Introduction

Only finite groups are considered. One of the important problems of group theory is the structural study of a finite group which can be factorized as a product of two or more pairwise permutable subgroups. The origin of this problem may be traced back to the well-known theorem of Burnside about the solvability of biprimary groups.

Formations closed under taking products of certain types (arbitrary [1], normal and subnormal [2], abnormal and contrnormal [3] and etc.) of subgroups were studied in many papers. An important generalization of the subnormality is the $\mathfrak{N}$-subnormality [4], [5]. Formations closed under taking products of $\mathfrak{N}$-subnormal subgroups were studied in [6]–[8] etc. Formations with the Shemetkov property play significant role in this research. Recall that a formation $\mathfrak{N}$ is called a formation with the Shemetkov property if every $\mathfrak{s}$-critical group for $\mathfrak{N}$ is either a Schmidt group or a group of prime order.

In 1938 Fitting [9] showed that a product of two normal nilpotent subgroups is again nilpotent. It means that there exists the unique maximal normal nilpotent subgroup $F(G)$ in every group $G$. This subgroup is called the Fitting subgroup. The Fitting subgroup has a great influence on the structure of a soluble group. That is why in the paper [10] authors introduced the following definition.

Definition 0.1. A subgroup $H$ of a group $G$ is called $F(G)$-subnormal if $H$ is subnormal in $HF(G)$. A subnormal subgroup of a group $G$ is obviously $F(G)$-subnormal. The following example shows that in the general case a $F(G)$-subnormal subgroup is not subnormal.

Example 0.2. Let $G \cong S_4$ be the symmetric group of degree 4. Let $H$ be a Sylow 2-subgroup of $G$. Then $H$ is a maximal subgroup of $G$ which is not normal in $G$. Note that $F(G) \leq H$. Hence $H$ is $F(G)$-subnormal in $G$. But $H$ is not subnormal in $G$.

Definition 0.3. Let $\mathfrak{N}$ and $\mathfrak{X}$ be classes of soluble groups. We say that $\mathfrak{N}$ is $F(G)$-radical in $\mathfrak{X}$ if $\mathfrak{N}$ is $S_\pi$-closed and contains every $\mathfrak{X}$-group $G = AB$ where $A$ and $B$ are $F(G)$-subnormal $\mathfrak{N}$-subgroups of $G$.

The following problem seems natural.

Problem A. Describe all classes (formations, Schunk classes, Fitting classes) of soluble groups which are $F(G)$-radical in the class $\mathfrak{S}$ of all soluble groups.

Definition 0.4. We shall call a class of groups $\mathfrak{X}$ $S_\pi$-closed if $\mathfrak{X}$ contains with every group $G$ all its Schmidt subgroups.

Recall that $S_\pi$ is the class of all soluble $\pi$-groups. Formations closed under products of $F(G)$-subnormal subgroups are described in the following theorem.
Theorem A. Let $\mathfrak{F}$ be a $S$-closed saturated formation of soluble groups and $\pi = \pi(\mathfrak{F})$. The following statements are equivalent:

1. $\mathfrak{F}$ is $F(G)$-radical in $\mathfrak{S}$.
2. $\mathfrak{F}$ contains every soluble group $G = AB$ where $A$ and $B$ are $F(G)$-subnormal $\mathfrak{F}$-subgroups of $G$.
3. $\mathfrak{F}$ is a hereditary formation and there exists a partition $\sigma = \{\pi_i | i \in I\}$ of $\pi$ into mutually disjoint subsets such that $\mathfrak{F} = \bigoplus_{i \in I} \mathfrak{S}_{\pi_i}$.

Corollary A.1 [10]. Let $G = AB$ be a product of nilpotent $F(G)$-subnormal subgroups. Then $G$ is nilpotent.

Corollary A.2. Let $\pi$ be a set of primes and a soluble group $G = AB$ be a product of $\pi$-decomposable $F(G)$-subnormal subgroups. Then $G$ is $\pi$-decomposable.

