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Recommended Citation

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Disciplines

Engineering | Mechanical Engineering

Comments

On the generation of smooth three-dimensional rigid body motions[*]

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Abstract

This paper addresses the problem of generating smooth trajectories between an initial and a final position and orientation in space. The main idea is to define a functional depending on velocity or its derivatives that measures the smoothness of a trajectory and find trajectories that minimize this functional. In order to ensure that the computed trajectories are independent of the parameterization of positions and orientations, we use the notions of Riemannian metric and covariant derivative from differential geometry and formulate the problem as a variational problem on the Lie group of spatial rigid body displacements, $SE(3)$. We show that by choosing an appropriate measure of smoothness, the trajectories can be made to satisfy boundary conditions on the velocities or higher order derivatives. Dynamically smooth trajectories can be obtained by incorporating the inertia of the system into the definition of the Riemannian metric. We state the necessary conditions for the shortest distance, minimum acceleration and minimum jerk trajectories. Analytical expressions for the smooth trajectories are derived for some special cases. We also provide several examples of the general case where the trajectories are computed numerically.

1   Introduction

There are many applications in which the problem of generating smooth trajectories for a rigid body in $\mathbb{R}^3$ is encountered. In robotics, it is frequently necessary to plan movements between a given (start) end-effector position and orientation and a desired (goal) position and orientation [2]. In general, we have to compute the actuator forces that achieve the specified displacement. But when the dynamic model of the system is not available or difficult to derive, it is better to separately plan the kinematic (task space) trajectory and use some other method to compute the corresponding actuator torques. Smooth trajectories are preferred because (a) the electro-mechanical system is limited by the size of the actuators and their control bandwidth so it cannot produce large velocities and accelerations; and (b) movements with high acceleration and/or jerk can excite the structural natural frequencies in the system. Planning of smooth task space trajectories is also employed in the programming of industrial robots for tasks such as welding and painting where a "teaching" process is employed to record intermediate positions and the final trajectory is obtained by interpolation [2]. Similarly, in computer animation it is necessary to generate a smooth trajectory passing through a set of key frames specifying positions and orientations [3]. In this case, smoothness is required to obtain realistic motions or motions that "look" natural.

There are several factors that need to be considered when developing a trajectory planning method. It is desirable that the trajectories are independent of the choice of coordinates for the space. In this way, computations performed with different choices of coordinates will produce consistent results. Further, to describe motion of a rigid body in space, an inertial and a body fixed reference frames must be chosen. We would therefore also like to find a planning method that does not depend on the choice of these two frames. And finally, the computed trajectories should have good performance for the chosen task.

Coordinate independence of the trajectories is assured if they are computed using the intrinsic geometric (i.e., coordinate free) properties of the space. Appropriate tools are provided by differential geometry and the theory of

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[*]Part of this paper was presented at the 1996 IEEE International Conference on Robotics and Automation [1].

Lie groups. Differential geometry offers a consistent way of extending the notion of differentiation from Euclidean space to an arbitrary manifold so that we can define acceleration, jerk and different measures of smoothness of the trajectories. The theory of Lie groups provides a framework for investigating the invariance of the trajectories with respect to the choice of the inertial and body fixed frames.

There is extensive literature on trajectory generation in kinematics, robotics and computer graphics. In order to generate a smooth motion for a robot arm from an initial to a final position, Whitney [4] and Pieper [5] advocated using a screw motion. Waldron [6] developed an algorithm that is based on a slight variation of Pieper's scheme so that the velocity profile along the trajectory is trapezoidal. In all these schemes, although the screw motion is invariant with respect to rigid body transformations, it does not optimize a meaningful cost function. Further, the translational part of a screw motion between two points is in general not a straight line. Paul [2] decomposes the desired displacement into a translation and two rotations each of which is smoothly parameterized with respect to time. The motion of the end-effector is obtained by a composition of these three displacements. He employs a fourth-order polynomial of time to obtain a smooth motion. Although there is some justification for the proposed trajectory, the approach will lead to different trajectories if different parameterization is chosen for the rotation or if the coordinate frames in which the trajectory is computed are changed. There is also no attempt to develop a measure of smoothness for three-dimensional motions.

Shoemake [7] proposed a scheme for interpolating rotations with Bezier curves. This idea was extended by Ge and Ravani [8] to spatial motions and proposed for computer-aided geometric design. In both cases, the interpolating curves are screw motions and therefore invariant with respect to the choice of reference frames. However, the interpolating scheme produces a motion that does not posses these invariance properties. Further, these motions are not of minimal length for any meaningful metric. In contrast, Park and Ravani [9] use a scale-dependent left invariant metric to design Bezier curves for three-dimensional rigid body motion interpolation.

In this paper, the trajectory planning problem is posed as finding maximally smooth trajectories between an initial and a final position and orientation. The measure of the lack of smoothness is chosen to be the integral over the trajectory of a cost function depending on velocity or its higher derivatives. Boundary conditions on the derivatives of desired order can be enforced by appropriately choosing the cost function. For example, by minimizing the norm of the velocity we obtain the shortest distance paths. The minimum acceleration (minimum jerk) trajectories can be made to satisfy boundary conditions on the velocities (accelerations). Dynamically smooth trajectories can be obtained by incorporating the inertia of the system into the cost function. A simple extension of the ideas in this paper allows the inclusion of intermediate positions and orientations and lends itself to motion interpolation.

Necessary conditions for smooth curves on general manifolds were derived by Noakes et $al.$ [10], and in parallel with our work by Camarinha et $al.$ [11] and Crouch and Silva Leite [12]. In [10], necessary conditions for cubic splines which correspond to our minimum acceleration curves are derived for an arbitrary manifold. These results are extended in [12] to the dynamic interpolation problem. In [11] necessary conditions for curves minimizing the integral of the norm of an arbitrary derivative of velocity are derived. None of these works deals specifically with computing the trajectories on $SE(3)$, nor do they address the choice of the metric for the space. Since there is no natural metric for $SE(3)$ [13, 14], the choice of metric for trajectory planning becomes an important issue.

The paper is organized as follows. We first review some preliminary concepts on Lie groups and space kinematics, including the ideas of a left invariant metric, connection and the covariant derivative. This material is standard and can be found in many texts [14, 15, 16]. In Section 3, we address the choice of metric for $SE(3)$. We propose a left invariant metric given by the kinetic energy of a rigid body and derive the expressions for the covariant derivative given by this metric. We use these geometric constructs to formalize the ideas of acceleration and jerk on $SE(3)$. Most of these results are presented here for the first time. In Section 4, we discuss the variational problems that need to be solved in order to calculate the shortest distance, minimum acceleration and minimum jerk trajectories. While some of these results were derived in [10] and [11], we present alternative proofs and specialize the results to $SE(3)$. In Section 5, we derive analytical solutions for the smooth trajectories in some special cases. For more general situations, we compute numerical solutions. Section 6 provides some concluding remarks.

2 Kinematics, Lie groups and differential geometry

2.1 The Lie group $SE(3)$

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body at point O' as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame {F} to frame {M}. The set of all such matrices is called $SE(3)$, the special Euclidean group of rigid body transformations in three-dimensions:

$$SE(3) = \left\{ \left[\begin{array}{cc} R & d \\ 0 & 1 \end{array} \right] \mid R \in \mathbb{R}^{3 \times 3}, d \in \mathbb{R}^3, R^T R = I, \det(R) = 1 \right\}. \tag{1}$$

It is easy to show [14] that $SE(3)$ is a group for the standard matrix multiplication and that it is a manifold. It is therefore a Lie group [16].
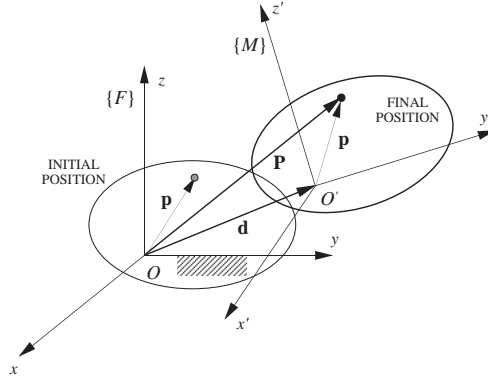
Figure 1: The inertial (fixed) frame and the moving frame attached to the rigid body

On a Lie group, the tangent space at the group identity defines a Lie algebra. The Lie algebra of $SE(3)$, denoted by $se(3)$, is given by:

$$se(3) = \left\{ \left[\begin{array}{cc} \Omega & v \\ 0 & 0 \end{array} \right] \mid \Omega \in \mathbb{R}^{3 \times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \tag{2}$$

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the vector cross product operation in $\mathbb{R}^3$. Each element $S \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Given a curve $A(t) : [-a, a] \to SE(3)$, an element $S(t)$ of the Lie algebra $se(3)$ can be associated to the tangent vector $\dot{A}(t)$ at an arbitrary point t by:

$$S(t) = A^{-1}(t)\dot{A}(t) = \left[\begin{array}{cc} R^T \dot{R} & R^T \dot{d} \\ 0 & 0 \end{array} \right]. \tag{3}$$

A curve on $SE(3)$ physically represents a motion of the rigid body. If $\{\omega(t), v(t)\}$ is the vector pair corresponding to $S(t)$, then ω physically corresponds to the angular velocity of the rigid body while v is the linear velocity of the origin O' of the frame {M}, both expressed in the frame {M}. In kinematics, elements of this form are called twists [14] and $se(3)$ thus corresponds to the space of twists. It is easy to check that the twist $S(t)$ computed from Eq. (3) does not depend on the choice of the inertial frame {F}. For this reason, $S(t)$ is called the left invariant representation of the tangent vector $\dot{A}$. Alternatively, the tangent vector $\dot{A}$ can be identified with a right invariant twist (invariant with respect to the choice of the body-fixed frame {M}). In this paper, right invariant twists will not be considered, but all the developments are parallel to those for the left invariant twists.

