates. These properties are measured for controlled experiments. Thus, to find the characteristics of other explosives, scaling laws are applied. This section describes the widely used scaling law- Hopkinson [4]. Hopkinson’s cube root scaling states that when two charges of the same explosive and geometry having different sizes are detonated in the same environment (atmosphere), the shock waves produced are similar in nature of the same-scaled distances. The scaled distance is:

$$Z = R / W^{1/3}$$  \hspace{1cm} (2.2)
Where $R$ is the distance from the centre of the explosive to a given location (or target) and $W$ is the weight of the explosive. The yield factor, $\lambda$ is also used in blast description and it is defined as:

$$\lambda = \left(\frac{W}{W_r}\right)^{1/3} \quad (2.3)$$

Where $W_r$ is the weight of the reference explosion. This implies that similar shock waves occur at the same-scaled distance. Thus, we have:

$$Z = \frac{R_r}{W_r^{1/3}} = \frac{R}{W^{1/3}} \quad (2.4)$$

Where $R_r$ is the distance of reference explosions which is related to $R$ as:

$$R = \lambda R_r \quad (2.5)$$

Scaling is also applied to other parameters such as:

$$\tau_{sc} = \frac{t}{W^{1/3}} \quad (2.6)$$

Where $\tau_{sc}$ is scale time. The values of arrival time and duration time for a blast wave, $t_a$ and $t_s$ respectively are related to corresponding $t_{ar}$ and $t_{sr}$ scaled values in:

$$t_a = \lambda t_{ar}, \quad t_s = \lambda t_{sr} \quad (2.7a, 2.7b)$$

### 2.2.4 Loading Associated with Airflow and Reflected Process

As has been discoursed, blast loads are generally characterised by peak overpressure. However, in some cases the very strong transient winds behind the shock waves can be of great significance. These drag forces depend on the size, shape of the structure and the value of the dynamic pressure resulting from the wind behind the shock front. In predicting the peak value of the dynamic pressure, the shock front velocity, peak wind velocity and the density of the air behind the shock front are needed. The front wave velocity, $U_s$ is calculated as Equations (2.8) and (2.9).

$$U_s = C_0 M_s \quad (2.8)$$

$$M_s = \left[1 + \left(\frac{\gamma_h + 1}{2\gamma_h}\right) \frac{P_s}{P_0}\right]^{1/2} \quad (2.9)$$
Where $C_0$ is the speed of sound in the undisturbed atmosphere, $M_s$ is the Mach number for the corresponding peak overpressure, $\gamma_h$ is the ratio of the specific heat of air, and $P_0$ is the ambient air pressure (atmospheric pressure).

The relationship between the wind velocity and the shock velocity is given as:

$$u_s = \frac{2}{1 + \gamma_h} \left( \frac{U_s^2 - C_0^2}{U_s} \right)$$

(2.10)

In terms of overpressure, peak velocity and air density, $\rho_s$, behind the shock wave we have:

$$u_s = \frac{C_0 P_s}{\gamma_h P_0} \left( 1 + \frac{(\gamma_h + 1)P_s}{2\gamma_h P_0} \right)^{-1/2}$$

(2.11)

$$\rho_s = \rho_0 \left( \frac{(\gamma_h + 1)P_s + 2\gamma_h P_0}{(\gamma_h - 1)P_s + 2\gamma_h P_0} \right)$$

(2.12)

The dynamic pressure is proportional to the square of the wind velocity and the density of air. The peak dynamic pressure, $P_d$ is:

$$P_d = \frac{1}{2} \rho_s u_s^2$$

(2.13)

It can be shown that, $P_d$ is also:

$$P_d = \frac{P_s^2}{2\gamma P_0 + (\gamma_h - 1)P_s}$$

(2.13)

Reflected blast waves are as result of a change in momentum when an incident air blast strikes a surface in the path of propagation. The reflection factor is the ratio of the reflected overpressure to the incident overpressure. Equation (2.14) shows the relationship between the peak reflected overpressure, $P_r$ and peak incident overpressure.

$$P_r = 2P_s + (1 + \gamma_h)P_d$$

(2.14)

For $P_s$ less than 10 bars, considering ideal gas conditions ($\gamma_h=1.4$) and substituting for $P_d$ from equation (2.14), the peak reflected overpressure is:

$$\frac{P_r}{P_s} = 2 + \frac{6P_s}{P_s + 7P_0}$$

(2.15)

For $P_s$ greater than 10 bars, the empirical relationship in Equation (2.16) holds
2.2.5 External Blast Loads on Rectangular Structure

The practice in blast load analysis is to assume that the actual blast effects in the incidental and reflected shock wave are approximated as equivalent triangular pulses of similar impulses. Each pulse is idealised to have a peak pressure value similar to the actual blast effect and idealised durations defined as a function of the impulse and peak pressure [1, 5].

Figure 2.4 shows the variation of pressure in the front phase of a rectangular structure. The graph illustrates the instantaneous rise in peak reflected pressure \( P_r \) followed by a rapid decay as the reflected pressure causes a flow around the structure. The \( t_c \) in the graph is referred to as the clearing time and it is:

\[
t_c = \frac{3S}{U_s} \tag{2.17}
\]

Where \( S \) is equal to the lesser of the height of the structure or one-half the width of the structure.

After the clearing time, the pressure decreases to zero with the decay on side-on overpressure and dynamic pressure. In this phase, the maximum pressure is:

\[
P = P_s + C_f P_d \tag{2.18}
\]

Where \( C_f \) is the drag coefficient for the front face (ranges from 0.8 to 1.5). The values of \( t_1 \) and \( t_1' \) are evaluated as the fictitious durations of the dynamic pressure and incident overpressure.
In the propagation of blast waves around the structure, no pressure is transmitted to the back face until the shock front reaches that face. Using the same time reference as for the front phase, average pressure begins to build up on the back face at a time equal the length of the structure (L) in the direction of the shock propagation divided by the velocity of the shock propagation [1]. After the back face has been engulfed in blast, the pressure becomes a maximum equal to the side-on overpressure reduced by the amount equal to the side-on overpressure, which acts as a suction on the rear surface. The time it takes the pressure to build up on the back face is expressed as:

\[
T_{rb} = \frac{L + 5S}{U_s}
\]  

(2.19)

As the blast wave goes through the roof of the structure, its value is equal to the incident overpressure reduced by a negative drag pressure or suction associated with the flow of air around the structure. Thus, the roof’s loading is assumed a triangular load with duration:

\[
T_{rr} = T_{rf} + L/U_s
\]  

(2.20)

Where, \(T_{rf}\) is the rise time to the peak pressure. The peak value of the pressure is expressed as:

\[
P = P_s - C_t P_d
\]  

(2.21)
\( C_t \) is the drag coefficient for the roof. In addition, the duration of the equivalent pulse is equal to:

\[
T_{ds} = T_{rr} + T_i
\]  \hspace{1cm} (2.22)

\( T_i \) corresponds to the component of the equivalent duration as result of the dynamic pressure.

### 2.3. Numerical Integration

Strictly speaking, linear or nonlinear dynamic problems can be classified as wave propagation problems or inertial problems. In wave propagation problems, the characteristic of the wave front is of significance and in some instances median or high frequency structural modes dominate the structural response. While for inertial problems (all dynamic problems excluding wave propagation problems), the response is governed by a small number of low frequency modes. Seismic response and large deformations of elastic-plastic structures under ramp or step loading can be viewed as inertial problems [6].

As a guideline, wave propagation problems referred here are efficiently solved by explicit integration techniques. While inertial problems are efficiently solved by implicit integration. Methods, which do not involve the solution of sets of linear equations at a time step but use the differential equation at time, \( t \) to solve the displacements at time \( t + \Delta t \) are referred to as explicit methods. On the other hand, implicit schemes attempt to solve the differential equation at time, \( t \) with results from solution from time, \( t - \Delta t \).

#### 2.3.1 Explicit Integration Schemes

This section highlights the explicit scheme for numerical integration. In this scheme, the solutions of the governing differential equation are not solved at each time step. In effect, the differential equation at time \( t \) is used to evaluate the solution at time, \( t + \Delta t \). A small time step is needed to execute this scheme. Stiffer structures require even smaller time steps. Thus, this method is conditionally stable with respect to the time step [7]. Some examples of this widely used schemes are: second order central difference method which has a very high accuracy and a maximum stability limit for any explicit method of the order of two [1]; other variant forms of central difference schemes and the fourth order Runge-Kutta method.
2.3.2 Implicit Integration Schemes

This section briefly highlights the implicit integration scheme for linear structural dynamics. A major feature of the implicit scheme is that it attempts to satisfy the dynamic differential equation at time, $t$, after a preceding solution at $t - \Delta t$ is known. Solution of a set of linear equations at each time step is required in this scheme. The method can be conditionally or unconditionally stable. Larger time steps when compared to the explicit method can be used [7]. Examples of these integration strategies are the Newmark Family of integration schemes, the Wilson’s $\theta$ Factor method and the Hilber, Hughes and Taylor $\alpha$ method.

2.4 Composite Damage

Composites constitute parts of the hybrid system in this thesis, it is imperative that a look at the mechanics of composites be undertaken. A reinforced composite is composed of three main constituents: the fibres, the matrix, and fine interface region - responsible for bonding between the matrix and fibre. The manner in which the material fractures depends upon the chemical and mechanical properties of these constituents. The failure modes include delamination, intra-laminar cracking, longitudinal matrix splitting, fibre/matrix debonding, fibre pullout, and fibre fracture. The relative energy absorbed by these fracture modes depend upon the properties of the constituent and the pattern or direction of loading [8].

The properties of fibre in a continuous fibre composite have a significant effect on the impact resistance and subsequent load-bearing capacity of components made from such materials. For low velocity impact loading, the ability of the fibres to store energy elastically seems to be the major factor influencing the impact response [8]. On the hand, the matrix properties play a significant role in determining the impact resistance and consequent load-bearing capacity of a composite.

In the study of composites, various methods are used in the characterisation of the onset and progression of damage. For instance, allowable stresses of composites can be used to characterise the initiation and growth of all types of damage. In this case, once the strength criterion is reached the material is assumed to have suffered an irreversible damage [9].

Classical fracture mechanics theory studies growth of existing defects. This has been applied to the study of delamination and debonding. The growth of a macroscopic defect is controlled by the strain energy released in propagation, as compared to a threshold maximum strain energy release rate for a material, which is measure of material toughness. The strain energy released in
crack propagation is typically split into the separate mechanisms of crack growth [9]: peeling, shearing and tearing as shown in Figure 2.5.

![Figure 2.5: Crack growth modes (a) peeling, (b) shearing (c) tearing](image)

Various methods have been developed to determine the strain energy release rate components from the results of FE analysis. The majority of these methods make some assumptions regarding the crack front geometry and crack growth behaviour. Some of these methods include the $J$-integral, equivalent domain integral, finite extension and virtual crack extension methods [10].

Equations (2.23) - (2.31) show some failure criteria used in composite analysis. The reader is referred to the work done by Orifici et al [9] for more damage initiation theories. In the presented equations, $X, Y,$ and $Z$ represent the three directions of anisotropy and inclusions of subscripts $C$ and $T$ represents their respective uniaxial compressive and tensile failure strengths. Shear strengths are represented by $S$ with subscripts to show its directions. Other conventions used in these equations are standard Voigt notations for stress/strain and strength representations. While $G_I, G_{II}$ and $G_{III}$ represents the fracture energies corresponding to peeling, shearing and tearing failure modes and $G_{Ic}, G_{IIc}$ and $G_{IIIc}$ represent threshold values.

1) **Failure criteria for fibre failure in tension**

Max stress fibre tension

$$\sigma_1 \geq X_T \quad (2.23a)$$
Max strain fibre tension

\[ \varepsilon_1 \geq \varepsilon_{1T} \quad (2.23b) \]


\[ \frac{\sigma_1}{X_T} + \frac{1}{S_{12}^2} (\tau_{12}^2 + \tau_{13}^2) \geq 1 \quad (2.23c) \]


\[ \left( \frac{\sigma_1}{X_T} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 \geq 1 \quad (2.23d) \]

Other failure criteria for fibre failure in tension include the 1987 Chang-Chang fibre tension (reader is referred to paper for details [12]) and 1998 Puck fibre Tension [13].

2) Failure criteria for fibre failure in compression

Max stress fibre compression

\[ \sigma_1 \geq X_C \quad (2.24a) \]

Max strain fibre compression

\[ \varepsilon_1 \geq \varepsilon_{1C} \quad (2.24b) \]


3) Failure criteria for fibre in tension and compression

1982 Lee fibre in tension and compression [16]

\[ \sigma_1 \geq \sigma_{FN} \text{ or } \sqrt{\left( \frac{\sigma_{12}^2 + \sigma_{13}^2}{2} \right)} \geq \sigma_{FS} \quad (2.25a) \]

\( \sigma_{FN} \): fibre normal strength, \( \sigma_{FS} \) fibre in – plane strength

1997 Christensen fibre in tension and compression [17]

\[ \sigma_1 k_2 \sigma_1 + \frac{1}{4} (1 + 2\alpha_2) - \frac{(1 + \sigma_2)^2}{2} \left( \frac{\sigma_2 + \sigma_2}{2} \right) \sigma_1 \leq k_2^2 \quad (2.25b) \]

\[ k_2 = \frac{X_T}{2}, \alpha_2 = \frac{1}{2} \left( \frac{X_T}{|X_C|} - 1 \right) \]
4) Failure criteria for matrix failure in tension

Maximum stress matrix tension

\[ \sigma_2 \geq Y_T \]  \hspace{1cm} (2.26a)

Maximum strain matrix tension

\[ \varepsilon_2 \geq \varepsilon_{\text{T2}} \]  \hspace{1cm} (2.26b)

1973 Hashin-Rotem matrix tension [18]

\[ \left( \frac{\sigma_2}{Y_T} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 \geq 1 \]  \hspace{1cm} (2.26c)


\[ \left( \frac{\sigma_2 + \sigma_3}{Y_T^2} \right)^2 + \frac{\tau_{23}^2 - \sigma_2 \sigma_3}{S_{23}^2} + \frac{\tau_{12}^2 - \tau_{13}^2}{S_{12}^2} \geq 1 \]  \hspace{1cm} (2.26d)

5) Failure criteria for matrix failure in compression

Maximum stress matrix compression

\[ \sigma_2 \geq Y_C \]  \hspace{1cm} (2.27a)

Maximum strain matrix compression

\[ \varepsilon_2 \geq \varepsilon_{\text{2c}} \]  \hspace{1cm} (2.27b)

1973 Hashin-Rotem matrix compression [18]

\[ \left( \frac{\sigma_2}{Y_C} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 \geq 1 \]  \hspace{1cm} (2.27c)


\[ \frac{\sigma_2}{Y_C} \left[ \left( \frac{Y_C}{S_{23}} \right)^2 - 1 \right] + \left( \frac{\sigma_2}{S_{23}} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 \geq 1 \]  \hspace{1cm} (2.27d)
6) **Failure criteria for matrix in tension and compression**


7) **Failure criteria for fibre matrix shear failure**

Matrix stress shear

\[ \tau_{12} \geq S_{12} \]  
(2.28a)

Max strain shear

\[ \gamma_{12} \geq \gamma_{12}^u \quad \gamma_{12}^u : \text{ultimate shear strain} \]  
(2.28b)


\[ \left( \frac{\sigma_1}{X_T} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 \geq 1 \]  
(2.28c)

Other failure criteria include the 1991 Chang-Lessard shear [15] criteria.

8) **Interactive failure criteria for ply failure**

1965 Tsai-Hill ply inter[19-20]

\[ \left( \frac{\sigma_1}{X} \right)^2 + \left( \frac{\sigma_2}{Y} \right)^2 + \left( \frac{\tau_{12}}{S_{12}} \right)^2 - \left( \frac{\sigma_1 \sigma_2}{X^2 Y} \right) \geq 1 \]  
(2.29a)

Where \( X \) and \( Y \) are either \( X_C, Y_C \) or \( X_T, Y_T \) depending on sign of \( \sigma_1, \sigma_2 \)

1967 Hoffman ply inter[21]

\[ \left( \frac{1}{X_T} - \frac{1}{X_C} \right) \sigma_1 + \left( \frac{1}{Y_T} - \frac{1}{Y_C} \right) \sigma_2 + \frac{\sigma_1^2}{X_C X_T} + \frac{\sigma_2^2}{Y_C Y_T} + \left( \frac{\tau_{12}}{S_{12}} \right)^2 + 2f_{12}\sigma_1 \sigma_2 \geq 1 \]  
(2.29b)

\[ f_{12} = -\frac{1}{2} \sqrt{f_{11}f_{22}} \quad \text{or} \quad f_{12} = -\frac{1}{2} \sqrt{\frac{1}{(X_C Y_T Y_{C_T})}} \]

9) **Failure criteria for delamination initiation**

Maximum stress delamination initiation

\[ \sigma_3 \geq Z_T, \tau_{31} \geq S_{31}, \tau_{23} \geq S_{23} \]  
(2.30a)

(2.30b) \frac{\sigma_3}{Z_T}^2 + \frac{\tau_{23}^2}{S_{23}} + \frac{\tau_{31}^2}{S_{31}} \geq 1

1982 Lee delamination initiation \[16\]

\sigma_3 \geq Z_T \text{ or } \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \geq S_{23} \tag{2.30c}

1987 Ochoa-Engblom delamination\[22\]

\frac{\sigma_3^2}{Z_T^2} + \frac{\tau_{23}^2 + \tau_{31}^2}{S_{23}^2} \geq 1 \tag{2.30d}

Other delamination initiation postulations can be obtained in \[9\].

10) Criteria for growth of an existing delamination

Single mode

\[ G_1 \geq G_{1c}, G_{II} \geq G_{IIc}, G_{III} \geq G_{IIIc} \tag{2.31a} \]

1981 Hahn delamination growth\[23\]

\[ G_T \geq G_{IIc} - (G_{IIc} - G_{Ic}) \sqrt{\frac{G_I}{G_{1c}}} \tag{2.31b} \]

1984 Power-Law delamination growth\[24\]

\[ \left( \frac{G_I}{G_{1c}} \right)^m + \left( \frac{G_{II}}{G_{IIc}} \right)^n + \left( \frac{G_{III}}{G_{IIIc}} \right)^p \geq 1 \tag{2.31c} \]

\[ m, n, p: \text{ curve fit i.e. linear: } m = n = p = 1; \text{ quadratic: } m = n = p = 2 \]

Other postulations for growth of existing delamination can be obtained in \[9\].

2.5 Interface Elements

Interface elements are separate FE quantities modelled between two materials as a means of inserting a damageable layer for delamination modelling. Thus, the interface elements connect two substructures and transfers all tractions across the interface, until a particular criterion is reached, after which the element stiffness properties decreases. These elements are determined by the damage mechanics constitutive relationship between the relative displacement of the two
connected substructures, and also the resultant traction between them [9]. There have been a number of research works on these elements [25-32].

Cohesive elements are used in modelling adhesives, bonded interfaces, and gaskets. The constitutive response of these elements depends on the specific application and is based on certain assumptions about the deformation and stress states that are appropriate for each application. The nature of the mechanical constitutive response are classified based on (a) a continuum description of the material, (b) a traction-separation description of the interface, and (c) a uniaxial stress state appropriate for modelling gaskets and/or unconstrained adhesive patches [33].

The continuum-based modelling involves the cases where two bodies are connected together a glue like material. A continuum-based modelling of adhesive is appropriate when the glue has a finite thickness. The experimentally derived properties of the adhesive are used directly for the modelling process. In three-dimensional problems the continuum-based constitutive model assumes one direct (through-thickness) strain, two transverse shear strains, and all (six) stress components to be active at a material point. In two-dimensional problems it assumes one direct (through-thickness) strain, one transverse shear strain, and all (four) stress components to be active at a material point [33].

The traction-separation-based modelling involves situations where the intermediate glue material is very thin and for all practical purposes may be considered to be zero thickness. In this case, the macroscopic material properties are not relevant directly, and the analyst must resort to concepts adopted from fracture mechanics such as the amount of energy required to create a new surface. In three-dimensional problems, the traction separation model assumes three components of separation i.e. one normal to the interface and two parallel to it; and the corresponding stress components are assumed active at a material point. In two dimensional problems the traction-separation-based model assumes two components of separation i.e. one normal to the interface and the other parallel to it; and the stress components are assumed active at a material point [33].

**Damage Model adopted in numerical modelling**

As show by the graph of typical traction separation response (Figure 2.6), the damage initiation refers to the beginning of degradation of the response of a material point. The damage initiation in damage models begins when the stress and/or strains meet some damage initiation criteria (equation (2.32) - (2.35)).
Maximum nominal stress criterion

$$\max \left\{ \frac{t_n}{t_n^0}, \frac{t_s}{t_s^0}, \frac{t_t}{t_t^0} \right\} = 1 \tag{2.32}$$

Maximum nominal strain criterion

$$\max \left\{ \frac{\varepsilon_n}{\varepsilon_n^0}, \frac{\varepsilon_s}{\varepsilon_s^0}, \frac{\varepsilon_t}{\varepsilon_t^0} \right\} = 1 \tag{2.33}$$

Quadratic nominal stress criterion

$$\left( \frac{t_n}{t_n^0} \right)^2 + \left( \frac{t_s}{t_s^0} \right)^2 + \left( \frac{t_t}{t_t^0} \right)^2 = 1 \tag{2.34}$$

Quadratic nominal strain criterion

$$\left( \frac{\varepsilon_n}{\varepsilon_n^0} \right)^2 + \left( \frac{\varepsilon_s}{\varepsilon_s^0} \right)^2 + \left( \frac{\varepsilon_t}{\varepsilon_t^0} \right)^2 = 1 \tag{2.35}$$

Where $t_n^0, t_s^0, t_t^0$ represent the peak values of the nominal stress when the deformation is either purely normal to the interface or purely in the first or the second shear direction respectively. Also $\varepsilon_n^0, \varepsilon_s^0, \varepsilon_t^0$ is the strain equivalent.
Damage evolution

The damage evolution law describes the rate at which the material stiffness is degraded once the corresponding initiation criterion is reached. There are various damage evolution laws or models. In the model used in this thesis, an evolution based on energy is used. Here, damage evolution is defined based on the energy that is dissipated because of the damage process; this energy is called fracture energy. The fracture energy is the area under the traction-separation curve. The fracture energy is a material property obtained from test. A linear or exponential behaviour of decay after the initiation is triggered can be specified. The area of the curve is always equal to the fracture energy in whichever case is used.

Equation (2.36) shows the dependency of the fracture energy on the mode mix. Here, the mode mix is assumed to be defined in terms of energies. Equation (2.36) is also called the power law dependency criterion, where $\alpha$ takes a desired value. This law states that failure under mixed-mode conditions is governed by a power interaction of the energies required to cause failure in the individual (normal and two shear) modes.

$$\left(\frac{G_n}{G^c_n}\right)^\alpha + \left(\frac{G_s}{G^c_s}\right)^\alpha + \left(\frac{G_t}{G^c_t}\right)^\alpha = 1$$  \hspace{1cm} (2.36)

Where the mixed mode fracture energy is $G_c = G_n + G_s + G_t$ when the above equation is satisfied. In the expression, the quantities $G_n$, $G_s$ and $G_t$ refer to the work done by the traction and its conjugate relative displacement in the normal, the first, and the second shear directions, respectively. The specified quantities $G^c_n$, $G^c_s$ and $G^c_t$ (usually material property) which refer to the critical fracture energies required to cause failure in the normal, the first, and the second shear direction respectively [33].

2.6 Background on Blast Walls

Blast walls are used in the offshore industry to mitigate the disastrous effects of a possible gas explosion by separating topside modules from each other. They are also sometimes employed on onshore platforms of oil and gas facilities to separate living quarters from the process units. They are designed to resist a particular explosion threat, usually defined in terms of peak overpressure in their lifetime and have different directional behaviours due to the particulars of section classification and connection arrangement. Blast walls consist of corrugated sheeting running top to bottom [34]. The design of blast walls in the UK is carried out using guidance issued by the
FABIG and the Steel Construction Institute (SCI) commonly referred to as Technical Note 5 (TN 5) [35].

Stainless steel has become the material of choice for most blast walls because of its superior engineering properties. For example, in thickness of 2-6 mm, high strength carbon steel is not readily available and localised thinning (pitting) due to corrosion is a common occurrence in thin carbon steels and painting is not always possible. In particular, the shape of the stress-strain curve of stainless steel in the plastic range ensures higher plastic moment resistance than carbon steel of equivalent strength. Stainless steel offers intrinsically greater fire resistance than carbon steel [36].

From the analytical point of view, a blast wall can be analysed using a single-degree-of-freedom (SDOF) system as proposed by Biggs [37]. In this method, a load mass factor and a resistance term are introduced which take into account inelastic deformation, membrane action and/or strain hardening. Numerical integration of the resultant equations provides the dynamic response.

Louca et al. [38] have worked on the design of blast walls and factors that influence the design against blast. This study revealed the advantages of SS316 in the design of blast walls. It showed that the material allows better dissipating energy capacity that can prevent the sudden onset of instability and thus present a better means of retaining the integrity of blast walls under high loads that can cause deformation.

Langdon and Schleyer [39-43] have analysed unstrengthened blast wall panels analytically, experimentally and numerically. Two sets of experiments were performed [39, 42]: one set was performed using a pulse pressure loading rig at the University of Liverpool [39], and a second set of tests on a limited number of panels at the BakerRisk test site using a shock tube assembly [42]. The first round of experiments resulted in quasi-static-dynamic loading, with a triangular pressure pulse with the rise time approximately half of the total load duration [39]. The experiments performed using the shock tube had similar load durations but with a much steeper rise time [42].

The analytical model proposed by Langdon and Schleyer [40] involved a simplification of the blast wall panels to a system of beam and springs. Energy methods were used to model flexural and membrane behaviour. The support connection of a blast wall was modelled with three springs (horizontal, vertical, and rotational springs). The elastic-plastic horizontal spring was used to model the variable membrane resistance and the connection pull-in. The elastic-plastic
rotational spring at the support modelled the rotation at the support while the perfectly elastic vertical spring modelled the rigid body motion of the beam. This was an extension of a model proposed by Schleyer and Hsu [44].

Regarding improving the energy absorption capacity of blast wall panels, Boh et al [45] proposed a method to increase the energy absorption capacity by introducing passive impact barriers. The research illustrated a technique where the response of blast walls is modified by the inclusion of a passive barrier system at a certain offset distance from the wall. It was observed in this study that the diagonal impact barriers increased the containment pressure of blast walls by reducing the membrane action and thus delaying the tearing of the horizontal welds. However, one major drawback of this scheme is the space that would be occupied by the barriers in a relatively tight platform.

Bambach et al [46] studied the energy absorption of thin-walled, spot-welded square hollow sections (SHS) with externally bonded CFRP under static and dynamic axial crushing. For low ductility, high strength sections, the application of externally bonded CFRP increased the mean crushing load and the energy absorbed under static and dynamic conditions. As expected, ductile spot-welded SHS exhibited extensive spot-weld and material fracture, which reduced the static and dynamic crash-worthiness efficiency. This was due to the spot weld being less ductile.

When static loads are concerned, many schemes have been proposed for strengthening structures. One that is relatively new involves the use of advanced composite plates, usually made of unidirectional carbon fibre reinforced polymers (CFRP) to strengthen beams by retrofitting them. The CFRP plates are adhesively bonded to the metallic beam to reduce the stresses in the tension area of the steel element or to defer buckling in the compression elements of the beam. Today, there is no single established code or standard for structural design of externally reinforced metallic beams in civil engineering [47]. Recently, researchers such as Haedir et al [48] have proposed methods of predicting the strength of steel circular hollow sections externally reinforced with CFRP sheets in bending. The method involves expressions which are functions of the amount of CFRP and its mechanical properties [49].

2.7 Background on Fibre Metal Laminates

Fibre metal laminated plates better known as Fibre Metal Laminates (FMLs) are a class of hybrid materials that have attracted interest due to their improved impact resistance and fatigue behaviour. A number of research have been done comparing FMLs to monolithic metallic plates.
under dynamic loads. Glare, an FML patented by a joint venture of Aluminium Corporation of America (ALCOA) and Akzo Nobel N.V, has been used in commercial applications in Airbus 380 [50]. It comprises thin aluminium 2024-T3 sheets and a unidirectional or a biaxial Glass-Fibre-Reinforced Epoxy composite (GFRP) interleaved in a periodic structure through the thickness [51-54].

Fleischer studied blast load resistance of light luggage containers based on Glare and reported the capacity to withstand a bomb blast greater than that of the Lockerbie air disaster. The FML container tested with a Lockerbie type explosion shows the enormous potential of Glare in the design of Blast-Resistant Aircraft Baggage Containers for in-service aircraft use [51-52, 55-56].

More recently, comprehensive investigations on the blast resistance of FML’s have been carried at the Impact Research Centre of the University of Liverpool and the Blast, Impact and Survivability Research Unit (BISRU) of the University of Cape Town. In a set of the experiments [53, 57-59], FML’s composed of 2024-0 aluminium alloy and glass fibre reinforced polypropylene (GFPP) subjected to localised blast loading were investigated. The test sample consisted of an FML based on twelve different configurations, having total laminate thickness of 2mm to 15mm. Failure modes in this work were characterised as Mode I failure: large inelastic deformation of the back face of the panel (furthest away from the blast); Mode II failure: complete tearing of the back face; and Mode II*: transition between two failure modes- this occurs at threshold impulses. It was observed that thicker panels exhibit smaller displacements for a given impulse than thinner panels and that the Mode I non-dimensional displacement of both the front and back faces are linearly related to the non-dimensional impulse. It was also shown that thinner panels exhibit behaviour closer to that of a monolithic plate. As the panel thickness increases, behaviour deviates from the monolithic plate response and debonding becomes more pronounced. Further numerical studies for mode I failure show that when expressed in terms of non-dimensional parameters, the difference between the mean displacements of the front and back face respectively fall within one plate thickness, following a linear trend, and the threshold impulse for the onset of tearing for mode II failure was shown to be linearly dependent on panel thickness [51-53].

As observed by Vo et al [51-52] experimental models and subsequent optimisation are quite expensive and there is need for the finite element (FE) models to depict the behaviour of FML’s through predictive capabilities rather than as post-test corroboration tools. Attempts in this line have been made to model FML’s by e.g. Vo et al [51-52]. However, the focus of the present study is to introduce additional parameters that would increase the confidence of the analyst in
modelling FML’s as well as proposing a robust material model to be used in blast loading scenarios usually characterised and accompanied by occurrence of damage and fracture.

Composite laminates composed of fibre-reinforced plies are integral parts of FML’s. Thus, a thorough modelling of composites is an integral part of modelling FML’s. Failure and damage in laminated structures (composites) can be studied using micro-mechanical approach that considers failure and damage at the consistent level or a Continuum Damage Mechanics (CDM) approach in which material properties of the composite are homogenised and failure and damage is studied at the ply/lamina level [60] through several damage parameters. It is important to note that damage studied at constituent level is both computationally expensive for practical structures and experimentally expensive, furthermore extensive characterisation is required to determine material parameters in the damage model.

CDM theories capture effects of microscopic damage using the theory of internal variables [61]. Ladeveze and Dantec [62] used this approach to degrade elastic properties of the composite due to fibre breakage and matrix cracking and plasticity theory to account for permanent deformations induced under shear loading. Hassan and Batra [63] used three internal variables to characterise damage due to fibre breakage, matrix cracking and fibre/matrix debonding. In their work, delamination between adjacent plies was analysed using a failure surface quadratic (and thus convex) in transverse normal and the transverse shear stresses. Puck and Schurmann [13] generalised Hashin’s [11] stressed-based failure criteria and proposed a technique to degrade elastic parameters of the lamina subsequent to the initiation of a failure mode. Xiao et al [64] validated experimental quasi-static punch-shear test results carried out on plain weave S-2 glass/SC-15 epoxy composite laminates. The numerical modelling was carried out using a computer code called MAT162, which was incorporated into LS-DYNA. MAT162 uses damage mechanics principles for progressive damage and material degradation. Matzenmiller et al studied composite degradation [65]. They proposed that when one of the Hashin’s failure criteria is satisfied at a point in the composite structure, damage ensues at that point and it is depicted by introducing damage variables for fibre breakage in tension and compression, matrix cracking in tension and compression, and in plane shear. It is shown that the evolution of these internal variables depends upon the values of stresses in Hashin’s failure criteria which are expressed in terms of stress invariants for a transversely isotropic body and strength parameters for the composite [60].
2.8 Lap Joints

More often than not engineering structures are composed of parts or components which must be assembled or joined together to form the system as a whole – a hybrid system is a perfect example. Joining components is sometimes absolutely necessary as a means of designing a load bearing integrated part with an efficient load transferring path. Several methods of joining structural components comprise bolting, welding, adhesive bonding, riveting, etc. Adhesive bonding is, in particular, a creative and very popular joining technique used in the realms of aeronautical, civil and mechanical engineering to join components of different materials to achieve an optimised design. Hybridisation through adhesive bonding has proven successful in defining sole or alternative load paths and is almost inescapable to any design taking advantage of optimal characteristics of different components. The Boeing 747 aircraft, for instance, has 62% of its surface area constructed with adhesive bonding [66] due to the advantages this method of joining elements provides [67].

One of the earliest work on the topic is due to Goland and Resissner [68] who in 1944 analysed a generic joint. They assumed that adherends behaved as beams bonded via a shear and transverse normal deformable adhesive layer in analysing adhesive lap joints (assuming shear and peel stresses to be constant through the thickness and across the adhesive). Other researchers went on to incorporate linear variation of stresses across the thickness of the adhesive[69-71]. Hart-Smith [72-78] improved on the these models by using the knowledge of continuum mechanics in analysing lap joints and incorporating elastic-plastic adhesives. He analysed single-lap joints, double-lap joints, scarf, stepped-lap joints and tapered-lap joints composed of metallic and composite adherends. Various researchers have included the non-linear effects and large displacement analysis of adherends under static in plane loading [79-80].

The study of the dynamic response of adhesive joints has received limited attention compared to that under quasi-static loading [66]. Rao and Croker [81] developed an analytical model, which was validated experimentally, to predict the natural frequencies and mode shapes of a bonded lap joint system. The importance of the work lies in the fact that in free-vibration only these modes contribute to the overall response. The model can also be used to predict the system modal damping values by properly choosing the material damping values for the beam and adhesive. He and Rao [82-83] derived and solved for the dynamic response the governing equations of motion of a lap-joint (with viscoelastic adhesive) using Hamilton’s principle. Vaziri et al [84] went further to investigate the dynamic response of an adhesively bonded

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3 Viscoelastic materials are those for which the relationship between stress and strain depends on time or, in the frequency domain on frequency
single-lap joint with a void subjected to a harmonic peeling edge load by modelling the adherends as Euler-Bernoulli beams. Sato and Ikegami [85] investigated the propagation of stress wave and the concentration of dynamic stress under impact loading in adhesively bonded single lap joints, tapered joints and scarf joints with viscoelastic adhesives (using rheological Kelvin-Voigt model). The stress distribution and the temporal variation of stresses and strains in the joints under tensile in-plane pulse loading were calculated using finite element method, considering the viscoelastic properties of the adhesive. Researchers such as Zgoul and Crocombe [86] have also gone on to develop models which included a rate dependent adhesive model.

