$S$-C-CONDITIONALLY PERMUTABLY EMBEDDED SUBGROUPS OF FINITE GROUPS

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A subgroup $H$ of a finite group $G$ is said to be $s$-conditionally permutably embedded (or in brevity, $s$-c-permutably embedded) in $G$ if for each $p \in \pi(H)$, every Sylow $p'$-subgroup of $H$ is a Sylow $p'$-subgroup of some $s$-conditionally permutable subgroup of $G$. In this paper, we use some $s$-c-permutably embedded subgroups to study the structure of some groups. Some known results are generalized.

Keywords: finite group, $s$-conditionally permutably embedded subgroup, formation, Sylow subgroup, maximal subgroup.

Introduction

Throughout this paper, all groups considered are finite and $G$ denotes a finite group. The terminology and notations are standard, as in [1] and [2].

Let $A$ and $B$ be subgroups of $G$. $A$ is said to be permutable with $B$ if $AB = BA$. If $A$ is permutable with all subgroups of $G$, then $A$ is said to be a permutable subgroup [1] (or quasinormal subgroup [3]) of $G$. The permutable subgroups have many interesting properties. For example, Ore [3] proved that every permutable subgroup of a finite group is subnormal. Itô and Szép [4] proved that for every permutable subgroup $H$ of a finite group $G$, $H/H_A$ is nilpotent.

However, in general, two subgroups $H$ and $T$ of $G$ may not be permutable in $G$ but $G$ maybe contain an element $x$ such that $HT^x = T^xH$. Based on the observations, Guo, Shum and Skiba introduced the concept of conditionally permutable subgroup (in more general, the concept of $X$-permutable subgroup) [5]–[7]: let $X$ be a non-empty subset of $G$. Then a subgroup $A$ of $G$ is said to be conditionally permutable ($X$-permutable) in $G$ if for every subgroup $T$ of $G$, there exists some $x \in G$ ($x \in X$ respectively) such that $AT^x = T^xA$. By using the conditionally permutable subgroups and $X$-permutable subgroups, authors have obtained some new elegant results on the structure of some groups (cf. [5]–[8]).

By considering some local conditionally permutable subgroups, Huang and Guo [9] introduced the concept of $s$-conditionally permutable subgroup: a subgroup $H$ of $G$ is said to be $s$-conditionally permutable in $G$ if, for every Sylow subgroup $T$ of $G$, there exists some $x \in G$ such that $HT^x = T^xH$. By Sylow’s theorem, we see that a subgroup $H$ of $G$ is $s$-conditionally permutable in $G$ if and only if for every $p \in \pi(G)$, there exists a Sylow $p'$-subgroup $T$ such that $HT^p = TH$. As a development of $s$-conditionally permutable subgroups, Chen and Guo [10] introduced the concept of $s$-c-permutably embedded subgroups:

Definition 0.1 [10, Definition 1.1]. A subgroup $H$ of $G$ is said to be $s$-conditionally permutably embedded (or in brevity, $s$-c-permutably embedded) in $G$ if every Sylow subgroup of $H$ is a Sylow subgroup of some $s$-conditionally permutable subgroup of $G$. 

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Clearly, all permutable subgroups, $s$-permutable subgroups and $s$-conditionally permutable subgroups are $s$-c-permutably embedded. But the converse is not true in general (see, for example, Example 1-2 in [10]).

The purpose of this paper is to go further into the influence of $s$-c-permutably embedded subgroups on the structure of finite groups. Some new results are obtained and some known results are generalized.

1 Preliminary results

In this section, we give the related concepts and some basic results which are useful in the sequel.

**Lemma 1.1** [10, Lemma 2.2]. Suppose that $G$ is a group, $K$ and $H \leq G$. Then:

(1) If $H$ is $s$-c-permutably embedded in $G$, then $HK/K$ is $s$-c-permutably embedded in $G/K$.

(2) If $K \leq H$ and $H/K$ is $s$-c-permutably embedded in $G/K$, then $H$ is $s$-c-permutably embedded in $G$.

(3) If $HK/K$ is $s$-c-permutably embedded in $G/K$ and $|H|/|K|=1$, then $H$ is $s$-c-permutably embedded in $G$.

(4) If $H$ is $s$-c-permutably embedded in $G$, then $H \cap K$ is $s$-c-permutably embedded in $K$.

**Lemma 1.2** [11, Lemma 3.1]. Let $N$ and $L$ be normal subgroups in $G$ such that $P/L$ is a Sylow $p$-subgroup of $NL/L$ and $M/L$ is a maximal subgroup of $P/L$. If $P_p$ is a Sylow $p$-subgroup of $P \cap N$, then $P_p$ is a Sylow $p$-subgroup of $N$ such that $D=M \cap N \cap P_p$ is a maximal subgroup of $P_p$ and $M=LD$.

**Lemma 1.3** [12, Lemma 4.1]. Let $p$ be a prime dividing the order of $G$. Suppose that $|G|/p!-1$ and the order of $G$ is not divisible by $p^2$ and $G$ is $A_4$-free. Then $G$ is $p$-nilpotent.