Let us note that the class $\mathfrak{S}_{\pi} = \{G | G = O_{\pi_1}(G) \times \cdots \times O_{\pi_n}(G)\}$ is a lattice formation. Recall that a formation $\mathfrak{F}$ is called lattice if the intersection and the join of two $\mathfrak{F}$-subnormal subgroups is again a $\mathfrak{F}$-subnormal subgroup. This formations were studied by many researchers [5, chapter 6].

There are examples [11, p. 8] of non-supersoluble groups which are products of supersoluble normal (subnormal) subgroups. So the formation $\mathfrak{U}$ of all supersoluble groups is not $F(G)$-radical in $\mathfrak{S}$. R. Baer [12] showed that if a group $G$ is the product of two normal supersoluble subgroups and $G'$ is nilpotent then $G$ is supersoluble. In [13] A.F. Vasil’ev and D.N. Simonenko generalized Baer’s theorem on arbitrary hereditary saturated formations.

These results are the motivations for the following

Problem B. Let $\mathfrak{X}$ be a hereditary saturated formation of soluble groups. Describe all hereditary saturated $F(G)$-radical in $\mathfrak{X}$ subformations $\mathfrak{F}$ of $\mathfrak{X}$.

Theorem B. Let $\mathfrak{X}$ be a hereditary saturated formation of soluble groups. The following statements are equivalent:

1. Every hereditary saturated subformation $\mathfrak{F}$ of $\mathfrak{X}$ is $F(G)$-radical in $\mathfrak{X}$.
2. Every group in $\mathfrak{X}$ has nilpotent derived subgroup.

K. Doerk [14] showed that a group is supersoluble if it contains four supersoluble subgroups of pairwise coprime indexes. This result was generalized by O.U. Kramer [15] on arbitrary saturated formations of metanilpotent groups.

Problem C. Let $n$ be a natural number, $n \geq 3$ and $\mathfrak{F}$ be a saturated formation of soluble groups such that $\mathfrak{F}$ contains every group $G$ which has $n$ $\mathfrak{F}$-subgroups of pairwise coprime indexes in $G$. Assume that a group $G$ contains $n-1$ $F(G)$-subnormal $\mathfrak{F}$-subgroups of pairwise coprime indexes in $G$. Does $G \in \mathfrak{F}$?

Partially answer on this problem is given in the following theorem.

Theorem C. Let $\mathfrak{F}$ be a hereditary saturated formation of metanilpotent groups with Sylow tower. If a group $G$ contains three $F(G)$-subnormal $\mathfrak{F}$-subgroups of pairwise coprime indexes in $G$ then $G \in \mathfrak{F}$.

Corollary C.1. If a group $G$ contains three $F(G)$-subnormal supersoluble subgroups of pairwise coprime indexes in $G$ then $G$ is supersoluble.

Corollary C.2. Let $\mathfrak{F}$ be the formation of groups with nilpotent derived subgroup and Sylow tower. If a group $G$ contains three $F(G)$-subnormal $\mathfrak{F}$-subgroups of pairwise coprime indexes in $G$ then $G \in \mathfrak{F}$.

D.K. Friesen [16] noted that if a group $G$ contains two normal (subnormal) supersoluble subgroups of coprime indexes in $G$ then $G$ is supersoluble. The following example shows that we can not replace the subnormality by the $F(G)$-subnormality in Friesen’s theorem.

Example 0.5. Let a group $G$ be isomorphic to the symmetric group of degree 3. Then there is a faithful irreducible $G$-module $V$ of dimension 2 over $F$. Let $T$ be the semidirect product of $V$ and $G$. Consider $A = VG_1$ and $B = VG_2$ where $G_p$ is a Sylow $p$-subgroup of $G$ and $p \in \{2, 3\}$. From $7 = 1 (\text{mod} \ p)$ and $p \in \{2, 3\}$ it follows that $A$ and $B$ are supersoluble. Since $V$ is a faithful irreducible $G$-module, $F(T) = V$. Now $A$ and $B$ are $F(T)$-subnormal supersoluble subgroups of $T$. Note that $T = AB$ is not supersoluble.

