NEW CHARACTERIZATION OF FINITE P-SUPERSOLUBLE GROUPS

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Abstract: In this paper new criteria of $p$-supersolubility of finite groups are given. A group is said to be a $p$-supersoluble group provided every chief $p$-subgroup of a group is Frattini. If $E$ is a non-Frattini chief $p$-factor of a group $G$, then $E$ covers or avoids the factor $H/K$, where $H/K$ is a chief factor of $G$. The following lemmas will be used in the paper.

Lemma 1.1. Let $E \leq G$, $p$ be a prime and $N$ a normal subgroup of $G$. If $E$ is a $\mathrm{CAP}_{p^\infty}$-subgroup of and either $N \leq E$ or $(|E|/|N|)=1$, then $E\cap N$ is a $\mathrm{CAP}_{p^\infty}$-subgroup of $G/N$.

Proof. Let $(H/N)/(K/N)$ be any non-Frattini chief $pd$-factor of $G/N$. Then $H/K$ is any non-Frattini chief $pd$-factor of $G/N$. Hence, by hypothesis, $E$ either covers or avoids the factor $H/K$, so in the case, where $N \leq E$, the lemma is true. Now suppose that $(|E|/|N|)=1$. First assume that $E$ covers the factor $H/K$, that is, $EH=EK$. Then $NEH=NEK$ and $(NE/N)(H/N)=(NE/N)(K/N)$. Hence $EN/N$ covers the factor $(H/N)/(K/N)$. Finally, assume that $E$ avoids the factor $H/K$, that is, $E \cap H \leq K$. Let $N$ be a $\pi$-group. Since $(|E|/|N|)=1$, it follows that $E$ is a Hall $\pi'$-subgroup of $NE$. Let $D=NE \cap H$. Then $D=(N \cap H)(E \cap H)$. Since $N \leq K$ and $E \cap H \leq K$ we have $D \leq K$. Hence $EN/N$ avoids the factor $(H/N)/(K/N)$. The lemma is proved.

Lemma 1.2 (see [3]). Let $P$ be a nilpotent normal subgroup of a group $G$. If $P \cap \Phi(G)=1$, then $P$ is a direct product of some minimal normal subgroups of $G$.

Lemma 1.3. Let $N$ be a non-identity normal $p$-subgroup of a group $G$. If $N \cap \Phi(G)=1$ and every maximal subgroup of $N$ is $\mathrm{CAP}_{p^\infty}$-subgroup

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of G, then some maximal subgroup of N is normal in G.

Proof. By Lemma 1.2, \( N = N_1 \times \cdots \times N_t \), where \( N_i \Phi(G) \) is a minimal normal subgroup of G for all \( i = 1, \ldots, t \).

Let L be a minimal normal subgroup of G contained in N. Suppose that \( L \neq N \) and let \( T/L \) be a chief factor of G, where \( T \leq N \). If \( T/L \leq \Phi(G/L) \), then by [1],
\[
T = L(T \cap \Phi(G)) = L,
\]
a contradiction. Hence \( T/L \Phi(G/L) \).

Therefore by Lemma 1.1 the hypothesis holds for \( G/L \), so by induction some maximal subgroup \( M/L \) of \( N/L \) is normal in G. Hence M is normal in G and maximal in N.

Now suppose that \( N = L \). Let M be any maximal subgroup of N. Then since M does not cover \( N/1, M = 1 \). The lemma is proved.

Recall that a subgroup H is said to be primitive [4] or meet-irreducible [5] in G if whenever
\[
H = X_1 \cap \cdots \cap X_s,
\]
for some subgroups \( X_1, \ldots, X_s \) of G, then \( H = X_i \) for some i. This is equivalent to say that H is a proper subgroup of the intersection of all subgroups of G which contain proper H.

Lemma 1.4 (see [4]). Let G be a group.
(1) Every subgroup of G is the intersection of some meet-irreducible subgroups of G.

(2) Let \( H \) be a subgroup of G and K a meet-irreducible subgroup of H. Then there exists a meet-irreducible subgroup \( X \) of G such that \( K = H \cap X \).