**Lemma 1.4** [2, Theorem 1.8.17]. Let $N$ be a non-trivial normal subgroup of $G$. If $N \cap \Phi(G)=1$, then the Fitting subgroup $F(N)$ of $N$ is the direct product of minimal normal subgroups of $G$ which are contained in $F(N)$.

**Lemma 1.5** [13, III, Lemma 3.3].

i) If $N \leq G$, $U \leq G$ and $N \leq \Phi(U)$, then $N \leq \Phi(G)$.

ii) If $M \leq G$, then $\Phi(M) \leq \Phi(G)$.

Recall that, a class $\mathcal{F}$ of groups is called a formation if it is closed under homomorphic image and subdirect product and every group $G$ has a smallest normal subgroup (called $\mathcal{F}$-residual) with quotient is in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if it contains every group $G$ with $G/\Phi(G) \in \mathcal{F}$. A class of groups $\mathcal{F}$ is said to be $S$-closed if every subgroup of $G$ belongs to $\mathcal{F}$ whenever $G \in \mathcal{F}$. We say a subgroup $H$ of $G$ is $\mathcal{F}$-supplemented in $G$ if $G$ has a subgroup $T \in \mathcal{F}$ such that $G=HT$. In this case, $T$ is said to be an $\mathcal{F}$-supplement of $H$ in $G$. In particular, if $\mathcal{F}$ is the class of all supersoluble groups ($p$-supersoluble groups), then an $\mathcal{F}$-supplement is said to be a supersoluble supplement (a $p$-supersoluble supplement). We use $\mathcal{U}$ to denote the formation of all supersoluble groups. The following Lemma is obvious.

**Lemma 1.6.** Let $\mathcal{F}$ be a formation of groups. Suppose that a subgroup $H$ of $G$ has an $\mathcal{F}$-supplement in $G$. Then:

(1) If $N \leq G$, then $HN/N$ has an $\mathcal{F}$-supplement in $G/N$.

(2) If $H \leq K \leq G$ and $\mathcal{F}$ is $S$-closed, then $H$ has an $\mathcal{F}$-supplement in $K$.

**Lemma 1.7** [14, Lemma 2.3]. Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and $G$ a group with a normal subgroup $F$ such that $G/F \in \mathcal{F}$. If $E$ is cyclic, then $G \in \mathcal{F}$.

**Lemma 1.8** [15, Theorem 3.1]. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ has a soluble normal subgroup $H$ such that $G/H \in \mathcal{F}$. If for any maximal subgroup $M$ of $G$, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F}=\mathcal{U}$.

**Lemma 1.9** [10, Theorem 3.2]. Let $G$ be a soluble group. If every maximal subgroup of every non-cyclic Sylow subgroup of $G$ having no supersoluble supplement in $G$ is $s$-c-permutably embedded in $G$, then $G$ is supersoluble.

Recall that a subgroup $H$ of $G$ is said to be a 2-maximal subgroup of $G$ if $H$ is a maximal subgroup of some maximal subgroup $M$ of $G$.

2 Main results

**Theorem 2.1.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ a group. Then $G \in \mathcal{F}$ if and only if $G$ has a soluble normal subgroup $H$ such that $G/H \in \mathcal{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $H$ having no supersoluble supplement in $G$ is $s$-c-permutably embedded in $G$.

**Proof.** The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $(G,H)$ be a counterexample with $|G||H|$ is minimal. Then:

(1) $G/R \in \mathcal{F}$, where $R$ is an arbitrary minimal normal subgroup of $G$. 

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Obviously, 
\[(G/R)/(H/R) = G/HR = (G/H)/(HR/H) \in \mathfrak{F}_G\]
and \(HR/R \cong H/(H \cap R)\) is soluble. Let \(P/R\) be a non-cyclic Sylow \(p\)-subgroup of \(HR/R\), where \(p\) is any prime divisor of \(|HR/R|\), and \(M/R\) a maximal subgroup of \(P/R\). If \(P_p\) is a Sylow \(p\)-subgroup of \(P \cap H\), then by Lemma 1.2, \(P_p\) is a Sylow \(p\)-subgroup of \(H\) such that \(L = M \cap H \cap P_p\) is a maximal subgroup of \(P_p\) and \(M = LR\). Clearly, \(P_p\) is non-cyclic. By hypothesis, either \(L\) is \(s\)-S-permutably embedded in \(G\) or \(L\) has a supersoluble supplement in \(G\). By Lemma 1.1 and Lemma 1.6, either \(M/R = LR/R\) is \(s\)-\(S\)-permutably embedded in \(G\) or \(M/R = LR/R\) has a supersoluble supplement in \(G\). By the choice of \(G\), \(G/R \in \mathfrak{F}_G\).

(2) \(G\) has a unique minimal normal subgroup \(N, G = [N]M\), where \(M\) is a maximal subgroup of \(G\), and \(N = O_p(G) = F(G) = C_G(N)\) for some prime \(p\).

