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Weyl asymptotics of bisingular operators and Dirichlet divisor problem

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Abstract We consider a class of pseudodifferential operators, with crossed vector valued symbols, defined on the product of two closed manifolds. We study the asymptotic expansion of the counting function of positive selfadjoint operators in this class. Using a general Theorem of J. Aramaki, we can determine the first term of the asymptotic expansion of the counting function and, in a special case, we are able to find the second term. We give also some examples, emphasizing connections with problems of analytic number theory, in particular with Dirichlet divisor function.

Keywords Weyl's law · Bisingular operators · Dirichlet divisor problem · Spectral analysis

Mathematics Subject Classification (2010) 35P20 · 58J40 · 47A10

Introduction

In [28] L. Rodino introduced bisingular operators: a class of pseudodifferential operators defined on the product of two closed manifolds $M_1 \times M_2$, related to the multiplicative property of Atiyah-Singer index, see [2]. A simple example of an operator in this class is the tensorial product $A_1 \otimes A_2$, where A_1, A_2 are pseudodifferential operators on the closed manifolds M_1, M_2. Another example, studied in [28], is the vector-tensor product $A_1 \boxtimes A_2$. In [26], in order to prove an index formula, F. Nicola and L. Rodino introduced classical, i.e. polyhomogeneous, bisingular operators and defined Wodzicki Residue for this class of operators. The two authors defined the residue, via holomorphic families, as in [9,25]. For the index of bisingular operators

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see also the work of V. S. Pilidi [27] and of R. V. Dudučava [5,6]. In [23], R. Melrose and F. Rochon introduced pseudodifferential operators of product type, a class of operators close to bisingular operators. Bisingular operators are an example of operators with vector valued symbols; pseudodifferential operators of this type have been meticulously studied, see, for example, Fedosov, Schulze, Tarkhanov [8] and the references therein.

The aim of this paper is to analyze the asymptotic behavior of the counting function of selfadjoint elliptic positive bisingular operators. Similarly to the the case of SG-calculus [3] (see e.g. [7,29] for more detail on *SG*-calculus), we use techniques related to complex powers of operators, ζ-function and Tauberian Theorems. This strategy, in the setting of closed manifolds, was first used by V. Guillemin [14] in order to get the so called *soft proof* of Weyl's formula.

Here, as in the case of *SG*-calculus, it turns out that the ζ-function can have poles of order two. Thus, using a refinement of Tauberian Theorem due to J. Aramaki [1], the asymptotic behavior of the counting function is determined. The presence of a pole of order two of the ζ-function implies that the counting functions can have asymptotic terms of order $\lambda^c \log\lambda$. Such a behavior appears in various setting: manifolds with conical singularities [9], *SG*-calculus on $\mathbb{R}^n$ [25], *SG*-calculus on manifolds with cylindrical ends [20]. See also Gramchev, Pilipović, Rodino, Wong [10,11] on the asymptotic expansion of the counting function in the case of twisted bi-Laplacian. Furthermore, in [24], S. Moroianu studied Weyl's law on manifolds with cusps, with an approach similar to the one used in this paper. In a special case, he showed that the growth rate of the counting function is $\lambda^c \log\lambda$.

We remark that it is not surprising that the ζ-function of a selfadjoint elliptic positive bisingular operator can have poles of order 2. Indeed, let us consider two positive elliptic pseudodifferential operators A,B defined on the closed manifolds M_1,M_2. From general theory of complex powers of pseudodifferential operators on closed manifolds [30], we know that the ζ-function of an operator P of this type is holomorphic for $\mathfrak{Re}(z) < -\frac{n}{m}$ ($n = \dim M$, m order of P) and it can be extended as a meromorphic function to the whole of $\mathbb{C}$ with poles of order 1. As we noticed at the beginning, the tensorial product $A \otimes B$ is a bisingular operator on $M_1 \times M_2$ and it is clearly positive and selfadjoint. One can prove the following

$$\zeta(A\otimes B,z) = \zeta(A,z)\zeta(B,z). \tag{1}$$

If one defines the ζ-function using the eigenvalues, equality (1) becomes more transparent. To this end, let $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{\mu_i\}_{i\in\mathbb{N}}$ be the eigenvalues of A and B, respectively. Then the eigenvalues of $A\otimes B$ turn out to be $\{\lambda_j\mu_i\}_{i,j\in\mathbb{N}^2}$. Therefore we have

$$\begin{aligned}
\zeta(A,z) &= \sum_{j\in\mathbb{N}} \lambda_j^z, \quad \mathfrak{Re}(z) < -\frac{n_1}{m_A};\\
\zeta(B,z) &= \sum_{i\in\mathbb{N}} \mu_i^z, \quad \mathfrak{Re}(z) < -\frac{n_2}{m_B};\\
\zeta(A\otimes B,z) &= \sum_{i,j\in\mathbb{N}^2} \lambda_j^z\mu_i^z = \zeta(A,z)\zeta(B,z), \quad \mathfrak{Re}(z) < -\max\Big\{\frac{n_1}{m_A},\frac{n_2}{m_B}\Big\};
\end{aligned}$$

where $n_1 = \dim M_1$, $n_2 = \dim M_2$ and m_A, m_B are the orders of A and B. Then the product structure of $\zeta(A\otimes B,z)$ implies that it can have poles of order two. Let us now focus on the special case $\frac{n_1}{m_A} = \frac{n_2}{m_B} = z_0$:

$$\begin{aligned}
\zeta(A,z) =& \frac{C_A}{(z+z_0)} + h_A(z), \quad \mathfrak{Re}(z) < -z_0+\varepsilon;\\
\zeta(B,z) =& \frac{C_B}{(z+z_0)} + h_B(z), \quad \mathfrak{Re}(z) < -z_0+\varepsilon;\\
\zeta(A\otimes B,z) =& \frac{C_AC_B}{(z+z_0)^2} + \frac{h_A(z)+h_B(z)}{(z+z_0)} + h_A(z)h_B(z), \quad \mathfrak{Re}(z) < -z_0+\varepsilon;
\end{aligned} \tag{2}$$

where C_A, C_B are constants that depend just on the principal symbol of A, B, while h_A, h_B are holomorphic functions which depend on the whole symbol of A, B. From (2), it is clear that $\zeta(A\otimes B,z)$ has a pole of order two. Moreover, we observe that the coefficient of the pole of order one depends on the whole symbol of A and B. Finally, applying J. Aramaki's Theorem 3.1, from (2) one obtains

$$N_{A\otimes B}(\lambda) \sim \frac{C_AC_B}{z_0}\lambda^{z_0}\log(\lambda) - \left(\frac{h_A(-z_0)-h_B(-z_0)}{z_0} + \frac{C_AC_B}{z_0^2}\right)\lambda^{z_0} + O(\lambda^{z_0-\delta}), \tag{3}$$

where $\delta > 0$. Simple examples of operators A and B for which (3) holds are $A = -\Delta_g + 1$, $B = -\Delta_{g'} + 1$, where Δ_g, $\Delta_{g'}$ are the Laplace Beltrami operators associated to Riemanniann structures of M_1, M_2 respectively. We will extend (3) to all positive bisingular elliptic operators, expressing the constants in the Weyl asymptotics in terms of the crossed vector-valued symbols.

The paper is organized as follows. In Section 1 we shortly recall basic properties of bisingular operators; we refer the reader to [26,28] for more details. Section 2 is devoted to the definition of complex powers of suitable bisingular operators; we introduce the ζ-function in this setting and we study its meromorphic extension. The main result, concerning the asymptotics of the counting function of selfadjoint elliptic positive bisingular operators, is stated in section 3. In section 4, we show the connection with Dirichlet divisor problem, which we reconsider from the point of view of Spectral Theory.

1 Bisingular operators

We start with the definitions of bisingular symbols and bisingular symbols with homogeneous principal symbol. In the following, Ω_i always denotes a bounded open domain of $\mathbb{R}^{n_i}$.

