scottish institute for research in economics

SIRE DISCUSSION PAPER

SIRE-DP-2008-07

Can a Lucas model with habit generate realistic conditional volatility in exchange rate returns?

Jingyi Liu

University of Edinburgh

www.sire.ac.uk

Can a Lucas model with habit generate realistic conditional volatility in exchange rate returns?*

Jingyi Liu
Department of Economics, University of Edinburgh, UK
E-mail: J.Liu-12@sms.ed.ac.uk

February 12, 2008

Abstract

In this paper, we attempt to give a theoretical underpinning to the well established empirical stylized fact that asset returns in general and the spot FOREX returns in particular display predictable volatility characteristics. Adopting Moore and Roche's habit persistence version of Lucas model we find that both the innovation in the spot FOREX return and the FOREX return itself follow "ARCH" style processes. Using the impulse response functions (IRFs) we show that the baseline simulated FOREX series has "ARCH" properties in the quarterly frequency that match well the "ARCH" properties of the empirical monthly estimations in that when we scale the x-axis to synchronize the monthly and quarterly responses we find similar impulse responses to one unit shock in variance. The IRFs for the ARCH processes we estimate "look the same" with an approximately monotonic decreasing fashion. The Lucas two-country monetary model with habit can generate realistic conditional volatility in spot FOREX return.

Keywords: asset pricing, CCAPM, conditional volatility, GARCH models, foreign exchange, habit persistence

*The paper comes from one of the chapters of my PhD dissertation. I am grateful to Prof. Andy Snell and Dr. Angelica Gonzalez for their kind guidance, encouragement, and insightful comments.

1 Introduction

Time series plots of financial returns display an important feature that is usually called volatility clustering. Volatility clustering describes the general tendency for financial markets to have some periods of high volatility and other periods of low volatility. High volatility produces more dispersion in returns than low volatility, so that returns are more spread out when volatility is higher. A high volatility cluster will contain several large positive returns and several large negative returns, but there will be few, if any, large returns in a low volatility cluster.

In terms of the boom of volatility literatures published in this decade, although much is known about the structure of volatility persistence, little is known about its causes (LeBaron (2005)). People have tried to explain volatility persistence using empirical findings in finance including rate of information arrivals, fashion of price analysis, released economic data, sensitivity of traders to information etc. However, it seems elusive to answer "which economic model or behavior is consistent with ARCH[1] ".

Theoretical asset pricing models, as acknowledged, can explain some variation in volatility while volatility changes explain some stylized facts for asset returns e.g. risk premium. There are some papers that do mention persistent volatility using theoretical asset pricing models for example Campbell and Cochrane (1999) and Moore and Roche (2006). In Campbell and Cochrane (1999) a consumption-based asset pricing model with external habits is able to explain dynamic behavior of stock prices even persistent volatility in stock returns. The habit defined by Campbell and Cochrane (1999) as an AR(1) process in which the lagged level of consumption is the "shock" is a solution to the equity premium puzzle. Moore and Roche (2006) do a pioneering work using a flexible-price two-country monetary model of Lucas (1982) representative agent theory with habit persistence to solve many FOREX puzzles[2] simultaneously and to mimic the volatilities of real and nominate exchange rates, the forward premium, expected spot returns, and expected forward profits. In Moore and Roche (2006) the utility function depends on surplus consumption. The log of the surplus consumption ratio, consumption and money growths following an AR(1) process is the solution to FOREX puzzles and the ability of mimicking volatility. As the habit defined to goods not countries, Moore and Roche (2006) conclude that the Lucas two-country, two-good, two-money economy model with habit is capable to capture and account for FOREX puzzles and some empirical stylized facts in FOREX markets e.g. persistent volatility. Moreover, Moore and Roche (2002, 2005, 2006) discuss that the surplus consumption ratio is very volatile in comparison to nominal fundamentals such as consumption and money. We uses the moment expressions in Moore and Roche (2006) to provide economic intuition to simulated results. We agree with Moore and Roche that "the volatility of the fundamentals is able to produce the volatility in the nominal exchange rate". But, the volatility mentioned in both Campbell and Cochrane (1999) and Moore and Roche (2006) is unconditional. Campbell and Cochrane (1999) and Moore and Roche (2006) do not say anything about conditional volatility, an important empirical fact in finance, while, most empirical work involves volatility clustering in returns last decade.

The volatility is conditional and asymmetric. There is little work done to investigate the ability of the theoretical asset pricing model with habit to generate volatility clustering in asset returns. McQueen and Vorkink (2004) is the one of few examples that applies the theoretical models to the issue of volatility clustering. In McQueen and Vorkink (2004), a preference-based equilibrium asset pricing model is developed to capture long-term stock predictability and excess volatility. The optimal proceeds are made from both consumption and financial utility. McQueen and Vorkink let utility depend on consumption plus the score coefficient times changes in wealth. They make the marginal utility of financial wealth an AR(1) process by adding a wealth term with a time varying score coefficient. McQueen and Vorkink (2004) show that the mental scorecard that records the market's sensitivity to news and affects the agents' level of risk aversion due to wealth changes and experience loss aversion is able to explain the conditional volatility even its asymmetric property. But, McQueen and Vorkink focus on asset returns rather than exchange rate changes. They do not give the details of the properties of volatility clustering produced by the preference-based equilibrium asset pricing model. For example, how persistent and asymmetric

[1] We refer "ARCH" afterwards for short to generic predictable conditional volatility.

