Integral representations for matter fields in quantum Einstein gravity

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Abstract

Integral representations for the gluon, electroweak, quark, lepton, and Higgs fields are presented in the manifestly covariant operator formalism of quantum gravity. These representations involve all the interactions in the standard model of particle physics, and satisfy the matter field equations. Several transformation properties of them are investigated.

1. Introduction

The d’Alembert equation for a free massless field $\varphi(x)$,

$$
\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi(x) = 0, \tag{1.1}
$$

can be solved in terms of an integral representation,

$$
\varphi(x) = \int d^3 z [D(x - z)^{\overrightarrow{\partial_0}} \cdot \varphi(z) - D(x - z)\partial_0^z \varphi(z)]. \tag{1.2}
$$

Here, $\eta^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$ and $D(x - z)$ is the Pauli–Jordan D function defined by the following Cauchy problem:

$$
\eta^{\mu\nu} \partial_\mu^x \partial_\nu^x D(x - z) = 0, \tag{1.3}
$$

$$
D(x - z)|_0 = 0, \tag{1.4}
$$

$$
\partial_0^z D(x - z)|_0 = -\delta^3, \tag{1.5}
$$
where \( \delta^3 \) denotes the spatial delta function \( \prod_{k=1}^{3} \delta(x^k-z^k) \) and the symbol \( |_0 \) means to set \( x^0 = z^0 \). The right-hand side of (1.2) is independent of \( z^0 \); thus it reduces to \( \varphi(x) \) at \( z^0 = x^0 \) via (1.4) and (1.5).

The Pauli–Jordan D function has been extended to a non-abelian version by Kanno and Nakanishi [1, 2]. This non-abelian D function is defined by the following q-number Cauchy problem:

\[
\eta^{\mu \nu} \partial_{\mu} \mathcal{D}^{ac}(x,z) + \lambda f^{abc} A_{\nu}^b(x) \mathcal{D}^{bc}(x,z) = 0, \tag{1.6}
\]

\[
\mathcal{D}^{ac}(x,z)|_0 = 0, \tag{1.7}
\]

\[
\partial^x_0 \mathcal{D}^{ac}(x,z)|_0 = -\delta^{ac} \delta^3. \tag{1.8}
\]

Here, \( A_{\mu}^a(x) \) denotes a non-abelian gauge field, \( \lambda \) a coupling constant, \( f^{abc} \) the structure constant of a non-abelian Lie algebra, and \( \mathcal{D}^{ac}(x,z) \) the non-abelian D function.

On the basis of (1.6)–(1.8), Kanno and Nakanishi [1] proved

\[
[\mathcal{D}^{ac}(x,z) \partial_0^x + \mathcal{D}^{ab}(x,z) \partial_0^x \cdot \mathcal{L} \mathcal{D}^{ac}(x,z) |_0 = 0, \tag{1.9}
\]

\[
\mathcal{D}^{ac}(x,z) \partial_0^x |_0 = \delta^{ac} \delta^3. \tag{1.10}
\]

Equations (1.6) and (1.9) show that \( \mathcal{D}^{ab}(x,z) \) satisfies the same equations as the following ones for the Faddeev–Popov ghost fields \( C^a(x) \) and \( \bar{C}^a(x) \),

\[
\eta^{\mu \nu} \partial_{\mu} C^a(x) + \lambda f^{abc} A_{\nu}^b(x) C^c(x) = 0, \tag{1.11}
\]

\[
\eta^{\mu \nu} \partial_{\mu} \bar{C}^a(x) + \lambda f^{abc} A_{\nu}^b(x) \partial_{\nu} \bar{C}^c(x) = 0, \tag{1.12}
\]

respectively.

Equations (1.6)–(1.12) lead us to integral representations for \( C^a(x) \) and \( \bar{C}^a(x) \) [1],

\[
C^a(x) = \int d^3 z \{ \mathcal{D}^{ab}(x,z) \partial_0^x \cdot C^b(z) - \mathcal{D}^{ab}(x,z) \delta^{bc} \partial_0^x + \lambda f^{bdc} A_{0}^d(z) C^c(z) \}, \tag{1.13}
\]
\[
\bar{C}^a(x) = \int d^3 z \{ \bar{C}^b(z) \bar{\partial}_0^z \cdot D^{ba}(z, x) \\
- \bar{C}^b(z) [\delta^{bc} \bar{\partial}_0^z + \lambda f^{bdc} A_0^d(z)] D^{ca}(z, x) \} .
\]

(1.14)

In these integral representations, the interactions with the non-abelian
gauge field are contained not only in the terms proportional to \( \lambda \) but also
in \( D^{ab}(x, z) \) via (1.6) and (1.9).

On the other hand, Abe and Nakanishi [3] have treated the Pauli–Jordan
D function separately from the electromagnetic interaction in a method for
solving quantum electrodynamics in the Heisenberg picture. They noted
that an inhomogeneous differential equation can be solved in terms of an
integral representation. For \( u(x) \) satisfying

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu u(x) = f(x) ,
\]

(1.15)
on one has

\[
u(x) = - \int d^4 z [\theta(x^0 - z^0) - \theta(y^0 - z^0)] D(x - z) f(z) \\
+ \int d^3 z [D(x - z) \bar{\partial}_0^z \cdot u(z) - D(x - z) \partial_0^z u(z)]|_{y^0 = y^0} ,
\]

(1.16)

with

\[
\theta(\zeta^0) \equiv \begin{cases} 1 & \text{for } \zeta^0 > 0 , \\
0 & \text{otherwise} .
\end{cases}
\]

(1.17)

The inhomogeneous term \( f(x) \) in (1.15) is involved in the integral repre-
sentation (1.16) while no inhomogeneous ones appear in (1.3). This formu-
lation differs from the one in (1.13) or (1.14) where the interaction terms
对应的 \( f(x) \) relate to \( D^{ab}(x, z) \) via (1.6) and (1.9).

By analogy with the treatment of the inhomogeneous term in (1.15)
and (1.16), it seems that we can form a Cauchy problem (1.6)–(1.8) with
setting \( \lambda = 0 \) to define another type of non-abelian D function. However,
we find that such a function is equivalent to the Pauli–Jordan D function:

\[ D^{ab}(x, z) = \delta^{ab} D(x - z). \]  

(1.18)

Therefore, alternative integral representations for \( C^a(x) \) and \( \bar{C}^a(x) \) are given by

\[
C^a(x) = \int d^4 z [\theta(x^0 - z^0) - \theta(y^0 - z^0)] D(x - z) \eta^{\mu\nu} \partial^z_\mu [\lambda f^{abc} A^b_\nu(z) C^c(z)] \\
+ \int d^3 z [D(x - z) \partial_0^z \cdot C^a(z) - D(x - z) \partial_0^z C^a(z)]|_{z^0 = y^0},
\]

(1.19)

\[
\bar{C}^a(x) = \int d^4 z [\theta(x^0 - z^0) - \theta(y^0 - z^0)] D(x - z) \eta^{\mu\nu} \lambda f^{abc} A^b_\nu(z) \partial^z_\mu \bar{C}^c(z) \\
+ \int d^3 z [D(x - z) \partial_0^z \cdot \bar{C}^a(z) - D(x - z) \partial_0^z \bar{C}^a(z)]|_{z^0 = y^0}.
\]

(1.20)

Of course, these satisfy the field equations (1.11) and (1.12), respectively.

The treatment in (1.15)–(1.20) can be extended to a quantum-gravity version with the use of the quantum-gravity Pauli–Jordan D function \( D(x, z) \) [4, 5]. Here, \( D(x, z) \) is defined on the basis of the manifestly covariant operator formalism of quantum gravity [6]. This formalism is a quantum field theory version of general relativity in which the gravitational field is an operator in the Heisenberg picture and the other fields are also similar ones. Thus, it is called quantum Einstein gravity.

On the analogy of the above extension, we [7] have applied the treatment of the inhomogeneous term in (1.15) and (1.16) to the matter fields with the electromagnetic interaction in quantum Einstein gravity. We regarded the terms proportional to the electromagnetic constant \( e \) in the equations for the electromagnetic field \( A_\mu(x) \) and the electron one \( \psi(x) \)
as inhomogeneous terms corresponding to that in (1.15). In addition to the quantum-gravity Pauli–Jordan D function, we defined the tensorial q-number commutator function $D_{\mu\nu}(x, z)$ [8] and the quantum-gravity S function $S(x, z; m)$ [7] without any terms proportional to $e$. Then, we incorporated the above electromagnetic interaction terms into the integral representations for $A_\mu(x)$ and $\psi(x)$.

Furthermore, the quantum-gravity version of (1.18)–(1.20) shows that we can use the quantum-gravity Pauli–Jordan D function without extending to any non-abelian versions in order to form integral representations for scalar fields with non-abelian indexes. This fact implies that we can also use the tensorial q-number commutator function or the spinorial q-number anti-commutator functions [9] without extending to any non-abelian versions in order to form integral representations for vector fields or spinor ones with non-abelian indexes.

The purpose of the present paper is to give integral representations for matter fields with non-abelian indexes in quantum Einstein gravity. For this purpose, we treat the gluon, electroweak, their auxiliary, quark, lepton, and Higgs fields with the gravitational interaction, then form their equations as in [7]: we collect all the interaction terms in the right-hand sides of the matter field equations and place the rest in the left-hand ones, because we expect that this formulation enable us to use the above functions.

The present paper is organized as follows. In the next section, we give the Lagrangian densities and the field equations in quantum gravi-chromo-electroweak dynamics; we treat all the matter fields. In Sect. 3, we provide various commutation and anti-commutation relations between the matter-field operators in the Heisenberg picture. In Sect. 4, we present integral representations for these matter fields, using the quantum-gravity Pauli–
Jordan D function, the tensorial q-number commutator function, and the spinorial q-number anti-commutator functions. We show several transformation properties of these representations in Sect. 5. Some remarks are made in the last section.

2. Lagrangian and field equations

In quantum Einstein gravity, all fields interacting with the gravitational field $g_{\mu\nu}(x)$ or the vierbein one $h_\mu{}^a(x)$ ($a = 0, 1, 2, 3$) are regarded as matter ones; the exceptions are the gravitational B-field $b_\rho(x)$, the gravitational Faddeev–Popov ghost fields $c_\rho(x)$ and $\bar{c}_\rho(x)$, the internal Lorentz B-field $s_{ab}(x)$, and the internal Lorentz Faddeev–Popov ghost fields $t^{ab}(x)$ and $\bar{t}_{ab}(x)$. We use Greek small letters for $GL(4)$ indexes, and italic small letters for internal Lorentz ones. On the basis of the standard model of particle physics [10], we deal with three groups of matter fields: non-abelian gauge fields or Yang–Mills fields, 2-component massless spinor ones or Weyl ones, and complex scalar ones or Higgs ones.

We treat both the gravitational field and the matter ones. Therefore, we consider the quantum coupled Einstein–Yang–Mills–Weyl–Higgs system, whose total Lagrangian density is constructed by combining the following two Lagrangian densities. The one contains $g_{\mu\nu}(x)$, the gluon field $A_\mu{}^a(x)$ ($a = 1, 2, \cdots, 8$), the electroweak ones $W_\mu{}^j(x)$ ($j = 1, 2, 3$) and $V_\mu(x)$, and the Higgs one $\Phi^r(x)$ ($r = 1, 2$); the other $h_\mu{}^a(x)$, the left-handed quark ones $\xi_u(x)$ and $\xi_d(x)$, the right-handed quark ones $\eta_u(x)$ and $\eta_d(x)$, the left-handed lepton ones $\xi_n(x)$ and $\xi_e(x)$, and the right-handed lepton ones $\eta_n(x)$ and $\eta_e(x)$.

Here, we use upper roman small letters $a, b, c, \cdots$ for color indexes of the gluon field, and $j, k, l, \cdots$ for weak ones of the electroweak field. All
the quark fields are triplets of the color $SU(3)_C$; their color indexes are always omitted. The left-handed quark, left-handed lepton, and Higgs fields are doublets of the weak $SU(2)_W$; their weak indexes are denoted by upper italic small letters $r, s, t, \ldots$. We deal with two types of quark up and down, and two types of lepton electron-neutrino and electron, for simplicity. We introduce the right-handed neutrino field $\eta_n$ because if it exists then it interacts with the gravitational one; usually, it is not treated within the standard model of particle physics [10].

So, we obtain the matter Lagrangian density as follows:

$$L_M = L_A + L_W + L_V + L_Q + L_L + L_H .$$

(2.1)

In the right-hand side of this equation, the 1st, 2nd, and 3rd terms are the Lagrangian densities [8, 11, 12] for the gluon and electroweak fields, respectively:

$$L_A = \frac{h}{4} g^{\kappa \mu} g^{\lambda \nu} G_{\kappa \lambda}^a G_{\mu \nu}^a - h g^{\lambda \mu} A_{\mu}^a \partial_{\lambda} B^a + \alpha_C \frac{h}{2} B^a B^a$$

$$- i h g^{\mu \nu} \partial_{\mu} \bar{C}^a \cdot (\partial_{\nu} C^a + \lambda_C f^{abc} A_{\nu}^b C^c) ,$$

(2.2)

$$L_W = \frac{h}{4} g^{\kappa \mu} g^{\lambda \nu} F_{\kappa \lambda}^i F_{\mu \nu}^i - h g^{\lambda \mu} W_{\mu}^i \partial_{\lambda} B^i + \alpha_W \frac{h}{2} B^i B^i$$

$$- i h g^{\mu \nu} \partial_{\mu} \bar{C}^i \cdot (\partial_{\nu} C^i + \lambda_W \varepsilon^{ijkl} W_{\nu}^j C^l) ,$$

(2.3)

$$L_V = - \frac{h}{4} g^{\kappa \mu} g^{\lambda \nu} (\partial_{\nu} V_{\lambda} - \partial_{\lambda} V_{\nu})(\partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu}) - h g^{\lambda \mu} V_{\mu} \partial_{\lambda} B$$

$$+ \alpha_Y \frac{h}{2} B^2 - i h g^{\mu \nu} \partial_{\mu} \bar{C} \cdot \partial_{\nu} C ,$$

(2.4)

with $h \equiv \det h_{\mu}^a$. The Lagrangian density (2.2) relates to the $SU(3)_C$ gauge symmetry; $G_{\kappa \lambda}^a \equiv \partial_{\kappa} A_{\lambda}^a - \partial_{\lambda} A_{\kappa}^a + \lambda_C f^{abc} A_{\kappa}^b A_{\lambda}^c$, the coupling constant $\lambda_C$, the structure constant $f^{abc}$, the B-field $B^a$, the Faddeev–Popov ghost ones $C^a$ and $\bar{C}^a$, and the gauge parameter $\alpha_C$. The Lagrangian
density (2.3) does to the $SU(2)_{W}$ gauge symmetry; $F_{\kappa\lambda}^{j} \equiv \partial_{\kappa}W_{\lambda}^{j}-\partial_{\lambda}W_{\kappa}^{j}+\lambda_{W}e^{ijkl}W_{\kappa}^{k}W_{\lambda}^{l}$, $\lambda_{W}$, $e^{ijkl}$, $B^{i}$, $C^{j}$ and $\bar{C}^{j}$, and $\alpha_{W}$ are the corresponding ones. Also, the Lagrangian density (2.4) does to the $U(1)_{Y}$ gauge symmetry; $B$, $C$ and $\bar{C}$, and $\alpha_{Y}$ are the corresponding ones.