Definition 1.1 We define $S^{m_1,m_2}(\Omega_1,\Omega_2)$ as the set of $C^\infty(\Omega_1\times\Omega_2\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ functions such that, for all multiindex α_i, β_i and for all compact subset $K_i \subseteq \Omega_i$, $i = 1,2$, there exists a positive constant $C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2}$ so that

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}a(x_1,x_2,\xi_1,\xi_2)| \le C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2}\langle\xi_1\rangle^{m_1-|\alpha_1|}\langle\xi_2\rangle^{m_2-|\alpha_2|},$$

for all $x_i \in K_i$, $\xi_i \in \mathbb{R}^{n_i}$, $i = 1,2$. As usual, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

$S^{-\infty,-\infty}(\Omega_1,\Omega_2)$ is the set of smoothing symbols. Following [28], we introduce the subclass of bisingular operators with homogeneous principal symbol.

Definition 1.2 Let $a \in S^{m_1,m_2}(\Omega_1,\Omega_2)$; a has a homogeneous principal symbol if

i) there exists $a_{m_1,\cdot}(x_1,x_2,\xi_1,\xi_2) \in S^{m_1,m_2}(\Omega_1,\Omega_2)$ such that

$$a(x_1,x_2,t\xi_1,\xi_2) = t^{m_1}a(x_1,x_2,\xi_1,\xi_2), \quad \forall x_1,x_2,\xi_2, \quad \forall|\xi_1|>1, t>0,$$
$$a-\psi_1(\xi_1)a_{m_1,\cdot} \in S^{m_1-1,m_2}(\Omega_1,\Omega_2), \quad \psi_1 \text{ cut-off function of the origin.}$$

Moreover, $a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2) \in L^{m_2}_{\mathrm{cl}}(\Omega_2)$, so, being a classical symbol on Ω_2, it admits an asymptotic expansion w.r.t. the ξ_2 variable.

ii) there exists $a_{\cdot,m_2}(x_1,x_2,\xi_1,\xi_2) \in S^{m_1,m_2}(\Omega_1,\Omega_2)$ such that

$$a(x_1,x_2,\xi_1,t\xi_2) = t^{m_2}a(x_1,x_2,\xi_1,\xi_2), \quad \forall x_1,x_2,\xi_1, \quad \forall|\xi_2|>1, t>0,$$
$$a-\psi_2(\xi_2)a_{\cdot,m_2} \in S^{m_1,m_2-1}(\Omega_1,\Omega_2), \quad \psi_2 \text{ cut-off function of the origin.}$$

Moreover, $a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2) \in L^{m_1}_{\mathrm{cl}}(\Omega_1)$, so, being a classical symbol on Ω_1, it admits an asymptotic expansion w.r.t. the ξ_1 variable.

iii) The symbols $a_{m_1,\cdot}$ and $a_{\cdot,m_2}$ have the same leading term, so there exists a_{m_1,m_2} such that

$$a_{m_1,\cdot}-\psi_2(\xi_2)a_{m_1,m_2} \in S^{m_1,m_2-1}(\Omega_1,\Omega_2),$$
$$a_{\cdot,m_2}-\psi_1(\xi_1)a_{m_1,m_2} \in S^{m_1-1,m_2}(\Omega_1,\Omega_2),$$

and

$$a-\psi_1 a_{m_1,\cdot}-\psi_2 a_{\cdot,m_2}+\psi_1\psi_2 a_{m_1,m_2} \in S^{m_1-1,m_2-1}(\Omega_1,\Omega_2).$$

The set of symbols with homogeneous principal symbol is denoted as $S^{m_1,m_2}_{\mathrm{pr}}(\Omega_1,\Omega_2)$. We will shortly write that the principal symbol of a is $\{a_{m_1,\cdot},a_{\cdot,m_2}\}$.

We can observe a similarity, at least formal, between bisingular symbols with homogeneous principal symbol and SG- classical symbols, see, e.g.. [7,25].

We define bisingular operators via their left quantization. A linear operator $A: C^\infty_c(\Omega_1\times\Omega_2) \to C^\infty(\Omega_1\times\Omega_2)$ is a bisingular operator if it can be written in the form

$$\begin{aligned} A(u)(x_1,x_2) =& \mathrm{Op}(a)(x_1,x_2) \\ =& \frac{1}{(2\pi)^{n_1+n_2}}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} e^{ix_1\cdot\xi_1+ix_2\cdot\xi_2}a(x_1,x_2,\xi_1,\xi_2)\hat{u}(\xi_1,\xi_2)d\xi_1 d\xi_2. \end{aligned}$$

If $a \in S^{m_1,m_2}(\Omega_1,\Omega_2)$ or $a \in S^{m_1,m_2}_{\mathrm{pr}}(\Omega_1,\Omega_2)$, then we write $A \in L^{m_1,m_2}(\Omega_1,\Omega_2)$ and $A \in L^{m_1,m_2}_{\mathrm{pr}}(\Omega_1,\Omega_2)$ respectively. The above definition can be extended to the product of closed manifolds; we refer to [28] for the details of the construction of global operators and the corresponding calculus.

Definition 1.2 implies that, for every operator $A \in L^{m_1,m_2}_{\mathrm{pr}}(\Omega_1,\Omega_2)$, we can define functions $\sigma^{m_1}, \sigma^{m_2}, \sigma^{m_1,m_2}$ such that

$$\begin{aligned}
\sigma_1^{m_1}(A) &: T^*\Omega_1 \setminus \{0\} \to L^{m_2}_{\mathrm{cl}}(\Omega_2) \\
&\quad (x_1,\xi_1) \mapsto a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2), \\
\sigma_2^{m_2}(A) &: T^*\Omega_2 \setminus \{0\} \to L^{m_1}_{\mathrm{cl}}(\Omega_1) \\
&\quad (x_2,\xi_2) \mapsto a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2), \\
\sigma^{m_1,m_2}(A) &: T^*\Omega_1 \setminus \{0\} \times T^*\Omega_2 \setminus \{0\} \to \mathbb{C} \\
&\quad (x_1,x_2,\xi_1,\xi_2) \mapsto a_{m_1,m_2}(x_1,x_2,\xi_1,\xi_2).
\end{aligned} \tag{4}$$

Moreover, denoting by $\sigma(P)(x,\xi)$ the principal symbol of a preudodifferential operator P on a closed manifold, the following *compatibility relation* holds

$$\begin{aligned}
\sigma(\sigma_1^{m_1}(A)(x_1,\xi_1))(x_2,\xi_2) &= \sigma(\sigma_2^{m_2}(A)(x_2,\xi_2))(x_1,\xi_1) \\
&= \sigma^{m_1,m_2}(A)(x_1,x_2,\xi_1,\xi_2) = a_{m_1,m_2}(x_1,x_2,\xi_1,\xi_2).
\end{aligned} \tag{5}$$

Remark 1 If we consider the product of closed manifolds $M_1 \times M_2$, then the whole symbol is a local object, in general. Nevertheless, similarly to the calculus on closed manifolds, it is possible to give an invariant meaning to the functions (4) as functions defined on the cotangent bundle, see [28].

As in the case of the calculus on closed manifolds, it is possible to define adapted Sobolev spaces and then to prove some continuity results.

Definition 1.3 Let M_1, M_2 be two closed manifolds. The Sobolev space $H^{m_1,m_2}(M_1 \times M_2)$ is defined by

$$H^{m_1,m_2}(M_1 \times M_2) = \{u \in \mathscr{S}'(M_1 \times M_2) \mid \mathrm{Op}(\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2})(u) \in L^2(M_1 \times M_2)\}.$$

If $u \in H^{m_1,m_2}(M_1 \times M_2)$ then $\|u\|_{m_1,m_2} = \|\mathrm{Op}(\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2})(u)\|_2$. Using the formalism of tensor product, we can also write[1]

$$H^{m_1,m_2}(M_1 \times M_2) = H^{m_1}(M_1)\widehat{\otimes}_\pi H^{m_2}(M_2).$$

Similarly to Sobolev spaces $H^s(M)$, we have

i) $H^{m_1,m_2}(M_1 \times M_2) \hookrightarrow H^{m_1',m_2'}(M_1 \times M_2)$ is a continuous immersion if $m_i \geq m_i'$, $i = 1,2$.
ii) $H^{m_1,m_2}(M_1 \times M_2) \hookrightarrow H^{m_1',m_2'}(M_1 \times M_2)$ is a compact immersion if $m_i > m_i'$, $i = 1,2$.