Since \(\mathfrak{F}_G\) is a saturated formation, by (1), \(G\) has a unique minimal normal subgroup \(N\) and \(\Phi(G) = 1\). Hence, there exists a maximal subgroup \(M\) of \(G\) such that \(G = [N]M\). Since \(H\) is soluble, \(N\) is an abelian elementary \(p\)-group for some prime \(p\). Clearly, \(N \subseteq O_p(G) \leq F(G) \leq C_G(N)\). Let \(C = C_G(N)\). It is easy to see that \(C \cap M \leq G\). Hence \(C = C \cap NM = (C \cap M) = N\). Thus (2) holds.

(3) \(N\) is a non-cyclic Sylow \(p\)-subgroup of \(H\).

By Lemma 1.1, Lemma 1.6 and Lemma 1.9, we know that \(H\) is supersoluble. By the choice of \(G\), \(H < G\). Let \(q\) be the largest prime divisor of \(|H|\) and \(Q \in \text{Syl}_q(H)\). Then \(Q = O_q(H) \leq G\). Since \(N\) is the unique minimal normal subgroup of \(G\), \(q = p\). Hence, by (2), we see that \(N \subseteq Q = O_p(G) \subseteq O_p(G) = N\). By (1) and Lemma 1.7, we see that \(N\) is not cyclic. Thus (3) holds.

(4) Final contradiction.

Let \(G_p\) be a Sylow \(p\)-subgroup of \(G\). Since \(\mathfrak{F}_G\), \(G_p \notin \Phi(G)\) by Lemma 1.5. So there exists a maximal subgroup \(P\) of \(G_p\) such that \(\mathfrak{F}_G\). Clearly, \(N_1 = P \cap N\) is a maximal subgroup of \(N\).

If \(N_1\) has a supersoluble supplement in \(G\), then there exists a supersoluble subgroup \(T\) of \(G\) such that \(G = NT\). It is easy to see that \(N \cap T \leq NT \leq G\). Hence \(N \cap T = 1\) or \(N \cap T = N\).

If \(N \cap T = 1\), then \(G = N \cap T = T\) is supersoluble, a contradiction. If \(N \cap T = 1\), then \(N = N_1\), which is impossible. Hence we assume that \(N_1\) is \(s\)-\(S\)-\(p\)-permutably embedded in \(G\), that is, there exists an \(s\)-conditionally permutable subgroup \(A\) of \(G\) such that \(N_1\) is a Sylow \(p\)-subgroup of \(A\). In this case, for every \(q \in \pi(G)\) and \(q \neq p\), there exists a Sylow \(q\)-subgroup \(Q\) of \(G\) such that \(AQ = QA\). Then \(N_1 = N \cap P = N \cap AQ \leq AQ\) and consequently \(Q \leq N_1\). On the other hand, \(N_1 = N \cap P \leq G_p\). Thus, \(N_1 \leq G\). It follows that \(N \cap P = 1\) and so \(|N| = p\). Then by (1) and Lemma 1.7, we obtain that \(G \in \mathfrak{F}_G\). This contradiction completes the proof.

Theorem 2.2. Let \(\mathfrak{F}\) be a saturated formation containing \(\Delta\) and \(G\) a group. Then \(G \in \mathfrak{F}\) if and only if \(G\) has a soluble normal subgroup \(H\) such that \(G/H \in \mathfrak{F}\) and every maximal subgroup of every non-cyclic Sylow subgroup of \(F(H)\) having no supersoluble supplement in \(G\) is \(s\)-\(c\)-\(p\)-permutably embedded in \(G\).

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let \((G/H, H)\) be a counterexample with \(|G/H|\) is minimal.

Let \(M\) be a maximal subgroup of \(G\) if \(F(H) \not\leq M\), then there exists a prime \(p\) dividing \(|F(G)|\) such that \(O_p(H) \not\leq M\). Thus \(G = O_p(H)M\).

It is clear that \(\Phi(G) \cap F(H) = 1\). If not, we choose a minimal normal subgroup \(R\) of \(G\) contained in \(\Phi(G) \cap F(H)\), then \((G/R \cap H)/R\) satisfies the hypothesis. The minimal choice of \((G, H)\) implies that \(G/R \in \mathfrak{F}_G\). Then, since \(\mathfrak{F}_G\) is a saturated formation, we have that \(G \in \mathfrak{F}_G\), a contradiction. By Lemma 1.5, \(\Phi(O_p(H)) \subseteq \Phi(G) \cap F(H)\). Hence \(\Phi(O_p(H)) = 1\).

If \(|O_p(H)| = p\), then \(|F(H): F(H) \cap M| = |G: M| = p\).