1 Preliminary results

We use standard notation and terminology that if necessary can be found in [17]. Recall some of them that are important in this paper. By $\mathfrak{P}$ is denoted the set of all primes; $\pi(G)$ is the set of all prime divisors of the order of $G$; $\pi(\mathfrak{S}) = \bigcup_{\mathfrak{S}} \pi(G)$; a group $G$ is called $\pi$-group if $\pi(G) \subseteq \pi$; $Z_p$ is the cyclic group of order $p$; $O_p(G)$ is the greatest normal $\pi$-subgroup of $G$; $G'$ is the derived subgroup of $G$; $G^3$ is the $\mathfrak{F}$-residual for a formation $\mathfrak{F}$; $O_{\pi, p}(G)$ is the $p$-nilpotent radical of $G$ for $p \in \mathfrak{P}$ it also can be defined by $O_{\pi, p}(G) = O_{\pi, p}(G) = O_p(G / O_{\pi, p}(G))$; $\Phi(G)$ is the Frattini subgroup of $G$; $\ast$ is the regular wreath product of groups $A$ and $B$; $G = N \ast M$ is the semidirect product of groups $M$ and $N$ ($N < G$ and $N \cap M = 1$); $\Phi_{\pi}(\mathfrak{S})$ is the class of all (nilpotent) $\pi$-groups, where

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On the influence of the Fitting subgroup on products of finite soluble groups

Let $\pi = \{\pi_i | i \in I\}$ be a partition of $\pi$ into mutually disjoint subsets then the class of all groups which are direct products of their (soluble) $\pi_i$-subgroups is denoted by $\times_{i \in I} \pi_i = \times_{i \in I} \pi_i$. Let $F$ and $X$ be formations then $F \subseteq X$.

A class of groups $\mathfrak{F}$ is called a formation if from $G \in \mathfrak{F}$ and $N \unlhd G$ it follows that $G/N \in \mathfrak{F}$ and from $H/A \in \mathfrak{F}$ and $H/B \in \mathfrak{F}$ it follows that $H/A \cap B \in \mathfrak{F}$.

A class of groups $X$ is called hereditary ($\mathfrak{H}$-closed) if from $G \in X$ and $H \leq G$ ($H \lhd G$) it follows that $H \in X$.

A class of groups $X$ is called weakly hereditary if from $p \in \pi(X)$ it follows that $Z_p \in X$.

A class of groups $X$ is called saturated if from $G/\Phi(G) \in X$ it follows that $G \in X$.

A function $f : \mathcal{P} \to \{\text{formations}\}$ is called a formation function.

By well known Gashütz – Lubeseder – Schmid theorem saturated formations are exactly local formations, i.e. formations $\mathfrak{F} = LF(f)$ defined by a formation function $f : LF(f) = \{G \in \mathbb{P} | \text{if } H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K) \text{ then } G/\varphi_p(H/K) \in f(p)\}$.

Among all local definitions of a local formation $\mathfrak{F}$ there is exactly one, denoted by $F$, such that $F$ is integrated ($F(p) \subseteq \mathfrak{F}$ for all $p \in \mathcal{P}$) and full ($\forall p, F(p) = F(p)$ for all $p \in \mathcal{P}$). The function $F$ is called the canonical local definition of $\mathfrak{F}$.

**Lemma 1.1** ([17, p. 357]). Let $f$ be a local definition of a formation $\mathfrak{F}$. A group $G$ belongs $\mathfrak{F}$ if and only if $G/O_{\varphi_p}(G) \in f(p)$ for all $p \in \pi(G)$.

Recall some properties of Schmidt groups.

**Lemma 1.2** ([4, p. 243]). Let $G$ be a Schmidt group. Then

1. $G = P \vartriangleleft Q$ where $P$ is the normal $p$-subgroup of $G$ and $Q$ is a cyclic Sylow $q$-subgroup of $G$ that is not normal in $G$.
2. $G/\Phi(G)$ is a Schmidt group.
3. $P\Phi(G)/\Phi(G)$ is an elementary abelian $p$-subgroup and $\lvert \Phi(G)/\Phi(G) \rvert = q$.