Proposition 1.1 *A pseudodifferential operator $A \in L^{m_1,m_2}(M_1 \times M_2)$ can be extended to a continuous operator*

$$A : H^{s,t}(M_1 \times M_2) \to H^{s-m_1,t-m_2}(M_1 \times M_2).$$

Furthermore, the norm of the operator can be estimated using the seminorms of the symbol. It is also possible to prove the following proposition:

[1] For definition of $\widehat{\otimes}_\pi$ see [32].

Proposition 1.2 *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be a bisingular operator; if $m_i \leq 0$ ($i = 1,2$), then there exists $N \in \mathbb{N}$ such that $\|A\|_{0,0} \leq \sup |\sum_{i \leq N} p_i(a(x_1,x_2,\xi_1,\xi_2))|$, where $\{p_i(\cdot)\}_{i\in\mathbb{N}}$ are the seminorms of the Fréchet space $S^{m_1,m_2}(M_1,M_2)$.*

An operator $A \in L^{m_1,m_2}(M_1 \times M_2)$ is elliptic if $a_{m_1,\cdot}, a_{\cdot,m_2}, a_{m_1,m_2}$, the three components of its principal symbol, are invertible in their domain of definition. Explicitly:

Definition 1.4 Let $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$; A is elliptic if

i) $\sigma^{m_1,m_2}(A)(v_1,v_2) \neq 0$ for all $(v_1,v_2) \in T^*M_1 \setminus \{0\} \times T^*M_2 \setminus \{0\}$;
ii) $\sigma_1^{m_1}(A)(v_1) \in L^{m_2}_{\mathrm{cl}}(M_2)$ is invertible for all $v_1 \in T^*M_1 \setminus \{0\}$;
iii) $\sigma_2^{m_2}(A)(v_2) \in L^{m_1}_{\mathrm{cl}}(M_1)$ is invertible for all $v_2 \in T^*M_2 \setminus \{0\}$;

where $\sigma^{m_1,m_2}(A), \sigma_1^{m_1}(A)$, $\sigma_2^{m_2}(A)$ are as in (4).

In [28], it is proved that, if A satisfies Definition 1.4, then A is a Fredholm operator. This property is a corollary of the following theorem:

Theorem 1.1 *Let $A \in L^{m_1,m_2}_{\mathrm{pr}}(M_1 \times M_2)$ be elliptic; then there exists an operator $B \in L^{-m_1,-m_2}_{\mathrm{pr}}(M_1 \times M_2)$ such that*

$$AB = \mathrm{Id} + K_1,$$
$$BA = \mathrm{Id} + K_2,$$

where Id *is the identity map and K_1, K_2 are compact operators. Moreover, the symbol of B is $b = \{\sigma_1^{m_1}(A)^{-1}, \sigma_2^{m_2}(A)^{-1}\}$.*

The proof of Theorem 1.1 is an easy consequence of the global version of the following lemma:

Lemma 1.1 *Let $A \in L^{m_1,m_2}(\Omega_1 \times \Omega_2)$ and $B \in L^{m_1',m_2'}(\Omega_1 \times \Omega_2)$, then*

$$\{(a \circ b)_{m_1+m_1',\cdot}, (a \circ b)_{\cdot,m_2+m_2'}\} = \{a_{m_1,\cdot} \circ_{\xi_2} b_{m_1',\cdot}, a_{\cdot,m_2} \circ_{\xi_1} b_{\cdot,m_2'}\}$$

where

$$(a \circ_{\xi_1} b)(x_1,x_2,D_1,\xi_2)(u) = a(x_1,x_2,D_1,\xi_2) \circ b(x_1,x_2,D_1,\xi_2)(u) \quad \forall u \in C_c^\infty(\Omega_1),$$
$$(a \circ_{\xi_2} b)(x_1,x_2,\xi_1,D_2)(v) = a(x_1,x_2,\xi_1,D_2) \circ b(x_1,x_2,\xi_1,D_2)(v) \quad \forall v \in C_c^\infty(\Omega_2).$$

In first row the composition is in the space $L^\infty(\Omega_1)$ of pseudodifferential operators on Ω_1, in second row, it is in the space $L^\infty(\Omega_2)$.

2 Complex powers of bisingular operators

In this section we define complex powers of a subclass of elliptic bisingular operators. The first step is to give a suitable definition Λ-elliptic operators w.r.t. a sector of the complex plane Λ.

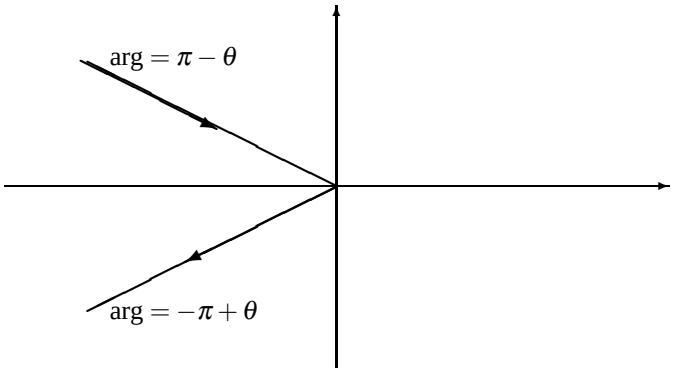
Definition 2.1 Let Λ be a sector of $\mathbb{C}$; we say that $a \in S^{m_1,m_2}_{\mathrm{pr}}(M_1,M_2)$ is Λ-elliptic w.r.t. Λ if there exists a positive constant R such that

i)
$$\big(\sigma^{m_1,m_2}(A)(v_1,v_2)-\lambda\big)^{-1}\in S^{-m_1,-m_2}(M_1,M_2),$$
for all $|v_i|>R$, $i=1,2$, and for all $\lambda\in\Lambda$.

ii)
$$\sigma_1^{m_1}(A)(v_1)-\lambda\ \mathrm{Id}_{M_2}\in L^{m_2}_{\mathrm{cl}}(M_2),$$
is invertible for all $|v_1|>R$ and for all $\lambda\in\Lambda$.

iii)
$$\sigma_2^{m_2}(A)(v_2)-\lambda\ \mathrm{Id}_{M_1}\in L^{m_1}_{\mathrm{cl}}(M_1),$$
is invertible for all $|v_2|>R$ and for all $\lambda\in\Lambda$.

In the following, in order to define the complex power of A, we assume that Λ is a sector of the complex plane with vertex at the origin, that is
$$\Lambda=\{z\in\mathbb{C}\mid \arg(z)\in[\pi-\theta,-\pi+\theta]\}.$$



Lemma 2.1 *Let $a\in S^{m_1,m_2}(\Omega_1,\Omega_2)$ be Λ-elliptic. For all $K_i\subseteq\Omega_i$, $i=1,2$, there exist $c_0>1$ and a set*
$$\Omega_{\xi_1,\xi_2}:=\{z\in\mathbb{C}\setminus\Lambda\mid \frac{1}{c_0}\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2}<|z|<c_0\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2}\}\tag{6}$$
such that
$$\mathrm{spec}(a(x_1,x_2,\xi_1,\xi_2))=\{\lambda\in\mathbb{C}\mid a(x_1,x_2,\xi_1,\xi_2)-\lambda=0\}\subseteq\Omega_{\xi_1,\xi_2},\qquad\forall x_i\in\Omega_i,\xi_i\in\mathbb{R}^{n_i};$$
moreover,
$$\begin{aligned}&|\big(\lambda-a_{m_1,m_2}(x_1,x_2,\xi_1,\xi_2)\big)^{-1}|\le C(|\lambda|+\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2})^{-1},\\&|\big(a_{m_1,\cdot}-\lambda\,\mathrm{Id}_{\Omega_1}\big)^{-1}|\le C(|\lambda|+\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2})^{-1},\\&|\big(a_{\cdot,m_2}-\lambda\,\mathrm{Id}_{\Omega_2}\big)^{-1}|\le C(|\lambda|+\langle\xi_1\rangle^{m_1}\langle\xi_2\rangle^{m_2})^{-1},\\&\forall x_i\in K_i,\xi_i\in\mathbb{R}^{n_i},\lambda\in\mathbb{C}\setminus\Omega_{\xi_1,\xi_2},i=1,2,\end{aligned}$$
where $\big(a_{m_1,\cdot}-\lambda\,\mathrm{Id}_{\Omega_1}\big)^{-1}$ stands for the symbol of the operator $(a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)-\lambda\,\mathrm{Id}_{\Omega_1})^{-1}$, and similarly for $\big(a_{\cdot,m_2}-\lambda\,\mathrm{Id}_{\Omega_2}\big)^{-1}$.

