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Abstract

This paper investigates arbitrage chains involving four currencies and four foreign exchange trader-arbitrageurs. In contrast with the three-currency case, we find that arbitrage operations when four currencies are present may appear periodic in nature, and not involve smooth convergence to a "balanced" ensemble of exchange rates in which the law of one price holds. The goal of this article is to understand some interesting features of sequences of arbitrage operations, features which might well be relevant in other contexts in finance and economics.

Keywords: Limits to arbitrage, Four currencies, Recurrent sequences, Asynchronous systems

JEL Classification: C60, F31, D82

1. Introduction

An arbitrage operation involves buying some good or asset for a lower price than that for which it can be sold, taking advantage of any imbalance in the quoted prices. The "law of one price" is a statement of a key implication of the absence of arbitrage opportunities. In turn arbitrage is often the process invoked to explain why goods or assets that are in some sense "identical" should have a common price.

A study of commodity prices since 1273 concluded that "... despite the steady decline in transportation costs over the past 700 years, the repeated intrusion of wars and disease,

and the changing fashions of commercial policy, the volatility and persistence of deviations in the law of one price have remained quite stable" [1, p. 18]. The present paper investigates a relatively neglected complication regarding arbitrage operations, namely the order in which information about arbitrage opportunities is presented, illustrating this in relation to arbitrage chains involving four currencies. The key finding is that arbitrage operations can be periodic in nature, rather than involving a smooth convergence to a law of one price.

The early literature on the law of one price is coeval with the purchasing power parity explanation of foreign exchange rates. The terminology was coined in [2], involving arbitrage between relatively homogeneous goods priced in different currencies [1, 3]. Empirical tests suggest that arbitrage operations in goods do not exert a strong influence on exchange rates until the price index deviations involved exceed about 25% [4, 5]. Innovations that were expected to reduce price dispersion, such as the European Single Market legislation coming into effect in 1992, and the Economic and Monetary Union project beginning in 1999, have had little effect on price level disparities [6]. The degree of price level dispersion between US cities has displayed no marked trend over time [7]. A study of the prices charged for identical products in IKEA stores in twenty-five countries revealed typical price divergences of 20–50%, differences that could not be attributed to just country or location-specific factors [8]. Among the most cited reasons for deviations from the law of one price are transaction costs, taxes, transport costs, trade barriers, the costs of searching for price differences, nominal price rigidities, customer market pricing, nominal exchange rate rigidities and differences in market power [9].

In relation to assets, an early application of the law of one price was to the interest rate parity theory of the forward exchange rate, whereby the ratio of the forward to spot exchange rate between two currencies is equal to the ratio of the interest rates in the two currencies over the forward period in question [10, p. 130]. An arbitrage opportunity in relation to assets can be defined as "an investment strategy that guarantees a positive payoff in some contingency with no possibility of a negative payoff and with no net investment" [11, online]. The absence of such arbitrage opportunities has been seen as the unifying concept underlying mainstream theories in finance, no-arbitrage principles being applied in the Modigliani–Miller theorem of corporate capital structure, in the Black–Scholes model of option pricing and in the arbitrage pricing model of asset prices [12]. Actual arbitrage operations in relation to assets often involve net investment and risk and/or uncertainty, in addition to the complications arising in relation to arbitrage in goods. Notable deviations from the law of one price in financial markets have been documented in relation to comparable circumstances applying to closed-end country funds, American Depository Receipts, twin shares, dual share classes and corporate spin-offs [13]. Among the limits to arbitrage in financial markets are those arising from transactions costs [14], and those involving the capital requirements of conducting arbitrage operations [15]. A spectacular illustration of the capital limits to arbitrage was provided by the demise of the Long-Term Capital Management (LTCM) hedge funds. The arbitrage discrepancies being exploited in LTCM's "convergence trades" widened in 1998. LTCM attempted unsuccessfully to raise new capital to finance its arbitrage positions. To avoid a major financial collapse the New York Federal Reserve Board organised a bail-out by creditors [16].

In what follows we focus on the limits to arbitrage arising from the order in which information is disseminated to arbitrage traders. The illustration used is for a foreign

exchange (FX) market with four FX traders and four currencies, see Sections 2 and 3. An Arbiter, the metaphorical equivalent of an unpaid auctioneer in a Walrasian system, knows all the actual exchange rates. The individual FX traders, however, initially know only the exchange rates involving their own, domestic currencies. So the US FX trader knows the exchange rates for the dollar against the euro, sterling and yen, but not the cross exchange rates for the non-dollar currencies. There are no transactions costs, no net capital requirements and no risks involved in the arbitrage operations. Instead we focus on the information dissemination problem, and show that the order in which information about cross exchange rate discrepancies, and hence arbitrage opportunities, is presented makes an important difference to the sequences of arbitrage operations conducted.

A general discussion of arbitrage dynamics is given in Section 4. An unexpected feature of the processes considered in this paper is that, *rather than there being a smooth convergence to an ensemble of exchange rates with no arbitrage opportunities, the arbitrage operations may display periodicity and no necessary convergence on a cross exchange rate law of one price.* See Proposition 6 in Section 5 for a rigorous explanation. A further unexpected feature is that, *starting at an ensemble of exchange rates which is not balanced, and using special periodic sequences of arbitrages, the Arbiter can achieve any balanced (satisfying the law of one price) exchange rate ensemble.* See, in particular, Theorem 1 in Section 5 and Theorem 2 in Section 7. These counter-intuitive results are new, as far as we are aware. In line with the renowned "impossibility theorem" of [17] these results suggest an "arbitrage impossibility theorem". Proofs are relegated to Section 8.

The mathematical approach taken in this paper to the analysis of arbitrage operation chains may be understood as a typical example of the asynchronous interactions that are important in systems theory and in control theory, see the monographs [18–20] and the surveys [21, 22]. The arbitrage chains are particularly relevant to desynchronised systems theory, see [19]. Presence of an asynchronous interaction often leads to a dramatic complication of the related mathematical problems. [23, 24] proved that many asynchronous problems cannot be solved algorithmically, and also [25–27] and [28] demonstrated that, even in the cases when the problem is algorithmically solvable, it is typically as hard to solve numerically as the famous "Travelling salesman problem," see [29] (that is, in the mathematical language, the problem is NP-complete, see [30]). In this context the fact that the principal questions that arise in analysis of arbitrage operation chains admit straightforward combinatorial analysis came to the authors as a pleasant surprise. Our construction uses a geometrical approach to visualisation of arbitrage chains presented in Sections 6–8, which may be useful in relation to other problems in mathematical economics.

2. The Three Currency Case

Consider a foreign exchange (FX) market that involves only three currencies: Dollars (\$), Euros (€) and Sterling (£). This FX market involves three pair-wise exchange operations:

$$Dollar \leftrightarrows Euro, \quad Dollar \rightleftarrows Sterling, \quad Euro \rightleftarrows Sterling.$$

The currencies are measured in natural currency units, and the corresponding (strictly positive) exchange rates, $r_{\$€}$, $r_{\$£}$, $r_{€£}$, are well defined. For instance, one dollar can be

exchanged for $r_{\$\euro}$ euros. The rates related to the inverted arrows are reciprocal:

$$r_{\euro\$} = \frac{1}{r_{\$\euro}}, \quad r_{\pounds\$} = \frac{1}{r_{\$\pounds}}, \quad r_{\pounds\euro} = \frac{1}{r_{\euro\pounds}} \ . \tag{1}$$

We treat the triplet

$$(r_{\$\euro}, \ r_{\$\pounds}, \ r_{\euro\pounds}) \tag{2}$$

as the ensemble of *principal exchange rates.*

We suppose that, prior to a reference time moment 0, each FX trader knows only the exchange rates involving his domestic currency. So the dollar trader does not know the value of $r_{\euro\pounds}$, the euro trader is unaware of $r_{\$\pounds}$, and the sterling trader is unaware of $r_{\$\euro}$. We are interested in the case where the initial rates are unbalanced in the following sense. By assumption, the dollar trader can exchange one dollar for $r_{\$\euro}$ euros. Let us suppose that unbeknownst to him the exchange rate between sterling and euro is such that the the dollar trader could make a profit by first exchanging a dollar for $r_{\$\pounds}$ units of sterling and then exchanging these for euros. The inequality which guarantees that dollar trader can take advantage of this arbitrage opportunity is that the product $r_{\$\pounds}r_{\pounds\euro}$ is greater than $r_{\$\euro}$:

$$r_{\$\pounds} \cdot r_{\pounds\euro} > r_{\$\euro}. \tag{3}$$

Let us consider the situation where the inequality (3) holds, and, after the reference time moment 0, one of the three traders becomes aware of the third exchange rate. The evolution of this FX market depends on *which trader is the first to discover the information concerning the third exchange rate.* The following three cases are relevant.

2.1. Case 1.

The dollar trader becomes aware of the value of the rate $r_{\euro\pounds}$. Therefore, the dollar trader contacts the euro trader and makes a request to increase the rate $r_{\$\euro}$ to the new fairer value

$$r_{\$\euro}^{new} = r_{\$\pounds} \cdot r_{\pounds\euro} = \frac{r_{\$\pounds}}{r_{\euro\pounds}}.$$

The reciprocal exchange rate $r_{\euro\$}$ is also to be adjusted to the new level:

$$r_{\euro\$}^{new} = \frac{1}{r_{\$\euro}^{new}}.$$

The result is that the principal exchange rates become balanced at the levels:

$$r_{\$\euro}^{new} = \frac{r_{\$\pounds}}{r_{\euro\pounds}}, \quad r_{\$\pounds}, \quad r_{\euro\pounds}.$$

2.2. Case 2.

The euro trader is the first to discover the third exchange rate $r_{\$\pounds}$. By (1), inequality (3) may be rewritten as

$$\frac{r_{\$\pounds}}{r_{\euro\pounds}} < \frac{1}{r_{\euro\$}},$$

4

which is, in turn, equivalent to $r_{€\$} \cdot r_{\$£} > r_{€£}$. In this case the euro trader could do better by first exchanging euros for dollars, and then by exchanging the dollars for sterling. Therefore, the euro trader requests adjustment of the rate $r_{€£}$ to the value

$$r_{€£}^{new} = r_{€\$} \cdot r_{\$£} = \frac{r_{\$£}}{r_{\$€}}.$$

In terms of the principal exchange rates the outcome is that the FX market adjusts to the following balanced rates:

$$r_{\$€}, \quad r_{\$£}, \quad r_{€£}^{new} = \frac{r_{\$£}}{r_{\$€}}.$$

2.3. Case 3.

The sterling trader is the first to discover the third exchange rate $r_{\$€}$. The inequality (3) may be rewritten as $r_{£€} \cdot r_{€\$} > r_{£\$}$. Thus, the sterling trader requests adjustment of the rate $r_{£\$}$ to $r_{£\$}^{new} = r_{£€} \cdot r_{€\$}$. In this case the principal exchange rates become balanced at the levels:

$$r_{\$€}, \quad r_{\$£}^{new} = r_{\$€} \cdot r_{€£}, \quad r_{€£}.$$

After the adjustment of the principal exchange rates (2), following the new informa-tion being revealed, the exchange rates become balanced, and this is the end of the arbi-trage evolution of an FX market with three currencies. Having established the reasonably straightforward application of arbitrage to three currencies, we now turn to investigation what happens when the FX market contains four currencies and four currency traders.