(3) Suppose that \( H \) is a subgroup of G and N is a normal subgroup of G such that \( N \leq H \).
Then H is a meet-irreducible subgroup of G if and only if \( H \cap N \) is a meet-irreducible subgroup of \( G/N \).

2 Criteria of p-supersolubility of groups

Theorem 2.1. Let G be a group, \( p \) a prime. Then the following are equivalent:
(1) G is \( p \)-supersoluble.

(2) Every subgroup of G is a \( \text{CAP}_{p \Phi} \)-subgroup of G.

(3) G is \( p \)-soluble and G has a normal subgroup E with \( p \)-supersoluble quotient G/E such that every maximal subgroup of every Sylow p-subgroup of E is a \( \text{CAP}_{p \Phi} \)-subgroup of G.

(4) G is \( p \)-soluble and every meet-irreducible subgroup with order divisible by \( p \) of every maximal subgroup of G is a \( \text{CAP}_{p \Phi} \)-subgroup of G.

(5) G is \( p \)-soluble and G has a normal subgroup E with \( p \)-supersoluble quotient G/E such that every maximal subgroup of every Sylow p-subgroup of \( O_{p \Phi}(E) \) is a \( \text{CAP}_{p \Phi} \)-subgroup of G.

Proof. The implications (1) \( \Rightarrow \) (2), (2) \( \Rightarrow \) (3), (2) \( \Rightarrow \) (4), (5) \( \Rightarrow \) (5) are evident.

(3) \( \Rightarrow \) (1) Suppose that this assertion is false and consider a counterexample for which \( |G| \) is minimal. Let N be a minimal normal subgroup of G contained in E. Since G is \( p \)-soluble, N is either a \( p \)-subgroup or \( p' \)-subgroup. Note that the hypothesis is still true for \( G/N \) by Lemma 1.1, so \( G/N \) is \( p \)-supersoluble, by the choice of G. Hence N is the only minimal normal subgroup of G contained in E, \( p \) divides \( |N| \) and \( |N| \neq p \). Since G is not \( p \)-supersoluble, it follows that N is a \( p \)-group and so \( N \leq P \) where P is a Sylow \( p \)-subgroup of E. Besides, since the class of the \( p \)-supersoluble groups is a saturated formation, \( N\Phi(G) \). Thus for some maximal subgroup \( V \) of P we have \( P = VN \). Moreover, for some maximal subgroup \( M \) of G we have \( G = [N]M \). Let \( C = C_p(N) \). Then \( C = NM_G \), so \( C \cap E = [N](E \cap M_G) \). It follows that
\[
C_p(N) = N = O_{p \Phi}(E).
\]
Since by hypothesis V is a \( \text{CAP}_{p \Phi} \)-subgroup of G and \( P = VN \), \( |V \cap N| = 1 \), which implies \( |N| = p \), so G is \( p \)-supersoluble. This contradiction completes the proof of the implication.

(4) \( \Rightarrow \) (1) Suppose that this implication is false and let G be a counterexample with minimal order. Let N be a minimal normal subgroup of G. Then \( G/N \) satisfies (4) by Lemma 1.1. Hence \( G/N \) is \( p \)-supersoluble by the choice of G. Hence N is a \( p \)-group and \( |N| > p \). Let \( N \leq M \) and \( E \leq N \), where \( M \) is a maximal subgroup of G and E is a maximal subgroup of N. Then E is a meet-irreducible subgroup of N. By Lemma 1.4 for some meet-irreducible subgroup \( X \) of M we have \( E = X \cap N \). Clearly the order of \( X \) is a multiple of \( p \). Since the class of all \( p \)-supersoluble groups is a saturated formation, N is the only minimal normal subgroup of G and \( N\Phi(G) \). Hence by Condition (4), \( X \) either covers or avoids \( N/1 \). But since \( |N| > p \), \( E \neq 1 \), so \( X \cap N \neq 1 \), which implies \( N = N \cap X = M \), a contradiction.

(5) \( \Rightarrow \) (1) Suppose that this is false and consider a counterexample for which \( |G||E| \) is minimal. Then

(a) \( O_p(E) = 1 \).