The proof of Lemma 2.1 is essentially the same of the one of Lemma 3.5 in [22].

Next, we prove that, if A Λ-elliptic, then we can define a parametrix of $(A-\lambda\ \mathrm{Id})$. Actually, we prove that, for $|\lambda|$ large enough, the resolvent $(A-\lambda\ \mathrm{Id})^{-1}$ exists. Restricting ourselves to differential operators, we could follow formally the idea of Shubin ([31], ch. II) of parameter depending operators. For general pseudodifferential operators, it is well know that this idea does not work, see [12].

Theorem 2.1 *Let $A\in L^{m_1,m_2}_{\mathrm{pr}}(M_1\times M_2)$ be Λ-elliptic. Then there exists $R\in\mathbb{R}^+$, such that the resolvent $(A-\lambda\,\mathrm{Id})^{-1}$ exists for $\lambda\in\Lambda_R=\{\lambda\in\Lambda\mid|\lambda|\geq R\}$. Moreover,*

$$\|(A-\lambda\,\mathrm{Id})^{-1}\|=O(|\lambda|^{-1}),\quad \lambda\in\Lambda_R.$$

Proof First, we look for an inverse of $(A-\lambda\ \mathrm{Id})$ modulo compact operators, that is an operator $B(\lambda)$ such that:

$$\begin{aligned}(A-\lambda)\circ B(\lambda)&=\mathrm{Id}+R_1(\lambda),\quad \lambda R_1(\lambda)\in L^{-1,-1}(M_1\times M_2),\\ B(\lambda)\circ(A-\lambda)&=\mathrm{Id}+R_2(\lambda),\quad \lambda R_2(\lambda)\in L^{-1,-1}(M_1\times M_2),\end{aligned}\tag{7}$$

uniformly w.r.t. $\lambda\in\Lambda$. In order to find such an operator, we make the principal symbol explicit:

$$a-\lambda=\mathrm{psym}(a)-\lambda+c,\quad c\in S^{m_1-1,m_2-1}(M_1,M_2),$$

where $\mathrm{psym}(a)=\psi_1 a_{m_1,\cdot}+\psi_2 a_{\cdot,m_2}-\psi_1\psi_2 a_{m_1,m_2}$. As we have noticed in Theorem 1.1, we can write the symbol of the inverse (modulo compact operators) of an elliptic operator. In this case we need to be more careful because of the parameter λ. Following the same construction as in Theorem 1.1, we obtain

$$b(\lambda)=\{\big((\sigma_1^{m_1}(A)-\lambda\ \mathrm{Id}_{M_2})^{-1},(\sigma_2^{m_2}(A)-\lambda\ \mathrm{Id}_{M_1})^{-1}\}.\tag{8}$$

The above definition (8) is consistent in view of the Λ-ellipticity and of the following relation

$$\begin{aligned}\sigma\big((\sigma_1^{m_1}(A)-\lambda\ \mathrm{Id}_{M_2})^{-1}(x_1,\xi_1)\big)(x_2,\xi_2)&=(a_{m_1,m_2}-\lambda)^{-1}(x_1,x_2,\xi_1,\xi_2),\\ \sigma\big((\sigma_2^{m_2}(A)-\lambda\ \mathrm{Id}_{M_1})^{-1}(x_2,\xi_2)\big)(x_1,\xi_1)&=(a_{m_1,m_2}-\lambda)^{-1}(x_1,x_2,\xi_1,\xi_2).\end{aligned}$$

Using the rules of the calculus and Lemma 2.1, we can check that $B(\lambda)$ satisfies conditions (7). By parameter ellipticity, we get that $R_1(\lambda)$ and $R_2(\lambda)$ are compact operators for $\lambda\in\Lambda$, namely

$$\begin{aligned}(A-\lambda\ \mathrm{Id})\circ B(\lambda)&=\mathrm{Id}+R_1(\lambda),\\ (A-\lambda\ \mathrm{Id})\circ B(\lambda)&=\mathrm{Id}+R_2(\lambda),\end{aligned}\tag{9}$$

$\lambda R_1(\lambda),\lambda R_2(\lambda)\in S^{-1,-1}(M_1\times M_2)$ uniformly w.r.t. $\lambda\in\Lambda$. So $B(\lambda)$ is a parametrix and its symbol $b(\lambda)$ has the following form

$$\begin{aligned}b(\lambda)&=-(a_{m_1,m_2}(x_1,x_2,\xi_1,\xi_2)-\lambda)^{-1}\psi_1(\xi_2)\psi_2(\xi_1)\\ &+(a_{m_1,\cdot}-\lambda\ \mathrm{Id}_{M_2})^{-1}(x_1,x_2,\xi_1,\xi_2)\psi_1(\xi_1)\\ &+(a_{\cdot,m_2}-\lambda\ \mathrm{Id}_{M_1})^{-1}(x_1,x_2,\xi_1,\xi_2)\psi_2(\xi_2),\end{aligned}$$

where $(a_{m_1,\cdot}-\lambda\ \mathrm{Id}_{M_2})^{-1}(x_1,x_2,\xi_1,\xi_2)$ is the value of the symbol of the operator $(a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)-\lambda\ \mathrm{Id}_{M_2})^{-1}$ at (x_2,ξ_2), and similarly for $(a_{\cdot,m_2}-\lambda\ \mathrm{Id}_{M_1})^{-1}$. Furthermore, denoting by $r_1(\lambda)$ the symbol of $R_1(\lambda)$, we easily obtain

$$r_1(\lambda)=(a-\mathrm{psym}(a))\circ b(\lambda)+(\mathrm{psym}(a)\circ b(\lambda))-1, \tag{10}$$

hence $r_1(\lambda)\in S^{-1,-1}(M_1,M_2)$ is the asymptotic sum of terms of the type

$$\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}gD_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}b(\lambda)\quad g\in S^{m_1,m_2}(M_1,M_2).$$

Clearly $(a_{m_1,m_2}(x_1,x_2,\xi_1,\xi_2)-\lambda)^{-1}=O(|\lambda|^{-1})$. By the theory of pseudodifferential operators on closed manifolds, the same property holds for the symbols of the operators $(a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)-\lambda\ \mathrm{Id}_{M_2})^{-1}$ and $(a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2)-\lambda\ \mathrm{Id}_{M_1})^{-1}$ and their derivatives. Thus $r_1(\lambda)=O(|\lambda|^{-1})$, as a consequence of the calculus. By Proposition 1.2, this implies $\|R_1\|_{L^2}=O(|\lambda|^{-1})$, and the same is true for the operator R_2. So we can choose λ large enough such that R_1,R_2 have norm less than 1. In this way, using Neumann series, we prove that $(A-\lambda\ \mathrm{Id})$ is one to one and onto, therefore invertible, by the Open Map Theorem. Again, by Neumann series, we obtain $\tilde{B}(\lambda)$ such that (9) is fulfilled with $\tilde{R}_1,\tilde{R}_2$ smoothing and still with norm $O(\lambda^{-1})$. Now notice that $\lambda\left[B(\lambda)-\tilde{B}(\lambda)\right]\in S^{-m_1-1,-m_2-1}$ for all $\lambda\in\Lambda$. Furthermore, if we multiply both equations in (7) by $(A-\lambda\ \mathrm{Id})^{-1}$ we obtain

$$(A-\lambda\ \mathrm{Id})^{-1}=\tilde{B}(\lambda)+\tilde{B}(\lambda)R_1(\lambda)+R_2(\lambda)(\lambda-A)^{-1}R_1(\lambda).$$

Hence $\|(A-\lambda\ \mathrm{Id})^{-1}\|=O(|\lambda|^{-1})$ and $\lambda^2\left[(A-\lambda)^{-1}-\tilde{B}(\lambda)\right]$ is a smoothing operator in $L^{-\infty,-\infty}(M_1\times M_2)$, uniformly w.r.t. λ.