It has been assumed in literature that failure initiation criterion is stress-based, strain-based or displacement based [66, 87]. It has been observed that failure of an adhesively bonded joint depends upon the crack initiation position and the path of its propagation and can be classified as: (1) adhesive failure between adhesive and adherend where crack initiates and propagates along the interface, (2) adhesive failure within the adhesive, when the crack initiation and propagation are contained within the adhesive layer, and (3) crack initiation at joint edge due to peel stresses and its propagation in the adherend causing failure in the adherend [66].
2.9 Cited References


Chapter 3

Inelastic Deformation and Failure of Partially Strengthened Profiled Blast Walls

3.1 Introduction

This chapter studies a hybrid system of carbon-fibre-reinforced plastic (CFRP) retrofitted blast wall. Retrofitting is achieved with a centrally placed CFRP patch in the wall. The symmetrically strengthened region is enhanced with a CFRP patch, as this material is the most suitable type of composite for strengthening steel sections due to the similar elastic moduli of CFRP and steel, resulting in lower interfacial shear, and peeling stresses. An analytical model is developed to model the dynamic response of the hybrid system of strengthened blast wall. As the length of the strengthening part approaches zero, the model converges to the model developed by Langdon and Schleyer for an unstrengthened blast wall [1]. Numerical results obtained from a finite element model using Abaqus 6.9 for CFRP-strengthened blast walls are compared to the results of the proposed analytical model of the strengthened wall. The analytical model proposed for the CFRP strengthened blast wall serves as a rapid assessment tool for blast wall designers. It allows the design engineer to assess the potential influence of the CFRP-retrofitted patch on the dynamic response, and perform rapid assessments of multiple proposed designs (such as the influence of varying the length of the patch) [2]. The tool is based and is an improvement upon a previous analytical formulation for unstrengthened walls with realistic connection systems [1].

Damage in the composite patch is modelled using the Hashin damage model [3], and the numerical simulations showed that fibre fracture did not occur and that the small amounts of matrix failure occurring at the composite patch edges did not result in significant loss of in-plane stiffness and strength.

Most existing stainless steel blast walls are rated to a pressure of approximately 1 bar. However, some joint industry projects have shown the possibility of blast-induced overpressures as high as 4 bars [4]. Thus, it would be necessary to strengthen existing blast walls. The use of composites in strengthening metallic structures subject to blast loading is a recent innovation. Some of the reasons, which have made retrofitting using composites an attractive option for strengthening offshore structures, are corrosion resistance required in harsh environment, high specific strength, and stiffness (thus the ease of installation) and the elimination of welding when joining composites to metals. The last reason is particularly significant as welding of steel patches as an
alternative method of strengthening causes a fire hazard and requires platform shutdown, which is excessively costly. Some successful applications and potential utilisations of composites in the marine environment have been reviewed by Mouritz et al [5].

3.2. Analytical Modelling

3.2.1 Unstrengthened Blast Wall Modelling – Overview of the Approach

The analytical model proposed by Schleyer et al. [1, 6-7] is extended to the strengthened blast wall case. For complete details of the modelling approach, the reader is referred to reference [1] as the model is only described briefly here as a starting point. The structural model comprises various beam and spring elements [1] connected to form a continuous system that incorporates the blast wall profile response and the connection geometry. Each connection was idealised to consist of an elastic-plastic horizontal spring with strain hardening, an elastic-plastic rotational spring with strain hardening and an elastic vertical spring. Using this approach, shape functions with generalised displacements are used to represent the overall behaviour of the blast wall. Energy equations are developed using the assumed mode shapes, comprising total strain energy (flexural and membrane) and kinetic energy of the structural elements. A code was developed in Matlab to formulate the dynamic equilibrium equations in the various stages of motion of the beam, which represented the wall and the springs i.e. the connections. Characterisation of the connection geometry was performed experimentally in order to idealise its behaviour using spring elements [1, 6].
Figure 3.1a shows the schematic configuration of a blast wall on an offshore deck. The symbol $n$ represents the overall length of the blast wall panel and $m$ represents the overall length of the connections. The connections consist of an assembly of angles, which connect the blast wall panel to the offshore deck. Blast walls are designed for blast loads in the direction shown in Figure 3.1. Figure 3.1b shows the cross section of the blast wall panel. Figure 3.2 shows the direction of applied force used in determining the connection properties by Langdon and Schleyer [1]. When a tensile load is applied in direction 1, the connection is subjected to tension and bending causing it connection to open out. In the case of direction 2, the applied load causes the connection to be subjected to inward bending about the vertical angle in contact with the offshore deck.
Langdon and Schleyer [1] found that the analytical model captured the response of the blast walls reasonably well when correlated with experiments.

3.2.2 Strengthened Blast Wall Modelling – Overview of the Approach

The CFRP patch is assumed to behave as a rigid body and will prevent buckling of the blast wall in the central region. Hence, this model analytically required an extension of the approach adopted by Langdon and Schleyer [1]. The central region is represented by a rigid piece representing the CFRP, as shown in Figure 3.3 and Figure 3.4. The inclusion of a patch will also mean that the location of plastic hinges will be moved to the CFRP-steel interfaces at the ends of the patch and rigid-plastic rotational springs are incorporated at these locations, also shown in Figure 3.3. Another effect of modelling the CFRP patch as a rigid body is the assumption that no membrane stretching will occur in the retrofitted section. Thus, the structure is influenced by stretching in the unstrengthened part only [2].

Figure 3. 2: Use of springs to model connection response: (a) Direction 1: rotational spring and vertical spring, (b) Direction 2: horizontal spring
3.2.3 Strengthened Blast Wall Model Procedures

Equation (3.1) (assumed modes decomposition method) is used to describe the overall deformation of the structural member. Note the summation of coupled mode shapes represents the overall displaced figure of the beam element in all conditions. The symbol $C_i(t)$ represents the $i$-th generalised displacement of the system and $q_i(x)$ is an admissible function (shape function) corresponding to the $i$-th generalised coordinate similar to the $i$-th modal displacement.
The shape functions here are similar to those used by Langdon and Schleyer [1], adapted to include the CFRP patch influence. The analytical approach is a generalised one, and has various stages to account for the possibility of plasticity in the different spring elements. Figure 3.5 shows the various stages the beam goes through in the analytical model [2].

\[ w(x, t) = \sum_{i=1}^{n} q_i(x) \dot{C}_i(t) \]  

(3.1)

**Stage I**

![Diagram showing the stages of the beam](image-url)
Stage IIa

Figure 3.5: Assumed modes of beam response to uniform pressure loading
Stage I

Stage I, the first stage of motion, represents the elastic flexural deformations together with elastic or plastic membrane deformations without plastic hinge formation (no hinges at the supports or in the beam). The shape function for this stage is shown for \(0 \leq x \leq L_b\) by equation (3.2). The assumed mode equation below is basically a combination of the modal displacements of a fixed beam, a simply supported beam plus the vertical motion of the vertical spring (rigid body translation). The rotation at the support is controlled by the properties of the rotational support spring. The bending over the unstrengthened length, \(L_b\), influences the magnitude of rotation in the support.

\[
w_1(x, t) = \frac{C_1(t)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) + C_4(t) \quad 0 \leq x \leq L_b \tag{3.2}
\]

Thus, since region \(0 \leq x \leq \alpha L_b\) behaves as a rigid body, bending strain energy of the unstrengthened element \(L_b\) is given by equations (3.3a) and (3.3b).

\[
U_{be} = \int_0^{L_b} \frac{M^2}{2EI} \, dx = \frac{EI}{2} \int_0^{L_b} \left( \frac{d^2w}{dx^2} \right)^2 \, dx \tag{3.3a}
\]

\[
U_{be} = \frac{EI}{2} \int_0^{L_b} \left( -\frac{1}{2} \frac{C_1(t) \cos \left( \frac{\pi x}{L_b} \right) \pi^2}{L_b^2} - \frac{1}{4} \frac{C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) \pi^2}{L_b^2} \right)^2 \, dx
\]

\[
= \frac{1}{192} I E \pi (16C_1C_2 + 3C_2^2\pi + 12C_1^2\pi) \frac{L_b^3}{L_b^3} \tag{3.3b}
\]

The elastic strain energy of the support rotational spring element (with \(\theta < \theta_p\)) corresponds to elastic rotation prior to the formation of plastic hinges at the supports and is represented by \(U_{se}\), thus the rotation and elastic strain energy of the rotational spring are given by equations (3.4a) and (3.4b). Equation (3.4c) defines the angle of rotation when the moment reaches plastic yield moment at support.

\[
\theta = \left. \frac{\partial w}{\partial x} \right|_{x=0} = \int_0^{L_b} \left( \frac{d^2w}{dx^2} \right) \, dx = \int_0^{L_b} \left( -\frac{1}{2} \frac{C_1(t) \cos \left( \frac{\pi x}{L_b} \right) \pi^2}{L_b^2} - \frac{1}{4} \frac{C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) \pi^2}{L_b^2} \right) \, dx
\]

\[
= -\frac{1}{2} \frac{C_2\pi}{L_b} \tag{3.4a}
\]
The elastic membrane strain energy of the beam element is given by $U_{me}$. To calculate this energy, let $\Delta$ be the axial stretching of the beam length, $L_b$, during bending, $\delta_x$ the extension of the horizontal spring of stiffness, $K_{sx}$, due to the thrust from beam and $u$ the difference between the developed length and the projected length of the beam. Assuming that the axial deformation is linearly proportional to the thrust developed in the beam element and the thrust is constant along the entire unstrengthened length, the force, $F_x$, in the horizontal support spring is:

$$ F_x = \frac{\Delta AE}{L_b} = K_{xe} \delta_x $$

(3.5)

where the relation between $\Delta$, $\delta_x$, $u$ and $F_x$ is given by equations (3.6a) - (3.6d):

$$ \Delta = u - \delta_x $$

(3.6a)

$$ u = \frac{1}{2} \int_0^{L_b} \left( \frac{dw}{dx} \right)^2 dx $$

(3.6b)

$$ u = \int_0^{L_b} \left( -1 \frac{C_1(t)}{2} \sin \left( \frac{\pi x}{L_b} \right) \pi - \frac{1}{2} \frac{C_2(t)}{L_b} \sin \left( \frac{\pi x}{2L_b} \right) \pi \right)^2 dx $$

$$ = \frac{1}{48} \frac{\pi (16C_1(t)C_2(t) + 3C_2^2 C(t) \pi + 3C_1^2(t) \pi)}{L_b} $$

(3.6c)

$$ F_x = \frac{\Delta AE}{L_b} = \frac{1}{48} \frac{AE \pi (16C_1(t)C_2(t) + 3C_2^2 C(t) \pi + 3C_1^2(t) \pi)}{L_b^2 \left( 1 + \frac{AE}{K_{xe} L_b} \right)} $$

(3.6d)

Equilibrium dictates that the force on the horizontal spring is equal to the force on the beam (which is actually the force generated by the part of the beam that is unstrengthened)—see equation (3.5). The elastic membrane strain energy of the beam is given by equation (3.8) below.

$$ U_{me} = \frac{\Delta^2 AE}{2L_b} = \frac{1}{4608} \frac{\pi^2 (16C_1 C_2 + 3C_2^2 \pi + 12C_1^2 \pi) AE}{L_b^3 \left( 1 + \frac{AE}{K_{xe} L_b} \right)^2} $$

(3.8)
The elastic strain energy of the horizontal translational spring is given by \( U_{xe} \):

\[
U_{xe} = \frac{K_{xe}}{2} \delta_x^2 = \frac{K_{xe}}{2} \left( \frac{\pi (16C_1C_2 + 3C_2^2 \pi + 12C_1^2 \pi) AE}{L_b^2 \left( 1 + \frac{AE}{K_{xe}L_b} \right) K_{xe}} \right)^2
\]

\[
= \frac{1}{4608} \frac{\pi^2 (16C_1C_2 + 3C_2^2 \pi + 3C_1^2 \pi)^2 A^2 E^2}{K_{xe}L_b^4 \left( 1 + \frac{AE}{K_{xe}L_b} \right)}
\]

(3.9)

In instances where the spring yields before the beam element \( L_b \) the plastic strain energy in the horizontal spring element is given by:

\[
U_{xp} = R_{mx} (\delta_x - \delta_{px}) + \frac{K_{xp}}{2} (\delta_x - \delta_{px})^2
\]

\[
= R_{mx} \left( \delta_x - \frac{R_{mx}}{K_{xe}} \right) + \frac{K_{xp}}{2} \left( \delta_x - \frac{R_{mx}}{K_{xe}} \right)^2
\]

(3.10a)

\[
\delta_x = u = \frac{1}{48} \frac{\pi (16C_1(t)C_2(t) + 3C_2^2 C(t)\pi + 3C_1^2 (t)\pi)}{L_b}
\]

(3.10b)

The elastic strain energy in the vertical spring \( U_{ye} \) is as follows:

\[
U_{ye} = \frac{K_{ye}}{2} C_4^2
\]

(3.11)

The potential energy loss due to the pressure loading \( P(x, t) \) for the region, \( 0 \leq x \leq L_b \), which influences the motion of the system, is given by:

\[
PE = \int_0^{L_b} -P(x, t)w(x, t)dx
\]

\[
= \int_0^{L_b} -P(x, t) \left( \frac{C_1(t)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) \right.
\]

\[
+ C_4(t) \left. \right) dx
\]

(3.12)

From the law of conservation of energy, the total potential energy, \( V \), of the system is given by

\[
V = \sum U + PE \text{ where } \sum U = U_{be} + U_{me} + U_{xe} + U_{xp} + U_{ye}. \]

The total kinetic energy (which helps forming the terms of the mass matrix) of the beam on this stage is given by:
\[
T = \frac{1}{2} \int_0^{L_b} \rho A \dot{w}^2 dx = \frac{1}{2} \int_0^{L_b} \rho A \left( \frac{\dot{C}_1(t)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + \dot{C}_2(t) \cos \left( \frac{\pi x}{2L_b} \right) + \dot{C}_4(t) \right)^2 \ dx \quad (3.13)
\]

\[
= \frac{1}{48} \rho A L_b \left( 12 \dot{C}_2^2(t) \pi + 9 \dot{C}_1^2(t) \pi + 24 \dot{C}_1(t) \ddot{C}_4(t) + 64 \dot{C}_1(t) \dot{C}_2(t) + 24 \dot{C}_4^2(t) \pi + 96 \dot{C}_4(t) \ddot{C}_2(t) \right) \frac{1}{\pi}
\]

The motion of the beam in stage I is controlled by the generalised velocities \( \dot{C}_1, \dot{C}_2 \) and \( \dot{C}_4 \). The dynamic equilibrium equation for this stage is formulated by the Lagrange’s equation:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{C}_i} \right) + \frac{\partial V}{\partial C_i} = 0 \quad (3.14)
\]

The general conditions for stage I are: \( M_{sup} < M_0^o \) and \( M_{al} < M_0 \) and the terminating conditions for stage I are \( M_{sup} > M_0^o \) or \( M_{al} > M_0 \). Here it is understood that the operations and or have their logical implications. The rotation at the support is simply \( \phi \) and the moment \( M_{al} \) at the point of formation of plastic hinge within the beam is given by \( EI \) multiplied by the curvature of the beam at point O-see Figure 3.5. Equations (3.15) and (3.16) represent the support moment and moment in the beam at the point of formation of plastic hinge, respectively.

\[
M_{sup} = K_{pe} \left( -\frac{1}{2} \frac{C_2(t) \pi}{L_b} \right) \quad (3.15)
\]

\[
M_{al} = EI \left( -\frac{C_1(t) \pi^2}{4 L_b^2} - \frac{C_2(t) \pi^2}{8 L_b^2} \right) \quad (3.16)
\]

**Stage IIa**

This stage represents the elastic-plastic flexural deformation when plastic hinges have formed in the supports (but not at the CFRP/metal edge) and this is represented by the modal equation (3.17). The velocity field at this stage is shown by equation (3.18) which is obtained by differentiating equation (3.17):

\[
w_{IIa}(x,t) = \frac{C_1(t)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) + C_4(t) \quad 0 \leq x \leq L_b \quad (3.17)
\]

\[
\dot{w}_{IIa}(x,t) = \frac{\dot{C}_1(t)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + \dot{C}_2(t) \cos \left( \frac{\pi x}{2L_b} \right) + \dot{C}_4(t) \quad (3.18)
\]
Here $C_1$ in this stage is a constant term extracted from stage I. The rigid body motion represented by $C_4$ is unaffected by the phase change. The phase transition condition proposed by Symonds (called the Symonds minimum $\Delta_0$ technique) [8] based on the convergence theorem proposed by Martin [9] connects the velocity field from the end of stage I to state IIa. In this technique, the initial velocity magnitude of the mode describing the new stage is determined by the velocity field $v(x)$ at the end of the preceding phase. In equation (3.19), the new mode is written as $A_o\phi^*(x)$, where $\phi^*(x)$ is the shape function and $A_o$ denotes the amplitude at the start of this stage. $A_o^*$ and $A_o^{**}$ represent the components of the starting amplitude of velocity resulting from different modes and their summation yields the initial velocity of the phase. The specific density of the structure is represented by $\bar{\rho}$. The starting amplitude for state IIa is given by:

\[
A_0^{**} = \frac{\int_0^L \bar{\rho} v^* \sigma^2 \, dx}{\int_0^L \bar{\rho} \sigma^2 \, dx} = \frac{\int_0^{L_b} \bar{\rho} \frac{C_1(t_1)}{2} \left( 1 + \cos \left( \frac{\pi x}{L_b} \right) \cos \left( \frac{\pi x}{L_b} \right) \right) \, dx}{\int_0^{L_b} \frac{\bar{\rho}}{3} \left( \cos \left( \frac{\pi x}{L_b} \right) \right)^2 \, dx} = \frac{8 \dot{C}_1(t_1)}{3} \left( \frac{8 \dot{C}_1(t_1)}{\pi} + \dot{\dot{C}}_2(t_1) \right) = \frac{8 \dot{C}_1(t_1)}{\pi} + \dot{\dot{C}}_2(t_1)
\]

(3.19a)

\[
A_0^* = \frac{\int_0^L \bar{\rho} v^* \sigma \, dx}{\int_0^L \bar{\rho} \sigma^2 \, dx} = \frac{\int_0^{L_b} \bar{\rho} \dot{C}_2(t_1) \cos \left( \frac{\pi x}{L_b} \right) \cos \left( \frac{\pi x}{L_b} \right) \, dx}{\int_0^{L_b} \bar{\rho} \left( \cos \left( \frac{\pi x}{L_b} \right) \right)^2 \, dx} = \dot{\dot{C}}_2(t_1)
\]

(3.19b)

The plastic strain energy of the rotational spring element at the supports is given by equations (3.20a) and (3.20b). Since this stage represents formation of plastic hinges at the support:

\[
U_{op} = M_o^e \left( \theta_x - \theta_{px} \right) + \frac{K_{op}}{2} \left( \theta_x - \theta_{px} \right)^2
\]

(3.20a)

\[
U_{op} = M_o^e \left( \int_0^{L_b} -\frac{1}{2} \frac{C_1(t_1) \cos \left( \frac{\pi x}{L_b} \right) \pi^2}{L_b^2} - \frac{1}{4} \frac{C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) \pi^2}{L_b^2} \, dx - \frac{M_o^e}{K_{oe}} \right)
\]

\[
+ \frac{K_{op}}{2} \left( \int_0^{L_b} -\frac{1}{2} \frac{C_1(t_1) \cos \left( \frac{\pi x}{L_b} \right) \pi^2}{L_b^2} - \frac{1}{4} \frac{C_2(t) \cos \left( \frac{\pi x}{2L_b} \right) \pi^2}{L_b^2} \, dx - \frac{M_o^e}{K_{oe}} \right)^2
\]

(3.20b)
The general condition for stage IIa is: 
\[ M_{\text{sup}} = M_0 + K_{\text{sup}} (\phi - \phi_0) \text{ and } M_{\text{ul}} < M_0 \] 
and the terminating condition for state IIa is \( M_{\text{ul}} \geq M_0 \) at \( t = t_2 \). The time \( (t_2 - t_1) \), therefore, represents the total time for stage IIa. The equation of motion in this stage is developed with the Lagrange’s equation as in stage I. The kinetic energy of the beam throughout this stage is given by:

\[
T = \frac{1}{2} \int_0^L \rho A \dot{w}^2 \, dx = \frac{1}{2} \int_0^L \rho A \left( \dot{C}_2(t) \cos \left( \frac{\pi x}{2L_b} \right) + \dot{C}_4(t) \right)^2 \, dx
\]

(3.21)

**Stage IIb:**

This stage represents the elastic-plastic flexural deformations when plastic hinge forms in the beam at the ends of the CFRP before plastic hinge forms at the support. The shape function below approximates this stage

\[
w_{IIb}(x, t) = \frac{C_1(t_1)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + C_2(t_1) \cos \left( \frac{\pi x}{L_b} \right) + C_3(t) \left[ 1 - \frac{x}{L_b} \right] + C_4(t)
\]

(3.22)

The initial velocity field for this stage from Symonds [8] is:

\[
A_{0}^{**} = \frac{\int_0^L \dot{\rho} \omega \, dx}{\int_0^L \dot{\rho} \, dx} = \frac{\int_0^{L_b} \dot{\rho} \frac{C_1(t_1)}{2} \left( 1 + \cos \left( \frac{\pi x}{L_b} \right) \right) \left( 1 - \frac{x}{L_b} \right) \, dx}{\int_0^{L_b} \dot{\rho} \left( 1 - \frac{x}{L_b} \right)^2 \, dx} = \frac{4}{3} \frac{\dot{C}_1(t_1)(4 + \pi^2)}{\pi^2}
\]

(3.23a)

\[
A_{0}^{*} = \frac{\int_0^L \dot{\rho} \omega \, dx}{\int_0^L \dot{\rho} \, dx} = \frac{\int_0^{L_b} \dot{\rho} \frac{C_2(t_1)}{2} \cos \left( \frac{\pi x}{L_b} \right) \left( 1 - \frac{x}{L_b} \right) \, dx}{\int_0^{L_b} \dot{\rho} \left( 1 - \frac{x}{L_b} \right)^2 \, dx} = \frac{12}{\pi^2} \dot{C}_2(t_1)
\]

(3.23b)

\[
\dot{C}_3 = \frac{4}{3} \frac{\dot{C}_1(t_1)(4 + \pi^2)}{\pi^2} + \frac{12}{\pi^2} \dot{C}_2(t_1)
\]

(3.23c)

The motion of the beam in this stage is controlled by the generalised velocity \( \dot{C}_3 \) with starting amplitude \( \dot{C}_3 \) (\( \dot{C}_1 \) and \( \dot{C}_2 \) have no influence on the motion of the beam at this stage). Thus, the generalised displacement \( C_3 \) is the variable term and the generalised displacements \( C_1 \) and \( C_2 \) are treated as constant terms inherited from stage I above. The expression for strain energy follows the assumption as before. The spring in the beam has yielded, thus, plastic strain energy of the rotational spring at the point of formation of plastic hinge within the beam is given by:

\[
U_\theta = M_\theta \theta + \frac{1}{2} K_\theta = M_\theta \left( \frac{C_3(t)}{L_b} \right) + \frac{1}{2} K_\theta \left( \frac{C_3(t)}{L_b} \right)^2
\]

(3.24)
The kinetic energy of the beam on this stage is given by:

$$T = \frac{1}{2} \int_0^{L_b} \rho A \dot{w}^2 \, dx = \frac{1}{2} \int_0^{L_b} \rho A \left( \dot{C}_3(t) \left[ 1 - \frac{x}{L_b} \right] + \dot{C}_4(t) \right)^2 \, dx$$  \hspace{1cm} (3.25)

The general condition for stage IIb is: \( M_{sup} < M_0' \), \( M_{al} = M_\theta + K_\theta \theta \). The terminating condition for stage IIb is \( M_{sup} \geq M_0' \) at time \( t = t_2 \). Again, \((t_2 - t_1)\) is the total time for stage IIb. The equation of motion in this stage is developed with the Lagrange’s equation as in stage I.

**Stage III:**

This stage represents the elastic-plastic flexural deformations when a plastic hinge has been formed in the beam and at the supports and this is approximated by equation (3.26).

$$w_{III}(x,t) = \frac{C_1(t_1)}{2} \left[ 1 + \cos \left( \frac{\pi x}{L_b} \right) \right] + C_2(t_2) \cos \left( \frac{\pi x}{2L_b} \right) + C_3(t) \left[ 1 - \frac{x}{L_b} \right]$$  \hspace{1cm} (3.26)

The initial velocity field for this stage from Symonds [8] is:

$$A_0' = \frac{\int_0^L \rho \dot{\nu} \dot{\theta}^* \, dx}{\int_0^L \rho \theta^* \, dx}$$ \hspace{1cm} (3.27a)

$$A_0' = C_3 = \frac{\int_0^L \rho \nu \dot{\theta}^* \, dx}{\int_0^L \rho \theta^* \, dx} = \frac{\int_0^L \rho \frac{C_2(t_2)}{2} \cos \left( \frac{\pi x}{L} \right) \left( 1 - \frac{x}{L} \right) \, dx}{\int_0^L \rho \left( 1 - \frac{x}{L} \right)^2 \, dx} = \frac{12}{\pi^2} \hat{C}_2(t_2)$$ \hspace{1cm} (3.27b)

The motion of the beam in this stage is controlled by the generalised velocity \( \dot{C}_3 \) with starting amplitude \( \dot{C}_3 \) (\( \dot{C}_1 \) and \( \dot{C}_2 \) treated as constant terms from the previous stages). The expressions for strain energy are similar to the procedure in stages I, IIa and IIb above. Similarly, the generalised displacement \( C_3 \) is treated as a variable term and the generalised displacements \( C_1 \) and \( C_2 \) are treated as constants extracted from stage I and stage IIa or IIb.

The general condition for stage III is: \( M_{sup} = M_0' + K_\theta (\theta - \theta_0) \), \( M_{al} = M_\theta + K_\theta \theta \). The time \((t_3 - t_2)\) represents the total time for stage III. The terminating condition for stage III is reached when the sum of the generalised velocities equals zero, at which point the beam is assumed to respond in an elastic manner with residual deformation. Thus, zero velocity condition in the system will trigger the beam in stages IIa and IIb or III to enter into an elastic rebound state.
The kinetic energy (this forms the terms of the mass matrix) of the beam on this stage is given by:

\[
T = \frac{1}{2} \int_0^{L_b} \rho A \dot{w}^2 dx = \frac{1}{2} \int_0^{L_b} \rho A \left( \dot{C}_3(t) \left[ 1 - \frac{x}{L_b} \right] + \dot{C}_4(t) \right)^2 dx
\]  

(3.28)

The equation of motion in this stage is developed with the Lagrange’s equation as in stage I.

### 3.2.4 Application of the Models to Blast Wall Response

Blast walls are connected to the primary framework of a topside module in a particular orientation as shown in Figure 3.1 (known as the design direction). Langdon and Schleyer [1, 10-11] have shown that the wall response is superior in the design direction. All results reported herein are for the design direction of the blast wall only. After validation of the unstrengthened model, the springs used to model the current connection geometry in the analytical models were derived from a numerical finite element static Riks analysis conducted in Abaqus.

#### 3.2.4.1 Connection Characterisation

In order to determine the force-displacement and moment-rotation curves used to calculate the spring constants and resistance for the supports in the analytical model, a finite element procedure is adopted to depict similar experimental test carried out by Langdon and Schleyer [10].

Figures 3.6 (a) and (b) show the model of the support connection in Abaqus and its configuration respectively. A Static Riks Analysis\(^4\) is performed in Abaqus to determine the resistance functions of the springs used to depict the connection in the analytical model. The displacement \(\Delta x\) for the horizontal spring, rotation for the support rotational spring and the vertical displacement \(\Delta y\) for the vertical spring are measured in the FEA model in the orientation shown in Figure 3.2. The obtained resistance curves for the horizontal spring; support rotational spring and support vertical spring are shown in Figures 3.8(a)-(c). Table 3.1 shows the spring parameters (resistance curve parameters). The stiffness values of rotational spring, \(K_\theta\) in Table 3.1 represent the varying stiffness for the rigid plastic spring. As shown in equation 3.29, at the outset of bending in the rigid plastic spring, \(K_\theta = 238.6/2L_h = 6061\) Nm/rad while for the

---

\(^4\) Riks Analysis is a special method to capture the behaviour after the instability. Other static analysis types may fail to capture the non-linear force-displacement curve after a point of instability is reached. As the analysis proceeds uniformly, the displacements are incremented at a uniform rate, and the force required to cause the displacement are calculated. The calculated forces are usually nonlinear with displacement (or time). This load then creates a displacement, and this displacement is incremented at each step of the analysis.
unstrengthened model presented by Langdon and Schleyer $K_0 = \frac{238.6}{L_h}$ =12122Nm/rad [1]. Where $2L_h$ represents the length of the plastic hinge and $L_h$ is the thickness of an equivalent rectangular section to the panel, which is equal to 19.68mm.

Figure 3. 6: (a) FE model of the blast wall connection in Abaqus (b) Schematic configuration of the blast wall connection
Figure 3. 7: (a) Riks Analysis output showing opening out of 4mm thick angle under direction 1 loading (b) Riks Analysis output showing inward bending of connection under direction 2 loading
Figure 3.8: (a) Force-displacement curve for the support horizontal spring (b) Moment-rotation curve for the support rotational spring (c) Force-displacement curve for support vertical spring

Table 3.1: Mechanical properties of idealised springs

<table>
<thead>
<tr>
<th>Properties</th>
<th>Strengthened Model</th>
<th>Unstrengthened Model</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_\theta$</td>
<td>6061**, 3031*</td>
<td>1.21E4**, 6061*</td>
<td>Nm/rad</td>
</tr>
<tr>
<td>$K_{xe}$</td>
<td>259.7</td>
<td>259.7</td>
<td>kN/m</td>
</tr>
<tr>
<td>$K_{xp}$</td>
<td>37</td>
<td>37</td>
<td>kN/m</td>
</tr>
<tr>
<td>$K_y$</td>
<td>7500</td>
<td>7500</td>
<td>kN/m</td>
</tr>
<tr>
<td>$K_{\theta e}$</td>
<td>1311</td>
<td>1311</td>
<td>Nm/rad</td>
</tr>
<tr>
<td>$K_{\theta p}$</td>
<td>87</td>
<td>87</td>
<td>Nm/rad</td>
</tr>
<tr>
<td>$M_{\theta p}$</td>
<td>160</td>
<td>160</td>
<td>Nm</td>
</tr>
<tr>
<td>$R_{xy}$</td>
<td>850</td>
<td>850</td>
<td>N</td>
</tr>
</tbody>
</table>

*Rotational spring plastic stiffness at the onset of membrane phase
**Rotational spring stiffness at the start of bending
Buckling process in the beam

Plasticity in the corrugation occurs at the ends of the CFRP patch in the beam, where plastic hinges are formed. This phenomenon is controlled by the plastic hinge capacity. It is assumed in this model that buckling is initiated in the plastic response regime of the structure. Thus, once the bending moment at the end of the CFRP patch of the beam exceeds the plastic moment, $M_\theta$, plastic hinges form and non-zero rotation at their location is expected. Buckling sets in within the beam when the moment in the beam exceeds the moment capacity, $M_{\theta_p}$. The post-yield regime is characterised by a varying plastic stiffness, $K_\theta$. In calculating $K_\theta$, the additional moment, $M$, carried by the section once $M_\theta$ is exceeded at the point of formation of plastic hinge within the panel is evaluated by equation (3.29).

$$M = \int_0^{z_{eaa}} (\sigma_0 + E_h z k) A dz = M_\theta + 238.609 * k = M_\theta + 238.609 * \frac{\theta}{2L_h}$$

$$= M_\theta + K_\theta \theta$$

(3.29)

The buckling process is intricate and its correct treatment requires a rigorous mathematical model to be proposed. For the sake of analyses here, however; it is assumed that buckling occurs once the moment goes beyond $M_{\theta_p}$. In the simplified model here, $M_{\theta_p}$ is chosen to be 5% higher than $M_\theta$. Equation (3.29) corresponds to the moment for a singly symmetric section. The term $\sigma_0$ represents the yield stress, $E_h$ represents the hardening modulus, $A$ the area of the cross section, $z$ the distance of element and $z_{eaa}$ is the distance from the equal area axis. The term $(\sigma_0 + E_h z k)$ represents the flow stress, where $k$ is the curvature at the point of formation of plastic hinge (which is the end of the CFRP patch). The curvature of the beam at the point of formation of plastic hinge within the beam is given by equation (3.30) where $2L_h$ is the total plastic hinge length at the point of formation of plastic hinge (incipient hinge formation) and $\theta$ is the beam element rotation at that point. Figure 3.9 shows the formation of plastic hinges at the support and at end of the CFRP patch[2]. The relationship between the plastic stiffness, $K_\theta$, rotation, $\theta$, moment capacity of the section, $M_\theta$ and the total additional moment, $M$ carried by the section during the buckling process can be seen in equation (3.29).

$$\kappa = \frac{\theta}{2L_h}$$

(3.30)
According to Jones [12], the parameter $L_h$ is assumed to be equal to the thickness of the beam during the bending phase of deformation. Blast wall panels are corrugated sections, thus and effective thickness, $t_{eq}$, is used to represent the thickness of the wall in the equivalent beam model used in the analytical model. The parameter is obtained by equating the global second moment of area ($I$) of the actual cross-section to that of an equivalent rectangular section with the same magnitude of second moment of area ($I$) as shown in equation (3.31). In the buckling analysis of the model, $L_h$ and $K_\theta$ are functions of time (or of deformation). Since buckling initiates in the plastic regime after elastic bending phase, the plastic hinge length is assumed to vary linearly from $t_{eq}$ (equivalent thickness of panel) to $2t_{eq}$ as stress state varied from bending to tensile stresses.

$$I = \frac{B t_{eq}^3}{12} = 1.4 \times 10^{-7} m^4 ; B = 0.22m$$  (3.31)

In order to fully depict the nonlinear moment rotation relationship after moment capacity at the point of formation of plastic hinge is exceeded, the plastic stiffness, $K_\theta$ was decreased linearly with increasing plastic displacement ($C_3$) up to $C_3 = t_{eq}$, and then assumed constant at 3031Nm/rad thereafter.

At the start of bending $K_\theta = 6061 \text{ Nm/rad}$ (when $C_3 = 0$)  

At the onset of membrane phase $K_\theta = 3031 \text{ Nm/rad}$ (when $C_3 = t_{eq}$)  

Figure 3.9: Formation of plastic hinge in the beam and at the support
3.3. Numerical Modelling

3.3.1 Finite Element Modelling of a Blast Wall

An initial linear perturbation analysis was performed to obtain the natural frequencies and corresponding mode shapes of the structures under consideration in Abaqus 6.9-1. A strengthened panel and its unstrengthened counterpart were modelled in Abaqus. The thickness of the panel is 2mm. The unstrengthened and strengthened FE models consist of 3048 and 3772 S4R linear quadrilateral reduced integration shell elements, respectively. The additional 724 elements are a result of the CFRP patch. Figure 3.10 shows the finite element model of the strengthened blast wall panel.