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Since $se(3)$ is a vector space, any element can be expressed as a 6×1 vector of components corresponding to a chosen basis. The standard basis for $se(3)$ is:

$$
L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad
L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad
L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
L_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad
L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad
L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\tag{4}
$$

The twists L_1, L_2 and L_3 represent instantaneous rotations about and L_4, L_5 and L_6 instantaneous translations along the Cartesian axes x, y and z, respectively. The components of a twist $S \in se(3)$ in this basis are given precisely by the velocity vector pair, $\{\omega, v\}$.

The Lie bracket of two elements $S_1, S_2 \in se(3)$ is defined by:

$$
[S_1, S_2] = S_1 S_2 - S_2 S_1.
$$

It can be easily verified that if $\{\omega_1, v_1\}$ and $\{\omega_2, v_2\}$ are vector pairs corresponding to the twists S_1 and S_2, the vector pair $\{\omega, v\}$ corresponding to their Lie bracket $[S_1, S_2]$ is given by

$$
\{\omega, v\} = \{\omega_1 \times \omega_2, \omega_1 \times v_2 + v_1 \times \omega_2\}.
\tag{5}
$$

In kinematics, this product operation is called the motor product of the two twists.

The Lie bracket of two elements of a Lie algebra is an element of the Lie algebra and can be expressed as a linear combination of the basis vectors. The coefficients C_{ij}^k corresponding to the Lie brackets of the basis vectors are called *structure constants* of the Lie algebra [15]:

$$
[L_i, L_j] = \sum_k C_{ij}^k L_k.
\tag{6}
$$

2.2   Left invariant vector fields

A (differentiable) vector field on a manifold is a (smooth) assignment of a tangent vector to each element of the manifold. At each point, a vector field defines a unique *integral curve* to which it is tangent [16]. Formally, a vector field X is a (derivation) operator which, given a differentiable function f, returns its derivative (another function) along the integral curves of X. In other words, if $\gamma(t)$ is a curve tangent to a vector field X at point $p = \gamma(t_0)$, then:

$$
Xf|_p = \frac{df(\gamma(t))}{dt}\bigg|_{t_0}.
\tag{7}
$$

On a matrix Lie group, an example of a (differentiable) vector field, X, is obtained by setting:

$$
X(A) = \hat{T}(A) = AT,
\tag{8}
$$

where T belongs to the Lie algebra of the group. Such a vector field is called a left invariant vector field. We use the notation $\hat{T}$ to indicate that the vector field is obtained by left translating the Lie algebra element T. The set of all left invariant vector fields is a vector space and by construction it is isomorphic to the Lie algebra. Right invariant vector fields can be defined in analogous way. In general, a vector field need not be left or right invariant.

We now concentrate on the group $SE(3)$. Since $L_1, L_2, \ldots, L_6$ are a basis for the Lie algebra $se(3)$, the set of the left invariant vector fields $\left\{ \hat{L}_1, \hat{L}_2, \ldots, \hat{L}_6 \right\}$ is a basis of the space of the left invariant vector fields. In addition, we have [15]:

$$
[\hat{L}_i, \hat{L}_j] = [\widehat{L_i, L_j}] = \sum_k C_{ij}^k \hat{L}_k.
\tag{9}
$$

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Finally, because at any point $A \in SE(3)$ the vectors $\hat{L}_1(A), \ldots, \hat{L}_6(A)$ form a basis of the tangent space at that point, any vector field X can be expressed as

$$X = \sum_{i=1}^{6} X^i \hat{L}_i, \tag{10}$$

where the coefficients X^i vary over the manifold – if they are constant then X is left invariant. This implies that we can associate a vector pair $\{\omega, v\}$ defined by

$$\omega = [X^1, X^2, X^3]^T, \quad v = [X^4, X^5, X^6]^T.$$

to an arbitrary vector field X.

2.3   Exponential map and local coordinates

A motion of the rigid body in $\mathbb{R}^3$ is described by a curve, $A(t)$, on $SE(3)$. If $V = \frac{dA}{dt}$ is the vector field tangent to $A(t)$, the vector pair $\{\omega, v\}$ associated with V corresponds to the instantaneous twist (screw axis) for the motion. In general, the twist $\{\omega, v\}$ changes with time. Motions for which the twist $\{\omega, v\}$ is constant are known in kinematics as *screw motions*. In this case the twist $\{\omega, v\}$ can be identified with *the screw axis* of the motion. If the vector pair $\{\omega, v\}$ is interpreted as Plücker coordinates of a line in space, it is not difficult to see that the screw motion physically corresponds to a rotation about this line with a constant angular velocity and a concurrent translation along the line with a constant translational velocity.

Let the twist $S \in se(3)$ be represented by a vector pair $\{\omega, v\}$ and let $A(t)$ be a screw motion with the screw axis $\{\omega, v\}$ such that $A(0) = I$. We define *the exponential map* $\exp : se(3) \rightarrow SE(3)$ by:

$$\exp(tS) = A(t), \tag{11}$$

Using Eq. (3) we can show that the exponential map agrees with the usual exponentiation of the matrices in $\mathbb{R}^{4 \times 4}$:

$$\exp(tS) = \sum_{k=0}^{\infty} \frac{t^k S^k}{k!}, \tag{12}$$

where S denotes the matrix representation of the twist S. The sum of this series can be computed explicitly and the resulting expression, when restricted to $SO(3)$, is known as Rodrigues' formula. The formula for the sum in $SE(3)$ is derived in [14, pp. 413].

2.4   Riemannian metrics on Lie groups

If a smoothly varying, positive definite, bilinear, symmetric form is defined on the tangent space at each point on the manifold, we say the manifold is Riemannian. The bilinear form is an inner product on the tangent space at each point and is called a Riemannian metric.

On a Lie group, and thus on $SE(3)$, an inner product in the tangent space at the identity can be extended to a Riemannian metric (everywhere on the manifold) using the idea of left translation. Assume that the inner product of two elements $T_1, T_2 \in se(3)$ is defined by

$$< T_1, T_2 >_I = t_1^T W t_2, \tag{13}$$

where t_1 and t_2 are the 6×1 vectors of components of T_1 and T_2 with respect to some basis and W is a positive definite matrix. If V_1 and V_2 are tangent vectors at an arbitrary group element $A \in SE(3)$, the inner product $< V_1, V_2 >_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$< V_1, V_2 >_A = < A^{-1} V_1, A^{-1} V_2 >_I . \tag{14}$$

The metric obtained in such a way is called a *left invariant* metric [16]. Physically, left invariance corresponds to independence of the choice of the inertial frame. Let $A_1(t)$ and $A_2(t)$ represent two motions of a rigid body that pass through a point A at $t = t_0$ and let $V_1 = \frac{dA_1}{dt}$ and $V_2 = \frac{dA_2}{dt}$ be the corresponding velocity vector

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fields. Let C describe a displacement of the inertial reference frame. In new reference frame, the motions become $\tilde{A}_1(t) = CA_1(t)$ and $\tilde{A}_2(t) = CA_2(t)$, and the velocity vector fields $\tilde{V}_1 = CV_1$ and $\tilde{V}_2 = CV_2$. Then:

$$< \tilde{V}_1, \tilde{V}_2 >_{CA} = < A^{-1}C^{-1}\tilde{V}_1, A^{-1}C^{-1}\tilde{V}_2 >_I = < V_1, V_2 >_A . \tag{15}$$

We could similarly define a right invariant Riemannian metric and in this case the metric would be independent on the choice of the body-fixed frame.

2.5   Affine connection and covariant derivative

The motion of a rigid body is represented by a curve, $A(t)$, on $SE(3)$. The velocity at an arbitrary point is the tangent vector to the curve at that point. In order to obtain the acceleration, or to engage in a dynamic analysis, we need to be able to differentiate a vector field along the curve. At each point $A \in SE(3)$, the value of a vector field belongs to the tangent space $T_A SE(3)$ and to differentiate a vector field along a curve, we must be able to subtract vectors from tangent spaces at different points on the curve. But tangent spaces at different points are not related. We thus have to specify how to transport a vector along the curve from one tangent space to another. This process is called *parallel transport* and is formalized with the *affine connection* [16].