3. Four Currencies

Consider an FX market \$€£¥ that involves four currencies: Dollars (\$), Euros (€), Sterling (£) and Yen (¥). This FX market involves six exchange relationships:

$$Dollar \rightleftarrows Euro, \qquad Dollar \rightleftarrows Sterling, \qquad Dollar \rightleftarrows Yen,$$
$$Euro \rightleftarrows Sterling, \qquad Euro \rightleftarrows Yen, \qquad Sterling \rightleftarrows Yen.$$

The exchange rates are:

$$r_{\$€}, \quad r_{\$£}, \quad r_{\$¥}, \quad r_{€\$}, \quad r_{€¥}, \quad r_{£¥},$$
$$r_{£\$}, \quad r_{£€}, \quad r_{£¥}, \quad r_{¥\$}, \quad r_{¥€}, \quad r_{¥£}.$$

The rates relating to the inverted arrows are reciprocal:

$$r_{€\$} = \frac{1}{r_{\$€}}, \quad r_{£\$} = \frac{1}{r_{\$£}}, \quad r_{¥\$} = \frac{1}{r_{\$¥}},$$
$$r_{£€} = \frac{1}{r_{€£}}, \quad r_{¥€} = \frac{1}{r_{€¥}}, \quad r_{¥£} = \frac{1}{r_{£¥}}. \tag{4}$$

Our market may be described by the ensemble of six principal exchange rates

$$\mathcal{R} = (r_{\$€}, \ r_{\$£}, \ r_{\$¥}, \ r_{€£}, \ r_{€¥}, \ r_{£¥}) \tag{5}$$

together with the reciprocal exchange rates (4).

The following characterisation of balanced, no-arbitrage, exchange rates (5), that is the ensembles of exchange rates such that no trader could do better by trading indirectly, is convenient.

Proposition 1. *Ensemble (5) of the principal exchange rates is balanced if and only if the following relationships hold:*

$$r_{\math013£} = \frac{r_{\$£}}{r_{\$\math013}}, \quad r_{\math013¥} = \frac{r_{\$¥}}{r_{\$\math013}}, \quad r_{£¥} = \frac{r_{\$¥}}{r_{\$£}}. \tag{6}$$

Proof. This assertion can be proved by inspection. $\square$

4. Arbitrages

Let us suppose that initially each trader is aware only of the three exchange rates involving his domestic currency. For instance, the dollar trader knows only the rates $r_{\$\math013}$, $r_{\$£}$, $r_{\$¥}$.

We are interested in the case where the rates $r_{\$\math013}$, $r_{\$£}$, $r_{\$¥}$, $r_{\math013£}$, $r_{\math013¥}$, $r_{£¥}$ are unbalanced.

For instance, let us suppose that the dollar trader can make a profit by first exchanging one dollar for $r_{\$£}$ units of sterling, and then by exchanging this sterling for euros. This means that the product $r_{\$£} \cdot r_{£\math013}$ is greater than $r_{\$\math013}$:

$$r_{\$£} \cdot r_{£\math013} > r_{\$\math013}. \tag{7}$$

Suppose that the dollar trader becomes aware of the rate $r_{\math013£}$, and, therefore, about the inequality (7). The dollar trader then asks the euro trader to increase the exchange rate $r_{\$\math013}$ to the new fairer value

$$r_{\$\math013}^{new} = r_{\$£} \cdot r_{£\math013} = \frac{r_{\$£}}{r_{\math013£}}.$$

Along with the adjustment of the exchange rate $r_{\$\math013}$ the reciprocal rate $r_{\math013\$}$ would be adjusted to

$$r_{\math013\$}^{new} = \frac{1}{r_{\$\math013}^{new}}.$$

We call this procedure $\$\math013£$-*arbitrage*, and we use the notation $\mathcal{A}_{\$\math013£}$ to represent it. We denote by $\mathcal{RA}_{\$\math013£}$ the ensemble of the new principal exchange rates:

$$\mathcal{R}^{new} = \mathcal{RA}_{\$\math013£} = \left(r_{\$\math013}^{new}, \ r_{\$£}, \ r_{\$¥}, \ r_{\math013£}, \ r_{\math013¥}, \ r_{£¥} \right).$$

We also use the notation $\mathcal{RA}_{\$\math013£}$ in the case where the inequality (7) does not hold. In this case, of course, $\mathcal{RA}_{\$\math013£} = \mathcal{R}$, and we say that arbitrage $\mathcal{A}_{\$\math013£}$ is *not active* in this case. This particular arbitrage is an example of the 24 possible arbitrages listed in Table 1. We will also use, where convenient, the notation $\mathcal{A}^{(n)}$ for the arbitrage number n from this table: for instance, $\mathcal{A}^{(1)} = \mathcal{A}_{\$\math013£}$.

The principal distinction of the FX market with four currencies from that with only three currencies is that applying a single arbitrage operation does not bring the FX market to a balance in which no arbitrage opportunities exist, and in which the law of one price holds.

6

Table 1: List of arbitrages

Number	Arbitrage	Activation condition	Actions
1	$\mathcal{A}_{\$€£}$	$r_{€£} > r_{\$€} \cdot r_{\$£}$	$r^{new}_{\$€} = r_{\$£} \cdot r^{-1}_{€£}$
2	$\mathcal{A}_{\$€¥}$	$r_{\$¥} > r_{\$€} \cdot r_{€¥}$	$r^{new}_{\$€} = r_{\$¥} \cdot r^{-1}_{€¥}$
3	$\mathcal{A}_{\$£€}$	$r_{\$€} \cdot r_{€£} > r_{\$£}$	$r^{new}_{\$£} = r_{\$€} \cdot r_{€£}$
4	$\mathcal{A}_{\$£¥}$	$r_{\$¥} > r_{\$£} \cdot r_{£¥}$	$r^{new}_{\$£} = r_{\$¥} \cdot r^{-1}_{£¥}$
5	$\mathcal{A}_{\$¥€}$	$r_{\$€} \cdot r_{€¥} > r_{\$¥}$	$r^{new}_{\$¥} = r_{\$€} \cdot r_{€¥}$
6	$\mathcal{A}_{\$¥£}$	$r_{\$£} \cdot r_{£¥} > r_{\$¥}$	$r^{new}_{\$¥} = r_{\$£} \cdot r_{£¥}$
7	$\mathcal{A}_{€\$£}$	$r_{\$£} < r_{\$€} \cdot r_{€£}$	$r^{new}_{\$€} = r_{\$£} \cdot r^{-1}_{€£}$
8	$\mathcal{A}_{€\$¥}$	$r_{\$¥} < r_{\$€} \cdot r_{€¥}$	$r^{new}_{\$€} = r_{\$¥} \cdot r^{-1}_{€¥}$
9	$\mathcal{A}_{€£\$}$	$r_{\$£} > r_{€£} \cdot r_{\$€}$	$r^{new}_{€£} = r_{\$£} \cdot r^{-1}_{\$€}$
10	$\mathcal{A}_{€£¥}$	$r_{€¥} > r_{€£} \cdot r_{£¥}$	$r^{new}_{€£} = r_{€¥} \cdot r^{-1}_{£¥}$
11	$\mathcal{A}_{€¥\$}$	$r_{\$¥} > r_{€¥} \cdot r_{\$€}$	$r^{new}_{€¥} = r_{\$¥} \cdot r^{-1}_{\$€}$
12	$\mathcal{A}_{€¥£}$	$r_{€£} \cdot r_{£¥} > r_{€¥}$	$r^{new}_{€¥} = r_{€£} \cdot r_{£¥}$
13	$\mathcal{A}_{£\$€}$	$r_{\$€} \cdot r_{€£} < r_{\$£}$	$r^{new}_{\$£} = r_{\$€} \cdot r_{€£}$
14	$\mathcal{A}_{£\$¥}$	$r_{\$¥} < r_{\$£} \cdot r_{£¥}$	$r^{new}_{\$£} = r_{\$¥} \cdot r^{-1}_{£¥}$
15	$\mathcal{A}_{£€\$}$	$r_{\$£} < r_{€£} \cdot r_{\$€}$	$r^{new}_{€£} = r_{\$£} \cdot r^{-1}_{\$€}$
16	$\mathcal{A}_{£€¥}$	$r_{€¥} < r_{€£} \cdot r_{£¥}$	$r^{new}_{€£} = r_{€¥} \cdot r^{-1}_{£¥}$
17	$\mathcal{A}_{£¥\$}$	$r_{\$¥} > r_{£¥} \cdot r_{\$£}$	$r^{new}_{£¥} = r_{\$¥} \cdot r^{-1}_{\$£}$
18	$\mathcal{A}_{£¥€}$	$r_{€¥} > r_{£¥} \cdot r_{€£}$	$r^{new}_{£¥} = r_{€¥} \cdot r^{-1}_{€£}$
19	$\mathcal{A}_{¥\$€}$	$r_{\$€} \cdot r_{€¥} < r_{\$¥}$	$r^{new}_{\$¥} = r_{\$€} \cdot r_{€¥}$
20	$\mathcal{A}_{¥\$£}$	$r_{\$£} \cdot r_{£¥} < r_{\$¥}$	$r^{new}_{\$¥} = r_{\$£} \cdot r_{£¥}$
21	$\mathcal{A}_{¥€\$}$	$r_{\$¥} < r_{€¥} \cdot r_{\$€}$	$r^{new}_{€¥} = r_{\$¥} \cdot r^{-1}_{\$€}$
22	$\mathcal{A}_{¥€£}$	$r_{€£} \cdot r_{£¥} < r_{€¥}$	$r^{new}_{€¥} = r_{€£} \cdot r_{£¥}$
23	$\mathcal{A}_{¥£\$}$	$r_{\$¥} < r_{£¥} \cdot r_{\$£}$	$r^{new}_{£¥} = r_{\$¥} \cdot r^{-1}_{\$£}$
24	$\mathcal{A}_{¥£€}$	$r_{€¥} < r_{£¥} \cdot r_{€£}$	$r^{new}_{£¥} = r_{€¥} \cdot r^{-1}_{€£}$

5. Main Results

One can apply arbitrages from Table 1 sequentially in any order and to any initial exchange rates $\mathcal{R}$. The situation that we have in mind is the following. Suppose that there exists an *Arbiter* who knows current ensemble $\mathcal{R}$ of exchange rates. This *Arbiter* could provide information to the FX traders in any order he wants, thus activating the *chain* (or *superposition*) of corresponding arbitrages. The principal question is:

Question 1. *How powerful is the* Arbiter*?*

The short answer is: the *Arbiter is surprisingly powerful.*

Let us explain at a more formal level what we mean.

For a finite chain of arbitrages $\boldsymbol{A} = \mathcal{A}_1 \cdots \mathcal{A}_n$, and for a given ensemble $\mathcal{R}$ of initial exchange rates, we denote the resulting ensemble of principal exchange rates as

$$\mathcal{R}\boldsymbol{A} = \mathcal{R}\mathcal{A}_1 \cdots \mathcal{A}_n \tag{8}$$

If $\mathcal{R}$ is balanced, then $\mathcal{R}\mathcal{A} = \mathcal{R}$ for any individual arbitrage, and therefore $\mathcal{R}\boldsymbol{A} = \mathcal{R}$ for any chain (8). If, on the contrary, $\mathcal{R}$ is not balanced, then different arbitrage chains (8)

could result in different balanced or unbalanced ensembles of principal exchange rates. Denote by $S(\mathcal{R})$ the collection of the sets $\mathcal{R}\boldsymbol{A}$ related to all possible chains (8). Denote also by $S^{bal}(\mathcal{R})$ the subset of $S(\mathcal{R})$, that includes only balanced exchange rates ensembles. Our principal observation is the following:

For a typical unbalanced exchange rate ensemble $\mathcal{R}$, the set $S^{bal}(\mathcal{R})$ is unexpectedly rich; therefore the Arbiter, who prescribes a particular sequence of arbitrages, is an unexpectedly powerful figure.