Recall that $\mathfrak{S}$ is the greatest hereditary subclass of a class of groups $\mathfrak{S}$. Let $X$ be a class of groups. Recall that a group $G$ is called $s$-critical for $X$ if $G \in X$ but every proper subgroup of $G$ belongs in $X$. The class of all $s$-critical for $X$ groups is denoted by $\mathcal{M}(X)$. Note that $\mathcal{M}(\mathfrak{S}) = \mathcal{M}(\mathfrak{S}^2)$.

*2 $\mathfrak{S}_G$-closed formations*

In the sequel a Schmidt $(p, q)$-group is a Schmidt $(p, q)$-group with a normal Sylow $p$-subgroup.

**Lemma 2.1.** Let $\mathfrak{S}$ be a saturated formation and $S$ be a Schmidt $(p, q)$-group. If $S \in \mathfrak{S}$ then every Schmidt $(p, q)$-group belongs to $\mathfrak{S}$.

**Proof.** Let $f$ be a local definition of $\mathfrak{S}$. From lemmas 1.1 and 1.2 it follows that $S/O_{\varphi_p}(S) = 1 \in f(p)$ and $S/O_{\varphi_q}(S) = Z_q \in f(q)$. Now if $K$ is a Schmidt $(p, q)$-group then $K/O_{\varphi_p}(K) = 1$ and $K/O_{\varphi_q}(K) = Z_q$ by lemma 1.2. So $K \in \mathfrak{S}$ by lemma 1.1. \(\Box\)

**Theorem 2.2.** Let $\mathfrak{S} = LF(F)$ be a local formation of soluble groups and $F$ be the canonical local definition of $\mathfrak{S}$. Then $\mathfrak{S}$ is $\mathfrak{S}_G$-closed if and only if $F(p)$ is a weakly hereditary formation for every $p \in \pi(\mathfrak{S})$.

**Proof.** Let $\mathfrak{S} = LF(F)$ be a $\mathfrak{S}_G$-closed formation, $F$ be the canonical local definition of $\mathfrak{S}$ and $p \in \pi(\mathfrak{S})$. Then $F(p) \neq \emptyset$.

Consider $q \in \pi(F(p))$. If $q = p$ then $Z_p \in \mathfrak{S}_F \subseteq F(p)$.

Assume that $q \neq p$. Let $G$ be a group of minimal order such that $G \in F(p)$ and $q \in \pi(G)$. Note that $O_{\varphi_q}(G) = 1$. Let $R = Z_p \wr G = L \vartriangleleft G$ where $L = Z_p \times \ldots \times Z_p$ is the base of $R$.

From $G \in F(p)$, lemma 1.1 and the properties of the regular wreath product it follows that $R \in \mathfrak{S}$. Let $R_q$ be a Sylow $q$-subgroup of $R$. Consider $T = LR_q$. By the properties of the regular wreath product $C_q(L) = L$. That is why $T$ is non-nilpotent. Then $T$ has a Schmidt $(p, q)$-subgroup $S$. So $S \in \mathfrak{S}$.

Since $S/O_{\varphi_p}(S) = Z_q \times Z_q \in F(p)$. Q.E.D.

Let $F(p)$ be a weakly hereditary formation for all $p \in \pi(\mathfrak{S})$. Assume that the theorem is false and let $G$ be a minimal order counterexample. It means that $G \in \mathfrak{S}$ and $G$ has a Schmidt $(p, q)$-subgroup $S \notin \mathfrak{S}$. Since $G$ is soluble, we see that the order of every minimal normal subgroup of $G$ is the power of a prime.

Let $N$ be a minimal normal $r$-subgroup of $G$. Assume that $q \neq r$ and $p \neq r$. Then $N \cap S = 1$. It means that $S = SN / N \subseteq G / N$. By our assumption $S \in \mathfrak{S}$, a contradiction.