Let \( D = O_{p \Phi}(E) \neq 1 \). Then
\[
O_{p \Phi}(E/D) = O_{p \Phi}(E)/D
\]
and the hypothesis holds for \((G/D, E/D)\) (see the proof of (3) \(\Rightarrow (1)\)). Hence by the choice of \((G, E), G/D\) is \(p\)-supersoluble and so \(G\) does, which contradicts the choice of \(G\).

(b) \(O_{\phi,p}(E)\) is a \(p\)-group and
\[
\Phi(G) \cap O_{\phi,p}(E) = \{1\}.
\]

The first statement is a corollary of (a). Now suppose that
\[
D = \Phi(G) \cap O_{\phi,p}(E) \neq \{1\}.
\]
Then
\[
O_{\phi,p}(E/D) = O_{\phi,p}(E)/D,
\]
by [7], so the hypothesis holds for \((G/D, E/D)\), which implies the \(p\)-supersolubility of \(G\), a contradiction.

(c) Every subgroup of \(O_{\phi,p}(E)\) with prime order is not normal in \(G\).

Suppose that \(O_{\phi,p}(E)\) has a subgroup \(L\) such that \(|L| = p\) and \(L\) is normal in \(G\). Let \(C = C_p(L)\). We shall show that the hypothesis holds for \((G, C_p(L))\). It is clear that \(G/C_p(L)\) is \(p\)-supersoluble and
\[
O_{\phi,p}(E) \leq C_p(L).
\]
Hence
\[
O_{\phi,p}(C_p(L)) = O_{\phi,p}(E),
\]
so the hypothesis holds for \((G, C_p(L))\). It follows that \(E \leq C_p(L)\). Hence
\[
O_{\phi,p}(E/L) = O_{\phi,p}(E)/L
\]
and the hypothesis holds for \((G/L, E/L)\). Therefore \(G/L\) is \(p\)-supersoluble, which implies the \(p\)-supersolubility of \(G\), a contradiction.

Final contradiction. Let \(P = O_{\phi,p}(E)\). Then by (b), \(P\) is a \(p\)-group and \(P\) is normal in \(G\). Besides, by (b) and [3], \(P\) is the direct product of some minimal normal subgroups of \(G\). Hence by Lemma 1.2, \(P\) has a maximal subgroup \(M\) such which is normal in \(G\). Now by [1] for some minimal normal subgroup \(L\) of \(G\) contained in \(P\) we have \(|L| = p\), which contradicts (c).

We say that \(A\) is a (partial) \(\text{CAP}_{\phi}\) subgroup of \(G\) if \(A\) is a (partial) \(\text{CAP}_{\phi}\) subgroup of \(G\) for all primes \(p\).

From Theorem 2.1 we get

**Theorem 2.2.** Let \(G\) be a group. Then the following are equivalent:

1. \(G\) is supersoluble.
2. Every subgroup of \(G\) is a \(\text{CAP}_{\phi}\) subgroup of \(G\).
3. \(G\) is soluble and \(G\) has a normal subgroup \(E\) with supersoluble quotient \(G/E\) such that every maximal subgroup of every Sylow subgroup of \(E\) is a \(\text{CAP}_{\phi}\) subgroup of \(G\).
4. \(G\) is soluble and every meet-irreducible subgroup of every maximal subgroup of \(G\) is a \(\text{CAP}_{\phi}\) subgroup of \(G\).
5. \(G\) is soluble and \(G\) has a normal subgroup \(E\) with supersoluble quotient \(G/E\) such that every maximal subgroup of every Sylow subgroup of \(E\) is a \(\text{CAP}_{\phi}\) subgroup of \(G\).

From Theorem 2.2 we get

**Corollary 2.1** (Srinivasan [6]). Let \(G\) be a group. If \(G\) has a normal subgroup \(E\) with supersoluble quotient \(G/E\) such that every maximal subgroup of every Sylow subgroup of \(E\) is normal in \(G\), then \(G\) is supersoluble.

**Proof.** It is not difficult to show that \(G\) is soluble. Hence we may use Theorem 2.

**Corollary 2.2** (Ezquerro [2]). Let \(G\) be a group. If \(G\) has a normal subgroup \(E\) with supersoluble quotient \(G/E\) such that every maximal subgroup of every Sylow subgroup of \(E\) is a \(\text{CAP}\) subgroup of \(G\), then \(G\) is supersoluble.

**REFERENCES**