[2] Moore and Roche (2006) explain the exchange rates disconnect, forward bias, and Meese-Rogoff puzzles in details. With external consumption, inseparable utility and habit persistence, Moore and Roche think that the non-stationary surplus consumption is the radical reason for exchange rate disconnect puzzle; taking an account of the negative correlation between interest rate and expected exchange rate in the nation, Moore and Roche consider that preference for savings triggers forward bias problem.

conditional volatility is? What is the dynamic form conditional heteroscedasticity takes? Do its dynamics match those found from the actual data? McQueen and Vorkink (2004) refuse the capability of the theoretical model where utility depends on surplus consumption to explain volatility clustering in U.S. stock data. A clear and thorough literature of investigating the consumption-based equilibrium asset pricing model with habit to generate conditional volatility in FOREX returns is missing.

The paper's purpose is to investigate volatility clustering in FOREX returns. The paper is developed basing upon the theoretical model in Moore and Roche (2006) to fully investigate the idea of conditional volatility mentioned by McQueen and Vorkink (2004). The paper works in both the theoretical and empirical frameworks. In the theoretical framework, we use artificial data as in Campbell and Cochrane (1999), McQueen and Vorkink (2004) and Moore and Roche (2006)[3]. In the paper, we first overview the development of the theoretical model with introducing and discussing several important academic papers, and then deduce an implied ARMA(2,2) process for spot return from the model. Furthermore, we numerically solve the model and find ARCH effects in the spot return we simulate where the simulated data for spot return and for its innovation in the spot return process are definitely conditionally heteroscedastic. What is more, we estimate and establish the form of the conditional heteroscedasticity implied by the model. We find the same fit GARCH models for the best estimates for the simulated and empirical data, which is consistent with the results in our working paper of forecasting volatility. The estimates of conditional volatility for the innovation in the theoretical quarterly spot return of an ARMA(2,2) process are highly consistent with those for the real monthly spot return itself. We explain why empirical researchers tend to consider the FOREX spot return itself rather than its innovation, which is also the reason why we estimated and forecasted conditional volatility for the spot return previously. We show that the dynamics of the conditional heteroscedasticity implied by the model match those we found from the empirical data due to the "same looks" of the two impulse response functions (IRFs) for the theoretical and empirical ARCH processes we estimate. The Lucas model with habit can generate realistic conditional volatility in FOREX returns.

The paper is organized as follows. In Section 2, we review and develop the CCAPM with habit preferences, where we extend CCAPM to habit persistence using Campbell and Cochrane (1999). In Section 3, we review McQueenand and Vorkink (2004) that motivate the preference-based equilibrium asset pricing model to explain volatility clustering by revisions to wealth introduced in the utility function. In Section 4, we present Moore and Roche's habit version of Lucas and show the process in which the model can generate intrinsic conditional volatility in spot return. In Section 5, we numerically solve the model and test the spot return we simulate for the implied time series properties and conditional heteroscedasticity as well as assess sensitivity to parameter changes. In Section 6, we estimate the best fit GARCH model(s) and present IRFs to establish the exact dynamic form of conditional heteroscedasticity. Section 7 summarizes and concludes.

2 CCAPM with habit persistence

An investor must make a decision how much to consume and how much to save, and what portfolio of assets to hold. The basic idea of most pricing equations is to take the first-order condition (FOC) for that decision. The investor should always set the marginal utility loss of less consumption and more investment today equal to the marginal utility gain of more consumption of the asset's payoff tomorrow. The main theory of asset pricing is about how to use marginal utility to solve observable indicators. As known, a gross return is obtained by dividing the payoff by the price. For simplicity, we suppose that the price of the consumption good is unity. An intertemporal decision problem of an investor who maximizes the expectation of a time-separable utility is

$$Max\ E_t\left[\sum_{i=0}^{\infty}\delta^i u(C_{t+i})\right]$$

$$s.t.\ A_{t+i+1} = (1+r_{t+i})(A_{t+i}+Y_{t+i}-C_{t+i}),\quad i=1,2,3...$$

[3]Campbell and Cochrane (1999) and McQueen & Vorkink (2004) simulate stock data at a monthly frequency while Moore and Roche (2006) simulates FOREX data at a quarterly frequency with the different parameter settings.

where δ is time discount factor, C_{t+i} is consumption, A_{t+i} is asset, Y_{t+i} is income, r_{t+i} is interest rate in period $t+i$, and $u(C_{t+i})$ is the period utility of consumption at $t+i$. The FOC (or Euler equation) describing the investor's optimal consumption and portfolio choice is

$$u\acute{}(C_t) = \delta E_t \left[(1+r_t) u\acute{}(C_{t+1})\right]$$

If we divide both the left- and right-hand sides of the above equation by $u\acute{}(C_t)$, we get

$$1 = E_t \left[(1+r_t) M_{t+1}\right]$$

where the variable $M_{t+1} = \frac{\delta u\acute{}(C_{t+1})}{u\acute{}(C_t)}$ is known as the stochastic discount factor. It shows how an equation relates asset returns to the stochastic discount factor. Assuming that individual investors can be aggregated into a single representative investor so that aggregate consumption can be used in place of consumption of any particular individual, the equation, $1 = E_t \left[(1+r_t) M_{t+1}\right]$, with $M_{t+1} = \frac{\delta u\acute{}(C_{t+1})}{u\acute{}(C_t)}$, where C_t is aggregate consumption, is known as the consumption CAPM, or CCAPM[4] (Campbell, Lo and Mackinlay (1997), p304). It is emphasized that in the CCAPM the agents don't buy or sell any assets because there is no need to have traded portfolios as nobody else trades with this only representative investor for the only intertemporal optimization in one investor-consumer economy.