The middle one-third of the beam was strengthened in this model with CFRP. The composite patches were tied to the panels using the Tie feature of Abaqus thus eliminating any possibility of debonding of the patches from the panel. This is a restriction in capturing the actual failure modes, however; it can be imagined that a mixture of adhesive bonding and edge bolting can in reality satisfy this condition.

The connections where fully fixed, thus, modelling the fixity of blast wall connecting angles to the upper and lower decks of a platform. The edges of the panel were fixed in the global $U1$, $UR2$ and $UR3$ directions. This was to make the model behave as a continuous panel. Various triangular pulses ranging from 0.5 to 4 bars in amplitude and with rise time of 40 ms and duration of 100 ms (much greater than the average natural period of the models which averaged at 8 ms) were applied in the design direction of the model. Figure 3.11 shows the profile of the applied blast load. This was considered to be representative of an offshore hydrocarbon gas explosion, and is similar to the loading profiles explored in work of Langdon and Schleyer [1, 10, 13].
3.3.2 Material Characterisation

The material stress-strain curve inputted into the numerical model was generated using the Rasmussen model also called the modified Ramberg-Osgood model [14]. This model uses equation (3.34) which explicitly gives the direct nominal strain as a function of nominal stress. These are then converted to true stress vs. true strain curves as Abaqus requires. Table 3.2

![Finite Element Model of blast wall](image)

Figure 3. 10: Finite Element Model of blast wall

![Temporal profile of applied blast overpressure](image)

Figure 3. 11: Temporal profile of applied blast overpressure
provides the input for equation (3.34). The material properties of the wall are that of AISI316L, obtained from Rasmussen’s work [14]. The nominal true stress-strain conversion for 316L stainless steel was conducted using the standard equations and the results are shown in Figure 3.12 as true values were used in the Abaqus model. Table 3.3 shows the geometric properties of the blast wall section.

Table 3.2: Material properties of AISI316L [14]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Young Modulus $E_0$</td>
<td>2.0x10^5</td>
<td>N/mm$^2$</td>
</tr>
<tr>
<td>$\sigma_{0.2}$</td>
<td>316</td>
<td>N/mm$^2$</td>
</tr>
<tr>
<td>$n$</td>
<td>5.88</td>
<td>-</td>
</tr>
<tr>
<td>Ultimate stress $\sigma_u$</td>
<td>616</td>
<td>N/mm$^2$</td>
</tr>
<tr>
<td>$m$</td>
<td>2.8</td>
<td>-</td>
</tr>
<tr>
<td>$\varepsilon_u$</td>
<td>0.487</td>
<td>-</td>
</tr>
<tr>
<td>$E_{0.2}$</td>
<td>23541</td>
<td>N/mm$^2$</td>
</tr>
<tr>
<td>$\varepsilon_{0.2}$</td>
<td>0.00366</td>
<td>-</td>
</tr>
</tbody>
</table>

$$
\varepsilon = \begin{cases}
\frac{\sigma}{E_0} + 0.002 \left( \frac{\sigma}{\sigma_{0.2}} \right)^n & \sigma \leq \sigma_{0.2} \\
\left( \frac{\sigma - \sigma_{0.2}}{E_{0.2}} \right) + \varepsilon_u \left( \frac{\sigma - \sigma_{0.2}}{\sigma_u - \sigma_{0.2}} \right)^m + \varepsilon_{0.2} & \sigma > \sigma_{0.2}
\end{cases}
$$

(3.34)

Figure 3.12: Stress strain curve for AISI316L

0 0.05 0.1 0.15 0.2
0 100 200 300 400 500 600 700
Stress (N/mm$^2$) Strain
Nominal Stress-Strain True Stress-Strain
### Table 3.3: Geometric properties of blast wall model, from [4]

<table>
<thead>
<tr>
<th>Property</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of wall</td>
<td>915mm</td>
</tr>
<tr>
<td>Breadth of wall</td>
<td>220mm</td>
</tr>
<tr>
<td>Second moment area of cross section</td>
<td>$1.40 \times 10^{-7} \text{m}^4$</td>
</tr>
<tr>
<td>Position of neutral axis (relative to the tension flange)</td>
<td>17.27 cm</td>
</tr>
<tr>
<td>Equal area axis position (relative to the tension flange)</td>
<td>12.37 mm</td>
</tr>
<tr>
<td>Fully plastic moment capacity of section $M_\theta$</td>
<td>2273.40Nm</td>
</tr>
<tr>
<td>Young’s Modulus of steel $E$</td>
<td>200GPa</td>
</tr>
<tr>
<td>Hardening Modulus $E_h$</td>
<td>1.25GPa</td>
</tr>
</tbody>
</table>

### 3.3.3 Carbon Fibre Reinforced Plastics (CFRP) Modelling

#### 3.3.3.1 Constitutive modelling of composite patch

A ply in a composite laminate shows linear elastic behaviour before failure under load. Here, the term failure is defined as a complete separation or fracture of the laminate resulting in an inability to support loads acting on it. Thus, in the model adopted for composite failure in this chapter, failure results in the degradation of particular stiffness associated with the failure mode.

The composite patch stack used in this model is such that fibres in each lamina are in alternating orientations of 0 and 90 degrees. The laminates are assumed to be bonded together with the same matrix material. The modelled composites assume the basic behaviour of laminates which are: perfect bonding exists between fibres and matrix; fibres are parallel and uniformly distributed throughout; the matrix is free of voids or micro-cracks and in a stress free state; both fibres and matrix are isotropic and obey Hooke’s law; and the applied loads are either parallel or perpendicular to the fibre direction.

The constitutive modelling of composite consists of two phases. The first is the linear elastic phase, which terminates once the failure criterion (criteria) is reached. The second phase is the post failure for the particular damage mode.
The Hashin damage initiation criteria (for shell elements in Abaqus) are used to initiate the damage. Here, a uniaxial fibre-reinforced ply can be modelled as homogenous linear elastic transversely isotropic body with the fibre axis normal to the plane of transverse isotropy. Fibre, matrix, and shear failure are initiated when their respective criteria in equations (3.35a) – (3.35e) are satisfied:

Fibre tensile/compressive failure

\[
\left(\frac{\sigma_{11}}{X_T}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 \leq 1 \quad \text{for } \sigma_{11} > 0
\]

(3.35a)

\[
\left(\frac{\sigma_{11}}{X_C}\right)^2 \leq 1 \quad \text{for } \sigma_{11} < 0
\]

(3.35b)

Matrix tensile/compressive failure

\[
\left(\frac{\sigma_{22}}{Y_C}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 \leq 1 \quad \text{for } \sigma_{22} < 0
\]

(3.35c)

\[
\left(\frac{\sigma_{22}}{Y_T}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 \leq 1 \quad \text{for } \sigma_{22} > 0
\]

(3.35d)

Fibre-matrix shear failure

\[
\left(\frac{\sigma_{11}}{X_T}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 \leq 1 \quad \text{for } \sigma_{33} < 0
\]

(3.36e)

It is a normally accepted convention for a ply to make direction 1 the material principal axis to the fibre direction, direction 2 to the in-plane normal to direction 1 and direction 3 to the normal to direction 1 which lies in the direction of the thickness. There are 5 independent parameters in the material elastic constitutive relations corresponding to modulus in direction 1 ($E_1$), modulus in direction 2 ($E_2$), shear modulus in the 1-2 plane ($G_{12}$) and major and minor Poisson ratios ($\nu_{12}$) and ($\nu_{23}$), respectively. Laminates are defined by longitudinal strength $X$, transverse strength $Y$ and shear $S$. These properties are obtained experimentally [15]. Tables 3.4 and 3.5 show the mechanical properties of a typical CFRP ply and the geometrical properties and through-thickness architecture of the laminate, respectively. The subscripts $t$ and $c$ represent tension and compression respectively [2].
Table 3. 4: Mechanical properties of CFRP patch [16]

<table>
<thead>
<tr>
<th>Property</th>
<th>Graphite-Epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>207GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>5GPa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.25</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>2.6GPa</td>
</tr>
<tr>
<td>$X_t$</td>
<td>1000Mpa</td>
</tr>
<tr>
<td>$Y_t$</td>
<td>100MPa</td>
</tr>
<tr>
<td>$S$</td>
<td>250MPa</td>
</tr>
<tr>
<td>$X_c$</td>
<td>700MPa</td>
</tr>
<tr>
<td>$Y_c$</td>
<td>100MPa</td>
</tr>
</tbody>
</table>

Table 3. 5: Composite layup

<table>
<thead>
<tr>
<th>Ply Material</th>
<th>Orientation of Ply (degrees)</th>
<th>Thickness of Ply (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>CFRP</td>
<td>90</td>
<td>2</td>
</tr>
<tr>
<td>CFRP</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

A VUSDFLD subroutine, which defines field variables (FV) at material points, is used to model the damage. These field variables can be used to introduce solution-dependent material properties (since such properties can be easily defined as a function of field variables). In this subroutine, the field variable takes up the state variable from the Hashin damage initiation criteria. Consequently, a field variable of value 1 indicates failure at its related failure mode. Output of the user-defined field variables at the material points can be obtained with the element integration point output. Before user subroutine VUSDFLD is called, the values of the field variables at the material point are calculated by interpolation from the values defined at the nodes. Table 3.6 shows the degradation rule for the stiffness of the ply once each of the failure modes is reached adopted by the subroutine.

Table 3. 6: Property degradation rules for the composite patch

<table>
<thead>
<tr>
<th>Failure modes</th>
<th>FV1</th>
<th>FV2</th>
<th>FV3</th>
<th>E1</th>
<th>E2</th>
<th>$\nu_{12}$</th>
<th>$G_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No failure</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td>$\nu_{12}$</td>
<td>$G_{12}$</td>
</tr>
<tr>
<td>Matrix</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$E_1$</td>
<td>0</td>
<td>$\nu_{12}$</td>
<td>$G_{12}$</td>
</tr>
<tr>
<td>Fibre</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$G_{12}$</td>
</tr>
<tr>
<td>Shear failure</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

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3.4. Results

The results of the analytical analyses were compared with the output of the numerical model. The analytical model for the strengthened and unstrengthened cases showed good correlation with the Abaqus 6.9-1 models for the strengthened and unstrengthened models for the maximum displacement, respectively.

Figures 3.13-3.17 show the comparison of the midpoint displacements predicted by Abaqus for an unstrengthened model (i.e. before strengthening patched was fixed to the panel) and the analytical solutions predicted by the analytical model proposed by Langdon and Schleyer [1] using connection properties shown in Table 3.1. Thus, in the limit as the length of the strengthening patch approaches zero the strengthened panel behaves analogous to the unstrengthened panel.

Figures 3.18-3.22 represent the comparison between the Abaqus predictions for a strengthened model and the modified analytical solution for the strengthened walls. It was also observed that when the thickness of the patch was increased from its original 6mm there was no change in the lateral displacement time history of the panel predicted by Abaqus. This confirms the earlier assumption made in the analytical mode i.e. as long as the strengthened part is sufficiently enhanced the response of the panel is influenced by the unstrengthened part only. Table 3.7 shows the comparison between the maximum displacement predicted by the analytical and numerical models for the strengthened and unstrengthened walls[2].

![Image of numerical and analytical displacement time history curves for unstrengthened model under 4 bar blast load](image_url)

Figure 3.13: Numerical and analytical displacement time history curves for unstrengthened model under 4 bar blast load
Figure 3. 14: Numerical and analytical displacement time history curve for unstrengthened model under 3 bar blast load

Figure 3. 15: Numerical and analytical displacement time history curve for unstrengthened model under 2 bar blast load
Figure 3.16: Numerical and analytical displacement time history curve for unstrengthened model under 1 bar blast load

Figure 3.17: Numerical and analytical displacement time history curve for unstrengthened model under 0.5 bar blast load
Figure 3.18: Numerical and analytical displacement time history curve for CFRP strengthened model under 4 bar blast load

Figure 3.19: Numerical and analytical displacement time history curve for CFRP strengthened model under 3 bar blast load
Figure 3. 20: Numerical and analytical displacement time history curve for CFRP strengthened model under 2 bar blast load

Figure 3. 21: Numerical and analytical displacement time history curve for CFRP strengthened model under 1 bar blast load
Figure 3. 22: Numerical and analytical displacement time history curve for strengthened model under 0.5 bar blast load

Table 3. 7: Summary of maximum transverse displacements

<table>
<thead>
<tr>
<th>Load (bar)</th>
<th>Unstrengthened (mm)-Abaqus</th>
<th>Unstrengthened (mm)-Analytical</th>
<th>Strengthened (mm)-Abaqus</th>
<th>Strengthened (mm)-Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>142</td>
<td>145</td>
<td>102</td>
<td>102</td>
</tr>
<tr>
<td>3</td>
<td>125</td>
<td>125</td>
<td>88</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>98</td>
<td>58</td>
<td>60</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>10</td>
<td>6.5</td>
<td>7.0</td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>4</td>
<td>3.3</td>
<td>3.4</td>
</tr>
</tbody>
</table>

The somehow weaker correlation in the maximum displacements in Figures 3.16 and 3.21 is a result of the transition of elastic behaviour into plasticity. The graphs show predictions of elastic response with resultant small plastic deformation, which is about 1mm for the unstrengthened panel and less than 1mm for the strengthened panel. In this transition regime, acute sensitivity to loading, structural geometry and material behaviour affects the results predicted by finite element programs. This phenomenon was observed by researchers (e.g. by Symonds and Yu [17]). The surprisingly different frequency predicted by the numerical and analytical model subsequent to the point maximum displacement has been reached is a result of including a limited number of modes as well as the sensitivity of the structure to chaos and perturbation.
This effect apparently exacerbates the inherent mathematical sensitivities normally present in computational structural mechanics codes, so that small changes in solution strategy may lead to large changes in the output of various finite element code [17]. That same effect is expected to exhibit itself in analytical model which is necessarily a discrete parameter model.

The degradation of the composite patch was also investigated in the strengthened scheme and there was no evidence of fibre damage which is the detrimental failure mode. However, significant matrix failures were observed within the beam, at the points where the plastic hinges are formed (see Figures 3.23 and 3.24). Figures 3.25 - 3.28 show the strains and stresses in the connections and points of formation of plastic hinges within the panel. The maximum strains at the connections in the strengthened scheme were found to be much lower than that in the unstrengthened schemes for small overpressures. However, as the overpressures increased, causing inelastic behaviour and subsequent membrane effects in the strengthened scheme, the strain levels in the connections grow almost identical [2].

Lastly, it can be observed that the maximum transverse displacement occurs at about 45ms. Thus, the response of this system can be described as quasi-static (i.e. the structure reaches its maximum displacement before the blast load has undergone any significant decay).

Figure 3.23: Panel under 4 bar blast load showing buckling in the beam
Figure 3.24: Panel under 3 bar blast load showing buckling in the beam

Figure 3.25: Strengthened panel under 4 bar showing strains: show location of maximum strains (half span)
Figure 3.26: Unstrengthened panel under 4 bar, showing location of maximum strains (half span)

Figure 3.27: Strengthened panel under 3 bar, showing location of maximum strains (half span)
3.5. Conclusion

This work presents a quick assessment tool for the prediction of the dynamic response in partially strengthened blast walls. From the preliminary FE analyses, it is observed that the strengthened scheme was able to absorb more blast energy than the unstrengthened scheme. This is achieved by the formation of two symmetrically placed plastic hinges within the panel at a finite distance as compared to the formation of a single central plastic hinge for the unstrengthened scheme. The multiplicity of plastic hinges is a result of a centrally placed patch constituting an approximate rigid body in the patched region thereby preventing the formation of a plastic hinge at the mid span of the strengthened wall hence pushing the hinges to the ends of the CFRP patch. The strengthened concept gave reduced mid-span displacements and strains at the connections (i.e. reduced connection pull-in). While the sufficiently strengthened patched area behaves rigidly the bending and deformation of the un-patched area governs the rotation at the support and the overall behaviour of the beam. In the proposed analytical model, it is postulated that the strengthened blast wall tries to utilise the full ductility of the unstrengthened effective length, $L_b$. 

Figure 3. 28: Unstrengthened panel under 3 bar, showing location of maximum Strains (half span)
The results obtained by the analytical model were corroborated with FE. A good correlation was achieved for the maximum displacements for strengthened and unstrengthened blast walls with the idealised connection properties extracted from the modified Riks algorithm implemented in Abaqus. The maximum inelastic deflections predicted by Abaqus remained unchanged no matter how much the thickness of the strengthening patch is increased once sufficient strengthening had been achieved. This confirms the rigid body behaviour of the strengthened region as assumed in the analytical modelling. The perfectly trapezoidal shape of the deformed blast wall panels shows the behaviour of the wall when it has been sufficiently strengthened [2].

Based on the findings of the present study, the energy absorption capacity of the strengthened wall is more than of its unstrengthened counterpart. This implies that for a given impulse, amplitude or a combination of the two i.e. a pulse of specific magnitude and duration the energy absorbed is the same but at the cost of less deflection ensued in the strengthened scheme. In other words, for a given maximum displacement, the blast energy absorption capacity of the strengthened beam exceeds that of the unstrengthened scheme.

In this work, it has been observed that if possible failure modes such as debonding of the composite patch(es) from the steel beam/plate, tearing (rupture) in the steel beam/plate, fibre breakage or buckling in the FRP laminate, delamination of plies in the CFRP laminate and in-plane shear failure of a lamina do not occur, the deformation and ductility of the unstrengthened part of the blast wall governs the behaviour of the blast wall, thus, reducing the in-plane forces exerted on the connections and reducing overall displacement of the plate. It can be inferred from the behaviour observed in the strengthened scheme that strengthening the connections will also reduce displacements at the connections and therefore result in a reduced overall deformation. This is, however, at the cost of larger forces being exerted to the interface between the panel and connection. The allowable degree of strengthening in the connections is therefore contingent upon the level of strains developed at this critical point [2].

One final remark on strain-rate effects is in order. While most metallic structures exhibit strain-rate sensitive behaviour (visco-plasticity) when subjected to high rate loading these effects are neglected in the current work. These effects are local as the strain field has a temporal and spatial variation and the point of maximum strain rate corresponds to that of high deformation gradient. Unless an ad hoc model is proposed for a specific structural configuration and loading profile these effects are extremely difficult, if not impossible, to include in an analytical model. The finite element simulation is, however, capable of taking into account these effects with reasonable effectiveness.
It is worth mentioning that the fact that strain-rate effects were disregarded in the procedure presented in this chapter does not limit its usefulness. Langdon and Schleyer [18] reported that the moderate strain rate sensitivity in the yield region of stainless steel is too complex to be modelled accurately by a constitutive equation such as Cowper-Symonds. They concluded that the whole stress-strain curve, at a given strain rate, could be accurately described by a few (true stress, logarithmic stain) coordinate pairs as linear hardening was observed. Such procedure is in line with the procedure presented in this chapter. In addition, the design strength of stainless steel or any metal, as the case might be, can be further enhanced when blast loading is considered to a value, \( f_y^* \) (from \( f_y \)) to take advantage of the improvement in strength due to high strain rates. Such enhancement can be easily implemented in the scheme presented in this chapter [19-20].
### 3.6 Cited References


Chapter 4

Pressure-Impulse Diagrams for Blast Loaded Continuous Beams Based on Dimensional Analysis

4.1. Introduction

This chapter introduces and investigates a simplified system used in the analysis of continuous beams. The model adopted in this work to replicate a continuous beam is a single span beam with semi-rigid connections. A simplified analytical solution is presented and the maximum elastic displacement compared with the maximum elastic displacement predicted by the numerical and analytical models for the blast wall in Chapter 3. As a result of the simplification and assumptions made in the model presented in this chapter, it cannot model support shear hinges, membrane effects in the beam and connection pull-in at the supports. However, during the elastic response of the blast wall, when membrane effects are minimal, the maximum displacements predicted by the simplified model in this chapter correlates well with the maximum displacement for an unstrengthened blast wall presented in Chapter 3. A simple FE model (with minimal membrane effects) is developed to model a continuous beam with flexible supports and the maximum displacements predicted by the model correlates well with the maximum displacement predicted by the analytical model presented in this chapter.

Consequently, a traditional pressure-impulse (for pulses with finite rise time) diagram for the blast wall in Chapter 3 is developed. Pressure-impulse diagrams are commonly used in preliminary blast resistant design to assess the maxima of damage related parameter(s) in different types of structures as a function of pulse loading parameters. It is well known that plastic dynamic response of elastic-plastic structures is profoundly influenced by the temporal shape of applied pulse loading [1-3].

With the maximum structural deflection, being the controlling criterion for damage, pressure-impulse diagrams for high explosives (pulses with zero rise time) are developed in this chapter. Dimensionless parameters are introduced to develop a unique pulse-shape-independent pressure-impulse diagram for elastic and elastic-plastic responses for this type of pulse [4]. The beam is modelled as a single span with symmetrical semi-rigid support conditions. The rotational spring can assume different stiffness values ranging from 0 (zero) to $\infty$ (infinity). Analytical solutions for evaluating displacement time histories of the semi-rigidly supported (continuous) beam
subjected to pulse loads, which can be extendable to very high frequency pulses, is presented in this chapter.

4.2. Background on Pressure-Impulse Diagrams

During preliminary dynamic loading resistant design, structures are normally reduced to a single-degree-of-freedom (SDOF) model using equivalent mass, damping parameter and resistance function for simplicity. An SDOF model provides an approximation of the fundamental response mode for the overall structure [5]. Conventionally, maximum deflection at a point of particular significance, $y_{\text{max}}$, defines structural damage in engineering design. Damage occurs when $y_{\text{max}}$ reaches the critical deflection level (previously defined or agreed upon) of the structure, $y_c$. A pressure-impulse ($P$-$I$) diagram is an isodamage curve based on the maximum response criterion (which can be stress, displacement, moment, reaction force, etc.) for the system which is represented in the space of non-dimensional pressure and impulse of pulse loading [6-7]. $P$-$I$ diagrams are used to assess the level of damage in structures. Figure 4.1 shows a typical non-dimensional pressure-impulse diagram. $P$-$I$ diagram encompass three regimes of response viz. impulsive, dynamic and quasi-static. The impulse-controlled range is termed as “impulsive” and the peak load-controlled range as “quasi-static”. The response in the dynamic range is more intricate and contingent upon both parameters.

Over thirty years ago, Youngdahl [1] introduced correlation parameters in order to eliminate pulse loading shape effects on blast loaded structures, using simplified rigid, perfectly-plastic models and examining a wide range of structural elements comprising a circular plate, a reinforced circular cylindrical shell, a free-free beam and a circular shell. This was in sequel to the observation of the work conducted by Symonds [8]. Symonds observed that the final permanent deflection of a free beam when subjected to a concentrated pulse load depends only on the total impulse $I$ and peak load $P_{\text{max}}$. Youngdahl identified key parameters of pulse which, when equal for dissimilar shapes would predict approximately equal deflections. Youngdahl’s first parameter, effective impulse, is determined as:

$$ I_e = \int_{t_y}^{t_f} P(t)dt \quad (4.1) $$

where $P(t)$ is the loading function, $t_y$ and $t_f$ are the times when plastic deformation begins and completes, respectively. The second parameter, the effective load is defined by:
where \( t_{\text{mean}} \) is the time interval between the outset of plastic deformation and when the centroid of the pulse occurs and it is given as follows:

\[
P_e = \frac{I_e}{2t_{\text{mean}}}
\]  

(4.2)

\[
t_{\text{mean}} = \frac{1}{I_e} \int_{t_y}^{t_f} (t - t_y) P(t) \, dt
\]

(4.3)

In the first three structures tested, he observed that the maximum deflection, \( W_o(t_f) \), was found to be equal to \( I_e^2 f(P_e) \), where \( f(P_e) \) is a derived function different for each structure. This relationship was not very accurate for the circular shell whose equations were more complicated. However, when \( W_o P_y/I_e^2 \) was plotted versus \( P_e/P_y \) for each pulse shape of a particular structure where \( P_y \) is the static yield pressure, the five resultant curves were nearly indistinguishable. Youngdahl’s work successfully eliminated load shape effects from the prediction of the response for all four structures. Youngdahl went on to develop parameters for cases where the pulse load was not uniform over the loaded area and when the material posed strain-hardening behaviour. Li and Jones [9] went on to establish the theoretical foundation for empirically determined Youngdahl’s correlation parameters using bound theorems. They presented bounds for final displacement and structural response time for two-dimensional rigid-plastic structural members.

Amongst the earliest works on \( P-I \) diagrams is included the work of Abrahamson and Lindberg [10]. They developed \( P-I \) curves for rectangular, triangular (linearly decaying), and exponentially decaying loads acting on various structures - ranging from linear elastic to rigid plastic SDOF spring-mass systems, rigid plastic beams and plates. The \( P-I \) diagram produced by them was highly pulse-shape-dependent and there were pronounced discrepancies at middle curves. Zhu et al [11] used Youngdahl’s work to develop characteristic curves for rigid, perfectly-plastic models of a simply supported beam, a circular plate and a reinforced circular cylindrical shell subjected to uniformly distributed pulse loading. Their developed curves showed that a combination of the parameters developed by Youngdhal would lead to structural failure according to the Tresca yield criterion. Vaziro et al [12] produced ‘isoresponse’ curves which were similar to the characteristic curves of Abrahamson and Lindberg [10] and Zhu et al [11]. Their work examined rigid, perfectly plastic beams with either simply supported or clamped end conditions subjected to a rectangular pulse.

Schleyer and Langdon [13] developed \( P-I \) diagrams for blast walls using an SDOF model based on a tri-linear resistance curve. In their work, FEA was used to develop resistance curves for the
blast walls and subsequently an approximate tri-linear resistance curve based on engineering judgement was extrapolated from the FEA curves. Transformation factors were applied to the mass of the blast wall and the load on the wall to obtain the effective mass and load used in the SDOF model. Finally, they calculated the dynamic response of the SDOF system for all necessary pressure-impulse combinations and developed iso-damage curves for acceptable levels of damage. Schleyer and Landon went on state the limitations of obtaining iso-damage curve based on SDOF analyses.

Li and Meng [3, 5] have worked extensively on the subject of pressure-impulse diagrams**. In their works, they extended Youngdahl’s work to eliminate pulse load shape effects in elastic and elastic-plastic SDOF systems. By solving the general equation of motion for zero initial displacement and velocity with zero damping, which is the common practise in analysing blast-loaded structures, the following nondimensional quantities were derived:

\[
p = \frac{F_m}{y c K} = \frac{1}{\int_0^{\tau_m} f(\tau') \sin(\tau_m - \tau') d\tau'}
\] (4.4a)

\[
i = \frac{l}{y c \sqrt{MK}} = p \int_0^{\tau_d} f(\tau) d\tau
\] (4.4b)

where \(F_m\) is the amplitude (maximum value) of the load \(F(t)\), \(f(t) = F(t)/F_m\), \(y_c\) is a pre-defined critical displacement, and \(\tau_m\) and \(\tau_d\) are nondimensional values of time corresponding to maximum deflection and loading duration, respectively. The pressure-impulse diagram developed consisted of three regimes - Regime I corresponds to a small value of \(\tau_d\) i.e., \(\tau_d < \tau_1\), where \(\tau_1\) corresponds to loading time which is relatively short compared with the natural vibration period of the structure. In this regime, the structural response is considered an impulsive problem. Regime III corresponds \(\tau_d > \tau_2\), where \(\tau_2\) is normally greater than unity. Regime II corresponds to anything in-between Regimes I and III. Table 4.1 illustrates these regimes:

** P-I diagram developed by Li and Meng were for pulses with zero rise time.
Table 4.1: Regimes of a typical pressure-impulse diagram [5]

<table>
<thead>
<tr>
<th>Regime</th>
<th>Response identification ($\tau_d$)</th>
<th>Response type</th>
<th>Conditions for P-I regimes</th>
<th>P-I diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regime-I</td>
<td>($0, \tau_1$)</td>
<td>Impulsive</td>
<td>$p &gt; p_1$</td>
<td>$i$ controlled $i = 1$</td>
</tr>
<tr>
<td>Regime-II</td>
<td>($\tau_1, \tau_2$)</td>
<td>Dynamic</td>
<td>$p &lt; p_1$ and $i &lt; i_1$</td>
<td>($p,i$) controlled $g(p,i) = 1$</td>
</tr>
<tr>
<td>Regime-III</td>
<td>($\tau_2, \infty$)</td>
<td>Quasi-static</td>
<td>$i &gt; i_1$</td>
<td>$P$ controlled $P = 0.5$</td>
</tr>
</tbody>
</table>

Figure 4.1: A generic P-I diagram in the space of nondimensional pressure and impulse

Li and Meng [3, 5] proposed empirical formulas to eliminate the sensitivity of P-I diagrams to pulse shape. In their work, P-I diagrams were derived for three pulse shapes (linearly decaying, exponentially decaying and rectangular). In order to modify their proposed equations for the respective P-I diagrams, which were obtained from curve fitting, non-dimensional parameters are introduced to make the analysis suitable for a vast spectrum of blast loads. The quantities introduced are effective pressure and impulse shown in Equations (4.5a) and (4.5b).
\[
p_e = \frac{1}{i_e} + 0.5 \quad i_e = \frac{(i - 1)n_2}{n_1} \quad (4.5a \text{ and } 4.5b)
\]

where \(n_1\) and \(n_2\) are solutions of two separate least squares-derived quadratic equations involving a particular pulse shape’s centroid.

Li and Meng [3, 5] extended the P-I diagrams for elastic-plastic single degree of freedom (SDOF) systems. The resistance curve for the SDOF system was assumed to be a bilinear elastic-perfectly plastic resistance function, \(R(y)\), whose value at yielding was \(R_o\). They confirmed, theoretically, the validity of Youngdahl’s correlation parameters for rigid-plastic SDOF case. Li and Meng [3, 5] observed that the P-I diagram for elastic-plastic structure was influenced by a parameter, \(\eta\), and the pulse shape.

\[
\eta = \frac{R_0}{y_c K} \quad (4.6)
\]

Thus, they postulated that for a given loading shape, the explicit \(\alpha\) influence on the non-dimensional P-I diagram may be separated as in Equation (4.7) before satisfying the procedure in Equation (4.5).

\[
g = g \left( \frac{p}{h_1(\eta)}, \frac{i}{h_2(\eta)} \right) \quad (4.7)
\]

where \(g(p, i) = 1\) is the non-dimensional P-I diagram corresponding to a fixed loading shape and \(\alpha\) value. Fallah and Louca [14] introduced necessarily the same parameter, and called it “inverse ductility”.

Li and Meng [3, 5] went on to derive a closed form relation for P-I diagrams of a rigid-plastic single-degree-of-freedom (SDOF) model. Using a similar logic with Youngdahl’s a relations between \(i_e\) and \(p_e\) was developed as shown by Equation (4.8).

\[
\frac{y_f}{y_c} = \frac{i_e^2}{2} \left( \frac{1}{\alpha} - \frac{1}{p_e} \right) \quad (4.8)
\]

where \(y_f\) is the final and maximum deflection achieved and \(y_c\) is the critical deflection chosen for the P-I diagram.

The shortcoming of works on P-I diagrams in available literature necessitates the need for a fundamental work P-I diagrams for continuous systems and a simplified model which would comprise structural parameters such as mass, density, stiffness and the nature of end connections.
as presented in this chapter. For example, Shi et al [15] used numerical methods to generate $P$-$I$ diagrams for reinforced concrete. Using numerical methods might not make $P$-$I$ diagram generation a quick assessment tool. Hence, a fundamental method proposed in this chapter. Though, one of their aims was to show that damage criterion set by maximum deflection criterion was not sufficient to characterise damage especially in reinforced concrete columns, thus, introducing new damage criterion (some of which cannot be easily captured without numerical analysis). However, a maximum displacement criterion is sufficient to establish damage in many engineering cases. One very good example is on blast walls on offshore decks. The model and procedure proposed in this chapter comes in very handy in such scenarios. In the same vein, Krauthammer et al [16] developed analytical and numerically for generating $P$-$I$ diagrams for structural elements. Apart from using numerical methods, which might be expensive and not available for quick assessment, the analytical model proposed for developing $P$-$I$ diagrams was for an SDOF systems. However, their analytical and numerical work becomes very useful in cases where the resistance function is not linear elastic or elastic plastic or where multiple behaviour modes are present. Scherbatiuk et al [17] presented a simple model to calculate the time history of response and the subsequent $P$-$I$ diagram (based on maximum rotation of the wall about its support) for a free-standing soil filled HESCO Bastion (HB) concertainer™ wall subjected to blast load. However, the analytical model proposed for the analysed structure was based on rigid-body rotation. The rigid body model had uniform mass, density and a rotational inertia. Obviously, this rigid body model was best suited for the problem they studied, however, it cannot be applicable to wide range of other structures. In the wide range of literature presented by Florek and Benaroya [18] on $P$-$I$ diagrams, there is no mention of any fundamental work done on continuous systems. In the light of these, this work presents a fundamental development of $P$-$I$ diagram for continuous systems.

One major limitation of the procedure presented in this chapter is its inability to predict travelling plastic hinges, membrane effects and support shear hinges. However, it is important to bear in mind that the philosophy behind this work utilises a closed form approach in determining the velocity and displacement profiles of the beam system. When compared to the work of Ma et al [19] in the development of $P$-$I$ diagrams for simply supported and clamped rigid plastic beams, the velocity profiles where determined $a$ $priori$ and thus they could easily account for travelling plastic hinges and support shear hinges. Ma et al [19] presented five possible velocity profiles which took into account support shear hinges, bending hinges at the midspan and travelling plastic hinges. It is important to mention that this approach could only be achieved by a rigid plastic formulation as opposed to the procedure we have presented in this chapter, which
relies on the closed form solution of the displacement of a continuous beam with semi-rigid connections [20].

### 4.3. A Description of the Problem

Model of the system analysed in this chapter is shown in Figure 4.2. The support connections are represented by rotational springs. The spring parameter, $K_\theta$, varies between zero (for simply supported end conditions) to $\infty$ (corresponding to fixed supports), where intermediate values represent practical support conditions rendering the proposed model useful and of practical value. The closed form solution of the resultant fourth-order one-dimensional non-stationary equation encountered in transverse vibration of the postulated model in this work is presented. The procedure postulated by Polyanin [21] is used in deriving the closed form solution for the elastic response of the beam section based on boundary conditions derived in this chapter. The plastic response of the system is analysed by decomposing the system to an equivalent elastic beam and static beam with plastic end moments with initial conditions from the elastic response. The plastic deformation in the system is characterised by plastic hinges forming first at the span and then at the supports, or at the supports and then at the span or simultaneously at the supports and span. A dimensionless parameter $\alpha$ has been defined to determine the order of hinge formation. $P-I$ diagrams for elastic and elastic-plastic conditions are introduced using another set of non-dimensional parameters. $P-I$ diagrams are determined uniquely based for various structural configuration that are differentiated by various values of $\alpha$. However, the major limitation of the procedure presented is its inability to account for travelling plastic hinges, membrane effect and shear hinges at the support.