A derivative of a vector field along a curve $A(t)$ is defined through the parallel transport. Let X be a vector field defined along $A(t)$, and let $X(t)$ stand for $X(A(t))$. Denote by $X^{t_0}(t)$ the parallel transport of the vector $X(t)$ to the point $A(t_0)$. The *covariant derivative* of X along $A(t)$ is:

$$\left. \frac{DX}{dt} \right|_{t_0} = \lim_{t \to t_0} \frac{X^{t_0}(t) - X(t_0)}{t}. \tag{16}$$

By taking covariant derivatives along integral curves of a vector field Y, we obtain a covariant derivative of the vector field X with respect to the vector field Y. This derivative is also denoted by $\nabla_Y X$:

$$\nabla_Y X|_{A_0} = \left. \frac{DX}{dt} \right|_{t_0}, \tag{17}$$

where $\frac{DX}{dt}$ is taken along the integral curve of Y passing through A_0 at $t = t_0$. It is clear that in order to compute $\nabla_Y X$ at a point, we have to know how X changes in a neighborhood of that point. The affine connection, ∇, is therefore not a tensor.

The covariant derivative of a vector field is another vector field so it can be expressed as a linear combination of the basis vector fields. The coefficients Γ_{ji}^k of the covariant derivative of a basis vector field along another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \sum_k \Gamma_{ji}^k \hat{L}_k, \tag{18}$$

are called *Christoffel symbols* [1]. Note the reversed order of the indices i and j.

Given a Riemannian manifold, there exists a unique connection [16] which is compatible with the metric:

$$X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z > \quad [2] \tag{19}$$

and symmetric:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{20}$$

This connection is called the *Levi-Civita* or *Riemannian connection*.

The velocity, $V(t)$, of the rigid body moving along the curve $A(t)$ is given by the tangent vector field:

$$V(t) = \frac{dA(t)}{dt}.$$

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[1] In the literature, different definitions for the Christoffel symbols can be found. Some texts (e.g. [15]) reserve the term for the case of the coordinate basis vectors. We follow the more general definition from [16] in which the basis vectors can be arbitrary.

[2] Note that $X < Y, Z >$ is a derivative of the function $< Y, Z >$ along the integral curves of X (see Section 2.2).

The acceleration, $\mathcal{A}(t)$, is the covariant derivative of the velocity along the curve:

$$\mathcal{A} = \frac{\mathrm{D}}{dt}\left(\frac{dA}{dt}\right) = \nabla_V V. \tag{21}$$

Note that the acceleration depends on the choice of the connection. We can also define jerk, $\mathcal{J}$, as the covariant derivative of the acceleration:

$$\mathcal{J} = \frac{\mathrm{D}}{dt}\mathcal{A}(t) = \nabla_V \nabla_V V. \tag{22}$$

2.6   Curvature tensor

The *curvature* of a Riemannian manifold Σ is a correspondence R that associates to a pair of vector fields X and Y a mapping:

$$R(X,Y) : Z \mapsto \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{23}$$

where Z is a vector field and ∇ is the Riemannian connection on Σ[3]. Unlike the affine connection, curvature is a pointwise object. That is, the value of $R(X,Y)Z$ at a point A only depends on the vectors $X(A)$, $Y(A)$ and $Z(A)$, it is not important how the vector fields change in the neighborhood of A. The *curvature tensor* is a multi-linear mapping which maps a quadruple of vectors (X,Y,U,V) into a real number. The value of the curvature tensor on the quadruple (X,Y,U,V) is given by $< R(X,Y)U,V >$. If X_i is a basis, the components of the curvature tensor are given by:

$$R_{ijkl} = < R(X_i, X_j)X_k, X_l > \tag{24}$$

3   Riemannian structure on $SE(3)$

3.1   Choice of metric

A desired property of a planning method is that the generated trajectories are invariant with respect to the choice of the reference frames. One family of such invariant trajectories are screw motions [17]. But it can be shown [17] that screw motions are not the shortest length curves for any Riemannian metric so they do not minimize any physically meaningful cost function. Since the trajectories that we propose in the paper will depend on a Riemannian metric, another possibility to obtain invariant trajectories is to choose a metric that is bi-invariant (both, left and right invariant) and thus independent of the choice of the reference frames (see Section 2.4). However, $SE(3)$ does not admit a bi-invariant Riemannian metric (see [13] and in the context of robotics [18, 19]). For this reason we focus on the left invariant metrics that are independent of the choice of the inertial reference frame thus giving up the independence of the computed trajectories with respect to the choice of the body-fixed reference frame.

A metric that is attractive for trajectory planning can be obtained by considering the dynamic properties of the rigid body. The kinetic energy of a rigid body is a scalar that does not depend on the choice of the inertial reference frame. It thus defines a left invariant metric. For this metric, the matrix W in Eq. (13) is the inertia matrix and $\frac{1}{2} < V, V >$ corresponds to the kinetic energy of the rigid body moving with a velocity V. If the body-fixed reference frame is attached at the centroid and aligned with the principal axes, then we have:

$$W = \begin{bmatrix} H & 0 \\ 0 & mI \end{bmatrix}, \tag{25}$$

where m is the mass of the rigid body and H is the matrix:

$$H = \begin{bmatrix} H_{xx} & 0 & 0 \\ 0 & H_{yy} & 0 \\ 0 & 0 & H_{zz} \end{bmatrix},$$

with H_{xx}, H_{yy}, and H_{zz} denoting the moments of inertia about the x, y, and z axes, respectively. If $\{\omega, v\}$ is the vector pair associated with the vector V, this vector pair represents the instantaneous twist associated with

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[3]Sign convention in the definition of the curvature in the literature varies. Here we follow [15].

the motion, expressed in the body-fixed reference frame. The norm of the vector V thus assumes the familiar expression:

$$< V, V >= \omega^T H \omega + m v^T v. \tag{26}$$

Now assume that the body fixed frame {M} is displaced by the matrix:

$$C = \left[\begin{array}{cc} R & d \\ 0 & 1 \end{array} \right]$$

to a new frame {M}$_C$. The kinetic energy does not change if the body-fixed frame is changed. It is not difficult to check that this implies that the matrix W_C defining the energy metric for the new description of the motion of the rigid body is:

$$W_C = \left[\begin{array}{cc} R^T H R - m R^T D^2 R & -m R^T D R \\ m R^T D R & m I \end{array} \right], \tag{27}$$

where D is the skew-symmetric matrix corresponding to the vector d. This is therefore the most general form of the inertia matrix and can be viewed as a spatial version of Steiner's parallel-axis theorem.

If we desire a trajectory that can be used for different objects, we can abstract the inertial properties by setting $H = \alpha I$ and $m = \beta$ in Eq. (25), where α and β are two arbitrary positive scalars. In this way the matrix W becomes:

$$W = \left[\begin{array}{cc} \alpha I & 0 \\ 0 & \beta I \end{array} \right] \tag{28}$$

This was the metric proposed by Park and Brockett [20] for study of kinematic dexterity of robot mechanisms. In addition to being left invariant, this metric is also bi-invariant when restricted to the group of rotations, $SO(3)$. The two scalars, α and β, act like scaling factors for angular velocities and linear velocities. In *kinematic analysis* there is no *a priori* justification for choosing them.

Metrics (25) and (28) are not right invariant. Specifically, they will depend on the choice of the origin of the body-fixed reference frame.

Remark 3.1 In [17] it was shown that the matrix of metric coefficients, $G = \left[< \hat{L}_i, \hat{L}_j > \right]$, for a product metric on $SO(3) \times \mathbb{R}^3$ induced by a left invariant metric Q on $SO(3)$ and the standard Euclidean metric on $\mathbb{R}^3$, has the form:

$$G = \left[\begin{array}{cc} Q & 0 \\ 0 & \gamma I \end{array} \right]. \tag{29}$$

The metric (25) (and thus (28)) has this form and is therefore a product metric. In other words, there is an isometry between $SE(3)$ endowed with any of these metrics and the product space $SO(3) \times \mathbb{R}^3$ with appropriately defined metrics on $SO(3)$ and $\mathbb{R}^3$. Although (27) is not a product metric with respect to this splitting, it is isometric to a metric of the form (25). Consequently, any metric induced by the kinetic energy will be isometric to a product metric. These isometries do not preserve the group structure of $SE(3)$, they are isometries in the sense of manifolds. But since none of the functionals that we later use to define the smoothness of a curve depends on the group structure of $SE(3)$, the calculations in the examples could be simplified by performing them on the product space $SO(3) \times \mathbb{R}^3$. However, the key results in this paper are derived for a general metric and are not limited to product metrics. There are important applications of such general metrics. For example, if the metric is defined so that it reflects the dynamic properties of the mechanical system to which the object is attached, it will in general not be a product metric. For this reason, the product structure of $SE(3)$ equipped with the metric induced by the kinetic energy metric (25) will not be used in the derivations.