Hence by Lemma 1.8, \(G \in \mathfrak{F}_G\). This contradiction shows that \(O_p(H)\) is a non-cyclic Sylow \(p\)-subgroup of \(F(H)\). Let \(M_p\) be a Sylow \(p\)-subgroup of \(M\). Then \(G_p = O_p(H)M_p\) is a Sylow \(p\)-subgroup of \(G\). Let \(P\) be a maximal subgroup of \(G_p\) with \(M_p \leq P\) and \(P_2 = P \cap O_p(H)\). Then \(P = P_2 \cap O_p(H)M_p = (P_2 \cap O_p(H))M_p = (P_2 \cap O_p(H)) = P_2\) and \(P_2 \cap O_p(H) = O_p(H)M_p\).

Hence \(|O_p(H) : P_2| = |O_p(H)M_p| : P_2M_p| = |G_p : P_2| = p\), that is, \(P_2\) is a maximal subgroup of \(O_p(H)\). Since \(O_p(H) \cap M \leq G\), \(P_2(O_p(H) \cap M)\) is a subgroup of \(O_p(H)\). By the maximality of \(P_2\) in \(O_p(H)\), we know that \(P_2(O_p(H) \cap M) = P_2\) or \(P_2(O_p(H) \cap M) = O_p(H)\).
If \( P_2(O_p(H) \cap M) = O_p(H) \), then \( G = O_p(H)M = P_2M \). Since, obviously, \( O_p(H) \cap M = P_2 \cap M \), \( O_p(H) = P_2 \), a contradiction. Hence \( P_2(O_p(H) \cap M) = P_2 \). It follows that \( O_p(H) \cap M \leq P_2 \). Since \( O_p(H) \cap M \leq G \), \( O_p(H) \cap M \subseteq (P_2)_0 \). If \( (P_2)_0 \subseteq M \), then \( G = (P_2)_0 M = P_2 M \) and \( O_p(H) = P_2(O_p(H) \cap M) = P_2 \), a contradiction. Hence, \( (P_2)_0 \subseteq M \) and \( (P_2)_0 = O_p(H) \cap M \).

Suppose that \( P_2 \) has a supersoluble supplement \( N \) in \( G \), then \( G = P_2N = O_p(H)N \). If \( O_p(H) \cap N \leq M \), then \( O_p(H) \leq M \cap O_p(H) = (P_2)_0 \leq P_2 \). Therefore, \( O_p(H) = P_2(O_p(H) \cap N) = P_2 \), a contradiction. It follows that \( O_p(H) \cap N \neq M \).

Since \( O_p(H) \cap N \leq G \) and \( M \) is maximal in \( G \), we have that \( G = (O_p(H) \cap N)M \). By the modular law, \( N = (O_p(H) \cap N)(M \cap N) \). It follows that \( G = O_p(H)(M \cap N) \). By the modular law again, \( M = (P_2)_0(M \cap N) \). Hence, \( G = M(O_p(H) \cap N) = MN = (P_2)_0(N) \).

If \( M \cap N \) is not maximal in \( N \), then there exists a maximal subgroup \( N_1 \) of \( N \) such that \( M \cap N \leq N_1 \). Let \( L = (P_2)_0N_1 \). Since \( (P_2)_0 \leq M \), it follows that \( (P_2)_0 \cap N = (P_2)_0 \cap (N \cap M) \leq (P_2)_0 \cap N_1 \leq (P_2)_0 \cap N \). Hence, \( (P_2)_0 \cap N = (P_2)_0 \cap N_1 = (P_2)_0 \cap (M \cap N) \). Since \( G = (P_2)_0N \), \( L = (P_2)_0N_1 \), \( M = (P_2)_0(M \cap N) \), we have that \( M \leq L < G \), a contradiction. Therefore, \( M \cap N \) is a maximal subgroup of \( N \). Since \( N \) is supersoluble, it follows that \( |F(H) : F(H) \cap M| = |G : M| = N : M \cap N | = p \), a prime. This implies that \( F(H) \cap M \) is a maximal subgroup of \( F(H) \). Then by Lemma 1.8, we obtain that \( G \in \mathcal{F} \), a contradiction.

Hence, by hypothesis, \( P_2 \) is \( s \cdot c \)-permutable embedded in \( G \). Then there exists an \( s \)-conditionally permutable subgroup \( A \) of \( G \) such that \( P_2 \) is a Sylow \( p \)-subgroup of \( A \). Now, for every \( q \in \pi(G) \) and \( q \neq p \), there exists a Sylow \( q \)-subgroup \( Q \) of \( G \) such that \( AQ \leq G \). Because \( P_2 = AQ \cap O_p(H) \leq AQ \), we have that \( Q \leq N_q(P_2) \). On the other hand, since \( P_2 = P_2 \cap O_p(H)P_2 \) and \( O_p(H) \) is abelian,

\[
G_p = O_p(H)M_p = O_p(H)P_2 = N_q(P_2).
\]

Thus, \( P_2 \leq G \). This implies that \( P_2 = (P_2)_0 \leq M \) and so \( O_p(H) \cap M = P_2 \cap M = P_2 \). It follows that

\[
|F(H) : F(H) \cap M| = |G : M| = |O_p(H) : O_p(H) \cap M| = p.
\]

This indicates that \( F(H) \cap M \) is a maximal subgroup of \( F(H) \). By Lemma 1.8 again, we obtain that \( G \in \mathcal{F} \). The final contradiction completes the proof.