### 4.4. Analytical Model

#### 4.4.1. Elastic Response

The elastic response of the continuous beam, shown in Figure 4.2, is controlled by the parameters defining the model uniquely viz. density of the beam, $\rho$; elastic modulus, $E$; cross sectional area of the beam, $A$; the length, $L$, of the beam; second moment of inertia, $I$ and the elastic rotational constant, $K_\theta$, of the support springs [20]. Using Hamilton’s principle (the principle of least action) the equation of motion and natural boundary conditions are derived:
where $W$ is the non-conservative work done on the beam by externally applied loads, $U$ is the total potential energy, $T$ is the total kinetic energy and $t_1$ and $t_2$ represent times corresponding to known configuration of the beam i.e. $\delta w(x,t_1) = \delta w(x,t_2) = 0$. For an impulse load of $P(t)$ being applied to the semi-rigidly supported beam, the equation of motion is shown in Equation (4.10), and the mixed boundary conditions are given by Equations (4.11) and (4.12).

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = P(x,t)$$  \hfill (4.10)

![Figure 4.2: Idealised continuous beam system with semi-rigid supports (varying from simply supported, to fixed support)](image)

From the Handbook on Linear Partial Differential Equations for Engineers and Scientists [21], the closed form solution of Equation (4.10) which reduces to Equation (4.14) for simplicity is given in Equation (4.15).
where the Green function, \( G(x, \xi, t) \), is given by in Equation (4.16)

\[
G(x, \xi, t) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(\xi)}{\lambda_n^2 \| \varphi_n \|^2} \sin(\lambda_n^2 at) \tag{4.16}
\]

and the term \( \| \varphi_n \|^2 \) is the squared norm of \( n \)-th eigenfunction obtained as follows:

\[
\| \varphi_n \|^2 = \int_0^L \varphi_n^2(x) \, dx = \frac{L}{4} \varphi_n^2(L) + \frac{L}{4\lambda_n^2} [\varphi_n''(L)]^2 - \frac{L}{2\lambda_n^4} \varphi_n'(L) \varphi_n''(L) \tag{4.17}
\]

The terms \( \lambda_n \) and \( \varphi_n(x) \) which are eigenvalues and eigenfunctions, respectively, analogous to the natural frequencies and modes of the idealised structure, are determined by solving the self-adjoint eigenvalue problem in Equation (4.18).

\[
\varphi'''' - \lambda^4 \varphi = 0 \tag{4.18}
\]

Solving the self-adjoint Equation (4.18), we have the solution for eigenfunction as:

\[
\varphi(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} + C_3 \sin(\lambda x) + C_4 \cos(\lambda x) \tag{4.19}
\]

Substituting this eigenfunction into the boundary conditions in Equation (4.11) and (4.12), we obtain a system of four homogeneous equations written in matrix form as follows:

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
L^2 - a\lambda & e^{-\lambda L} & \sin(\lambda L) & \cos(\lambda L) \\
L^2 e^{\lambda L} + a\lambda & L^2 + a\lambda & -a\lambda & -L^2 \\
L^2 e^{-\lambda L} + a\lambda e^{-\lambda L} & L^2 e^{-\lambda L} - a\lambda e^{-\lambda L} & -\lambda^2 \sin(\lambda L) + a\lambda \cos(\lambda L) & -L^2 \cos(\lambda L) - a\lambda \sin(\lambda L)
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\tag{4.20}
\]

where \( \alpha = \frac{K_0 L}{EI} \)
Various values of $\lambda_n$ (i.e. $\lambda_1, \lambda_2, \lambda_3... \lambda_n$) corresponding to a specific value of $\alpha$ can be obtained by equating the determinant of the square matrix at the right hand side of Equation (4.20) to zero.

The constants $C_2$, $C_3$, and $C_4$ in terms of $C_1$ corresponding to particular structural configurations can be obtained for various eigenvalues, $\lambda_n$ from Equation (4.21). The exact eigenfunction is obtained by substituting the values of $C_1$, $C_2$, $C_3$, $C_4$ and corresponding $\lambda$ for a given structural configuration with parameter, $\alpha$, into Equation (4.19).

$$C_2 = \frac{C_1 \left( (a^2 - 2\lambda^2L^2 - \lambda L \alpha) \sin(\lambda L) + 3a \cos(\lambda L) \left( \lambda L + \frac{\alpha}{3} \right) \right) e^{\lambda L} + a \sin(\lambda L)^2 \cos(\lambda L)^2 (\lambda L - \alpha) }{ (2\lambda^2L^2 - \lambda L \alpha - a^2) \sin(\lambda L) - 3 \left( \lambda L - \frac{\alpha}{3} \right) a \cos(\lambda L) \right) e^{-\lambda L} - a \left( \sin(\lambda L)^2 + \cos(\lambda L)^2 \right) (\lambda L + \alpha) }$$

$$C_3 = \frac{C_1 \left( 2e^{\lambda L} \alpha \lambda L - 2 \left( \frac{-a}{2} \lambda L \right) \cos(\lambda L) + \frac{1}{2} \sin(\lambda L) \alpha \right) (\lambda L - \alpha) }{ \left( -3\lambda L \alpha + a^2 \right) \cos(\lambda L) + 2 \left( \frac{a}{2} \lambda L \right) \left( \lambda L - \alpha \right) \sin(\lambda L) \right) e^{\lambda L} - a \left( \sin(\lambda L)^2 + \cos(\lambda L)^2 \right) (\lambda L + \alpha) }$$

$$C_4 = \frac{C_1 \left( -2e^{\lambda L} \alpha^2 + \left( \frac{-a}{2} \lambda L \right) \sin(\lambda L) - \frac{1}{2} \cos(\lambda L) \alpha \right) (\lambda L - \alpha) }{ \left( 2\lambda^2L^2 - \lambda L \alpha - a^2 \right) \sin(\lambda L) - 3 \cos(\lambda L) \left( \lambda L - \frac{\alpha}{3} \right) a \right) e^{-\lambda L} - a \left( \sin(\lambda L)^2 + \cos(\lambda L)^2 \right) (\lambda L + \alpha) }$$
Table 4. 2: Various values of $\lambda$ corresponding to various structural configurations with parameter, $\alpha$

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 0$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1L$</td>
<td>3.14</td>
<td>3.40</td>
<td>3.58</td>
<td>3.71</td>
</tr>
<tr>
<td>$\lambda_2L$</td>
<td>6.28</td>
<td>6.43</td>
<td>6.55</td>
<td>6.65</td>
</tr>
<tr>
<td>$\lambda_3L$</td>
<td>9.43</td>
<td>9.52</td>
<td>9.61</td>
<td>9.69</td>
</tr>
<tr>
<td>$\lambda_4L$</td>
<td>12.57</td>
<td>12.64</td>
<td>12.71</td>
<td>12.78</td>
</tr>
<tr>
<td>$\lambda_5L$</td>
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<td>15.77</td>
<td>15.83</td>
<td>15.88</td>
</tr>
<tr>
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<td>18.85</td>
<td>18.90</td>
<td>18.95</td>
<td>19.00</td>
</tr>
<tr>
<td>$\lambda_7L$</td>
<td>21.99</td>
<td>22.036</td>
<td>22.08</td>
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<th>$\alpha = 7$</th>
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<td>3.97</td>
<td>4.03</td>
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<tr>
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<td>6.81</td>
<td>6.87</td>
<td>6.93</td>
<td>6.98</td>
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<tr>
<td>$\lambda_3L$</td>
<td>9.83</td>
<td>9.88</td>
<td>9.94</td>
<td>9.98</td>
</tr>
<tr>
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<td>12.89</td>
<td>12.94</td>
<td>12.99</td>
<td>13.03</td>
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<td>$\lambda_5L$</td>
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<td>16.02</td>
<td>16.06</td>
<td>16.10</td>
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<tr>
<td>$\lambda_6L$</td>
<td>19.08</td>
<td>19.12</td>
<td>19.16</td>
<td>19.19</td>
</tr>
<tr>
<td>$\lambda_7L$</td>
<td>22.19</td>
<td>22.23</td>
<td>22.26</td>
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<table>
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<tbody>
<tr>
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<td>4.12</td>
<td>4.16</td>
<td>4.64</td>
<td>4.68</td>
</tr>
<tr>
<td>$\lambda_2L$</td>
<td>7.03</td>
<td>7.07</td>
<td>7.71</td>
<td>7.78</td>
</tr>
<tr>
<td>$\lambda_3L$</td>
<td>10.03</td>
<td>10.07</td>
<td>10.81</td>
<td>10.89</td>
</tr>
<tr>
<td>$\lambda_4L$</td>
<td>13.07</td>
<td>13.17</td>
<td>13.89</td>
<td>14.01</td>
</tr>
<tr>
<td>$\lambda_5L$</td>
<td>16.14</td>
<td>16.17</td>
<td>16.99</td>
<td>17.12</td>
</tr>
<tr>
<td>$\lambda_6L$</td>
<td>19.22</td>
<td>19.26</td>
<td>20.09</td>
<td>20.24</td>
</tr>
<tr>
<td>$\lambda_7L$</td>
<td>22.32</td>
<td>22.35</td>
<td>23.19</td>
<td>23.35</td>
</tr>
</tbody>
</table>
Attempt has been made to provide a wide range of $\lambda_n$’s corresponding to various values of $\alpha$ for practical conceptual engineering design purposes. The truncation at $\lambda_7$ provides a conservative approximation for the displacement profile for most impulsively loaded beams and has been used in this work. However, it should be mentioned that there are cases where higher frequencies can be triggered. In such cases, care should be taken to determine the possible point of truncation. However, the robust procedure presented in this chapter still comes in handy.

### 4.4.2. Natural Frequency of the System

A harmonic motion of Equation (4.22a) gives the eigenvalue Equation (4.18) and the natural frequency in Equation (4.22b)

$$w(x, t) = \varphi(x)e^{i\omega t} \quad \omega = \frac{\lambda_n^2}{\rho A} \left(\frac{EI}{\rho A}\right)^{1/2} \quad (4.22a)\text{ and } (4.22b)$$

The blast pulse is transformed from time domain to frequency domain to determine the point of truncation in Equation (4.16) for accuracy of the closed form solution. Using the Fourier integral theorem in Equations (4.23) a pulse in the time domain ($-\infty < t < \infty$) can be transformed to a corresponding pulse in the frequency domain ($-\infty < \omega < \infty$). Figure 4.3a shows a rectangular pulse transformed from time to frequency domain in Figure 4.3b.

$$x(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt \quad x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\omega)e^{i\omega t} d\omega \quad (4.23)$$
Assuming frequencies with amplitudes less than 5% of the maximum amplitude have no influence on the maximum deflection, the point of truncation is determined based on the duration of pulse loading (See Figure 4.3). This procedure has been used in this chapter to determine a suitable point of truncation of $\lambda_n$.

### 4.4.3 Alternative Elastic Response Evaluation

Using the exact assumed modes for various semi-rigid supports, the temporal displacements (generalised coordinates) are obtained in this procedure. The modes can be obtained from Equations (4.18) to (4.21). The displacement field is derived as a series with each term being the product of a generalised coordinate and an exact shape function. The derived exact shape functions, which depend upon a set of dimensionless parameters, are obtained through an eigenvalue analysis and define the associated eigenfunctions of the generalised coordinates. Table 4.2 aids easy formulation of exact modes for varies beams using an intrinsic nondimensional parameter, $\alpha$. Using Galerkin’s weighted residual the equation of motion is transformed from a partial differential equation to an ordinary differential equation for easy calculations.
To solve the partial differential equation in Equation (4.10), the solution is formulated as a series expansion as follows:

$$w_i(x, t) = a_{i\beta}(t)\phi_\beta(x)$$  \hspace{1cm} (4.24)

Where $a_{i\beta}(t)$ are the generalised coordinates and $\phi_\beta(x)$ are the shape functions that satisfy boundary conditions (4.11) - (4.13). Summation convention is implied here.

The equation of motion (Equation (4.10)) can be re-written in the following form:

$$L_{ij}U_j + f_i = 0$$  \hspace{1cm} (4.25)

Using Galerkin’s method to convert the partial differential equations to ordinary differential equation for easy computation we have:

$$\int_0^L w_\beta (L_{ij}U_j) \, dx + \int_0^L w_\beta f_i \, dx = 0$$  \hspace{1cm} (4.26)

Where

$$w_\beta = \phi_\beta$$

For instance for $\alpha = 6$, the corresponding shape function are shown in Equations (4.27)

$$\phi_1(x) = e^{3.97x} + 52.76e^{-3.97x} + 122.56 \sin(3.97x) - 53.76 \cos(3.97x)$$

$$\phi_2(x) = e^{6.87x} - 964.99e^{-6.87x} - 3186.99 \sin(6.87x) + 963.99 \cos(6.87x)$$

$$\phi_3(x) = e^{9.88x} + 19562.32e^{-9.88x} + 84400.84 \sin(9.88x)$$
$$- 19563.32 \cos(9.88x)$$

$$\phi_4(x) = e^{12.94x} - 4.15 \times 10^5 e^{-12.94x} + 2.20 \times 10^6 \sin(12.94x) + 4.15$$
$$\times 10^5 \cos(12.94x)$$

$$\phi_5(x) = e^{16.02x} + 9.06 \times 10^6 e^{-16.02x} + 5.76 \times 10^7 \sin(16.02x) - 9.06 \times 10^6 \cos(16.02x)$$
\[
\phi_6(x) = e^{19.12x} + 2.01 \times 10^8 e^{-19.12x} - 2.82 \times 10^7 \sin(19.12x) + 2.01 \\
\times 10^8 \cos(19.12x)
\]

\[
\phi_7(x) = e^{22.22x} + 4.49 \times 10^9 e^{-22.22x} + 3.9 \times 10^{10} \sin(22.22x) - 4.49 \\
\times 10^9 \cos(22.22x)
\]

Therefore the solution, truncated at \( n = 7 \) is

\[
w(x, t) = a_1(t)\phi_1(x) + a_2(t)\phi_2(x) + a_3(t)\phi_3(x) + a_4(t)\phi_4(x) + a_5(t)\phi_5(x) + a_6(t)\phi_6(x) \\
+ a_7(t)\phi_7(x)
\]

Using Equation (4.26), the partial differential equation, PDE, in equation (4.10) can be solved by reducing it to an ordinary differential equation, ODE, using Galerkin’s method of weighted residuals. It can be seen that an infinite degree-of-freedom system (infinite series) is developed in analysing the system. Truncating at say \( n = 7 \) gives a 7 degree-of-freedom system which gives a high accuracy for most pulse loads [22]. With input variables are \( \alpha, K_\theta, L, EI, A, \rho, P(t) \) the system can be alternatively solved using this scheme.

### 4.4.4. Pulse Loading with Zero Rise Time

Loads from detonations (blast loads with zero rise time) can be idealised as linearly decaying, exponentially decaying, concave, and rectangular pulse shapes [14]. The idealised pulse load, \( P(x,t) \), starts with its maximum value, \( F_m \) (e.g. maximum value of uniformly distributed load on length \( L \)), and descends to zero at time \( t_d \) in the case of linear, exponentially decaying or concave pulses. This illustrates the basic feature of a uniformly distributed blast load (detonation) on the structure if the negative pressure phase is neglected [6]. Figure 4.4 illustrates the normalised pulse shapes (pulses with zero rise time). Equation (4.29) shows the various pulse time histories written as a single general equation using values of \( \eta \) and \( \gamma^* \) from Table 4.3.

\[
f(t) = \left(1 - \eta \frac{t}{t_d}\right) \exp\left(\gamma^* \frac{t}{t_d}\right)
\]

(4.29)
Table 4.3: Parameters to define various pulse loadings

<table>
<thead>
<tr>
<th>Parameters for blast load</th>
<th>Rectangular (LA)</th>
<th>Concave (LB)</th>
<th>Triangular (LC)</th>
<th>Exponential (LD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\gamma^*$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2.8</td>
</tr>
</tbody>
</table>

Figure 4.4: Four typical normalised zero rise time pulse-loading shapes: Rectangular, Concave, Exponential and Triangular

4.5. Plastic Response

The structural configuration of the system determines the point where plastic hinges first occur. Depending on values of the dimensionless quantity, $\alpha$ ($\alpha = K_0L/EI$), plastic hinges first occur either at the supports or midspan of the beam. At a transition value of $\alpha$, the plastic hinges form simultaneously at the supports and midspan. This model does not take into account the phenomena of travelling plastic hinge. As observed by Jones [2] for uniformly distributed loads of magnitude, $P_0$ ($P_c \leq P_0 \leq 3P_c$) plastic hinge forms at the midspan for simply supported beams and at the supports and span for fixed ended beams. For cases where $P_0 \geq 3P_c$, the phenomena of travelling plastic hinges is observed. Travelling plastic hinges form at a distance, $\epsilon$ from the supports and travel to the midspan. The term $P_c$ is the static collapse load and $P_0$ is the applied uniformly distributed load.

Irrespective of the point where the plastic hinge first forms, three plastic hinges could eventually form at the supports and spans as the structure is been loaded. The value of $\alpha$ at which plastic hinges would be formed simultaneously at the support and in the beam can be found easily when
the values of $M_{support}$ and $M_{span}$ are plotted against non-dimensional $\alpha$ for specific values impulses, for various values of non-dimensional $\frac{It}{\rho AL^2}$ where $t$ is any point in time in the simulation and $I$ is the total impulse. Figure 4.5 shows the span moment and support moment plotted against various non-dimensional values of $\alpha$ for $\frac{It}{\rho AL^2} = 0.3$ for a triangular pulse. Observe that at $\alpha = 0$, the support moment is zero and the span moment is at a maximum as expected. However, as $\alpha$ increases and the beam system approaches fully clamped configuration, the value of span moment becomes less than that of the support moment. The point $\alpha = 6$, which represents the transition point, the span moment equals the support moment [20].

Figure 4.5: Moment span and support moment versus non-dimensional value $\alpha$

Figure 4.6 shows the elastic-plastic response of the beam and support rotational spring studied.
The initial conditions for the plastic phase are \( f(x,t_e) \) and \( g(x,t_e) \) which correspond to the initial displacement and velocity profiles at the end of the elastic phase (i.e. when plasticity sets in). Equation (4.30) shows a quartic equation of a curve, which well describes the very lengthy equations for the displacement and velocity profiles at \( t_e \) for ease of calculation. Using error minimisation, as presented in Equations (4.31) and (4.32), we can solve for the constant \( A, B, C, D, \) and \( F \) in equation (4.30).

\[
\begin{align*}
    w &= Ax^4 + Bx^3 + Cx^2 + Dx + F \\
    E &= \text{error} = \sum_{i} (w_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F)^2
\end{align*}
\]  

(4.30)  

(4.31)
\[
\frac{\partial E}{\partial A} = -2 \sum_{i} x_i^4 (w_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F) = 0 ; \frac{\partial E}{\partial B} \\
= -2 \sum_{i} x_i^3 (y_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F) = 0 \\
\frac{\partial E}{\partial C} = -2 \sum_{i} x_i^2 (y_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F) = 0 ; \frac{\partial E}{\partial D} \\
= -2 \sum_{i} x_i (y_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F) = 0 \\
\frac{\partial E}{\partial F} = -2 \sum_{i} (y_i - Ax_i^4 - Bx_i^3 - Cx_i^2 - Dx_i - F) = 0 \\
(4.32)
\]

The variables \(x_i\) represents the various points along the length of the beam and \(w_i\) represents the corresponding displacements or velocity, as the case might be, from profile predictions of the presented model.

### 4.5.1 Plastic Hinge First Forms at Midspan (Parameter, \(\alpha < 6\))

This response for this structural configuration can be analysed by decomposing the response of the system into an equivalent elastic beam and statically loaded beam with initial span plastic moment, \(M_p\) at the midspan as shown in the Figure 4.7. Using the displacement and velocity profiles at time \(t_e\) (time at which plasticity sets in) as the initial condition for the elastic analysis of the equivalent structure in Figure 4.7a, the derived maximum displacement can be added the displacements from Figure 4.7b to give the total maximum plastic displacement. For cases where the moment at the support reaches \(M_p\) the procedure illustrated in the case where \(\alpha = 6\) is used to determine the resultant maximum displacement.

Natural and mixed boundary conditions, obtained from Hamilton’s principle are shown in Equations (4.33) to (4.35). This corresponds to the updated boundary conditions for the previously presented elastic boundary conditions as soon as the plastic hinge is formed at the span[20].

\[
w''(0, t) - \frac{K_0}{EI}w'(0, t) = 0 \\
(4.33)
\]
4.5.2. Plastic Hinges First Form at Supports (Parameter, $\alpha > 6$)

This response for this structural configuration can be analysed by decomposing the response of the system into an equivalent elastic beam and statically loaded beam with span plastic moment, $M_p$ at the supports as shown in the Figure 4.8. Using the displacement and velocity profiles at time $t_e$ (time at which plasticity sets in) as the initial condition for the elastic analysis of the elastic of the equivalent structure in Figure 4.8a, the derived maximum displacement can be added the displacements from Figure 4.8b to give the total maximum plastic displacement. For cases where the moment at the span reaches $M_p$ the procedure illustrated in the case where $\alpha = 6$ is used to determine the resultant maximum displacement [20].

The natural and essential boundary conditions are shown in equations (4.36) to (4.38). Coordinates used are same with Figure 4.2

$$w''(0, t) + \frac{M_p}{EI} = 0 \quad (4.36)$$

$$w''(L/2, t) = 0 \quad (4.35)$$
This case corresponds to the condition where plastic hinges form simultaneously at the supports and at the midspan. The beam goes from the elastic response to the plastic response comprising of the formation of three plastic hinges. In this configuration, the moments at the supports are always equal to the moment at the midspan in the elastic analysis. The natural boundary when three plastic hinges are formed in the beam is shown in Equations (4.39) to (4.41).

\[
\begin{align*}
    w''(0, t) + \frac{M_p}{EI} &= 0 \\ \\
    w''(L, t) + \frac{M_p}{EI} &= 0 \\
    w''(L_{1/2}, t) &= 0 \\
    w''(L_{1/2}, t) &= 0
\end{align*}
\]

(4.39) and (4.40)
The velocity profile for this stage is shown in Figure 4.9 and it is represented by Equation (4.42).

$$\dot{w} = \dot{W} \left(1 - \frac{x}{L_1}\right)$$  \hspace{1cm} (4.42)

Using the Symonds [23] ‘minimum delta’ approach, we can obtain the initial velocity amplitude for this stage. We express the initial velocity as $v(x) = \dot{w}(x, 0)$ and the desired mode solution for this stage as $A(t) \varphi^*(x)$, with arbitrary initial value $A_0 = A(0)$. Where $\rho$ is the specific mass and the integration is over the structure. The value of $A_0$, which minimises $\Delta_0$, is:

$$A_0 = \frac{\int_0^L \rho v \varphi^* \, dx}{\int_0^L \rho \varphi^* \, dx} \quad \varphi^*(x) = 1 - \frac{x}{L_1}$$  \hspace{1cm} (4.43)

The maximum displacement for this stage can easily be derived by the process presented by Jones [2] for the different pulse shapes [20].

4.6 Equivalent Finite Element Model of Continuous Beam

A continuous beam with semi-rigid supports can be modelled as shown in Figure 4.10 in Abaqus. Applying springs and hinges in Abaqus cannot eliminate membrane effects, which is the major limitation of the model presented in this chapter. In Figure 4.10, infinitely long $L_a$ and $L_b$ represent pinned connections at the ends of the beam of length, $L$. Details of this kind of
modelling can be found in [24]. From the schematic representation of the FE model presented, we can obtain

\[ K_\theta = \lambda \frac{EI}{L} \quad \text{where} \quad \lambda = \frac{3L}{L_a} \text{ or } \frac{3L}{L_b} \]

(4.44)

In this model, the overall beam model has a stiffness of \( EI \), thus, the moment capacity of the beam, \( L \), is equal to the moment capacity of the connections at the ends of beam. Table 4.4 shows the properties of the section modelled in Abaqus.

![Schematic representation of continuous beam with semi-rigid supports at connections.](image)

**Figure 4. 10**: Schematic representation of continuous beam with semi-rigid supports at connections.

**Table 4. 4**: Mechanical and geometric property of FEA model for continuous beam

<table>
<thead>
<tr>
<th>Properties</th>
<th>Value</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_\theta ) Support rotational spring stiffness</td>
<td>1562.5</td>
<td>kNm/rad</td>
</tr>
<tr>
<td>( B ) and ( d ), Breath / depth of section</td>
<td>50</td>
<td>mm</td>
</tr>
<tr>
<td>( L_a / L_b )</td>
<td>200</td>
<td>mm</td>
</tr>
<tr>
<td>( L ), Length of beam</td>
<td>600</td>
<td>mm</td>
</tr>
<tr>
<td>( \rho ) Density</td>
<td>7850</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>( E ) Young's Modulus</td>
<td>200</td>
<td>Gpa</td>
</tr>
<tr>
<td>( M_p ) of beam/connection</td>
<td>7812.5</td>
<td>Nm</td>
</tr>
</tbody>
</table>
Applying the procedure illustrated in Equations (4.11-4.35), for this model \((K_0 L/(EI) =9)\), the displacement history can be obtained.

Figure 4.10 shows the FE model of the beam under a blast load of 5 bars with a duration of 0.02s and rise time of 0.01s. Figure 4.12 shows a comparison of the displacement time history predicted by the simplified procedure and the FE model. The apparently weak correlation between the curve is as a result of inherent membrane stretching in the FE model.

Figure 4.11: FE model of equivalent continuous beam with plastic hinges at support and midspan

Figure 4.12: Displacement time histories for analytical and FEA predictions
4.7 Blast Wall Model

The blast wall system presented in Chapter 3, which consist of a vertical, horizontal and support rotational springs is simplified to a beam with semi-rigid supports as shown in Figure 4.2. Intuitively, it can be observed that this simplification would model, to high level of accuracy, the elastic response of a blast. This is because membrane effect is minimal in the elastic range of deformation. The inability of this idealisation (continuous system with rotational springs at supports) to model membrane effects, midspan buckling, connection pull-in at supports and strain hardening renders this model ineffective in the large permanent plastic deformation of blast walls. Buckling, connection pull-in and membrane effect become pronounced in large permanent plastic displacements. Also, the support springs in this model are elastic-perfectly plastic while the rotational springs in Chapter 3 have strain hardening incorporated in them.

The corrugated panel response studied in Chapter 3 is approximated to a rectangular beam with cross-section equivalent thickness, $t_{eq}$. This is based on equating the second moments of area of the two cross sections. The elastic properties for the support rotational springs are taken to be equal to the elastics properties of the support rotational springs in Chapter 3. Figures 4.13 and 4.14 show a comparison of the elastic responses predicted by the analytical model and the FE model in Chapter 3, and the presented analytical method in this chapter. A good correlation was achieved between the three curves.

Traditional $P-I$ diagram presented is for the elastic response of the blast wall in Chapter 3. The reason for presenting the elastic $P-I$ diagram is because of the limitations of the model as highlighted. Figure 4.15 shows a pressure-impulse diagram for a triangular pulse with rise time equal to half the duration of the pulse. The impulse corresponds to the impulse on single corrugation.
Figure 4. 13: Displacement time history for overpressure of 0.5 bar (elastic response)

Figure 4. 14: Displacement time history for overpressure of 1 bar (elastic response)
4.8. Elastic Pressure-Impulse diagrams for Pulse Loads with Zero Rise Time

It is clear, from the foregoing paragraphs that the maximum displacement, $w_{\text{max}}$, for any specific structural configuration with parameter, $\alpha$, depends on parameters $\rho$, $A$, $E$, $I$, $K_\theta$, $L$, $I_d$ and the maximum value of the pulse, $F_m$ (in N/m) for a particular pulse loading shape. From the equations of motion presented in Equations (4.14-4.21), we can obtain closed form solutions for specific configurations of a structure based on values of $\alpha$. Table 4.2 shows the natural frequencies associated for each structural configuration. Subsequently, the procedure for determining $P-I$ diagrams for each structural configuration is presented. Using Buckingham’s Pi-theorem approach [25], the following non-dimensional parameters are obtained. Non-dimensional pressure is:

$$p = \frac{F_m L^4}{E I y_c}$$  \hspace{1cm} (4.45)

Non-dimensional time, $\tau$, is given as

$$\tau = \frac{t}{\sqrt{\frac{\rho A L^3}{K_\theta}}}$$  \hspace{1cm} (4.46)

The ratio of maximum displacement to the critical displacement is

$$\frac{y_{\text{max}}}{y_c} = g(p,i)$$  \hspace{1cm} (4.47)
where \( i \) is the non-dimensional impulse as follows:

\[
i = \frac{I_{\text{impulse}}}{y_c \sqrt{\frac{EI \rho A}{dL^2}}} = p \int_0^{\tau_d} f(\tau) d\tau \quad I_{\text{impulse}} = \text{Impulse (Nm)} \quad (4.48)
\]

The non-dimension \( \tau_1 \) and \( \tau_2 \) corresponding to impulsive and quasi-static regime is shown in Table 4.5 for a structural configuration of \( \alpha = 1 \). The non-dimensional impulsive pressure and quasi-static impulse \( p \) and \( i \) is used in developing the impulsive and quasi-static asymptotes. A point in the dynamic regime is consequently obtained for each pulse shape respectively.

<table>
<thead>
<tr>
<th>Pulse Loading</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( p_1 )</th>
<th>( i_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>0.101</td>
<td>0.22</td>
<td>99.85</td>
<td>10.29</td>
</tr>
<tr>
<td>Concave</td>
<td>0.122</td>
<td>0.58</td>
<td>122.4</td>
<td>19.32</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.131</td>
<td>2.18</td>
<td>160.2</td>
<td>51.45</td>
</tr>
<tr>
<td>Exponential</td>
<td>0.138</td>
<td>5.10</td>
<td>333.6</td>
<td>59.4</td>
</tr>
</tbody>
</table>

An empirical formula as proposed by Li and Meng [5] is used to define the shape of \( P-I \) curve as shown in Equation (4.49). Table 4.5 shows values for non-dimensional time, impulsive pressure, and quasi-static impulse.

\[
p = \frac{n_1}{(i - 10)n_2} + 10 \quad (4.49)
\]

The polynomial Equation (4.49) is used to determine the values of \( n_1 \) and \( n_2 \). Using the least square method values for coefficients \( \beta_0, \beta_1, \beta_2 \) and \( \beta_3 \) are determined. For \( n_1, \beta_0 = 855.87, \beta_1 = -3202.01, \beta_2 = 4349.94, \beta_3 = -1981.55 \) and for \( n_2, \beta_0 = 2.43, \beta_1 = -10.65, \beta_2 = 16.48, \beta_3 = -7.52 \) are obtained. The parameter \( d \) in equation is related to the centroid of the specific pulse shape as shown in Table 4.6. The centroids of normalised pulses in Figure 4.4 for different pulses are defined by equations (4.50). Table 4.6 shows the centroid of various normalised pulses and the corresponding values of \( n_1 \) and \( n_2 \).
\[ n_{1,2} = \beta_0 + \beta_1 d + \beta_2 d^2 + \beta_3 d^3 \quad (4.50) \]

Table 4.6: Centroids of pulses and corresponding \( n_1 \) and \( n_2 \) values for \( \alpha = 1 \)

<table>
<thead>
<tr>
<th>Pulse Load</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( d = \sqrt{x_0^2 + y_0^2} )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular (LA)</td>
<td>0.5</td>
<td>0.5</td>
<td>0.707</td>
<td>53.2788</td>
<td>0.4345</td>
</tr>
<tr>
<td>Concave (LB)</td>
<td>0.393</td>
<td>0.416</td>
<td>0.572</td>
<td>76.205</td>
<td>0.3229</td>
</tr>
<tr>
<td>Triangular (LC)</td>
<td>0.333</td>
<td>0.333</td>
<td>0.467</td>
<td>106.704</td>
<td>0.2836</td>
</tr>
<tr>
<td>Exponential (LD)</td>
<td>0.209</td>
<td>0.266</td>
<td>0.338</td>
<td>194.1884</td>
<td>0.4244</td>
</tr>
</tbody>
</table>

\[
x_0 = \frac{\int_0^1 tf(t)dt}{\int_0^1 f(t)dt} \quad y_0 = \frac{\int_0^1 f^2(t)dt}{2\int_0^1 f(t)dt} \quad (4.51)
\]

Figure 4.16: Elastic P-I diagrams for \( \alpha = 1.0 \) for four typical descending loads

Pressure–impulse diagrams for pulses with zero rise time are pulse-shape-dependent as shown in Figure 4.16. As expected, the damage caused by the rectangular pulse is relatively larger than the damage caused by other pulses. This is because the total impulse of rectangular pulse is larger than the total impulse of other pulses with the same duration. To eliminate the effect of pulse shape an effective impulse \( (i_e) \) and pulse \( (p_e) \) are introduced in Equation (4.51) and (4.52). This is in line with the procedure adopted by Li and Meng [3, 5] for their P-I curving fitting. An elastic pulse-shape-independent P-I diagram is shown in Figure 4.17.
\[ i_e = \frac{(i - 10)^{n_2}}{n_1} \]  
(4.52)

\[ p_e = \frac{1}{i_e} + 10 \]  
(4.53)

Figure 4.17: Pulse shape independent elastic P-I diagrams for four typical descending loads

### 4.9. Elastic-plastic Pressure-impulse diagrams for Pulse Loads with Zero Rise Time

Maximum plastic displacement, \( w_{\text{max}} \), for elastic-plastic deformation depends on parameters \( \rho, A, E, I, K_\theta, M_p \) (moment capacity of beam and support springs), \( L \) and \( t_d \) for a particular pulse load shape. The response in this regime depends on the \( \sigma_y \), yield strength of the beam, which is related to \( M_p \). The moment capacity, \( M_p \), for the section is equal to \( \sigma_y b d^3/4 \), where \( b \) is the width of the beam and \( d \) is the depth of the beam. To determine the plastic pressure impulse diagram another non-dimensional parameter \( \chi = M_p L/EI \) is introduced which is derived from the plastic response equations. Non-dimensional parameters for pressure and impulse for elastic-plastic deformation are introduced in Equations (4.54) to (4.56).

\[ p = \frac{F_m L^4}{y_{cp} EI} \]  
(4.54)  
\[ y_{cp} = \text{critical plastic displacement} \]

Non-dimensional time, \( \tau \), is given as

\[ \tau_d = \frac{t_d}{\rho A L^3} \]  
\[ \sqrt{\frac{K_\theta}{\rho A L^3}} \]  
(4.55)
Non-dimensional impulse, $i$, is given as

$$i = \frac{l_{\text{impulse}}}{y_{cp} \sqrt{\frac{EI}{\alpha L^2}}} = p \int_0^{\tau_d} f(\tau) d\tau \quad l_{\text{impulse}} = \text{impulse (Ns)} \quad (4.56)$$

Like the elastic case, elastic-plastic $P-I$ diagrams are pulse-shape-dependent. Using the same procedure of curve fitting used in the elastic case, by applying Equations (4.55) to (4.57), we generate pulse shape independent $P-I$ diagrams for particular cases of $\alpha < 6$, $\alpha = 6$ and $\alpha > 6$ in Figure 4.18, 4.19 and 4.20 respectively.

$$p = \frac{n_1}{(i - 1)^{n_2}} + 1 \quad (4.55)$$

$$i_e = \frac{(i - 1)^{n_2}}{n_1} \quad p_e = \frac{1}{i_e} + 1 \quad (4.56) \text{ and } (4.57)$$

Figure 4. 18: Unique elastic-plastic $P-I$ (for blast with zero rise time) diagrams for four typical descending loads, $\alpha = 1$, $\chi = 0.087$. 
Figure 4. 19: Pulse shape independent (for blast with zero rise time) $P-I$ diagrams for, $\alpha = 6, \chi = 0.087$

Figure 4. 20: $P-I$ (for blast with zero rise time) diagrams for, $\alpha = 10, \chi = 0.087$

4.10. Conclusions

An analytical procedure for obtaining the maximum displacement of elastic and elastic-plastic continuous beams is presented in this chapter. Due to the limitations of the model presented (i.e. inability to model membrane effect, midspan buckling and connection pull-in), only the elastic response of the blast wall presented in Chapter 3 is validated in this chapter. A good correlation was achieved because the limitations of the model are not pronounced in the elastic response.
regime. In addition, the strain hardening in the rotational support springs in Chapter 3 cannot be modelled with the analytical model presented in this chapter. For plastic response, the model presents an ideal case, which is not always the case in practice. However, the analysis presented is intended to form the bedrock for future modification to include other practical effects in plastic deflection.