The 4th term in (2.1) is the Lagrangian density for the quark fields,

$$L_{Q} = \frac{i}{2}h\mu^{\alpha} \left[ \bar{\Psi}_{q} \frac{\hat{\sigma}}{a} \left( \partial_{\mu} + \frac{\hat{S}_{bc}}{2} \omega_{\mu}^{bc} \right) \Psi_{q} - \bar{\Psi}_{q} \left( \frac{\hat{S}_{bc}}{2} \omega_{\mu}^{bc} \right) \sigma_{a} \Psi_{q} \right]$$

$$+ h\mu^{\alpha} \bar{\Psi}_{q} \frac{\hat{\sigma}}{a} \left( \lambda_{C} \frac{t_{a}}{2} A_{\mu}^{a} + \lambda_{W} \frac{j_{W}}{2} W_{\mu}^{j} + \lambda_{Y} \frac{1}{6} V_{\mu} \right) \Psi_{q}$$

$$+ \frac{i}{2} h\mu^{\alpha} \sum_{f=u,d} \left[ \eta_{f}^{\dagger} \frac{\hat{\sigma}}{a} \left( \partial_{\mu} + \frac{\hat{S}_{bc}}{2} \omega_{\mu}^{bc} \right) \eta_{f} - \eta_{f}^{\dagger} \left( \frac{\hat{S}_{bc}}{2} \omega_{\mu}^{bc} \right) \sigma_{a} \eta_{f} \right]$$

$$+ h \left( G_{u} \bar{\Psi}_{q} \hat{\Phi}_{\eta_{u}} + G_{d} \bar{\Psi}_{q} \hat{\Phi}_{\eta_{d}} + G_{u}^{*} \eta_{u}^{\dagger} \bar{\Phi}^{\dagger} \Psi_{q} + G_{d}^{*} \eta_{d}^{\dagger} \bar{\Phi}^{\dagger} \Psi_{q} \right).$$

(2.5)

Here, $\Psi_{q}$ is a doublet of the weak $SU(2)_{W}$ whose components are the left-handed quark fields,

$$\Psi_{q} \equiv \begin{pmatrix} \xi_{u} \\ \xi_{d} \end{pmatrix},$$

(2.6)

$\sigma_{a}$ and $\hat{\sigma}_{a}$ consist of the unit matrix $\sigma_{0}$ and the Pauli spin matrices,

$$(\sigma_{a})_{AB} \equiv (\sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3})_{AB},$$

(2.7)

$$(\hat{\sigma}_{a})^{A\hat{B}} \equiv (\sigma_{0}, -\sigma_{1}, -\sigma_{2}, -\sigma_{3})^{A\hat{B}},$$

(2.8)

and these give two matrices [9],

$$\left( \hat{S}_{ab} \right)_{A}^{B} \equiv \frac{1}{4} \left( \sigma_{a} \sigma_{b} - \sigma_{b} \sigma_{a} \right)_{A}^{B},$$

(2.9)

$$\left( \hat{S}_{ab} \right)^{A}_{\hat{B}} \equiv \frac{1}{4} \left( \sigma_{a} \sigma_{b} - \sigma_{b} \sigma_{a} \right)^{A}_{\hat{B}} = -\left[ \left( \hat{S}_{ab} \right)^{\dagger} \right]^{A}_{\hat{B}}.$$
We use capital italic letters $A, B, {\dot{A}}, {\dot{B}}$ for Weyl spinor indexes, and raise or lower them by
\[
\epsilon^{AB} = \epsilon_{AB} = \epsilon^{{\dot{A}}{{\dot{B}}} = \epsilon_{{\dot{A}}{\dot{B}}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .
\]
(2.11)

The symbol $\omega^{ab}_\mu$ denotes the spin connection [6] given by
\[
\omega^{ab}_\mu \equiv \frac{1}{2} [ h^{\rho a} (\partial_\mu h^b_\rho - \partial_\rho h^b_\mu) - h^{\rho b} (\partial_\mu h^a_\rho - \partial_\rho h^a_\mu) \\
+ h^c_\mu h^{a \sigma b} (\partial_\sigma h_{\rho c} - \partial_\rho h_{\sigma c}) ] .
\]
(2.12)

In (2.5), $\frac{t^c}{2}$ $(a = 1, 2, \cdots, 8)$ is the generator of $SU(3)_C$, $\frac{\tau^j_W}{2}$ $(j = 1, 2, 3)$ the one of $SU(2)_W$, and $\lambda_Y$ the coupling constant with respect to the $U(1)_Y$ gauge symmetry. In the last term of (2.5), the Yukawa coupling constants $G_u$ and $G_d$ of the Higgs and quark fields are complex-valued, and $\Phi$ and $\bar{\Phi}$ are doublets of the weak $SU(2)_W$.
\[
\Phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \quad \bar{\Phi} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^* = \begin{pmatrix} \phi^{0\ast} \\ -\phi^{\ast\ast} \end{pmatrix} ,
\]
(2.13)
respectively. We omit the Weyl spinor indexes and the weak ones of the quark and Higgs fields in (2.5) for simplicity.

The 5th term in (2.1) is the Lagrangian density for the lepton fields,
\[
\mathcal{L}_L = \frac{i}{2} h h^{\mu a} \left[ \bar{\psi}_1^a \sigma_a \left( \partial_\mu + \frac{\partial_\rho}{2} \omega^\rho_\mu \right) \psi_1 - \bar{\psi}_1^a \left( \frac{\partial_\mu}{2} - \frac{\partial_\sigma h_{\rho c}}{2} \right) \sigma_a \psi_1 \right] \\
+ h h^{\mu a} \bar{\psi}_1^a \sigma_a \left( \lambda_W \frac{\tau^j_W}{2} W^j_\mu - \lambda_Y \frac{1}{2} V_\mu \right) \psi_1 \\
+ \frac{i}{2} h h^{\mu a} \sum_{f=n,e} \left[ \eta_f^a \sigma_a \left( \partial_\mu + \frac{\partial_\rho}{2} \omega^\rho_\mu \right) \eta_f - \eta_f^a \left( \frac{\partial_\mu}{2} - \frac{\partial_\sigma h_{\rho c}}{2} \right) \sigma_a \eta_f \right] \\
- h h^{\mu a} \eta_e^a \lambda_Y V_\mu \eta_e \\
- h (G_n \bar{\psi}_1^a \eta_n + G_e \bar{\psi}_1^a \eta_e + G_n^* \eta_n^a \bar{\Phi} \psi_1 + G_e^* \eta_e^a \bar{\Phi} \psi_1) .
\]
(2.14)
Here, $\Psi_1$ is a doublet of the weak $SU(2)_W$ whose components are the left-handed lepton fields,

$$\Psi_1 \equiv \begin{pmatrix} \xi_n \\ \xi_e \end{pmatrix}, \quad (2.15)$$

the Yukawa coupling constants $G_n$ and $G_e$ of the Higgs and lepton fields are complex-valued. As in (2.5), we omit the Weyl spinor and weak indexes.

The last term in (2.1) is the Lagrangian density for the Higgs field,

$$\mathcal{L}_H = h g^{\mu\nu} \left[ \left( \partial_\mu - i \lambda_W \frac{\tau^j_W}{2} W^j_\mu - i \frac{\lambda_Y}{2} V_\mu \right) \Phi \right]^\dagger \times \left[ \left( \partial_\nu - i \lambda_W \frac{\tau^k_W}{2} W^k_\nu - i \frac{\lambda_Y}{2} V_\nu \right) \Phi \right] + h \left[ \mu_H^2 \Phi^\dagger \Phi - \lambda_H (\Phi^\dagger \Phi)^2 \right], \quad (2.16)$$

where $\mu_H^2 > 0$ and $\lambda_H > 0$; we omit the weak indexes of the Higgs field.

The matter Lagrangian density (2.1) yields the equations for the Yang–Mills, their auxiliary, Weyl, and Higgs fields.

For the gluon field $A_\mu^a$, and for its auxiliary ones $B^a$, $C^a$, and $\bar{C}^a$, we have

$$\partial_\kappa \left[ h (g^{\kappa\mu} g^{\lambda\nu} - g^{\kappa\nu} g^{\lambda\mu}) \partial_\mu A_\nu^a \right] = -h g^{\lambda\nu} \partial_\nu B^a$$

and

$$\partial_\lambda (h g^{\lambda\mu} A_\mu^a) + \alpha_C h B^a = 0, \quad (2.19)$$

where

$$\lambda_C J^a_\lambda \equiv \frac{\partial \mathcal{L}_Q}{\partial A_\lambda^a} = \lambda_C h h^\lambda_a \sum_{f=u,d} \sum_{a} \left( \xi_f^\dagger \sigma_a \frac{t^a_C}{2} \xi_f + \eta_f^\dagger \sigma_a \frac{t^a_C}{2} \eta_f \right), \quad (2.18)$$
\[ \partial_\lambda (h g^{\lambda \mu} \partial_\mu B^a) = -\lambda_C f^{abc} h g^{\lambda \mu} [A_\lambda^b \partial_\mu B^c - i \bar{e}_\lambda \tilde{C}^b \cdot (\partial_\mu C^c + \lambda_C f^{cde} A_\mu^d C^e)] , \]  
\[ \text{ (2.20)} \]

\[ \partial_\lambda (h g^{\lambda \mu} \partial_\mu C^a) = -\lambda_C f^{abc} \partial_\lambda (h g^{\lambda \mu} A_\mu^b C^c) , \]  
\[ \text{ (2.21)} \]

\[ \partial_\lambda (h g^{\lambda \mu} \partial_\mu \tilde{C}^a) = -\lambda_C f^{abc} A_\lambda^b h g^{\lambda \mu} \partial_\mu \tilde{C}^c . \]  
\[ \text{ (2.22)} \]

In the right-hand sides of (2.17), (2.20)–(2.22), we place only the terms proportional to \( \lambda_C \). Using (2.17), (2.20), (2.22), and

\[ f^{abe} f^{ecd} + f^{bce} f^{ead} + f^{cae} f^{ebd} = 0 , \]  
\[ \text{ (2.23)} \]

we obtain the four-divergence of (2.18) as follows:

\[ \partial_\lambda J^a_C = -\lambda_C f^{abc} A_\lambda^b J^c_C . \]  
\[ \text{ (2.24)} \]

For the electroweak fields \( W^j_\mu \) and \( V_\mu \), and for their auxiliary ones, we have

\[ \partial_\kappa [h (g^{\kappa \mu} g^{\lambda \nu} - g^{\kappa \nu} g^{\lambda \mu}) \partial_\mu W^j_\nu] - h g^{\lambda \nu} \partial_\nu B^j \]

\[ = -\lambda_W \{ J^j_W + e^{ikl} [\partial_\kappa (h g^{\kappa \mu} g^{\lambda \nu} W^k_\mu W^{l}_\nu) + h g^{\kappa \mu} g^{\lambda \nu} W^k_\nu F^{l}_\mu \]

\[ + i h g^{\lambda \nu} \partial_\nu \tilde{C}^k \cdot C^l] \} , \]  
\[ \text{ (2.25)} \]

with

\[ \lambda_W J^j_W \equiv \frac{\partial}{\partial W^j_\lambda} (\mathcal{L}_Q + \mathcal{L}_L + \mathcal{L}_H) \]

\[ = \lambda_W h \left[ h^\lambda \left( \bar{\Psi}_q^\dagger \tilde{\sigma} a \frac{\tau^j_W}{2} \Psi_q + \bar{\Psi}_l^\dagger \tilde{\sigma} a \frac{\tau^j_W}{2} \Psi_l \right) \right. \]

\[ \left. + i g^{\lambda \mu} \left( \Phi^\dagger \frac{\tau^j_W}{2} \partial_\mu \Phi - \Phi^\dagger \frac{\tau^j_W}{2} \cdot \Phi \right) \right. \]

\[ \left. + g^{\lambda \mu} \Phi^\dagger \left( \frac{\lambda_W}{2} W^j_\mu + \lambda_Y V_\mu \frac{\tau^j_W}{2} \right) \Phi \right] , \]  
\[ \text{ (2.26)} \]
\[
\partial_\kappa [h(g^{\kappa \mu} g^{\lambda \nu} - g^{\kappa \nu} g^{\lambda \mu}) \partial_\mu V_\nu] - h g^{\lambda \nu} \partial_\nu B = -\lambda_Y J_\lambda^Y, \quad (2.27)
\]

with
\[
\lambda_Y J_\lambda^Y \equiv \frac{\partial}{\partial V_\lambda} (\mathcal{L}_Q + \mathcal{L}_L + \mathcal{L}_H)
= \lambda_Y h \left[ h^a \left( \frac{1}{6} \psi_q^\dagger \vec{\sigma}_a \psi_q + \frac{2}{3} \eta_u^\dagger \vec{\sigma}_a \eta_u - \frac{1}{3} \eta_d^\dagger \vec{\sigma}_a \eta_d \\
- \frac{1}{2} \psi_1^\dagger \vec{\sigma}_a \psi_1 - \eta_e^\dagger \vec{\sigma}_a \eta_e \right)
+ \frac{i}{2} g^{\lambda \mu} (\Phi^\dagger \partial_\mu \Phi - \Phi^\dagger \vec{\sigma}_a \cdot \Phi)
+ g^{\lambda \mu} \Phi^\dagger \left( \lambda_W \frac{\tau^k_\mu}{2} W^k_\mu + \frac{\lambda_Y}{2} V_\mu \right) \Phi \right],
\quad (2.28)
\]

and also,
\[
\partial_\lambda (h g^{\lambda \mu} W^j_\mu) + \alpha_W h B^j = 0, \quad (2.29)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu B^j)
= -\lambda_W e^{ijkl} h g^{\lambda \mu} [W^k_\lambda \partial_\mu B^l - i \partial_\lambda \vec{C}^k \cdot (\partial_\mu C^l + \lambda_W e^{lmn} W^m_\mu C^n)], \quad (2.30)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu C^j) = -\lambda_W e^{ijkl} \partial_\lambda (h g^{\lambda \mu} W^k_\mu C^l), \quad (2.31)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu \vec{C}^j) = -\lambda_W e^{ijkl} W^k_\lambda h g^{\lambda \mu} \partial_\mu \vec{C}^l, \quad (2.32)
\]
\[
\partial_\lambda (h g^{\lambda \mu} V_\mu) + \alpha_Y h B = 0, \quad (2.33)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu B) = 0, \quad (2.34)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu C) = 0, \quad (2.35)
\]
\[
\partial_\lambda (h g^{\lambda \mu} \partial_\mu \vec{C}) = 0. \quad (2.36)
\]

As in the gluon field equation (2.17), we place only the terms proportional to \(\lambda_W\) or \(\lambda_Y\) in the right-hand sides of (2.25), (2.27), (2.30)–(2.32). Using (2.25), (2.27), (2.30), (2.32), (2.34), and
\[
e^{jkn} e^{nlm} + e^{kln} e^{njm} + e^{ljn} e^{nkm} = 0, \quad (2.37)
\]
we obtain the four-divergences of (2.26) and (2.28) as follows:

\[ \partial_\lambda J_\lambda^\lambda = -\lambda_W e^{ikl} W_\lambda^k J_\lambda^l, \]

(2.38)

\[ \partial_\lambda J_\lambda^\gamma = 0. \]

(2.39)

For the quark fields, we have

\[ i h h^\mu a \bar{\sigma}_a \left( \partial_\mu + \frac{\hat{S}_{bc} \omega_{bc}^\mu}{2} \right) \Psi_q = -h h^\mu a \bar{\sigma}_a \left( \lambda_C t^a c A_\mu^a + \frac{\lambda_W \tau^j W_\mu^j + \lambda_Y \frac{1}{6} V_\mu}{2} \right) \Psi_q + h (G_u \tilde{\Phi}_u + G_d \Phi_d), \]

(2.40)

\[ i h h^\mu a \bar{\sigma}_a \left( \partial_\mu + \frac{\hat{S}_{bc} \omega_{bc}^\mu}{2} \right) \eta_u = -h h^\mu a \bar{\sigma}_a \left( \lambda_C t^a c A_\mu^a + \lambda_Y \frac{2}{3} V_\mu \right) \eta_u + h G_u^* \tilde{\Phi}^\dagger \Psi_q, \]

(2.41)

\[ i h h^\mu a \bar{\sigma}_a \left( \partial_\mu + \frac{\hat{S}_{bc} \omega_{bc}^\mu}{2} \right) \eta_d = -h h^\mu a \bar{\sigma}_a \left( \lambda_C t^a c A_\mu^a - \lambda_Y \frac{1}{3} V_\mu \right) \eta_d + h G_d^* \Phi^\dagger \Psi_q. \]

(2.42)

We place only the terms proportional to \( \lambda_C, \lambda_W, \lambda_Y, G_u, G_d, G_u^*, \) or \( G_d^* \) in the right-hand sides of these equations.