In order to define complex powers of an elliptic bisingular operator, we introduce some natural assumptions.

Assumptions 1 *1. $A\in S^{m_1,m_2}(M_1,M_2)$ is Λ-elliptic.*
2. $\sigma(A)\cap\Lambda=\emptyset$ (in particular A is invertible).
3. A has homogeneous principal symbols.

Remark 2 If we consider a Λ-elliptic operator $A\in L^{m_1,m_2}_{\mathrm{pr}}(M_1\times M_2)$ with $m_i>0$ $(i=1,2)$, then $\sigma(A)$ is either discrete or the whole of $\mathbb{C}$, because the resolvent is a compact operator ([31], Ch. I). Since by Theorem 2.1 we know that for large λ the resolvent is well defined, it turns out that the spectrum $\sigma(A)$ is discrete. Then, modulo a shift of the operator, we can find a suitable sector such that Assumptions 1 is fulfilled.

Definition 2.2 Let A be an operator fulfilling Assumptions 1. Then, we can define

$$A_z:=\frac{i}{2\pi}\int_{\partial\Lambda_\varepsilon^+}\lambda^z(A-\lambda\ \mathrm{Id})^{-1}d\lambda,\quad \mathfrak{Re}(z)<0, \tag{11}$$

where $\Lambda_\varepsilon=\Lambda\cup\{z\in\mathbb{C}\mid |z|\le\varepsilon\}$.
The Dunford integral in (11) is convergent because $\|(A-\lambda\ \mathrm{Id})^{-1}\|=O(|\lambda|^{-1})$ for λ large enough. As usual, we next define

$$A^z:=A_{z-k}\circ A^k,\quad \mathfrak{Re}(z-k)<0.$$

Remark 3 In Assumptions 1 we require $\Lambda \cap \sigma(A) = \emptyset$, therefore in particular the operator must be invertible. It is possible to define complex powers of non invertible operator as well, provided the origin is an isolated point of the spectrum, see, e.g., [4]. For example, one can define the complex powers of $A = -\Delta \otimes -\Delta$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$, even if A has an infinite dimensional kernel.

Theorem 2.2 *If $A \in L^{m_1,m_2}(M_1,M_2)$ satisfies Assumptions 1, then $A^z \in L^{m_1 z,m_2 z}(M_1 \times M_2)$ and it has homogeneous principal symbol. Moreover, by Cauchy Theorem*[2]

$$\begin{aligned} a^z_{m_1 z,m_2 z} &= (a_{m_1,m_2})^z, \\ a^z_{m_1 z,\cdot} &= (a_{m_1,\cdot})^z, \\ a^z_{\cdot,m_2 z} &= (a_{\cdot,m_2})^z. \end{aligned} \tag{12}$$

Proof As a consequence of a general version of Fubini's Theorem, denoting by a^z the symbol of A^z, we obtain

$$a^z = \frac{i}{2\pi} \int_{\partial^+ \Lambda_\varepsilon} \lambda^z (a - \lambda \,\mathrm{Id})^{-1}) d\lambda, \quad \mathfrak{Re}(z) < 0.$$

where $(a - \lambda \,\mathrm{Id})^{-1}$ is the symbol of the operator $(A - \lambda \,\mathrm{Id})^{-1}$. By Theorem 2.1, we know that $\lambda^2 \Big[(A - \lambda \,\mathrm{Id})^{-1} - B(\lambda) \Big] \in L^{-\infty,-\infty}(M_1 \times M_2)$ so, up to smoothing symbols, we have

$$\begin{aligned} a^z =& \frac{i}{2\pi} \int_{\partial^+ \Lambda_\varepsilon} \lambda^z (\tilde{b}(\lambda)) d\lambda \\ =& \frac{i}{2\pi} \int_{\Omega_{\xi_1,\xi_2}} \lambda^z (\tilde{b}(\lambda)) d\lambda, \end{aligned} \tag{13}$$

where Ω_{ξ_1,ξ_2} is as in Lemma 2.1 and the second equality in (13) follows by Cauchy integral formula. Now, by Lemma 2.1 and by the explicit form of $\tilde{b}(\lambda)$, we get $A^z \in L^{m_1 z,m_2 z}(M_1 \times M_2)$. In order to show that A^z has homogeneous principal symbol, we write

$$\begin{aligned} (\tilde{b}(\lambda)) =& \psi_1 (\sigma^{m_1}(A) - \lambda \,\mathrm{Id}_{M_2})^{-1} + \psi_2 (\sigma^{m_2}(A) - \lambda \,\mathrm{Id}_{M_1})^{-1} \\ & - \psi_1 \psi_2 (\sigma^{m_1,m_2}(A) - \lambda)^{-1} + c(\lambda), \end{aligned}$$

where $\lambda c(\lambda) \in S^{-m_1-1,-m_2-1}(M_1,M_2)$, $\forall \lambda \in \Lambda$. We split integral in (13) so that

$$a^z = \quad \tfrac{i}{2\pi} \textstyle\int_{\partial^+ \Lambda_\varepsilon} \lambda^z \psi_1 (\sigma^{m_1}(A) - \lambda \,\mathrm{Id}_{M_2})^{-1} \tag{14}$$

$$+ \tfrac{i}{2\pi} \textstyle\int_{\partial^+ \Lambda_\varepsilon} \lambda^z \psi_2 (\sigma^{m_2}(A) - \lambda \,\mathrm{Id}_{M_1})^{-1} d\lambda \tag{15}$$

$$- \tfrac{i}{2\pi} \textstyle\int_{\partial^+ \Lambda_\varepsilon} \lambda^z \psi_1 \psi_2 (\sigma^{m_1,m_2}(A) - \lambda)^{-1} d\lambda \tag{16}$$

$$+ \tfrac{i}{2\pi} \textstyle\int_{\partial^+ \Lambda_\varepsilon} \lambda^z c(\lambda) d\lambda. \tag{17}$$

The theorem follows from theory of complex powers on closed manifolds for the integrals (14) and (15), and from Cauchy Theorem for integral (16). Finally, we notice that integral (17) gives a symbol of order $(m_1 z - 1, m_2 z - 1)$.

[2] In equation (12) $a^z_{m_1 z,\cdot}, a^z_{\cdot,m_2 z}, a^z_{m_1 z,m_2 z}$ represent respectively $\sigma_1^{m_1 z}(A^z), \sigma_2^{m_2 z}(A^z), \sigma^{m_1 z,m_2 z}(A^z)$, while $(a_{m_1,\cdot})^z, (a_{\cdot,m_2})^z$ are complex powers of the operators $\sigma_2^{m_1}(A), \sigma_2^{m_2}(A)$ and $(a_{m_1,m_2})^z$ is the complex power of the function $\sigma^{m_1,m_2}(A)$.

We now introduce the function $\zeta(A,z)$ of an elliptic operator that satisfies Assumptions 1. The proof of the following property is similar to the case of compact manifolds (see [31], ch. II).