Since the dynamic response of blast walls can either be elastic or elastic-plastic with plastic deflection, this model can be applied to elastic response of blast walls. Since class 2, 3, or 4 cross-sections cannot sustain plastic deflections without loss of moment resistance and consequently are limited to blast walls, which are designed to respond elastically, the procedure presented in this chapter is well suited for this family of cross-sections. The bending moments at the supports and midspan can easily obtained from the solution presented in Equation 4.15. This would aid the analyst in making sure plasticity does not occur at these points in an intended design of a blast wall to respond elastically. The procedure presented in this chapter complements design guides for the elastic design of blast walls, which uses the concept of dynamic load factors (DLF) to evaluate the maximum displacement of an elastic blast wall [26-27].

In other to buttress the usefulness of the model presented in this chapter, it is important to note that 90% of installed stainless steel blast walls respond elastically to blast load [26-27]. In the offshore industry, suppliers will not offer a plastic blast wall unless specifically requested by the client. Elastically designed blast walls are usually lighter and therefore more economic than walls, which respond plastically, because it is difficult to obtain an economic section of sufficient depth whilst maintaining low enough $b/t$ ratios to ensure plastic cross-section.

In Chapter 3, a 33% reduction in the plastic deformation of the blast wall strengthened by the proposed scheme (i.e. hybrid system) was observed. This is important from an engineering point view because sometimes the reduction in dynamic load factor (DLF) arising from allowing plastic deformation can outweigh the economic advantage of an elastic wall and can also lead to the incorporation of strength reserves that would not be available in elastic walls. In addition, plastic designed blast walls are much easier to upgrade in service than elastic walls because they have large strength reserves for local effects. Elastic blast walls are often governed by their resistance to local effects; in such cases, upgrades are more expensive [26].
In the light of this, a traditional elastic pressure-impulse diagram for the blast wall studied in Chapter 3 is presented in this chapter for a pulse shape with a finite rise time (i.e. a typical hydrocarbon explosion).

To further extend the use of the model presented in this chapter, a pressure impulse diagram for pulses with zero rise time is presented. Though this kind of pulse shape does not correspond to a typical hydrocarbon explosion, it is a possible type of blast load than can be experienced by a continuous system. The displacements are shown to be functions of non-dimensional parameters extracted based on model parameters and by the application of Buckingham’s Pi-theorem for pulses with zero rise time. Different non-dimensional parameters are obtained for plastic and elastic deformations for this pulse type. Subsequently, a fundamental study of the $P$-$I$ diagrams of a continuous system is conducted. It is shown the three distinct $P$-$I$ regimes on a continuous system exist, i.e., (1) pulse-controlled, (2) peak load and impulse-controlled, and (3) peak load-controlled critical conditions depending on the values of non-dimensional pulse and non-dimensional impulse. This three regimes are common to traditional pressure impulse diagrams. This represents a complementary solution to exiting analytically derived $P$-$I$ diagrams available in the literature, which simplify continuous systems and complex engineering systems to single-degree- of-freedom systems [3, 5, 16-17]. Dimensionless parameters based on the complete set of quantities that define the transverse dynamic response of a beam with semi-rigid supports for pulses with zero rise time uniquely are presented in this chapter. These dimensionless parameters represent the non-dimensional impulse and pressure of the system. In the elastic-plastic response, it is observed that the point of formation of plastic hinge can be controlled by a non-dimensional quantity, $K_0L/EI$. The structural geometrical and mechanical parameters and critical deflection levels that represent particular damage levels have to be identified using experimental methods or prescribed as input parameters [20].

For elastic and elastic-plastic $P$-$I$ diagrams a procedure is presented for eliminating the loading shape dependency of the $P$-$I$ diagrams. This procedure is analogous the procedure adopted by Li and Meng in eliminating pulse load dependency on $P$-$I$ diagram [3, 5].
4.11 Cited References


[20] A. Soleiman Fallah, E. Nwankwo, Louca LA. Pressure-impulse diagrams for blast loaded continuous beams based on dimensional analysis


[27] ABS. Design, materials and connection for blast-loaded structures. Health & Safety Executive (HSE); 2006.
Chapter 5

Numerical Modelling of the Dynamic Response in Pulse Loaded Fibre Metal Laminated Plates

5.1 Introduction

This chapter presents a three-dimensional finite element model to replicate the dynamic response up to failure and beyond for fibre metal laminates (FMLs) made of a 2024-O aluminium alloy and a woven glass-fibre/polypropylene composite (GFPP). Strength and modelling parameters for the GFPP and the blast loading configuration are taken from the work of Karagiozova et al [1]. The underlying layers of composite and metal in an FML provide structural support for layers above them [2], thus, proper modelling of the failure and degradation of these constituent layers, accompanied by correct simulation of their interface, results in a model able to capture the tearing of either the back face or front face of the FML. The state of the constituent parts of the FML (i.e. whether damaged or not) is captured by the proposed model, thus, complimenting the work of Vo et al. [3-4] which only predicts the maximum displacements of the back and front faces of the FML’s aluminium plates.

A model is presented to analyse damage initiation, damage progression, and failure of the three-dimensional solid woven composite material within the FML. The model incorporates strain rate effects in composites and a mesh-objectivity algorithm for strain softening to control energy dissipation associated with each failure mode regardless of mesh refinement and topology. Damage is initiated when a modified Hashin damage criteria [5] is satisfied for fibre tensile/shear failure modes, compression failure modes, in-plane shear failure mode and out-of-plane failure modes. Damage evolution is modelled by an empirical relationship analogous to the model proposed by Matzenmiller et al [6].

A good correlation is observed between experimental results presented by Langdon et al [7] and the numerical results predicted by the model. The displacements of the back and front faces of the FML’s obtained from experiments where compared to the results predicted by the numerical model. Again, the model was able to predict accurately the impulses, which caused tearing of the aluminium plates. It is important that the analyst is able to predict the integrity of the constituent parts of the FML and its maximum deflections.
In Chapter 3, debonding which is a possible failure mode between the composite patch and the mild steel blast wall was assumed not to occur. Thus, Debonding was prevented by using the Abaqus ‘TIE’ feature. The implication of this is that the model in Chapter 3 behaves as predicted if the stresses within the adhesive layer between the patch and steel blast wall do not exceed values that would result in debonding. In the light of this, a study is deemed necessary in order to give an insight into the development of debonding in a hybrid system. Due to the fact that experimental results on blast loaded fibre metal laminates (FMLs) is available in literature, this thesis, thus, develops a numerical model to validate this experiment in order to increase the understanding of the debonding process in the metal-composite interface. An interface element is introduced between the composite layers and the aluminium layers of the FML. The characteristics of the cohesive layer used to represent the interface between the aluminium layers and composite layers correspond with the properties of the adhesive in the experimentally tested FML. The motivation for this chapter was to observe the sequence of failure modes, which was unavailable from the experimental results for the failure of the FML. An observation of the failure modes and sequence would show when debonding occurred (i.e. at beginning of the blast response or at the end of response). Obviously if debonding is a first failure mode, assumptions made in Chapter 3 would be invalid because the structure would not behave as presented. It was observed that debonding of the back face occurred after a large plastic deformation of the panel. This early stage debonding was attributed to the large difference in in-place stiffness between the aluminium layer and composite patch. Due to the non-availability of experimental data for other types of composite/aluminium arrangement, the numerical model was not continued in order to investigate the effect of in-plane and through-thickness stiffness on debonding between aluminium and composite layer.

5.2. Geometry and Modelling

Various 400 x 400mm panels of FML’s, were manufactured and tested under local blast at the University of Cape Town from 0.025in thick sheets of 2024-O aluminium alloy and a woven glass fibre/propylene composite. The FML panels tested had exposed areas of 300x300 mm unanimously and were labelled AXTYZ-#. Where A = aluminium, X = number of aluminium layers, T = GFPP, Y = number of blocks of GFPP, Z = number of plies of GFPP per block and # indicates the panel number.

Due to the symmetrical in-plane architecture of the panel only one-quarter of the panel was modelled in Abaqus 6.9 with appropriate boundary conditions i.e. symmetry and fully clamped
on internal and external edges, respectively. The FML is modelled as a four-part structure i.e. comprising aluminium alloy, composite, interface adhesive layer (cohesive layer representing the adhesive between composite and aluminium alloy) and laminate adhesive layer (cohesive layer representing the adhesive layer between plies of composite). In this model, a laminate adhesive layer is introduced after every two consecutive plies. Figure 5.1 illustrates the through-thickness structure of the FML. The aluminium part is meshed using linear brick elements i.e. C3D8R elements, which are eight-noded, linear hexahedral elements with reduced integration formulation and hourglass control. The adhesive layers in the composite and interface between composite and aluminium are modelled with 3-D cohesive element with direct traction-separation formulation (COH3D8).

Figure 5.1: Schematic representation of through thickness architecture in an FML plate

5.3. Material Modelling

5.3.1 Composite Modelling

Continuum Damage Mechanics (CDM) has been adopted in this model, which assumes a linear elastic orthotropic response up to the point of damage initiation. Modified Hashin damage initiation criteria must be satisfied for damage to initiate as this model is adopted. These damage criteria are similar to the ones proposed by Xiao et al [5] for multi-axially loaded composites. The initial elastic constants of the undamaged material are the elastic moduli $E_1$, $E_2$ and $E_3$, shear moduli $G_{12}$, $G_{23}$ and $G_{31}$ and the Poisson’s coefficients $\nu_{12}$, $\nu_{23}$ and $\nu_{31}$, where 1, 2 and 3 denote local axes of the material in the in-plane fill (warp), transverse in-plane (weft) and out-of plane directions, respectively.
The constitutive relation for undamaged composite ply is therefore:

\[ \varepsilon = C_0 \sigma \]  \hspace{1cm} (5.1)

Where \( \sigma \) is the stress tensor with respect to the material principal axes and compliance matrix \( C_0 \) is expressed as follows in this system of coordinates:

\[
 C_0 = \begin{bmatrix}
 \frac{1}{E_1} & -\frac{v_{21}}{E_2} & -\frac{v_{31}}{E_3} & 0 & 0 & 0 \\
 -\frac{v_{12}}{E_1} & \frac{1}{E_2} & -\frac{v_{32}}{E_3} & 0 & 0 & 0 \\
 -\frac{v_{13}}{E_1} & -\frac{v_{23}}{E_2} & \frac{1}{E_3} & \frac{1}{G_{12}} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}} \\
 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{31}} \\
 \end{bmatrix} \]  \hspace{1cm} (5.2)

The components of the stress tensor \( \sigma \) and strain tensor, \( \varepsilon \) are, respectively:

\[
 \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T \hspace{0.5cm} \varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})^T \]  \hspace{1cm} (5.3)

Equations (5.3) are further simplified in the relationship shown in equation (5.4).

\[
 \sigma_{11} = \frac{(-1 + v_{23}v_{32})}{T} E_1 \varepsilon_{11} + \frac{(v_{21} + v_{23}v_{31})}{T} E_2 \varepsilon_{22} - \frac{(v_{21} + v_{23}v_{31})}{T} E_2 \varepsilon_{22} \\
 \sigma_{22} = \frac{(v_{12} + v_{13}v_{32})}{T} E_2 \varepsilon_{22} + \frac{(-1 + v_{13}v_{31})}{T} E_2 \varepsilon_{22} - \frac{(v_{32} + v_{12}v_{31})}{T} E_2 \varepsilon_{22} \\
 \sigma_{33} = \frac{(v_{31} + v_{23}v_{12})}{T} E_3 \varepsilon_{33} - \frac{(v_{23} + v_{13}v_{21})}{T} E_3 \varepsilon_{33} - \frac{(v_{23} + v_{13}v_{21})}{T} E_3 \varepsilon_{33} \\
 \sigma_{12} = 2G_{12} \varepsilon_{12} \\
 \sigma_{23} = 2G_{23} \varepsilon_{23} \\
 \sigma_{31} = 2G_{31} \varepsilon_{31} \\
\]

where

\[
 T = -1 + v_{12}v_{21} + v_{13}v_{31} + v_{23}v_{32} + v_{23}v_{12}v_{31} + v_{13}v_{12}v_{32} \]  \hspace{1cm} (5.4)

**5.3.1.1 Composite Material Properties**

The mechanical properties of composite shown in Table 5.1 are measurable using standard test procedures. Donodan et al [8] have highlighted various procedures for measuring modal fracture
energies. The directional strengths of the tested composites (GFPP) were obtained from the work of Karagiozova et al [1]. In order to determine the fracture energy associated with each fracture mode, tests can be undertaken using three different pre-cracked geometries i.e. Overhead Compact Tension (OCT) (Tension/Compression), Double Edge Notch (DEN) (Tension) and Four-point –Bending specimens. Procedures and Techniques employed for the determination of these parameters were highlighted by Donadon et al [8]. Due to lack of availability of values for intra-laminar fracture energies in open literature, values presented in Table 5.3 are calibrated to validate the experimental results for the test samples.

Table 5. 1: Mechanical properties of GFPP [1, 3-4]

<table>
<thead>
<tr>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
<th>$E_3$ (GPa)</th>
<th>$v_{12}$</th>
<th>$v_{23}$</th>
<th>$v_{31}$</th>
<th>$G_{12}$ (GPa)</th>
<th>$G_{23}$ (GPa)</th>
<th>$G_{31}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>13</td>
<td>4.8</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>1.72</td>
<td>1.72</td>
<td>1.69</td>
</tr>
</tbody>
</table>

Table 5. 2: Strength of composite (GFPP) [1]

<table>
<thead>
<tr>
<th>$S_{1T}$ (MPa)</th>
<th>$S_{2T}$ (MPa)</th>
<th>$S_{1C}$ (MPa)</th>
<th>$S_{2C}$ (MPa)</th>
<th>$S_{3T}$ (MPa)</th>
<th>$S_{FC}$ (MPa)</th>
<th>$S_{FS}$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>300</td>
<td>200</td>
<td>140</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_{12}$ (MPa)</th>
<th>$S_{23}$ (MPa)</th>
<th>$S_{31}$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>140</td>
<td>140</td>
</tr>
</tbody>
</table>
Table 5.3: Intra-laminar fracture energies

<table>
<thead>
<tr>
<th></th>
<th>Fracture Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{f_{11}}$ (tension in direction-1,warp)</td>
<td>30kJ/m²</td>
</tr>
<tr>
<td>$G_{f_{22}}$ (tension in weft direction-2,weft)</td>
<td>30kJ/m²</td>
</tr>
<tr>
<td>$G_{f_{11}}$ (compression in direction-1,warp)</td>
<td>15kJ/m²</td>
</tr>
<tr>
<td>$G_{f_{22}}$ (compression in direction-2,weft)</td>
<td>15kJ/m²</td>
</tr>
<tr>
<td>$G_{f_{33}}$ (through thickness direction)</td>
<td>10kJ/m²</td>
</tr>
<tr>
<td>$G_{s_{12}}$ (in-plane shear)</td>
<td>9kJ/m²</td>
</tr>
<tr>
<td>$G_{s_{23}}$ (out-plane shear)</td>
<td>9kJ/m²</td>
</tr>
<tr>
<td>$G_{s_{31}}$ (out-plane shear)</td>
<td>9kJ/m²</td>
</tr>
</tbody>
</table>

5.3.1.2 Elastic Damage Energy and Dissipation

Using a strain equivalent damage mechanics formulation, a damage elastic compliance matrix $S$ is assumed with the general form:

$S = \begin{bmatrix}
\frac{1}{(1 - d_1)}E_1 & -v_{21}/E_2 & -v_{31}/E_3 & 0 & 0 & 0 \\
-v_{12}/E_1 & \frac{1}{(1 - d_2)}E_2 & -v_{32}/E_3 & 0 & 0 & 0 \\
-v_{13}/E_1 & -v_{23}/E_3 & \frac{1}{(1 - d_3)}E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{(1 - d_4)}G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{(1 - d_5)}G_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{(1 - d_6)}G_{31}
\end{bmatrix}$ (5.5)

The above compliance matrix has six scalar damage parameters $d_1, d_2, d_3, d_4, d_5, d_6$. Intrinsic to this formulation is the assumption that symmetry of the compliance matrix is preserved throughout the analysis and is never violated. These six damage parameters have values between zero and one corresponding to intact (virgin) and fully damaged (fractured) material, respectively. Values of $d_1, d_2$ and $d_3$ are associated with damage in the warp (fibre direction, direction 1), weft (direction 2) and through thickness directions respectively while, $d_4, d_5, d_6$ correspond to the in-plane and out-of-plane shear failures. Parameters $d_1 - d_6$ can be viewed as invariant functions that represent physical parameters. Note there are no additional damage parameters to model independent degradation in Poisson’s ratios. The general damage
mechanics formulation is based on an internal energy function $\varphi$ for an orthotropic solid. The function $\varphi$ is shown in the relationship in equation (5.6).

$$\varphi = \frac{1}{2} \sigma^T S \sigma$$  \hspace{1cm} (5.6)

We introduce thermodynamic forces ($Y = Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$) which act as driving forces for damage development. It can be shown that the strain tensor $\varepsilon$ and the thermodynamic forces $Y$ can be derived from the strain energy function as:

$$\varepsilon = \frac{\partial \varphi}{\partial \sigma} = S \sigma \quad \text{and} \quad Y = \frac{\partial \varphi}{\partial d}$$  \hspace{1cm} (5.7)

It implies that for elastic material:

$$Y_1 = \frac{\sigma_{11}^2}{2(1 - d_1)^2 E_1} \quad Y_2 = \frac{\sigma_{22}^2}{2(1 - d_2)^2 E_2}$$

$$Y_3 = \frac{\sigma_{33}^2}{2(1 - d_3)^2 E_3} \quad Y_4 = \frac{\sigma_{12}^2}{2(1 - d_4)^2 G_{12}}$$

$$Y_5 = \frac{\sigma_{23}^2}{2(1 - d_5)^2 G_{23}} \quad Y_6 = \frac{\sigma_{31}^2}{2(1 - d_6)^2 G_{31}}$$  \hspace{1cm} (5.8)

The variables $Y_i$ are referred to as damage energy rates in particular failure modes. It can be inferred, as we will see later that:

$$d_i = f_i(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$$  \hspace{1cm} (5.9)

Where $f_i$ ($f_1, f_2, f_3, f_4, f_5, f_6$) are to be determined from a multi-axial failure or interaction between damage states. Further model assumptions are- (a) The six damage modes in all stress directions are decoupled and determined by $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$. This assumption was made by A.F. Johnson [9] in modelling 2-D plane elements for reinforced composites. (b) Ply material is non-healing thus on unloading after been damaged the damage parameter remains constant until a larger damaging load is applied, in other words each damage parameter is an absolutely ascending function of time irrespective of loading history and multi-axial pattern of damage. Thus, the evolution function depends on the maximum values of $Y_i$ attained. (c) The inelastic behaviour of the matrix is ignored and was found in this particular case not to affect the correlation between the experiments with the numerical model.
In order to justify assumption (b), we introduce a parameter $Y_i(t)$ related to maximum value of damage forces reached during the previous loading history.

$$Y_i(t) = \max(Y_i(\tau)) \quad \text{where} \quad \tau \leq t \quad (5.10)$$

Thus, imagining an elastic domain without damage at the outset of loading and subsequent evolution to a lower modal damage initiation parameter, and full separation or cut off at an upper damage threshold we deduce that:

$$d_1 = 0 \text{ for } \bar{Y}_1 < Y_{10} \quad ; \quad d_1 = f(\bar{Y}_1 - Y_{10}) \text{ for } Y_{10} < \bar{Y}_1 < Y_{1f} \quad \text{or} \quad d_1 = 1$$

$$d_2 = 0 \text{ for } \bar{Y}_2 < Y_{20} \quad ; \quad d_2 = f(\bar{Y}_2 - Y_{20}) \text{ for } Y_{20} < \bar{Y}_2 < Y_{2f} \quad \text{or} \quad d_2 = 1$$

$$d_3 = 0 \text{ for } \bar{Y}_3 < Y_{30} \quad ; \quad d_3 = f(\bar{Y}_3 - Y_{30}) \text{ for } Y_{30} < \bar{Y}_3 < Y_{3f} \quad \text{or} \quad d_3 = 1$$

$$d_4 = 0 \text{ for } \bar{Y}_4 < Y_{40} \quad ; \quad d_4 = f(\bar{Y}_4 - Y_{40}) \text{ for } Y_{40} < \bar{Y}_4 < Y_{4f} \quad \text{or} \quad d_4 = 1$$

$$d_5 = 0 \text{ for } \bar{Y}_5 < Y_{50} \quad ; \quad d_5 = f(\bar{Y}_5 - Y_{50}) \text{ for } Y_{50} < \bar{Y}_5 < Y_{5f} \quad \text{or} \quad d_5 = 1$$

$$d_6 = 0 \text{ for } \bar{Y}_6 < Y_{60} \quad ; \quad d_6 = f(\bar{Y}_6 - Y_{60}) \text{ for } Y_{60} < \bar{Y}_6 < Y_{6f} \quad \text{or} \quad d_6 = 1 \quad (5.11)$$

In this model, we assume nonlinear (exponential) forms for function $f$. i.e. for $d_1, d_2, d_3, d_4, d_5, d_6$. Thus, the evolution equations of a three-dimensional composite element requires the relation between $d_i$ and $(\bar{Y}_i - Y_{i0})$, where $i = 1, 2, 3, 4, 5, 6$ as shown in the equation above. In the above equation, the threshold parameters, defined as $Y_{10}, Y_{20}, Y_{30}, Y_{40}, Y_{50}, Y_{60}, Y_{1f}, Y_{2f}, Y_{3f}, Y_{4f}, Y_{5f}, Y_{6f}$ determine the bounds for the damage parameters.

### 5.3.1.3 Damage Progression Criteria

The evolution function for the damage proposed in the relationship of equation (5.12) is empirically determined following a procedure similar to the one adopted by Matzenmiller et al [9-10]. The damage evolution at a material point is defined in terms of an internal variable $Q_m$ ($m = 1, 2, 3, 4, 5, 6, 7, 8$) associated with failure index $r_m$ by the empirical relationship in equation (5.13). The threshold values of $r_m$ where damage sets in is represented as $r_0$. The eight failure modes shown by the modified Hashin criteria in equations (5.19) determines the value of the failure index $r_0$. The value $H_m$ measures the rate of damage evolution.

$$Q_m = 1 - \frac{r_0}{r_m} \exp \left\{ -2H_m \left( \frac{r_m - r_0}{r_0} \right) \right\} \quad (m = 1, 2, 3, 4, 5, 6, 7, 8) \quad r_m \geq 1 \quad (5.12)$$
Equation (5.14) shows how the various failure modes (internal variable, $Q_m$) affects the properties of the composite in all directions (i.e. the values of $d_i$).

$$d_1 = \max\{Q_1, Q_3, Q_5\}$$
$$d_2 = \max\{Q_2, Q_4, Q_5\}$$
$$d_3 = Q_5$$
$$d_4 = \max\{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$$
$$d_5 = \max\{Q_3, Q_4, Q_5, Q_7\}$$
$$d_6 = \max\{Q_1, Q_2, Q_5, Q_8\}$$

(5.13)

In the relationship above, it can be seen that more than one failure indexes, $Q_m$, affects the various individual values of failure parameter, $d_i$. Fibre damage in either the fill or warp directions results in or influences the reduction of stiffness in the loading direction and in the related shear direction. For failure in shear direction, notice that $Q_5$ (related to fibre crushing mode i.e. as a resultant of high through thickness compressive stress – equation (5.19e)) influences all six components of the damage vector, $d_i$.

In this model, elastic properties are degraded as damage evolves and the damaged and undamaged constitutive equations are checked and updated in the model per increment. The irreversibility of the internal variables is accounted for by requiring that $dQ_m = 0$ whenever $dr_m \leq 0$ where $dr_m$ represents an increment in $r_m$ for an increment in the applied load.

### 5.3.1.4 Fracture Energy (Energy Release Rate) and Mesh Objectivity

Equation (5.13) proposed above defines the softening evolution in composites after the damage is initiated. We assume the crack growth direction is parallel to one edge of the finite element and the composite meshes are structured. Consider a multi-axial state of stress in which the strain components increase monolithically from an unstressed state to another in which full degradation takes place, the specific energy dissipated for each of the damage modes in equations (5.19) can be written as:

$$\int_{t=0}^{t=\infty} \dot{D} \, dt = \int_{t=0}^{t=\infty} W^e \dot{Q} \, dt = \frac{S_m}{2E_m} \int_{r_m=r_0}^{r_m=\infty} r_m Q_m \dot{r}_m \, dr_m \text{ where } \dot{Q} = Q' \dot{r} \quad (5.14)$$
Here $E_m$ corresponds to shear or elastic modulus, $S_m$ is the effective strength of the composite in the particular direction in question with reference to the failure modes in equation (5.19) and $l_{elm}$ represents the characteristic length of the finite element. The term $G_m$ corresponds to the fracture energy associated with each failure mode in equation (5.19). After calibration with experimental results, the respective fracture energy is given in Table 5.3. It is convenient to express the effective stress damage evolution as:

$$\sigma(r) = S_m(1 - d(r))r \quad r_0 \leq r$$  \hspace{1cm} (5.15)

Here, the mechanical free energy term associated with each damage mode for the damage model is defined in the from equation (5.16a) and the rate of mechanical dissipation is valid if the damage indexes increase monotonically, thus:

$$W = (1 - d)W^o(\varepsilon) \quad \dot{D} = W^o(\varepsilon)\dot{\varepsilon} \geq 0$$  \hspace{1cm} (5.16a and b)

We propose from the above analysis a value for $H_m$ as:

$$H_m = \frac{\bar{H}_s l_{elm}}{1 - \bar{H}_s l_{elm}} \geq 0 \quad \bar{H}_s = \frac{S_m^2}{2E_m G_m}$$  \hspace{1cm} (5.17)

Where $G_m$ is the specific fracture energy corresponding to each damage mode, $g_m$ is the energy dissipated per unit volume, $d(r)$ depicts that the damage parameter as a function of $r$ where $r_0$ corresponds to the point of damage initiation.

The value $H_m$ measures the brittleness of the finite element. As observed by Cervera et al [11], in finite element analysis, the state variables of the local model are computed at the integration points in terms of the local stress or strain history. Thus, the characteristic length $l_{elm}$ is related to the volume or area of the finite element. This implies that for a simple beam element, for instance, the characteristic length can be taken as the size of the element. For equilateral plane triangular element with area $A_e$ we have the characteristic length as equation (5.18a). In addition, for a tetrahedral and a cubic element, respectively with volumes equal to $V_e$, we have the characteristic lengths as shown respectively in equations (5.18b) and (5.18c).

$$l_{elm}^2 = \left(\frac{4}{\sqrt{3}}\right)A_e \quad l_{elm}^3 = \left(\frac{12}{\sqrt{2}}\right)V_e \quad l_{elm}^3 = V_e$$  \hspace{1cm} (518a – 518c)
5.3.1.5 Damage Initiation Criteria

As mentioned earlier, the composite is assumed to behave elastically in an orthotropic manner until the onset of damage. The Modified Hashin criteria shown in equations (5.19a)-(5.19h) determine the point of damage initiation. The modified Hashin criteria can be used in predicting damage in woven composites [5]. Damage initiates in any of the failure modes when \( r_m = 1 \).

**Fill and warp fibre tensile/shear failure mode**

The failure surface that characterises this mode is given by the quadratic interaction between the associated axial and through thickness shear strains, i.e.

For fill-direction (direction 1),

\[
(r_1)^2 = \left( \frac{E_1 \epsilon_1}{S_{1T}} \right)^2 + \left( \frac{G_{31} \epsilon_{31}}{S_{1FS}} \right)^2
\]  

(5.19a)

For warp-direction (direction 2),

\[
(r_2)^2 = \left( \frac{E_2 \epsilon_2}{S_{2T}} \right)^2 + \left( \frac{G_{23} \epsilon_{23}}{S_{2FS}} \right)^2
\]  

(5.19b)

Where for the woven composite ply, 1, 2, and 3 represent in-plane fill (direction 1), in-plane warp (direction 2), and out-of-plane (direction 3) directions, respectively. \( E \) and \( G \) are tensile and shear moduli, respectively. \( S_{1T} \) and \( S_{2T} \) are tensile strengths in the fill and warp directions. \( S_{1FS} \) and \( S_{2FS} \) are fibre shear strengths in 1-3 and 2-3 directions, \( \epsilon_1 \) and \( \epsilon_2 \) are failure tensile strains in \( a \) and \( b \) directions, \( \epsilon_{31} \) and \( \epsilon_{23} \) are shear strains in 1-3 and 2-3 planes. We assume that \( S_{1FS} = S_{FS} \) and \( S_{2FS} = S_{FS} S_{2T}/S_{1T} \). Where \( S_{FS} \) is the fibre shear strength in Table 5.1.

**Fibre compressive failure mode**

The in-plane compressive damage in directions 1 and 2 are given by the following failure criteria:

For fill direction (direction 1)

\[
(r_3)^2 = \left( \frac{E_1 \epsilon'_1}{S_{1C}} \right)^2 \quad \text{where} \quad \epsilon'_1 = -\epsilon_1 - \epsilon_3 \frac{E_3}{E_1}
\]  

(5.19c)

For fill direction (direction 2)

\[
(r_4)^2 = \left( \frac{E_2 \epsilon'_2}{S_{2C}} \right)^2 \quad \text{where} \quad \epsilon'_2 = -\epsilon_2 - \epsilon_3 \frac{E_3}{E_2}
\]  

(5.19d)
Where $S_{1C}$ and $S_{2C}$ are in-plane compressive strengths in directions 1 and 2, respectively.

**Fibre crush failure mode**

The crush damage mode as a result of high through thickness compressive pressure from blast waves can be modelled with the surface

$$ (r_5)^2 = \left( \frac{E_3 \varepsilon_3}{S_{FC}} \right)^2 $$

Where $S_{FC}$ is the fibre crush strength.

**Fibre in-plane shear failure mode (plane 1-2)**

A woven layer can damage under in-plane shear stress without occurrence of fibre-breakage. The in-plane matrix damage mode is given by

$$ (r_6)^2 = \left( \frac{G_{12} \varepsilon_{12}}{S_{12}} \right)^2 $$

Where $S_{12}$ is the layer shear strength due to matrix shear failure.

**Fibre out-of-plane shear failure mode (plane 2-3)**

The in-plane matrix damage mode is given by

$$ (r_7)^2 = \left( \frac{G_{bc} \varepsilon_{bc}}{S_{23}} \right)^2 $$

Where $S_{23}$ is the corresponding layer shear strength due to matrix shear failure.

**Fibre in-plane shear failure mode (plane 1-3)**

The in-plane matrix damage mode is given by

$$ (r_8)^2 = \left( \frac{G_{31} \varepsilon_{31}}{S_{31}} \right)^2 $$

Where $S_{31}$ is the corresponding layer shear strength due to matrix shear failure.

5.3.1.6 Strain rate effect

The effect of strain rate on the strength values of composite failure modes is modelled by multiplying the associated strength values $S_0$ by a scale factor:

$$ S_{RT} = S_0 \left( 1 + C_{rate} \ln \left( \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} \right) \right) $$

(5.20)
Where $C_{rate}$ is the strain rate constant, and $S_0$ are the available strength values of $S_{RT}$ at the reference strain rate $\dot{\varepsilon}_0 = 1s^{-1}$. In this model $C_{rate} = 0.35$ [3-4].