For the lepton fields, we have

\[ i h h^\mu a \bar{\sigma}_a \left( \partial_\mu + \frac{\hat{S}_{bc} \omega_{bc}^\mu}{2} \right) \Psi_l = -h h^\mu a \bar{\sigma}_a \left( \lambda_W \frac{\tau^j W_\mu^j - \lambda_Y \frac{1}{2} V_\mu}{2} \right) \Psi_l + h (G_n \tilde{\Phi}_n + G_e \Phi_e), \]

(2.43)

\[ i h h^\mu a \bar{\sigma}_a \left( \partial_\mu + \frac{\hat{S}_{bc} \omega_{bc}^\mu}{2} \right) \eta_n = h G_n^* \tilde{\Phi}^\dagger \Psi_l, \]

(2.44)
\[ i \hbar h^{\mu a} \sigma_a \left( \partial_\mu + \frac{\mathcal{S}_{bc}}{2} \omega_\mu^{bc} \right) \eta_e = h h^{\mu a} \sigma_a \lambda_Y V_\mu \eta_e + h G_e^* \Phi^r \Psi_1. \] (2.45)

The right-hand sides of these equations are sums of the terms proportional to \( \lambda_W, \lambda_Y, G_u, G_e, G_n^*, \) or \( G_e^*. \)

We have the Higgs field equation as follows:

\[
\begin{align*}
\partial_\mu (h g^{\mu \nu} \partial_\nu \Phi) &= h (\mu_H^2 - 2 \lambda_H \Phi^r \Phi) + i \partial_\mu \left[ h g^{\mu \nu} \left( \lambda_W \frac{\tau^j_W}{2} W_\nu^j + \frac{\lambda_Y}{2} V_\nu \right) \Phi \right] \\
&\quad + i h g^{\mu \nu} \left( \lambda_W \frac{\tau^j_W}{2} W_\mu^j + \frac{\lambda_Y}{2} V_\mu \right) \cdot \left( \partial_\nu - i \lambda_W \frac{\tau^k_W}{2} W_\nu^k - i \frac{\lambda_Y}{2} V_\nu \right) \Phi \\
&\quad - h [G_u \tilde{\Psi}_q \eta_u + G^*_u \eta_d \tilde{\Psi}_q + G_n \tilde{\Psi}_l \eta_n + G^*_n \eta_e \tilde{\Psi}_l],
\end{align*}
\] (2.46)

with

\[
\begin{align*}
\tilde{\Psi}_q &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_u^\dagger \\ \xi_d \end{pmatrix} = \begin{pmatrix} -\xi_u^\dagger \\ \xi_d \end{pmatrix}, \\
\tilde{\Psi}_l &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_n^\dagger \\ \xi_e \end{pmatrix} = \begin{pmatrix} -\xi_e^\dagger \\ \xi_n \end{pmatrix}.
\end{align*}
\] (2.47) (2.48)

The right-hand side of (2.46) is a sum of the terms proportional to \( \mu_H^2, \lambda_H, \lambda_W, \lambda_Y, G_u, G_d^*, G_n, \) or \( G_e^*. \)

### 3. Commutation and anti-commutation relations

We introduce the matter-field operators in the Heisenberg picture, and present various commutation and anti-commutation relations.

#### 3.1. Canonical variables

In order to adopt \( A_\mu^a, C^a, \bar{C}^a, W_\mu^j, C^j, \bar{C}^j, V_\mu, C, \bar{C}, \Psi_q^r, \eta_u, \eta_d, \Psi_l^r, \eta_n, \eta_e, \) and \( \Phi^r \) as the canonical variables, we replace \( \mathcal{L}_M \) in (2.1) by
\[ \mathcal{L}_M \equiv \mathcal{L}_M + \partial_\lambda \left[ h g^{\lambda \mu} (A^a_\mu B^a + W^{a}_\mu B^a + V_\mu B) \right. \]
\[ \left. + \frac{i}{2} \ h h^{\lambda a} \left( \psi_q^\dagger \ \bar{\sigma}_a \ \psi_q + \psi_l^\dagger \ \bar{\sigma}_a \ \psi_l + \sum_{f=u,d,n,e} \eta_f^\dagger \ \bar{\sigma}_a \ \eta_f \right) \right], \]

with the use of

\[ \partial_\mu (h h^{\mu a}) \cdot \bar{\sigma}_a = h h^{\mu a} \left( \bar{\sigma}_a \ \frac{S_{bc} \omega_{bc}}{2} - \frac{\bar{S}_{bc} \omega_{bc}}{2} \bar{\sigma}_a \right), \] (3.2)
\[ \partial_\mu (h h^{\mu a}) \cdot \bar{\sigma}_a = h h^{\mu a} \left( \bar{\sigma}_a \ \frac{S_{bc} \omega_{bc}}{2} - \frac{\bar{S}_{bc} \omega_{bc}}{2} \bar{\sigma}_a \right), \] (3.3)

and of

\[ \bar{\sigma}_a \bar{S}_{bc} - \bar{S}_{bc} \bar{\sigma}_a = \eta_{ab} \ \bar{\sigma}_c - \eta_{ac} \ \bar{\sigma}_b, \] (3.4)
\[ \bar{\sigma}_a \bar{S}_{bc} - \bar{S}_{bc} \bar{\sigma}_a = \eta_{ab} \ \bar{\sigma}_c - \eta_{ac} \ \bar{\sigma}_b. \] (3.5)

The canonical conjugates of the above Yang–Mills fields and the related Faddeev–Popov ghost ones are defined by

\[ \pi_{A^\lambda a} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial A^\lambda_a} = -h g^{0\mu} g^{\lambda \nu} G^{a}_{\mu \nu} + h g^{0\lambda} B^a, \] (3.6)
\[ \pi_{C^a} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial C^a} = i h g^{0\mu} \partial_\mu C^a, \] (3.7)
\[ \pi_{\bar{C}^a} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial \bar{C}^a} = -i h g^{0\nu} (\partial_\nu C^a + \lambda_C f^{abc} A^b_\nu C^c), \] (3.8)
\[ \pi_{W^{\lambda j}} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial W^{\lambda j}_j} = -h g^{0\mu} g^{\lambda \nu} F^{\mu \nu} + h g^{0\lambda} B^j, \] (3.9)
\[ \pi_{\bar{C}^j} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial \bar{C}^j} = i h g^{0\nu} \partial_\nu \bar{C}^j, \] (3.10)
\[ \pi_{C^j} \equiv \frac{\partial \tilde{\mathcal{L}}_M}{\partial C^j} = -i h g^{0\nu} (\partial_\nu C^j + \lambda_W \epsilon^{jkl} W^{k \lambda \nu} C^l), \] (3.11)
\[
\pi_V^\lambda \equiv \frac{\partial \tilde{L}_M}{\partial V^\lambda} = -h g^{0\mu} \epsilon^\lambda_{\nu} (\partial_\mu V_\nu - \partial_\nu V_\mu) + h g^{0\lambda} B, \tag{3.12}
\]

\[
\pi_C \equiv \frac{\partial \tilde{L}_M}{\partial C} = i h g^{\mu 0} \partial_\mu \bar{C}, \tag{3.13}
\]

\[
\pi_{\bar{C}} \equiv \frac{\partial \tilde{L}_M}{\partial \bar{C}} = -i h g^{0\nu} \partial_\nu C, \tag{3.14}
\]

respectively. Here, the functional derivatives with respect to the Faddeev–Popov ghost fields are made from the left of all operands. For the above Weyl fields, their canonical conjugates are defined by

\[
\pi_q^r \equiv \frac{\partial \tilde{L}_M}{\partial \Psi_q^r} = -i h h^{0a} (\Psi_q^\dagger)^r \bar{\sigma}_a, \tag{3.15}
\]

\[
\pi_u \equiv \frac{\partial \tilde{L}_M}{\partial \eta_u} = -i h h^{0a} \eta_u \bar{\sigma}_a, \tag{3.16}
\]

\[
\pi_d \equiv \frac{\partial \tilde{L}_M}{\partial \eta_d} = -i h h^{0a} \eta_d \bar{\sigma}_a, \tag{3.17}
\]

\[
\pi_l^r \equiv \frac{\partial \tilde{L}_M}{\partial \Psi_l^r} = -i h h^{0a} (\Psi_l^\dagger)^r \bar{\sigma}_a, \tag{3.18}
\]

\[
\pi_n \equiv \frac{\partial \tilde{L}_M}{\partial \eta_n} = -i h h^{0a} \eta_n \bar{\sigma}_a, \tag{3.19}
\]

\[
\pi_e \equiv \frac{\partial \tilde{L}_M}{\partial \eta_e} = -i h h^{0a} \eta_e \bar{\sigma}_a, \tag{3.20}
\]

respectively. The functional derivatives with respect to the Weyl fields are made from the left of all operands. In addition, the canonical conjugate of the Higgs field is defined by

\[
\pi_\Phi \equiv \frac{\partial \tilde{L}_M}{\partial \Phi} = h g^{0\nu} \left[ \Phi^\dagger \left( \frac{\partial}{\partial \nu} + i \lambda W^j \frac{\tau^j}{2} W_\nu \right) + i \frac{\lambda}{2} V_\nu \right]^r. \tag{3.21}
\]

The equal-time canonical commutation and anti-commutation relations are set as follows:

\[
[\pi_A^{\lambda a}, A^b_\mu] = -i \delta^{ab} \delta_\mu^\lambda \delta^3, \tag{3.22}
\]
\[ \{ \pi_{C^a}, C^{b'} \} = -i \delta^{ab} \delta^3, \quad (3.23) \]
\[ \{ \pi_{\bar{C}^a}, \bar{C}^{b'} \} = -i \delta^{ab} \delta^3, \quad (3.24) \]
\[ [\pi_{W^\lambda_j}, W_{\mu}^{k'}] = -i \delta^{jk} \delta^\lambda \delta^\mu \delta^3, \quad (3.25) \]
\[ \{ \pi_{C^j}, C^{k'} \} = -i \delta^{jk} \delta^3, \quad (3.26) \]
\[ \{ \pi_{\bar{C}^j}, \bar{C}^{k'} \} = -i \delta^{jk} \delta^3, \quad (3.27) \]
\[ [\pi_{V^\lambda}, V_{\mu}'] = -i \delta^\lambda \delta^\mu \delta^3, \quad (3.28) \]
\[ \{ \pi_{C}, C' \} = -i \delta^3, \quad (3.29) \]
\[ \{ \pi_{\bar{C}}, \bar{C}' \} = -i \delta^3, \quad (3.30) \]
\[ \{ \pi_{q^r}, \Psi_{q'} \} = -i \delta^{rs} \delta^3, \quad (3.31) \]
\[ \{ \pi_{u}, \eta'_{u} \} = -i \delta^3, \quad (3.32) \]
\[ \{ \pi_{d}, \eta'_{d} \} = -i \delta^3, \quad (3.33) \]
\[ \{ \pi_{l^r}, \Psi_1^{s'} \} = -i \delta^{rs} \delta^3, \quad (3.34) \]
\[ \{ \pi_{n}, \eta'_{n} \} = -i \delta^3, \quad (3.35) \]
\[ \{ \pi_{e}, \eta'_{e} \} = -i \delta^3, \quad (3.36) \]
\[ [\pi_{\Phi^r}, \Phi^{s'}] = -i \delta^{rs} \delta^3. \quad (3.37) \]

In the above and the next subsection, a prime attached to a spacetime function means that its argument is not \( x^\lambda \) but \( z^\lambda \) where it is understood that \( x^0 = z^0 \).

### 3.2. Between matter fields

Using the field equations, the canonical conjugates, and the equal-time (anti-)commutation relations, we have various (anti-)commutation relations.
For the gluon and its auxiliary fields, we obtain

\[
[A^a_{\mu}, B^{b'}] = i\delta^{ab} \delta^0 \frac{\delta^3}{h g^{00}},
\]
(3.38)

\[
[A^a_{\mu}, A^{b'}_\nu] = i\delta^{ab} \left[ g_{\mu\nu} - (1 - \alpha_C) \frac{\delta^0 \delta^0}{g^{00}} \right] \frac{\delta^3}{h g^{00}},
\]
(3.39)

\[
[\dot{B}^a, A^{b'}_\nu] = i\delta^{ab} \left[ \partial_\nu \left( \frac{1}{h g^{00}} \right) \cdot \delta^3 - \left( \delta^m_\nu - 2\delta^0 \frac{g^m_0}{g^{00}} \right) \partial^x_m \left( \frac{\delta^3}{h g^{00}} \right) \right] + i\lambda_C f^{abc} A^c_m \left( \delta^m_\nu - \delta^0 \frac{g^m_0}{g^{00}} \right) \frac{\delta^3}{h g^{00}},
\]
(3.40)

\[
[\dot{B}^a, B^{b'}] = 0,
\]
(3.41)

\[
[\dot{B}^a, C^{b'}] = i\lambda_C f^{abc} C^c \frac{\delta^3}{h g^{00}},
\]
(3.42)

\[
[\dot{C}^a, \bar{C}^{b'}] = 0,
\]
(3.43)

\[
\{ \dot{C}^a, \bar{C}^{b'} \} = \delta^{ab} \frac{\delta^3}{h g^{00}},
\]
(3.44)

\[
[\dot{B}^a, \Psi^{r'}] = -\lambda_C \frac{t^a_C}{2} \Psi^r \frac{\delta^3}{h g^{00}},
\]
(3.45)

\[
[\dot{B}^a, \eta_u'] = -\lambda_C \frac{t^a_C}{2} \eta_u \frac{\delta^3}{h g^{00}},
\]
(3.46)

\[
[\dot{B}^a, \eta_d'] = -\lambda_C \frac{t^a_C}{2} \eta_d \frac{\delta^3}{h g^{00}}.
\]
(3.47)

In the right-hand side of (3.40), the argument \(x\) of each field is omitted for simplicity.