Assume that $q = r$. From lemma 1.2 it follows that $N \cap S \leq \Phi(S)$. It means that $SN / N \in \mathfrak{S}$. So $SN / N$ is a Schmidt group by lemma 1.2. By lemma 2.1 $S \in \mathfrak{S}$, a contradiction.
Thus $O_p(G) = 1$. Now
\[ q \in \pi(G / O_{p^*}(G)) \subseteq \pi(F(p)). \]
From $S / O_{p^*}(S) = 1$, $S / O_{p^*}(T) = Z_p \subseteq F(p)$ and lemma 1.1 it follows that $S \subseteq \bar{S}$, the final contradiction. □

**Lemma 2.3.** Every $S_{ch}$-closed formation of soluble groups is weakly hereditary.

**Proof.** Let $\bar{S}$ be a $S_{ch}$-closed formation of soluble groups, $G \subseteq \bar{S}$ and $p \in \pi(G)$. Since $G$ is soluble, there is a chief factor $H / K$ of $G$ such that $p(H / K) = \{p\}$. Since $\bar{S}$ is a $S_{ch}$-closed formation, $H / K \subseteq \bar{S}$. From $H / K = Z_p \times \ldots \times Z_p$ it follows that $Z_p \subseteq \bar{S}$. □

**Corollary 2.4.** Let $\bar{S}$ be a saturated $S_{ch}$-closed formation of soluble groups. Then $\bar{S}$ is $S_{ch}$-closed.

**Proof.** According to [17, p. 365] $\bar{S}$ has the canonical local definition $F$ such that $F(p)$ is a $S_{ch}$-closed formation for every prime $p$. By lemma 2.3 $F(p)$ is a weakly hereditary formation for every prime $p$. By theorem 2.2 formation $\bar{S}$ is $S_{ch}$-closed. □

The converse to corollary 2.4 is false. Let $\bar{S}$ be the formation generated by the symmetric group $S_4$ of degree 4 and cyclic groups of orders 2 and 3. According to [18, p. 44] the alternating group $A_4$ of degree 4 does not belong $\bar{S}$. It is well known that $\mathfrak{F}_{\bar{S}}$ is a local formation with the canonical local definition $F$ where $F(p) = \mathfrak{F}_{p,\bar{S}}$ for all $p \in \mathbb{P}$. Since $\bar{S}$ is a weakly hereditary formation, it is clear that $\mathfrak{F}_{p,\bar{S}}$ is also weakly hereditary for all $p \in \mathbb{P}$. By theorem 2.2 formation $\mathfrak{F}_{\bar{S}}$ is $S_{ch}$-closed. By theorem 10.3B [17] there is a faithful irreducible $S_4$-module $V$ over $F_2$. Let $G = V \ltimes S_4$. Then $C_2(V) = V$ and $V = F(G)$. It means that $G \in \mathfrak{F}_{\bar{S}}$. Note that $H = V A_4 \triangleleft G$. Since $C_2(V) = V$, $O_{2^*}(H) = A_4 \not\subseteq \mathfrak{F}_{\bar{S}}$. It follows that $H \not\subseteq \mathfrak{F}_{\bar{S}}$. Thus $\mathfrak{F}_{\bar{S}}$ is a $S_{ch}$-closed but not $S_{ch}$-closed formation.

**Theorem 2.5.** Let $\bar{S}$ be a saturated $S_{ch}$-closed formation with the Shemetkov property. Then $\bar{S}$ is a hereditary formation.