Using a convenient power utility function, $u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$[5], we can obtain the risk free interest rate

$$\widetilde{r}_{ft} = -\log \delta + \gamma E_t \left(\triangle c\right) - \frac{\gamma^2}{2}\sigma_c^2$$

where we denote lowercase letters for logs and set the log risk free return $\widetilde{r}_{ft} \equiv \log(1+r_{ft})$ and the log consumption growth $c \equiv \triangle c$, and get the risk premium

$$E_t\left(\widetilde{r}_t\right) - \widetilde{r}_{ft} = -\frac{1}{2}\sigma_{\widetilde{r}_t}^2 + \gamma \sigma_{\widetilde{r}_t,c}$$

where the log return is $\widetilde{r}_t \equiv \log(1+r_t)$, and the notation $\sigma_{\widetilde{r}_t,c}$ is the unconditional covariance of asset returns with consumption growth.

We took a glance at the CCAPM. We give some intuitions: (a) investors expect high returns on almost all assets that are associated with high consumption growth. Investors are "happy" with relatively low consumption today and relatively high consumption tomorrow only when rates are high and vice versa. (b) with uncertain rates investors require a risk premia. Assets that have negative covariance with consumption (growth) are very valuable and may even earn a negative risk premium, where bad consumption tomorrow is likely to be offset by high returns because of negative covariance. The covariance of asset payoffs with consumption drives risk correction to asset prices. As an intertemporal equilibrium model, the CCAPM is helpful to understand changes of financial asset returns over time, the relationship between saving and consumption, and an investor's risk aversion for a optimal portfolio decision.

Cochrane (2001) gives an explanation for the predictability of returns from price/dividend ratios that "people get less risk averse as consumption and wealth increase in a boom, and more risk averse as consumption and wealth decrease in a recession". Equity premia does not decline as risk aversion increases. There is no way to fix risk aversion to the level of consumption and wealth. The idea is to make "a model in which risk aversion depends on the level of consumption or wealth relative to some trend or the recent past". Following this idea, Campbell and Cochrane (1999) develop the "trend" in consumption using a consumption-based model while Mcqueen and Vorkink (2004) investigate the "trend" in wealth level in the recent past. We talk Campbell and Cochrane (1999) in this section and Mcqueen and Vorkink (2004) in the next section.

Campbell and Cochrane (1999) emphasize that people slowly develop habits for more or less consumption so that the habits form the "trend" in consumption. They specify a habit which is externally determined by the history of aggregate consumption, slowly moves and responds to consumption, and nonlinearly adjusts to the history of consumption. Following Abel (1990), Campbell and Cochrane (1999) feature this external habit for a technical convenience[6].

[4]Breeden (1979) develops the CCAPM by defining risk with respect to aggregate consumption.

[5]As γ approaches one, the power utility function approaches the log utility function $u(C_t) = \log(C_t)$. The first derivative of the power utility function is $u\acute{}(C_t) = C_t^{-\gamma}$ for $\gamma \neq 1$ and $u\acute{}(C_t) = \frac{1}{C_t}$ for $\gamma = 1$.

[6]External habit persistence implies positive serial correlation in consumption changes, which also holds for internal habits. We argue that it does not make much difference to the results for aggregate consumption and asset prices. See Cochrane (2001) for the details.

Campbell and Cochrane (1999) start to model an endowment economy with independent and identically distributed (i.i.d.) consumption growth in a lognormal process

$$\triangle c_{t+1} = g + v_{t+1}, \quad v_{t+1} \sim i.i.d.\ N(0, \sigma^2)$$

where log consumption is assumed to follow a random walk with drift g and innovation v_{t+1}. They replace the utility function $u(C)$ with $u(C-H)$ in terms of nonseparable utility over time, where H is the habit level, to maximize the utility function for identical agents

$$E\sum_{t=0}^{\infty}\delta^t \frac{(C_t - H_t)^{1-\gamma} - 1}{1-\gamma}$$

Habits should move slowly in response to consumption and may be written

$$h_t = \phi h_{t-1} + \lambda c_t$$

(Small letters denote the logs of large letters throughout this section e.g. $c_t = \ln C_t$, $h_t = \ln H_t$, etc.)

Campbell and Cochrane (1999) define the surplus consumption ratio $X_t = \frac{C_t - H_t}{C_t}$ to capture the relation between consumption and habit conveniently. They let the surplus consumption ratio of consumption to habit follow an AR(1)

$$x_{t+1} = (1-\phi)\overline{x} + \phi x_t + \lambda(x_t)(c_{t+1} - c_t - g)$$

Here, the equation specifies how h responds nonlinearly to c because x is associated with c and h, which means that consumption can never fall below habit since $X = e^x \succ 0$, although it is approximately the same as a traditional habit-formation model[7]. The nonlinear adjustment of habit to consumption guarantees habit always below consumption with finite and positive marginal utility while in other habit models in endowment economy habit can be above consumption with undesirable infinite or negative marginal utility. Campbell and Cochrane (1999) also allow consumption to affect habit differently in different states by featuring a square root type process

$$\begin{aligned} \lambda(x_t) &= \frac{1}{\overline{X}}\sqrt{1 - 2(x_t - \overline{x})} - 1 \\ \overline{X} &= \sigma\sqrt{\frac{\gamma}{1-\phi}} \end{aligned}$$

X_t becomes the single state variable in this economy. Time-varying expected returns, price/dividend ratios, etc. are all functions of this state variable.