For the electroweak and their auxiliary fields, we obtain

\[
[W^j_\mu, B^{k'}] = i\delta^{jk} \delta^0 \frac{\delta^3}{h g^{00}},
\]
(3.48)

\[
[\dot{W}^j_\mu, W^{k'}_\nu] = i\delta^{jk} \left[ g_{\mu\nu} - (1 - \alpha_W) \frac{\delta^0 \delta^0}{g^{00}} \right] \frac{\delta^3}{h g^{00}},
\]
(3.49)
\[ [\dot{B}^i, W_{\nu}^{kn}] = i \delta^{jk} \left[ \partial_{\nu} \left( \frac{1}{h g^{00}} \right) \cdot \delta^3 - \left( \delta^m_{\nu} - 2 \delta^0_{\nu} \frac{g^{0m}}{g^{00}} \right) \partial^x_m \left( \frac{\delta^3}{h g^{00}} \right) \right] \\
+ i \lambda W e^{jkl} W^{-1}_m \left( \delta^m_{\nu} - \delta^0_{\nu} \frac{g^{0m}}{g^{00}} \right) \frac{\delta^3}{h g^{00}}, \tag{3.50} \]

\[ [\dot{B}^i, B^{kn}] = \frac{i}{2} \delta^{jk} \lambda^2 \Phi^\dagger \Phi \frac{\delta^3}{h g^{00}}, \tag{3.51} \]

\[ [\dot{B}^i, C^{kn}] = i \lambda W e^{jkl} C^l \frac{\delta^3}{h g^{00}}, \tag{3.52} \]

\[ [\dot{B}^i, \bar{C}^{kn}] = 0, \tag{3.53} \]

\[ \{ \dot{C}^j, \bar{C}^{kn} \} = \delta^{jk} \frac{\delta^3}{h g^{00}}, \tag{3.54} \]

\[ [\dot{B}^i, \Psi^{q\nu}] = - \lambda W \left( \frac{r_i^j \Psi_q}{2} \right)^r \frac{\delta^3}{h g^{00}}, \tag{3.55} \]

\[ [\dot{B}^i, \Psi^{l\nu}] = - \lambda W \left( \frac{r_i^j \Psi_l}{2} \right)^r \frac{\delta^3}{h g^{00}}, \tag{3.56} \]

\[ [\dot{B}^i, \eta^f] = 0 \, \text{ (f = u, d, n, e)}, \tag{3.57} \]

\[ [\dot{B}^i, \Phi^{r\nu}] = - \lambda W \left( \frac{r_i^j \Phi_r}{2} \right)^r \frac{\delta^3}{h g^{00}}, \tag{3.58} \]

\[ [V_{\mu}, B^i] = i \delta^{0\mu} \frac{\delta^3}{h g^{00}}, \tag{3.59} \]

\[ [\dot{V}_{\mu}, V^{\nu}] = i \left[ g_{\mu \nu} - (1 - \alpha_Y) \delta_{\mu}^0 \delta_{\nu}^0 \frac{\delta^3}{g^{00}} \right] \frac{\delta^3}{h g^{00}}, \tag{3.60} \]

\[ [\dot{B}, V^{\nu}] = i \left[ \partial_{\nu} \left( \frac{1}{h g^{00}} \right) \cdot \delta^3 - \left( \delta^m_{\nu} - 2 \delta^0_{\nu} \frac{g^{0m}}{g^{00}} \right) \partial^x_m \left( \frac{\delta^3}{h g^{00}} \right) \right], \tag{3.61} \]

\[ [\dot{B}, B^i] = \frac{i}{2} \lambda^2 \Phi^\dagger \Phi \frac{\delta^3}{h g^{00}}, \tag{3.62} \]

\[ [\dot{B}, C^i] = 0, \tag{3.63} \]

\[ [\dot{B}, \bar{C}^i] = 0, \tag{3.64} \]

\[ \{ \dot{C}, \bar{C}^i \} = \frac{\delta^3}{h g^{00}}, \tag{3.65} \]
\[ [\dot{B}, \Psi^r_q] = -\lambda_Y \frac{1}{6} \Psi^r_q \frac{\delta^3}{hg^{00}}, \quad (3.66) \]
\[ [\dot{B}, \eta_u^r] = -\lambda_Y \frac{2}{3} \eta_u \frac{\delta^3}{hg^{00}}, \quad (3.67) \]
\[ [\dot{B}, \eta_d^r] = \lambda_Y \frac{1}{3} \eta_d \frac{\delta^3}{hg^{00}}, \quad (3.68) \]
\[ [\dot{B}, \Psi^r_l] = \lambda_Y \frac{1}{2} \Psi^r_l \frac{\delta^3}{hg^{00}}, \quad (3.69) \]
\[ [\dot{B}, \eta_e^r] = 0, \quad (3.70) \]
\[ [\dot{B}, \eta_{\nu}^r] = \lambda_Y \eta_{\nu} \frac{\delta^3}{hg^{00}}, \quad (3.71) \]
\[ [\dot{B}, \Phi^r] = -\lambda_Y \frac{1}{2} \Phi^r \frac{\delta^3}{hg^{00}}. \quad (3.72) \]

We apply the de Donder condition [6],
\[ \partial_{\mu}(hg^{\mu\nu}) = 0, \quad (3.73) \]

to (3.39), (3.40), (3.49), (3.50), (3.60), and (3.61).

For the quark, lepton, and Higgs fields, we obtain
\[ \{ \Psi^r_q, (\Psi^t_q)^{s'} \} = \delta^{rs} h^{0a} \frac{\bar{\sigma}_a}{hg^{00}} \frac{\delta^3}{hg^{00}}, \quad (3.74) \]
\[ \{ \Psi^r_l, (\Psi^t_l)^{s'} \} = \delta^{rs} h^{0a} \frac{\bar{\sigma}_a}{hg^{00}} \frac{\delta^3}{hg^{00}}, \quad (3.75) \]
\[ \{ \eta_f, \eta_{\nu}^{t'} \} = h^{0a} \frac{\bar{\sigma}_a}{hg^{00}} \frac{\delta^3}{hg^{00}} \quad (f = u, d, n, e), \quad (3.76) \]
\[ [\dot{\Phi}^r, (\Phi^t)^{s'}] = -i \delta^{rs} \frac{\delta^3}{hg^{00}}. \quad (3.77) \]

### 3.3. BRST charges and matter fields

In our system described by (3.1), there are five types of BRST symmetry: the gravitational, internal Lorentz, \( SU(3)_C \), \( SU(2)_W \), and \( U(1)_Y \) BRST
symmetries. The corresponding BRST charges are defined as follows. The gravitational BRST charge [6] is

$$Q_G \equiv \int d^3x h g^{0\nu}(b_\mu \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho).$$  (3.78)

The internal Lorentz BRST charge [6] is

$$Q_L \equiv \int d^3x h g^{0\nu}[s_{ab}(D_\nu t)^{ab} - \partial_\nu s_{ab} \cdot t^{ab} + i\partial_\nu \bar{t}_{ab} \cdot t^{bc}t^{a}c],$$  (3.79)

with

$$(D_\mu t)^{ab} \equiv \partial_\mu t^{ab} + \omega^{ac}_\mu t^{b}c - \omega^{bc}_\mu t^{a}c.$$  (3.80)

Here, $s_{ab}$, $t^{ab}$ and $\bar{t}_{ab}$ are anti-symmetric with respect to the indexes $a$ and $b$. The $SU(3)_C$ and $SU(2)_W$ BRST charges are the quantum-gravity version of the ones in [6]:

$$Q_C \equiv \int d^3x h g^{0\nu}\left[B^a \partial_\nu C^a - \partial_\nu B^a \cdot C^a + \lambda_C f^{abc} \left(B^a A^b_\nu + \frac{i}{2} \partial_\nu \bar{C}^a \cdot C^b\right)C^c\right],$$  (3.81)

$$Q_W \equiv \int d^3x h g^{0\nu}\left[B^i \partial_\nu C^i - \partial_\nu B^i \cdot C^i + \lambda_W \epsilon^{ijkl} \left(B^i W^k_\nu + \frac{i}{2} \partial_\nu \bar{C}^j \cdot C^k\right)C^l\right].$$  (3.82)

The $U(1)_Y$ BRST charge [8, 12] is

$$Q_Y \equiv \int d^3x h g^{0\nu}(B \partial_\nu C - \partial_\nu B \cdot C).$$  (3.83)

The (anti-)commutation relations between these BRST charges and the matter fields yield the BRST transformations of them. Using $Q_G$, we have, for example,

$$[iQ_G, A^a_\mu] = -\kappa(\partial_\mu c^\rho \cdot A^a_\rho + c^\rho \partial_\rho A^a_\mu),$$  (3.84)

$$[iQ_G, B^a] = -\kappa c^\rho \partial_\rho B^a,$$  (3.85)
\{ iQ_G, C^a \} = -\kappa c^\rho \partial_\rho C^a, \quad (3.86)
\{ iQ_G, \tilde{C}^a \} = -\kappa c^\rho \partial_\rho \tilde{C}^a, \quad (3.87)
\{ iQ_G, \Psi^r_q \} = -\kappa c^\rho \partial_\rho \Psi^r_q, \quad (3.88)
\{ iQ_G, \Psi_1^r \} = -\kappa c^\rho \partial_\rho \Psi_1^r, \quad (3.89)
\{ iQ_G, \eta_f \} = -\kappa c^\rho \partial_\rho \eta_f \quad (f = u, d, n, e), \quad (3.90)
\{ iQ_G, \eta_f \} = -\kappa c^\rho \partial_\rho \Phi^r, \quad (3.91)

because the gluon and electroweak fields are world vectors [6], and the other matter ones are world scalars. Here \( \kappa \) is Einstein’s gravitational constant.

Using \( Q_L \), we have, for example,

\[ [ iQ_L, A^a_\mu ] = 0, \quad (3.92) \]
\[ [ iQ_L, B^a ] = 0, \quad (3.93) \]
\[ \{ iQ_L, C^a \} = 0, \quad (3.94) \]
\[ \{ iQ_L, \Psi^r_q \} = -\frac{\bar{S}_{ab}}{2} t^{ab} \Psi^r_q, \quad (3.95) \]
\[ \{ iQ_L, \Psi_1^r \} = -\frac{\bar{S}_{ab}}{2} t^{ab} \Psi_1^r, \quad (3.96) \]
\[ \{ iQ_L, \eta_f \} = -\frac{\bar{S}_{ab}}{2} t^{ab} \eta_f \quad (f = u, d, n, e), \quad (3.97) \]
\[ [ iQ_L, \Phi^r ] = 0, \quad (3.98) \]

because the quark and lepton fields are Lorentz spinors [6], and the other matter ones are Lorentz scalars.

Furthermore, using \( Q_C \), \( Q_W \), and \( Q_Y \), we have the following transformations:

\[ [ iQ_C, A^a_\mu ] = \partial_\mu C^a + \lambda_C f^{abc} A^b_\mu C^c, \quad (3.99) \]
\[ [iQ_C, B^a] = 0, \quad (3.100) \]
\[ \{iQ_C, C^a\} = -\frac{1}{2} \lambda_C \epsilon^{abc} C^b C^c, \quad (3.101) \]
\[ \{iQ_C, \bar{C}^a\} = iB^a, \quad (3.102) \]
\[ \{iQ_C, \Psi_q^r\} = i\lambda_C \frac{t^a}{2} C^a \Psi_q^r, \quad (3.103) \]
\[ \{iQ_C, \eta_f\} = i\lambda_C \frac{t^a}{2} C^a \eta_f \quad (f = u, d), \quad (3.104) \]
\[ [iQ_W, W_{\mu}^j] = \partial_\mu C^j + \lambda W e^{jkl} W_{\mu}^k C^l, \quad (3.105) \]
\[ [iQ_W, B^j] = 0, \quad (3.106) \]
\[ \{iQ_W, C^j\} = -\frac{1}{2} \lambda_W e^{jkl} C^k C^l, \quad (3.107) \]
\[ \{iQ_W, \bar{C}^j\} = iB^j, \quad (3.108) \]
\[ \{iQ_W, \Psi_q^r\} = i\lambda_W \left( \frac{\tau^j}{2} C^j \Psi_q^r \right), \quad (3.109) \]
\[ \{iQ_W, \Psi_1^r\} = i\lambda_W \left( \frac{\tau^j}{2} C^j \Psi_1^r \right), \quad (3.110) \]
\[ \{iQ_W, \eta_f\} = 0 \quad (f = u, d, n, e), \quad (3.111) \]
\[ [iQ_W, \Phi^r] = i\lambda_W \left( \frac{\tau^j}{2} C^j \Phi \right)^r, \quad (3.112) \]
\[ [iQ_Y, V_\mu] = \partial_\mu C, \quad (3.113) \]
\[ [iQ_Y, B] = 0, \quad (3.114) \]
\[ \{iQ_Y, C\} = 0, \quad (3.115) \]
\[ \{iQ_Y, \bar{C}\} = iB, \quad (3.116) \]
\[ \{iQ_Y, \Psi_q^r\} = i\lambda_Y \frac{1}{6} C \Psi_q^r, \quad (3.117) \]
\[ \{iQ_Y, \eta_u\} = i\lambda_Y \frac{2}{3} C \eta_u, \quad (3.118) \]
\[ \{iQ_Y, \eta_d\} = -i\lambda_Y \frac{1}{3} C \eta_d, \quad (3.119) \]
\{iQ_Y, \Psi_1^r \} = -i\lambda_Y \frac{1}{2} C \Psi_1^r, \quad (3.120)

\{iQ_Y, \eta_\lambda \} = 0, \quad (3.121)

\{iQ_Y, \eta_e \} = -i\lambda_Y C \eta_e, \quad (3.122)

[ iQ_Y, \Phi^r ] = i\lambda_Y \frac{1}{2} C \Phi^r. \quad (3.123)

These five BRST charges give us the subsidiary conditions,

\begin{align*}
Q_G |_{\text{phys}} &= 0, \\
Q_L |_{\text{phys}} &= 0, \\
Q_C |_{\text{phys}} &= 0, \quad (3.124) \\
Q_W |_{\text{phys}} &= 0, \\
Q_Y |_{\text{phys}} &= 0,
\end{align*}

\noindent to define the physical subspace of the indefinite-metric Hilbert space.

4. Integral representations

In this section, we have integral representations for all the fields interacting with the gravitational one.

4.1. Yang–Mills fields and their auxiliary fields

As in [7], we define the following bilocal currents relating to the gluon field $A_\mu^a(x)$, and to the electroweak ones $W_\mu^j(x)$ and $V_\mu(x)$:

\begin{align*}
\mathcal{J}_\mu^{a\lambda}(x, z) &= D_{\mu\nu}(x, z) \bar{\partial}_{\rho} \cdot h(z) [g^{\lambda \rho}(z)g^{\nu \sigma}(z) - g^{\lambda \nu}(z)g^{\rho \sigma}(z)] A^a_\sigma(z) \\
&\quad - D_{\mu\nu}(x, z) h(z) [g^{\lambda \rho}(z)g^{\nu \sigma}(z) - g^{\lambda \sigma}(z)g^{\nu \rho}(z)] \partial_{\rho} A^a_\sigma(z) \\
&\quad + D_{\mu\nu}(x, z) h(z) g^{\lambda \nu}(z) B^a_\rho(z) - \partial_{\mu} D(x, z) \cdot h(z) g^{\lambda \sigma}(z) A^a_\sigma(z), \quad (4.1)
\end{align*}

\( \mathcal{J}_\mu^\lambda(x, z) \)
\[ \equiv D_{\mu\nu}(x, z) \partial_\sigma \cdot h(z)[g^{\lambda\rho}(z)g^{\nu\sigma}(z) - g^{\lambda\nu}(z)g^{\rho\sigma}(z)]W_\sigma^j(z) \]
\[ - D_{\mu\nu}(x, z)h(z)[g^{\lambda\rho}(z)g^{\nu\sigma}(z) - g^{\lambda\sigma}(z)g^{\nu\rho}(z)]\partial_\rho W_\sigma^j(z) \]
\[ + D_{\mu\nu}(x, z)h(z)g^{\lambda\nu}(z)B_j(z) - \partial_\nu D(x, z) \cdot h(z)g^{\lambda\sigma}(z)W_\sigma^j(z), \] (4.2)

\( J_\mu^\lambda(x, z) \)
\[ \equiv D_{\mu\nu}(x, z)\partial_\rho \cdot h(z)[g^{\lambda\rho}(z)g^{\nu\sigma}(z) - g^{\lambda\nu}(z)g^{\rho\sigma}(z)]V_\sigma(z) \]
\[ - D_{\mu\nu}(x, z)h(z)[g^{\lambda\rho}(z)g^{\nu\sigma}(z) - g^{\lambda\sigma}(z)g^{\nu\rho}(z)]\partial_\rho V_\sigma(z) \]
\[ + D_{\mu\nu}(x, z)h(z)g^{\lambda\nu}(z)B(z) - \partial_\nu D(x, z) \cdot h(z)g^{\lambda\sigma}(z)V_\sigma(z). \] (4.3)

In these right-hand sides, the tensorial q-number commutator function \( D_{\mu\nu}(x, z) \) and the quantum-gravity Pauli–Jordan D function \( D(x, z) \) are defined by the following Cauchy problems [4, 5, 6, 8]:

\[ \partial_\kappa [h(g^{\kappa\lambda}g^{\sigma\mu} - g^{\kappa\mu}g^{\sigma\lambda})\partial_\nu D_{\mu\nu}(x, z)] - hg^{\sigma\tau} \cdot \partial_\tau D(x, z)\partial_\nu \bar{V} = 0, \] (4.4)

\[ \partial_\kappa [h g^{\lambda\mu} D_{\mu\nu}(x, z)] + \alpha h \cdot D(x, z)\partial_\nu \bar{V} = 0, \] (4.5)

\[ D_{\mu\nu}(x, z)|_0 = 0, \] (4.6)

\[ \partial_0^x D_{\mu\nu}(x, z)|_0 = - \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta_\mu^0 \delta_\nu^0}{g^{00}} \right] \frac{\delta^3}{hg^{00}}, \] (4.7)

and

\[ \partial_\mu^x [hg^{\mu\nu} D(x, z)] = 0, \] (4.8)

\[ D(x, z)|_0 = 0, \] (4.9)

\[ \partial_0^x D(x, z)|_0 = - \frac{\delta^3}{hg^{00}}. \] (4.10)

In (4.5) and (4.7), the gauge parameter \( \alpha \) stands for \( \alpha_C, \alpha_W, \) and \( \alpha_Y; \) they relate to \( A_\mu^a, W_\mu^j, \) and \( V_\mu \) via (4.1)–(4.3).