### 5.3.2 Aluminium Constitutive Model

The aluminium component of the FML was modelled using the Johnson-Cook plasticity failure model, which is a special type of Mises plasticity model with analytical forms of the hardening law and rate dependency. Its suitability for high-strain rate deformation of many materials (including most metals) makes it an attractive option in the development of this model. In using this model, adiabatic transient response is usually assumed. The Johnson-Cook’s hardening model (a type of isotropic hardening) is:

$$
\sigma^0 = [A + B(\bar{\varepsilon}^{pl})^n][1 - \theta^m]
$$

(5.21)

Where $\sigma^0$ is the static yield stress, $\bar{\varepsilon}^{pl}$ is the equivalent plastic strain, and $A$, $B$, $C$, $n$ and $m$ are material constants for Aluminium 2024-O, properties which are determined by experiment and are measured at or below the ambient temperature ($\theta_{\text{transition}}$ or transition temperature). $\hat{\theta}$ is non-dimensional and is defined as:

$$
\hat{\theta} = \begin{cases} 
\frac{\theta - \theta_{\text{transition}}}{\theta_{\text{melt}} - \theta_{\text{transition}}} & \text{for } 0 < \theta_{\text{transition}} \\
0 & \text{for } 0 \leq \theta_{\text{transition}} \leq \theta_{\text{melt}} \\
1 & \text{for } 0 > \theta_{\text{melt}}
\end{cases}
$$

(5.22)

Here, $\theta$ is the current temperature, $\theta_{\text{transition}}$ is the temperature defined as the one at or below which there is no temperature dependency in the expression of the yield stress, and $\theta_{\text{melt}}$ is the melt temperature. Adiabatic conditions are assumed such that all internal plastic work is converted into heat thus temperature changes accordingly i.e.

$$
\Delta T = \frac{\bar{\sigma}\bar{\varepsilon}^{pl}}{\rho C_v}
$$

(5.23)

In cases where $\theta \geq \theta_{\text{melt}}$, the material will be melted and would be fluid-like, thus, there would be no shear resistance ($\sigma^0 = 0$). Abaqus, in this scenario, removes the hardening memory by setting the equivalent plastic strain to zero. The strain rate assumption of the Johnson-Cook material model postulates that:

$$
\bar{\sigma} = \sigma^0(\bar{\varepsilon}^{pl}, \theta)R(\dot{\varepsilon}^{pl}) \quad \text{where } \dot{\varepsilon}^{pl} = \dot{\varepsilon}_0 \left[\frac{1}{C} (R - 1)\right]
$$

(5.24)
\( \bar{\sigma} \) is the dynamic yield stress i.e. yield stress at nonzero strain rate; \( \dot{\varepsilon}_{\text{pl}} \) is the equivalent plastic strain rate; \( \dot{\varepsilon}_0 \) and \( C \) are material parameters at or below the transition temperature, \( \theta_{\text{transition}} \); \( \sigma^0(\dot{\varepsilon}_{\text{pl}}, \theta) \) is the static yield stress; \( R(\dot{\varepsilon}_{\text{pl}}) \) is the ratio of the yield stress at non-zero strain rate to the static yield stress (such that \( R(\dot{\varepsilon}_0) = 1.0 \)). Thus, the yield stress is:

\[
\bar{\sigma} = [A + B(\dot{\varepsilon}_{\text{pl}})^n] \left[ 1 + C \ln \left( \frac{\dot{\varepsilon}_{\text{pl}}}{\dot{\varepsilon}_0} \right) \right] \left[ 1 - \dot{\theta}^m \right]
\]  
(5.25)

The effective plastic strain \( \varepsilon_{\text{pl}} \) (PEEQ) is defined as:

\[
\varepsilon_{\text{pl}} = \varepsilon_{\text{pl}|0} + \int_0^t \frac{2}{3} \frac{\dot{\varepsilon}_0}{\varepsilon_{\text{pl}|0} + \dot{\varepsilon}_p} \dot{\varepsilon}_{\text{pl}} \, dt
\]  
(5.26)

Where \( \varepsilon_{\text{pl}|0} \) is the initial is plastic strain and is usually taken as zero. The effective stress \( \bar{\sigma} \) is defined based on \( J_2 \)-plasticity model by:

\[
\bar{\sigma} = \sqrt{\frac{3}{2} \sigma_{ij} \sigma_{ij}}
\]  
(5.27)

The Johnson-Cook failure model is used in conjunction with its plasticity model in the numerical modelling of failure of the aluminium sheet metal. The failure mechanism is based on the value of the effective (or equivalent) plastic strain at element integration points. Fracture occurs in the aluminium sheet when the damage parameter \( \omega \) exceeds 1.0. The evolution of \( \omega \) is given by the accumulated incremental effective plastic strains divided by the current strain at fracture.

\[
\omega = \sum \frac{\Delta \varepsilon_{\text{pl}}}{\dot{\varepsilon}_{\text{f}}}
\]  
(5.28)

where \( \Delta \varepsilon_{\text{pl}} \) is an increment of the equivalent plastic strain, \( \varepsilon_{\text{f}} \) is the strain at failure, \( \dot{\varepsilon}_{\text{f}} \), is assumed to be dependent on a non-dimensional plastic strain rate, \( \frac{\dot{\varepsilon}_{\text{pl}}}{\dot{\varepsilon}_0} \); a dimensionless pressure-deviatoric stress ratio, \( \frac{p}{q} \) where \( p \) is the pressure stress (hydrostatic component of stress tensor corresponding to the spherical, purely dilatational, and irrotational part of the deformation) and \( q \) is the Mises stress (related to the second invariant of the stress deviatoric tensor \( J_2 \) representing equivoluminal distortion); and the non-dimensional temperature, \( \dot{\theta} \), defined earlier in the Johnson-Cook hardening model. The dependencies are assumed separable and are of the form:

\[
\dot{\varepsilon}_{\text{f}} = \left[ D_1 + D_2 \exp \left( D_3 \frac{p}{q} \right) \right] \left[ 1 + D_4 \ln \left( \frac{\dot{\varepsilon}_{\text{pl}}}{\dot{\varepsilon}_0} \right) \right] \left[ 1 + D_5 \dot{\theta} \right]
\]  
(5.29)
Where $D_1$ to $D_5$ are non-dimensional failure parameters measured at or below the transition temperature. It is important to note that the failure parameter $D_3$ is reported as being negative in literature for aluminium alloy studied. However, Abaqus’s general expression for strain at fracture expects this parameter to be positive for most materials. It is worthy of mentioning that failure to properly account for the sign of $D_3$ will result in an inaccurate response. Choosing a zero value for the fracture energy, which is used as a data parameter for the damage evolution law, completes the setting of the fracture model. Elements are deleted by default upon reaching maximum degradation according to the usual rules of Abaqus progressive damage framework. Table 5.4 shows the values of the plasticity and failure parameters used for Al 2024-O in this chapter.

Table 5.4: Mechanical properties of Aluminium (Al2024-O)

<table>
<thead>
<tr>
<th>Al 2024-O</th>
<th>$\rho$, kg/m$^3$</th>
<th>$E$ (Gpa)</th>
<th>$A$ (MPa)</th>
<th>$B$ (MPa)</th>
<th>$n$</th>
<th>$C$</th>
<th>$\varepsilon_0$, s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2700</td>
<td>73.4</td>
<td>85</td>
<td>325</td>
<td>0.4</td>
<td>0.001</td>
<td>0.0083</td>
<td></td>
</tr>
<tr>
<td>UTS (MPa)</td>
<td>$D_1$</td>
<td>$D_2$</td>
<td>$D_3$</td>
<td>$D_4$</td>
<td>$D_5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>186</td>
<td>0.13</td>
<td>0.13</td>
<td>-1.5</td>
<td>0.011</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.3.3 Cohesive Elements and Interface Simulation

The constitutive behaviour of the adhesive layer in this chapter is described in terms of direct traction vs. separation model implemented in Abaqus (usually preferable for bonded interface where the thickness of the adhesive is negligible and the path of fracture lies essentially within the adhesive) [12]. This usually assumes an initial linear elastic model followed by the initiation and evolution of the damage. The nominal stress vector relates to the nominal strains across the interface with an elastic constitutive matrix. The nominal stresses are the force component divided by the original area at each integration point while the nominal strains are the separations divided by the original thickness at each integration point. However, the default choice of the constitutive thickness in terms of traction separation is 1.0 (irrespective of the thickness of the adhesive layer). Thus, the diagonal terms of elastic matrix shown in equation (5.30) are:
Where \( t_n \) represents the normal traction, and \( t_s \) and \( t_t \) are the two shear tractions (i.e. in the first and second shear directions). The term \( \rho_c \) is the actual density of the cohesive element and \( \rho \) is the inputted density. The elasticity matrix in equation (5.30) provides a fully coupled behaviour between all components of the traction vector and the separation vector (strain vector). Note the off diagonal terms in the matrix can be set to zero if the uncoupled behaviour between stress and strain is desired. This is the case for the model used in this chapter.

As shown by the graph of typical traction separation response in Figure 5.2, the damage initiation refers to the beginning of degradation in material constitutive response at a material point. The damage initiation in damage models begins when the stress and/or strains meet some damage initiation criteria, in this case a quadratic nominal stress criterion as shown in equation (5.31).

\[
\left( \frac{t_n}{t_n^0} \right)^2 + \left( \frac{t_s}{t_s^0} \right)^2 + \left( \frac{t_t}{t_t^0} \right)^2 = 1
\]

(5.31)

Where \( t_n^0, t_s^0, t_t^0 \) represent the peak values of the nominal stress when the deformation is either purely normal to the interface or purely in the first or the second shear direction, respectively. Equation (5.32) shows the dependency of the fracture energy on the mode mix, where \( \alpha = 1 \). This law states that failure under mixed-mode conditions is governed by a power interaction of the energies required to cause failure in the individual (normal and two shears) modes.

\[
\left\{ \frac{G_n}{G_n^c} \right\}^\alpha + \left\{ \frac{G_s}{G_s^c} \right\}^\alpha + \left\{ \frac{G_t}{G_t^c} \right\}^\alpha = 1
\]

(5.32)

Where the mixed mode fracture energy is \( G_c = G_n + G_s + G_t \) when the above equation is satisfied. In the expression, the quantities \( G_n, G_s \) and \( G_t \) refer to the work done by the traction and its conjugate relative displacement in the normal, the first, and the second shear directions, respectively. The specified quantities \( G_n^c, G_s^c \) and \( G_t^c \) refer to the critical fracture energies required to cause failure in the normal, the first, and the second shear direction, respectively. In this chapter, the thickness of the interface adhesive between aluminium and GFPP is taken as 0.001mm and the thickness of adhesive between the GFPP plies is taken as 0.0005mm. The
interface and laminate adhesives are assumed to have the same mechanical properties in this model. The properties of the cohesive layer are shown in Table 5.5.

![Diagram of traction-separation response of cohesive element]

Figure 5.2: Traction-separation response of cohesive element

Table 5.5: Properties of cohesive layer

<table>
<thead>
<tr>
<th>Elastic properties</th>
<th>Damage Initiation</th>
<th>Damage Evolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_c$ (kg/m$^3$)</td>
<td>$E_n$ (GPa)</td>
<td>$\sigma_n$ (MPa)</td>
</tr>
<tr>
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<td>2.05</td>
<td>140</td>
</tr>
<tr>
<td></td>
<td>0.72</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>0.72</td>
<td>300</td>
</tr>
</tbody>
</table>

5.4. Localised Blast

Simulation of the dynamic response in localised blast loaded FML’s is of importance due to the differences this type of loading attributes as compared to global blast. The apparently peculiar nature of this kind of loads renders the dynamic solution, damage pattern, thresholds, and critical conditions intrinsically different. When localised blast loads are applied to plates, it is normal practice to idealise the localised blast as pressure loading relating to impulse obtained from
actual experiment. For localised blast at the centre of a plate, the pressure loading is a function of time and distance from the centre of the plate [1].

However, for FML’s, like for other plated system the response to localised blast includes two regimes: one associated with the initial through-thickness compression phase, the subsequent reflected tensile from the back face, and the other related to the overall response. These characteristics make the time history of localised blast load very important.

This work has been validated by the experimental work conducted by other researchers at the University of Cape Town, the loading comprises a disk of 8g explosive (cylindrical charge) with a diameter of $d_0 = 30$mm and a leader of 1 g which was placed at the centre of the panel. Result from AUTODYN simulation shows that the pulse is well described by an exponentially decaying function $\exp\left(-2t/t_0\right)$ in time with $t_0=0.008\text{ms}$ [1]. The spatial distribution of the loading shows that the leader influences the shape of the pressure pulse by producing a non constant pressure within the area bounded by the disk of explosive, $r \leq d_0/2$ [1]. Figure 5.3 shows the temporal and spatial distributions of the localised blast load.

![Pulse shape approximation](image)
The pressure function, \( P(r, t) \) used in this model is:

\[
P(r, t) = p_1(r)p_2(t)
\]  \hspace{1cm} (5.33)

Where,

\[
p_1(r) = \begin{cases} 
P_0 & r \leq r_0 \\ 
P_0 \exp[-k(r - r_0)] & r_0 < r \leq r_b \\ 
0 & r > r_b
\end{cases}
\]

\[
p_2(t) = \exp(2t/t_0)
\]  \hspace{1cm} (5.34a)

In the above equation, \( r_0 = 15 \text{mm} \) which is the radius of the explosive disc used in the experiments, \( [k] = [\text{m}^{-1}] \) is an exponential decay constant, which models the pressure distribution over the exposed area of the plate, \( r_b < L/2 \), where \( L \) is the length of the panel and \( t_0 \) is the characteristic decay time for the pulse \([1]\). The decay constant, \( k \), depends on the ratio \( d_0/L \) through the relationship:

\[
k = 130 - 261(d_0/L) + 948(d_0/L)^2 \quad 0.15 < \frac{d_0}{L} < 0.6
\]  \hspace{1cm} (5.35)
The range $d_0/L$ was restricted by the experimental configurations used, however, the above expression is still valid for current configuration, $d_0/L = 0.1$. Thus $k \approx 114 \text{m}^{-1}$. The total impulse is given by the expression:

$$I = 2\pi \int_0^{\infty} \int_0^{r_b} P(r,t) \, dr \, dt$$  

(5.36)

Figure 5.4 shows the spatial distribution of local blast on the centre of the FML plate. Table 5.6 shows the values of calculated peak pulse, $P_0$.

Figure 5. 4: Panel showing distribution of applied localised blast load
Table 5. 6: Computed peak pulse pressure

<table>
<thead>
<tr>
<th>Lay-ups</th>
<th>Thickness (mm)</th>
<th>Impulse (Ns)</th>
<th>Pressure, $P_0$(MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2T14-2</td>
<td>3.36</td>
<td>5.89</td>
<td>741.5</td>
</tr>
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<td>5.35</td>
<td>6.17</td>
<td>775.1</td>
</tr>
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<td>A2T18-4</td>
<td>5.6</td>
<td>7.94</td>
<td>1000.1</td>
</tr>
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<td>948.2</td>
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<td>A3T22-4</td>
<td>4.13</td>
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<td>975.1</td>
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<tr>
<td>A3T24-5</td>
<td>6.08</td>
<td>10.58</td>
<td>1326.3</td>
</tr>
<tr>
<td>A3T24-7</td>
<td>6.27</td>
<td>3.76</td>
<td>471</td>
</tr>
<tr>
<td>A3T24-8</td>
<td>6.06</td>
<td>7.85</td>
<td>984</td>
</tr>
<tr>
<td>A3T26-1</td>
<td>8.49</td>
<td>7.8</td>
<td>980</td>
</tr>
<tr>
<td>A3T26-3</td>
<td>8.1</td>
<td>9.54</td>
<td>1196</td>
</tr>
<tr>
<td>A3T26-4</td>
<td>8.41</td>
<td>11.29</td>
<td>1415.5</td>
</tr>
<tr>
<td>A3T28-4</td>
<td>9.84</td>
<td>12.43</td>
<td>1560.1</td>
</tr>
<tr>
<td>A3T28-5</td>
<td>9.82</td>
<td>10.34</td>
<td>1296.2</td>
</tr>
</tbody>
</table>

5.5. Results and Discussion

The experimental localised blast load as illustrated in equations (5.33-5.36) varies temporarily and spatially. Temporal and spatial variation of load cannot be inputted directly into Abaqus. Thus, a simple user defined subroutine for this type of load is developed to depict the localised blast load illustrated in equations (5.33-5.36). The loads are applied to the developed FML model and the results compared with the experimental results presented by Langdon et al [13]. The significant back face deflection was observed as the first response regime of the structures in the numerical models. This can be attributed to the through-thickness reflected tensile wave propagation in the FML. Fig 5.5 (b) and (c) show large back face deflection with no tearing in the composites. This can be compared to the diamond shaped deflected back face of panel A2T18-3 shown in Figure 5.5a. The numerical models comprising of 1mm x 1mm size with
appropriate softening modulus calculated from equations 5.17 predicted same level of front and back face displacement of the aluminium plate when compared to models with mesh size 1.5mm x 1.5mm. Thus, validating the mesh size algorithm postulation.

Figure 5.6 (b) and (c) show the large inelastic displacement of the back face of panel A3T28-4 and the level of tensile damage in direction 1 in the composite panels. The models predict similar level of damage in the composite layer and a good correlation with the back and front face maximum displacement of the aluminium layers.

Table 5.7 shows a summary of the predicted displacements of the back and front faces of the aluminium panel.

The numerical models in Figures 5.7 and 5.8 are able to capture the tearing of the aluminium layers and composite layers of panels for panels A2T14-1 and A3T24-3.
Figure 5. (a) Photographs of the back faces of the FML panels from experiment for A2T18-3 and A2T14-1 configurations [13]  (b) Numerical simulation of A2T18-3 panel with mesh size 1.5mm x 1.5mm (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel) (c) Numerical simulation of A2T18-3 panel with mesh size 1mm x 1mm (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel), $t=0.15$ms
Figure 5. 6: (a) Photographs of the back faces of the FML panels from experiment for A3T28-4 configurations [13] (b) Numerical simulation of A3T28-4 panel with 1.5mm x 1.5mm mesh size (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel) (c) Numerical simulation of A3T28-4 panel with 1mm x 1mm mesh size, (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel) $t = 0.15\text{ms}$
Figure 5. 7 (a) Photographs of the back faces of the FML panels from experiment for A2T14-1 configurations [13] (b) Numerical simulation of A2T14-1 panel with 1.5mm x 1.5mm mesh size (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel) (c) Numerical simulation of A2T14-1 panel with 1mm x 1mm mesh size, $t = 0.15\text{ms}$ (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel)
Figure 5. 8 (a) Photographs of the back faces of the FML panels from experiment for A3T24-3 configurations [13] (b) Numerical simulation of A3T24-3 panel with 1.5mm x 1.5mm mesh size (one quarter of panel was modelled with appropriate boundary conditions to depict the response of full panel), $t = 0.15$ms

Table 5. 7: Comparison of displacement of FML panels

<table>
<thead>
<tr>
<th>Panel</th>
<th>Mass PE 4(g)</th>
<th>Mean Thickness in (mm)</th>
<th>Impulse (Ns)</th>
<th>Experimental Back face deflection</th>
<th>Experimental Front face deflection</th>
<th>Numerical Back face deflection</th>
<th>Numerical Front face deflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT14-1</td>
<td>2.5</td>
<td>1.01</td>
<td>3.38</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>A2T18-4</td>
<td>2.8</td>
<td>5.60</td>
<td>7.94</td>
<td>16.3</td>
<td>9.1</td>
<td>16.0</td>
<td>8.1</td>
</tr>
<tr>
<td>A3T28-4</td>
<td>5.0</td>
<td>9.84</td>
<td>12.43</td>
<td>26.7</td>
<td>4.1</td>
<td>26</td>
<td>4.9</td>
</tr>
<tr>
<td>A2T24-3</td>
<td>5.3</td>
<td>1.74</td>
<td>6.06</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

5.5.1 Efficiency of the Charge and Effectiveness of the Target

The efficiency of localised blast on the target can be represented in a non-dimensional form for correlation studies and investigation into the effect of such charges on the target. We can rewrite the distribution of the above local blast load on the FML as:

$$p_s(r) = \begin{cases} \frac{p_0}{P_0 \exp[-k(r - R_0)]} & r \leq R_0 \\ \frac{p_0}{P_0} & R_0 < r \leq +\infty \end{cases}$$
\[ p_t(t) = \begin{cases} 
  e^{-\omega t} & 0 \leq t < t_d \\
  0 & t > t_d 
\end{cases} \] (5.37)

Thus, for \( r < R_0 \) we can show that the total impulse \( I_{r<R_0}(r, t_d) \) is:

\[
\lim_{t_d \to 0^+} I_{r<R_0}(r, t_d) = \pi P_0 r^2 \lim_{t_d \to 0^+} \left( \frac{1 - e^{-\omega t_d}}{\omega} \right) = \pi r^2 P_0 t_d
\] (5.38)

In addition, for \( R_0 < r < \infty \) we can show that the total impulse, \( I_{R_0<r<\infty}(r, t_d) \) is:

\[
\lim_{t_d \to 0^+} I_{R_0<r<\infty}(r, t_d) = \left( \frac{2\pi P_0}{b^2} + \frac{2\pi P_0 R_0}{b} - \frac{2\pi P_0 e^{b R_0}}{b^2 e^{b r}} - \frac{2\pi P_0}{b e^{b r}} \right) \lim_{t_d \to 0^+} \left( \frac{1 - e^{-\omega t_d}}{\omega} \right)
\]

\[
= \left( \frac{2\pi P_0}{b^2} + \frac{2\pi P_0 R_0}{b} - \frac{2\pi P_0 e^{b R_0}}{b^2 e^{b r}} - \frac{2\pi P_0}{b e^{b r}} \right) P_0 t_d
\] (5.39)

Introducing non-dimensional parameters, we have

\[ \lambda = \frac{r}{R_0} \quad \text{and} \quad I_r(\lambda) = \frac{I_r(r, t_d)}{\pi R_0^2 P_0 t_d}, \eta = \frac{I_r(\lambda)}{I_r(\infty)} \] (5.40)

Where \( \eta \) referred to as the efficiency of the charge and \( I_r \) is the non-dimensional total impulse on the FML plate. After simplification, \( \eta \) can be written as:

\[
\eta = \frac{I_r(\lambda)}{I_r(\infty)} = \frac{1 + \frac{2}{R_0^2 b^2} + \frac{2}{R_0 b} - \frac{2 e^{b R_0}}{R_0^2 b^2 e^{b R_0}} - \frac{2 e^{b R_0}}{R_0 b e^{b R_0}}}{1 + \frac{2}{R_0^2 b^2} + \frac{2}{R_0 b}}
\] (5.41)

It can be shown for that for the configuration studied; the efficiency of the charge is 99.5%. This indicates that in these scenarios the full impulse of the charge is taken by the system.

**5.6. Conclusion**

The response observed in the numerical simulation of the FML gives an insight into debonding, which was assumed not to occur in Chapter 3. First, a large plastic displacement of the global form of the FML was observed in the initial stage of the response. Secondly, debonding of the back face aluminium panel was observed as the simulation progressed. Other phases such as tearing and stretching of the back face aluminium panel after debonding; pitting of the front face (tearing in the model); and debonding of internal aluminium panels were observed in subsequent
order. This response process gives an insight into debonding. It can be inferred that the initial global large plastic deformation introduces high strains and consequently stresses in the adhesive layer (cohesive elements in this case). The difference in in-plane stiffness of the composite layer \(E_1=13\ \text{Gpa}\) and aluminium panel \(E = 73\ \text{Gpa}\) results in an increase in stress gradient across the adhesive. Thus, high stresses are generated in the adhesive region. This explains the initial debonding of the unsupported back face aluminium panel in the model. Thus the selection of a Boron-Epoxy composite with \(E_1 = 207\\text{GPa}\) which is almost equal to the Young’s modulus of steel is in order (i.e. composite patch in Chapter 3). The assumption that interfacial stresses (i.e. stresses in the cohesive element layer representing the adhesive) is at a minimum when the stiffness of the composite and the metal panel are close reiterates the validity of the model in Chapter 3. Due to the unavailability of experimental investigations for different stiffness (i.e. thickness and Young’s modulus values of aluminium panels and composites), further investigation could not be conducted beyond this point to investigate the influence of these values on debonding.

Furthermore, the work presented in this chapter validates several aspects of the experimental response of localised blast loaded fibre-metal laminates (FMLs) using numerical simulations thus, providing a cost-effective predictive and efficient way of studying the response of FMLs without having to go through the rigours of blasting panels. The response of FML’s has been found to be highly dependent on the thickness of the loaded plate and the characteristics of underlying layers. Thus, it is of crucial significance that in the behaviour of FML’s highly nonlinear transient dynamic phenomena (i.e. the large plastic deformation, tearing and failure of the constituent components of the plate) be captured in order to be able to describe the full response of the FML in such a loading environment.

Layers at the back in a typical FML support layers at the front providing the reason why the back face easily de-bonds from the rest of the plate when loaded. Thus, the failure of the intermediate composites in the FML’s needs to be adequately captured in order to be able to predict the health of the top aluminium plates. If the composites are modelled improperly and do not fail when they ought to fail as observed in the experiments, the maximum deflection of the system would be predicted wrongly, and thus, the overall response of the system rendered under-predicted. As a result of this, subsequent tearing of the aluminium plate might not be captured in the model. The developed model captures the state/health of the aluminium and composite in the FML and the maximum displacements of the back and front faces.
One final remark regarding the constitutive models used here is in order. While several nonlinear phenomena have been considered and the corresponding material models incorporated, the hydrodynamic equations of state have not been used for the target. The loading is obtained as a result of a full FSI simulation in AUTODYN, however material equations of state (EoS) such as those of Rankine-Hugoniot or Mie-Grueneissen are not included. Such models, while necessary to simulate high rate phenomena as ballistic perforation are of little use for charge sizes and stand-offs of the present study. A more comprehensive study would be needed to show with certainty the little relevance of these EoS’s in a blast loading scenario replicated here as well as to give bounds of relevance on charge sizes, stand-offs or related dimensionless parameters.

Since the constitutive models incorporate strain rate effects in metal and composite parts and an objectivity algorithm for strain softening to control energy dissipation associated with each failure mode regardless of mesh refinement and topology the confidence of the analyst in determining the response of the intermediate composite layers as well as the global response is improved.
5.7 Cited References


Chapter 6

An Investigation of Stresses in Adhesively-Bonded Single Lap Joints Subject to Pulse Loading

6.1 Introduction

The investigation carried out in Chapter 5, which gave an insight into debonding had one major limitation. The inability to obtain peel and shear stresses from the output of the analysis in Abaqus for the interface elements, to an extent, limits the analyst from knowing the exact point and the values of strains and stresses where debonding initiates. Despite the fact that crack initiation (i.e. debonding) is a stress/strain based phenomenon in cohesive elements in Abaqus 6.9, the values of stresses in cohesive elements cannot be obtained. Therefore, the analyst is left with just being able to observe the physical crack in the model as the analysis progresses. As a result of this limitation and in order to have understanding into debonding, an analytical model is developed to give an insight on how the mechanical properties of layers of a hybrid system affect debonding. Thus, in this case, a lap joint model is analysed. However, because of the very complex relationship between the stresses in the adhesive and layers of panels bonded together, an attempt to develop an analytical model for a metal-adhesive-composite system would be extremely complicated. However, for completeness a numerical model in Abaqus for a metal-adhesive-composite system is developed in Section 6.7 to increase the understanding of the distribution of stresses within the adhesive layer of a metal-adhesive-composite system. Most importantly, the relationship obtained for the stresses in the adhesive layer gives an insight on how the displacements of bonded layers affect the magnitude of strains and stresses in the adhesive layer. This goes further in increasing the understanding of interfacial stresses, which were assumed to be minimal, in Chapter 3.

In this chapter, an analytical model is developed to estimate the peel and shear stresses in an isotropic elastic adhesive in a single lap joint subjected to transverse pulse loads. The proposed analytical model is an extension of the mathematical models developed by He and Rao [1-2] to study the coupled transverse and longitudinal vibrations of a bonded lap joint system. The adhesive, in this chapter, is modelled as an elastic isotropic material implemented in Abaqus 6.9-1. The interfacial stresses obtained by finite element simulations were used to validate the proposed analytical model. The maximum peel and shear stresses in the adhesive as predicted by the analytical model were found to correlate well with the maximum stresses predicted by the
corresponding numerical models. The peel stresses in the adhesive were found to be higher than shear stresses, a result which is consistent with intuition for transversally loaded joints. The analytical model is able to predict the maximum stresses in the edges where debonding initiates due to the highly asymmetrical stress distribution as observed in the finite element simulations and experiment. This phenomenon is consistent with observations made by Uday K. Vaidya et al [3]. The stress distribution under uniformly distributed transverse pulse loading was observed to be similarly asymmetric.

The accurate prediction of maximum interfacial stresses, rather than the detailed stress time-history, is essential to determining upper bounds for impulsive loads that could safely be applied or lower bounds to loading levels which can cause debonding of the adherends and are thus non-transferable. This renders the herein developed procedure useful in the design of pulse loaded lap joints. The analytical model is defined in Section 6.2 using a set of simplifying assumptions. The equations of motions are then derived in the space of relevant displacement components and by the application of the principle of least action (Hamilton’s principle). These partial differential equations (PDE’s) are then reduced to ordinary differential equations (ODE’s) using the Galerkin’s weighted residual method in Section 6.3. These are then solved using Central Difference Method. Transverse dynamic loads arise from blast, crashes, bullets, fragments, tool drops or flying debris [3]. While solutions for the dynamic response of joints under in-plane loading are available in the literature the present work involves transverse pulse loads which have not been looked into by researchers [4].

6.2 Analytical Modelling of Transversely Loaded Lap Joint

6.2.1 Formulation of the Analytical Model

The schematic of the system chosen for the present study is shown in Figure 6.1. The objective is to formulate an analytical model to study temporal and spatial stress distribution in the adhesive layer under transverse distributed pulse load. The unjointed (unbonded) ends of both adherends modelled as beams are simply supported. The upper and lower adherends are considered as similar in every manner in this model. The governing equations of motion for the system are derived using the principle of least action i.e. Hamilton’s principle. The system is partitioned into two parts. Part one consists of the bonded overlap region, and part two consists of the unbonded regions. Arbitrary as this partitioning scheme may seem it confers some benefit upon the simplicity of the solution to the derived partial differential equations. Figures 6.2 and 6.5 and show the unbonded region, while Figure 6.3 shows the bonded region. Equations of
motion are derived separately for the upper and lower adherends in the overlap region. The exact solution of the equations of the system combines the equations of motion of the unbonded regions with the bonded overlapped subdomain and satisfies the natural boundary conditions, essential boundary conditions and continuity conditions.

The coordinate system chosen for this study is shown in Figure 6.1. The following assumptions are made in the definition of the analytical model presented in this study and thus are related to the mathematical model of the problem: (a) Rotary inertia and transverse shear effects are neglected when calculating the kinetic and strain energies of the adherends (Euler-Bernoulli assumption of the beam theory), (b) In the adhesive layer, the strain energy is composed of terms related to transverse and longitudinal displacements as well as to shear deformation, (c) The effect of wave propagation is neglected in the bonded overlap region in the through-thickness direction. The effect of wave propagation through the thickness might prove important in problems dealing with high-rate severe shock and impact loading and might result in premature failure by spalling-like modes usually termed ‘early stage response’ [5], (d) Rate dependent material behaviour is neglected in the adherends and the adhesive, (e) The adherends and adhesive are linearly elastic and isotropic. The latter assumption is found feasible when considered vis-a-vis experimental observations of researchers e.g. of Vaidya et al [3] who observed debonding was the dominant mode of failure and preceded other modes i.e. failure of the adherends due to plastic deformation or fracture and prior to any non-linear elastic behaviour. Also, Having the adhesive as an elastic material is in line with the assumptions made by Fallah et al [5], (f) Both the adherends and adhesive are assumed to be in the plane stress state, the shear strain and longitudinal strain in the adhesive are assumed to vary linearly through the thickness of the adhesive, but the transverse strain is assumed to be constant through the thickness of adhesive [4].
6.2.1.1 Analysis of Upper Beam Region

The equations of motion for the part of the upper adherend not bonded through the adhesive layer to the lower part are derived as follows. Equation (1) illustrates Hamilton’s principle where $T_{11}$ is the kinetic energy, $U_{11}$ the strain energy and $W_{11nc}$ external work in the upper beam region. The free body diagram for the unbonded upper beam part of the system is shown in Figure 6.2.

$$\int_{t_0}^{t_1} \delta(T_{11} - U_{11}) dt + \int_{t_0}^{t_1} \delta W_{11nc} dt = 0 \quad (6.1)$$
The kinetic and strain energies of the beam are given by equations (6.2) and (6.3) as follows:

\[ T_{11} = \frac{1}{2} \int_0^{L_a} \rho A \left[ \dot{w}_{11}^2 + \dot{u}_{11}^2 \right] \, dx, \quad U_{11} = \frac{1}{2} \int_0^{L_a} E \left[ I \left( w_{11}'' \right)^2 + A (u_{11}')^2 \right] \, dx \]  

(6.2, 6.3)

where \( \rho, A, w_{11}(x,t) \) and \( u_{11}(x,t) \) (denoted as \( w_{11} \) and \( u_{11} \), respectively, for brevity) represent the density, cross-sectional area, transverse and longitudinal displacement of the unbonded region of the upper adherend. \( E \) and \( I \) are the elastic modulus and second moment of inertia of the two adherends.

Introducing the Lagrangian function \( F \),

\[ F = \rho A \left[ \dot{w}_{11}^2 + \dot{u}_{11}^2 \right] - E \left[ I \left( w_{11}'' \right)^2 + A (u_{11}')^2 \right] \]  

(6.4)

and assuming the related functional dependence on generalised displacements and velocities the variation of the terms of Lagrangian contingent upon internal energies i.e. excluding external work in the principle of least action is manifested as follows:

\[ \int_{t_0}^{t_1} \delta (T_{11} - U_{11}) \, dt = \int_{t_0}^{t_1} \int_0^{L_a} \delta F \left( u_{11}', u_{11}, w_{11}, w_{11}', w_{11}'' \right) \, dx \, dt \]  

(6.5)

From the calculus of variations, we obtain
Equations (6.5) and (6.6) deal with the variation of the total energy of the system shown in Figure 6.2. The variation of work done by the external load in the time interval between \( t_0 \) and \( t_1 \) is given in equation (6.7). \( W_{1nc} \) represents work done by external forces in this region of the system.

\[
\int_{t_0}^{t_1} \delta(T_{11} - U_{11})\,dt = \int_{t_0}^{t_1} \left\{ \int_{0}^{L_a} \left\{ \left[ -\frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_{11}} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{u}_{11}} \right) \right] \delta u_{11} \\
+ \left[ \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{11}''} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{w}_{11}} \right) \right] \delta w_{11} \right\} \,dx \right\} \,dt \\
+ \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial u_{11}} \delta u_{11} - \left( \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_{11}''} \right) \right) \delta w_{11} + \frac{\partial F}{\partial \dot{w}_{11}} \delta \dot{w}_{11} \right]_{0}^{L_a} \,dt \\
+ \int_{0}^{L_a} \left[ \frac{\partial F}{\partial u_{11}} \delta u_{11} + \frac{\partial F}{\partial \dot{w}_{11}} \delta \dot{w}_{11} \right]_{t_0}^{t_1} \,dx
\]

(6.6)

Equations (6.5) and (6.6) deal with the variation of the total energy of the system shown in Figure 6.2. The variation of work done by the external load in the time interval between \( t_0 \) and \( t_1 \) is given in equation (6.7). \( W_{1nc} \) represents work done by external forces in this region of the system.

\[
\int_{t_0}^{t_1} \delta W_{11nc} \,dt = \int_{t_0}^{t_1} \int_{0}^{L_a} (q_{11}(x,t) \delta w_{11}) \,dx \,dt \\
+ \int_{t_0}^{t_1} \left[ \left\{ N_{11} \delta u_{11} + S_{11} \delta w_{11} - M_{11} \delta \dot{w}_{11} \right\} \right]_{0}^{L_a} \,dt
\]

(6.7)

Using Euler’s equations, and applying Hamilton’s principle the equations of motion are derived as:

\[
- \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_{11}} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{u}_{11}} \right) = EA \frac{\partial^2 u_{11}}{\partial x^2} - \rho A \frac{\partial^2 u_{11}}{\partial t^2} = 0
\]

(6.8)

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{11}''} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{w}_{11}} \right) + q_{11}(x,t) = EI \frac{\partial^4 w_{11}}{\partial x^4} + \rho A \frac{\partial^2 w_{11}}{\partial t^2} - q_{11}(x,t) = 0
\]

(6.9)

From equations (6.7) and (6.7), the natural boundary conditions can be written as

\[\text{†† These equations are called Euler’s equations for the problem of determining the extrema of an integral of the form of Eq. (6.7). In mechanics these equations are called Lagrange equations.}\]
The following conditions can be essential or natural depending on the zero multiplier being associated with force or displacement.

\[ [EAu_{11}]_{x=L_a} - N_{11} = 0, [EIw_{11}^{''}]_{x=L_a} + S_{11} = 0, [EIw_{11}^{''}]_{x=L_a} + M_{11} = 0, [w_{11}^{''}]_{x=0} = 0, \]

\[ (6.10a) \]

The essential boundary conditions are

\[ [u_{11}]_{x=0} = 0; \quad [w_{11}]_{x=0} = 0 \]

Essential continuity conditions with the bonded part are

\[ [u_{11}]_{x=L_a} = [u_1]_{x=L_a}; \quad [w_{11}]_{x=L_a} = [w_1]_{x=L_a}, \quad [w_{11}]_{x=L_a} = [w_1]_{x=L_a} \]

\[ (6.12) \]

The natural boundary conditions are not satisfied.