3.2 The Riemannian connection

In this section we find the Riemannian connections that correspond to the left invariant metrics (25) and (28). We start with an elementary result relating the Christoffel symbols and the structure constants for an arbitrary Lie group. It can be shown [15] that if ∇ is the Riemannian connection then for any three vector fields X, Y and Z:

$$< Z, \nabla_X Y >= \frac{1}{2} \{ Y < X, Z > + X < Z, Y > - Z < X, Y > +$$
$$+ < [Z, Y], X > + < [Z, X], Y > + < [X, Y], Z > \} \tag{30}$$

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This immediately implies:

Proposition 3.2 *If ∇ is the Riemannian connection compatible with a left invariant metric described by a matrix $W = [w_{ij}]$, the Christoffel symbols for the basis $\hat{L}_i$ are given by*

$$\Gamma^k_{ji} = \frac{1}{2} \sum_m w^{-1}_{km} \left(C^s_{ij} w_{sm} + C^s_{mj} w_{si} + C^s_{mi} w_{sj} \right), \tag{31}$$

where C^k_{ij} are the structure constants of the Lie algebra and $w^{-1}_{km} = (W^{-1})_{km}$.

Any vector field on $SE(3)$ can be expressed as a linear combination of left invariant vector fields (with possibly varying coefficients) according to Equation (10). If $X = \sum_{i=1}^6 X^i \hat{L}_i$ and $Y = \sum_{i=1}^6 Y^i \hat{L}_i$ are any two vector fields, then

$$\nabla_X Y = \nabla_{X^j \hat{L}_j} Y^i \hat{L}_i = \frac{dY^i}{dt} \hat{L}_i + X^i Y^j \nabla_{\hat{L}_i} \hat{L}_j = \frac{dY^i}{dt} \hat{L}_i + X^i Y^j \Gamma^k_{ji} \hat{L}_k, \tag{32}$$

where $\frac{d}{dt}$ is the derivative along the integral curve of X and Γ^k_{ji} are obtained from Equation (31)[4]. But instead of computing the Γ^k_{ji}, we derive expressions for the Riemannian connection directly from Equation (30). First, we prove the following lemma for $SE(3)$:

Lemma 3.3 *Let $X = X^i \hat{L}_i$, $Y = Y^i \hat{L}_i$ and $Z = Z^i \hat{L}_i$ be three arbitrary vector fields and let the corresponding vector pairs be $\{\omega_x, v_x\}$, $\{\omega_y, v_y\}$, and $\{\omega_z, v_z\}$, respectively. If ∇ is the Riemannian connection corresponding to a left-invariant Riemannian metric $< ., . >$, then:*

$$
\begin{aligned}
< Z, \nabla_X Y > \ = \ & < Z, X(Y^i)\hat{L}_i > \\
& + \frac{1}{2} [< \{(\omega_z \times \omega_y), (\omega_z \times v_y + v_z \times \omega_y)\}, \{\omega_x, v_x\} > \\
& + \ < \{(\omega_z \times \omega_x), (\omega_z \times v_x + v_z \times \omega_x)\}, \{\omega_y, v_y\} > \\
& + \ < \{(\omega_x \times \omega_y), (\omega_x \times v_y + v_x \times \omega_y)\}, \{\omega_z, v_z\} >]
\end{aligned}
\tag{33}
$$

Proof: The result of the Lemma follows directly from Equation (30). The Lie bracket of any two vector fields is:

$$[X, Y] = X^i Y^j [\hat{L}_i, \hat{L}_j] + X(Y^i)\hat{L}_i - Y(X^i)\hat{L}_i,$$

where $X(f)$ denotes the action of the vector field on a scalar function f (See Section 2.2). Rewritten in terms of the pairs $\{\omega_x, v_x\}$ and $\{\omega_y, v_y\}$, the first term becomes

$$X^i Y^j [\hat{L}_i, \hat{L}_j] = \{\omega_x \times \omega_y, \omega_x \times v_y + v_x \times \omega_y\}$$

Thus, in Equation (30),

$$
\begin{aligned}
< Z, [X, Y] > \ = \ & < \{\omega_z, v_z\}, \{(\omega_x \times \omega_y), (\omega_x \times v_y + v_x \times \omega_y)\} > \\
& + \ < Z, X(Y^i)\hat{L}_i > - < Z, Y(X^i)\hat{L}_i >
\end{aligned}
$$

Furthermore, if W_{ij} are the entries of W in Equation (13),

$$X < Y, Z > = X(Y^i W_{ij} Z^j) = X(Y^i) W_{ij} Z^j + Y^i W_{ij} X(Z^j) = < X(Y^i)\hat{L}_i, Z > + < Y, X(Z^i)\hat{L}_i >$$

If we similarly expand all terms in Equation (30) and add them, the result in Equation (33) follows. $\square$

---

[4]Starting from this point we use the Einstein summation convention to simplify the notation.

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Proposition 3.4 *Let $X = X^i \hat{L}_i$ and $Y = Y^i \hat{L}_i$ be two arbitrary vector fields. If ∇ is the Riemannian connection corresponding to the Riemannian metric (25), then*

$$\nabla_X Y = \left[\begin{array}{c} \frac{d\omega_y}{dt} + \frac{1}{2}\left[(\omega_x \times \omega_y) + H^{-1}(\omega_x \times (H\omega_y)) + H^{-1}(\omega_y \times (H\omega_x)) \right] \\ \frac{dv_y}{dt} + \omega_x \times v_y \end{array} \right] \tag{34}$$

where $\frac{d}{dt}$ is the derivative along the integral curve of X. The translational component of the expression $\nabla_X Y$ is independent of the choice of matrix H and thus independent of the choice of the metric on $SO(3)$.

Proof: We use Lemma 3.3 and compute the right hand side of Eq. (33) using the metric (25):

$$\begin{aligned}
< Z, \nabla_X Y > \; = \;\; & < Z, X(Y^i)\hat{L}_i > \\
& + \frac{1}{2}[(\omega_z \times \omega_y) \cdot (H\omega_x) + m(\omega_z \times v_y + v_z \times \omega_y) \cdot v_x \\
& + (\omega_z \times \omega_x) \cdot (H\omega_y) + m(\omega_z \times v_x + v_z \times \omega_x) \cdot v_y \\
& + (\omega_x \times \omega_y) \cdot (H\omega_z) + m(\omega_x \times v_y + v_x \times \omega_y) \cdot v_z] \\
= \;\; & < Z, X(Y^i)\hat{L}_i > + \frac{1}{2}\left[2\, m(\omega_x \times v_y) \cdot v_z \right. \\
& \left. - ((H\omega_x) \times \omega_y) \cdot \omega_z + (\omega_x \times (H\omega_y)) \cdot \omega_z + (\omega_x \times \omega_y) \cdot (H\omega_z) \right] \\
= \;\; & < Z, X(Y^i)\hat{L}_i > + \frac{1}{2}\left[2\, m(\omega_x \times v_y) \cdot v_z - (H^{-1}((H\omega_x) \times \omega_y)) \cdot (H\omega_z) \right. \\
& \left. + (H^{-1}(\omega_x \times (H\omega_y))) \cdot (H\omega_z) + (\omega_x \times \omega_y) \cdot (H\omega_z) \right] \\
= \;\; & < Z, X(Y^i)\hat{L}_i > \\
& + < Z, \{ \frac{1}{2}\left[(\omega_x \times \omega_y) + H^{-1}(\omega_x \times (H\omega_y)) + H^{-1}(\omega_y \times (H\omega_x)) \right], (\omega_x \times v_y) \} >
\end{aligned}$$

Since the above is true for an arbitrary Z, this proves the proposition. $\qquad\square$

By substituting $H = \alpha I$, we obtain the following corollary:

Corollary 3.5 *Let $X = X^i \hat{L}_i$ and $Y = Y^i \hat{L}_i$ be two arbitrary vector fields. If ∇ is the Riemannian connection corresponding to the Riemannian metric (28), then*

$$\nabla_X Y = \{ \frac{d\omega_y}{dt} + \frac{1}{2}\omega_x \times \omega_y, \frac{dv_y}{dt} + \omega_x \times v_y \}, \tag{35}$$

where $\frac{d}{dt}$ is the derivative along the integral curve of X.

Remark 3.6 Note that the expression for the Riemannian connection corresponding to the metric (28) is independent of the scaling constants, α and β.

3.3 The curvature

In the subsequent sections we will also need expressions for the Riemannian curvature of $SE(3)$ for the metric (28).

Proposition 3.7 *If X, Y and Z are three arbitrary vector fields on $SE(3)$ with the associated vector pairs $\{\omega_x, v_x\}$, $\{\omega_y, v_y\}$, and $\{\omega_z, v_z\}$, and $SE(3)$ has the Riemannian connection defined in Equation (35), then the Riemannian curvature $R(X,Y)Z$ is*

$$R(X,Y)Z = \{ \frac{1}{4}(\omega_x \times \omega_y) \times \omega_z, 0 \} \tag{36}$$

Proof: The result directly follows from Equations (23) and (35). $\qquad\square$

3.4 Acceleration and jerk in three-dimensional motions

Having a formula for the covariant derivative, we can compute the expressions for the acceleration and jerk. We use the scale-dependent left invariant metric from Equation (28) to illustrate this. Since the connection coefficients and the covariant derivative are independent of the choice of the constants α and β, the resulting expressions for acceleration and jerk will also be independent of these scale factors.