Using the Hermitian conjugates of these functions [5, 8],

\[ [D_{\mu\nu}(x, z)]^\dagger = - D_{\nu\mu}(z, x), \quad [D(x, z)]^\dagger = - D(z, x), \] (4.11)
we obtain

\[ [\mathcal{D}_{\mu\nu}(x, z)\vec{\partial}_{\rho} \cdot (g^\rho\sigma g^{\nu\kappa} - g^{\nu\sigma} g^\rho_\kappa)h] \vec{\partial}_{\sigma} - \partial_{\mu}^{x}\mathcal{D}(x, z)\vec{\partial}_{\sigma} \cdot h g^{\sigma\kappa} = 0 , \]  

\[ [\mathcal{D}_{\mu\nu}(x, z)hg^{\nu\rho}] \vec{\partial}_{\rho} + \alpha \partial_{\mu}^{x}\mathcal{D}(x, z) \cdot h = 0 , \]  

\[ \mathcal{D}_{\mu\nu}(x, z)\vec{\partial}_0 |_{0} = \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta_{\mu}^{0}\delta_{\nu}^{0}}{g_{00}} \right] \frac{\delta^3}{h g_{00}} , \]  

and

\[ [\mathcal{D}(x, z)\vec{\partial}_{\nu} \cdot hg^{\nu\rho}] \vec{\partial}_{\rho} = 0 , \]  

\[ \mathcal{D}(x, z)\vec{\partial}_0 |_{0} = \frac{\delta^3}{h g_{00}} . \]  

The four-divergences of the bilocal currents (4.1)–(4.3) with respect to \( z \) are given by

\[ J_{\mu}^{a\lambda}(x, z)\vec{\partial}_{\lambda}^{z} \]  

\[ = -\mathcal{D}_{\mu\nu}(x, z)\{ \partial_{\lambda}^{z}[h(\epsilon^{\lambda\rho} g^{\nu\sigma} - g^{\lambda\sigma} g^{\nu\rho})\partial_{\rho} A_{\sigma}^{a}] - hg^{\nu\lambda} \partial_{\lambda}^{z} B^{a} \} , \]  

\[ J_{\mu}^{j\lambda}(x, z)\vec{\partial}_{\lambda}^{z} \]  

\[ = -\mathcal{D}_{\mu\nu}(x, z)\{ \partial_{\lambda}^{z}[h(\epsilon^{\lambda\rho} g^{\nu\sigma} - g^{\lambda\sigma} g^{\nu\rho})\partial_{\rho} W_{\sigma}^{j}] - hg^{\nu\lambda} \partial_{\lambda}^{z} B^{j} \} , \]  

\[ J_{\mu}^{\lambda}(x, z)\vec{\partial}_{\lambda}^{z} \]  

\[ = -\mathcal{D}_{\mu\nu}(x, z)\{ \partial_{\lambda}^{z}[h(\epsilon^{\lambda\rho} g^{\nu\sigma} - g^{\lambda\sigma} g^{\nu\rho})\partial_{\rho} V_{\sigma}] - hg^{\nu\lambda} \partial_{\lambda}^{z} B \} , \]  

on the basis of (2.19), (2.29), (2.33), (4.12), and (4.13).

By analogy with the treatment in [7], we obtain the following integral representations for \( A_{\mu}^{a}(x) \), \( W_{\mu}^{j}(x) \), and \( V_{\mu}(x) \) using (2.17), (2.25), (2.27), (4.1)–(4.3), (4.17)–(4.19):
\[ A^a_\mu(x) = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D_{\mu\nu}(x, z) \]
\[ \times \lambda_C \left\{ J_{C}^{\nu} + f^{abc} \left[ \partial^{z}_{\lambda}(h g^{\lambda \rho} g^{\nu \sigma} A^b_\rho A^c_\sigma) + h g^{\lambda \rho} g^{\nu \sigma} A^b_\lambda G_{\rho \sigma}^c \right. \right. \]
\[ \left. \left. \left. + i h g^{\nu \rho} \partial^z_C \bar{C}^b \cdot C^c \right] \right\} \]
\[ + \int d^3 z \mathcal{J}_{\mu}^{a0}(x, z) \big|_{z^0 = y^0} , \tag{4.20} \]

\[ W^j_\mu(x) = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D_{\mu\nu}(x, z) \]
\[ \times \lambda_W \left\{ J^j_{W} + e^{jkl} \left[ \partial^{z}_{\lambda}(h g^{\lambda \rho} g^{\nu \sigma} W^k_\rho W^l_\sigma) + h g^{\lambda \rho} g^{\nu \sigma} W^k_\lambda F^l_\rho \right. \right. \]
\[ \left. \left. \left. + i h g^{\nu \rho} \partial^z_C \bar{C}^k \cdot C^l \right] \right\} \]
\[ + \int d^3 z \mathcal{J}_{\mu}^{j0}(x, z) \big|_{z^0 = y^0} , \tag{4.21} \]

\[ V_\mu(x) = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D_{\mu\nu}(x, z) \lambda_Y J^\nu_Y \]
\[ + \int d^3 z \mathcal{J}_{\mu}^{\nu0}(x, z) \big|_{z^0 = y^0} . \tag{4.22} \]

Differentiating the right-hand sides of these equations with respect to \( y^0 \), we find that they vanish because of (4.17)–(4.19). Namely, the expressions in (4.20)–(4.22) are independent of \( y^0 \); they reduce to \( A^a_\mu(x) \), \( W^j_\mu(x) \), and \( V_\mu(x) \) via (4.10) and (4.14) when we set \( y^0 = x^0 \).

Using (3.38), (3.39), (3.48), (3.49), (3.59), and (3.60), we obtain the 4D commutation relations between the integral representations (4.20)–(4.22), and \( A^b_\nu(y) \), \( W^k_\nu(y) \), and \( V_\nu(y) \) as follows:

\[ [A^a_\mu(x), A^b_\nu(y)] = -i \delta^{ab} D_{\mu\nu}(x, y) + C^{ab}_{\mu\nu}(x, y; A^a_\sigma, B^a; \lambda_C) , \tag{4.23} \]

\[ [W^j_\mu(x), W^k_\nu(y)] = -i \delta^{jk} D_{\mu\nu}(x, y) + C^{jk}_{\mu\nu}(x, y; W^j_\sigma, B^j; \lambda_W) , \tag{4.24} \]

\[ [V_\mu(x), V_\nu(y)] = -i D_{\mu\nu}(x, y) + C_{\mu\nu}(x, y; V_\sigma, B; \lambda_Y) . \tag{4.25} \]
In these right-hand sides, \( C^a_{\mu\nu}, C^i_{\mu\nu}, \) or \( C_{\mu\nu} \) is a functional with respect to \( A^a_{\sigma}, B^a, \) and the right-hand side of (2.17), to \( W^i_{\sigma}, B^i, \) and the one of (2.25), or to \( V_{\sigma}, B, \) and the one of (2.27), respectively. Each functional contains commutators \([G, X]\) where \( G = D_{\mu\nu}, \partial D_{\mu\nu}, \) and \( X = A^b_{\nu} \) for (4.23), \( = W^k_\nu \) for (4.24), and \( = V_\nu \) for (4.25).

We next define bilocal currents relating to the auxiliary fields \( B^a(x), C^a(x), \) and \( \bar{C}^a(x) \) for the gluon field, and to the auxiliary ones \( B^i(x), C^i(x), \) \( \bar{C}^i(x), B(x), C(x), \) and \( \bar{C}(x) \) for the electroweak ones as follows:

\[ J^a_\mu(x, z) \equiv D(x, z) \partial^\mu z \cdot h(z) g^{\lambda\rho}(z) F^a(z) - D(x, z) h(z) g^{\lambda\rho}(z) \partial^\rho z F^a(z), \quad (4.26) \]

\[ J^i_\mu(x, z) \equiv D(x, z) \partial^\mu z \cdot h(z) g^{\lambda\rho}(z) F^i(z) - D(x, z) h(z) g^{\lambda\rho}(z) \partial^\rho z F^i(z), \quad (4.27) \]

\[ J^F_\mu(x, z) \equiv D(x, z) \partial^\mu z \cdot h(z) g^{\lambda\rho}(z) F(z) - D(x, z) h(z) g^{\lambda\rho}(z) \partial^\rho z F(z). \quad (4.28) \]

Here, each symbol \( F^b, F^k, \) or \( F \) stands for \( B^b, C^b, \) and \( \bar{C}^b, \) for \( B^k, C^k, \) and \( \bar{C}^k, \) or for \( B, C, \) and \( \bar{C}. \) The four-divergences of these bilocal currents with respect to \( z \) are given by

\[ J^a_\mu(x, z) \partial^\mu z = -D(x, z) \partial^\rho z (h g^{\lambda\rho} \partial^\rho z F^a), \quad (4.29) \]

\[ J^i_\mu(x, z) \partial^\mu z = -D(x, z) \partial^\rho z (h g^{\lambda\rho} \partial^\rho z F^i), \quad (4.30) \]

\[ J^F_\mu(x, z) \partial^\mu z = -D(x, z) \partial^\rho z (h g^{\lambda\rho} \partial^\rho z F), \quad (4.31) \]

via (4.15).
In parallel with (4.20)–(4.22), we obtain the following integral representations for the auxiliary fields using (2.20)–(2.22), (2.30)–(2.32), (2.34)–(2.36), (4.26)–(4.31):

\[ B^a(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \]
\[ \times \lambda_C f^{abc} h g^{\lambda \nu} [A_\lambda^b \partial_\nu B^c - i \partial_\nu \bar{C}^b \cdot (\partial_\nu C^c + \lambda_C f^{cde} A_\nu^d C^e)] + \int d^3z J_B^{a0}(x, z) |_{z^0 = y^0} , \]

\[ C^a(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \lambda_C f^{abc} \partial_\nu (h g^{\lambda \nu} A_\nu^b C^c) \]
\[ + \int d^3z J_C^{a0}(x, z) |_{z^0 = y^0} , \]

\[ \bar{C}^a(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \lambda_C f^{abc} h g^{\lambda \nu} A_\lambda^b \partial_\nu \bar{C}^c \]
\[ + \int d^3z J_C^{a0}(x, z) |_{z^0 = y^0} , \]

\[ B^0(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \]
\[ \times \lambda_W e^{ijkl} h g^{\lambda \nu} [W_\nu^k \partial_\mu B^l - i \partial_\nu \bar{C}^k \cdot (\partial_\nu C^l + \lambda_W e^{lmn} W_\nu^m C^n)] + \int d^3z J_B^{j0}(x, z) |_{z^0 = y^0} , \]

\[ C^0(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \lambda_W e^{ijkl} \partial_\nu (h g^{\lambda \nu} W_\nu^k C^l) \]
\[ + \int d^3z J_C^{j0}(x, z) |_{z^0 = y^0} , \]

\[ \bar{C}^0(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \lambda_W e^{ijkl} h g^{\lambda \nu} W_\nu^k \partial_\nu \bar{C}^l \]
\[ + \int d^3z J_C^{j0}(x, z) |_{z^0 = y^0} , \]
\[ B(x) = \int d^3z J_B^0(x, z), \quad (4.38) \]
\[ C(x) = \int d^3z J_C^0(x, z), \quad (4.39) \]
\[ \bar{C}(x) = \int d^3z J_C^0(x, z). \quad (4.40) \]

Differentiating the right-hand sides of (4.32)–(4.37) with respect to \( y^0 \) lead us to the fact that they vanish because of (4.29) and (4.30). Namely, the expressions in (4.32)–(4.37) are independent of \( y^0 \); they reduce to \( B^a(x), C^a(x), \bar{C}^a(x), B^i(x), C^i(x), \) and \( \bar{C}^i(x) \) via (4.16) when we set \( y^0 = x^0 \).

On the other hand, differentiating the right-hand sides of (4.38)–(4.40) with respect to \( z^0 \), we find that they are independent of \( z^0 \). So, these expressions yield \( B(x), C(x), \) and \( \bar{C}(x) \) via (4.16) at \( z^0 = x^0 \).

Using (3.38), (3.40), (3.48), (3.50), (3.59), and (3.61), we obtain the 4D commutation relations between the integral representations (4.32), (4.35), and (4.38), and \( A_{\nu}^b(y), W_{\nu}^k(y), \) and \( V_{\nu}(y) \) as follows:

\[ [B^a(x), A_{\nu}^b(y)] = -i\delta^{ab}\partial_{\nu} + C_{\nu}^{ab}(x, y; B^a; \lambda_C), \quad (4.41) \]
\[ [B^i(x), W_{\nu}^k(y)] = -i\delta^{ik}\partial_{\nu} + C_{\nu}^{ik}(x, y; B^i; \lambda_W), \quad (4.42) \]
\[ [B(x), V_{\nu}(y)] = -i\partial_{\nu} + C_{\nu}(x, y; B; \lambda_Y). \quad (4.43) \]

In the right-hand side of (4.41) or (4.42), \( C_{\nu}^{ab} \) or \( C_{\nu}^{ik} \) is a functional with respect to \( B^a \) and the right-hand side of (2.20), or to \( B^i \) and the one of (2.30), respectively. Each functional contains commutators \([G, X]\) where \( G = \partial_D \) and \( \partial_D \), and \( X = A_{\nu}^b \) for (4.41) and = \( W_{\nu}^k \) for (4.42). In the right-hand side of (4.43), \( C_{\nu} \) is a linear functional with respect to \( B \) and contains commutators \([G, V_{\nu}]\) where \( G = \partial_D \) and \( \partial_D \).