Proposition 2.1 *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$, $m_i > 0$, $i = 1,2$, be a selfadjoint operator satisfying Assumptions 1. Then we have*

$$A^z(u) = \sum_{i\in\mathbb{N}} \lambda_j^z (f_i, u),$$

where $\{\lambda_j\}_{j\in\mathbb{N}}$ is the spectrum of A, and $\{f_j\}_{j\in\mathbb{N}}$ are the corresponding orthonormal eigenfunctions. We define

$$\zeta(A,z) := \sum_{j\in\mathbb{N}} \lambda_j^z, \quad \mathfrak{Re}(z) < \min\big\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\big\}.$$

The definition of $\zeta(A,z)$ in the general case is the following:

Definition 2.3 Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be an operator satisfying Assumptions 1 then

$$\zeta(A,z) := \int_{M_1\times M_2} K_{A^z}(x_1,x_2,x_1,x_2)dx_1dx_2, \quad \mathfrak{Re}(z)m_1 < -n_1, \mathfrak{Re}(z)m_2 < -n_2,$$

where K_{A^z} is the kernel of A^z. The integral is well defined if $\mathfrak{Re}(z)m_1 < -n_1$ and $\mathfrak{Re}(z)m_2 < -n_2$ since, in this case, A^z is trace class.

Theorem 2.3 *$K_{A^z}(x_1,x_2,y_1,y_2)$ is a smooth function outside the diagonal. Furthermore, $K_{A^z}(x_1,x_2,x_1,x_2)$ restricted to the diagonal can be extended as a meromorphic function on the half plane $\{z \in \mathbb{C} \mid \mathfrak{Re}(z) < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \varepsilon\}$ with, at most, poles at the point $z_{\text{pole}} = \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$. The pole can be of order two if $\frac{n_1}{m_1} = \frac{n_2}{m_2}$, otherwise it is a simple pole.*

Proof By definition, the kernel of A^z has the form

$$K_{A^z}(x_1,x_2,x_1,x_2) = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} a^z(x_1,x_2,\xi_1,\xi_2)d\xi_1 d\xi_2. \tag{18}$$

First, let us consider the case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$. Then, if $\mathfrak{Re}(z) < -\frac{n_1}{m_1}$, $A^z \in L^{m_1 z, m_2 z}(M_1 \times M_2) \subseteq L^{-n_1-\varepsilon,-n_2-\varepsilon}(M_1 \times M_2)$; hence it is trace class and the integral of the kernel is finite. We can write $a^z = a^z_{m_1 z,\cdot} + a^z_r$, $a^z_r \in S^{m_1 z-1, m_2 z}(M_1, M_2)$ and we have then

$$\begin{aligned} K_{A^z}(x,x) =& \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_2}} \int_{|\xi_1|\geq 1} \big(a^z_{m_1 z,\cdot} + a^z_{r,\cdot}\big)d\xi_1 d\xi_2 \\ &+ \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_2}} \int_{|\xi_1|\leq 1} \big(a^z_{m_1 z,\cdot} + a^z_{r,\cdot}\big)d\xi_1 d\xi_2. \end{aligned} \tag{19}$$

The second integral in (19) is an holomorphic function for $\mathfrak{Re}(z) \leq -\frac{n_1}{m_1} + \varepsilon$ since we integrate w.r.t. the ξ_1 variable on a compact set. The same conclusion holds for the integral of $a^z_{r,\cdot}$ on the set $\{(\xi_1,\xi_2) \mid |\xi_1| \geq 1, \xi_2 \in \mathbb{R}^{n_2}\}$ because it has order $(m_1 z -$

$1, m_2 z)$. In order to analyze the integral of $a^z_{m_1,\cdot}$, we switch to polar coordinates and we obtain

$$\int_{\mathbb{R}^{n_2}}\int_{|\xi_1|\geq 1} a^z_{m_1 z,\cdot} d\xi_1 d\xi_2 = -\frac{1}{m_1 z + n_1}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{S}^{n_1-1}} a_{m_1 z,\cdot} d\theta_1 d\xi_2. \tag{20}$$

Clearly (20) can be extended as a meromorphic function on $\{z \in \mathbb{C} \mid \mathfrak{Re}(z) < -\frac{n_1}{m_1} + \varepsilon\}$, and, moreover, by (12), we get

$$\lim_{z\to -\frac{n_1}{m_1}} \left(z + \frac{n_1}{m_1}\right) K_{A^z}(x_1, x_2) = -\frac{1}{(2\pi)^{n_1+n_2} m_1}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{S}^{n_1-1}} a^{-\frac{n_1}{m_1}}_{m_1,\cdot} d\theta_1 d\xi_2.$$

The case $\frac{n_1}{m_1} < \frac{n_2}{m_2}$ is equivalent, by exchanging m_1 and m_2.

The case $\frac{n_1}{m_1} = \frac{n_2}{m_2}$ is a bit more delicate, since we have to analyze the whole principal symbol. First we write

$$\begin{aligned} K_{A^z}(x,x) = \frac{1}{(2\pi)^{n_1+n_2}}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \big(a^z_{m_1 z,\cdot} + a^z_{\cdot,m_2 z} - a^z_{m_1 z, m_2 z}\big) + \\ \big(a^z - a^z_{m_1 z,\cdot} - a^z_{\cdot,m_2 z} + a^z_{m_1 z, m_2 z}\big) d\xi_1 d\xi_2. \end{aligned} \tag{21}$$

The definition of principal symbol implies that the second term in (21) belongs to $S^{m_1 z - 1, m_2 z - 1}(M_1, M_2)$, hence the second integral is well defined for $\mathfrak{Re}(z) < -\frac{n_1}{m_1} + \varepsilon$ and holomorphic for $\mathfrak{Re}(z) < -\frac{n_1}{m_1} + \varepsilon$. Now we have to analyze the integral of the principal symbol. Splitting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ into the following four regions

$$\begin{aligned} &\{(\xi_1,\xi_2) \mid |\xi_1| < \tau, |\xi_2| < \tau\}, \quad \{(\xi_2,\xi_2) \mid |\xi_1| \leq \tau, |\xi_2| \geq \tau\}, \\ &\{(\xi_1,\xi_2) \mid |\xi_1| \geq \tau, |\xi_2| \leq \tau\}, \quad \{(\xi_2,\xi_2) \mid |\xi_1| > \tau, |\xi_2| > \tau\}, \end{aligned}$$

one gets

$$\begin{aligned} &\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \big(a^z_{m_1 z,\cdot} + a^z_{\cdot,m_2 z} - a^z_{m_1 z, m_2 z}\big) d\xi_1 d\xi_2 = \\ &\frac{\tau^{(m_1+m_2)z+n_1+n_2}}{(m_1 z + n_1)(m_2 z + n_2)}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}} a^z_{m_1 z, m_2 z} d\theta_1 d\theta_2 \\ &- \frac{\tau^{m_1 z + n_1}}{(m_1 z + n_1)}\int_{|\xi_2|\leq\tau}\int_{\mathbb{S}^{n_1-1}} a^z_{m_1 z,\cdot} d\theta_1 d\xi_2 \\ &- \frac{\tau^{m_2 z + n_2}}{(m_2 z + n_2)}\int_{|\xi_1|\leq\tau}\int_{\mathbb{S}^{n_2-1}} a^z_{\cdot,m_2 z} d\theta_1 d\xi_1 \\ &- \frac{\tau^{m_1 z + n_1}}{(m_1 z + n_1)}\int_{|\xi_2|>\tau}\int_{\mathbb{S}^{n_1-1}} \big(a^z_{m_1 z,\cdot} - a^z_{m_1 z, m_2 z}\big) d\theta_1 d\xi_1 \\ &- \frac{\tau^{m_2 z + n_2}}{(m_2 z + n_2)}\int_{|\xi_1|>\tau}\int_{\mathbb{S}^{n_2-1}} \big(a^z_{\cdot,m_2 z} - a^z_{m_1 z, m_2 z}\big) d\theta_1 d\xi_1 \\ &+ h(z), \end{aligned} \tag{22}$$

where $h(z)$ is an holomorphic function for $\mathfrak{Re}(z) \leq z_{\text{pole}} + \varepsilon$. The evaluation of the integrals in (22) are similar to Proposition 3.3 in [25], and Theorem 2.2 in [3]. This concludes the proof.