If V is the velocity (tangent to the curve) associated with the motion $A(t)$ of a rigid body and if $\{\omega, v\}$ is the corresponding velocity pair, it immediately follows from Equations (21) and (35) that the acceleration is given by

$$\mathcal{A} = \nabla_V V = \begin{bmatrix} \dot{\omega} \\ \dot{v} + \omega \times v \end{bmatrix} \tag{37}$$

The third derivative of motion, jerk, can be computed from Equations (22) and (35):

$$\mathcal{J} = \nabla_V \nabla_V V = \begin{bmatrix} \frac{d\dot{\omega}}{dt} + \frac{1}{2}\omega \times \dot{\omega} \\ \frac{d(\dot{v}+\omega \times v)}{dt} + \omega \times (\dot{v} + \omega \times v) \end{bmatrix} \tag{38}$$

Remark 3.8 The resulting expression for the acceleration corresponds to the acceleration that is used in kinematics. The same is true for the jerk. Given that the acceleration and jerk depend on the connection and therefore on the metric, this result is due to the special choice of the metric (28) and does not hold, for example, for a general form of the metric (25). See [17] for discussion of this phenomenon.

4 Necessary conditions for smooth trajectories

4.1 Variational calculus on manifolds

In this section, we consider trajectories between a starting and a final position and orientation that minimize an integral cost functions while possibly satisfying additional boundary conditions on the velocities and/or accelerations. The cost functions can be the kinetic energy of the rigid body, or some other measure of smoothness involving velocity or its higher derivatives. More specifically, we will be interested in curves $A : [a, b] \to SE(3)$ that minimize integrals of the form

$$J = \int_a^b < h(\frac{dA}{dt}), h(\frac{dA}{dt}) > dt \tag{39}$$

where boundary conditions on $A(t)$ and its derivatives may be specified at the end points a and b. The function h returns a vector field and usually involves one or more recursive applications of the covariant derivative. To obtain trajectories that are independent of the choice of the inertial reference frame $\{F\}$, we will use a left invariant metric and the corresponding Riemannian connection.

We adapt methods from the classical calculus of variations to the differential geometric setting [15]. Noakes *et al.* [10] use such a framework to derive expressions for cubic splines on a general manifold and they provide the formulas for the group of rotations $SO(3)$. The cubic splines correspond to our minimum acceleration curves and we derive the results from [10] using more direct approach. We will illustrate this approach by deriving the necessary conditions for minimum jerk curves. These necessary conditions were independently obtained by Camarinha *et al.* [11], who extended the results by Noakes *et al.* to higher order splines.

In the calculus of variations, the first-order necessary conditions for the minimal solution are derived by studying variations of the optimal trajectory. Let $A(t)$ be a curve on $SE(3)$ and let $f : (-\epsilon, \epsilon) \times [a, b] \to SE(3)$ be a differentiable mapping such that $f(0, t) = A(t)$. Such mapping is called a variation of the curve $A(t)$ [15]. A variation is called proper, if any curve $f_s(t) = f(s, t)$ satisfies the given boundary conditions at $t = a$ and $t = b$. For a variation f, we can define the vector fields $V = \frac{\partial f(s,t)}{\partial t}$ and $S = \frac{\partial f(s,t)}{\partial s}$. The value of the cost functional on a curve $f_s(t)$ is

$$J(s) = \int_a^b < h(\frac{\partial f(s,t)}{\partial t}), h(\frac{\partial f(s,t)}{\partial t}) > dt, \quad s \in (-\epsilon, \epsilon). \tag{40}$$

If the curve $A(t) = f(0, t)$ is a stationary point of J then the first variation $\frac{dJ(s)}{ds}$ must vanish for $s = 0$ and this gives us the first order necessary condition for the optimal trajectories.

4.2 Minimum distance curves - geodesics

Given a Riemannian metric, the length of a curve $A(t)$ between the points $A(a)$ and $A(b)$ is defined to be:

$$L(A) = \int_a^b < \frac{dA}{dt}, \frac{dA}{dt} >^{\frac{1}{2}} dt \tag{41}$$

We are usually interested in finding the shortest curve (the curve that minimizes L) between two points. It can be shown [15], that if there exist a curve that minimizes the functional L, this curve also minimizes so called *energy functional*:

$$J = E(s) = \int_a^b < \frac{df(s,t)}{dt}, \frac{df(s,t)}{dt} > dt = \int_a^b < V, V > dt \tag{42}$$

The critical points of the energy functional are called *geodesics* and they are given by the following equation [15]:

$$\nabla_V V = 0, \tag{43}$$

where $V = \frac{dA(t)}{dt}$.

To solve Equation (43) and find the geodesics on $SE(3)$ for the metric (25), we express V as a linear combination of left invariant vector fields $\hat{L}_1, \ldots, \hat{L}_6$ according to Equation (10). The coefficients of the linear combination form the vector pair $\{\omega, v\}$ which in general varies over the manifold.

Proposition 4.1 *A curve $A(t)$ is a geodesic on $SE(3)$ equipped with the metric (25) if and only if the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ satisfies the equations:*

$$\begin{aligned} \frac{d\omega}{dt} &= -H^{-1}(\omega \times (H\omega)) \\ \frac{dv}{dt} &= -\omega \times v. \end{aligned} \tag{44}$$

The second equation in (44) can be simplified to the equation:

$$\ddot{d} = 0. \tag{45}$$

Proof: A curve $A(t)$ is a geodesic if and only if Equation (43) is satisfied. Substituting for $\nabla_V V$ from Equation (34), and letting $\{\omega_x, v_x\} = \{\omega_y, v_y\} = \{\omega, v\}$ we get the Equation (44). The second equation in (44) can be written as:

$$\dot{v} + \omega \times v = 0.$$

By writing $\Omega = R^T \dot{R}$ and $v = R^T \dot{d}$ and using the identity $\dot{R}^T = -R^T \dot{R} R^T$, we obtain

$$\dot{v} + \omega \times v = \dot{v} + \Omega v = (\dot{R}^T \dot{d} + R^T \ddot{d}) + R^T \dot{R} R^T \dot{d} = R^T \ddot{d} = 0,$$

which proves $\ddot{d} = 0$. □

Remark 4.2 According to Hamilton's principle, the trajectory that minimizes the kinetic energy is obtained by solving the dynamic equations of motion. It therefore comes as no surprise that the first equation in (44) are the Euler equations while Equation (45) is the Newton's equation in the absence of external forces.

Corollary 4.3 *A curve*

$$A(t) = \begin{bmatrix} R(t) & d(t) \\ 0 & 1 \end{bmatrix}$$

is a geodesic on $SE(3)$ equipped with the metric (28) if and only if the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ satisfies the equations:

$$\begin{aligned} \frac{d\omega}{dt} &= 0 \\ \frac{dv}{dt} &= -\omega \times v. \end{aligned} \tag{46}$$

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The second equation in (46) can be simplified to the equation:

$$\ddot{d} = 0.$$

Remark 4.4 It is worth noting that the above result is independent of the choice of scale factors α and β. The necessary conditions for minimum acceleration and minimum jerk curves derived in subsequent subsections will also have the same property. However, the curves do depend on the choice (of the origin) of the body-fixed reference frame.

4.3 Minimum acceleration curves

We derive the necessary conditions for the curves that minimize the square of the L^2 norm of the acceleration by considering the first variation of the acceleration functional

$$L_a = \int_a^b < \nabla_V V, \nabla_V V > dt, \tag{47}$$

where $V(t) = \frac{dA(t)}{dt}$ and $A(t)$ is a curve on the manifold. The initial and final point as well as the initial and final velocity for the motion are prescribed. Noakes *et al.* [10] derived the following theorem:

Theorem 4.5 (Noakes *et al.* [10]) *Let $A(t)$ be a curve on a Riemannian manifold that satisfies the boundary conditions (that is, it starts and ends at the prescribed points with the prescribed velocities) and let $V = \frac{dA}{dt}$. If $A(t)$ minimizes the functional L_a, then:*

$$\nabla_V \nabla_V \nabla_V V + R(V, \nabla_V V)V = 0. \tag{48}$$

Proof: The proof of the theorem is similar to the proof of Theorem 4.8 and it will be omitted in the interest of space. Noakes *et al.* use slightly different approach and their proof is more involved. □

We can directly apply Theorem 4.5 to $SE(3)$ with the Riemannian connection computed from the metric (28).

Proposition 4.6 *Let*

$$A(t) = \left[\begin{array}{cc} R(t) & d(t) \\ 0 & 1 \end{array} \right]$$

be a curve between two prescribed points on $SE(3)$ that has prescribed initial and final velocities. If $\{\omega, v\}$ is the vector pair corresponding to $V = \frac{dA}{dt}$, the curve minimizes the cost function L_a derived from the metric (28) only if the following equations hold:

$$\omega^{(3)} + \omega \times \ddot{\omega} = 0$$

$$d^{(4)} = 0, \tag{49}$$

where $(\cdot)^{(n)}$ denotes the nth derivative of $(\cdot)$.