To avoid cumbersome notation and technical details when providing a rigorous formulation of this observation, we concentrate on the simplest initial ensemble. Let us consider the ensemble

$$\bar{\mathcal{R}}_\alpha = (\alpha \cdot \bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}), \tag{9}$$

where $\alpha > 0, \alpha \neq 1$ and $\bar{\mathcal{R}}$ is a given balanced ensemble of principal exchange rates. The ensemble (9) is not balanced. The ensemble (9) may have emerged as follows. Let us suppose that the underlying balanced rates

$$\bar{\mathcal{R}} = (\bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}) \tag{10}$$

had been in operation up to a certain reference time moment 0. At this moment the dollar trader has decided to increase his price for euros by a factor $\alpha > 1$. A natural respecification of Question 1 is the following:

Question 2. *To which balanced exchange rates can the Arbiter now bring the foreign exchange market?*

The possible general structure of elements from the corresponding sets $S(\bar{\mathcal{R}}_\alpha)$ and $S^{bal}(\bar{\mathcal{R}}_\alpha)$ is easy to describe. To this end we denote by $T_\alpha(\bar{\mathcal{R}})$ the collection of all sextuples of the form

$$(\alpha^{n_1} \cdot \bar{r}_{\$€}, \ \alpha^{n_2} \cdot \bar{r}_{\$£}, \ \alpha^{n_3} \cdot \bar{r}_{\$¥}, \ \alpha^{n_4} \cdot \bar{r}_{€£}, \ \alpha^{n_5} \cdot \bar{r}_{€¥}, \ \alpha^{n_6} \cdot \bar{r}_{£¥}), \tag{11}$$

where n_i are integer numbers (positive, negative or zero). We also denote by T_α^{bal} the subset of elements of T_α, which satisfy the relationships

$$n_4 = n_2 - n_1, \quad n_5 = n_3 - n_1, \quad n_6 = n_3 - n_2.$$

Proposition 2. *The following inclusions hold:*

$$S(\bar{\mathcal{R}}_\alpha) \subset T_\alpha(\bar{\mathcal{R}}), \tag{12}$$
$$S^{bal}(\bar{\mathcal{R}}_\alpha) \subset T_\alpha^{bal}(\bar{\mathcal{R}}). \tag{13}$$

Proof. The ensemble (10) belongs to T. To verify (12) we show that the set T_α is invariant with respect to each arbitrage $\mathcal{A}$ from Table 1. This statement can be checked by inspection. Let us, for instance, apply to a sextuple (11) the first arbitrage $\mathcal{A}_{\$€£}$. Then, by definition, either this arbitrage is inactive, or it changes the first component $\alpha^{n_1} \cdot \bar{r}_{\$€}$ of (11) to the new value

$$r_{\$€}^{new} = \frac{\alpha^{n_2} \cdot \bar{r}_{\$£}}{\alpha^{n_4} \cdot \bar{r}_{€£}} = \alpha^{n_2 - n_4} \cdot \frac{\bar{r}_{\$£}}{\bar{r}_{€£}}. \tag{14}$$

8

However, the ensemble $\bar{\mathcal{R}}$ is balanced, and, by the first equation (6), $\frac{\bar{r}_{\$\pounds}}{\bar{r}_{\in\pounds}} = \bar{r}_{\$\in}$. Therefore, (14) implies that the ensemble $\bar{\mathcal{R}}\mathcal{A}_{\$\in\pounds}$ also may be represented in the form (11). We have proved the first part of the proposition, related to the set $S(\bar{\mathcal{R}}_\alpha)$. The inclusion (13) follows now from Proposition 1. $\qquad\square$

Proposition 2 in no way answers Question 2. This proposition, however, allows us to reformulate this question in a more constructive form:

Question 3. *How big is the set $S^{bal}(\bar{\mathcal{R}}_\alpha)$, compared with the collection $T_\alpha^{bal}(\bar{\mathcal{R}})$ of all elements that satisfy the restrictions imposed by Proposition 2?*

The naive expectation would be that the set $S^{bal}(\bar{\mathcal{R}}_\alpha)$ is finite and, at least for values of α close to 1, that all elements of $S^{bal}(\bar{\mathcal{R}}_\alpha)$ are close to $\bar{\mathcal{R}}$. However, the following statement, describing an unexpected feature of the power of the *Arbiter*, is true.

Theorem 1. *The set $S^{bal}(\bar{\mathcal{R}}_\alpha)$ coincides with $T_\alpha^{bal}(\bar{\mathcal{R}})$:*

$$S^{bal}(\bar{\mathcal{R}}_\alpha) = T_\alpha^{bal}(\bar{\mathcal{R}}). \tag{15}$$

Moreover each balanced ensemble (11) *may be achieved via a chain of arbitrage operations no longer than*

$$N(n_1, n_2, n_3) = 3(|n_1 - 1| + |n_2| + |n_3|) + 3. \tag{16}$$

Loosely speaking, this theorem means that the *Arbiter* is extremely powerful. An assertion similar to Theorem 1 was formulated as a hypothesis in [31]. We describe the algorithms corresponding to this theorem in the next section.

The following assertion certifies that the estimate (16) from Theorem 1 is pretty close to the optimal.

Proposition 3. *The inequalities*

$$|n_1 - n_2 + n_4|, \ |n_1 - n_3 + n_5)|, \ |n_2 - n_3 + n_6| \leq 1$$

hold for any $\mathcal{R} \in S(\bar{\mathcal{R}}_\alpha)$. Here n_i are the integers from representation (11) *of $\mathcal{R}$.*

Proof. This assertion is a special case of Lemma 7 which will be considered below. $\quad\square$

Note that the set $S(\bar{\mathcal{R}}_\alpha)$ is, in contrast to (15), much smaller than the totality $T_\alpha(\bar{\mathcal{R}})$ of all ensembles of the form (11). In particular, the following assertion holds:

Proposition 4. *Let $\boldsymbol{A}$ denote a chain of arbitrages of length N, and $\mathcal{R} = \bar{\mathcal{R}}_\alpha \boldsymbol{A}$. Then $3(|n_1 - 1| + |n_2| + |n_3|) \leq N + 8$, where n_1, n_2, n_3 are the integers from the representation* (11) *of $\mathcal{R}$.*

Let us consider an infinite arbitrage chain:

$$\boldsymbol{A} = \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\cdots\mathcal{A}_n\cdots. \tag{17}$$

This chain is periodic with minimal period p if $\mathcal{A}_n = \mathcal{A}_{n+p}$ for $n = 1, 2, \ldots$, and p is the minimal positive integer with this property. Various periodic chains of arbitrage play a special role in context of this article, and we summairise below some interesting

9

features of such periodic arbitrage chains. For a periodic chain (17) and for an initial (unbalanced) exchange rate ensemble $\mathcal{R}_0$ we consider the sequence

$$\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n, \ldots \tag{18}$$

defined by $\mathcal{R}_n = \mathcal{R}_{n-1}\mathcal{A}_n$, $n = 1, 2, \ldots$.

Proposition 5. *Either (i) the sequence (18) is periodic for $n \geq 36p$; or (ii) this sequence is diverging: at least one of the following six relationships hold:*

$$r_{\$€_n} \to 0, \quad r_{\$£_n} \to 0, \ r_{\$¥_n} \to 0, \ r_{\$€_n} \to \infty, \ r_{\$£_n} \to \infty, \ r_{\$¥_n} \to \infty.$$

Moreover, in Case (i) the minimal period of the sequence is a divisor of $24p$; in Case (ii) there exist a divisor q of $24p$ and factors $\gamma_{\$€}, \ldots, \gamma_{£¥}$ such that the relationships $r_{\$€_{n+q}} = \gamma_{\$€} r_{\$€_n}$, $\ldots$, $r_{£¥_{n+q}} = \gamma_{£¥} r_{\$€_n}$ hold for $n \geq 36p$.

Proof. This statement follows from Lemmas 3 and 4. $\square$

To conclude this discussion, we note one more unexpected feature of periodic chains of arbitrage. A chain (17) is *regular for the initial ensemble* $\mathcal{R}_0$ if this chain includes all 24 arbitrages, and each arbitrage is active infinitely many times while generating the sequence (18). By analogy with typical results from the desynchronised systems theory, one could expect a regular chain of arbitrage elements of the corresponding sequence (18) should be balanced for sufficiently large n. However, this is not the case: the sequences (18) may be both periodic (after some transient period) or diverging.

As an instructive example consider the 24-periodic chain $\boldsymbol{A}_*$ which is defined by the following equations:

$$\mathcal{A}_1 = \mathcal{A}^{(15)}, \quad \mathcal{A}_2 = \mathcal{A}^{(10)}, \quad \mathcal{A}_3 = \mathcal{A}^{(3)}, \quad \mathcal{A}_4 = \mathcal{A}^{(21)},$$
$$\mathcal{A}_5 = \mathcal{A}^{(11)}, \quad \mathcal{A}_6 = \mathcal{A}^{(8)}, \quad \mathcal{A}_7 = \mathcal{A}^{(24)}, \quad \mathcal{A}_8 = \mathcal{A}^{(17)},$$
$$\mathcal{A}_9 = \mathcal{A}^{(6)}, \quad \mathcal{A}_{10} = \mathcal{A}^{(9)}, \quad \mathcal{A}_{11} = \mathcal{A}^{(16)}, \quad \mathcal{A}_{12} = \mathcal{A}^{(13)},$$
$$\mathcal{A}_{13} = \mathcal{A}^{(12)}, \quad \mathcal{A}_{14} = \mathcal{A}^{(22)}, \quad \mathcal{A}_{15} = \mathcal{A}^{(14)}, \quad \mathcal{A}_{16} = \mathcal{A}^{(18)},$$
$$\mathcal{A}_{17} = \mathcal{A}^{(23)}, \quad \mathcal{A}_{18} = \mathcal{A}^{(15)}, \quad \mathcal{A}_{19} = \mathcal{A}^{(5)}, \quad \mathcal{A}_{20} = \mathcal{A}^{(7)},$$
$$\mathcal{A}_{21} = \mathcal{A}^{(4)}, \quad \mathcal{A}_{22} = \mathcal{A}^{(19)}, \quad \mathcal{A}_{23} = \mathcal{A}^{(1)}, \quad \mathcal{A}_{24} = \mathcal{A}^{(5)}.$$

Proposition 6. *For the initial ensemble $\mathcal{R}_0 = \bar{\mathcal{R}}_\alpha$ the corresponding sequence (18) is periodic with minimal period 24, and all arbitrages from $\boldsymbol{A}_*$ are active.*

Proof. By inspection. $\square$

This proposition demonstrates that arbitrage operation chains may display periodicity and no necessary convergence on a cross exchange rate law of one price. See Figs. 5, 7 and formula (31) below for an explanation of the geometrical meaning of the arbitrage chain $\boldsymbol{A}_*$.

6. The Basic Algorithm

Introduce the following chains of arbitrages of length 3:

$$\boldsymbol{A}_+^{(1)} = \mathcal{A}^{(21)}\mathcal{A}^{(16)}\mathcal{A}^{(1)}, \quad \boldsymbol{A}_+^{(2)} = \mathcal{A}^{(3)}\mathcal{A}^{(17)}\mathcal{A}^{(10)}, \quad \boldsymbol{A}_+^{(3)} = \mathcal{A}^{(5)}\mathcal{A}^{(18)}\mathcal{A}^{(12)},$$
$$\boldsymbol{A}_-^{(1)} = \mathcal{A}^{(8)}\mathcal{A}^{(9)}\mathcal{A}^{(11)}, \quad \boldsymbol{A}_-^{(2)} = \mathcal{A}^{(15)}\mathcal{A}^{(18)}\mathcal{A}^{(14)}, \quad \boldsymbol{A}_-^{(3)} = \mathcal{A}^{(21)}\mathcal{A}^{(23)}\mathcal{A}^{(20)}.$$

It is convenient to define the mapping $\sigma(n)$ which corresponds to a non-negative integer n by the symbol "+", and by the symbol "−" for a negative integer.

