On GNA-subgroups in locally finite groups

A.A. Рурка, N.A. Турабай

A new generalization of pronormal subgroups, namely GNA-subgroups is introduced. It is proved that if $G$ is a locally finite group with all subgroups GNA-subgroups then $G$ is a $\bar{T}$-group.

**Keywords:** abnormal subgroup, pronormal subgroup, GNA-subgroup, locally finite group. $\bar{T}$-group.

1. **Introduction.** The investigation of influence of some important systems of subgroups on the structure of the whole group is one of the classical questions in Group Theory. Normal subgroups and their generalizations (subnormal subgroups, ascendant subgroups, almost normal subgroups, nearly normal subgroups and others) have a strong influence on the group structure, a question studied by many past and recent researchers (S.N. Chernikov, R. Baer, P. Hall, L.A. Kurdachenko, B.H. Neumann, S.E. Stonehewer, J. Wiegold, D.J.S. Robinson, D.I. Zaitsev, M.R. Dixon, H. Smith, C. Casolo, I.Ya. Subbotin, N.F. Kuzennyi, N.N. Semko, F.N. Liman and many others). Another important subgroups that have a significant effect on the group structure are antipodes of normal and generalized normal subgroups. Some of the most common types of such subgroups are abnormal, self-normalizing, contranormal subgroups and their generalizations. Recall that a subgroup $H$ of a group $G$ is called abnormal in $G$ if $g \in H, H^g >$ for each element $g \in G$. Note also that a subgroup $H$ of a group $G$ is both normal and abnormal in $G$ if $H = G$. A very important generalization of normal and abnormal subgroups is pronormal subgroups. A subgroup $H$ of a group $G$ is called pronormal in $G$ if for each element $g \in G$ the subgroups $H$ and $H^g$ are conjugate in $< H, H^g >$, i.e. $H^g = H^u$ for some element $u \in < H, H^g >$. The investigation of pronormal subgroups was initiated by P. Hall more than 75 years ago. The first non-trivial results about pronormal subgroups in finite groups were obtained by T.A. Peng [1], [2], J.S. Rose [3], A. Mann [4] and G.J. Wood [5]. With [6] M.S. Ba and Z.I. Borevich began the study of pronormal subgroups in infinite groups.

It was natural to consider the groups with all subgroups pronormal. In [1] T.A. Peng investigated some finite groups of such type. In particular, he proved that if $G$ is a finite group whose subgroups of prime power order are all pronormal in $G$ then $G$ is a soluble $T$-group. Recall that a group $G$ is called a $T$-group if every subnormal subgroup of $G$ is normal. A group $G$ is called a $\bar{T}$-group if every subgroup of $G$ is a $T$-group. Note that W. Gaschütz [7] proved that every finite soluble $T$-group is a $\bar{T}$-group. In [8] N.F. Kuzennyi and I.Ya. Subbotin studied infinite groups with all subgroups pronormal. In particular, they described locally soluble non-periodic and locally graded periodic groups with all subgroups pronormal. A special case of the above situation is the investigation of groups’ structure whose all subgroups are normal. R. Dedekind in [9] obtained a description of the finite groups with all normal subgroups. Later, R. Baer in [10] obtained a description of all infinite groups of such type. Note that a group (not necessarily finite) with all subgroups normal is called a Dedekind group.

On the other hand, as we noted above, every normal subgroup and abnormal subgroup is pronormal. It is interesting to consider the situation when a group has only subgroups with certain property and the antipode of this property. Many articles are devoted to the discussion of this problem with one of the first ones written by A. Fattahi [11]. A. Fattahi obtained the description of finite groups with all subgroups normal and abnormal. For infinite groups this situation was studied by I.Ya. Subbotin in [12] and L.A. Kurdachenko and I.Ya. Subbotin in [13].

In this paper we consider the «local version» of pronormality. It is well-known that a normalizer of pronormal subgroup is abnormal. Also note that every abnormal subgroup is self-normalizing (i.e. coin-
Consider a pronormal subgroup of a group $G$. If $x \in N_G(H)$, then, obviously, $H^x = H$. On the other hand $N_G(H)$ is abnormal in $G$, and then it is self-normalizing in $G$. Therefore, $N_G(N_G(H)) = N_G(H)$. Thus, we naturally obtain the following generalization of pronormal subgroups.