Proof: We start by using Equations (37) and (36) to compute the second term in Equation (48):

$$\nabla_V \nabla_V \nabla_V V + R(V, \nabla_V V)V = \nabla_V \nabla_V \nabla_V V + \left[\begin{array}{c} \frac{1}{4}(\omega \times \dot{\omega}) \times \omega \\ 0 \end{array} \right] = 0 \tag{50}$$

By repeated application of Equation (35) the term $\nabla_V \nabla_V \nabla_V V$ can be simplified. The rotational part of the above equation thus reduces to the first equation in (49). To simplify the translational component, we first observe that the translational component of $\nabla_V V$ can be written as (see Proposition 4.1):

$$\dot{v} + \omega \times v = R^T \ddot{d}.$$

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It follows that the translational component of $\nabla_V \nabla_V V$ is:

$$\frac{d}{dt}(R^T \ddot{d}) + \omega \times (R^T \ddot{d}) = (\frac{d}{dt}(R^T \ddot{d}) + R^T \dot{R}(R^T \ddot{d}) = (\dot{R}^T \ddot{d} + R^T d^{(3)}) + R^T \dot{R}R^T \ddot{d} = R^T d^{(3)}.$$

Similarly, the translational component of $\nabla_V \nabla_V \nabla_V V$ can be simplified to get

$$R^T d^{(4)} = 0$$

from which the second equation in (49) follows. □

Remark 4.7 As observed in [10], the first equation (49) can be integrated to obtain

$$\omega^{(2)} + \omega \times \dot{\omega} = \text{constant} \tag{51}$$

However, this equation cannot be further integrated analytically for arbitrary boundary conditions. In Section 5.2 we will show how to obtain the solution for special choice of the initial and final velocities.

4.4 Minimum jerk curves

The minimum jerk curves between two points are obtained by minimizing the L^2 norm of the Cartesian jerk, provided that the appropriate boundary conditions are given. In particular, it is possible to solve for minimum jerk trajectories when the initial and final velocities and the initial and final accelerations are specified. Minimum jerk trajectories are particularly useful in robotics where one is generally able to control the acceleration of the end effector of a robot (and therefore the velocity and position) but the electro-mechanical actuators cannot produce sudden changes in the acceleration.

The jerk cost functional is:

$$L_j = \int_a^b < \nabla_V \nabla_V V, \nabla_V \nabla_V V > dt \tag{52}$$

where $V = \frac{dA(t)}{dt}$. The curve must start and end at the desired points on the manifold and with the desired velocities and accelerations. We arrive at the necessary conditions for the solution by following the same approach as in the previous subsection[5].

Theorem 4.8 *Let $A(t)$ be a curve on a Riemannian manifold that satisfies the boundary conditions (that is, it starts and ends at the prescribed points with the prescribed velocities and the prescribed accelerations) and let $V = \frac{dA}{dt}$. If $A(t)$ minimizes the functional L_j, then:*

$$\nabla_V^5 V + R(V, \nabla_V^3 V)V - R(\nabla_V V, \nabla_V^2 V)V = 0. \tag{53}$$

Proof: See Appendix A. □

The expressions for the minimum jerk trajectories on $SE(3)$ for the metric (28) immediately follow.

Proposition 4.9 *Let $A(t)$ be a curve between two prescribed points on $SE(3)$ that has prescribed initial and final velocities and initial and final accelerations. If $\{\omega, v\}$ is the vector pair corresponding to $V = \frac{dA}{dt}$, the curve minimizes the cost function L_j for the metric (28) only if the following equations hold:*

$$\omega^{(5)} + 2\,\omega \times \omega^{(4)} + \tfrac{5}{4}\omega \times (\omega \times \omega^{(3)}) + \tfrac{5}{2}\dot{\omega} \times \omega^{(3)}$$
$$+ \tfrac{1}{4}\omega \times (\omega \times (\omega \times \ddot{\omega})) + \tfrac{3}{2}\omega \times (\dot{\omega} \times \ddot{\omega}) - (\omega \times \ddot{\omega}) \times \dot{\omega}$$
$$-\tfrac{1}{4}(\omega \times \dot{\omega}) \times \ddot{\omega} - \tfrac{3}{8}\omega \times ((\omega \times \dot{\omega}) \times \dot{\omega}) - \tfrac{1}{8}(\omega \times (\omega \times \dot{\omega})) \times \dot{\omega} = 0$$
$$d^{(6)} = 0. \tag{54}$$

Proof: The proof follows the same lines as the proof for Proposition 4.6. We use formulas (35) and (36) to evaluate the three terms in Equation (53) and the result follows in a straightforward manner. □

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[5] A generalized version of Theorem 4.8 was derived in parallel with our work by Camarinha *et al.* [11].

5 Solutions for optimal trajectories

5.1 Shortest distance path on $SE(3)$

According to Proposition 4.1, the rotational components for the minimum distance curves corresponding to the metric (25) are the Euler equations (see Remark 4.2). In general, these equations do not have an analytical solution and must be solved numerically. However, for the metric (28), the equations simplify and the minimum distance curves can be computed analytically. Using properties of the Riemannian covering maps, Park showed [18] that for the metric (28), the geodesics can be obtained by lifting the geodesics from $SO(3)$ (zero pitch screw motions) and $\mathbb{R}^3$ (straight lines). We come to the same result constructively using Equation (46).

Proposition 5.1 (Park [18]) *Given two configurations*

$$A_1 = \left[\begin{array}{cc} R_1 & d_1 \\ 0 & 1 \end{array} \right] \qquad\qquad A_2 = \left[\begin{array}{cc} R_2 & d_2 \\ 0 & 1 \end{array} \right]$$

a shortest distance path (minimal geodesic)

$$A(t) = \left[\begin{array}{cc} R(t) & d(t) \\ 0 & 1 \end{array} \right]$$

between them with respect to the metric (28) is given by

$$\begin{array}{rcl} R(t) & = & R_1 \exp(\Omega_0 t) \end{array} \qquad\qquad (55)$$
$$\begin{array}{rcl} d(t) & = & (d_2 - d_1)\, t + d_1 \end{array} \qquad\qquad (56)$$

where

$$\Omega_0 = \log(R_1^T R_2).$$

The path is unique unless $\mathrm{Trace}(R_1^T R_2) = -1$ when there exist two geodesics of equal minimum length (see Remark 5.2).

Proof: The result follows from Proposition (4.3). The first equation in (46) can be readily integrated to obtain

$$\omega(t) = \omega_0. \qquad\qquad (57)$$

Let Ω be the skew symmetric matrix representation of the vector ω. From Equation (3) we have

$$\Omega = R^T \dot{R} \qquad\qquad (58)$$

Equation (57) can be thus integrated:

$$R^T \dot{R} = \Omega_0 \Rightarrow R(t) = R_0 \exp(\Omega_0 t). \qquad\qquad (59)$$

From the initial condition we get $R_0 = R_1$ and from the boundary condition $\Omega_0 = \log(R_1^T R_2)$. The function log is the inverse of the exponential function. See [14, pp. 414] for the formula on $SE(3)$.

The expression for the vector $d(t)$ is obtained by integrating the equation $\ddot{d} = 0$ twice. As a result, we get

$$d(t) = c_1 t + c_0$$

and Equation (56) immediately follows from the initial and final conditions on d. $\qquad\qquad \square$

Remark 5.2 The log function on $SO(3)$ is multi-valued. If $\log(R)$ yields a solution (u, θ), where u is a unit vector along the axis of rotation and θ is the angle of rotation, then $(u, \theta + 2k\pi)$ is also a solution for any integer k. The multiplicity of the solution can be avoided by restricting θ to lie in the interval $[0, \pi]$ (the interval $[-\pi, 0]$ is covered by using axis $-u$). The geodesic computed by restricting θ to lie in the interval $[0, \pi]$ can be shown to give the unique minimal-length geodesic [18] unless $\theta = \pi$. If we think of the representation of $SO(3)$ as a unit hyper-sphere in $\mathbb{R}^4$ with antipodal points identified, the minimal-length geodesic is unique between any two general points, R_1 and R_2, except when $\mathrm{Trace}(R_1^T R_2) = -1$ and there exist two geodesics of equal minimum length.

According to Chasles' theorem there is a unique[6] screw motion between any two given positions and orientations. A screw motion will be a geodesic for metric (28) only in the special case in which the screw axis for the screw displacement from the initial position and orientation to the final position and orientation passes through the origin O. In [17] we show that there is no Riemannian metric whose geodesics are screw motions. Furthermore, it is shown that there is a family of non-degenerate (but not positive-definite) bi-invariant 2-forms for which the screw motions satisfy the geodesic equation (43). These forms can be viewed as generalizations of the Klein and Killing forms and they are the only 2-forms for which the geodesic equation is satisfied by the screw motions.
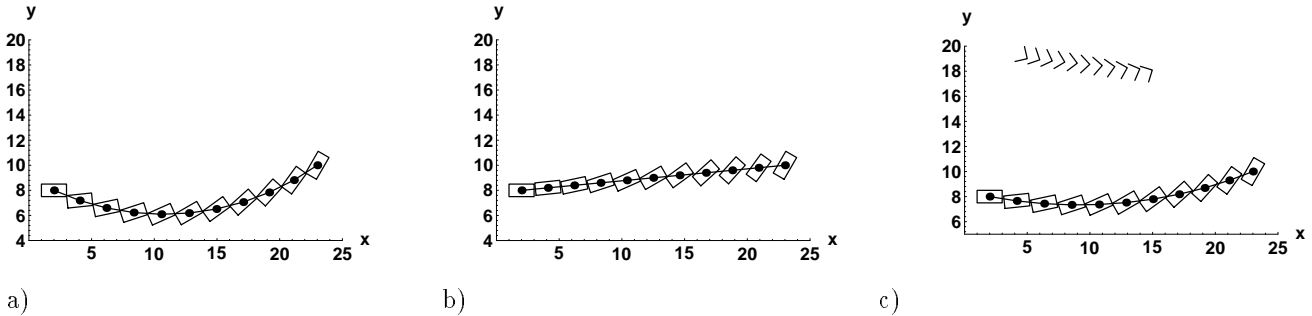
Figure 2: Motions in a plane: (a) a screw motion; (b) a geodesic for metrics (28) and (25); and (c) a geodesic after the body-fixed frame $\{M\}$ is displaced.