Proposition 7. *The chain*

$$\boldsymbol{A}(n_1, n_2, n_3) = \left(\boldsymbol{A}_{\sigma(n_3)}^{(3)}\right)^{|n_3|}\left(\boldsymbol{A}_{\sigma(n_2)}^{(2)}\right)^{|n_2|}\mathcal{A}^{(15)}\mathcal{A}^{(18)}\left(\boldsymbol{A}_{\sigma(n_1)}^{(1)}\right)^{|n_1-1|}\mathcal{A}^{(5)} \tag{19}$$

satisfies Theorem 1: the ensemble $\bar{\mathcal{R}}_\alpha \boldsymbol{A}(n_1, n_2, n_3)$ coincides with

$$\left(\alpha^{n_1} \cdot \bar{r}_{\$€}, \ \alpha^{n_2} \cdot \bar{r}_{\$£}, \ \alpha^{n_3} \cdot \bar{r}_{\$¥}, \ \alpha^{n_1-n_2} \cdot \bar{r}_{€£}, \ \alpha^{n_1-n_3} \cdot \bar{r}_{€¥}, \ \alpha^{n_2-n_3} \cdot \bar{r}_{£¥}\right),$$

and the length N of the chain (19) satisfies $N \le 3(|n_1 - 1| + |n_2| + |n_3|) + 3$.

The legitimacy of this algorithm may be verified by induction. However a simple geometric proof is much more instructive. This proof will be given later on. This chain is not always the shortest: for instance, in the case $n_1 = n_2 = n_3 = 0$ the shortest chain $\boldsymbol{A}$ is of length one: $\boldsymbol{A} = \mathcal{A}_7$.

7. General case

7.1. Direct Generalisation

We begin with the following comment. The ensemble (9) is the first item in the list

$$\bar{\mathcal{R}}_\alpha^1 = (\alpha \cdot \bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}),$$
$$\bar{\mathcal{R}}_\alpha^2 = (\bar{r}_{\$€}, \ \alpha \cdot \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}),$$
$$\bar{\mathcal{R}}_\alpha^3 = (\bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \alpha \cdot \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}),$$
$$\bar{\mathcal{R}}_\alpha^4 = (\bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \alpha \cdot \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \bar{r}_{£¥}), \tag{20}$$
$$\bar{\mathcal{R}}_\alpha^5 = (\bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \alpha \cdot \bar{r}_{€¥}, \ \bar{r}_{£¥}),$$
$$\bar{\mathcal{R}}_\alpha^6 = (\bar{r}_{\$€}, \ \bar{r}_{\$£}, \ \bar{r}_{\$¥}, \ \bar{r}_{€£}, \ \bar{r}_{€¥}, \ \alpha \cdot \bar{r}_{£¥}).$$

A natural "relabelling" procedure confirms that the main results described in Section 5 hold without any changes for first initial ensemble from the list (20). In particular, Theorem 1 implies

Corollary 1. *The equality $S^{bal}(\bar{\mathcal{R}}_\alpha^i) = T_\alpha^{bal}(\bar{\mathcal{R}})$ holds for $i = 2, 3$. Moreover each balanced ensemble (11) may be achieved via a chain of arbitrage operations no longer than $N^i(n_1, n_2, n_3)$, where*

$$N^2(n_1, n_2, n_3) = 3(|n_1| + |n_2 - 1| + |n_3|) + 3,$$
$$N^3(n_1, n_2, n_3) = 3(|n_1| + |n_2| + |n_3 - 1|) + 3,$$

To describe the corresponding algorithms we introduce the auxiliary chains

$$\tilde{\boldsymbol{A}}_+^{(1)} = \mathcal{A}^{(1)}\mathcal{A}^{(21)}\mathcal{A}^{(16)}, \quad \tilde{\boldsymbol{A}}_+^{(2)} = \mathcal{A}^{(13)}\mathcal{A}^{(23)}\mathcal{A}^{(16)}, \quad \tilde{\boldsymbol{A}}_+^{(3)} = \mathcal{A}^{(24)}\mathcal{A}^{(12)}\mathcal{A}^{(19)}.$$

$$\tilde{\boldsymbol{A}}_-^{(1)} = \mathcal{A}^{(9)}\mathcal{A}^{(11)}\mathcal{A}^{(8)}, \quad \tilde{\boldsymbol{A}}_-^{(2)} = \mathcal{A}^{(9)}\mathcal{A}^{(34)}\mathcal{A}^{(4)}, \quad \tilde{\boldsymbol{A}}_-^{(3)} = \mathcal{A}^{(6)}\mathcal{A}^{(11)}\mathcal{A}^{(17)};$$

$$\tilde{\tilde{\boldsymbol{A}}}_+^{(1)} = \mathcal{A}^{(18)}\mathcal{A}^{(12)}\mathcal{A}^{(5)}, \quad \tilde{\tilde{\boldsymbol{A}}}_+^{(2)} = \mathcal{A}^{(23)}\mathcal{A}^{(16)}\mathcal{A}^{(13)}, \quad \tilde{\tilde{\boldsymbol{A}}}_+^{(3)} = \mathcal{A}^{(18)}\mathcal{A}^{(12)}\mathcal{A}^{(5)}.$$

$$\tilde{\tilde{\boldsymbol{A}}}_-^{(1)} = \mathcal{A}^{(20)}\mathcal{A}^{(21)}\mathcal{A}^{(23)}, \quad \tilde{\tilde{\boldsymbol{A}}}_-^{(2)} = \mathcal{A}^{(4)}\mathcal{A}^{(9)}\mathcal{A}^{(24)}, \quad \tilde{\tilde{\boldsymbol{A}}}_-^{(3)} = \mathcal{A}^{(20)}\mathcal{A}^{(21)}\mathcal{A}^{(23)}.$$

The equation (19) can be modified to the form

$$\boldsymbol{A}_2(n_1, n_2, n_3) = \left(\tilde{\boldsymbol{A}}_{\sigma(n_1)}^{(1)}\right)^{|n_1|}\mathcal{A}^{(24)}\mathcal{A}^{(12)}\left(\tilde{\boldsymbol{A}}_{\sigma(n_3)}^{(3)}\right)^{|n_3|}\left(\tilde{\boldsymbol{A}}_{\sigma(n_2)}^{(2)}\right)^{|n_2-1|}\mathcal{A}^{(1)}$$

for $i = 2$, and to the form

$$\boldsymbol{A}_3(n_1, n_2, n_3) = \left(\tilde{\tilde{\boldsymbol{A}}}_{\sigma(n_2)}^{(2)}\right)^{|n_2|}\left(\tilde{\tilde{\boldsymbol{A}}}_{\sigma(n_1)}^{(1)}\right)^{|n_1|}\mathcal{A}^{(12)}\mathcal{A}^{(10)}\left(\tilde{\tilde{\boldsymbol{A}}}_{\sigma(n_3)}^{(3)}\right)^{|n_3-1|}\mathcal{A}^{(3)}$$

for $i = 3$.

Let us turn to the initial ensembles $\bar{\mathcal{R}}_\alpha^i$, $i = 4, 5, 6$.

Proposition 8. *The equality* $S^{bal}(\bar{\mathcal{R}}_\alpha^i) = T_\alpha^{bal}(\bar{\mathcal{R}})$ *holds for* $i = 4, 5, 6$. *Moreover each balanced ensemble* (11) *may be achieved via a chain of arbitrage no longer than* $N^i(n_1, n_2, n_3)$, *where*

$$N^{4,5,6}(n_1, n_2, n_3) = 3(|n_1| + |n_2| + |n_3|) + 4.$$

The corresponding chains $\boldsymbol{A}_4(n_1, n_2, n_3)$, $i = 4, 5, 6$, may be defined by the following equations:

$$\boldsymbol{A}_4(n_1, n_2, n_3) = \mathcal{A}^{(12)}\boldsymbol{A}(n_1 + 1, n_2, n_3)$$
$$= \mathcal{A}^{(12)}\left(\boldsymbol{A}_{\sigma(n_3)}^{(3)}\right)^{|n_3|}\left(\boldsymbol{A}_{\sigma(n_2)}^{(2)}\right)^{|n_2|}\mathcal{A}^{(15)}\mathcal{A}^{(18)}\left(\boldsymbol{A}_{\sigma(n_1)}^{(1)}\right)^{|n_1|}\mathcal{A}^{(5)},$$

$$\boldsymbol{A}_5(n_1, n_2, n_3) = \mathcal{A}^{(16)}\boldsymbol{A}(n_1 + 1, n_2, n_3)$$
$$= \mathcal{A}^{(16)}\left(\boldsymbol{A}_{\sigma(n_3)}^{(3)}\right)^{|n_3|}\left(\boldsymbol{A}_{\sigma(n_2)}^{(2)}\right)^{|n_2|}\mathcal{A}^{(15)}\mathcal{A}^{(18)}\left(\boldsymbol{A}_{\sigma(n_1)}^{(1)}\right)^{|n_1|}\mathcal{A}^{(3)},$$

$$\boldsymbol{A}_6(n_1, n_2, n_3) = \mathcal{A}^{(16)}\boldsymbol{A}_2(n_1 + 1, n_2, n_3)$$
$$= \mathcal{A}^{(10)}\left(\tilde{\boldsymbol{A}}_{\sigma(n_1)}^{(1)}\right)^{|n_1|}\mathcal{A}^{(24)}\mathcal{A}^{(12)}\left(\tilde{\boldsymbol{A}}_{\sigma(n_3)}^{(3)}\right)^{|n_3|}\left(\tilde{\boldsymbol{A}}_{\sigma(n_2)}^{(2)}\right)^{|n_2-1|}\mathcal{A}^{(1)}.$$

Proof. This assertion may be proved analogously to Theorem 1. $\qquad\square$

7.2. Arbitrage Discrepancies

To formulate further generalisations we need an additional notion. To each ensemble $\mathcal{R} = (r_{\$\euro}, r_{\$\pounds}, r_{\$\yen}, r_{\euro\pounds}, r_{\euro\yen}, r_{\pounds\yen})$ we attach an *arbitrage discrepancies ensemble,* using the relationships for balanced principal exchange rates given in (6) above

$$\mathcal{D}(\mathcal{R}) = (d_{\euro\pounds}(\mathcal{R}), d_{\euro\yen}(\mathcal{R}), d_{\pounds\yen}(\mathcal{R}))$$

as follows:

$$
\begin{aligned}
d_{\euro\pounds}(\mathcal{R}) &= \log r_{\euro\pounds} - \log r_{\$\pounds} + \log r_{\$\euro}, \\
d_{\euro\yen}(\mathcal{R}) &= \log r_{\euro\yen} - \log r_{\$\yen} + \log r_{\$\euro}, \\
d_{\pounds\yen}(\mathcal{R}) &= \log r_{\pounds\yen} - \log r_{\$\yen} + \log r_{\$\pounds}.
\end{aligned}
\tag{21}
$$

For instance

$$
\begin{aligned}
&\mathcal{D}(\bar{\mathcal{R}}_\alpha^1) = a(1,1,0), \quad \mathcal{D}(\bar{\mathcal{R}}_\alpha^2) = a(-1,0,1), \quad \mathcal{D}(\bar{\mathcal{R}}_\alpha^3) = a(0,-1,-1), \\
&\mathcal{D}(\bar{\mathcal{R}}_\alpha^4) = a(1,0,0), \quad \mathcal{D}(\bar{\mathcal{R}}_\alpha^5) = a(0,1,0), \qquad \mathcal{D}(\bar{\mathcal{R}}_\alpha^6) = a(0,0,1),
\end{aligned}
\tag{22}
$$

where $a = \log \alpha$.

Proposition 9. *The ensemble $\mathcal{R}$ is balanced, if and only if $\mathcal{D}(\mathcal{R}) = 0$.*

Proof. Follows from Proposition 1 and equations (21). $\qquad\square$

7.3. Case A

The case where two of the discrepancies (21) are equal to zero was implicitly considered in Section 7.1: see the second line in (22) and Proposition 8.

7.4. Case B

Consider now the case when one of the discrepancies in (21) is equal to zero, while two others are not. We will be particularly interested in the situation where two nonzero discrepancies are different. This situation may have emerged, for instance, as follows. Let us suppose that the underlying balanced rates (10) had been in operation up to a certain reference time moment 0. At this moment the Euro trader has decided to change two of three his rates, namely $r_{\euro\pounds}$ and $r_{\euro\yen}$, by different factors α and β. Then at this moment the two discrepancies would acquire different non-zero values, while the third discrepancy remains equal to zero.