4.2. Weyl fields

As in [9], we define the following bilocal currents relating to the quark fields
\[ \Psi^r_q(x), \eta_u(x), \text{ and } \eta_d(x), \text{ and to the lepton ones } \Psi^r_1(x), \eta_u(x), \text{ and } \eta_e(x): \]

\[ \mathcal{J}^r_q(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \Psi^r_q(z), \quad (4.44) \]

\[ \mathcal{J}^\lambda_u(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \eta_u(z), \quad (4.45) \]

\[ \mathcal{J}^\lambda_d(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \eta_d(z), \quad (4.46) \]

\[ \mathcal{J}^\lambda_1(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \Psi^r_1(z), \quad (4.47) \]

\[ \mathcal{J}^{\lambda}_\eta(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \eta_u(z), \quad (4.48) \]

\[ \mathcal{J}^{\lambda}_\eta(x, z) \equiv i \mathcal{S}(x, z) h(z) h^{\lambda a}(z) \tilde{\sigma}^a \eta_e(z). \quad (4.49) \]

Here, the spinorial q-number anti-commutator functions \( \mathcal{S}(x, z) \) and \( \tilde{\mathcal{S}}(x, z) \) are defined by the following Cauchy problems [9]:

\[ i h h^{\mu a} \tilde{\sigma}^a \left( \partial^x - \frac{\mathcal{S}_{bc}}{2} \omega_{bc} \right) \mathcal{S}(x, z) = 0, \quad (4.50) \]

\[ \mathcal{S}(x, z)|_0 = -i h^{0 a} \tilde{\sigma}^a \frac{\delta^3}{h g^{00}}, \quad (4.51) \]

and

\[ i h h^{\mu a} \tilde{\sigma}^a \left( \partial^x + \frac{\mathcal{S}_{bc}}{2} \omega_{bc} \right) \mathcal{\tilde{S}}(x, z) = 0, \quad (4.52) \]

\[ \mathcal{\tilde{S}}(x, z)|_0 = -i h^{0 a} \tilde{\sigma}^a \frac{\delta^3}{h g^{00}}. \quad (4.53) \]

Using the Hermitian conjugates of these functions [9],

\[ [\mathcal{S}(x, z)]^\dagger = -\mathcal{S}(z, x), \quad [\mathcal{\tilde{S}}(x, z)]^\dagger = -\mathcal{\tilde{S}}(z, x), \quad (4.54) \]

we obtain

\[ \mathcal{S}(x, z) \left( \frac{\overleftarrow{\partial}^z}{\partial^x} - \frac{\mathcal{S}_{bc}}{2} \omega_{bc} \right) \tilde{\sigma}^a h^{\mu a} \dot{h} = 0, \quad (4.55) \]

\[ \mathcal{\tilde{S}}(x, z) \left( \frac{\overleftarrow{\partial}^z}{\partial^x} - \frac{\mathcal{S}_{bc}}{2} \omega_{bc} \right) \tilde{\sigma}^a h^{\mu a} \dot{h} = 0, \quad (4.56) \]
The four-divergences of the bilocal currents (4.44)–(4.49) with respect to \( z \) are given by

\[
\mathcal{J}_{q}^{r \lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^{\omega_{bc}}}{2} \right) \Psi_q^r ,
\]

(4.57)

\[
\mathcal{J}_{u}^{\lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^\mu_{bc}}{2} \right) \eta_u ,
\]

(4.58)

\[
\mathcal{J}_{d}^{\lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^{\omega_{bc}}}{2} \right) \eta_d ,
\]

(4.59)

\[
\mathcal{J}_{1}^{r \lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^{\omega_{bc}}}{2} \right) \Psi_1^r ,
\]

(4.60)

\[
\mathcal{J}_{n}^{\lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^{\omega_{bc}}}{2} \right) \eta_n ,
\]

(4.61)

\[
\mathcal{J}_{e}^{\lambda}(x, z) \partial_{\lambda}^z = S(x, z) i h h^{\mu a} \overset{\circ}{\sigma}_a \left( \partial_{\mu}^z + \frac{\overset{\circ}{S}_{bc}^{\omega_{bc}}}{2} \right) \eta_e ,
\]

(4.62)

on the basis of (3.2), (3.3), (4.55), and (4.56).

By analogy with the treatment in the previous subsection, we obtain the following integral representations for \( \Psi_q^r (x) \), \( \eta_u (x) \), \( \eta_d (x) \), \( \Psi_1^r (x) \), \( \eta_n (x) \), and \( \eta_e (x) \) using (2.40)–(2.45), (4.44)–(4.49), (4.57)–(4.62):

\[
\Psi_q^r (x) = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \left[ S(x, z) \times h \left[ -h^{\mu a} \overset{\circ}{\sigma}_a \left( \lambda_C \frac{t_a^C}{2} A_{\mu}^a + \lambda_W \frac{\tau^j_W}{2} W^j_{\mu} + \lambda_Y \frac{1}{6} V_{\mu} \right) \Psi_q + G_u \Phi \eta_u + G_d \Phi \eta_d \right] + \int d^3 z \mathcal{J}_q^{r0}(x, z) \right]_{z^0 = y^0} ,
\]

(4.63)
\[ \eta_u(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x,z) \]
\[ \times h \left[ -h^\mu a \langle \frac{t^a_C}{2} A_\mu + \lambda_Y \frac{2}{3} V_\mu \rangle \eta_u + G_u^* \bar{\Phi} \Psi_q \right] \]
\[ + \int d^3z \, \bar{J}_u^0(x,z) |_{z^0 = y^0}, \tag{4.64} \]

\[ \eta_d(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x,z) \]
\[ \times h \left[ -h^\mu a \langle \frac{t^a_C}{2} A_\mu - \lambda_Y \frac{1}{3} V_\mu \rangle \eta_d + G_d^* \Phi \Psi_q \right] \]
\[ + \int d^3z \, \bar{J}_d^0(x,z) |_{z^0 = y^0}, \tag{4.65} \]

\[ \Psi_1^r(x) \]
\[ = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x,z) \]
\[ \times h \left[ -h^\mu a \langle \frac{t^a_C}{2} W_\mu j + \lambda_Y \frac{1}{2} V_\mu \rangle \Psi_1 + G_n^* \bar{\Phi} \eta_n + G_e \Phi \eta_e \right] \]
\[ + \int d^3z \, \bar{J}_1^0(x,z) |_{z^0 = y^0}, \tag{4.66} \]

\[ \eta_n(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x,z) h G_n^* \bar{\Phi} \Psi_1 \]
\[ + \int d^3z \, \bar{J}_n^0(x,z) |_{z^0 = y^0}, \tag{4.67} \]

\[ \eta_e(x) = \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x,z) \]
\[ \times h \left( h^\mu a \langle \frac{t^a_C}{2} V_\mu \eta_e + G_e^* \Phi \Psi_1 \right) \]
\[ + \int d^3z \, \bar{J}_e^0(x,z) |_{z^0 = y^0}. \tag{4.68} \]

Differentiating the right-hand sides of these equations with respect to \( y^0 \), we find that they vanish because of (4.57)–(4.62). Namely, the expression in (4.63)–(4.68) are independent of \( y^0 \); they reduce to \( \Psi_1^r(x), \eta_u(x), \eta_d(x), \Psi_1^r(x), \eta_n(x), \) and \( \eta_e(x) \) via (4.51) and (4.53) when we set \( y^0 = x^0 \).
Using (3.74) and (3.75), we obtain the 4D anti-commutation relations between the integral representations (4.63) and (4.66), and $(\Psi_q^\dagger)^s(y)$ and $(\Psi_1^\dagger)^s(y)$ as follows:

$$\{\Psi_q^r(x), (\Psi_q^\dagger)^s(y)\} = i\delta^{rs}S(x, y) + \mathcal{R}^{rs}(x, y; \Psi_q^r; \lambda_C, \lambda_W, \lambda_Y, G_u, G_d), \quad (4.69)$$

$$\{\Psi_1^r(x), (\Psi_1^\dagger)^s(y)\} = i\delta^{rs}S(x, y) + \mathcal{R}^{rs}(x, y; \Psi_1^r; \lambda_W, \lambda_Y, G_n, G_e). \quad (4.70)$$

Here, $\mathcal{R}^{rs}$ is a functional with respect to $\Psi_q^r$ or $\Psi_1^r$, and to the right-hand side of (2.40) or (2.43). It contains a commutator $[S, X]$ where $X = (\Psi_q^\dagger)^s$ for (4.69) or $=(\Psi_1^\dagger)^s$ for (4.70).

Likewise, using (3.76), we obtain the 4D anti-commutation relations between the integral representations (4.64), (4.65), (4.67), and (4.68), and $\eta_f^\dagger(y)$ as follows:

$$\{\eta_f(x), \eta_f^\dagger(y)\} = i\bar{S}(x, y) + \bar{\mathcal{R}}(x, y; \eta_f; \lambda_C, \lambda_Y, G_f^* \eta_f)(f = u, d), \quad (4.71)$$

$$\{\eta_n(x), \eta_n^\dagger(y)\} = i\bar{S}(x, y) + \bar{\mathcal{R}}(x, y; \eta_n; G_n^* \eta_n), \quad (4.72)$$

$$\{\eta_e(x), \eta_e^\dagger(y)\} = i\bar{S}(x, y) + \bar{\mathcal{R}}(x, y; \eta_e; \lambda_Y, G_e^* \eta_e). \quad (4.73)$$

Here, $\bar{\mathcal{R}}$ is a functional with respect to $\eta_f$ ($f = u, d, n, e$) and to the right-hand side of (2.41), (2.42), (2.44), or (2.45). It contains a commutator between $\bar{S}$ and $\eta_f^\dagger$ ($f = u, d, n, e$).

### 4.3. Higgs field

We define a bilocal current relating to the Higgs field as

$$\mathcal{J}_{\Phi}^{r\lambda}(x, z) = D(x, z)\tilde{\partial}_\rho \cdot h(z)g^{\lambda\rho}(z)\Phi^r(z) - D(x, z)h(z)g^{\lambda\rho}(z)\tilde{\partial}_\rho \Phi^r(z). \quad (4.74)$$
The four-divergence of this current is given by
\[
\mathcal{J}_\Phi^\lambda(x, z) \partial_\lambda^z = -\mathcal{D}(x, z) \partial_\lambda^z(h g^{\lambda\mu} \partial_\mu^z \Phi^r),
\] via (4.15).

Using (2.46), (4.74), and (4.75), we obtain an integral representations for \(\Phi^r(x)\) as follows:
\[
\Phi^r(x) = -\int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \mathcal{D}(x, z)
\times h \left[ (\mu_H^2 - 2\lambda_H \Phi \Phi^\dagger) \Phi - G_u \bar{\Psi}_q \eta_u - G_d^* \eta_d^\dagger \Phi\Psi_q - G_n \bar{\Psi}_n \eta_n - G_e^* \eta_e^\dagger \Phi \right.
\]
\[
+ ig^{\mu\nu} \partial_\mu^z \left( \lambda_W \frac{\tau_j^j}{2} W_{\nu}^j + \lambda_Y \frac{1}{2} V_{\nu} \right) \Phi
\]
\[
+ 2ig^{\mu\nu} \left( \lambda_W \frac{\tau_j^j}{2} W_{\mu}^j + \lambda_Y \frac{1}{2} V_{\mu} \right) \partial_\nu^z \Phi
\]
\[
+ g^{\mu\nu} \left( \lambda_W \frac{\tau_j^j}{2} W_{\mu}^j + \lambda_Y \frac{1}{2} V_{\mu} \right) \cdot \left( \lambda_W \frac{\tau_k^k}{2} W_{\nu}^k + \lambda_Y \frac{1}{2} V_{\nu} \right) \Phi^r
\]
\[
+ \int d^3 z \mathcal{J}_\Phi^{r0}(x, z) |_{z^0 = y^0}.
\] (4.76)

Here, we apply the de Donder condition (3.73) to the first term. Of course, the right-hand side of (4.76) is independent of \(y^0\) and thus reduces to \(\Phi^r(x)\) via (4.16) when we set \(y^0 = x^0\).

The 4D commutation relation between the integral representation (4.76) and \((\Phi^\dagger)^s(y)\) is given by
\[
[\Phi^r(x), (\Phi^\dagger)^s(y)]
\]
\[
= i\delta^{rs} \mathcal{D}(x, y) + \mathcal{C}^{rs}(x, y; \Phi^r; \mu_H^2, \lambda_H, G_u, G_d^*, G_n, G_e^*, \lambda_W, \lambda_Y),
\] (4.77)
with the use of (3.77). The second term \(\mathcal{C}^{rs}\) is a functional with respect to \(\Phi^r\) and to the right-hand side of (2.46). It contains commutators \([\mathcal{G}, (\Phi^\dagger)^s]\) where \(\mathcal{G} = \mathcal{D}\) and \(\partial\mathcal{D}\).
5. Transformation properties

We investigate properties of the integral representations for the matter fields in Sect. 4 under the affine, gravitational BRST, internal Lorentz BRST, and $SU(3)_C \times SU(2)_W \times U(1)_Y$ BRST transformations.

5.1. Affine transformation

Let $\hat{P}_\lambda$ and $\hat{M}_\lambda^\kappa$ be the translation generator and the $GL(4)$ one [6], respectively. In order to obtain the commutation relations between these generators and the integral representations for the matter fields, we need the following commutation relations [4, 5, 6, 8, 9]:

$$[i \hat{P}_\lambda, h] = \partial_\lambda h, \quad [i \hat{M}_\lambda^\kappa, h] = x^\kappa \partial_\lambda h + \delta_\lambda^\kappa h,$$

$$[i \hat{P}_\lambda, g^{\mu\nu}] = \partial_\lambda g^{\mu\nu}, \quad [i \hat{M}_\lambda^\kappa, g^{\mu\nu}] = x^\kappa \partial_\lambda g^{\mu\nu} - \delta_\lambda^\mu g^{\kappa\nu} - \delta_\lambda^\nu g^{\mu\kappa},$$

$$[i \hat{P}_\lambda, h^{\mu a}] = \partial_\lambda h^{\mu a}, \quad [i \hat{M}_\lambda^\kappa, h^{\mu a}] = x^\kappa \partial_\lambda h^{\mu a} - \delta_\lambda^\mu h^{\kappa a},$$

$$[i \hat{P}_\lambda, \omega^{ab}_\mu] = \partial_\lambda \omega^{ab}_\mu, \quad [i \hat{M}_\lambda^\kappa, \omega^{ab}_\mu] = x^\kappa \partial_\lambda \omega^{ab}_\mu + \delta_\lambda^\kappa \omega^{ab}_\mu,$$

$$[i \hat{P}_\lambda, D(x, z)] = (\partial^x_\lambda + \partial^z_\lambda)D(x, z),$$

$$[i \hat{M}_\lambda^\kappa, D(x, z)] = (x^\kappa \partial^x_\lambda + z^\kappa \partial^z_\lambda)D(x, z),$$

$$[i \hat{P}_\lambda, D_{\mu\nu}(x, z)] = (\partial^x_\lambda + \partial^z_\lambda)D_{\mu\nu}(x, z),$$

$$[i \hat{M}_\lambda^\kappa, D_{\mu\nu}(x, z)] = (x^\kappa \partial^x_\lambda + z^\kappa \partial^z_\lambda)D_{\mu\nu}(x, z) + \delta_\mu^\kappa D_{\lambda\nu}(x, z) + \delta_\nu^\kappa D_{\mu\lambda}(x, z),$$

$$[i \hat{P}_\lambda, S(x, z)] = (\partial^x_\lambda + \partial^z_\lambda)S(x, z),$$

$$[i \hat{M}_\lambda^\kappa, S(x, z)] = (x^\kappa \partial^x_\lambda + z^\kappa \partial^z_\lambda)S(x, z).$$
\[
[i \hat{P}_\lambda, S(x, z)] = (\partial^x_\lambda + \partial^z_\lambda)\bar{S}(x, z), \quad (5.11)
\]

\[
[i \hat{M}^\kappa_\lambda, S(x, z)] = (x^\kappa \partial^x_\lambda + z^\kappa \partial^z_\lambda)\bar{S}(x, z). \quad (5.12)
\]

First, we have the following commutation relations between \(i \hat{P}_\lambda\) and \(i \hat{M}^\kappa_\lambda\), and the integral representations for the gluon field (4.20) with the use of (5.1), (5.2), (5.5)–(5.8):

\[
[i \hat{P}_\lambda, A^a_\mu(x)] = \partial^x_\lambda A^a_\mu(x) \\
+ \int d^4 z \partial^z_\lambda \{[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^{a, \rho}_\mu(x, z) \bar{\partial}^z_\rho\} \\
+ \int d^3 z \partial^z_\rho [\delta^\rho_\lambda \mathcal{J}^{a, 0}_\mu(x, z) - \delta^0_\lambda \mathcal{J}^{a, \rho}_\mu(x, z)]|_{z^0 = y^0}, \quad (5.13)
\]

\[
[i \hat{M}^\kappa_\lambda, A^a_\mu(x)] = x^\kappa \partial^x_\lambda A^a_\mu(x) + \delta^\kappa_\mu A^a_\lambda(x) \\
+ \int d^4 z \partial^z_\lambda \{z^\kappa [\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^{a, \rho}_\mu(x, z) \bar{\partial}^z_\rho\} \\
+ \int d^3 z \partial^z_\rho \{z^\kappa [\delta^\rho_\lambda \mathcal{J}^{a, 0}_\mu(x, z) - \delta^0_\lambda \mathcal{J}^{a, \rho}_\mu(x, z)]\}|_{z^0 = y^0}. \quad (5.14)
\]

In these right-hand sides, the 4D and 3D integral terms reduce to 3D surface and surface integral ones, respectively. Hence they vanish, and we obtain the affine transformation of \(A^a_\mu(x)\). Likewise, the commutation relations between \(i \hat{P}_\lambda\) and \(i \hat{M}^\kappa_\lambda\), and the integral representations for the electroweak fields (4.21) and (4.22) yield the affine transformations of \(W^j_\mu(x)\) and \(V^\mu_\mu(x)\).