Figure 2 shows various trajectories for a motion in the plane $z = 0$. Figure 2-a shows a screw motion which in the planar case corresponds to rotation about a fixed point in the plane. A geodesic for the metric (28) is shown in Fig. 2-b. For planar motions, the geodesic for the metric (25) will be the same. Since the trajectory is computed by using a left invariant metric, it does not change if the inertial reference frame $\{F\}$ is moved. But the trajectory changes if we change the body-fixed frame $\{M\}$. The trajectory for a different body-fixed reference frame is shown in Fig. 2-c and is different from the curve shown in Fig. 2-b. We also show the motion of the new body-fixed frame. The figure clearly shows that the new body-fixed frame follows a geodesic for metric (28), but the rigid body will move along a curve that is different from the geodesics on Fig. 2-b. Examples of three dimensional motions can be found in [1, 21].

It is also interesting to compare the geodesics for the metric (25), which are products of geodesics on $SO(3)$ and $\mathbb{R}^3$, with geodesics for a non-product metric. For illustrative purposes we present motions in the plane and thus the geodesics on $SE(2)$. A generalized form of metric (25) for $SE(2)$ is:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \end{bmatrix}. \tag{60}$$

The rows correspond to components ω_z, v_x and v_y, respectively. When $\beta_1 \neq \beta_2$, this metric is not a product metric (see Remark 3.1). Such a metric might be used, for example, to plan the end-effector trajectories for a gantry mechanism that has different dynamic characteristic for motions in the x and y directions.

Figure 3 shows the geodesics for different choices of β_1 and β_2. Figure 3-a shows a geodesic when $\beta_1 = \beta_2$. In this case the metric becomes the same as metric (28) and the geodesic is a product of geodesics on $S(1)$ and $\mathbb{R}^2$. The other two figures show geodesics for the cases when $\beta_1 \neq \beta_2$ and the metrics are not product metrics. In this case the rotational and the translational components of the motion are coupled. In particular, the translational motion does not follow a straight line. These geodesics were computed numerically.

Remark 5.3 To obtain trajectories satisfying the necessary conditions for a general variational problem, it is necessary to solve a two-point boundary value problem. To solve these boundary-value problems numerically, we used a finite-difference method [22]. Typically, the solution for approximation with a grid of 100 points takes less than 5 seconds to compute and is very robust with respect to the choice of the initial guess. More details, including some three-dimensional examples, are presented in [21].

---

[6]An argument similar to that in Remark 5.2 shows that the screw motion is unique if we limit the angle of rotation to $[0, \pi)$.
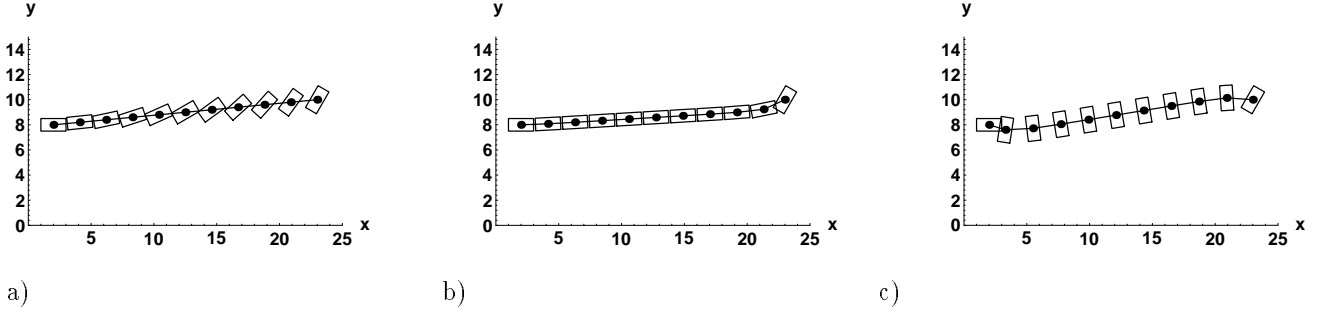
Figure 3: Geodesics for different metrics on $SE(2)$**: (a) product metric,** $\beta_1 = \beta_2 = 1$**; (b) a metric with** $\beta_1 = 1$**,** $\beta_2 = 10$**; and (c) a metric with** $\beta_1 = 5$**,** $\beta_2 = 1$**.**

5.2   Minimum acceleration and minimum jerk trajectories

In general, the rotational components of the minimum acceleration curves (Equation 49) and minimum jerk curves (Equation 54) can not be computed analytically. However, in the special case when the initial velocities and accelerations are collinear with the initial velocity of the geodesic between the two endpoints, and the final velocities and accelerations are collinear with the final velocity of the geodesic, it is easy to obtain a solution for these trajectories in terms of the geodesic curve. If the geodesic curve can be computed analytically, so can minimum acceleration and minimum jerk curves. This is true not only for $SE(3)$ with the metric (28) but for any geodesically complete Riemannian manifold.

Proposition 5.4 *Given an initial point q_0 and a final point q_1 on a Riemannian manifold, let $\gamma : [0, 1] \to \Sigma$ be a geodesic connecting these two points so that $\gamma(0) = q_0$ and $\gamma(1) = q_1$. Let $V_0 = \left.\frac{d\gamma}{dt}\right|_{t_0}$ and $V_1 = \left.\frac{d\gamma}{dt}\right|_{t_1}$. If the boundary conditions for the minimum acceleration curve are of the form:*

$$V(t_0) = \eta_1 V_0 \qquad\qquad V(t_1) = \rho_1 V_1, \tag{61}$$

then the minimum acceleration curve is given by $\gamma(p(t))$, where $p(t)$ is a third degree polynomial that satisfies:

$$\begin{aligned} p(0) &= 0, & p(1) &= 1 \\ p'(0) &= \eta_1, & p'(1) &= \rho_1, \end{aligned} \tag{62}$$

where $p' = \frac{dp}{dt}$.

Proof: Assume that the minimum acceleration curve α has the form $\alpha(t) = \gamma(p(t))$, where p is an arbitrary scalar function, $p : \mathbb{R} \to \mathbb{R}$. It is easy to see that $V = \frac{d\alpha}{dt} = p'\frac{d\gamma}{dt}$. Let $T = \frac{d\gamma}{dt}$. Since γ is a geodesic, $\nabla_T T = 0$. It then follows:

$$\nabla_V V = V(p')T + p'p'\nabla_T T = V(p')T. \tag{63}$$

But $V(p')$ is a derivative of p' along α, so $V(p') = p''$. It immediately follows that:

$$\nabla_V^n V = p^{(n+1)}T. \tag{64}$$

Using the linearity of the curvature, we also get:

$$R(V, \nabla_V V) = R(p'T, p''T) = p'p''R(T, T) = 0. \tag{65}$$

Equation (48) therefore reduces to:

$$p^{(4)}T = 0. \tag{66}$$

Since T is a tangent vector for a geodesic and therefore never vanishes, we must have:

$$p^{(4)} = 0. \tag{67}$$

Solution of this differential equation is a polynomial of degree 3 and the boundary conditions transform into Eq. (62).