Suppose, for example that $d_{\pounds\yen} = 0$, while $d_{\euro\pounds}, d_{\euro\yen} \neq 0$. We introduce the ratio

$$q(\mathcal{R}) = \frac{d_{\euro\yen}(\mathcal{R})}{d_{\euro\pounds}(\mathcal{R})}.
\tag{23}$$

Theorem 2. *Let the number (23) be irrational. Then set $S^{bal}(\mathcal{R})$ is dense in the totality T^{bal} of all possible balanced ensembles.*

A proof of this assertion will be given later on.

Consider also the case where $q = q(\mathcal{R})$ is a rational number: $q = m/n$ with co-prime integers m, n (including the possibilities $m = 1$ or $n = 1$). Denote also

$$\alpha = \exp(d_{\euro\yen}/n).$$

The following assertion is a straightforward analog of Proposition 2.

Proposition 10. *The inclusions* $S(\mathcal{R}) \subset T_\alpha(\mathcal{R})$ *and* $S^{bal}(\mathcal{R}) \subset T_\alpha^{bal}(\mathcal{R})$ *hold.*

The following is an analog of Theorem 1:

Proposition 11. *The equality* $S^{bal}(\mathcal{R}) = T^{bal}(\mathcal{R})$ *holds.*

A proof of this assertion will be given later on.

Note that the expressions like (16) are not valid in general. Similar expressions may be established, however, for the cases $m = 1$ or $n = 1$. Note also that the case when the discrepancy triplet is of one the forms $(a, a, 0)$ or $(a, 0, -a)$ or $(0, a, a)$, $a \neq 0$, was implicitly considered in Section 7.1: see the first line in (22) and Proposition 8.

7.5. Case C

Consider the case where all three arbitrage discrepancies (21) are not equal to zero.

Corollary 2. *Let at least one of the ratios*

$$q_1(\mathcal{R}) = \frac{d_{\text{€¥}}(\mathcal{R})}{d_{\text{€£}}(\mathcal{R})}, \quad q_2(\mathcal{R}) = \frac{d_{\text{£¥}}(\mathcal{R})}{d_{\text{€£}}(\mathcal{R})} \tag{24}$$

be irrational. Then the set $S^{bal}(\mathcal{R})$ *is dense in the totality* T^{bal} *of all possible balanced ensembles.*

Suppose now that both ratios (24) are rational:

$$q_1(\mathcal{R}) = \frac{m_1}{n_1}, \quad q_2(\mathcal{R}) = \frac{m_2}{n_2}.$$

Denote by $\text{lcm}(n_1, n_2)$ the least common multiple of the corresponding denominators. Denote

$$\alpha(\mathcal{R}) = \exp\left(\frac{d_{\text{€£}}(\mathcal{R})}{\text{lcm}(n_1, n_2)}\right).$$

Proposition 12. *The relationships* $S(\mathcal{R}) \subset T_\alpha(\mathcal{R})$ *and* $S^{bal}(\mathcal{R}) \subset T_\alpha^{bal}(\mathcal{R})$ *hold.*

Corollary 3. *Let*

$$\text{lcm}(n_1, n_2) = n_1 \cdot n_2. \tag{25}$$

Then $S^{bal}(\mathcal{R}) = T_\alpha^{bal}(\mathcal{R})$.

Proof. This assertion as well as formulated below Corollary 4 follows from Proposition 11 together with Lemma 17. □

Consider finally the case when the ratios $q_1(\mathcal{R})$ and $q_2(\mathcal{R})$ are rational, but (25) does not hold. In this case we introduce the number γ such that $d_i = k_i \gamma$ where the numbers k_i are integers and their greatest common divisor, $\gcd(k_1, k_2, k_3)$, is equal to 1. Consider also the following six numbers:

$$
\begin{aligned}
a_1 &= \gcd(k_1, k_2), & a_2 &= \gcd(k_1, k_3), \\
a_3 &= \gcd(k_2, k_3), & a_4 &= \gcd(k_1, k_2 - k_3), \\
a_5 &= \gcd(k_2, k_1 + k_3), & a_6 &= \gcd(k_3, k_1 - k_2).
\end{aligned} \tag{26}
$$

Introduce also the numbers $\alpha_i = \exp a_i$, $i = 1, \ldots, 6$.

14

Corollary 4. *The equation $S^{bal}(\mathcal{R}) = \cup_{i=1}^{6} T_{\alpha_i}^{bal}(\mathcal{R})$ holds.*

Note that all six numbers in (26) may indeed be greater than one. For instance, consider: $k_1 = 595$, $k_2 = 1683$, $k_3 = 308$. By inspection, $\gcd(k_1, k_2, k_3) = 1$, and

$$a_1 = \gcd(k_1, k_2) = 17, \qquad a_2 = \gcd(k_1, k_3) = 7,$$
$$a_3 = \gcd(k_2, k_3) = 11, \qquad a_4 = \gcd(k_1 - k_2, k_3) = 4,$$
$$a_5 = \gcd(k_1 + k_3, k_2) = 3, \quad a_6 = \gcd(k_1, k_2 - k_3) = 5.$$

8. Proofs

From this point onward we discuss the proofs of the theorems formulated above. This part of the paper is organised as follows. In Section 8.1 we introduce, as a useful auxiliary tool, stronger arbitrage procedures. Using strong arbitrages, we "linearise the problem", reducing it to investigation of all possible products of 12 explicitly written 6×6-matrices. Afterwards, in Section 8.2 we separate a family of 12 3×3-matrices $G^{(i)}$ such that the products of these matrices completely describe the dynamics of the discrepancy triplets. The properties of such products appear to be of key importance, and these are investigated in Section 8.3. The results are applied in Section 8.4. Sections 8.5 and 8.6 are dedicated to finalising the proof of Theorem 1. Finally, in Sections 8.7–8.9 we provide proofs for Theorem 2 and Proposition 11.

8.1. Strong Arbitrages

We use, as an auxiliary tool, stronger arbitrage procedures. Let us begin with an example. Consider the currencies triplet ($\$€£$). For a given $\mathcal{R}$ we define the strong arbitrage $\hat{\mathcal{A}}_{\$€£}\mathcal{R}$ as $\mathcal{A}_{\$€£}$ if the inequality (7) holds, and as $\mathcal{A}_{€\$£}$, otherwise. Note that in both cases the result in terms of principal exchange rates is the same: the rate $r_{\$€}$ is changed to $_{\$€}^{new} = \frac{r_{\$¥}}{r_{€£}}$.

The strong arbitrage $\hat{\mathcal{A}}_{\$€¥}$ is the second entry in Table 2 of the possible 12 strong arbitrages. The meaning of a strong arbitrage is simple. This is an arbitrage balancing a sub-FX market such as $\$€¥$ by changing the exchange rate for a pair such as *Dollar $\leftrightarrows$ Euro*. We will use, where convenient, the notation $\hat{\mathcal{A}}^{(n)}$ for the arbitrage number n from this table.

Proposition 13. *For any arbitrage chain* (8), *and any initial exchange rates $\mathcal{R}$, there exists a chain $\hat{\boldsymbol{A}} = \hat{\mathcal{A}}_1 \cdots \hat{\mathcal{A}}_n$ of strong arbitrages such that $\mathcal{R}\hat{\boldsymbol{A}} = \mathcal{R}\boldsymbol{A}$. Conversely, for any chain $\hat{\boldsymbol{A}} = \hat{\mathcal{A}}_1 \cdots \hat{\mathcal{A}}_n$ of strong arbitrages, and any initial exchange rates $\mathcal{R}$, there exists a chain of arbitrages such that $\mathcal{R}\hat{\boldsymbol{A}} = \mathcal{R}\boldsymbol{A}$.*

This proposition reduces investigation of the questions from the previous section to investigation of analogous questions related to chains of strong arbitrages.

Now we relate each strong arbitrage to a 6×6 matrix $B(\mathcal{A})$ as follows:

$$B_{\$€£} = B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{\$€¥} = B^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

Number	Strong arbitrage	Action	Numbers of arbitrages
1	$\hat{\mathcal{A}}_{\$€£}$	$r_{\$€}^{new} = r_{\$£} \cdot r_{€£}^{-1}$	1, 7
2	$\hat{\mathcal{A}}_{\$€¥}$	$r_{\$€}^{new} = r_{\$¥} \cdot r_{€¥}^{-1}$	2, 8
3	$\hat{\mathcal{A}}_{\$£€}$	$r_{\$£}^{new} = r_{\$€} \cdot r_{€£}$	3, 13
4	$\hat{\mathcal{A}}_{\$£¥}$	$r_{\$£}^{new} = r_{\$¥} \cdot r_{£¥}^{-1}$	4, 14
5	$\hat{\mathcal{A}}_{\$¥€}$	$r_{\$¥}^{new} = r_{\$,€} \cdot r_{€¥}$	5, 19
6	$\hat{\mathcal{A}}_{\$¥£}$	$r_{\$¥}^{new} = r_{\$£} \cdot r_{£¥}$	6, 20
7	$\hat{\mathcal{A}}_{€£\$}$	$r_{€£}^{new} = r_{\$£} \cdot r_{\$€}^{-1}$	9, 15
8	$\hat{\mathcal{A}}_{€£¥}$	$r_{€£}^{new} = r_{€¥} \cdot r_{£¥}^{-1}$	10, 16
9	$\hat{\mathcal{A}}_{€¥\$}$	$r_{€¥}^{new} = r_{\$¥} \cdot r_{\$€}^{-1}$	11, 21
10	$\hat{\mathcal{A}}_{€¥£}$	$r_{€¥}^{new} = r_{€£} \cdot r_{£¥}$	12, 22
11	$\hat{\mathcal{A}}_{£¥\$}$	$r_{£¥}^{new} = r_{\$¥} \cdot r_{\$£}^{-1}$	17, 23
12	$\hat{\mathcal{A}}_{£¥€}$	$r_{£¥}^{new} = r_{€¥} \cdot r_{€£}^{-1}$	18, 24

$$
B_{\$£€} = B^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{\$£¥} = B^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},
$$

$$
B_{\$¥€} = B^{(5)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{\$¥£} = B^{(6)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
B_{€£\$} = B^{(7)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{€£¥} = B^{(8)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
B_{€¥\$} = B^{(9)} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad B_{€¥£} = B^{(10)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
B_{£¥\$} = B^{(11)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad B_{£¥€} = B^{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

For any ensemble $\mathcal{R} = (r_{\$€}, r_{\$£}, r_{\$¥}, r_{€£}, r_{€¥}, r_{£¥})$ we denote

$$\log \mathcal{R} = (\log r_{\$€},\ \log r_{\$£},\ \log r_{\$¥},\ \log r_{€£},\ \log r_{€¥},\ \log r_{£¥}).$$

Proposition 14. *The equation* $\log(\mathcal{R}\hat{\mathcal{A}}^{(i)}) = (\log \mathcal{R})B^{(i)}$ *holds for* $i = 1, \ldots, 12$.

Proof. Follows from definitions. $\qquad\square$

8.2. A Special Coordinate System

In the six-dimensional real coordinate space $\mathbb{R}^6$ we introduce the vectors

$$\boldsymbol{v}_1 = (1, -1, 0, 1, 0, 0),\ \boldsymbol{v}_2 = (1, 0, -1, 0, 1, 0),\ \boldsymbol{v}_3 = (0, 1, -1, 0, 0, 1).$$

By definition for any ensemble $\mathcal{R}$

$$\langle \boldsymbol{v}_1, \log \mathcal{R} \rangle = d_{\text{€£}}(\mathcal{R}), \ \ \langle \boldsymbol{v}_2, \log \mathcal{R} \rangle = d_{\text{€¥}}(\mathcal{R}), \ \ \langle \boldsymbol{v}_3, \log \mathcal{R} \rangle = d_{\text{£¥}}(\mathcal{R}),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^6$.