6.2.1.2 Analysis of Bonded Region

The free body diagram of the overlapped part is shown in Figure 6.3. As in the upper adherend, \( T_1 \) and \( T_2 \) represent the kinetic energies of the top and bottom adherends, and \( U_1 \) and \( U_2 \) represent the potential energies of the bottom and top adherends in the system. The vertical and transverse displacements of the top and bottom adherends are represented by \( w \) and \( u \) and with the subscript 1 and 2 denoting top and bottom, respectively. \( (L_b - L_a) \) corresponds to the length of the bonded region[4].

The kinetic energies of the upper and lower adherends are:

\[ T_1 = \frac{1}{2} \int_{L_a}^{L_b} \rho A [\dot{w}_1^2 + \dot{u}_1^2] \, dx \quad T_2 = \frac{1}{2} \int_{L_a}^{L_b} \rho A [\dot{w}_2^2 + \dot{u}_2^2] \, dx \]

\[ (6.13, 6.14) \]

whereas before the overdot denotes, as is customary, differentiation with respect to the time and prime differentiation with respect to space i.e.

\[ \dot{w}_i = \frac{\partial w_i}{\partial t}, \quad \ddot{u}_i = \frac{\partial u_i}{\partial t}, \quad \dot{w}_i = \frac{\partial^2 w_i}{\partial x^2}, \quad \dddot{u}_i = \frac{\partial u_i}{\partial x} \quad i = 1,2 \]
The strain energies of the upper and lower beams are

\[ U_1 = \frac{1}{2} \int_{L_b}^{L_b} E \left[ I (w_1')^2 + A (u_1')^2 \right] dx \quad U_2 = \frac{1}{2} \int_{L_a}^{L_b} E \left[ I (w_2')^2 + A (u_2')^2 \right] dx \]  

(6.15, 6.16)

For the adhesive layer, we assume that the velocity field \( V \) varies linearly through the thickness (see Figure 6.4)
Mathematically, the variation of velocity $V$ across the thickness of the adhesive is shown by equation (6.17) as follows:

$$V(\xi) = \left[ \dot{u}_1 + \frac{\xi}{h_c} (\dot{u}_2 - \dot{u}_1) \right] \mathbf{i} + \left[ \dot{u}_1 + \frac{\xi}{h_c} (\dot{w}_2 - \dot{w}_1) \right] \mathbf{j}$$  \hspace{1cm} (6.17)

where $\mathbf{i}$ and $\mathbf{j}$ denote the first and the second unit vectors in the two-dimensional Cartesian coordinate system, respectively. After simplification, the kinetic energy of the adhesive layer is rewritten as follows:

$$T_c = \frac{\rho_c A_c}{6} \int_{L_a}^{L_b} \left( \dot{u}_2^2 + \dot{u}_1^2 + \dot{w}_2^2 + \dot{w}_1^2 + \ddot{u}_1 \dot{u}_2 + \ddot{w}_1 \dot{w}_2 \right) \, dx$$  \hspace{1cm} (6.18)

Where $\rho_c$ and $A_c$ are the density and cross-sectional area of the adhesive, respectively, and $h_c$ is the thickness of the adhesive layer. The longitudinal strain $\varepsilon_{c,\text{long}}$, transverse strain $\varepsilon_{c,\text{tran}}$ and shear strain $\gamma_c$ in the adhesive layer can be simplified as:

$$\varepsilon_x = \varepsilon_{c,\text{long}} = u'_1 + \frac{\xi}{h_c} (u'_2 - u'_1), \quad \gamma_c = \gamma'_1 + \frac{\xi}{h_c} (\gamma'_2 - \gamma'_1), \quad 0 \leq \xi \leq h_c \quad (6.19, 6.20)$$

$$\varepsilon_x = \varepsilon_{c,\text{tran}} = \frac{(w_2 - w_1)}{h_c} \quad (6.21)$$

$$\gamma'_1 = w'_1 \left( 1 + \frac{h_1}{2h_c} \right) + \frac{h_2 w'_2}{2h_c} + \frac{u_2 - u_1}{h_c}$$  \hspace{1cm} (6.22)

$$\gamma'_2 = w'_2 \left( 1 + \frac{h_2}{2h_c} \right) + \frac{h_1 w'_1}{2h_c} + \frac{u_2 - u_1}{h_c}$$  \hspace{1cm} (6.23)

The strain energy of the adhesive layer is

$$U_c = U_1 = \frac{1}{2} \int \sigma_{ij} \varepsilon_{ij} \, dZ \quad (6.24)$$

where $Z$ represents the volume of the domain.

In line with the assumption that the adhesive is in a state of plane stress, the constitutive matrix $C$ is as below
On simplification, the strain energy in the adhesive is given as follows:

\[ \sigma = [C] \varepsilon \]

where

\[ C = \frac{E_c}{1 - v_c^2} \begin{bmatrix} 1 & v_c & 0 \\ v_c & 1 & 0 \\ 0 & 0 & \left( \frac{1 - v_c}{2} \right) \end{bmatrix} \]

and \( G_c \)

\[ \frac{E_c}{2(1 + v_c)} \]  \hspace{1cm} (6.25)

On simplification, the strain energy in the adhesive is given as follows:

\[ U_c = \frac{b}{2} \int_{L_a}^{L_b} \left\{ \int_0^{h_c} \left[ \frac{E_c}{(1 - v_c^2)} \left( \varepsilon_x^2 + 2v_c \varepsilon_x \varepsilon_y + \varepsilon_y^2 \right) + G_c \gamma_c^2 \right] d\xi \right\} dx \]  \hspace{1cm} (6.26)

On further simplification and after combining equations (6.19 - 6.25) the strain energy is

\[ U_c = \frac{1}{2} \int_{L_a}^{L_b} \left\{ \frac{E_c}{(1 - v_c^2)} \left[ \left( \frac{w_2 - w_1}{h_c} \right)^2 + \frac{1}{3} \left( u_1^2 + u_2^2 + u_1' u_2' \right) + \frac{v_c}{h_c} (u_2' - u_1')(w_2 - w_1) \right] \\
+ \frac{G_c A_c}{3} \left[ (\gamma_2 - \gamma_1)^2 \\
+ 3 \gamma_1' \gamma_2' \right] \right\} dx \]  \hspace{1cm} (6.27)

where \( b \) is the width of the lap joint system and \( E_c \) and \( G_c \) are the elastic and shear moduli of the adhesive, respectively. The total kinetic energy of the system is \( T = T_1 + T_2 + T_3 \), and the total potential energy of the system is \( U = U_1 + U_2 + U_c \) given these parameters are additive integrals of the motion.

Assuming the appropriate related functional dependence the variation in the first part of the Lagrangian is as follows:

\[ \int_{t_0}^{t_1} \delta (T - U) dt \]

\[ = \int_{t_0}^{t_1} \int_{L_a}^{L_b} \delta F \left( u_1, u_2, u_1', u_2', u_1, u_2, w_1, w_2, w_1', w_2', w_1'', w_2'', w_1', w_2' \right) dx dt \]  \hspace{1cm} (6.28)

Where \( F \) is the Langrangian functional expressed in terms of generalised coordinates and velocities as follows:
The work done by the external forces acting on the adhesively bonded part is expressed in equation (6.30), where $W_{nc}$ represents work done by external forces in the bonded region of the system.

\[
F = \frac{\rho_1 A_1}{2} [\dot{\omega}_1^2 + \dot{\gamma}_1^2] + \frac{\rho_2 A_2}{2} [\dot{\omega}_2^2 + \dot{\gamma}_2^2] \\
+ \frac{\rho_c A_c}{6} [\dot{u}_2^2 + \dot{u}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_1^2 + \dot{u}_1 \dot{u}_2 + \dot{w}_1 \dot{w}_2] - E_1 \left[ I_1 (w_1')^2 + A_1 (u_1')^2 \right] \\
- E_2 \left[ I_2 (w_2')^2 + A_2 (u_2')^2 \right] \\
- \frac{E_c}{2(1 - \nu_c^2)} \left[ \left( \frac{w_2 - w_1}{h_c} \right)^2 + \frac{1}{3} \left( u_1^2 + u_2^2 + u_1' u_2' \right) + \frac{\nu_c}{h_c} (u_2' - u_1')(w_2 - w_1) \right] \\
+ \frac{G_c A_c}{3} [(\gamma_2' - \gamma_1')^2 + 3 \gamma_1' \gamma_2'] \] (6.29)

By the principles of calculus of variations, we have:

\[
W_{nc} = \int_{L_a}^{L_b} q_1 w_1 \, dx - \left[ (N_1 u_1 + S_1 w_1 - M_1 w_1') \right]_{L_a} \\
+ \left[ (N_2 u_2 + S_2 w_2 - M_2 w_2') \right]_{L_b} \] (6.30)

The variation of the work done in time interval between $t_0$ and $t_1$ is

\[
\int_{t_0}^{t_1} \delta W_{nc} \, dt = \int_{t_0}^{t_1} \int_{L_a}^{L_b} (q_1(x, t) \delta w_1) \, dx \, dt \\
+ \int_{t_0}^{t_1} \left[ \left\{ N_2 \delta u_2 + S_2 \delta w_2 - M_1 \delta w_2' \right\}_{L_b} - \left\{ N_1 \delta u_1 + S_1 \delta w_1 - M_1 \delta w_1' \right\}_{L_a} \right] \, dt \] (6.31)

By the principles of calculus of variations, we have:
The governing equations of motion of the adhesively bonded part obtained from calculus of variations are:

\[
\begin{align*}
\int_{t_0}^{t_1} & \left\{ \int_{L_a}^{L_b} \left[ \frac{\partial F}{\partial \delta u_1} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{u}_1} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \ddot{u}_1} \right) \right] \delta u_1 + \left[ \frac{\partial F}{\partial \delta u_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{u}_2} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \ddot{u}_2} \right) \right] \delta u_2 \\
+ & \left[ \frac{\partial F}{\partial \delta w_1} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{w}_1} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_1} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \ddot{w}_1} \right) \right] \delta w_1 \\
+ & \left[ \frac{\partial F}{\partial \delta w_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{w}_2} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_2} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \ddot{w}_2} \right) \right] \delta w_2 \right\} \, dx \, dt \\
+ & \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial \delta u_1} \delta u_1 + \frac{\partial F}{\partial \delta u_2} \delta u_2 + \left( \frac{\partial F}{\partial \delta w_1} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \delta w_1} \right) \right) \delta w_1 + \left( \frac{\partial F}{\partial \delta w_2} + \frac{\partial F}{\partial \delta w_2} \right) \delta w_2 \right\}^{L_b}_{L_a} \, dt \\
+ & \int_{L_a}^{L_b} \left[ \frac{\partial F}{\partial \delta u_1} + \frac{\partial F}{\partial \delta u_2} + \frac{\partial F}{\partial \delta w_1} + \frac{\partial F}{\partial \delta w_2} \right]^{t_1}_{t_0} \, dx \\
= & 0
\end{align*}
\]

(6.32)

\[
\begin{align*}
\frac{\partial F}{\partial u_1} - & \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{u}_1} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \ddot{u}_1} \right) \\
= & \left( \frac{G_c A_c}{2 h_c^2} (h + h_c) - \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) w_1' + \left( \frac{G_c A_c}{2 h_c^2} (h + h_c) + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) w_2' \\
+ & \frac{G_c A_c}{h_c^2} (u_2 - u_1) + \left( EA + \frac{E_c A_c}{3 (1 - v_c^2)} \right) u_1'' + \frac{E_c A_c}{6 (1 - v_c^2)} u_2'' \\
- & \left( \rho A + \frac{\rho_c A_c}{3} \right) \frac{\partial^2 u_1}{\partial t^2} - \frac{\rho_c A_c}{6} \frac{\partial^2 u_2}{\partial t^2} \\
= & 0
\end{align*}
\]

(6.33)
\[
\frac{\partial F}{\partial u_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_2} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial u_2} \right) = \left( \frac{G_c A_c}{2 h_c^2} (h_c + h) + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) w_1 + \left( \frac{G_c A_c}{2 h_c^2} (h_c + h) - \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) w_2 + \frac{G_c A_c}{h_c^2} \frac{E_c A_c}{6 (1 - v_c^2)} u'' \right) (u_2 - u_1) - \left( \frac{E_A + \frac{E_c A_c}{3 (1 - v_c^2)}}{6 (1 - v_c^2)} \right) u'' - \frac{E_c A_c}{6 (1 - v_c^2)} u'' + \left( \rho A + \frac{\rho_c A_c}{3} \right) \frac{\partial^2 u_2}{\partial t^2} + \frac{\rho_c A_c \partial^2 u_1}{6 \frac{\partial t^2}{}
\]

\[
= 0 \quad (6.34)
\]

\[
\frac{\partial F}{\partial w_1} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_1} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_1} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial w_1} \right) + q_1 (x, t) = \frac{E_c A_c}{(1 - v_c^2) h_c^2} (w_1 - w_2) + \left( \frac{G_c A_c}{2 h_c^2} (h_c + h) - \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) u_1^1' - \left( \frac{G_c A_i}{2 h_c^2} (h_c + h) + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) u_2^1' - \frac{G_c A_i h^2}{2 h_c^2} \left( \frac{1}{3} h_c^2 + h \left( \frac{h_c}{2} \right) \right) w_2^1 + \rho A + \frac{\rho_c A_c}{3} \frac{\partial^2 w_1}{\partial t^2} + \frac{\rho_c A_c \partial^2 w_2}{6 \frac{\partial t^2}{}
\]

\[
= 0 \quad (6.35)
\]

\[
\frac{\partial F}{\partial w_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_2} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_2} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial w_2} \right) = \frac{E_c A_c}{(1 - v_c^2) h_c^2} (w_2 - w_1) + \left( - \frac{G_c A_c}{2 h_c^2} (h_c + h) + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) u_2^1' + \left( \frac{G_c A_c}{2 h_c^2} (h_c + h) + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} \right) u_1^1' - \frac{G_c A_i h^2}{2 h_c^2} \left( \frac{1}{3} h_c^2 + h \left( \frac{h_c}{2} \right) \right) w_1^1 + \rho A + \frac{\rho_c A_c}{3} \frac{\partial^2 w_1}{\partial t^2} + \frac{\rho_c A_c \partial^2 w_1}{6 \frac{\partial t^2}{}
\]

\[
= 0 \quad (6.36)
\]

From equations (6.31) and (6.32) the natural boundary conditions are:

\[
\left. \frac{\partial F}{\partial u_1} \right|_{x=L_a} + N_1 = \left[ \left( \frac{E_A + \frac{E_c A_c}{3 (1 - v_c^2)}}{6 (1 - v_c^2)} \right) u_1' + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} w_1 - \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} w_2 \right] = 0 ;
\]

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\[
\left[ \frac{\partial F}{\partial w_1} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_1} \right) \right]_{x=L_a} + S_1
\]

\[
= \left[ -\left( \frac{G_cA_c}{2 \, h_c} + \frac{G_cA_c \, h}{2 \, h_c^2} \right) u_2 - \left( -\frac{G_cA_c}{2 \, h_c} - \frac{G_cA_c \, h}{2 \, h_c^2} \right) u_1 \right.
\]

\[
- \left( \frac{G_cA_c h^2}{4 h_c^2} + \frac{G_cA_c h}{2 h_c} + \frac{G_cA_c}{3} \right) w'_1 - \left( \frac{G_cA_c h^2}{4 h_c^2} + \frac{G_cA_c h}{2 h_c} + \frac{G_cA_c}{6} \right) w'_2
\]

\[
+ E I w''_1 \right]_{x=L_a} + S_1 = 0;
\]

\[
\left[ \frac{\partial F}{w_1'} \right]_{x=L_a} - M_1 = -[E I w''_1]_{x=L_a} - M_1 = 0;
\]

\[
\left[ \frac{\partial F}{w_2'} \right]_{x=L_a} = -[E I w''_2]_{x=L_a} = 0;
\]

\[
\frac{\partial F}{\partial u'_2} \bigg|_{x=L_a} = \left[ -\left( E A + \frac{E_cA_c}{3 \, (1 - v_c^2)} \right) u_2' + \frac{E_cA_c v_c}{2 \, (1 - v_c^2) h_c} w_1' - \frac{E_cA_c v_c}{2 \, (1 - v_c^2) h_c} w_2' \right.
\]

\[
- \frac{E_cA_c}{6 \, (1 - v_c^2) u_1'} \right]_{x=L_a} = 0;
\]

\[
\left[ \frac{\partial F}{\partial w_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_2} \right) \right]_{x=L_a}
\]

\[
= \left[ -\left( \frac{G_cA_c}{2 \, h_c} + \frac{G_cA_c \, h}{2 \, h_c^2} \right) u_2 - \left( -\frac{G_cA_c}{2 \, h_c} - \frac{G_cA_c \, h}{2 \, h_c^2} \right) u_1 \right.
\]

\[
- \left( \frac{G_cA_c h^2}{4 h_c^2} + \frac{G_cA_c h}{2 h_c} + \frac{G_cA_c}{6} \right) w'_1 - \left( \frac{G_cA_c h^2}{4 h_c^2} + \frac{G_cA_c h}{2 h_c} + \frac{G_cA_c}{3} \right) w'_2
\]

\[
+ E I w''_2 \right]_{x=L_a}
\]

\[
\frac{\partial F}{\partial u'_2} \bigg|_{x=L_b} + N_2
\]

\[
= \left[ -\left( E A + \frac{E_cA_c}{3 \, (1 - v_c^2)} \right) u_2' + \frac{E_cA_c v_c}{2 \, (1 - v_c^2) h_c} w_1' - \frac{E_cA_c v_c}{2 \, (1 - v_c^2) h_c} w_2' \right.
\]

\[
- \frac{E_cA_c}{6 \, (1 - v_c^2) u_1'} \right]_{x=L_b} + N_2 = 0;
\]
\[
\left[ \frac{\partial F}{\partial w_2} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_2} \right) \right]_{x=L_b} + S_2
\]

\[
= \left[ \left( - \frac{G_c A_c}{2 h_c} + \frac{G_c A_c h}{2 h_c^2} \right) u_2 - \left( - \frac{G_c A_c}{2 h_c} - \frac{G_c A_c h}{2 h_c^2} \right) u_1 \right. \\
- \left( \frac{G_c A_c h^2}{4 h_c} + \frac{G_c A_c h}{2 h_c} + \frac{G_c A_c}{6} \right) w'_1 - \left( \frac{G_c A_c h^2}{4 h_c^2} + \frac{G_c A_c h}{2 h_c} + \frac{G_c A_c}{3} \right) w'_2 \\
+ \left. E I w''_2 \right]_{x=L_b} + S_2 = 0;
\]

\[
\frac{\partial F}{\partial w_2} \bigg|_{x=L_b} - M_2 = -[E I w''_2]_{x=L_b} - M_2 = 0; \\
\left[ \frac{\partial F}{\partial w_1'} \right]_{x=L_b} = -[E I w'_1]_{x=L_b} = 0;
\]

\[
\frac{\partial F}{\partial u_1'} \bigg|_{x=L_b} = \left[ - \left( E A + \frac{E_c A_c}{3 (1 - v_c^2)} \right) u'_1 + \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} w_1 - \frac{E_c A_c v_c}{2 (1 - v_c^2) h_c} w_2 \\
- \frac{E_c A_c}{6 (1 - v_c^2)} u'_2 \right]_{x=L_b} = 0;
\]

\[
\left[ \frac{\partial F}{\partial w_1'} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_1'} \right) \right]_{x=L_b}
\]

\[
= \left[ \left( - \frac{G_c A_c}{2 h_c} + \frac{G_c A_c h}{2 h_c^2} \right) u_2 - \left( - \frac{G_c A_c}{2 h_c} - \frac{G_c A_c h}{2 h_c^2} \right) u_1 \right. \\
- \left( \frac{G_c A_c h^2}{4 h_c} + \frac{G_c A_c h}{2 h_c} + \frac{G_c A_c}{3} \right) w'_1 - \left( \frac{G_c A_c h^2}{4 h_c^2} + \frac{G_c A_c h}{2 h_c} + \frac{G_c A_c}{3} \right) w'_2 \\
+ \left. E I w''_1 \right]_{x=L_b} = 0
\]  \hspace{1cm} (6.37)

### 6.2.1.3 Analysis of Lower Beam Region

The free body diagram for the unbonded lower beam region of the lap joint system is shown in Figure 6.5.
Figure 6.5: Free body diagram of the unbounded lower beam of the system

Assuming the functional dependences as before:

\[
\int_{t_0}^{t_1} \delta(T_{22} - U_{22}) \, dt = \int_{t_0}^{t_1} \int_{L_b}^{L_c} \delta F\left(u'_{22}, \dot{u}_{22}, w'_{22}, \ddot{w}_{22}, \dddot{w}_{22}\right) \, dx \, dt
\]

(6.38)

Where \(T_{22}\) and \(U_{22}\) are the kinetic and potential energy of the lower beam region of the system.

From the calculus of variations, the expression for the lower beam is obtained as follows:

\[
\int_{t_0}^{t_1} \left\{ \int_{L_b}^{L_c} \left\{ \left(-\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u'_{22}}\right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial \dot{u}_{22}}\right) \right) \delta u_1 + \left(\frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w'_{22}}\right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial \ddot{w}_{22}}\right) \right) \delta w_1 \right\} \, dx \right\} \, dt
\]

\[
+ \int_{t_0}^{t_1} \left[ \frac{\partial F}{\partial u_{22}} \delta u_{22} - \left(\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial x'_{22}}\right) \right) \delta w_{22} + \frac{\partial F}{\partial w_{22}} \delta w_{22} \right]_{L_b}^{L_c} \, dt
\]

\[
+ \int_{L_b}^{L_c} \left[ \frac{\partial F}{\partial u_{22}} \delta u_{22} + \frac{\partial F}{\partial w_1} \delta w_{22} \right]_{t_0}^{t_1} \, dx
\]

(6.39)

The variation of the work done is given in equation (6.40), where \(W_{22nc}\) is the external work done in this region.
The equations of motion for this region are then as follows:

$$\int_{t_0}^{t_1} \delta W_{22nc} \, dt = \int_{t_0}^{t_1} \int_{L_b}^{L_c} (q_{22}(x,t) \delta u_1) \, dx \, dt$$

$$+ \int_{t_0}^{t_1} \left\{ \left[ (N_{22} \delta u_{22} + S_{22} \delta w_{22} - M_{22} \delta w_{22}') \right]_{L_c} - \left[ (N_{22} \delta u_{22} + S_{22} \delta w_{22} - M_{22} \delta w_{22}') \right]_{L_b} \right\} \, dt$$

(6.40)

The equations of motion for this region are then as follows:

$$- \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_{22}} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial u_{22}'} \right) = EA \frac{\partial^2 u_{22}}{\partial x^2} - \rho A \frac{\partial^2 u_{22}}{\partial t^2} = 0$$

(6.41)

$$\frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{22}''} \right) - \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial w_{22}'} \right) + q_{22}(x,t) = EI \frac{\partial^4 w_{22}}{\partial x^4} + \rho A \frac{\partial^2 w_{22}}{\partial t^2} - q_{22}(x,t) = 0$$

(6.42)

The natural boundary conditions are:

$$[EAu_{22}]_{x=L_b} = 0, [EIw_{22}']_{x=L_b} + S_{22} = 0, [EIw_{22}'']_{x=L_b} + M_{22} = 0, [w_{22}'']_{x=L_c} = 0$$

(6.43a)

The following equations can represent either natural or essential boundary conditions:

$$M_{11} \delta w_{22}' \Big|_{x=L_c} = 0, \quad \frac{\partial F}{\partial u_{22}} \delta u_{22} \Big|_{x=L_c} = 0$$

(6.43b)

Kinematic boundary conditions for the two ends of the system are:

$$[w_{22}]_{x=L_c} = -h_{m}, [u_{22}]_{x=L_c} = 0$$

(6.44)

Continuity conditions for the unbonded and bonded regions are:

$$[w_{22}']_{x=L_b} = [w_{22}']_{x=L_b}, [w_{22}]_{x=L_b} = [w_{22}]_{x=L_b}, [u_{22}]_{x=L_b} = [u_{22}]_{x=L_b}$$

(6.45)

### 6.3 Solution of the Governing Equations of Motion for Transversely Load Lap Joint

Equations (6.8), (6.9), (6.33), (6.34), (6.35), (6.36), (6.41), and (6.42) encompass the full set of governing partial differential equations of motion for the system under consideration. Since the stresses in the adhesive are of interest in this research, the motion of the system in the bonded
The equations of motion of the bonded part show a system of coupled equations in \( u_1, u_2, w_1 \) and \( w_2 \). These are the displacement functions for the relevant wave propagation problem. It is important to note at this juncture that if \( u_1(x,t), u_2(x,t), w_1(x,t) \) and \( w_2(x,t) \) oscillate with different frequencies, we cannot eliminate the time variable from the governing equations, thus we have no way of finding complete algebraic solutions. Consequently, a numerical solution to the governing differential equations is sought. To make the numerical calculations convenient, we non-dimensionalize the above equations of motion using the following non-dimensional parameters obtained by Buckingham’s Pi-theorem:

\[
\tilde{x} = \frac{x}{L_{12}}, \quad \tilde{u} = \frac{u}{h}, \quad \tilde{h}_c = \frac{h_c}{h}, \quad \tilde{\rho}_c = \frac{\rho_c h_c}{E}, \quad \tilde{w} = \frac{w}{h}, \quad \tilde{E}_c = \frac{E_c}{E}, \quad \tilde{\tau} = \frac{t}{\tau}
\]

where \( L_{12} = L_1 + L_2 + L_3 \) and \( \tau = h/c \) (where \( c \) is the wave velocity through the adherends).

After some mathematical manipulation, the following is obtained as the governing set of equations for the overlap region:

\[
\begin{align*}
\left(-\frac{\tilde{\rho}_c}{2} \left(1 + \frac{\tilde{\rho}_c E}{3E(1 - \tilde{v}_c^2)}\right) \right) \frac{\partial^2 \tilde{W}_1}{\partial \tilde{x}^2} &+ \left(-\frac{\tilde{\rho}_c}{2} \left(1 + \frac{\tilde{\rho}_c E}{3E(1 - \tilde{v}_c^2)}\right) \right) \frac{\partial^2 \tilde{W}_2}{\partial \tilde{x}^2} \\
&+ \frac{\tilde{\rho}_c}{\tilde{h}_c} (\tilde{U}_2 - \tilde{U}_1) - \left(1 + \frac{\tilde{E}_c \tilde{h}_c}{3(1 - \tilde{v}_c^2)}\right) \frac{\partial^2 \tilde{U}_1}{\partial \tilde{x}^2} \left(\frac{h}{L_{12}}\right)^2 - \frac{\tilde{E}_c \tilde{h}_c}{6(1 - \tilde{v}_c^2)} \frac{\partial^2 \tilde{U}_2}{\partial \tilde{x}^2} \left(\frac{h}{L_{12}}\right)^2 \\
&+ \left(\frac{\rho}{E} + \frac{\rho_c h_c}{3E}\right) \frac{\partial^2 \tilde{U}_1}{\partial \tilde{\tau}^2} \left(\frac{h}{\tau}\right)^2 + \frac{\rho_c h_c}{6E} \frac{\partial^2 \tilde{U}_2}{\partial \tilde{\tau}^2} \left(\frac{h}{\tau}\right)^2 = 0 \\
\end{align*}
\]

\[
\begin{align*}
\left(\frac{\tilde{\rho}_c}{2} \left(1 + \frac{\tilde{\rho}_c E}{3E(1 - \tilde{v}_c^2)}\right) \right) \frac{\partial^2 \tilde{W}_1}{\partial \tilde{x}^2} &+ \left(\frac{\tilde{\rho}_c}{2} \left(1 + \frac{\tilde{\rho}_c E}{3E(1 - \tilde{v}_c^2)}\right) \right) \frac{\partial^2 \tilde{W}_2}{\partial \tilde{x}^2} \\
&+ \frac{\tilde{\rho}_c}{\tilde{h}_c} (\tilde{U}_2 - \tilde{U}_1) - \left(1 + \frac{\tilde{E}_c \tilde{h}_c}{3(1 - \tilde{v}_c^2)}\right) \frac{\partial^2 \tilde{U}_1}{\partial \tilde{x}^2} \left(\frac{h}{L_{12}}\right)^2 - \frac{\tilde{E}_c \tilde{h}_c}{6(1 - \tilde{v}_c^2)} \frac{\partial^2 \tilde{U}_2}{\partial \tilde{x}^2} \left(\frac{h}{L_{12}}\right)^2 \\
&+ \left(\frac{\rho}{E} + \frac{\rho_c h_c}{3E}\right) \frac{\partial^2 \tilde{U}_1}{\partial \tilde{\tau}^2} \left(\frac{h}{\tau}\right)^2 + \frac{\rho_c h_c}{6E} \frac{\partial^2 \tilde{U}_2}{\partial \tilde{\tau}^2} \left(\frac{h}{\tau}\right)^2 = 0
\end{align*}
\]
For the sake of simplicity, the displacement variables for the whole system are represented by the vector \( \textbf{D} \).

\[
\begin{align*}
\frac{1}{12} \frac{\partial^4 \bar{W}_1}{\partial x^4} \left( \frac{h}{L_{12}} \right)^4 & - \frac{\bar{c}_c}{2} \left( \frac{2}{3H_c} + 1 + \frac{1}{H_c} \right) \frac{\partial^2 \bar{W}_1}{\partial x^2} \left( \frac{h}{L_{12}} \right)^2 + \frac{\bar{E}_c}{(1 - v_c^2)H_c} (\bar{W}_1 - \bar{W}_2) \\
+ & \left( \frac{\rho E}{3E} \right) \frac{\partial^2 \bar{W}_1}{\partial t^2} \left( \frac{h}{\tau} \right)^2 - \frac{\bar{c}_c}{2} \left( \frac{1}{3H_c} + 1 + \frac{1}{H_c} \right) \frac{\partial^2 \bar{W}_2}{\partial x^2} \left( \frac{h}{L_{12}} \right)^2 + \frac{\rho_c H_c}{6E} \frac{\partial^2 \bar{W}_2}{\partial t^2} \left( \frac{h}{\tau} \right)^2 \\
+ & \left( \frac{\bar{c}_c H_c}{2} \left( \frac{1}{H_c} + 1 \right) \right) \frac{\partial \bar{U}_1}{\partial x} \frac{h}{L_{12}} - \left( \frac{\bar{c}_c H_c}{2} \left( \frac{1}{H_c} + 1 \right) \right) \frac{\partial \bar{U}_2}{\partial x} \frac{h}{L_{12}} \end{align*}
\]

\[
\begin{align*}
&= q(\tau) h 
\end{align*}
\]

(6.48)

\[
\begin{align*}
\frac{1}{12} \frac{\partial^4 \bar{W}_2}{\partial x^4} \left( \frac{h}{L_{12}} \right)^4 & - \frac{\bar{c}_c H_c}{2} \left( \frac{2}{3H_c} + 1 + \frac{1}{H_c} \right) \frac{\partial^2 \bar{W}_2}{\partial x^2} \left( \frac{h}{L_{12}} \right)^2 - \frac{\bar{E}_c}{(1 - v_c^2)H_c} (\bar{W}_1 - \bar{W}_2) \\
+ & \left( \frac{\rho E}{3E} \right) \frac{\partial^2 \bar{W}_2}{\partial t^2} \left( \frac{h}{\tau} \right)^2 - \frac{\bar{c}_c H_c}{2} \left( \frac{1}{3H_c} + 1 + \frac{1}{2H_c^2} \right) \frac{\partial^2 \bar{W}_1}{\partial x^2} \left( \frac{h}{L_{12}} \right)^2 + \frac{\rho_c H_c}{6E} \frac{\partial^2 \bar{W}_1}{\partial t^2} \left( \frac{h}{\tau} \right)^2 \\
+ & \left( \frac{\bar{c}_c H_c}{2} \left( \frac{1}{H_c} + 1 \right) \right) \frac{\partial \bar{U}_1}{\partial x} \frac{h}{L_{12}} - \left( \frac{\bar{c}_c H_c}{2} \left( \frac{1}{H_c} + 1 \right) \right) \frac{\partial \bar{U}_2}{\partial x} \frac{h}{L_{12}} \end{align*}
\]

\[
\begin{align*}
&= 0 
\end{align*}
\]

(6.49)

For the sake of simplicity, the displacement variables for the whole system are represented by the vector \( \textbf{D} \).

\[
\textbf{D} = \begin{bmatrix}
  u_{11} \\
  w_{11} \\
  u_1 \\
  w_1 \\
  u_2 \\
  w_2 \\
  u_{22} \\
  w_{22}
\end{bmatrix}
\]

The differential equations from (6.8), (6.9), (6.33), (6.34), (6.35), (6.36), (6.41) and (6.42) can be re-written in matrix form as:

\[
\textbf{L} \textbf{D} + \textbf{A} = \textbf{0}
\]

(6.50)

Where \( \textbf{L} \) is the matrix of differential operator and \( \textbf{A} \) is the vector of known values/functions that is related to \( q(x,t) \). Using indicial notation equation (6.50) can be re-written as:

\[
L_{ij} u_j + \lambda_i = 0
\]

(6.51)

Using the assumed mode approach for the system of equations above, each \( u_i \) is approximated such that:
Where \( \varphi^{(i)}_\alpha(x) \)'s are shape functions that satisfy the essential boundary conditions and natural boundary conditions. \( a_{ia} \)'s are the generalised co-ordinates that must satisfy initial conditions: thus, in this problem \( a_{ia} \)'s along with their first temporal derivative must vanish at \( t = 0 \). The initial conditions of the problem are shown in equation (6.53).

\[
\begin{align*}
  u_{11}(x, 0) &= 0; \quad w_{11}(x, 0) = 0; \quad u_1(x, 0) = 0; \quad w_1(x, 0) = 0; \quad w_2(x, 0) = 0; \\
  u_{21}(x, 0) &= 0; \quad w_{21}(x, 0) = 0; \quad \dot{u}_{11}(x, 0) = 0; \quad \dot{w}_{11}(x, 0) = 0; \quad \dot{u}_1(x, 0) = 0; \quad \dot{w}_1(x, 0) = 0; \\
  \dot{w}_1(x, 0) &= 0; \quad \ddot{w}_1(x, 0) = 0; \quad \dot{w}_2(x, 0) = 0; \quad \ddot{w}_2(x, 0) = 0
\end{align*}
\]  
\( (6.53) \)

The weighted residual formulation of the problem leads to the following system of simultaneous equations:

\[
\int_{R(\Omega)} W^T (L\ddot{\mathbf{u}} + \Lambda) \, d\Omega = 0
\]  
\( (6.54) \)

where \( R(\Omega) \) is the relevant part of the domain. In this analysis, \( \Omega_1 \) and \( \Omega_3 \) represent the unbonded regions and \( \Omega_2 \) represents the bonded region. An approximate solution \( w(x, t) \) can be obtained in domains \( \Omega_1, \Omega_2 \) and \( \Omega_3 \). Besides the primary objective of the present study is to determine interfacial stress profile rather than the displacement profile. \( W \) is a vector of weighting functions which if selected to be the same as shape the functions vector, the method is known as Galerkin’s method. In indicial form Equation (6.54) is:

\[
\int_{R(\Omega)} w_\beta \left( L_{ij} \dot{a}_{j\alpha} \varphi^{(j)}_\alpha \right) \, d\Omega + \int_{R(\Omega)} w_\beta A_\alpha \, d\Omega = 0
\]  
\( (6.55) \)

where \( w_\alpha = \varphi_\alpha \) (Galerkin method)

In order to obtain an approximate solution, the approximating functions for the particular case of lap joint geometry where \( (L_1/L_2 = L_3/L_2 = 2) \) the following nondimensional shape functions in equation (6.56) are used to define the transversal and longitudinal vibration modes (modes 1-3):

Mode 1 (transversal)

\[
\varphi_{11} = -[\sin(2.4) \sinh(4.8x) + \sinh(2.4) \sin(4.8x)] \quad 0 \leq x \leq L_b
\]
\[
\varphi_{12} = - \left[ \sin(2) \sinh(4.8(1 - x)) + \sinh(2.4) \sin(4.8(1 - x)) \right] - \frac{h_m}{h} \quad L_a \leq x \leq L_c
\]

Mode 2 (transversal)
\[
\varphi_{21} = - \left[ \sin(5.9) \sinh(9.8x) + \sinh(5.9) \sin(9.8x) \right] + 500x \quad 0 \leq x \leq L_b
\]
\[
\varphi_{22} = - \left[ \sin(5.9) \sinh(9.8(1 - x)) + \sinh(5.9) \sin(9.8(1 - x)) \right] - 100 \frac{h_m}{h} \quad L_a \leq x \leq L_c
\]

Mode 3 (transversal)
\[
\varphi_{31} = - \left[ \sin(9.5) \sinh(15.8x) + \sinh(9.5) \sin(15.8x) \right] + 8000x \quad 0 \leq x \leq L_b
\]
\[
\varphi_{32} = - \left[ \sin(9.5) \sinh(15.8(1 - x)) + \sinh(9.5) \sin(15.8(1 - x)) \right] - 1000 \frac{h_m}{h} \quad L_a \leq x \leq L_c
\]

Mode 1 (longitudinal)
\[
\varphi_{51} = - \left[ \sin(2.4) \sinh(4.8x) + \sinh(2.4) \sin(4.8x) \right] \quad 0 \leq x \leq L_b
\]
\[
\varphi_{52} = - \left[ \sin(2) \sinh(4.8(1 - x)) + \sinh(2.4) \sin(4.8(1 - x)) \right] \quad L_a \leq x \leq L_c
\]

Mode 2 (longitudinal)
\[
\varphi_{61} = - \left[ \sin(5.9) \sinh(9.8x) + \sinh(5.9) \sin(9.8x) \right] + 500x \quad 0 \leq x \leq L_b
\]
\[
\varphi_{62} = - \left[ \sin(5.9) \sinh(9.8(1 - x)) + \sinh(5.9) \sin(9.8(1 - x)) \right] \quad L_a \leq x \leq L_c
\]

Mode 3 (longitudinal)
\[
\varphi_{71} = - \left[ \sin(9.5) \sinh(15.8x) + \sinh(9.5) \sin(15.8x) \right] + 8000x \quad 0 \leq x \leq L_b
\]
\[
\varphi_{72} = - \left[ \sin(9.5) \sinh(15.8(1 - x)) + \sinh(9.5) \sin(15.8(1 - x)) \right] \quad L_a \leq x \leq L_c
In the above equations, $\varphi_{11}$, $\varphi_{21}$ and $\varphi_{31}$ represent the shape functions (transversal vibration modes) of the upper adherend and $\varphi_{12}$, $\varphi_{22}$ and $\varphi_{32}$ represent the shape functions of the lower adherend. For the longitudinal modes, $\varphi_{51}$, $\varphi_{61}$ and $\varphi_{71}$ represent the shape functions (longitudinal vibration modes) of the upper adherend and $\varphi_{52}$, $\varphi_{62}$ and $\varphi_{72}$ represent the shape functions of the lower adherend. The overall non-dimensional length, $L$, of the lap joint is denoted as 1 and length along the lap joint is denoted as $x/L$. The non-dimensional centre-to-centre distance between the upper and lower adherends is denoted as $h_{m}/h$.