**Theorem 2.3.** A group \( G \) is \( p \)-supersoluble if and only if \( G \) has a normal \( p \)-soluble subgroup \( H \) such that \( G/H \) is \( p \)-supersoluble and every maximal subgroup of every Sylow \( p \)-subgroup of \( H \) having no \( p \)-supersoluble supplement in \( G \) is \( s \cdot c \)-permutable embedded in \( G \).

**Proof.** The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let \( (G, H) \) be a counterexample with \( |G : H| \) is minimal. We proceed the proof via the following steps:

1. If \( R \) is a minimal normal subgroup of \( G \), then \( G/R \) is \( p \)-supersoluble.

Clearly, \( (G/R)/(HR/R) = G/HR = (G/H)/(HR/H) \) is \( p \)-supersoluble and \( HR/R = H/H \cap R \) is \( p \)-soluble. Let \( P/R \) be a Sylow \( p \)-subgroup of \( HR/R \) and \( M/R \) a maximal subgroup of \( P/R \). If \( P_p \) is a Sylow \( p \)-subgroup of \( P \cap H \), then by Lemma 1.2, \( P_p \) is a Sylow \( p \)-subgroup of \( H \) such that \( L = M \cap H \cap P_p \) is a maximal subgroup of \( P_p \) and \( M = LR \). By hypothesis, either \( L \) is \( s \cdot c \)-permutable embedded in \( G \) or \( L \) has a \( p \)-supersoluble supplement in \( G \). By Lemma 1.1 and Lemma 1.6, we see that either \( M/R = LR/R \) is \( s \cdot c \)-permutable embedded in \( G \) or \( M/R = LR \) has a \( p \)-supersoluble supplement in \( G \). By the choice of \( (G, H) \), \( G/R \) is \( p \)-supersoluble.

2. \( O_p(G) = 1 \) and \( G \) has a unique minimal normal subgroup \( N \) such that \( N = C_o(N) = O_p(G)\Phi(G) \) and \( |N| = p \).

In fact, if \( O_p(G) \neq 1 \), then, by (1), \( G/O_p(G) \) is \( p \)-supersoluble. It follows that \( G \) is \( p \)-supersoluble, a contradiction. Hence, \( O_p(G) = 1 \). Since the class of all \( p \)-supersoluble groups is a saturated formation, \( G \) has a unique minimal normal subgroup \( N \) and \( N \not\subseteq \Phi(G) \). Obviously, \( N = C_o(N) = O_p(G) \). By (1) and Lemma 1.7, \( |N| = p \).

3. If \( H \leq D \leq G \) and \( D < G \), then \( D \) is \( p \)-supersoluble.

It is clear that \( D/H \) is \( p \)-supersoluble and \( (D, H) \) satisfies the hypothesis by Lemma 1.1 (4) and Lemma 1.6. Hence, by the choice of \( (G, H) \), \( D \) is \( p \)-supersoluble.

4. Let \( H_p \) be a Sylow \( p \)-subgroup of \( H \). Then \( 1 \neq H_p \neq N \) and so \( H_p \) is not normal in \( G \).

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By hypothesis, obviously, \( H_p \neq 1 \). If \( H_p = N \), then, by (2), \( |H_p| > p \). Since \( H_p \not\subseteq \Phi(G) \) and \( H_p \subseteq G \), \( H_p \not\subseteq \Phi(G_p) \) by Lemma 1.5, where \( G_p \) is a Sylow \( p \)-subgroup of \( G \). Hence, there exists a maximal subgroup \( P_1 \) of \( G_p \) such that \( H_p \not\subseteq P_1 \). Let \( E = H_p \cap P_1 \). Then \( E \) is a maximal subgroup of \( H_p \). If \( E \) has a \( p \)-supersoluble supplement \( T \) in \( G \), then \( |G:T| \leq |E| \). Since \( H_p \supseteq ET = G \) and \( H_p \) is an abelian minimal normal subgroup of \( G \), \( G = [H_p,T] \). This implies that \( |G:T|=|H_p| \), a contradiction. Hence \( E \) is \( s-c \)-permutably embedded in \( G \), that is, there exists an \( s \)-conditionally permutably normal subgroup \( A \) of \( G \) such that \( E \) is a Sylow \( p \)-subgroup of \( A \). So for every \( q \in \pi(G) \) and \( q \neq p \), there exists a Sylow \( q \)-subgroup \( Q \) of \( G \) such that \( QA = AQ \). Thus \( E = H_p \cap P_1 = H_p \cap AQ \subseteq AQ \). It follows that \( Q \leq N_E(E) \). Besides, \( E = H_p \cap P_1 \subseteq G_p \). Therefore \( E \subseteq G \). This induces that \( E = 1 \) and so \( |H_p| = p \), a contradiction. Thus (4) holds.