Campbell and Cochrane (1999) give marginal utility for an external habit

$$u'(C_t) = u_c(C_t, H_t) = (C_t - H_t)^{-\gamma} = X_t^{-\gamma} C_t^{-\gamma}$$

The external habit, like Abel (1990)'s "catching up with the Joneses" formulation, simplifies analysis and eliminates terms in marginal utility by which current consumption has an impact on future habits since an individual's habit is determined by "the history of aggregate consumption". With marginal utility, the stochastic discount factor is

$$M_{t+1} \equiv \delta \frac{u_c(C_{t+1}, H_{t+1})}{u_c(C_t, H_t)} = \delta\left(\frac{X_{t+1}}{X_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

The stochastic process is associated with X and C, and each is lognormal. Campbell and Cochrane (1999) evaluate the risk free rate by evaluating the conditional mean of the stochastic discount factor. The risk free rate is related to the stochastic discount factor by $1 + r_{ft} = 1/E_t(M_{t+1})$[8]. Taking logs, and using the expressions of x_{t+1} and M_{t+1}, the log risk free rate is

$$r_t^f = -\log E_t(M_{t+1}) = -\log(\delta) + \gamma g - \frac{1}{2}\gamma(1-\phi)$$

[7] We call $h_t = \phi h_{t-1} + \lambda c_t$ a traditional habit-formation model. The problem with the traditional model is that it allows consumption to fall below habit, resulting in infinite or imaginary marginal utility.

[8] See Chapter 3 of the author's PhD dissertation for the details.

Using the basic pricing relation $1 = E_t\left(M_{t+1}R_{t+1}\right)$ and the definition of returns $R_{t+1} \equiv \frac{P_{t+1}+D_{t+1}}{P_t}$, Campbell and Cochrane (1999) evaluate the price/consumption[9] (or the price/dividend) ratio as a function of the state variable by iteration on a grid

$$\frac{P_t}{C_t}\left(x_t\right) = E_t\left[M_{t+1}\frac{C_{t+1}}{C_t}\left(1+\frac{P_{t+1}}{C_{t+1}}\left(x_{t+1}\right)\right)\right]$$

The surplus consumption ratio x_t is the only state variable for the economy, so the price/consumption ratio is a function only of x_t. With the price/consumption ratio, Campbell and Cochrane (1999) can calculate returns, expected returns, the conditional standard deviation of returns, etc.

We extend the CCAPM to habit persistence using Campbell and Cochrane (1999). The original motivation of Campbell and Cochrane (1999) was to show that "habit" preferences can generate large equity risk premia but we also show that these same preferences lead to ARCH behavior in asset returns in the following sections.

3 CCAPM with wealth change

Volatility clustering in returns, one of new important empirical facts in finance, has been involved in most empirical work published last decade. There are different explanations for why volatility clustering. Those explanations are mainly divided into two groups: one is an exogenous explanation such as clustered news of economic fundamentals; another is an endogenous explanation such as heterogeneous traders where trading process plays the role of triggering volatility autocorrelation. McQueen and Vorkink (2004) argue that clustered economic fundamental news in the first group is problematic and heterogeneous traders in the second group is incomplete to which they complement a preference explanation using a preference-based equilibrium asset pricing model to explain low frequency volatility clustering. Also, there are other explanations using leverage models or state-uncertainty models but both models inappropriately predict low volatility after good news.

McQueen and Vorkink (2004) motivate their preference-based equilibrium asset pricing model to explain volatility clustering basing upon preference models capable of explaining long-run stock predictability and excess volatility. Their model is structured as an extension of both the preference model of Barberis, Huang and Santos (2001) and the volatility feedback model of Campbell and Hentschel (1992), where the preference model explains features of volatility clustering and the volatility feedback model explains asymmetry in volatility autocorrelation.

In McQueen and Vorkink (2004), a unique mental scorecard that records wealth changes and affects investors' level of risk aversion induces sufficient variation in aversion and sensitivity to news causing subsequent stock volatility. Investors' wealth-varying risk aversion is about loss aversion and scorecard dependence. Investors are more attentive and sensitive to financial news when their expected wealth is perturbed. McQueen and Vorkink (2004) propose a four-stage behavioral process: 1) investors measure their portfolio using the mental scorecard of past investment performance. 2) Investors are more risk averse when their portfolio has an unexpected investment performance. More risk aversion investors have, more sensitive to news investors are. Stock prices react more to news than when investors are not sensitive. 3) Return shocks cause greater return volatility that dies out slowly because of investors' persistent attention and sensitivity to news. 4) Investors recover their normal sensitivity to news when they are used to the new level of wealth. Time-varying sensitivity to news endogenously generates clustered returns and volatility clustering and, therefore, state-dependent sensitivity to news is the reason behind volatility clustering in McQueen and Vorkink (2004).

Following Lucas (1978), McQueen and Vorkink (2004) start to maximize expected lifetime utility by allocating wealth between consumption and investment as follows

$$\max\ \mathrm{E}\left[\sum_{t=0}^{\infty}\delta^t U\left(C_t\right) + b_0\overline{C}^{-\gamma}\delta^{t+1}F\left(W_{t+1}\right)\right] \qquad ([1])$$

where thereafter numbers in brackets [] denote the numbered equations in McQueen and Vorkink (2004). Financial utility assumptions are made

$$F\left(W_{t+1}\right) = \lambda\left(z_t, O_{t+1}\right)W_{t+1} \qquad ([3])$$

[9] The price-consumption ratio is an exponentially-weighted average of the expected dividend share by Cochrane, Longstaff, and Santa-Clara (2003), "TWO TREES: ASSET PRICE DYNAMICS INDUCED BY MARKET CLEARING".

$$\lambda(z_t, O_{t+1}) = k(a_0 - a_1 z_t) \quad \text{for } O_{t+1} \prec 0 \qquad ([4])$$
$$= a_0 - a_1 z_t \quad \text{for } O_{t+1} \succeq 0$$

Notation[10] is given as follows: δ^t - subjective discount rate, C_t - consumption, b_0 - scale parameter of importance of financial utility relative to consumption utility, $\overline{C}$ - aggregate per capita consumption, γ - constant relative risk aversion parameter, W_{t+1} - wealth, z_t - the scorecard, O_{t+1} - return shock, $F(W_{t+1})$ - financial wealth, $\lambda(z_t, O_{t+1})$ - investors' level of risk aversion, k - degree of loss aversion, a_0- investors' baseline level of financial utility derived from gains, a_1 - parameter of how the past performance affecting the magnitude of the utility derived from gains and losses.