Then, using (5.1), (5.2), (5.5), and (5.6), we have the commutation relations between \(i \hat{P}_\lambda\) and \(i \hat{M}^\kappa_\lambda\), and the integral representation for the auxiliary fields (4.32)–(4.34) as follows:

\[
[i \hat{P}_\lambda, F^a(x)] = \partial^x_\lambda F^a(x) \\
+ \int d^4 z \partial^z_\lambda \{[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^{a, \rho}_F(x, z) \bar{\partial}^z_\rho\} \\
+ \int d^3 z \partial^z_\rho [\delta^\rho_\lambda \mathcal{J}^{a, 0}_F(x, z) - \delta^0_\lambda \mathcal{J}^{a, \rho}_F(x, z)]|_{z^0 = y^0}, \quad (5.15)
\]
where $F^a$ stands for $B^a$, $C^a$, and $\bar{C}^a$ as in Sect. 4. In these right-hand sides, the integral terms vanish by virtue of the similar reasons in (5.13) and (5.14). Thus, these commutation relations reduce to the affine transformations of $B^a(x)$, $C^a(x)$, and $\bar{C}^a(x)$. Replacing $F^a$ with $F^j$ in (5.15) and (5.16), we obtain the commutation relations between $\hat{P}_\lambda$ and $\hat{M}^\kappa\lambda$, and the integral representations for the auxiliary fields (4.35)–(4.37); of course, $F^j$ stands for $B^j$, $C^j$, and $\bar{C}^j$. These commutation relations reduce to the affine transformations of $B^j(x)$, $C^j(x)$, and $\bar{C}^j(x)$.

In parallel with (5.15) and (5.16), using (5.1), (5.2), (5.5), and (5.6), we have the commutation relations between $\hat{P}_\lambda$ and $\hat{M}^\kappa\lambda$, and the integral representations for the auxiliary fields (4.38)–(4.40) as follows:

\[
[i\hat{P}_\lambda, F^a(x)] = x^\kappa \partial^\kappa_x F^a(x) + \int d^3z \partial^{\lambda}_z \{ [\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}_F^{a\rho}(x, z) \bar{\mathcal{J}}^{\rho}_F \}
\]

\[
[i\hat{M}^\kappa\lambda, F^a(x)] = x^\kappa \partial^\kappa_x F^a(x) + \int d^3z \partial^{\lambda}_z \{ [\delta^{\lambda}_x \mathcal{J}_F^0(x, z) - \delta^\lambda_0 \mathcal{J}_F^0(x, z)] \},
\]

where $F$ stands for $B$, $C$, and $\bar{C}$. Since these integral terms vanish, Eqs. (5.17) and (5.18) yield the affine transformations of $B(x)$, $C(x)$, and $\bar{C}(x)$.

Let $\Xi$ stand for the quark fields $\Psi^r_q$, $\eta_u$, and $\eta_d$, or the lepton ones $\Psi^r_l$, $\eta_n$, and $\eta_e$. Using (5.1), (5.3), (5.4), (5.9)–(5.12), we have the commutation relations between $\hat{P}_\lambda$ and $\hat{M}^\kappa\lambda$, and the integral representations for the
quark fields (4.63)–(4.65) or for the lepton ones (4.66)–(4.68) as follows:

\[
[i \hat{P}_\lambda, \Xi(x)] = \partial^x_\lambda \Xi(x) \\
+ \int d^4z \partial^x_\lambda \{[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^\rho_\Xi(x, z) \hat{\partial}_\rho^z \} \\
+ \int d^3z \partial^z_\rho [\delta^\rho_\lambda \mathcal{J}^0_\Xi(x, z) - \delta^\lambda_\rho \mathcal{J}^0_\Xi(x, z)]|_{z^0 = y^0}, \tag{5.19}
\]

\[
[i \hat{M}_\kappa, \Xi(x)] = x^\kappa \partial^x_\lambda \Xi(x) \\
+ \int d^4z \partial^x_\lambda \{z^\kappa[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^\rho_\Xi(x, z) \hat{\partial}_\rho^z \} \\
+ \int d^3z \partial^z_\rho [z^\kappa[\delta^\rho_\lambda \mathcal{J}^0_\Xi(x, z) - \delta^\lambda_\rho \mathcal{J}^0_\Xi(x, z)]|_{z^0 = y^0}. \tag{5.20}
\]

Because of the similar reasons in (5.13) and (5.14), these commutation relations reduce to the affine transformations of \( \Psi^r_q(x), \eta_u(x), \eta_d(x), \Phi^r_{\psi^r}(x), \eta_n(x), \) and \( \eta_e(x) \).

Also, the commutation relations between \( \hat{P}_\lambda \) and \( \hat{M}_\kappa \), and the integral representations for the Higgs field (4.76) are given by

\[
[i \hat{P}_\lambda, \Phi^r(x)] = \partial^x_\lambda \Phi^r(x) \\
+ \int d^4z \partial^x_\lambda \{[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^\rho_\phi^r(x, z) \hat{\partial}_\rho^z \} \\
+ \int d^3z \partial^z_\rho [\delta^\rho_\lambda \mathcal{J}^0_\phi^r(x, z) - \delta^\lambda_\rho \mathcal{J}^0_\phi^r(x, z)]|_{z^0 = y^0}, \tag{5.21}
\]

\[
[i \hat{M}_\kappa, \Phi^r(x)] = x^\kappa \partial^x_\lambda \Phi^r(x) \\
+ \int d^4z \partial^x_\lambda \{z^\kappa[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot \mathcal{J}^\rho_\phi^r(x, z) \hat{\partial}_\rho^z \} \\
+ \int d^3z \partial^z_\rho [z^\kappa[\delta^\rho_\lambda \mathcal{J}^0_\phi^r(x, z) - \delta^\lambda_\rho \mathcal{J}^0_\phi^r(x, z)]|_{z^0 = y^0}. \tag{5.22}
\]

with the use of (5.1), (5.2), (5.5) and (5.6). Of course, these equations yield the affine transformation of \( \Phi^r(x) \).
5.2. Gravitational BRST and internal Lorentz BRST transformations

In order to obtain the (anti-)commutation relations between the gravitational BRST charge $Q_G$ and the integral representations for the matter fields, we need the following commutation relations [4, 5, 6, 8, 9]:

\[
\begin{align*}
\{iQ_G, h\} &= -\kappa \partial_\mu (c^\mu h), \quad (5.23) \\
\{iQ_G, g^{\mu\nu}\} &= \kappa (\partial_\rho c^\rho \cdot g^{\mu\nu} + \partial_\rho c^\nu \cdot g^{\mu\rho} - c^\rho \partial_\rho g^{\mu\nu}), \quad (5.24) \\
\{iQ_G, h^{\mu a}\} &= \kappa (\partial_\rho c^\rho \cdot h^{\mu a} - c^\rho \partial_\rho h^{\mu a}), \quad (5.25) \\
\{iQ_G, \omega_{\rho}^{\mu} c^{\rho} \cdot \omega_{\rho}^{\mu} c^{\rho}, + c^\rho \partial_\rho \omega_{\rho}^{\mu},\} \quad (5.26)
\end{align*}
\]

Using (5.23), (5.24), (5.27), and (5.28), we have the following (anti-)commutation relations between $Q_G$, and the integral representations for the gluon field (4.20) and for its auxiliary ones (4.32)–(4.34):

\[
\begin{align*}
\{iQ_G, A_a^{\mu}(x)\} &= -\kappa [\partial_\mu c^\rho (x) \cdot A^a_\rho (x) + c^\rho (x) \partial_\rho A^a_\mu (x)] \\
&\quad -\kappa \int d^4 z \partial_\Lambda \{[\theta(x^0 - z^0) - \theta(y^0 - z^0)] \cdot J^{a\mu}(x, z) \partial_\rho c^\rho(z)\} \\
&\quad -\kappa \int d^3 z \partial_\Lambda [J^{a0}(x, z) c^\rho(z) - J^{a0}(x, z) c^0(z)] \bigg|_{z^0 = y^0}, \quad (5.31)
\end{align*}
\]
\[ [iQ_G, B^a(x)] \]
\[ = -\kappa c^\rho(x) \partial^\rho_x B^a(x) \]
\[ -\kappa \int d^4z \partial^z [\{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J_B^{a\rho}(x, z) \delta^z_\rho \cdot c^\lambda(z) - \{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J^a_{B\lambda}(x, z) \cdot c^0(z)] \big|_{z^0 = y^0}, \quad (5.32) \]

\[ \{iQ_G, C^a(x)\} \]
\[ = -\kappa c^\rho(x) \partial^\rho_x C^a(x) \]
\[ +\kappa \int d^4z \partial^z [\{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J_C^{a\rho}(x, z) \delta^z_\rho \cdot c^\lambda(z) - \{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J^a_{C\lambda}(x, z) \cdot c^0(z)] \big|_{z^0 = y^0}, \quad (5.33) \]

\[ \{iQ_G, \bar{C}^a(x)\} \]
\[ = -\kappa c^\rho(x) \partial^\rho_x \bar{C}^a(x) \]
\[ +\kappa \int d^4z \partial^z [\{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J_{\bar{C}}^{a\rho}(x, z) \delta^z_\rho \cdot c^\lambda(z) - \{\theta(x^0 - z^0) - \theta(y^0 - z^0)\} \cdot J^a_{\bar{C}\lambda}(x, z) \cdot c^0(z)] \big|_{z^0 = y^0}. \quad (5.34) \]

In these right-hand sides, the 4D and 3D integral terms vanish. Therefore, these equations yield the gravitational BRST transformations of \( A^a_\mu(x), B^a(x), C^a(x), \) and \( \bar{C}^a(x). \) Likewise, the (anti-)commutation relations between \( Q_G, \) and the integral representations for the electroweak fields (4.21) and (4.22), and for their auxiliary ones (4.35)–(4.40) reduce to the gravitational BRST transformations of \( W^j_\mu(x), B^j(x), C^j(x), \bar{C}^j(x), V_\mu(x), B(x), C(x), \) and \( \bar{C}(x). \)

Using (5.23), (5.25), (5.26), (5.29), and (5.30), we have the following anti-commutation relations between \( Q_G, \) and the integral representations
for the quark fields (4.63)–(4.65) and for the lepton ones (4.66)–(4.68):

\[
\{ iQ_G, \Xi(x) \} = -\kappa \bar{c}^\rho(x) \partial_\rho \Xi(x) \\
+ \kappa \int d^4z \partial_\lambda \{ \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \cdot \mathcal{J}_{\Xi}^\rho(x,z) \bar{c}^\lambda(z) \} \\
+ \kappa \int d^3z \partial_\lambda \left[ \mathcal{J}_{\Xi}^0(x,z) c^\lambda(z) - \mathcal{J}_{\Xi}^\lambda(x,z) c^0(z) \right] |_{z^0 = y^0} .
\]  

(5.35)

These equations yield the gravitational BRST transformations of \( \Psi_q^r(x) \), \( \eta_u(x) \), \( \eta_d(x) \), \( \psi_l(x) \), \( \eta_n(x) \), and \( \eta_e(x) \).

The commutation relation between \( Q_G \) and the integral representation for the Higgs field (4.76) is given by

\[
[ iQ_G, \Phi^r(x) ] = -\kappa \bar{c}^\rho(x) \partial_\rho \Phi^r(x) \\
- \kappa \int d^4z \partial_\lambda \{ \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \cdot \mathcal{J}_{\Phi}^\rho(x,z) \bar{c}^\lambda(z) \} \\
- \kappa \int d^3z \partial_\lambda \left[ \mathcal{J}_{\Phi}^0(x,z) c^\lambda(z) - \mathcal{J}_{\Phi}^\lambda(x,z) c^0(z) \right] |_{z^0 = y^0} ,
\]  

(5.36)

with the use of (5.23), (5.24), and (5.27). This commutation relation reduces to the gravitational BRST transformation of \( \Phi^r(x) \).

In order to obtain the (anti-)commutation relations between the internal Lorentz BRST charge \( Q_L \) and the integral representations for the matter fields, we need the following commutation relations [4, 5, 6, 8, 9]:

\[
[ iQ_L, h ] = 0 ,
\]

(5.37)

\[
[ iQ_L, h^{\mu a} ] = -t^{ab} h^\mu_{ b} ,
\]

(5.38)

\[
[ iQ_L, \omega^{bc}_\mu ] = (D_\mu t)^{bc} ,
\]

(5.39)

\[
[ iQ_L, S(x, z) ] = -\frac{1}{2} \left[ t^{ab}(x) \bar{S}_{ab} S(x, z) - S(x, z) \bar{S}_{ab} t^{ab}(z) \right] ,
\]

(5.40)
\[ [iQ_L, \tilde{S}(x, z)] = -\frac{1}{2} [t^{ab}(x) \tilde{S}_{ab} \tilde{S}(x, z) - \tilde{S}(x, z) \tilde{S}_{ab} t^{ab}(z)] . \] (5.41)

We easily confirm that \( Q_L \) (anti-)commutes with the integral representations for \( A_\mu^a(x), B^a(x), C^a(x), \bar{C}^a(x), W_\mu^i(x), B^i(x), C^i(x), \bar{C}^i(x), V_\mu(x), B(x), C(x), \bar{C}(x), \) and \( \Phi^r(x) \). Consequently, these representations are invariant under the internal Lorentz BRST transformations. On the other hand, we find that the anti-commutation relations between \( Q_L \) and the integral representations for \( \Psi^r_q(x), \eta_u(x), \eta_d(x), \Psi^r_l(x), \eta_n(x), \) and \( \eta_e(x) \) reduce to the internal Lorentz BRST transformations \((3.95)-(3.97)\) via \((5.37)-(5.41)\).

### 5.3. SU(3)_C × SU(2)_W × U(1)_Y BRST transformations

In order to obtain the (anti-)commutation relations between the BRST charges \( Q_C, Q_W, \) and \( Q_Y, \) and the integral representations for the matter fields, we use \((3.99)-(3.123)\) because each of \( D_{\mu\nu}(x,z), D(x,z), S(x,z), \) and \( \tilde{S}(x,z) \) is invariant under the \( SU(3)_C, SU(2)_W, \) and \( U(1)_Y \) BRST transformations \([8, 9]\).