Propositions 1 and 14 together imply

Corollary 5. *The three-dimensional subspace* $\langle \boldsymbol{v}_1, \boldsymbol{v} \rangle = \langle \boldsymbol{v}_2, \boldsymbol{v} \rangle = \langle \boldsymbol{v}_3, \boldsymbol{v} \rangle = 0$ *is invariant with respect to each linear operator* $\boldsymbol{v} \to \boldsymbol{v}B^{(i)}$, $i = 1, \ldots, 12$.

We introduce in $\mathbb{R}^6$ the new basis

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\};$$

here $\boldsymbol{e}_1 = (1,0,0,0,0,0)$, $\boldsymbol{e}_2 = (0,1,0,0,0,0)$, $\boldsymbol{v}_3 = (0,0,1,0,0,0)$. By the last corollary in this basis the matrices of the linear operators $\boldsymbol{v} \to \boldsymbol{v}B^{(i)}$ have the block-triangular form:

$$D^{(i)} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ H^{(i)} & G^{(i)} \end{pmatrix}.$$

Here

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $G^{(i)}, H^{(i)}$ are some 3×3-matrices.

Denote

$$\boldsymbol{v}(\mathcal{R}) = (\log r_{\text{\$€}}, \ \log r_{\text{\$£}}, \ \log r_{\text{\$¥}}, \ d_{\text{€£}}(\mathcal{R}), \ d_{\text{€¥}}(\mathcal{R}), \ d_{\text{£¥}}(\mathcal{R})).$$

Proposition 15. *The equality* $\boldsymbol{v}(\mathcal{R}\hat{A}^{(i)}) = \boldsymbol{v}(\mathcal{R})D^{(i)}$ *holds for* $i = 1, \ldots, 12$.

Proof. Follows from Lemma 1 and Proposition 14. $\qquad\square$

The matrices $D^{(i)}$ may be written explicitly as

$$QB^{(i)}Q^{-1}, \tag{27}$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

Lemma 1. *The following equations are valid:*

$$G^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad G^{(5)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$G^{(7)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{(8)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad G^{(9)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^{(10)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad G^{(11)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G^{(12)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$H^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H^{(4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad H^{(5)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$H^{(i)} = \mathbf{0}, \quad i = 7, \dots, 12.$$

Proof. Follows by inspection from (27), (28). □

Proposition 16. *The discrepancy ensemble* $\mathcal{D}(\mathcal{R}\hat{\mathcal{A}})$ *depends only on* $\mathcal{D}(\mathcal{R})$ *and* $\hat{\mathcal{A}}$, *and may be written as follows:* $\mathcal{D}(\mathcal{R}\hat{\mathcal{A}}^{(i)}) = \mathcal{D}(\mathcal{R})G^{(i)}$. *Here* i *is the number of a strong arbitrage as listed in Table 2.*

Proof. Follows from Proposition 15. □

By the last proposition a discrepancy ensemble $\mathcal{D}(\mathcal{R}\hat{\boldsymbol{A}})$ related to an arbitrage chain $\hat{\boldsymbol{A}} = \hat{\mathcal{A}}_1 \cdots \hat{\mathcal{A}}_n$ may be written as

$$\mathcal{D}(\mathcal{R}\hat{\boldsymbol{A}}) = \mathcal{D}(\mathcal{R}) \prod_{i=1}^{n} G_i.$$

Therefore the set $\mathbb{G}$ of all possible products of the matrices $G^{(i)}$ is of interest.

8.3. Structure of the Set $\mathbb{G}$

The following assertion is the key observation of our paper:

Lemma 2. *The set* $\mathbb{G}$ *consists of 229 elements.*

Proof. By inspection □

Denote by $\hat{\mathbb{A}}$ the totality of all finite chains of strong arbitrages.

Corollary 6. *For a given* $\mathcal{R}$ *the set* $\mathbb{D}(\mathcal{R}) = \{\mathcal{D}(\mathcal{R}\hat{\boldsymbol{A}}) : \boldsymbol{A} \in \mathbb{A}\}$ *consists of less than 230 elements.*

Let us discuss briefly the structure of the set $\mathbb{G}$. A subset $\boldsymbol{G}$ of $\mathbb{G}$ is called a connected component, if for any $G_1, G_2 \in \boldsymbol{G}$ there exists $G \in \mathbb{G}$ satisfying $G_2 = G_1 G$. By the definition different connected components do not intersect.

Lemma 3. *The set* $\mathbb{G}$ *is partitioned into 14 connected components* $U_1, \dots U_{14}$. *Each of the first six connected components includes 24 matrices of range 2; each of the connected components* $U_7, \dots U_{13}$ *includes 12 matrices of range one; the last component contains a single zero matrix.*

18

The sets $U_1, \ldots, U_6$ may be characterised by the following inclusions:

$$G^{(2i-1)}, \ G^{(2i)} \in U_i, \quad i = 1, \ldots, 6.$$

To identify the connected components $U_7, \ldots U_{13}$ we list below the smallest lexicographical matrices from these components

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in U_7, \qquad \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in U_8, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} \in U_9,$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in U_{10}, \qquad \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} \in U_{11}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in U_{12},$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \in U_{13}.$$

One can move from one connected component U_i to another component U_j applying a matrix $G^{(i)}$, $i = 1, \ldots, 12$. Let us describe the set of possible transitions. We will use the notation $U_i \succ U_j$ if such a transition is possible.

Lemma 4. *The following relationships hold:*

$$U_1 \succ U_9, U_{10}, U_{13}, \quad U_2 \succ U_8, U_{11}, U_{13}, \quad U_3 \succ U_7, U_{12}, U_{13},$$
$$U_4 \succ U_8, U_9, U_{12}, \quad U_5 \succ U_7, U_9, U_{11}, \quad U_6 \succ U_7, U_8, U_{10}.$$

Also $U_i \succ U_{14}$, $i = 1, \ldots, 13$.

Proof. By inspection. $\square$

Lemma 5. *For any $G \in \mathbb{G}$ either G or G^2 or G^3 is a projector.*

Proof. By inspection. $\square$

8.4. Discrepancy Dynamics

The structure of the set $\mathbb{G}$ explained above induces structuring of the set of discrepancies, which we discuss below. We say that a set $\boldsymbol{D}$ of discrepancies is a connected component if for any $\mathcal{D}_1, \mathcal{D}_2 \in \boldsymbol{D}$ there exists an arbitrage chain $\boldsymbol{A}$ satisfying $\mathcal{D}_1 \boldsymbol{A} = \mathcal{D}_2$. For a given reals a, b we denote by $\boldsymbol{D}(a, b)$ the set of different triplets from the collection

$$
\begin{aligned}
&\mathcal{D}_1(a, b) = (a, b, -a + b), && \mathcal{D}_2(a, b) = (-a + b, b, a), \\
&\mathcal{D}_3(a, b) = (a, a - b, -b), && \mathcal{D}_4(a, b) = (-a + b, -a, -b), \\
&\mathcal{D}_5(a, b) = (-b, a - b, a), && \mathcal{D}_6(a, b) = (-b, -a, -a + b), \\
&\mathcal{D}_7(a, b) = (0, b, -a + b), && \mathcal{D}_8(a, b) = (a, 0, -a + b), \\
&\mathcal{D}_9(a, b) = (a, b, 0), && \mathcal{D}_{10}(a, b) = (0, b, a), \\
&\mathcal{D}_{11}(a, b) = (-a + b, 0, a), && \mathcal{D}_{12}(a, b) = (-a + b, b, 0), \\
&\mathcal{D}_{13}(a, b) = (0, -a, -b), && \mathcal{D}_{14}(a, b) = (-a + b, 0, -b), \\
&\mathcal{D}_{15}(a, b) = (-a + b, -a, 0), && \mathcal{D}_{16}(a, b) = (0, a - b, -b), \\
&\mathcal{D}_{17}(a, b) = (a, 0, -b), && \mathcal{D}_{18}(a, b) = (a, a - b, 0), \\
&\mathcal{D}_{19}(a, b) = (0, a - b, a), && \mathcal{D}_{20}(a, b) = (-b, 0, a), \\
&\mathcal{D}_{21}(a, b) = (-b, a - b, 0), && \mathcal{D}_{22}(a, b) = (0, -a, -a + b), \\
&\mathcal{D}_{23}(a, b) = (-b, 0, -a + b), && \mathcal{D}_{24}(a, b) = (-b, -a, 0).
\end{aligned}
\tag{29}
$$

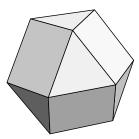**Lemma 6.** *Each set $\boldsymbol{D}(a,b)$ is a connected component, and each connected component coincides with a certain set $\boldsymbol{D}(a,b)$.*

Proof. This statement may be proved by inspection. □

Let us discuss in brief the structure of the sets $\boldsymbol{D}(a,b)$ for different values a, b. Clearly, $\boldsymbol{D}(0,0)$ consists of the single zero triplet $\mathcal{D}_0 = (0,0,0)$. The connected components $\boldsymbol{D}(\pm a, 0)$, $\boldsymbol{D}(0, \pm a)$, $\boldsymbol{D}(a,a)$, $\boldsymbol{D}(-a,-a)$ coincide and include the following 12 elements:

$$
\begin{aligned}
\mathcal{D}_1(a) &= a(\ \ 0,\ \ 0,\ \ 1), & \mathcal{D}_2(a) &= a(-1,\ \ 0,\ \ 1), \\
\mathcal{D}_3(a) &= a(-1,\ \ 0,\ \ 0), & \mathcal{D}_4(a) &= a(-1,-1,\ \ 0), \\
\mathcal{D}_5(a) &= a(\ \ 0,-1,\ \ 0), & \mathcal{D}_6(a) &= a(\ \ 0,-1,-1), \\
\mathcal{D}_7(a) &= a(\ \ 0,\ \ 0,-1), & \mathcal{D}_8(a) &= a(\ \ 1,\ \ 0,-1), \\
\mathcal{D}_9(a) &= a(\ \ 1,\ \ 0,\ \ 0), & \mathcal{D}_{10}(a) &= a(\ \ 1,\ \ 1,\ \ 0), \\
\mathcal{D}_{11}(a) &= a(\ \ 0,\ \ 1,\ \ 0), & \mathcal{D}_{12}(a) &= a(\ \ 0,\ \ 1,\ \ 1).
\end{aligned}
\tag{30}
$$

We use notation $\boldsymbol{D}(a)$ for this set. Geometrically the set $\boldsymbol{D}(a)$ represents vertices of a partly distorted truncated cuboctahedron, or triangular orthobicupola, shown in Fig. 1. The structure of this component will be explained in more detail in Section 8.6. The set $\boldsymbol{D}(a,-a)$, $\boldsymbol{D}(a, 2a)$, $\boldsymbol{D}(a, a/2)$, also consists of 12 elements. Geometrically these sets $\boldsymbol{D}(a)$ represent vertices of a distorted truncated tetrahedron, shown in Fig. 2. Otherwise, a set $\boldsymbol{D}(a,b)$, consists of 24 elements, and represents vertices of a distorted truncated octahedron, shown in Fig. 3. The structure of this component will be explained in more detail in Section 8.7.
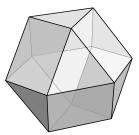

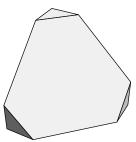Figure 1: Left: the form of a polyhedron with vertices $\boldsymbol{D}(a)$, $a \neq 0$; Right: the same polyhedron transparent.

We formulate also a corollary of Proposition 4. For a set $\boldsymbol{D}$ of discrepancies we denote by $G(\boldsymbol{D})$ the collection of elements of the form $\mathcal{D}G^{(i)}$, $\mathcal{D} \in \boldsymbol{D}$, $i = 1, \ldots, 12$.