Since M_1, M_2 are closed manifolds, Theorem 2.3 implies the following:

Corollary 2.1 *Let $A \in L^{m_1,m_2}(M_1 \times M_2)$ be an operator satisfying Assumptions 1; then $\zeta(A,z)$ is holomorphic for $\mathfrak{Re}(z) < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ and can be extended as a meromorphic function on the half plane $\mathfrak{Re}(z) < \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \varepsilon$. Moreover, the Laurent coefficients of $\zeta(A,z)$ at $z = z_{\text{pole}} = \min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ are*

$$\lim_{z \to -\frac{n_1}{m_1}} \left(z + \frac{n_1}{m_1}\right) \zeta(A,z) = -\frac{1}{(2\pi)^{n_1+n_2} m_1} \iint_{M_1 \times M_2} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{S}^{n_1-1}} a_{m_1,\cdot}^{-\frac{n_1}{m_1}} d\theta_1 d\xi_2, \quad (23)$$

if $\frac{n_1}{m_1} > \frac{n_2}{m_2}$.

$$\lim_{z \to -\frac{n_2}{m_2}} \left(z + \frac{n_2}{m_2}\right) \zeta(A,z) = -\frac{1}{(2\pi)^{n_1+n_2} m_2} \iint_{M_1 \times M_2} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{S}^{n_2-1}} a_{\cdot,m_2}^{-\frac{n_2}{m_2}} d\theta_2 d\xi_1, \quad (24)$$

if $\frac{n_2}{m_2} > \frac{n_1}{m_1}$.

$$\begin{aligned} res^2(A) = \lim_{z \to -l} (z+l)^2 \zeta(A,z) = \\ \frac{1}{(2\pi)^{n_1+n_2}(m_1 m_2)} \iint_{M_1 \times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} (a_{m_1,m_2})^{-l} d\theta d\theta', \end{aligned} \quad (25)$$

$$\lim_{z \to -l} (z+l)\big(\zeta(A,z) - \frac{res^2(A)}{(z+l)^2}\big) = -TR_{1,2}(A) + TR_\theta(A), \quad (26)$$

where

$$\begin{aligned} &TR_{1,2}(A) := \\ &\frac{1}{(2\pi)^{n_1+n_2}} \lim_{\tau \to \infty} \big(\frac{1}{m_1} \iint_{M_1 \times M_2} \int_{|\xi_2| \le \tau} \int_{\mathbb{S}^{n_1-1}} (a_{m_1,\cdot})^{-l} - res^2(A) \log \tau\big) \\ &+ \frac{1}{(2\pi)^{n_1+n_2}} \lim_{\tau \to \infty} \big(\frac{1}{m_2} \iint_{M_1 \times M_2} \int_{|\xi_1| \le \tau} \int_{\mathbb{S}^{n_2-1}} (a_{\cdot,m_2})^{-l} - res^2(A) \log \tau\big) \end{aligned} \quad (27)$$

and

$$TR_\theta(A) := \frac{1}{(2\pi)^{n_1+n_2}(m_1 m_2)} \int_{M_1 \times M_2} \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} a_{m_1,m_2}^{-l} \log a_{m_1,m_2} d\theta_1 d\theta_2, \quad (28)$$

if $\frac{n_1}{m_1} = \frac{n_2}{m_2} = l$.

In (27), $(a_{m_1,\cdot})^l$ and $(a_{\cdot,m_2})^l$ are the symbols of the complex powers of the operators $a_{m_1,\cdot}(x_1,x_2,\xi_1,D_2)$ and $a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2)$. In order to obtain the terms in (26), (27), (28), we notice that the constant τ in (22) is arbitrary and the Laurent coefficients clearly do not change if we change the partition of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, therefore we can let τ tend to infinity. In this way both the fourth and fifth integral in (22) vanish, due to the continuity of the integral w.r.t. the domain of integration. The evaluation is similar to the proof of Theorem 2.9 in [3] and of Proposition 3.3 in [25].

3 Weyl's formula for bisingular operators

In this section we study Weyl's formula for positive selfadjoint bisingular operators that satisfy Assumptions 1. In the sequel we use the following Theorem, proved by J. Aramaki [1]:

Theorem 3.1 *Let P be a positive selfadjoint operator satisfying Assumptions 1. If $\zeta(P,z)$ has the first left pole at the point $-z_0$ and*[3]

$$\zeta(P,z)+\sum_{j=1}^{p}\frac{A_j}{(j-1)!}\left(\frac{d}{dz}\right)^{j-1}\frac{1}{z+z_0},$$

extends to an holomorphic function on the half plane $\{z\in\mathbb{C}\mid \mathfrak{Re}(z)<-z_0+\varepsilon\}$, then, setting

$$N_P(\lambda)=\sum_{t\in\sigma(P),\,t\leq\lambda}1,$$

we have

$$N_P(\lambda)\sim\sum_{j=1}^{p}\frac{A_j}{(j-1)!}\left(\frac{d}{ds}\right)^{j-1}\left(\frac{\lambda^s}{s}\right)|_{s=z_0}+O(\lambda^{z_0-\delta}),\quad \lambda\to\infty,$$

for a certain $\delta>0$.

Theorem 3.2 *Let $A\in L^{m_1,m_2}(M_1\times M_2)$ be a positive selfadjoint bisingular satisfying Assumptions 1, then*

$$N_A(\lambda)\sim\begin{cases}C_1\lambda^l\log(\lambda)+C_1'\lambda^l+O(\lambda^{l-\delta_1}) & \text{for } \frac{n_1}{m_1}=\frac{n_2}{m_2}=l\\ C_2\lambda^{\frac{n_2}{m_2}}+O(\lambda^{\frac{n_2}{m_2}-\delta_2}) & \text{for } \frac{n_2}{m_2}>\frac{n_1}{m_1}\\ C_3\lambda^{\frac{n_1}{m_1}}+O(\lambda^{\frac{n_2}{m_2}-\delta_2}) & \text{for } \frac{n_2}{m_2}<\frac{n_1}{m_1}\end{cases},\quad \lambda\to\infty, \tag{29}$$

for certain $\delta_i>0$, $i=1,2,3$. The constants C_1,C_1',C_2,C_3 depend only on the principal symbol of A.

Proof We use J. Aramaki's Theorem 3.2, which gives the asymptotic of $N_A(\lambda)$ knowing the first left pole of the zeta function. As a simple application we get (29) with

$$\begin{aligned}
C_1=&\frac{1}{(2\pi)^{n_1+n_2}(n_1m_2)}\iint_{M_1\times M_2}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}(a_{m_1,m_2})^{-l}d\theta_1d\theta_2\\
=&\frac{1}{(2\pi)^{n_1+n_2}(n_2m_1)}\iint_{M_1\times M_2}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}(a_{m_1,m_2})^{-l}d\theta_1d\theta_2;\\
C_1'=&\frac{TR_{1,2}(A)+TR_\theta(A)}{l}-\frac{1}{n_1n_2}\iint_{M_1\times M_2}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}(a_{m_1,m_2})^{-l}\log(a_{m_1,m_2})d\theta_1d\theta_2;\\
C_2=&\frac{1}{(2\pi)^{n_1+n_2}n_2}\iint_{M_1\times M_2}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{S}^{n_2-1}}(a_{\cdot,m_2})^{-\frac{m_2}{n_2}}d\theta_2d\xi_1;\\
C_3=&\frac{1}{(2\pi)^{n_1+n_2}n_1}\iint_{M_1\times M_2}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{S}^{n_1-1}}(a_{m_1,\cdot})^{-\frac{n_1}{m_1}}d\theta_1d\xi_2.
\end{aligned}\tag{30}$$

[3] The Aramaki's Theorem actually requires another assumption on the decay of $\Gamma(z)\zeta(P,z)$ on vertical strips. In this case such condition is fulfilled, in view of the relationship between ζ-function, heat trace and gamma function, see [13,21].

Remark 4 In this paper we are focused just on bisingular operators with homogeneous principal symbol, since our aim is the study of the corresponding Weyl's formulae. We do not introduce classical bisingular operators and we do not investigate the relationship between the poles of the ζ-function and Wodzicki Residue defined in [26]. Nevertheless, extending the results of section 2 to classical bisingular operators, one can prove that, for a classical elliptic bisingular operator $A \in L^{m_1,m_2}(M_1 \times M_2)$ that admits complex powers,

$$\operatorname{Wres}(A) := m_1 m_2 \lim_{z\to 1}(z-1)^2 \zeta(A,z),$$

where $\operatorname{Wres}(A)$ is the bisingular Wodzicki residue defined by Nicola and Rodino in [26].