In the McQueen and Vorkink (2004) model, investors maximize expected lifetime utility not only from changes in consumption C_t and but also from financial wealth $F(W_{t+1})$, where unexpected fluctuations in the value of investors' financial wealth depends on investors' level of risk aversion $\lambda(z_t, O_{t+1})$. The mental scorecard z_t that affects investors' level of risk aversion remembers the prior portfolio shocks as follows:

$$z_{t+1} = \phi z_t + h(z_t) O_{t+1} \qquad ([5])$$

where ϕ is a memory parameter $(0 < \phi < 1)$ and $h(z_t)$ is the scorecard's sensitivity to wealth shocks. Return shocks drive changes in the scorecard. When investors' scorecard is perturbed, investors become more attentive and sensitive to subsequent financial news, which is the unique feature of their model. McQueen and Vorkink (2004) call this the law of motion for z_t.

Taking the first-order conditions of their equation (1) and substituting the marginal utility of financial wealth into the objective[11], McQueen and Vorkink (2004) obtain a wealth-varying risk aversion version of Euler equation

$$1 = E_t[m_{t+1} R_{t+1}] \qquad ([8])$$

where the pricing kernel is

$$m_{t+1} = \kappa_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} + \lambda(z_t, O_{t+1}) \right]$$

$$\kappa_t = \frac{\delta}{1 + \delta E_t[\lambda(z_t, O_{t+1})] E_t(R_{t+1})}$$

From their wealth version Euler equation, the pricing equation of price-dividend ratios presented is

$$\frac{P_t}{D_t}(z_t) = E_t \left[m_{t+1} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \frac{D_{t+1}}{D_t} \right] \qquad ([10])$$

McQueen and Vorkink (2004) discuss the model's "qualitative intuition" in which the model has symmetric sensitivity but generates asymmetric responses to news.

McQueen and Vorkink (2004) numerically solve the resulting asset pricing model for the equilibrium, where a solution to the pricing model (Equation [10]) as well as the scorecard's law of motion (Equation [5]) is required. Taking into account the endogenous nature of price-dividend ratios and the scorecard, McQueen and Vorkink (2004) use an iterative method and update the pricing model in their equation ([10]) as follows

$$\frac{P_t}{D_t}(z_t) = E_t \left\{ \left(\kappa_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} + \lambda(z_t, O_{t+1}) \right] \right) \frac{D_{t+1}}{D_t} \left(\frac{P_{t+1}}{D_{t+1}}(z_{t+1}) + 1 \right) \right\} \qquad ([11])$$
$$= \kappa_t E_t \left\{ \left[\left(G^{-\gamma} e^{-\gamma\rho \frac{\sigma_v}{\sigma_\epsilon} \varepsilon_{t+1} + \frac{\gamma^2}{2}(1-\rho^2)\sigma_v^2} \right) + \lambda(z_t, O_{t+1}) \right] G e^{\epsilon_{t+1}} \left(1 + \frac{P_{t+1}}{D_{t+1}}(z_{t+1}) \right) \right\}$$

where $G = ln(g)$. Now the pricing models can be solved numerically by using parameter values given in their Table 1 and conditional moments of return and volatility can be solved by using

[10] Notation in section 3 is almost identical to that in McQueen and Vorkink (2004) except the one for return shock. In order to make notation homogeneous throughout, we use O_{t+1} to replace X_{t+1} for return shock in McQueen and Vorkink (2004).

[11] Equation (7) in McQueen and Vorkink (2004) is $1 = \delta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} + F'(W_{t+1}) \right]$.

the price-dividend function and the scorecard's law of motion. McQueen and Vorkink (2004) define the conditional expected stock return as

$$E_t\left(r_{t+1}\right) = E_t\left[\left(\frac{\frac{P_{t+1}}{D_{t+1}}\left(z_{t+1}\right)+1}{\frac{P_t}{D_t}\left(z_t\right)}\right)\frac{D_{t+1}}{D_t}\right] \qquad ([12])$$

which is required to be solved simultaneously with the price-dividend function and the scorecard law of motion until convergence. Furthermore, basing upon the calculations and solutions of conditional expected returns, McQueen and Vorkink (2004) use numerical integration techniques to calculate the expected standard deviation of returns, $E_t\left(\sigma_{t+1}\right)$, in terms of the relevant parameter values given in their Table 1. As showed in their Figure 3 and 4, the expected standard deviation of returns is asymmetrically conditional on the scorecard.

McQueen and Vorkink (2004) conduct simulation practices of monthly returns to investigate if their preference-based equilibrium asset pricing model can internally explain conditional volatility. From their simulation and sensitivity analysis they argue that their model can generate consistent conditional volatility found in empirical facts. Moreover, they conclude that their model performs better than the traditional consumption-only model and the consumption-based model with external habits in Campbell and Cochrane (1999).

McQueen and Vorkink (2004) test the theoretical model about conditional moments. They first test their scorecard's ability to predict conditional volatility. Then they compare their scorecard with other preference scorecards and test its ability to predict conditional excess returns and skewness. Here, we briefly summarize some relevant results for our interests (e.g. conditional volatility and competing scorecards). In the tests of the scorecard's predictive ability of conditional volatility, McQueen and Vorkink (2004) use two regression models with monthly empirical data. The models are run by an estimate of conditional return volatility on an estimate of the lagged scorecard (and on predictions of conditional volatility) as displayed in their Table 3. In the tests of competing scorecards' prediction ability of conditional moments, McQueen and Vorkink (2004) compare their scorecard of past investment performance with the scorecard of the log consumption-aggregate wealth ratio in Lettau and Ludvigson (2001) and the scorecard of the surplus consumption ratio in Campbell and Cochrane (1999). They employ another two different empirical regression models with empirical data at both monthly and quarterly frequencies as showed in their Table 4. The results (in their Table 3 and 4) show that their scorecard can predict conditional volatility, excess returns and skewness. It performs better on conditional volatility and skewness than the scorecard in Lettau and Ludvigson (2001), which is better at predicting excess returns, and the scorecard in Campbell and Cochrane (1999). McQueen and Vorkink (2004) conclude that the preference-based equilibrium asset pricing model in which the utility is obtained from consumption and wealth changes is capable to explain many stylized facts about conditional volatility, even the new empirical facts in finance including excess returns, high risk premium and skewness.