**Proof.** Since $\bar{S}$ is saturated, $\bar{S}$ is weakly hereditary. Let us show that $\bar{S} = \mathfrak{F}_{\bar{S}}$. Assume that the set $\mathfrak{F}_{\bar{S}} \setminus \bar{S}$ is not empty. Let $G$ be a group of minimal order from it. Since $G \not\subseteq \mathfrak{F}_{\bar{S}}$, there is an s-critical for $\mathfrak{F}_{\bar{S}}$ subgroup $H$ of $G$. Since $\mathcal{M}(\bar{S}) = \mathcal{M}(\mathfrak{F}_{\bar{S}})$, $H$ is an s-critical for $\bar{S}$ Schmidt group. From $G \subseteq \mathfrak{F}_{\bar{S}}$ it follows that $H \subseteq \bar{S}$, the contradiction. □

### 3 Final remarks and problems

Note that the $F(G)$-subnormality is not a hereditary property, i.e. if $H$ is a $F(G)$-subnormal subgroup of a group $G$ and $H \subseteq K \subseteq G$ then $H$ is not $F(K)$-subnormal in general. Also note that from the $F(G)$-subnormality of $H$ does not follow the $F(G) / N$ -subnormality of $HN / N$ in $G / N$.

The main idea of the proof of theorem A (from (3) follows (1)) is to show that $F(G) \subseteq Z_4(G)$. It was achieved by the result of [19] where the author showed that $Z_4(G)$ coincides with the intersection of all normalizers of all $\pi_i$-maximal subgroups of $G$ for all $i \in I$ for any group $G$ where $\bar{S} = \times_{i \in I} A_{\pi_i}$.

This result generalizes the well known theorem of R. Baer [20] that claims that the hypercenter of a group is the intersection of all normalizers of Sylow subgroups.

**Problem 3.1.** Describe all soluble $F(G)$-radical in $\mathfrak{S}$ formations. Is there soluble non-saturated $F(G)$-radical in $\mathfrak{S}$ formation?

**Problem 3.2.** Describe all soluble (local) $F(G)$-radical in $\mathfrak{S}$ Fitting classes.

**Problem 3.3.** Is every soluble $F(G)$-radical in $\mathfrak{S}$ Fitting class a formation?

**Problem 3.4.** Describe all soluble $F(G)$-radical in $\mathfrak{S}$ Shemetkov classes.

Theorem B shows that the class of all groups with nilpotent derived subgroup is the greatest formation of soluble groups such that every its hereditary saturated subformation is $F(G)$-radical in it.

**Problem 3.5.** Describe all saturated $F(G)$-radical in $\mathfrak{S}^*$ formations.

The two main ideas of the proof of theorem C is the induction on a Sylow tower and the following lemma:

**Lemma** [19]. Let $\bar{S}$ be the formation of all $p$-decomposable groups. Then $G^\bar{S} = \{(a, b) | a, b \in G, \text{ where } a \text{ is a } p\text{-element, } b \text{ is a } q\text{-element and } q \neq p\}$.

**Problem 3.6.** Let a group $G$ contain three $F(G)$-subnormal metanilpotent subgroups with pairwise coprime indexes in $G$. Is $G$ metanilpotent?

In the universe of all groups there are a lot of groups $G$ with $F(G) = 1$. In this universe the quasinilpotent radical $F^*(G)$ and the Shemetkov – Schmid
subgroup $\bar{F}(G)$ are the generalizations of the Fitting subgroup [21].

**Definition 3.7.** A subgroup $H$ of a group $G$ is called $F'(G)$-subnormal ($\bar{F}(G)$-subnormal) if $H$ is subnormal in $H^F(G)$ ($H\bar{F}(G)$).

**Definition 3.8.** Let $\mathcal{F}$ be a class of groups. We say that $\mathcal{F}$ is $F'(G)$-radical ($\bar{F}(G)$-radical) if $\mathcal{F}$ is $S_p$-closed and contains every group $G = AB$ where $A$ and $B$ are $F'(G)$-subnormal ($\bar{F}(G)$-subnormal) $\mathcal{F}$-subgroups of $G$.

It is natural to consider the following problems.

**Problem 3.9.** Describe all hereditary $F'(G)$-radical ($\bar{F}(G)$-radical) formations.

**Problem 3.10.** Is every hereditary $F'(G)$-radical ($\bar{F}(G)$-radical) formation composition (saturated)?

**REFERENCES**