(5) \( G = [N]M \), where \( M \) is a \( p \)-supersoluble maximal subgroup of \( G \) such that \( p \parallel |M| \) and \( O_p(M) = 1 \).

By (1) and (2), \( G \) has a \( p \)-supersoluble maximal subgroup \( M \) such that \( G = [N]M \). By [2, Lemma 1.7.11], \( O_p(G/C_p(N)) = O_p(G/N) = 1 \). Hence \( O_p(M) = 1 \). Assume that \( p \parallel |M| \). Then \( p \) does not divide \( |G/N| \). Since \( N \not\subseteq H \), \( H/N \) is a \( p' \)-group, which contradicts (4).

(6) \( H = G \).

Assume that \( H \neq G \). Consider the subgroup \( H \cap M \). Since \( H = H \cap NM = N(H \cap M) \) and \( N \not\subseteq H \), \( H \cap M \neq 1 \). By (2) and (3), \( H \) is a \( p \)-supersoluble and \( O_p(H) = 1 \). It follows from [11, Lemma 3.3] that \( H \) is supersoluble. This implies that \( p \) is the largest prime divisor of \( |H| \) and so the Sylow \( p \)-subgroup \( P \) of \( H \cap M \) is normal in \( H \cap M \). Hence \( P \) char \( H \cap M \leq M \). Since \( O_p(M) = 1 \), \( P = 1 \). It follows that \( N \) is a Sylow \( p \)-subgroup of \( H \), which contradicts (4).

(7) Every maximal subgroup of every Sylow \( p \)-subgroup of \( G \) has a \( p \)-supersoluble supplement in \( G \).

Let \( G_p \) be a Sylow \( p \)-subgroup of \( G \) and \( P_1 \) a maximal subgroup of \( G_p \). If \( N \subseteq P_1 \), then, by (5), \( P_1 \) has a \( p \)-supersoluble supplement \( M \) in \( G \). Assume that \( N \not\subseteq P_1 \) and \( P_1 \) is \( s-c \)-permutably embedded in \( G \). Then there exists an \( s \)-conditionally permutably normal subgroup \( A \) of \( G \) such that \( P_1 \) be a Sylow \( p \)-subgroup of \( A \). By the same discussion as in (4), we obtain that \( P_1 \not\subseteq G \) and consequently \( N \subseteq P_1 \), a contradiction.

(8) Final contradiction.

By (7) and [11, Theorem 3.4], we obtain that \( G \) is \( p \)-supersoluble. This final contradiction completes the proof.

**Theorem 2.4.** Let \( p \) be the smallest prime dividing the order of a \( p \)-soluble group \( G \) and \( p \) a Sylow \( p \)-subgroup of \( G \). If every \( 2 \)-maximal subgroup of \( P \) is \( s-c \)-permutably embedded in \( G \) and \( G \) is \( A_p \)-free, then \( G \) is \( p \)-nilpotent.

**Proof.** Suppose that the assertion is false and let \( G \) be a counterexample of minimal order. We proceed with our proof as follows:

(1) \( G/N \) is \( p \)-nilpotent, for every non-trivial normal subgroup \( N \) of \( G \).

If some Sylow \( p \)-subgroup of \( G \) is contained in \( N \), then, obviously, \( G/N \) is \( p \)-nilpotent. Hence, we may assume that \( N \) does not contain any Sylow \( p \)-subgroup of \( G \). Let \( PN/N \) be a Sylow \( p \)-subgroup of \( G/N \), where \( P \) is a Sylow \( p \)-subgroup of \( G \), and \( M_2/N \) a \( 2 \)-maximal subgroup of \( PN/N \).

It is easy to see that \( M_2 = PGN \cap M_2 = (P \cap M_2)N \).

Let \( P_2 = P \cap M_2 \). Since \( P \cap M_2 \cap N = P \cap N \), \( p^2 = |PN/N:M_2/N|\leq|PN/(P \cap M_2)N|\leq|P:P_2| \).

Hence \( P_2 \) is a \( 2 \)-maximal subgroup of \( P \) and \( M_2 = P_2N \). By Lemma 1.1, \( M_2/N = P_2N/N \) is \( s-c \)-permutably embedded in \( G/N \). This shows that \( G/N \) satisfies the hypothesis. The minimal choice of \( G \) implies that \( G/N \) is \( p \)-nilpotent.

(2) \( G \) has a unique minimal normal subgroup \( H = C_o(H) \) and \( \Phi(G) = 1 \).

Since the class of all \( p \)-nilpotent groups is a saturated formation, \( G \) has a unique minimal normal subgroup, say \( H \), and \( \Phi(G) = 1 \). Because \( G \) is a \( p \)-soluble group, \( H \) is a \( p \)-group or a \( p' \)-group. If \( H \) is a \( p' \)-group, then \( G \) is \( p \)-nilpotent. Hence \( H \) is an elementary abelian \( p \)-group. Now, by the similar argument as in the proof (2) of Theorem 2.1, we can know that \( H = C_o(H) \).