4 Examples

First we consider the operator $A = -\Delta \otimes -\Delta$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$. We clearly have $\sigma(A) = \{n^2 m^2\}_{(n,m)\in\mathbb{N}^2}$. Hence the spectrum is countable and consists only of eigenvalues. The eigenvalue $\{0\}$ has an infinite dimensional eigenspace, while all other eigenspaces have dimension four. Therefore we get

$$N_A(\lambda) = \sum_{0<n^2m^2\le\lambda} 4. \tag{31}$$

Let us define the function $d(h) : \mathbb{N} \to \mathbb{N}$ so that $d(h)$ is equal to the number of ways we can write $h = m \cdot n$, with m,n natural positive numbers or, equivalently, it is equal to the number of divisors of h. This function is often called Dirichlet divisor function. By a simple computation, we obtain

$$N_A(\lambda^2) = 4D(\lambda) = 4\sum_{n\le\lambda} d(n). \tag{32}$$

Noticing that $\zeta(A) = 4\zeta_R(2z)\zeta_R(2z)$, where $\zeta_R(z)$ is Riemann zeta-function, we can easily find the coefficients of the asymptotic expansion and we have

$$D(\lambda) \sim \lambda\log(\lambda) + (2\gamma-1)\lambda + O(\lambda^{1-\delta}), \quad \lambda\to\infty, \tag{33}$$

where

$$\gamma := \lim_{\tau\to\infty}\left[\sum_{i=1}^{[\tau]}\frac{1}{i} - \log\tau\right] \tag{34}$$

is the well known Euler-Mascheroni constant. The asymptotic expansion (33) is well known (see [17] for an overview on Dirichlet divisor problem; see also [18,19]). It is still an open question to understand the behavior of remainder. In [15], G. H. Hardy proved that $O(\lambda^{\frac{1}{4}})$ is a lower bound for the third term. The best approximation, found by M. Huxley in [16], is $O(\lambda^c(\log\lambda)^d)$, where

$$c := \frac{131}{416} \sim 0,3149038462 \quad d := \frac{18627}{8320} + 1 \sim 3,238822115.$$

The conjecture is that the remainder is $O(\lambda^{\frac{1}{4}})$.

It is nevertheless interesting to investigate the link between Dirichlet divisor function and the above results on the spectral properties of a suitable operators. Let us notice that in (31) we have a slight abuse of notation, since $N(\lambda)$ was only defined for positive operators. In this case $A = -\Delta \otimes -\Delta$ is non-negative, but has a non trivial kernel. In other words we actually consider

$$N_A := N_{A\circ(\mathrm{Id}-P_{\ker A})}$$

where $P_{\ker A}$ is the projection on the kernel of A. This definition is compatible with the definition of complex powers of non invertible operators in [4]. The variant of our theory to such a setting, which is possible, will be not detailed here. Rather, let us now consider the operator $A_c := (-\Delta + c) \otimes (-\Delta + c)$, $c > 0$, defined on the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Clearly, A_c satisfies Assumptions 1; thus we can apply Theorem 3.2. It is easy to see that the eigenvalues of A_c are $\{(n^2+c)(m^2+c)\}_{(n,m)\in\mathbb{N}^2}$, each one with multiplicity four. Hence

$$\begin{aligned} N(A_c;\lambda^2) = 4\,\sharp\{ &\text{ real numbers of the form } (n^2+c)(m^2+c) \mid \\ &(n^2+c)(m^2+c) \le \lambda,\ n,m \in \mathbb{N}\} = 4\,D_c(\lambda). \end{aligned}$$

By Theorem 2.2, we know that $\sigma^{-1,-1}(A_c^{-\frac{1}{2}}) = (\sigma^{2,2}(A_c))^{-\frac{1}{2}}$ so the constant C_1 in (30) can be easily evaluated

$$C_1 = \frac{1}{2}\frac{1}{(2\pi)^2}(2\pi)^2\,4 = 2. \tag{35}$$

Since in this case we know the eigenvalue of the operator, $TR(A_c)$ turns into

$$\begin{aligned} TR_{1,2}(A_c) &= 2\lim_{\tau\to\infty}\left[\sum_{i=-[\tau]}^{[\tau]}\frac{1}{(c+i^2)^{\frac{1}{2}}} - 2\log\tau\right] \\ &= 4\lim_{\tau\to\infty}\left[\sum_{i=0}^{[\tau]}\frac{1}{(c+i^2)^{\frac{1}{2}}} - \log\tau\right] = 4\gamma_c. \end{aligned} \tag{36}$$

We have named this constant γ_c because of the link with the usual constant of Euler-Mascheroni γ in (34). Notice that, letting c tend to 0, γ_c goes to $+\infty$; while, if c tends to infinity, γ_c goes to $-\infty$. Finally, we obtain

$$\begin{aligned} D_c(\lambda) &= \frac{1}{4}N(A_c;\lambda^2) \\ &\sim \lambda\log(\lambda) + (2\gamma_c - 1)\lambda + O(\lambda^{1-\delta}), \quad \lambda\to\infty. \end{aligned} \tag{37}$$

In this case, knowing exactly the eigenvalues of the operator, we can check our estimate with a numerical experiment. We have checked (37) for $D_c(\lambda)$ with $\lambda = 10.000.000$. In the second column of the Table 1 there is the estimate of the coefficient of first term of the asymptotic expansion obtained with the software Maple 15, in the third the coefficient obtained by (37), and in the fourth the error. We can notice that the error increases with c. This is not surprising, since (35) does not depend on c.

Table 1 1st. term approximation

c	1st. term with Maple	1st. term in (37)	error
2	1,024846785	1	0,024846785
3	0,9916281891	1	0,008371811
4	0,968979304	1	0,031020696
5	0,951859819	1	0,048140181
6	0,938130598	1	0,061869402
7	0,926687949	1	0,073312051
8	0,916888721	1	0,083111279
9	0,908326599	1	0,091673401
10	0,900728511	1	0,099271489
11	0,893902326	1	0,106097674
12	0,887707593	1	0,112292407
13	0,882038865	1	0,117961135
14	0,876815128	1	0,123184872
15	0,871972341	1	0,128027659
16	0,867459966	1	0,132540034
17	0,863235614	1	0,136764386
18	0,859265437	1	0,140734563
19	0,855520776	1	0,144479224
20	0,851977951	1	0,148022049

Table 2 2nd. term approximation

c	2nd. term with Maple	2nd. term in (37)	error
2	0,40048285	0,401484386	0,001001536
3	-0,13493765	-0, 1339381238	0,000999526
4	-0,499994550	-0,498993281	0,001001269
5	-0,775928050	-0,774926584	0,001001466
6	-0,997216950	-0,996213733	0,001003217
7	-1,181650650	-1,180647904	0,001002746
8	-1,339595550	-1,3385899520	0,001005598
9	-1,477600650	-1,476592538	0,001008112
10	-1,600067350	-1,599058126	0,001009224
11	-1,710092450	-1,7090842470	0,001008203
12	-1,809939750	-1,808931287	0,001008463
13	-1,901308850	-1,9002985710	0,001010279
14	-1,985505550	-1,9844949070	0,001010643
15	-2,063562050	-2,0625496430	0,001012407
16	-2,136292950	-2,1352865400	0,001006410
17	-2,204381450	-2,2033750580	0,001006392
18	-2,268373150	-2,2673662890	0,001006861
19	-2,328729950	-2,3277195600	0,001010390
20	-2,385833550	-2,3848212840	0,001012266

In order to make the error smaller, we should increase the number of digits at which we truncate the series $D_c(\lambda)$. In Table 2 we analyze the coefficient of the second term. In this case the error is essentially independent of c, this is due to the fact that (36) does depend on c.

Our spectral approach to Dirichlet Divisor function suggests that others *Weyl's formula techniques* (e. g. Fourier Integral Operator) could be useful to attack the Dirichlet Divisor conjecture.

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