We extend the CCAPM to derive utility from wealth changes using McQueen & Vorkink (2004), which is able to explain conditional volatility found in US stock data. McQueen & Vorkink (2004) employ a preference-based asset pricing model to capture long-term stock predictability and excess volatility. The model includes wealth-varying degrees of risk aversion and sensitivity to news. They show that the mental scorecard that records the market's sensitivity to news and affects the agents' level of risk aversion due to wealth changes and experience loss aversion is able to explain the conditional volatility, even its asymmetric property. The original motivation of McQueen & Vorkink (2004) is to stress the fact that revisions to wealth introduced in the utility function can lead to ARCH behavior.

4 Model

4.1 Background

The aim of the paper is to investigate if the Lucas two-country monetary model with habit in Moore and Roche (2006) can generate generic predictable conditional volatility in spot returns $\triangle s_{t+1}$ and not in the risk premium or foreign asset returns or in asset returns in general.

In CCAPM, under one non-storable consumption good, a single representative consumer's aggregate consumption becomes equal to the total economy consumption so that total expected consumption (growth) in the economy is linked to expected returns. If there is no capital stock and perishable consumption goods as in the Lucas model, then things become even simpler with individual consumption becoming equal to economy wide consumption equal to economy wide exogenous income.

Before we show how the Lucas two-country monetary model with habit may generate predictable ARCH, we discuss the ability of the single Lucas model to produce conditional volatility.

Lucas (1982) sets up a dynamic general equilibrium model of endowment economy with a complete market. The Lucas model prices foreign exchange that depends on preference. In the Lucas model, representative agents in two countries are provided with identical preferences over two consumption goods but with different stochastic endowments of these. The assumption of the Lucas model is that securities markets are complete so that there is complete pooling of risks. With identical preferences agents will consume exactly one half of the endowment of each good in each period and maximize their expected infinite utility function for each country.

We discuss our initial ideas. Since the Lucas model in its simplest form can never generate ARCH effects in spot returns s_t, the real exchange rate in any Lucas model is just the relative price of home to foreign goods. In competitive models this relative price is always equal to the ratio of the goods' marginal utilities (marginal utility of one good divided by marginal utility of the other). Hence we have that for domestic good L and foreign good F

$$q_t \quad \equiv \frac{S_t P_t^*}{P_t} = \frac{\partial U(.)/\partial L_t}{\partial U(.)/\partial F_t}$$

where t is time; q_t is the relative price of home to foreign goods; S_t is spot rate; P_t is the price of domestic good; P_t^* is the price of foreign good; ∂ is the derivative and $U(.)$ denotes utility. Note that we have not made any assumptions about time separability of U yet. So this condition is true for all types of the Lucas model whatever the form of $U(.)$ is. We denote that lower case is logs and upper is levels. Now we can use the cash in advance constraints that say that all $L(F)$ goods must be paid for with domestic(foreign) money $M(N)$, where $M_t = P_t L_t$ and $N_t = P_t^* F_t$, to substitute out for prices. In terms of monies, we get the condition

$$q_t \quad \equiv \frac{S_t N_t L_t}{M_t F_t} = \frac{\partial U(.)/\partial L_t}{\partial U(.)/\partial F_t}$$

We now solve for S in terms of all the other conditions

$$S_t = \frac{M_t F_t}{N_t L_t} \frac{\partial U(.)/\partial L_t}{\partial U(.)/\partial F_t}$$

This equation shows that the exchange rate depends on the relative monies but also depends on the marginal utilities and the home and domestic (exogenous) endowment streams arising from the Lucas "trees".

There are several potential ways to get ARCH[12] in S. One is to fix the relative money processes to make them have ARCH. But by doing this we would be putting ARCH in to get ARCH out. It is not reasonable. Another way is to fix the U or u functions in such a way as to make their derivatives have ARCH time series properties.

First, if the simple Lucas model is taken to be Lucas plus standard time separable power utility then the marginal utility ratio is just a simple function of the ratio of home to foreign consumptions C_{l_t}/C_{f_t} and this ratio is just l_t/f_t because of the endowment economy where all outputs must be consumed. Substituting this into the formula for S it is found that S is just proportional to relative money supplies, exactly as in the simple "ad hoc" monetary model. The simple Lucas model cannot get ARCH for S out of this unless we assume ARCH for monies which, we know, is anyway counterfactual.

However if we specify an exotic U, for example, which depends on a "habit", a ratio of marginal utilities is obtained to give an ARCH behavior. A habit defined by Campbell and Cochrane (1999) to solve the equity premium puzzle in Section 2 is an AR(1). The lagged level of consumption is the "shock" in the AR(1) process. Solving the AR(1) backwards gives the habit H as something like

$$h_t \approx \frac{1-\phi}{\phi} \sum_{i=1}^{\infty} \phi^i c_{lt-i}$$

which is equivalent to $h_t = \phi h_{t-1} + \lambda c_t$ given in Section 2[13]. This is thought to be a reasonable

[12] We refer "ARCH" afterwards for short to generic predictable conditional volatility.