Corollary 7. *The equality*

$$
G(\boldsymbol{D}(a,b)) = \boldsymbol{D}(a,b)) \bigcup \boldsymbol{D}(a) \bigcup \boldsymbol{D}(b) \bigcup \boldsymbol{D}(a - b),
$$

holds for $a \neq b$. Also $G(\boldsymbol{D}(a)) = \boldsymbol{D}(a) \bigcup (0,0,0)$.

Some discrepancy triplets do not belong to any connected component; however any element of the form $\mathcal{D}G^{(i)}$ must belong to a connected component. More precisely:
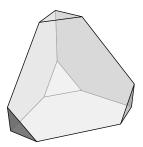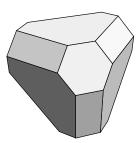
Figure 2: Left: the form of polyhedrons with vertices $\boldsymbol{D}(a,-a)$, $\boldsymbol{D}(a,2a)$, or $\boldsymbol{D}(a,a/2)$, $a \neq 0$; Right: the same polyhedron transparent.
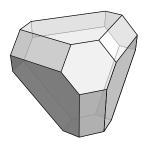

Figure 3: Left: a typical form of a generic polyhedron with vertices $\boldsymbol{D}(a,b)$; Right: the same polyhedron transparent.

Proposition 17. *The following inclusions hold:*

$$(a,b,c)G^{(1,2)} \in \boldsymbol{D}(c,-a+b), \qquad (a,b,c)G^{(3,4)} \in \boldsymbol{D}(a-c,b),$$
$$(a,b,c)G^{(5,6)} \in \boldsymbol{D}(-b+c,a), \qquad (a,b,c)G^{(7,8)} \in \boldsymbol{D}(c,b),$$
$$(a,b,c)G^{(9,10)} \in \boldsymbol{D}(a,-c), \qquad (a,b,c)G^{(11,12)} \in \boldsymbol{D}(a,b).$$

Proof. This assertion may be proved by inspection. $\qquad\square$

8.5. Incremental Dynamics

For a given sextuple $\mathcal{R}$ we denote by $\mathcal{R}'$ the triplet of the first three components of $\mathcal{R}$: $\mathcal{R}' = (r_{\$€}, r_{\$£}, r_{\$¥})$. Denote further $\mathcal{I}(\mathcal{R}, \hat{\mathcal{A}}) = \log(\mathcal{R}\hat{\mathcal{A}})' - \log \mathcal{R}'$, where $\hat{\mathcal{A}}$ is a strong arbitrage.

Proposition 18. $\mathfrak{I}(\mathcal{R}, \hat{A})$ *depends only on* $\hat{A}$ *and* $\mathcal{D}(\mathcal{R})$ *and may be described as follows:*

$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(1)}) = d(\mathcal{R})H^{(1)} = -d_{\text{€£}}(\mathcal{R})\,(1, 0, 0)\,,$$
$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(2)}) = d(\mathcal{R})H^{(2)} = -d_{\text{€¥}}(\mathcal{R})\,(1, 0, 0)\,,$$
$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(3)}) = d(\mathcal{R})H^{(3)} = \quad d_{\text{€£}}(\mathcal{R})\,(0, 1, 0)\,,$$
$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(4)}) = d(\mathcal{R})H^{(4)} = -d_{\text{£¥}}(\mathcal{R})\,(0, 1, 0)\,,$$
$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(5)}) = d(\mathcal{R})H^{(5)} = \quad d_{\text{€¥}}(\mathcal{R})\,(0, 0, 1)\,,$$
$$\mathfrak{I}(\mathcal{R}, \hat{A}^{(6)}) = d(\mathcal{R})H^{(6)} = \quad d_{\text{£¥}}(\mathcal{R})\,(0, 0, 1)\,.$$

Also the equalities $\mathfrak{I}(\mathcal{R}, \hat{A}^{(i)}) = d(\mathcal{R})H^{(i)} = (0, 0, 0)$ *hold for* $i = 7, 8, 9, 10, 11, 12$.

Proof. Follows from Corollary 15. $\qquad\square$

8.6. Proof of Theorem 1

This proceeds by graphing the detailed dynamics of the arbitrage discrepancies. In this section we use the shorthand notation $\mathcal{D}_i$ instead of $\mathcal{D}_i(a)$.

Lemma 7. *For any initial exchange rate ensemble belonging to the list* (20)*, and for any arbitrage chain, the corresponding sequence of discrepancies includes only elements from the union* $\mathcal{D}_0 \bigcup \boldsymbol{D}(a)$*,* $a = \log \alpha$*, see* (30)*. The possible transition paths, arising from the strong arbitrages listed in Table 1, are given in Table 3.*

Figure 4 plots the corresponding graph. Figure 5 plots a similar graph, where the numbers of the arbitrages from Table 1 are included, instead of the numbers of strong arbitrages.

Proof. By inspection follows from Proposition 16. $\qquad\square$

Ignoring the zero vertex $\mathcal{D}_0$, the edges that lead to this vertex and directions of the edges, another, polyhedral, representation of the graph plotted in Fig. 4 is given in Fig. 6. The corresponding polyhedron is a distorted triangular orthobicupola, shown in Fig. 1. The incidence matrix I of the graph plotted in Fig. 6 is as follows:

$$I = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix} .$$

Now let us deal with the coupled discrepancies and the incremental dynamics.

Corollary 8. *For any arbitrage chain the corresponding sequence of increments includes only the zero triplet* $\mathfrak{I}_0 = (0, 0, 0)$ *or one of the following six triplets:*

$$\mathfrak{I}_1 = a(1, \;\; 0, \;\; 0), \quad \mathfrak{I}_2 = a(-1, \;\; 0, \;\; 0), \quad \mathfrak{I}_3 = a(0, \;\; 1, \;\; 0),$$
$$\mathfrak{I}_4 = a(0, -1, \;\; 0), \quad \mathfrak{I}_5 = a(\;\; 0, \;\; 0, \;\; 1), \quad \mathfrak{I}_6 = a(0, \;\; 0, -1).$$

Table 3: Transition, caused by strong arbitrages from Table 1

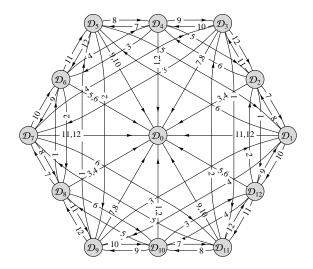|	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(1)}$	$\mathcal{D}_1$	$\mathcal{D}_{12}$	$\mathcal{D}_{11}$	$\mathcal{D}_0$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_6$	$\mathcal{D}_5$	$\mathcal{D}_0$	$\mathcal{D}_{11}$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(2)}$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_0$	$\mathcal{D}_9$	$\mathcal{D}_8$	$\mathcal{D}_7$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_0$	$\mathcal{D}_3$	$\mathcal{D}_2$
$\hat{\mathcal{A}}^{(3)}$	$\mathcal{D}_1$	$\mathcal{D}_0$	$\mathcal{D}_7$	$\mathcal{D}_6$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_0$	$\mathcal{D}_1$	$\mathcal{D}_{12}$	$\mathcal{D}_{11}$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(4)}$	$\mathcal{D}_1$	$\mathcal{D}_0$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_4$	$\mathcal{D}_3$	$\mathcal{D}_0$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_{10}$
$\hat{\mathcal{A}}^{(5)}$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_2$	$\mathcal{D}_1$	$\mathcal{D}_0$	$\mathcal{D}_7$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_8$	$\mathcal{D}_7$	$\mathcal{D}_0$
$\hat{\mathcal{A}}^{(6)}$	$\mathcal{D}_5$	$\mathcal{D}_4$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_0$	$\mathcal{D}_{11}$	$\mathcal{D}_{10}$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_0$
$\hat{\mathcal{A}}^{(7)}$	$\mathcal{D}_1$	$\mathcal{D}_1$	$\mathcal{D}_0$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_7$	$\mathcal{D}_0$	$\mathcal{D}_{11}$	$\mathcal{D}_{11}$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(8)}$	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_0$	$\mathcal{D}_4$	$\mathcal{D}_4$	$\mathcal{D}_6$	$\mathcal{D}_8$	$\mathcal{D}_8$	$\mathcal{D}_0$	$\mathcal{D}_{10}$	$\mathcal{D}_{10}$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(9)}$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_3$	$\mathcal{D}_0$	$\mathcal{D}_7$	$\mathcal{D}_7$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_9$	$\mathcal{D}_0$	$\mathcal{D}_1$
$\hat{\mathcal{A}}^{(10)}$	$\mathcal{D}_{12}$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_3$	$\mathcal{D}_0$	$\mathcal{D}_6$	$\mathcal{D}_6$	$\mathcal{D}_8$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_0$	$\mathcal{D}_{12}$
$\hat{\mathcal{A}}^{(11)}$	$\mathcal{D}_0$	$\mathcal{D}_3$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_0$	$\mathcal{D}_9$	$\mathcal{D}_9$	$\mathcal{D}_{10}$	$\mathcal{D}_{11}$	$\mathcal{D}_{10}$
$\hat{\mathcal{A}}^{(12)}$	$\mathcal{D}_0$	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_4$	$\mathcal{D}_6$	$\mathcal{D}_6$	$\mathcal{D}_0$	$\mathcal{D}_8$	$\mathcal{D}_8$	$\mathcal{D}_{10}$	$\mathcal{D}_{12}$	$\mathcal{D}_{12}$

Figure 4: Graph of the transitions caused by the strong arbitrages.

The dynamics of the increments $\mathcal{I}$ is conveniently visualised in Fig. 7.

The correctness of this description of the dynamics of the increments follows immediately from Corollary 7 and Proposition 18. The legitimacy of the algorithms relevant to Theorem 1 and, therefore, proof of Theorem 1 and Proposition 8 follows from Figs. 5 and 7.

We note also that the 24-periodic chain of arbitrage from Proposition 6 was also found looking at Fig. 5 and 7. The corresponding route is quite natural from this perspective,
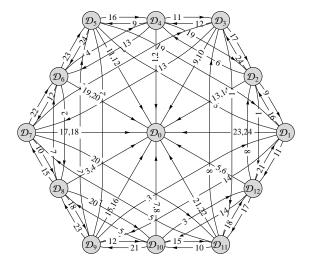
Figure 5: The previous graph with the arbitrage numbers, instead of the strong arbitrage numbers.
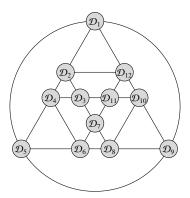

Figure 6: The polyhedral representation of the principal graph.

and is given by

$$
\begin{aligned}
&\mathcal{D}_{10} \to \mathcal{D}_{11} \to \mathcal{D}_{10} \to \mathcal{D}_{12} \to \mathcal{D}_{1} \to \mathcal{D}_{12} \to \mathcal{D}_{2} \to \ \mathcal{D}_{3} \to \\
&\mathcal{D}_{2} \to \ \mathcal{D}_{4} \to \ \mathcal{D}_{5} \to \ \mathcal{D}_{4} \to \mathcal{D}_{6} \to \ \mathcal{D}_{7} \to \mathcal{D}_{6} \to \ \mathcal{D}_{8} \to \\
&\mathcal{D}_{9} \to \ \mathcal{D}_{8} \to \mathcal{D}_{10} \to \ \mathcal{D}_{8} \to \mathcal{D}_{6} \to \ \mathcal{D}_{4} \to \mathcal{D}_{2} \to \mathcal{D}_{12} \to \mathcal{D}_{10}.
\end{aligned}
\tag{31}
$$

8.7. Commuters, Terminals and Knots

Now we move to a proof of Theorem 2 and Proposition 11. The case $d_{\in\pounds} = d_{\in\yen}$ has been considered in Section 7.1. Thus we can assume that $d_{\in\pounds} \neq d_{\in\yen}$.