**Definition.** Let $G$ be a group and $H$ be a subgroup of $G$. A subgroup $H$ of a group $G$ is called GNA-subgroup (generalized normal and abnormal) of $G$ if for every element $x \in G$ either $H^x = H$ or $N_K(N_K(H)) = N_K(H)$, where $K = <H, x>$, $x \in G$.

As we see from the definition, every pronormal subgroup is a GNA-subgroup, but the converse is not true in general. The following example shows this.

**Example.** Let $G$ be a symmetric group of degree 6. Suppose that $H$ is a subgroup of $G$, which is generated by the elements $(4, 5, 6)$ and $(2, 3)$. A subgroup $H$ is a GNA-subgroup of $G$, but it is not pronormal in $G$. Note also, that $H$ has order 6, and $H$ is nilpotent.

Thus, GNA-subgroups are a non-trivial generalization of pronormal subgroups. It is natural to consider the groups with all subgroups GNA-subgroups. In this paper we investigate locally finite groups of such type. The following theorem is the main result of this paper.

**Theorem.** Let $G$ be a locally finite group such that every subgroup of $G$ is GNA-subgroup. Then $G$ is a $\bar{T}$-group.

**2. Preliminaries and lemmas.**

**Lemma 2.1.** Let $G$ be a group such that every subgroup of $G$ is GNA-subgroup.

1) If $H$ is a subgroup of $G$, then every subgroup of $H$ is GNA-subgroup.

2) If $H$ is a normal subgroup of $G$, then every subgroup of the factor-group $G/H$ is GNA-subgroup.

**Proof.** The assertion follows from the definition.

**Lemma 2.2.** Let $G$ be a group such that every subgroup of $G$ is GNA-subgroup. If $H$ is a nilpotent subgroup of $G$, then $H$ is a Dedekind group.

**Proof.** Let $H$ be a nilpotent subgroup of $G$ and $x, y \in H$. Put $K = <x, y>$. Since $K$ is nilpotent, $H$ has no proper self-normalizing subgroups, therefore, $N_K(N_K(<x>)) \neq N_K(<x>)$. Thus, $<x>^y = <x>$, which means that every cyclic subgroup of $H$ is normal in $H$. Therefore every subgroup of $H$ is normal in $H$. Thus $H$ is a Dedekind group.

**Corollary 2.3.** Let $G$ be a group such that every subgroup of $G$ is GNA-subgroup. If $H$ is a locally nilpotent subgroup of $G$, then $H$ is a Dedekind group.

**Proof.** Let $H$ be a locally nilpotent subgroup of $G$ and $x, y \in H$. Put $K = <x, y>$. Since $H$ is a locally nilpotent subgroup, $K$ is nilpotent. By Lemma 2.2 $<x>^y = <x>$. It follows that $H$ is a Dedekind group.

**Lemma 2.4.** Let $G$ be a group such that every subgroup of $G$ is GNA-subgroup. Suppose that $H$ be a subgroup of $G$ and $A$ be a normal abelian subgroup of $H$. Then every subgroup of $A$ is $G$-invariant.

**Proof.** Let $a \in A$. If $a = <a>$ then the proof is obvious. Let $<a>$ be a proper subgroup of $A$ (i.e. $<a> \neq A$). Put $K = <A, y>$, where $y \in H \backslash A$. Obviously, in this case $A \leq N_K(<a>)$ and the factor-group $K/A$ is abelian. This means that $N_K(<a>)$ is normal subgroup of $K$. Thus $N_K(<a>) = K$, since every subgroup of $G$ is GNA-subgroup. Therefore every proper cyclic subgroup $<a>$ is $H$-invariant and then $G$-invariant, since $H$ is an arbitrary subgroup of $G$. Hence every subgroup of $A$ is $G$-invariant.

**Corollary 2.5.** Let $G$ be a group such that every subgroup of $G$ is GNA-subgroup. Suppose that $H$ be a nilpotent subgroup of $G$ and $A$ be an abelian minimal $H$-invariant subgroup of $H$. Then $A$ is a cyclic subgroup of prime order.