The commutation relations between \( Q_C \) and the integral representations for the gluon field \((4.20)\) is given by

\[ [iQ_C, A_\mu^a(x)] = -\int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] D_{\mu\nu}(x,z) \]
\[ \times \partial^z [h(g^{\lambda\rho} g^{\nu\sigma} - g^{\lambda\sigma} g^{\nu\rho}) \partial^z (g^{\lambda\sigma} C^a + \lambda_C f^{abc} A_\sigma^b C^c)] \]
\[ + \int d^3z \left[ D_{\mu\nu}(x,z) \partial^z \cdot h(g^{\rho\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}) (\partial^z (g^{\lambda\sigma} C^a + \lambda_C f^{abc} A_\sigma^b C^c)) \right. \]
\[ - D_{\mu\nu}(x,z) h(g^{\rho\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\sigma\nu}) (\partial^z g^{\lambda\sigma} C^a + \lambda_C f^{abc} A_\sigma^b C^c) \]
\[ - \partial^x D(x,z) \cdot h(g^{\rho\sigma} (\partial^z C^a + \lambda_C f^{abc} A_\sigma^b C^c)) \big|_{z^0 = y^0} , \] (5.42)

with the use of \((3.99), (3.100), \) and \((4.17)\). In the right-hand side of this equation, we integrate the first term by parts to cancel out the part of
the second one via (2.21), (4.6), (4.12), and (4.14). Thus, we obtain
\[ \partial_\mu^x C^a + \lambda_C f^{abc} A_\mu^b C^c \]
which is the \( SU(3)_C \) BRST transformation of \( A_\mu^a(x) \).
The BRST charge \( Q_C \) commutes with the integral representation for the B-field (4.32). Of course, both \( Q_W \) and \( Q_Y \) commute with the integral representations (4.20) and (4.32).

The anti-commutation relation between \( Q_C \) and the integral representations for the Faddeev–Popov ghost field (4.33) is given by

\[
\{ iQ_C, C^a(x) \} = \int d^4 z \left[ (x^0 - z^0) - \theta(y^0 - z^0) \right] D(x, z) \partial^z \left[ h g^{\lambda \mu} \partial_\mu^z \left( \frac{\lambda_C}{2} f^{abc} C^b C^c \right) \right] \\
- \int d^3 z \left[ D(x, z) \partial^{\nu} \cdot h g^{\lambda \nu} \frac{\lambda_C}{2} f^{abc} C^b C^c \right. \\
\left. \Big| \frac{\partial^z}{\partial y^0} \right|_{z^0 = y^0}, \tag{5.43}\]

with the use of (3.101) and (4.29). In the right-hand side of this equation, we integrate the first term by parts to cancel out the part of the second one via (4.9), (4.15), and (4.16). So, we obtain \(-\frac{\lambda_C}{2} f^{abc} C^b C^c\) which is the \( SU(3)_C \) BRST transformation of \( C^a(x) \). On the other hand, the anti-commutation relation between \( Q_C \) and the integral representation for the Faddeev–Popov anti-ghost field (4.34) reduces to (3.102) with the integral representation for the B-field (4.32). Both \( Q_W \) and \( Q_Y \) anti-commute with the integral representations for \( C^a(x) \) and \( \bar{C}^a(x) \).

In parallel with the above (anti-)commutation relations, the ones between \( Q_W \) and the integral representations for the electroweak field (4.21) and for their auxiliary ones (4.35)–(4.37) yield the right-hand sides of (3.105)–(3.108). Obviously, both \( Q_C \) and \( Q_Y \) (anti-)commute with these integral representations.

The commutation relation between \( Q_Y \) and the integral representation
for the electroweak field (4.22) is given by

\[
[iQ_Y, V_\mu(x)] = \int d^3z \left[ D_{\mu\nu}(x, z) \partial^z \cdot h(g^{0\rho}g^{0\sigma} - g^{0\nu}g^{0\rho}) \partial^z C \right.
\]
\[
- \partial^x D(x, z) \cdot hg^{0\rho} \partial^z C \bigg|_{z^0 = y^0},
\]

(5.44)

with the use of (3.113), (3.114), and (4.19). Since the first term of this integrand does not involve \( \partial_0 C \), we integrate it by parts. Using (4.12) and (4.39), we consequently reduce the right-hand side of (5.44) to \( \partial_\mu C \). Also, the (anti-)commutation relations between \( Q_Y \), and the integral representations for the auxiliary fields (4.38)–(4.40) yield the right-hand sides of (3.114)–(3.116). Of course, both \( Q_C \) and \( Q_W \) (anti-)commute with these integral representations.

Next, using (3.103), (3.109), and (3.117), we have the anti-commutation relations between the three BRST charges, and the integral representations for the left-handed quark fields (4.63) as follows:

\[
\{iQ_C, \Psi^r_q(x)\} = - \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z)
\]
\[
\times hh^{a}\sigma^a \left( \partial^{x} + \frac{\Sigma_{bc}}{2} \omega_{\mu}^{bc} \right) \left( \lambda_C \frac{t^a}{2} C^{a}\Psi^r_q \right)
\]
\[
- \int d^3z S(x, z) hh^{0a}\bar{\sigma}^a \lambda_C \frac{t^a}{2} C^{a}\Psi^r_q \bigg|_{z^0 = y^0},
\]

(5.45)

\[
\{iQ_W, \Psi^r_q(x)\} = - \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z)
\]
\[
\times hh^{a}\sigma^a \left( \partial^{x} + \frac{\Sigma_{bc}}{2} \omega_{\mu}^{bc} \right) \left( \lambda_W \frac{\tau^j}{2} C^{j}\Psi^r_q \right)^r
\]
\[
- \int d^3z S(x, z) hh^{0a}\bar{\sigma}^a \left( \lambda_W \frac{\tau^j}{2} C^{j}\Psi^r_q \right)^r \bigg|_{z^0 = y^0},
\]

(5.46)
\[
\{ iQ_Y, \Psi^r_q (x) \} = - \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z) \\
\times h \hbar^{\mu a} \frac{\sigma}{\sigma_a} \left( \partial_{\mu} + \frac{S_{bc}}{2} \omega^{bc}_{\mu} \right) \left( \lambda Y \frac{1}{6} C \Psi^r_q \right) \\
- \int d^3 z \left. S(x, z) h h^{0 a} \frac{\sigma}{\sigma_a} \lambda Y \frac{1}{6} C \Psi^r_q \right|_{z^0 = y^0}.
\]

These right-hand sides reduce to the ones of (3.103), (3.109), and (3.117) when we integrate their first terms by parts and apply (3.2), (4.51), and (4.55) to them.

Using (3.104), (3.118), and (3.119), we have the anti-commutation relations between \(Q_C\) and \(Q_Y\), and the integral representations for the right-handed quark fields (4.64) and (4.65) as follows:

\[
\{ iQ_C, \eta_f (x) \} = - \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \bar{S}(x, z) \\
\times h \hbar^{\mu a} \frac{\sigma}{\sigma_a} \left( \partial_{\mu} + \frac{S_{bc}}{2} \omega^{bc}_{\mu} \right) \left( \lambda_C \frac{t^a}{2} C a \eta_f \right) \\
- \int d^3 z \left. \bar{S}(x, z) h h^{0 a} \frac{\sigma}{\sigma_a} \lambda_C \frac{t^a}{2} C a \eta_f \right|_{z^0 = y^0}.
\] (5.48)

\[
\{ iQ_Y, \eta_u (x) \} = - \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \bar{S}(x, z) \\
\times h \hbar^{\mu a} \frac{\sigma}{\sigma_a} \left( \partial_{\mu} + \frac{S_{bc}}{2} \omega^{bc}_{\mu} \right) \left( \lambda Y \frac{2}{3} C \eta_u \right) \\
- \int d^3 z \left. \bar{S}(x, z) h h^{0 a} \frac{\sigma}{\sigma_a} \lambda Y \frac{2}{3} C \eta_u \right|_{z^0 = y^0},
\] (5.49)

\[
\{ iQ_Y, \eta_d (x) \} = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \bar{S}(x, z) \\
\times h \hbar^{\mu a} \frac{\sigma}{\sigma_a} \left( \partial_{\mu} + \frac{S_{bc}}{2} \omega^{bc}_{\mu} \right) \left( \lambda Y \frac{1}{3} C \eta_d \right) \\
+ \int d^3 z \left. \bar{S}(x, z) h h^{0 a} \frac{\sigma}{\sigma_a} \lambda Y \frac{1}{3} C \eta_d \right|_{z^0 = y^0}.
\] (5.50)
These right-hand sides reduce to the ones of (3.104), (3.118), and (3.119) when we integrate their first terms by parts and apply (3.3), (4.53), and (4.56) to them.

Also, using (3.110) and (3.120), we have the following anti-commutation relations between $Q_W$ and $Q_Y$, and the integral representations for the left-handed lepton fields (4.66):

\[
\{iQ^r_W, \Psi_1^r(x)\} = -\int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z) \\
\times h h^\mu a \overset{\circ}{\sigma}_a \left( \partial_{\mu} + \overset{\circ}{S}_{bc} \omega_{\mu}^{bc} \right) \left( \lambda_W \frac{\tau^j_W}{2} C^{ij} \Psi_1^r \right)^r \\
- \int d^3 z S(x, z) h h^{0a} \overset{\circ}{\sigma}_a \left( \lambda_W \frac{\tau^j_W}{2} C^{ij} \Psi_1^r \right)^r \bigg|_{z^0 = y^0}, \tag{5.51}
\]

\[
\{iQ^r_Y, \Psi_1^r(x)\} = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z) \\
\times h h^\mu a \overset{\circ}{\sigma}_a \left( \partial_{\mu} + \overset{\circ}{S}_{bc} \omega_{\mu}^{bc} \right) \left( \lambda_Y \frac{\tau^j_Y}{2} C^{ij} \Psi_1^r \right)^r \\
+ \int d^3 z S(x, z) h h^{0a} \overset{\circ}{\sigma}_a \lambda_Y \frac{1}{2} C \Psi_1^r \bigg|_{z^0 = y^0}. \tag{5.52}
\]

These right-hand sides reduce to the ones of (3.110) and (3.120) when we integrate their first terms by parts and apply (3.2), (4.51), and (4.55) to them.

In relation to the right-handed lepton fields, we have only the following anti-commutation relation between $Q_Y$ and the integral representation (4.68):

\[
\{iQ^r_Y, \eta_e(x)\} = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] S(x, z) \\
\times h h^\mu a \overset{\circ}{\sigma}_a \left( \partial_{\mu} + \overset{\circ}{S}_{bc} \omega_{\mu}^{bc} \right) (\lambda_Y C \eta_e) \\
+ \int d^3 z S(x, z) h h^{0a} \overset{\circ}{\sigma}_a \lambda_Y C \eta_e \bigg|_{z^0 = y^0}. \tag{5.53}
\]
This right-hand side reproduces the one of (3.122) with the similar treatment of that in (5.49) and (5.50).

The commutation relations between $Q_W$ and $Q_Y$, and the integral representation for the Higgs field (4.76) are given by

\[
[iQ_W, \Phi^r(x)] = - \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \mathcal{D}(x, z) \partial^z_\lambda \left[ h g^{\lambda \mu} \partial^z_\mu \left( i \lambda_W \frac{\tau^r_j}{2} C^j \Phi \right) \right] + \int d^3z \left[ \mathcal{D}(x, z) \partial^z_\nu \cdot h g^{0 \nu} \left( i \lambda_W \frac{\tau^r_j}{2} C^j \Phi \right) \right] \bigg|_{z^0 = y^0},
\]

(5.54)

\[
[iQ_Y, \Phi^r(x)] = - \int d^4z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right] \mathcal{D}(x, z) \partial^z_\lambda \left[ h g^{\lambda \mu} \partial^z_\mu \left( i \frac{\lambda_Y}{2} C \Phi^r \right) \right] + \int d^3z \left[ \mathcal{D}(x, z) \partial^z_\nu \cdot h g^{0 \nu} \left( i \frac{\lambda_Y}{2} C \Phi^r \right) \right] \bigg|_{z^0 = y^0},
\]

(5.55)

with the use of (3.112) and (3.123). In these right-hand sides, we integrate their first terms by parts to cancel out the parts of the second ones via (4.9), (4.15), and (4.16). Thus, these right-hand sides reduce to the ones of (3.112) and (3.123).

6. Summary and remarks

In the present paper, we have reviewed the (anti-)commutation relations with respect to the matter-field operators in the Heisenberg picture on the basis of quantum Einstein gravity. We have regarded the terms proportional to the coupling constants in the field equations (2.17), (2.20)–(2.22),
(2.25), (2.27), (2.30)–(2.32), (2.40)–(2.46) as inhomogeneous ones corresponding to that in (1.15). Then, we have solved these 16 equations and (2.34)–(2.36) in terms of the integral representations for the matter fields using the bilocal operators $D_{\mu\nu}(x, z)$, $D(x, z)$, $S(x, z)$, and $\bar{S}(x, z)$. These integral representations contain the gravitational, strong, and electroweak interactions. We have also shown the properties of these integral representations under the affine, gravitational BRST, internal Lorentz BRST, $SU(3)_C$ BRST, $SU(2)_W$ BRST, and $U(1)_Y$ BRST transformations.

Because of the inhomogeneous terms in the above 16 equations, the 4D integral terms with the factor $[\theta(x^0 - z^0) - \theta(y^0 - z^0)]$ appear in (4.20)–(4.22), (4.32)–(4.37), (4.63)–(4.68), and (4.76). These terms are sums of “retarded” superpositions on $x^0 > z^0 > y^0$ for $x^0 > y^0$, and “advanced” ones on $x^0 < z^0 < y^0$ for $x^0 < y^0$. The factor $[\theta(x^0 - z^0) - \theta(y^0 - z^0)]$ in these integral representations yield “time-order” between $x^0$ and $z^0$, while the bilocal operators $D_{\mu\nu}(x, z)$, $D(x, z)$, $S(x, z)$, and $\bar{S}(x, z)$ are not functions of $x - z$ alone. Therefore, these inhomogeneous terms at the point $z$ contribute to the field operators at the point $x$ via both these bilocal operators and the “time-order”. So, we can regard these inhomogeneous terms as origins of “time-evolution” in the operator level.

In addition, the 4D integral terms in the integral representations for the Yang–Mills fields contain not only the interactions with the Weyl and Higgs fields but also the self-interaction and the interaction with the Faddeev–Popov ghost fields. This fact means that the non-abelian properties of Yang–Mills fields and the existence of Faddeev–Popov ghost ones are the parts of origins of “time-evolution”. The 4D integral terms in the integral representations for the Weyl fields contain the interactions with the Yang–Mills fields, and the mixing of the right-handed Weyl fields and the left-
handed ones with the Higgs one. These are origins of “time-evolution”. Also, in the 4D integral term for the Higgs field, the interactions with the Yang–Mills fields and the Weyl ones, the self-interaction, and the squared “mass” are origins of “time-evolution”.

If a physical vacuum $|0\rangle$ is translationally invariant,

$$\hat{P}_\lambda |0\rangle = 0, \quad (6.1)$$

then the vacuum expectation values of all the matter fields are independent of $x^\lambda$. For example, the vacuum expectation value of the translation of the gluon field $A_\mu^a(x)$ is given by

$$\langle 0 | [i\hat{P}_\lambda, A_\mu^a(x)] |0\rangle = \partial^x_\lambda \langle 0 | A_\mu^a(x) |0\rangle = 0. \quad (6.2)$$

Consequently, taking the vacuum expectation value of the integral representation (4.20),

$$\langle 0 | A_\mu^a(x) |0\rangle = \int d^4 z \left[ \theta(x^0 - z^0) - \theta(y^0 - z^0) \right]$$

$$\times \langle 0 | D_{\mu\nu}(x, z) \lambda_C \{ J^{\nu a}_C + f^{abc} \left[ \partial^x_\lambda \left( h g^{\lambda\rho} g^{\nu\sigma} A^b_\rho A^c_\sigma \right) \right.$$

$$+ h g^{\lambda\rho} g^{\nu\sigma} A^b_\lambda G^{c}_{\rho\sigma}$$

$$\left. + i h g^{\nu\rho} \partial^x_\rho \bar{C}^b \cdot C^c \right] \} |0\rangle$$

$$+ \int d^3 z \langle 0 | J^{a0}_\mu(x, z) |0\rangle |_{z^0 = y^0}, \quad (6.3)$$

we find that the $x^\lambda$-dependence of the first term in the right-hand side cancels out that of the second one. In relation to the integral representations for the other matter fields except for (4.38)–(4.40), we can see similar facts for their vacuum expectation values.
References