[13] See Campbell, Lo, and Mackinlay (1997), p330-331, for the details.

assumption. Now instead of U being in terms of actual consumption it is written in terms of consumption relative to habit - i.e. the surplus consumption ratio of the form $\frac{C_{lt}-H_t}{C_{lt}}$. It turns out that if the habit H is as above then surplus consumption will be an AR(1) but with a sensitivity to shocks parameter λ whose value depends on the previous level of surplus consumption. This is the Moore and Roche (2006) surplus consumption evolution equation.

4.2 A Lucas two-country monetary model with habit

Moore and Roche (2006) present a Lucas (1982) two-country monetary model with an external habit persistence as in Campbell and Cochrane (1999) for solving many exchange rate puzzles (disconnect, forward bias, and Meese-Rogoff) and mimicking unconditional volatilities of real and nominal exchange rates, forward premium, expected spot returns and expected forward profits.

Moore and Roche (2006) assume that consumption growth and money growth follow an AR(1) processes

$$\Delta c^j_{t+1} = (1-\rho_\mu)\overline{\mu} + \rho_\mu \Delta c^j_t + v^j_{t+1}\ , \quad v^j_{t+1} \sim N(0,\sigma^2_v)\ , \quad j=1,2 \qquad ([17],4.01)$$

$$\Delta m^j_{t+1} = (1-\rho_\pi)\overline{\pi} + \rho_\pi \Delta m^j_t + u^j_{t+1}\ , \quad u^j_{t+1} \sim N(0,\sigma^2_u)\ , \quad j=1,2 \qquad ([18],4.02)$$

where $\overline{\mu}$ ($\overline{\pi}$) is the unconditional mean of consumption (money) growth; v^j_{t+1} (u^j_{t+1}) are the shocks to consumption (money) growth while the shocks to consumption and money growth are uncorrelated; σ^2_v (σ^2_u) is the variances of shocks to consumption (money) growth. We write, next to our numbered equation in (), a bracket [] in which the number written corresponds to the serial number of the equation in Moore and Roche (2006). All equations with identical notation in this section are cited from Moore and Roche (2006).

Moore and Roche (2006) define habit persistence using an aggregate consumption externality. They give the maximized utility function (with the assumed identical parameters for both countries)

$$\sum_{t=0}^{\infty}\beta^t \left\{ \frac{\left(C^1_{it}-H^1_{it}\right)^{1-\gamma}}{1-\gamma} + \frac{\left(C^2_{it}-H^2_{it}\right)^{1-\gamma}}{1-\gamma} \right\}, \qquad i=1,2 \qquad ([5],4.03)$$

$$s.t.\ W_{t+1} = S_{t+1}B^2_t + B^1_t + \ P^1_t Y^1_t \qquad ([6])$$

and the wealth constraint

$$W_t = P^1_t C^1_t + S_t P^2_t C^2_t + q^1_t B^1_t + S_t q^2_t B^2_t \qquad ([7])$$

where β is the discount factor, γ is a curvature parameter, C^j_{it} is the consumption of goods and services of country j by the household of country i, H^j_{it} is the subsistence consumption / habit of goods and services of country j by the household of country i, W_{t+1} is the next-period wealth, S_t is the level of the spot exchange rate, B^j_t is the amount of one-period discount bonds from country j, P^j_t is the price level in country j, Y^j_t is the endowment in country j and q^j_t is the nominal bond price in country j. As showed above, the next-period wealth consists of a monetary transfer ($S_{t+1}B^2_t$), dividends (B^1_t) and market value of securities ($P^1_t Y^1_t$) while three parts in the wealth constraint are goods ($P^1_t C^1_t + S_t P^2_t C^2_t$), equity ($q^1_t B^1_t$) and a money transfer ($S_t q^2_t B^2_t$).

Moore and Roche (2006) assume that the cash-in-advance constraint is

$$\frac{M^j_t}{P^j_t} = C^j_t\ , \qquad j=1,2 \qquad ([8])$$

and define the surplus consumption ratio (SCR) as

$$X^j_t = \frac{\overline{C}^j_t - H^j_t}{\overline{C}^j_t}\ , \qquad j=1,2 \qquad ([9])$$

where X^j_t is SCR of country j, M^j_t is money in country j and $\overline{C}^j_t$ is aggregate consumption per capita of goods and services of country j.

Moore and Roche (2006) let the log of the surplus consumption ratios follow an AR(1) process

$$x^j_{t+1} = (1-\phi)\overline{x} + \phi x^j_t + \lambda(x^j_t)(v^j_{t+1})\ , \quad j=1,2 \qquad ([10],4.04)$$

where $\phi(<1)$ is the habit persistence parameter, $\overline{x}$ is the steady state value for the logarithm of the surplus consumption ratio.

Moore and Roche (2006) also allow consumption to affect habit differently throughout states by featuring a squared root type process as in Campbell and Cochrane (1999), where they define the sensitivity function $\lambda(x_t^j)$ of the log surplus consumption ratio to endowment innovations to non-linearly depend on the current log surplus consumption ratio.

$$\lambda(x_t^j) = \frac{\sqrt{1-2\left(x_t^j - \overline{x}\right)}}{\overline{X}} - 1 \quad for\ x_t^j \leq x_{\max} \qquad j = 1,2$$

$$= 0 \qquad\qquad for\ x_t^j \succ x_{\max} \qquad ([11],4.05)$$

$$where\ \ x_{\max} = \overline{x} + \frac{1-\overline{X}^2}{2} \qquad \text{and} \qquad \overline{X} = \frac{\gamma\sigma_v}{\sqrt{\gamma(1-\phi)-\delta}} \qquad ([12])$$

where $\overline{X}$ is the steady state value of the surplus consumption ratio, δ is the parameter in steady state surplus consumption and $\gamma(1-\phi)-\delta > 0$.