Let $G$ be a group, $H, K$ be normal subgroups of $G$ such that $H \leq K$. The factor $K/H$ is called central (respectively, eccentric), if $C_G(K/H) = G$ (respectively, $C_G(K/H) \neq G$).

**Lemma 2.6.** Let $G$ be a finite group such that every subgroup of $G$ is GNA-subgroup. Then $G$ is supersoluble.

**Proof.** Let $S_p$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing the order of $G$. Put $L = N_G(S_p)$. By Corollary 2.5 $S_p$ has a chief series with cyclic factors of prime order. Denote by $A/B$ an arbitrary chief factor of $S_p$. The order of the factor-group $L/C_L(A/B)$ divides $p - 1$. But $p$ is the smallest prime number, which implies $C_L(A/B) = N_G(S_p) = L$. Thus the chief
factor $A/B$ is central. Hence $N_G(S_p) = S_p \times S_p$. If $p > 2$ then $S_p$ is abelian and $S_p \leq \zeta(N_G(S_p))$. Then by Burnside's Theorem $S_p$ has a normal complement in $G$. If $p = 2$ then by Lemma 2.2 $S_2$ is Dedekind and by Theorem 1 from [14] $S_2$ also has a normal complement in $G$. Thus in both cases there is a normal subgroup $H$ such that $G = H a S_p$. Using the similar arguments we choose a Sylow $q$-subgroup $S_q$ in $H$, where $q$ is the smallest prime dividing the order of $H$. Thus we obtain that $S_q$ has a normal complement $K$ in $H$, i.e. $H = K a S_q$. After a finite number of steps we obtain a Sylow series with cyclic factors of prime order. It follows that $G$ is supersoluble.

**Corollary 2.7.** Let $G$ be a locally finite group such that every subgroup of $G$ is GNA-subgroup. Then $G$ is locally supersoluble.

**Proof.** Let $H$ be a finitely generated subgroup of $G$. Since $G$ is locally finite, $H$ is finite. Note that by Lemma 2.1 every subgroup of $H$ is GNA-subgroup. By Lemma 2.6 a subgroup $H$ is supersoluble. Thus every finitely generated subgroup of $G$ is supersoluble. Therefore $G$ is locally supersoluble.

Further we need the following result due to R. Baer.

**Theorem 2.8** [15]. Let $G$ be a finite group and $A$ be an abelian normal subgroup of $G$. If the factor-group $G/A$ is nilpotent then

$$A = Z \times E,$$

where all chief factors of $Z$ are central and all chief factors of $E$ are eccentric.

In this case we will say that a subgroup $A$ has a $Z$-decomposition in $G$.

Let $G$ be a group. Recall that the locally nilpotent residual $L$ of $G$ is the intersection of all normal subgroups $H$ of $G$ such that $G/H$ is locally nilpotent.

**Lemma 2.9.** Let $G$ be a locally finite group such that every subgroup of $G$ is GNA-subgroup. Then all $G$-chief factors of locally nilpotent residual $L$ are eccentric.

**Proof.** Suppose that the locally nilpotent residual $L$ of $G$ has central factors. Let $A$ be an abelian normal subgroup of $L$. Denote by $S$ the set of all finite subgroups of $G$. Let $H \in S$ and denote by $H$ the set of all finite $H$-invariant subgroups of $A$. Let $B \in H$. By Theorem 2.8 a subgroup $B$ has a $Z$-decomposition $B = Z_b \times E_b$, where all chief factors of $Z_b$ are central and all chief factors of $E_b$ are eccentric. Note that if $B \leq C$ for some finite subgroup $C \in S$ then $Z_b \geq Z_c$, $E_b \leq E_c$. Put $E = \bigcup_{H \in S} E_H$.

Note that $E$ is a normal subgroup of $G$. All chief factors of $E$ are eccentric and all chief factors of the factor-group $A/E$ are central. Thus $A/E \leq \zeta(G/E)$. The factor-group $G/A$ is locally nilpotent, then $G/E$ is also locally nilpotent. A subgroup $E$ is a proper subgroup of $A$, i.e. $A \neq E$. On the other hand $G/E$ is nilpotent and then $A \leq E$. And we obtain a contradiction. Thus $L$ doesn’t have central factors.