(3) \( |H| \geq p^3 \).

If \( |H| = p \), then \( G/H = G/C_o(H) \leq Aut(H) \) is a cyclic group of order \( p-1 \). Since \( p \) is the smallest prime of \( |G| \), \( G = C_o(H) \), that is, \( H \leq Z(G) \). This induces that \( G \) is \( p \)-nilpotent, a contradiction. Thus (3) holds.

(4) Final contradiction.
By (2), we see that there exists a maximal subgroup $M$ of $G$ such that $G = [H]M$. Let $M_p$ be a Sylow $p$-subgroup of $M$. Then $G_p = M_pH$ is a Sylow $p$-subgroup of $G$. By Lemma 1.3, we see that $|G_p| \geq p^\nu$. Let $G_1$ be a $2$-maximal subgroup of $G_p$ with $M_p \subseteq G_1$ and $H_1 = G_1 \cap H$. Then $|H : H_1| = |H : G_1 \cap H|$. Hence $H_1$ is a $2$-maximal subgroup of $H$. By hypothesis, $G_1$ is $s$-c-permutably embedded in $G$. Hence there exists an $s$-conditionally permutable subgroup $A$ of $G$ such that $G_1 = A$ is a Sylow $p$-subgroup of $A$. Let $q$ be an arbitrary prime divisor of $|G|$ with $q \neq p$. Since $A$ is an Sylow $q$-subgroup in $G$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $AQ = QA$. As $H_1$ is a $2$-maximal subgroup of $H$ and $H_1 = G_1 \cap H \subseteq AQ \cap H \subseteq H$, we have that $H_1 = AQ \cap H$ or $AQ \cap H = H$ or $H_1 \subseteq AQ \cap H \subseteq H$. If $AQ \cap H = H$, then $H \subseteq AQ$ and so $G_p = M_pH \subseteq AQ$, which is clearly impossible. If $H \subseteq AQ \cap H \subseteq H$, then $AQ \cap H$ is a maximal subgroup of $H$. Hence $H_1 = AQ \cap H$. It follows that $H_1 \leq G$. However, because $H$ is the minimal normal subgroup of $G$, we have that $H_1 = 1$. It follows that $|H| = p$, a contradiction. Hence $H \leq AQ \cap H$. It follows that $AQ \subseteq N_\nu(AQ)$. On the other hand, since $H_1 = G_1 \cap H \leq G_1$ and $H$ is an abelian group, $G_1 \leq G_1 \cap H \subseteq N_\nu(AQ)$. This shows that $H_1 \leq G_1$. Consequently, $H_1 = 1$ and so $|H| = p^2$. It follows that $|Aut(H)| = (p + 1)p(p - 1)$. Since $q > p$ and $G/H = G/C_p(H) \leq Aut(H)$, $q = p + 1$. This induces that $p = 2$, $q = 3$. Let $x$ be an element of order $3$. Thus $[H]/\langle x \rangle$ is a subgroup of $G$, which contradicts the fact that $G$ is $A_4$-free. The final contradiction completes the proof.

**Remark 2.4.1.** In Theorem 2.4, we cannot omit the assumption that $G$ is $A_4$-free in general. For example, $G = A_4$. It is clear that every $2$-maximal subgroup of the Sylow $2$-subgroups of $G$ is the identity subgroup and of course, is $s$-c-permutably embedded in $G$. But $G$ is not $2$-nilpotent.

**Corollary 2.4.1.** Let $G$ be a soluble group. Suppose that for each prime divisor $p$ of $|G|$ and $P \in Syl_p(G)$, every $2$-maximal subgroup of $P$ is $s$-c-permutably embedded in $G$ and $G$ is $A_4$-free, then $G$ is a Sylow tower group (see [2, p. 49]).

**Theorem 2.5.** Let $G$ be a group and $N$ a soluble normal subgroup of $G$ such that $G/N$ is a Sylow tower group. If, for every prime $p$ dividing the order of $N$ and $P \in Syl_p(N)$, every $2$-maximal subgroup of $P$ is $s$-c-permutably embedded in $G$ and $G$ is $A_4$-free, then $G$ is a Sylow tower group.

**Proof.** By Lemma 1.1 (4) and Corollary 2.4.1, we can see that $N$ is a Sylow tower group by induction. Let $r$ be the largest prime number in $\pi(N)$ and $R \in Syl_p(N)$. Then $R$ is a $2$-maximal subgroup of $G$. By the hypothesis, $G$ is $s$-c-permutably embedded in $G$. Hence there exists a $s$-conditionally permutable subgroup $A$ of $G$ such that $G_1 = A$ is a Sylow $p$-subgroup of $A$. Let $q$ be the largest prime divisor of $|G|$ and $Q$ a Sylow $q$-subgroup of $G$. Then $R/\langle q \rangle \leq G/R$ and thereby $R/\langle q \rangle \leq G$. If $q = r$, then, obviously, $G$ is a Sylow tower group by induction. Hence, we assume that $r < q$.