$\square$

The following proposition can be proved along similar lines:

Proposition 5.5 *Given an initial point q_0 and a final point q_1 on a Riemannian manifold, let $\gamma : [0,1] \to \Sigma$ be a geodesic connecting these two points so that $\gamma(0) = q_0$ and $\gamma(1) = q_1$. Let $V_0 = \left.\frac{d\gamma}{dt}\right|_{t_0}$ and $V_1 = \left.\frac{d\gamma}{dt}\right|_{t_1}$. If the boundary conditions for the minimum jerk curve are of the form:*

$$
\begin{aligned}
V(t_0) &= \eta_1 V_0, & V(t_1) &= \rho_1 V_1, \\
\nabla_V V|_{t_0} &= \eta_2 V_0, & \nabla_V V|_{t_1} &= \rho_2 V_1,
\end{aligned}
\tag{68}
$$

then the minimum jerk curve is given by $\gamma(p(t))$, where $p(t)$ is a fifth degree polynomial that satisfies:

$$
\begin{aligned}
p(0) &= 0, & p(1) &= 1 \\
p'(0) &= \eta_1, & p'(1) &= \rho_1, \\
p''(0) &= \eta_2, & p''(1) &= \rho_2.
\end{aligned}
\tag{69}
$$

For this special form of the boundary conditions, the minimum acceleration, minimum jerk and minimum distance paths are therefore the same, only the parameterization along the path varies. Figure 4 shows that for more general boundary conditions the path of the minimum acceleration curve does not follow a geodesic. Further, the path changes with the boundary conditions. The figure shows minimum acceleration motions in the plane $z = 0$ for different choices of the initial and final velocities. We consider $SE(3)$ equipped with the metric (28). In Fig. 4-a, the initial and final velocities are 0, so the object follows the geodesic path shown in Fig. 3-a, but with a different velocity profile. The initial and final velocities for Figs. 4-b,c are not collinear with the initial and final velocities of the geodesic in the figure 3-a and the paths are different.

a)   b)   c)
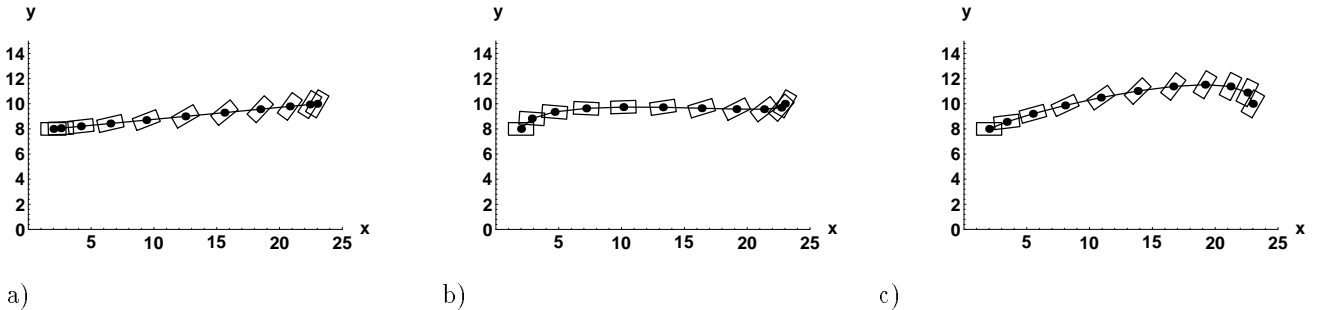
Figure 4: Minimum acceleration motions in the plane for different boundary conditions: (a) $V(0) = V(1) = \{0,0,0\}^T$; **(b)** $V(0) = \{-1,3,10\}^T$, $V(1) = \{2,2,5\}^T$; **and (c)** $V(0) = \{1,10,5\}^T$, $V(1) = \{-1,-10,-5\}^T$. **The triple** $V = \omega_z, v_x, v_y$ **denotes the velocity components for the planar motion.**

6   Concluding remarks

This paper addressed the problem of generating smooth trajectories for a rigid body between an initial and a final position and orientation. The main idea was to define a measure of the smoothness of a trajectory in the form of a functional and find trajectories that minimize this cost functional. Using some basic tools from differential geometry, the problem was formulated as a variational problem on the Lie group of rigid body displacements $SE(3)$. We defined an inner product on the Lie algebra $se(3)$ leading to a left invariant Riemannian metric on $SE(3)$. This metric gave rise to a Riemannian connection and a covariant derivative. We derived analytical expressions for the covariant derivative and the curvature of $SE(3)$. The covariant derivative was used to define acceleration and jerk for spatial rigid body motions. We stated the necessary conditions for minimum distance,

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minimum acceleration and minimum jerk trajectories and specialized these conditions for $SE(3)$. We computed the analytical solutions for the minimum distance trajectories by choosing an appropriate basis for the space of the vector fields. We also found analytical solutions for the minimum acceleration and minimum jerk trajectories for a special class of boundary conditions.

In addition to these results, we show how $SE(3)$ can be naturally endowed with a product metric or with metrics that are isometric to a product metric. We provide several numerical examples to illustrate how the generated solutions are affected by (a) the metric; (b) the choice of the body-fixed reference frame; and (c) the boundary conditions. A simple extension of the ideas in this paper allows the inclusion of intermediate positions and orientations and lends itself to motion interpolation (see [11]). The presented methods also have immediate applications in computer graphics and planning of the trajectories for robots and other machines.

Acknowledgment

A   Proof of Theorem 4.8

The proof is similar to the derivation of the first variation for the energy functional [15] and the same reasoning could be also used to prove Theorem 4.5. We will use the following identities:

(1) $\frac{df(s)}{ds} = Sf$

(2) $< \nabla_V S, U > = V < S, U > - < S, \nabla_V U >$

(3) $\nabla_V S = \nabla_S V + [V, S] = \nabla_S V$ (since S and V are derivatives with respect to coordinate curves t and s, $[V, S] = 0$).

(4) for $V = \frac{\partial}{\partial t}$, $\int_a^b V(f) dt = \int_a^b \frac{df(t)}{dt} dt = f(t)|_a^b$

(5) $\nabla_S \nabla_T U = \nabla_T \nabla_S U + R(T, S) U$.

(6) $< R(X, Y) Z, T > = < R(Z, T) X, Y >$.

The first and the fourth identity express the fact that a vector field is a differential operator. The second and the third identity state that ∇ is a Riemannian connection, thus compatible with the metric (2) and symmetric (3). Identity (5) is just the definition of the curvature operator when $[S, T] = 0$, while (6) is one of the symmetry properties of the curvature tensor [15]. In the proof, the numbers above the equal signs indicate which identities were employed.

We first obtain the expression for the first variation of the functional L_j:

$$
\begin{aligned}
\frac{1}{2} L_j'(s) &= \frac{1}{2} \frac{d}{ds} \int_a^b < \nabla_V^2 V, \nabla_V^2 V > dt \\
&\overset{1}{=} \frac{1}{2} S \int_a^b < \nabla_V^2 V, \nabla_V^2 V > dt \\
&\overset{2}{=} \int_a^b < \nabla_S \nabla_V^2 V, \nabla_V^2 V > dt \\
&\overset{5}{=} \int_a^b (< \nabla_V \nabla_S \nabla_V V + R(V, S) \nabla_V V, \nabla_V^2 V >) dt \\
&\overset{2,6}{=} \int_a^b (V < \nabla_S \nabla_V V, \nabla_V^2 V > - < \nabla_S \nabla_V V, \nabla_V^3 V > + < R(\nabla_V V, \nabla_V^2 V) V, S >) dt
\end{aligned}
$$

$$
\overset{4,5}{=} \quad < \nabla_S \nabla_V V, \nabla_V^2 V >\big|_a^b + \int_a^b (- < \nabla_V \nabla_S V + R(V,S)V, \nabla_V^3 V >
$$
$$
+ < R(\nabla_V V, \nabla_V^2 V)V, S >)dt
$$
$$
\overset{3}{=} \quad < \nabla_S \nabla_V V, \nabla_V^2 V >\big|_a^b + \int_a^b (- < \nabla_V^2 S, \nabla_V^3 V > - < R(V,S)V, \nabla_V^3 V >
$$
$$
+ < R(\nabla_V V, \nabla_V^2 V)V, S >)dt
$$
$$
\overset{2,6}{=} \quad < \nabla_S \nabla_V V, \nabla_V^2 V >\big|_a^b + \int_a^b (-V < \nabla_V S, \nabla_V^3 V > + < \nabla_V S, \nabla_V^4 V >
$$
$$
- < R(V, \nabla_V^3 V)V, S > + < R(\nabla_V V, \nabla_V^2 V)V, S >)dt
$$
$$
\overset{4,5}{=} \quad [< \nabla_V \nabla_S V + R(V,S)V, \nabla_V^2 V > - < \nabla_V S, \nabla_V^3 V >]_a^b
$$
$$
+ \int_a^b (< \nabla_V S, \nabla_V^4 V > - < R(V, \nabla_V^3 V)V, S > + < R(\nabla_V V, \nabla_V^2 V)V, S >)dt
$$
$$
\overset{2,3,6}{=} \quad [< \nabla_V \nabla_V S, \nabla_V^2 V > + < R(V, \nabla_V^2 V)V, S > - < \nabla_V S, \nabla_V^3 V >]_a^b
$$
$$
+ \int_a^b (V < S, \nabla_V^4 V > - < S, \nabla_V^5 V > + < -R(V, \nabla_V^3 V)V + R(\nabla_V V, \nabla_V^2 V)V, S >)dt
$$
$$
\overset{4}{=} \quad [< \nabla_V \nabla_V S, \nabla_V^2 V > + < R(V, \nabla_V^2 V)V, S > - < \nabla_V S, \nabla_V^3 V > + < S, \nabla_V^4 V >]_a^b
$$
$$
+ \int_a^b (< [-\nabla_V^5 V - R(V, \nabla_V^3 V)V + R(\nabla_V V, \nabla_V^2 V)V], S >)dt \tag{70}
$$

Since the initial and final positions, velocities and accelerations are fixed, S, $\nabla_S V = \nabla_V S$ and $\nabla_S \nabla_V V = \nabla_V \nabla_S V + R(V,S)V = \nabla_V \nabla_V S$ vanish at the endpoints. Thus the integral in the above equation must vanish for an arbitrary variation (that preserves the boundary conditions). But this is only possible if Equation (53) holds so the Theorem is proved.

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