The transversal mode shapes are shown in Figures 6.6. Modes 1 and 2 are similar to the mode shapes predicted by He and Rao[1-2].

![Figure 6.6: Transversal vibration modes of an adhesive bonded lap joint (a) mode 1 (b) mode (c) mode 3](image-url)
The shape functions are thus:

\[ u_{11} = \tilde{u}_{11} = a_{11}(t)\varphi_{s1}(x) + b_{11}(t)\varphi_{s2}(x) + c_{11}(t)\varphi_{s3}(x) \]
\[ w_{11} = \tilde{w}_{11} = a_{12}(t)\varphi_{t1}(x) + b_{12}(t)\varphi_{t2}(x) + c_{12}(t)\varphi_{t3}(x) \]
\[ u_1 = \tilde{u}_1 = a_{21}(t)\varphi_{s1}(x) + b_{21}(t)\varphi_{s2}(x) + c_{21}(t)\varphi_{s3}(x) \]
\[ w_1 = \tilde{w}_1 = a_{22}(t)\varphi_{t1}(x) + b_{22}(t)\varphi_{t2}(x) + c_{22}(t)\varphi_{t3}(x) \]
\[ u_2 = \tilde{u}_2 = a_{23}(t)\varphi_{s1}(x) + b_{23}(t)\varphi_{s2}(x) + c_{23}(t)\varphi_{s3}(x) \]
\[ w_2 = \tilde{w}_2 = a_{24}(t)\varphi_{t1}(x) + b_{24}(t)\varphi_{t2}(x) + c_{24}(t)\varphi_{t3}(x) \]
\[ u_{22} = \tilde{u}_{22} = a_{31}(t)\varphi_{s1}(x) + b_{31}(t)\varphi_{s2}(x) + c_{31}(t)\varphi_{s3}(x) \]
\[ w_{22} = \tilde{w}_{22} = a_{32}(t)\varphi_{t1}(x) + b_{32}(t)\varphi_{t2}(x) + b_{32}(t)\varphi_{t3}(x) \]  \( (6.57) \)

An increase in the number of terms increases the accuracy of the formulation at the expense of computational effort.

In solving the above partial differential equations, these functions are inputted in the PDE’s. Using Galerkin’s method of weighted residuals, the relevant ODE’s formed are solved using integration. The equations are then solved using a numerical technique in MATLAB or using the Laplace transform method, which applies to linear problems.

### 6.4 Finite Element Analysis of Transversely Loaded Lap Joint System

The finite element analyses of the model of the single lap joint were conducted using the model developed in Abaqus 6.9-1/Explicit. Due to the fact that stress components of the adhesive layer cannot be obtained directly from Abaqus 6.9-1 using cohesive elements, the adhesive is modelled as a linear elastic isotropic material in-between the aluminium adherends. This is in tandem with the assumptions made in the analytical model. Figure 6.7 shows the Abaqus model set up. The finite element model has been meshed with eight-noded C3D8R continuum elements with reduced integration and hourglass controlled formulation – which is well suited for this kind of analysis. The model comprises 4 elements through the 2mm thickness of adherend. A mesh sensitivity analysis was conducted with 2, 4 and 8 elements through the thickness of the 0.5mm thick adhesive layer and the results were roughly the same. The parameters of linear and quadratic bulk viscosity were set to zero to ensure accurate wave propagation capturing in the
response by the model. The boundary conditions were pinned at both ends. Table 6.1 shows the properties of the modeled single lap joint [4].

![Finite element model of adhesively bonded single lap joint](image)

Table 6.1: Material and geometric properties of lap joint

<table>
<thead>
<tr>
<th>Material/geometrical property</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>1mm</td>
</tr>
<tr>
<td>$h$</td>
<td>3mm</td>
</tr>
<tr>
<td>$h_c$</td>
<td>0.3mm</td>
</tr>
<tr>
<td>$E$</td>
<td>70000 N/mm²</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$2.77 \times 10^{-9}$ ton/mm³</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>$5 \times 10^{-13}$ ton/mm³</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>0.35</td>
</tr>
<tr>
<td>$E_c$ *</td>
<td>2000 N/mm²</td>
</tr>
<tr>
<td>$L_1$</td>
<td>50mm</td>
</tr>
<tr>
<td>$L_2$</td>
<td>25mm</td>
</tr>
<tr>
<td>$L_3$</td>
<td>50mm</td>
</tr>
</tbody>
</table>

*The adhesive is assumed to have similar elastic properties with Magnabond 6398 (a paste adhesive with high thermal stability) [3]. This adhesive is supplied by Magnolia Plastics Inc.*
6.5 Results

A triangular pulse load of amplitude 1200kPa and duration 0.001 seconds and a rise time of 0.0005 seconds, (an isosceles triangular pulse shape) was inputted into the analytical model and applied to the numerical FE model. Triangular pulses represent blast over-pressure (reflected) time history on the sides of a chamber under blast loading [6]. Figures 6.8 and 6.9 show the comparison of the average maximum peel and shear stresses predicted by the models. The dotted line represents stress time history predicted by the analytical model and continuous line represents the stress history predicted by the numerical model. Figure 6.10 shows the variation of stresses along the length of the adhesive. The maximum tensile peel stress occurs at the edge of the lower adherend and debonding starts from this region. Figure 6.11 shows an Abaqus model of the variation of peel and shear stresses along the length of the adhesive[4].

Figure 6.8: Numerical and analytical output for peel stress at the edge of lower adherend in overlap region

Figure 6.9: Numerical and analytical output for shear stress at the edge of lower adherend in overlap region
Figure 6. 10: (a) Peel stress distribution along the overlap at $t = 0.00055$ seconds predicted by Abaqus (b) Shear stress distribution along the overlap at $t = 0.00055$ seconds predicted by Abaqus

Figure 6. 11: (a) Abaqus model showing location of maximum peel stress (b) Abaqus model showing location of maximum shear stress

6.7 In-Plane blast Loads on Lap Joints

The preceding section studies the interfacial stresses of lap joints subjected to transverse pulse loads. However, the major shortcoming of the analytical model proposed is its inability to account for the propagation of through thickness stress waves. In order to compare the stresses in
a metal-adhesive-composite lap joint with metal-adhesive-metal lap joint, it is deemed necessary to compare the stresses in the adhesive when the lap joint is subjected to in-plane blast loadings.

6.7.1. Numerical Model

The FE analyses of in-plane loaded single lap joints were conducted in Abaqus 6.9-1/Explicit. Two models were created - the first model consists of a steel–adhesive–steel system while the second model consists of steel-adhesive-composite system. The FE model is meshed with eight-noded C3D8R continuum elements with reduced integration and hourglass controlled formulation – which is well suited for this kind of analysis. The model comprises 4 elements through the 3mm thickness of adherend. A mesh sensitivity analysis was conducted with 2, 4 and 8 elements through the thickness of the 0.003mm thick adhesive layer and the results were roughly the same. Figure 6.12 shows the configuration of the two models created‡‡.

‡‡ Young’s Modulus of adhesive in the two models is 3.5Gpa and Poisson ratio is 0.34.
A subroutine was developed to model the damage in the composites adherends in Abaqus. Initiation criteria for the 3-D Hashin damage are shown in equation (6.58). Table 6.2 shows the properties of the adherends.

Fibre tensile/compressive failure

\[
\left( \frac{\sigma_{11}}{X_T} \right)^2 + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S} \leq 1 \quad \text{for } \sigma_{11} > 0 \tag{6.58a}
\]

\[
\left( \frac{\sigma_{11}}{X_c} \right)^2 \leq 1 \quad \text{for } \sigma_{11} < 0 \tag{6.58b}
\]

Lamina crush

\[
\left( \frac{\sigma_{33}}{Z_c} \right)^2 \leq 1 \quad \text{for } \sigma_{33} < 0 \tag{6.58d}
\]

Matrix tensile/compressive failure

\[
\left( \frac{\sigma_{22} + \sigma_{33}}{Y_T} \right)^2 + \frac{\sigma_{22}^2 + \sigma_{33}^2}{S_T^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} \leq 1 \quad \text{for } (\sigma_{22} + \sigma_{33}) > 0 \tag{6.58e}
\]
\[
\left( \frac{Y_c}{2S_T} - 1 \right)^2 \left( \frac{\sigma_{22} + \sigma_{33}}{Y_c} \right) + \left( \frac{\sigma_{22} + \sigma_{33}}{4S_T} \right)^2 + \frac{\sigma_{23}^2 - \sigma_{22}\sigma_{33}}{S_T^2} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{S^2} \leq 1
\]

Delamination

\[
\left( \frac{\sigma_{23}}{Z_T} \right)^2 + \left( \frac{\sigma_{23}}{S_{23}} \right)^2 + \left( \frac{\sigma_{13}}{S_{23}} \right)^2 \leq 1 \quad \text{for } \sigma_{33} < 0
\] 

(6.59f)

Table 6. 2: Properties of composite adherend

<table>
<thead>
<tr>
<th>Property</th>
<th>Graphite-Epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>207GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>5GPa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.25</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\nu_{31}$</td>
<td>0.3</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>2.6GPa</td>
</tr>
<tr>
<td>$G_{23}$</td>
<td>2.6GPa</td>
</tr>
<tr>
<td>$G_{31}$</td>
<td>2.6GPa</td>
</tr>
<tr>
<td>$X_T$</td>
<td>1000Mpa</td>
</tr>
<tr>
<td>$Y_T$</td>
<td>100MPa</td>
</tr>
<tr>
<td>$S$</td>
<td>250MPa</td>
</tr>
<tr>
<td>$Z_C$</td>
<td>100MPa</td>
</tr>
<tr>
<td>$X_C$</td>
<td>700MPa</td>
</tr>
<tr>
<td>$Y_C$</td>
<td>100MPa</td>
</tr>
</tbody>
</table>

Here $Z_T$ is the through thickness tensile strength, $X_T$ ($X_c$) is the axial tensile (compressive) strength along the $x_1$-axis, $Z_c$ the lamina crush strength, $Y_T$ ($Y_c$) and $S_T$, respectively, the transverse tensile (compressive) strengths along the $x_2$-axis and the shear strengths in the $x_2x_3$-plane with fibres aligned along the $x_1$-axis. The shear strength in the $x_1x_3$-plane or the $x_1x_2$ plane is denoted by $S$. Note fibre failure due to kinking and buckling are not considered.
6.7.2. Results

A tensile pulse load of magnitude 300N/mm$^2$ with a rise time of 0.1ms and duration of 0.2ms was applied to the models. It was observed that the maximum interfacial stresses occurred at 0.4ms in both models. However, significant reduction in shear stresses were observed in the steel-adhesive-composite configuration and the peel stresses were similar in both cases. The change in distribution of stresses in the adhesive can be attributed to different in-plane vibration mode shapes for the steel-adhesive-steel and steel-adhesive-composite configurations. Kaya et al [7] studied the effect of lap joint configuration on the natural frequency of a lap joint. However, the aim of this section is to get an insight into the stress distribution of a metal-adhesive-metal lap joint and a metal-adhesive-composite lap joint.

![Stress distributions along overlap in steel-adhesive-steel lap joint system](image)

Figure 6. 13: Stress distributions along overlap in steel-adhesive-steel lap joint system
6.8 Conclusions

The first part of the present study presents a semi-analytical procedure to evaluate the time history and spatial variation of interfacial stresses in the adhesive layer of a single lap joint of similar adherends. The equations of motion for the joint have been derived using Hamilton’s principle which gives not only the governing PDE’s but also the natural boundary conditions. These nondimensionalised equations reduced to ODE’s in the time domain using Galerkin’s weighted residual method and are in turn solved using Central Difference Method in MATLAB. The results of analyses are correlated with Abaqus and feasible correlation is observed.

The peel strains in the adhesive layer is found to be related to the difference in the transverse displacement of the top and bottom materials been bonded together from the results obtained in the analytical model. The shear strains is function of the transverse displacement of the top and bottom adherends as well as the derivative with respect to distance of the longitudinal displacements of the bottom and top adherends. Interestingly, transverse and longitudinal displacements are functions of $EI$ (product of Young’s modulus and second moment of area) and $EA$ (product of Young’s modulus and cross-sectional area) respectively. This observations reiterate the assumption in Chapter 3. The implication of this is that different values of $E$ (Young’s modulus) for top and bottom adherends with approximately the same thickness results in higher strains within the adhesive layer. Since debonding is strain/stress based, such configuration is susceptible to debonding. Since the values Young’s moduli for stainless steel and
the composite patch in Chapter 3 are approximately the same, the strains in the adhesive layer is minimized in the composite strengthened blast wall.

From the present study the following conclusion can also be drawn:

The proposed analytical model gave a good correlation with the numerical model for maximum shear and peel stresses in the adhesive. The adhesive maximum shear and peel stresses predicted by the analytical and numerical models occured at the edge of the lower adherend in the bonded region. This was the position failure initiated as observed by Vaidya et al [3]. The prediction of maximum values of interfacial stresses is vital in predicting the point of initiation of debonding as debonding initiation is a flag parameter with respect to a convex hypersurface in the space of normalised interfacial stresses.

Although the maximum peel and shear stresses correlated very well, likewise the time to maximum response, stress histories predicted by the analytical model did not perfectly match the stress histories predicted by the numerical model. The possible reason for this is the possibility of presence of higher modes and through-thickness stress wave propagation in the full dynamic response. Higher modes can be included by requiring the displacement fields to be represented with more terms. However, this modification will increase the level of complexity accordingly. The study of through-thickness stress waves which cause spalling-like failure modes and occur at times of orders of magnitude lower than the global dynamic response falls beyond the scope of this work.

The effect of density of adhesive on interfacial stresses is deemed minimal as variations in this parameter did not affect predicted values of maximum stresses in the analytical model. Alteration of the adhesive density had no pronounced effect on the maximum peel or shear stresses predicted by the numerical model, either. Since the cross sectional area of adhesive is very small compared to the cross sectional area of the adherends, the dominant terms in the mass matrix of the coupled motion of the bonded part is $\rho A$. The magnitude of $\rho A$ is much larger than the adhesive components of the mass matrix i.e. $\rho c A_c/3$ and $\rho c A_c/6$.

The numerical model predicted an asymmetrical distribution of stresses along the overlapped zone in blast loaded single lap joints. This is expected because of the change of stresses from tensile to compressive on the two edges of the bonded region. Increasing the number of terms in the displacement fields approximation functions for the adherends would increase the level of accuracy in the prediction of the stress distribution in the adhesive layer. However, the scope of this work concerns the prediction of maximum stresses which occur at the edge of the lower
adherend in the bonded region, a parameter which is predicted accurately by the proposed model[4].

As stated in conclusion 4 the analytical model is intended to predict the maximum stresses at points of occurrence. This limitation renders the study of subsequent crack propagation impossible using the present model. Study of initiation is necessary since as soon as damage initiation occurs at this point, there is a stress redistribution and the following fracture/damage mechanics analysis falls within the realm of damage evolution which is not in the scope of the present work.
6.7 Cited References


Chapter 7

Conclusion

7.1 Introduction
This chapter presents the general conclusion of the dynamic response of the various hybrid systems studied in this work. Though three systems were studied, a general conclusion is drawn in the concluding part of this chapter.

First, a summary of the specific contribution to the knowledge of the individual systems studied is presented. A study of debonding, which was prevented in Chapter 3 and was continued in Chapter 4 and 5.

Second, a general conclusion is drawn from the analysis of the three specific hybrid systems studied. These conclusions are behaviours, which are common to hybrid systems and can be inferred from the study of the schemes presented.

Obviously, this research throws up some questions on hybrid systems that need further investigations. Thus, this chapter presents a discussion on possible future research works.

7.2 Specific Conclusions

7.2.1. Strengthened Blast Walls
A strengthened system of stainless blast wall absorbs more blast energy than an unstrengthened blast wall panel. The hybrid system formed by strengthening shows an improved mechanical performance under hydrocarbon blast loads as long as there is no debonding between the stainless steel blast wall and the strengthening patch. It is assumed that strengthening patches with similar in-plane stiffness with the blast wall results in minimal interfacial stresses between the stainless steel panel and composite patch i.e. minimum stresses in the adhesive between the two components. It well known that debonding is stress/strain based. If stresses are controlled in the adhesive, debonding can be prevented. The hybridisation with composite patches offers improved advantages when compared to other proposed strengthening schemes. The resultant lightweight of the hybrid system in this scheme and ease of installation (i.e. welding is not
required during installation) further reinforces the advantages of hybrid systems of metal and composites presented in Chapter 1.

This section presents a quick assessment tool for the prediction of the dynamic response of a partially strengthened blast wall. The developed model shows that for a sufficiently CFRP centrally strengthened blast wall, the bending, and deformation of the un-patched area govern the rotation at the support and overall behaviour of the panel. The strengthened scheme was able to absorb more blast energy by the formation of two symmetrically placed plastic hinges within the panel at a finite distance as compared to the formation of a single central plastic hinge in an unstrengthened panel. A good correlation was obtained when the results from analytical model where compared with FE models of the strengthened scheme. No substantial damage occurred in the composite strengthening patch when the strengthened region is sufficiently rigid.

On a final note, the fact that the reduction in the maximum displacement is more pronounced in the plastic response than in the elastic response does not limit the industrial application of the proposed strengthening scheme. Though suppliers of blast walls would prefer an elastic blast wall as against a plastic blast wall except specifically requested by a client. Plastically designed blast walls offer a wide range of advantages. For example, a reduction in dynamic load factor (DLF) arising from allowing plastic deformations can lead to the incorporation of strength reserves that would not be available in elastic walls [1-3]. The effect of strengthening is more pronounced in the plastic response because of the full trapezoidal mode shape formed in the strengthened panel as opposed to the triangular mode shape formed in the unstrengthened panel.

### 7.2.1.1. Future Work on Strengthened Blast Walls

Though the investigations on FMLs and lap joints carried out in Chapters 5 and 6 was to give an insight into debonding in order to substantiate (not necessarily to validate) the assumptions made on the strengthened blast wall, an experimental study is recommended to validate the model proposed. It would be important to conduct an experiment to investigate the effect of mechanical properties of adhesive on debonding of the hybrid system proposed. Debonding in composite strengthened structures takes place in regions of high stress concentrations. In addition, an investigation into the mode of debonding (i.e. theoretically debonding can take place within or at the interfaces of materials that form the strengthening system, favouring a propagation path that requires the least amount of energy) would be very useful in understanding the response of strengthened members under blast and other high velocity loadings. A fracture mechanics approach comes in handy in understanding this problem.
7.2.2. Simplified Model for Continuous Beams

This section presents a fundamental model for the dynamic analysis of continuous beams. Due to the limitations of the model presented in this section (i.e. inability to model membrane effects, connection pull-in and shear hinges at support), the elastic response predicted by this model was compared with the elastic response of the unstrengthened blast wall. It is obvious that the model would not be able to simulate plastic response where membrane effect, for instance, becomes pronounced. However, the practical application of the model proposed in this section cannot be obliterated. This is because from an offshore industry point of view, most clients would prefer elastic blast walls to plastic blast walls. One of the reasons for this is that elastic blast walls are usually lighter and therefore more economic than walls, which respond plastically because of the difficulty in obtaining a sufficient depth whilst maintaining low enough $b/t$ ratios to ensure plastic sections.

Furthermore, this procedure provides a quick assessment tool (an ideal case where the limitations of the model are ignored) for obtaining the response of continuous beam system bearing in mind the formation plastic hinges i.e. when plastic hinges form at the supports, mid span or at both points simultaneously. Consequently, a fundamental Pressure-Impulse ($P-I$) diagram for pulses with zero rise time, which is an indispensible tool for blast-loaded systems, is developed. The $P-I$ diagram presented marks a fundamental improvement on the existing fundamental work on $P-I$ diagrams which addresses SDOF system. Other existing works on $P-I$ diagrams for pulses with zero rise time attempts to use semi- analytical methods for continuous systems. This section of the research expresses the transverse dynamic responses in terms of dimensional parameters, based on a full set of quantities inherent in the system. This section further develops non-dimensional impulse and pressure based on these dimensional parameters. Based on critical deflection levels, Pressure-Impulse ($P-I$) diagrams are constructed for continuous beams. Further, a procedure for eliminating the loading shape dependency of the proposed $P-I$ diagram is adopted to make the $P-I$ diagrams pulse shape independent.

Obviously, this model provides an invaluable tool for the analysis of practical engineering structures. For ease of computation, simplifying a blast-loaded wall to a continuous beam system is acceptable. A very good example is in the analysis of blast wall systems in offshore applications.
7.2.3. Fibre Metal Laminates (FMLs)

In order to gain an insight into debonding, a numerical model of an FML is developed because of the availability of published experimental data on its blast response. Large plastic displacement of the global form of the FML followed by debonding of the back face aluminium panel was observed as the simulation progressed. This response gives an insight into debonding. It can be inferred that the initial global large plastic deformation introduces high strains and consequently stresses in the adhesive layer (cohesive elements in this case). The difference in in-plane stiffness in composite layer \( (E_1 = 13 \text{ Gpa}) \) and aluminium panel \( (E = 73 \text{ Gpa}) \) causes an increase in strain gradient across the adhesive. Thus, causing the debonding of the back face aluminium panel that was noticed. This observation led to the development of the lap joint model in Chapter 5 in order to have an understanding on how the elastic properties of bonded materials affect stresses in an adhesive region.

Furthermore, the work presented on FMLs in this research provides a cost effective method of investigating the response of FMLs and determining the integrity of constituent parts of the FMLs, thus, increasing the confidence of designers. The procedure presented extends already proposed numerical model for predicting the response of FMLs developed by Vo et al [4-5].

The section of FML presents an improved numerical model for analysing damage initiation, damage progression, and failure of a three-dimensional solid woven composite based on the principle of thermodynamics. Proper modelling the failure of constituent composite layers ensures appropriate idealisation of the response of the whole FML system. The constituent aluminium was model using the Johnson-Cook plasticity and damage model. The developed numerical model correlates well with experimental observations.

7.2.3.1. Future Work on Fibre Metal Laminates

The response of FMLs under the combined effects of blast and fragment loading is worth investigating – to have a full understanding of the blast response of FMLs. Fragments arise from the detonation of explosive filler in a cased bomb. When the bomb is initiated, the inside temperature and pressure increase rapidly and the casing expands until it breaks up into fragments. Thus, in this scenario a structure experiences (a) impulse from blast waves (b) impulse from striking fragments (c) impact from striking fragments [6]. The effect of such combined loading is more onerous than a single effect [7]. The retardation of the blast wave is
higher than that of the fragments while at larger distances the fragments will arrive before the blast [6].

**7.2.4. Lap Joint**

In an attempt to investigate the effect of the mechanical properties of bonded materials on the strain and stresses in an adhesive region, an analytical model for predicting the interfacial stresses in a transversely loaded lap joint was developed. The peel strains in the adhesive layer are found to be related to the difference in the transverse displacement of the top and bottom materials being bonded together from the results obtained in the analytical model. While the shear strains are a function of the transverse displacement of the top and bottom adherends plus the derivative with respect to distance of the longitudinal displacements of the bottom and top adherends. Interestingly, transverse and longitudinal displacements are functions of $EI$ (product of Young’s modulus and second moment of inertia) and $EA$ (product of Young’s modulus and cross-sectional area) respectively. These observations reiterates the observation in Chapter 3. The implication of this is that different values of $E$ (Young’s modulus) for top and bottom adherends with approximately the same thickness results in higher strains within the adhesive layer. Since debonding is strain/stress based, such configuration is susceptible to debonding.

The study on lap joint in this research presents a semi-analytical procedure to evaluate the maximum interfacial stresses in the adhesive layer of a single lap joint of similar adherends under transverse blast loads. The developed equation of motion is derived using Hamilton’s principle, which gives the governing PDE’s and the natural boundary conditions of the system. The equations is reduced to relevant ODE’s and solved appropriately.

The proposed analytical model gives a good correlation with the numerical model for maximum shear and peel stresses in the adhesive.

**7.3.4.1. Future Work on Lap Joints**

It would be interesting to extend the analytical procedure formulated in this work to lap joints with composite adherends. For composite joints, the inherent material heterogeneity and relatively low transverse stiffness imposes a greater complexity on its stress distribution when compared to lap joints with isotropic adherends [8]. The distinct features of lap joints with composite adherends are: (a) Coupling extension and bending behaviour in cases of adherends
with unsymmetrical stacking sequence. The effect of coupling extension can be fully appreciated when a close look is made on the compliance matrix of an anisotropic material. While the only coupling that occurs in an isotropic material is the extension-extension coupling (Poisson effect), composites with arbitrary stacking configuration could experience shear-extension coupling. Thus, making the analytical model for composites fundamentally different from that of isotropic materials. In addition, the compliance matrix of composite materials suggests different bending behaviour, (b) Considerable effect of staking sequence on stress distribution. These effects would make the analytical modelling of such joints quite rigorous.

On a final note, an experimental investigation of a pulse-loaded lap joint would further improve the understanding of the interfacial stress development in a lap joint.

7.3 Design Implications of Models

From a design point of view, two major conclusions can be drawn on the general performance of hybrid systems from this work.

1) In determining the maximum displacement for partially strengthened corrugated plates (i.e. composite strengthened) subjected to blast loadings, an analytical method which involves an equivalent beam section of the corrugated plate can be developed and analysed using assumed mode decomposition. An alternative way this can be done would be using an equivalent SDOF using Biggs method. This would involve developing a resistance curve for the hybrid systems using FEA and effective mass factors [2]. The design merit of the method proposed would not require an FEA analysis in the initial stage of design. Note that commercial FE software might not be available in the initial stage of the project, which involves preliminary sizing.

2) On the debonding failure mode of hybrid systems, it can be inferred from Chapters 5 and 6 that the strains developed in the adhesive layer connecting two components (say of different stiffness) is a function of the transverse displacements and longitudinal displacements of the components. In order to reduce the interfacial strains in the adhesive layer (debonding is a function of interfacial strain in the adhesive), materials of similar stiffness are recommended to be bonded together. This allows the full hybrid material to function properly without debonding reducing its efficiency.
7.4 General Conclusions

This section links observations made in this thesis as it relates to the performance of hybrid systems in general. As mentioned in Chapter 1, individual hybrid structural systems might have to be studied to obtain full insight into specific structural behaviours. However, this thesis studies three possible configurations and draws up a summary of behaviours that are related to hybrid systems in general.

- The blast performance of an engineering system can be improved by the combination of a high strength composite. Recent advances in the manufacture of composite materials have led to the development of high composites with relatively high in-plane stiffness. Due to the high in-plane stiffness and strengths of engineered composites, significant amount of energy are absorbed by them (i.e. Hoo Fatt showed that in a ballistic loaded Glare, 85% of the blast energy was absorbed by bending and membrane effects [9-12]). This scenario can be observed in the strengthened scheme, which behaved well in bending as result of the high stiffness of the composite patch in the hybrid system. Compston et al [13] have shown that FMLs with alternating 2024 aluminium and E-glass/polypropylene layers can increase the blast performance for given areal weight density of the aluminium by 50%. The sequence of failure observed in the numerical simulation of FMLs (i.e. in the following order: large global plastic deformation, debonding of back face aluminium, stretching of the aluminium and tearing, debonding of internal aluminium alloys and tearing of front face aluminium) obviously accounts for the increased blast performance when compared to monolithic aluminium. Monolithic aluminium would have only large deformation and stretching as the only energy-absorbing mode. In addition, it has been observed that fracture toughness of composites increases significantly with strain rate [13]. Thus, the increased performance of FMLs can also be attributed to the contribution of the strain rate effect of the constituent composites.

- It was observed that debonding of the back face aluminium sheet of an FML was the second failure mode observed in the simulated blast response of an FML. The implication of this is that, if debonding failure is prevented as assumed in the strengthened scheme, the energy absorption capacity of composites can be harnessed to contribute significantly the its blast performance. For a high hydrocarbon load of 4 bar, no fibre damage was observed in the composite patch of the strengthened scheme presented in Chapter 3. Fibre damage is the most detrimental failure mode.
The analytical model presented for a metal-adhesive-metal lap joint shows that the peel strains in the adhesives are a function of the difference in transverse displacements between the top and bottom adherends while shear strains in adhesives are a function of the longitudinal displacement. Thus, the assumption that interfacial stresses are kept to a minimum in hybrid systems with constituents having similar in-plane stiffness holds. The selection of Boron-epoxy as strengthening patch which has an in-plane stiffness of 207GPa in the fibre direction would result in minimal interfacial stresses, hence, the assumption made in Chapter 3 that debonding does not occur in the model is in line. On a general note, the analytical systems presented in Chapter 3, though for a blast wall can be applied to hybrid systems of composite strengthened thin wall panels. The beam models in chapters 3 and 4 can be generalised for any structure as the case might be.

The most important conclusion to be drawn from this work is the weight saving in the overall structure as a result of the efficient mechanical performance achieved by the introduction of composite systems compared to an alternative system of equivalent metal. An FML has been shown to have an improved mechanical performance than equivalent aluminium steel of same weight [13-14]. In addition, lap joints with composite adherends subjected to in-plane blast loads gave relatively reduced stresses in the adhesive layer when compared to lap joints with steel-steel adherends. The implication of this, as mentioned in Chapter 1 is a cheaper construction because of reduced weight of component materials. This can also be seen in the composite strengthened scheme of blast wall studied, an alternative strengthening scheme such as the one presented by Boh et al [15] would obviously result to a higher overall weight of the structure compared to the hybrid system proposed in this thesis. Boh et al [15] proposed a technique of increasing the energy absorption of blast walls by the inclusion of a passive impact barrier system placed at a certain offset behind the wall. The density of composite used in strengthening is one-third that of mild steel. Weight saving in construction is always desirable in offshore installations.

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8 Assumption was made in Chapter 3 – debonding is assumed not to occur
7.3 Cited References


Appendix

Copyright Authorization 1: Pressure-Impulse Diagrams for Blast Loaded Continuous Beams based on Dimensional Analysis

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