The focus is again on the dynamics of the exchange rate discrepancies. The set of all discrepancies that may be achievable from $\mathcal{D} = (a, b, 0)$ contains altogether 61 different elements, see Corollary 7. The corresponding connected component $\boldsymbol{D}(a, b)$,
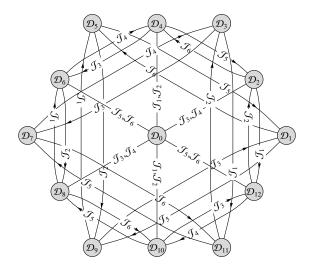
Figure 7: The increment dynamics graph

which contains $\mathcal{D} = (a, b, 0)$, see (29), contains 24 elements listed in (29). To describe the detailed structure of this set we will introduce a new notation. The set $\boldsymbol{D}(a, b)$, contains six elements that have all three components that are non-zero, and we re-denote these elements by

$$
\begin{aligned}
C_1 &= (a, b, -a + b), & C_2 &= (-a + b, b, a), & C_3 &= (a, a - b, -b), \\
C_4 &= (-a + b, -a, -b), & C_5 &= (-b, a - b, a), & C_6 &= (-b, -a, -a + b).
\end{aligned}
$$

We call these ensembles *commuters* by way of analogy with passenger travel.

We call an element with two non-zero components a *terminal*, if $d_1 \neq \pm d_2$. There are altogether 18 terminals in $\boldsymbol{D}(a, b)$. To each commuter C_i, $i = 1, \ldots, 6$, we relate three *terminals* T_i^j, $j = 1, 2, 3$, as follows:

$$
\begin{aligned}
T_1^1 &= (0, b, -a + b), & T_1^2 &= (a, 0, -a + b), & T_1^3 &= (a, b, 0); \\
T_2^1 &= (0, b, a), & T_2^2 &= (-a + b, 0, a), & T_2^3 &= (-a + b, b, 0); \\
T_3^1 &= (0, -a, -b), & T_3^2 &= (-a + b, 0, -b), & T_3^3 &= (-a + b, -a, 0); \\
T_4^1 &= (0, a - b, -b), & T_4^2 &= (a, 0, -b), & T_4^3 &= (a, a - b, 0); \\
T_5^1 &= (0, a - b, a), & T_5^2 &= (-b, 0, a), & T_5^3 &= (-b, a - b, 0); \\
T_6^1 &= (0, -a, -a + b), & T_6^2 &= (-b, 0, -a + b), & T_6^3 &= (-b, -a, 0).
\end{aligned}
$$

Lemma 8. *The equalities*

$$
\begin{aligned}
C_i G^{(7)} &= T_i^1, & C_i H^{(7)} &= (0, 0, 0), \\
C_i G^{(9)} &= T_i^2, & C_i H^{(9)} &= (0, 0, 0), \\
C_i G^{(11)} &= T_i^3, & C_i H^{(11)} &= (0, 0, 0)
\end{aligned}
$$

hold for $i = 1, \ldots, 6$. Also the following equalities hold: $T_i^j G^{(k)} = C_i$, for $i = 1, \ldots, 6$, $j = 1, 2, 3$, $k = 8, 10, 12$.

We group the commuters and terminals in six knots, $K_1, \ldots, K_6$ as follows:

$$K_i = \left\{ C_i, T_i^1, T_i^2, T_i^3 \right\}, \quad i = 1, \ldots, 6.$$

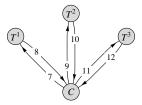Figure 8 illustrates behaviour at a knot.

Figure 8: The "commuter–terminals" graph of a knot

8.8. Travel Between Knots

Departing from a particular terminal, and applying some arbitrages with numbers $k = 7, \ldots, 12$, one can travel to another terminal belonging to a different knot, simultaneously "loading some cargo" upon the corresponded triplet $\mathcal{R}'$. Details are given in the following proposition.

Proposition 19. *The following groups of equalities hold:*

$$\begin{cases} T_1^1 G^{(3)} = T_2^1, & T_1^1 H^{(3)} = (0, a, 0), & T_1^1 G^{(5)} = T_4^2, & T_1^1 H^{(5)} = (0, 0, b); \\ T_1^2 G^{(1)} = T_6^1, & T_1^2 H^{(1)} = (-a, 0, 0), & T_1^2 G^{(6)} = T_3^3, & T_1^2 H^{(6)} = (0, 0, -a); \\ T_1^3 G^{(2)} = T_6^2, & T_1^3 H^{(2)} = (-b, 0, 0), & T_1^3 G^{(4)} = T_2^3, & T_1^3 H^{(4)} = (0, a - b, 0); \end{cases}$$

$$\begin{cases} T_2^1 G^{(2)} = T_5^2, & T_2^1 H^{(2)} = (-b, 0, 0), & T_2^1 G^{(4)} = T_1^3, & T_2^1 H^{(4)} = (0, -a, 0); \\ T_2^3 G^{(1)} = T_2^2, & T_2^3 H^{(1)} = (a - b, 0, 0), & T_2^3 G^{(6)} = T_3^3, & T_2^3 H^{(6)} = (0, 0, -a); \end{cases}$$

$$\begin{cases} T_3^1 G^{(2)} = T_4^2, & T_3^1 H^{(2)} = (a, 0, 0), & T_3^1 G^{(4)} = T_6^3, & T_3^2 H^{(4)} = (0, b, 0); \end{cases}$$

$$\begin{cases} T_4^1 G^{(2)} = T_3^2, & T_4^1 H^{(2)} = (-a + b, 0, 0), & T_4^1 G^{(4)} = T_5^3, & T_4^1 H^{(4)} = (0, b, 0); \\ T_4^3 G^{(1)} = T_3^1, & T_4^3 H^{(1)} = (-a, 0, 0), & T_4^3 G^{(6)} = T_1^3, & T_4^3 H^{(6)} = (0, 0, -b); \end{cases}$$

$$\begin{cases} T_5^1 G^{(2)} = T_2^2, & T_5^1 H^{(2)} = (-a + b, 0, 0), & T_5^1 G^{(4)} = T_4^3, & T_5^1 H^{(4)} = (0, -a, 0); \\ T_5^3 G^{(1)} = T_2^1, & T_5^3 H^{(1)} = (b, 0, 0), & T_5^3 G^{(6)} = T_6^3, & T_5^3 H^{(6)} = (0, 0, a); \end{cases}$$

$$\begin{cases} T_6^1 G^{(2)} = T_1^2, & T_6^1 H^{(2)} = (a, 0, 0), & T_6^1 G^{(4)} = T_3^3, & T_6^1 H^{(4)} = (0, a - b, 0). \end{cases}$$

We introduce the "travel between knots" directed graph Γ as follows. This graph has 6 vertices that correspond to the knots $K_1, \ldots, K_6$. A knot K_i is connected by an arrow with another knot K_j if one of terminals belonging to K^j figures in the rows belonging to the i-th subset of equalities from Proposition 19. For instance, the knot K_1 is connected with K_2, K_3, K_4, K_6. Moreover each arrow corresponds to the three dimensional "cargo vector(s)": these vectors are related in a natural way to the increment vectors in the equalities above. For instance, we attach the cargo-vectors $(0, a - b, 0)$ and $(0, a, 0)$ to the $K_1 \to K_2$ arrow. The incidence matrix of this graph is written below.

$$I(\Gamma) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
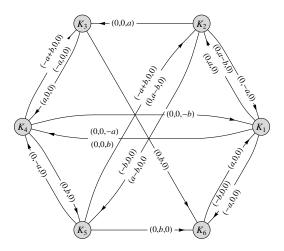

Figure 9: The "travel between knots" graph Γ

8.9. Finalising the proof of Theorem 2 and Proposition 11

If the single transition $K_i \to K_j$ is possible we use $W^{i \to j}$ for the corresponding cargo; we will use $W_1^{i \to j}, W_2^{i \to j}$ if two transitions are possible. In the latter case $W_1^{i \to j}$ refers to the upper vector indicated at graph Γ. For instance, $W_1^{1 \to 2} = (0, a - b, 0)$, $W_2^{1 \to 2} = (0, a, 0)$, $W^{2 \to 1} = (0, -a, 0)$, etc.

Lemma 9. *For any positive integers N_1, N_2, N_3 there exists a chain $\hat{A}$ of strong arbitrages such that $\Re \hat{A}$ has the form*

$$(r_{\$ \euro} + m_1 a - N_1 b, r_{\$ \pounds} + m_2 a + N_2 b, r_{\$ \yen} + m_3 a - N_3 b)$$

where m_1, m_2, m_3 are some positive integer numbers,

27

Proof. Since the moves from one terminal to another, within a particular knot, are always possible and do not change $\mathcal{R}'$ (see Lemma 8), any route allowed by the graph Γ can be performed, and any combination of corresponding cargo can be loaded. For the cycle $K_1 \to K_2 \to K_5 \to K_6 \to K_1$ we have

$$W_1^{1\to2} + W_1^{2\to5} + W^{5\to6} + W^{6\to1} = (a - b, a, 0).$$

For the cycle $K_1 \to K_2 \to K_3 \to K_6 \to K_1$ we have

$$W_2^{1\to2} + W^{2\to3} + W^{3\to6} + W^{6\to1} = (a, a + b, a).$$

For the cycle $K_1 \to K_2 \to K_3 \to K_4 \to K_1$ we have

$$W_2^{1\to2} + W^{2\to3} + W^{3\to4} + W^{4\to1} = (a, a, a - b).$$

$\square$

Corollary 9. *For any non-negative integers N_1, N_2, N_3 and M_1, M_2, M_3 there exists a chain $\hat{A}$ of strong arbitrages such that $\mathcal{R}\hat{A}$ has the form $(r_{\$\mathord{\in}} + M_1 a - N_1 b, r_{\$\pounds} + M_2 a + N_2 b, r_{\$\yen} + M_3 a - N_3 b)$.*

Proof. From the lemma above it follows that we can achieve the state

$$(r_{\$\mathord{\in}} + m_1 a - (N_1 - 1)b, r_{\$\pounds} + m_2 a + (N_2)b, r_{\$\yen} + m_3 a - (N_3 + 1)b, a, b, -a + b).$$

Then moving to the terminal T_1^3 and applying arbitrage $\hat{A}^{(6)}$ we arrive at

$$(r_{\$\mathord{\in}} + (m_1 - 1)a - (N_1)b, r_{\$\pounds} + m_2 a + (N_2)b, r_{\$\yen} + m_3 a - N_3 b, 0, a, 0).$$

However, from this state we can, by Proposition 1, adjust the numbers m_1, m_2, m_3 to the targets M_1, M_2, M_3. $\square$

Theorem 2 and Proposition 11 follow immediately from this last corollary.

9. Concluding Remarks

The key contribution of this paper is to ask what happens to arbitrage sequences when the number of goods or assets under consideration is four, rather than the two, or occasionally three, usually considered. The model is illustrated with regard to a foreign exchange market with four currencies and traders, so there are $C_4^2 = 6$ principal exchange rates. Despite abstracting from various complications – such as transaction costs, capital requirements and risk – that are often invoked to explain the limits to arbitrage, we find that the arbitrage operations conducted by the FX traders can generate periodicity or more complicated behaviour in the ensemble of exchange rates, rather than smooth convergence to a "balanced" ensemble where the law of one price holds.

We use the fiction of an Arbiter, who knows all the actual exchange rates and what a balanced ensemble would be, to bring out the information problem. FX traders tend to specialise in particular currencies, so the assumption that the FX traders are initially aware only of the exchange rates for their own "domestic" currencies is not entirely implausible. We show that the order in which the Arbiter reveals information to individual

traders regarding discrepancies in exchange rate ensembles makes a key difference to the arbitrage sequences that will be pursued. The sequences are periodic in nature, and show no clear signs of convergence on a balanced ensemble of exchange rates. The Arbiter might know the law of one price exchange rate ensemble, but the traders have little chance of stumbling onto such an ensemble by way of their arbitrage operations.

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