Case 1. $R/\langle q \rangle < G$. In this case, $R/\langle q \rangle$ is a Sylow tower group by Theorem 2.4 and induction. It follows that $Q \leq R/\langle q \rangle$ and so $Q \leq G$. Thus $G$ is a Sylow tower group.

Case 2. $G = R/\langle q \rangle$. Let $L$ be a minimal normal subgroup of $G$ with $L \leq R$. Then the quotient group $G/L$ (with respect to $N/L$) satisfies the hypothesis. Hence, by induction, $G/L$ is a Sylow tower group. Since the class of all Sylow tower groups is a saturated formation, $L \leq \Phi(G)$ and $L$ is the unique minimal normal subgroup of $G$ which is contained in $R$. Therefore, $L = F(R) = R$ by Lemma 1.4. In particular, $R$ is an elementary abelian group.

If $R$ is a cyclic subgroup of order $r$, then $r < q$ implies that $G$ is $r$-nilpotent by [16, (10.1.9)] and so $R = R \times Q$. Hence $G$ is a Sylow tower group. Now assume that $|R| \geq r^2$. Let $R$ be a $2$-maximal subgroup of $R$. By hypothesis, $R$ is $s$-c-permutably embedded in $G$. Hence there exists an $s$-conditionally permutable subgroup $A$ of $G$ such that $R$ is a Sylow $r$-subgroup of $A$. Then, for some $R \leq Syl_r(G)$, we have $AQ \leq G$. Since $R = R \cap AQ \leq AQ$, $AQ \subseteq N_\nu(R)$. This implies that $R \leq G$. But, because $R$ is the minimal normal subgroup of $G$, we have that $R = 1$ and so $|R| \geq r^2$. Since $Q \leq Aut(R)$ and $|Aut(R)| = (r + 1)r(r + 1)^2$, $q = 3$ and $r = 2$, which contradicts the fact that $G$ is $A_4$-free. The proof is completed.

### 3 Some applications of the results

Theorems 2.1–2.3 have many corollaries. We state only some special cases of theorems which can be found in the literature.

Theorem 2.1 immediately implies
Corollary 3.1 (Huang, Guo [9]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $H$ is $s$-conditionally permutable in $G$.

Corollary 3.2 (Chen, Guo [10]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if $G$ has a soluble normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of $H$ is $s$-conditionally permutable in $G$.

Recall that, let $X$ be a non-empty subset of $G$. Then a subgroup $H$ of $G$ is $c$-semipermutable ($X$-semipermutable) in $G$ if there is a minimal supplement $T$ of $H$ in $G$ such that $H$ is $T$-permutable ($X$-permutable) with all subgroups of $T$ (see [8], [17]). Clearly, if a subgroup $H$ of $G$ of prime power order is $c$-semipermutable ($X$-semipermutable) in $G$, then $H$ is $s$-conditionally permutable in $G$ and consequently is $s$-c-permutably embedded in $G$. Hence we immediately have the following corollary.

Corollary 3.3 (Hu, Guo [17]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of $H$ is $c$-semipermutable in $G$.

From Theorem 2.3, we have

Corollary 3.4 (Zha, Guo, Li [18]). Let $G$ be a $p$-soluble group. Then $G$ is $p$-supersoluble if and only if $G$ has a normal subgroup $N$ such that $G/N$ is $p$-supersoluble and every maximal subgroup of every Sylow $p$-subgroup of $N$ having no $p$-supersoluble supplement in $G$ is $s$-conditionally permutable in $G$.

From Theorem 2.2, we obtain

Corollary 3.5 (Ramadan [19]). Let $G$ be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(G)$ are normal in $G$, then $G$ is supersoluble.

Corollary 3.6 (Ramadan [19]). Let $G$ be a soluble group, and $E$ a normal subgroup of $G$ such that $G/E$ is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in $G$, then $G$ is supersoluble.

Corollary 3.7 (Asaad, Ramadan, Shaalan [20]). Suppose that $G/H$ is supersoluble. If $H$ is supersoluble and all maximal subgroups of any Sylow subgroup of $F(H)$ are $s$-permutable in $G$, then $G$ is supersoluble.

Corollary 3.8 (Asaad [21]). Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a soluble group with a normal subgroup $H$ such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are $s$-permutable in $G$, then $G \in \mathfrak{F}$.

Corollary 3.9 (Huang, Guo [9]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is $s$-conditionally permutable in $G$.

Corollary 3.10 (Chen, Guo [10]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of Sylow subgroups $F(H)$ is $s$-c-permutably embedded in $G$.

Corollary 3.11 (Hu, Guo [17]). Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of Sylow subgroups $F(H)$ is $c$-semipermutable in $G$.

Corollary 3.12 (Chen, Li [22]). A group $G$ is supersoluble if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H$ is supersoluble and every maximal subgroup of every Sylow subgroup of the Fitting subgroup $F(H)$ of $H$ is $F(H)$-semipermutable in $G$.

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