3. **Proof of the main result.** Since $G$ is a locally finite group such that every subgroup of $G$ is GNA-subgroup, by Corollary 2.7 $G$ is locally supersoluble. This means that $G$ has a descending Sylow series (not necessarily finite)

$$G = G_0 \geq G_1 \geq G_2 \geq \ldots,$$

where $G/G_1$ is a Sylow $p_1$-subgroup, $G_1/G_2$ is a Sylow $p_2$-subgroup etc., where $p_1 < p_2 < \ldots$. In particular, the factor-group $G/G_1$ can be a 2-group. Let $L$ be a locally nilpotent residual of $G$. Since $G/G_1$ is a locally nilpotent 2-group, $G_1 \geq L$, and then $2 \not\in \Pi(L)$. Thus Sylow $p$-subgroups of $L$ are abelian.

Since $G$ is a supersoluble group, the derived subgroup $[G, G]$ is locally nilpotent. This means that $L$ is also locally nilpotent, because $[G, G]$ contains $L$. Since $2 \not\in \Pi(L)$, a locally nilpotent residual $L$ is abelian.

Since the factor-group $G/L$ is locally nilpotent, then by Corollary 2.3 it is Dedekind. A subgroup $L$ is abelian and normal in $G$, then by Lemma 2.4 every subgroup of $L$ is $G$-invariant.

Finally, we need to show that $\Pi(L) \cap \Pi(G/L) = \emptyset$. Suppose that it is not true. Then there is a prime number $p$ such that $p \in \Pi(L) \cap \Pi(G/L)$. Note that from $p \in \Pi(L)$ and $2 \notin \Pi(L)$ implies that $p \neq 2$. Since $L$ is abelian, $L = S_p \times S_p$. Put $H = G/S_p$ and $K = L/H$. Then $K$ is a $p$-subgroup and $H/K$ is nilpotent. Choose in $H/K$ a Sylow $p$-subgroup $P$ (i.e. $P \in Syl_p(H/K)$). Note that $P$ contains $K$. Since $H/K$ is nilpotent, $\zeta(P/K) \neq 1$. By Lemma 2.9 all $G$-chief factors of $L$ are eccen-
tric. Put \( Z = \langle zK \rangle \). Since the factor \( P/K \) is central, \( [H, Z] \leq K \). By Theorem 2.8 and last inclusion we have \( Z = C \times K \) for some subgroup \( C \). On the other hand by Lemma 2.4 every subgroup of \( Z \) is \( G \)-invariant. Pick an element \( x \in \Omega_1(K) = \{x \in K \mid x^n = 1\} \). Then there is an integer \( k \) such that \( (k, p) = 1 \) and \( x^k = g \in G \). If \( c \in C \) then \( c^k = c \). Thus we have \( (xc)^s = x^s c^s = x^s c \neq (x^s)^1 \) for some \( s \). Therefore \( Z \) has an element which is not \( G \)-invariant. And we obtain a contradiction. Applying the results of the paper [16] we obtain that \( G \) is a \( \overline{E} \)-group.

Note that the last Theorem can’t be generalized to the case of arbitrary periodic groups. The following example illustrates this.

**Example.** Let \( G \) be a Tarski monster group, i.e. \( G \) is an infinite group such that every proper non-identity subgroup is cyclic of prime order \( p \). Note that A.Yu. Ol’shanskii proved the existence of such groups for any prime number \( p > 10^{15} \) [17, § 28]. Every element of \( G \) generates cyclic subgroup of prime order \( p \) and every two non-commuting elements generate the whole group \( G \). If \( y \in <x> \) then \( <x>^y =<y> \). If \( y \not\in <x> \) then \( <x, y> = G \) and \( <x> = N_G(<x>) \). Thus \( N_G(<x>) = N_G(N_G(<x>)) \). Therefore \( G \) is a periodic group such that every subgroup of \( G \) is \( G \)-subgroup, but \( G \) is not a \( \overline{E} \)-group.

**References**