Using the FOCs the optimization problem is solved and Moore and Roche express the nominal exchange rate

$$S_t = \frac{\left(C_t^2\right)^{1-\gamma}\left(X_t^2\right)^{-\gamma}}{\left(C_t^1\right)^{1-\gamma}\left(X_t^1\right)^{-\gamma}}\frac{M_t^1}{M_t^2} \qquad ([A10],4.06)$$

The log of the nominal exchange rate is

$$s_t = -(1-\gamma)(c_{t+1}^1 - c_{t+1}^2) + \gamma(x_{t+1}^1 - x_{t+1}^2) + (m_{t+1}^1 - m_{t+1}^2) \qquad ([20],4.07)$$

The volatility of nominal exchange rates is given by the variance of spot returns

$$Var(s_{t+1} - s_t) = 2\left[\frac{\sigma_u^2}{1-\rho_\pi^2} + (1-\phi)^2\gamma^2\sigma_x^2 + \sigma_v^2\left(1-\frac{\gamma}{\overline{X}}\right)^2\right] \qquad ([21],4.08)$$

where σ_x^2 is the variance of the surplus consumption ratio. Equation (4.08) is helpful for using the moment expressions to provide some intuition to understand the simulated results in Moore and Roche (2006).

4.3 Implications of the model

We aim to investigate the model's ability to generate conditional volatility in spot returns. Hence, we explore the properties of spot returns Δs_t and calculate innovations $\widehat{\zeta_t}$. We analyze the theoretical model (Moore and Roche's model) and then derive an implied ARMA(2,2) process of spot returns Δs_t. We filter the data for removing AR components to obtain the filtered spot returns Δs_t^f. We calculate innovations $\widehat{\zeta_t}$, subject to conditional volatility, after estimating an MA(2) model for the filtered spot returns Δs_t^f.

Using Moore and Roche's equation (10), of an AR(1) log surplus consumption ratio, and equation (A12), of the change in spot rates s, we give the analysis of an ARMA(2,2) process of spot returns.

We take the difference between country one and two in surplus consumption ratios using Equation (4.04)

$$x_t^1 - x_t^2 = \phi(x_{t-1}^1 - x_{t-1}^2) + \lambda_{t-1}^1 v_t^1 - \lambda_{t-1}^2 v_t^2 \qquad (4.09)$$

We also take the difference between country one and two in money growths using Equation (4.02) where both money growths are about an AR(1) with the identical AR(1) coefficient ρ_π.

$$\Delta m_t^1 - \Delta m_t^2 = \rho_\pi(\Delta m_{t-1}^1 - \Delta m_{t-1}^2) + u_t \qquad (4.10)$$

Denote

$$x_t = x_t^1 - x_t^2$$

$$m_t = \Delta m_t^1 - \Delta m_t^2$$

$$w_t = \lambda_{t-1}^1 v_t^1 - \lambda_{t-1}^2 v_t^2$$

where $var(w_t) = \sigma^2_{wt}$ is conditionally heteroscedastic but $E(w_t w_{t-i}) = 0$ for $i = 1, 2, 3.....$ and u_t is *iid*. Now Equations (4.09) and (4.10) may be written as

$$x_t = \phi x_{t-1} + w_t \tag{4.11}$$

$$m_t = \rho_\pi m_{t-1} + u_t \tag{4.12}$$

Using the lag operator and noting that the roots are less than one, thus they are invertible, we rewrite Equations (4.11) and (4.12) respectively as

$$x_t = \frac{w_t}{1 - \phi L}$$

$$m_t = \frac{u_t}{1 - \rho_\pi L}$$

The process of spot returns described in Moore and Roche's equation (A12) is

$$\begin{aligned} \Delta s_{t+1} &= \left(\Delta m^1_{t+1} - \Delta m^2_{t+1}\right) - \gamma(1-\phi)\left(x^1_t - x^2_t\right) \\ &\quad - \left\{1 - \gamma\left[1 + \lambda\left(x^1_t\right)\right]\right\} v^1_{t+1} + \left\{1 - \gamma\left[1 + \lambda\left(x^2_t\right)\right]\right\} v^2_{t+1} \end{aligned} \tag{[A12]}$$

Here, we write down the form for time t instead of Moore and Roche's time $t + 1$ and thus, in this notation, Moore and Roche's equation (A12) becomes

$$\Delta s_t = m_t + a x_{t-1} + z_t \tag{4.13}$$

where $var(z_t) = \sigma^2_{zt}$ is related to w_t above and it is likewise conditionally heteroscedastic but with $E(z_t z_{t-i}) = 0$ for $i = 1, 2, 3...$, and $a = -\gamma(1 - \phi)$.

Substituting x in Equation (4.11) and m in Equation (4.12) into Equation (4.13), we get

$$\Delta s_t = \frac{u_t}{1 - \rho_\pi L} + a\frac{w_{t-1}}{1 - \phi L} + z_t \tag{4.14}$$

Multiplying both sides by $(1 - \rho_\pi L)(1 - \phi L)$ gives

$$(1 - \rho_\pi L)(1 - \phi L)\Delta s_t = (1 - \phi L)u_t + a(1 - \rho_\pi L)w_{t-1} + (1 - \phi L)(1 - \rho_\pi L)z_t$$

It is noted that the error structure on the right hand side (RHS) is an MA(2) and because on the left hand side (LHS) we have an AR(2) structure the s series is an ARMA(2,2). The composite error has no autocovariances above 2 so it can be therefore written as

$$(1 - \phi L)u_t + a(1 - \rho_\pi L)w_{t-1} + (1 - \phi L)(1 - \rho_\pi L)z_t = \zeta_t + \theta_1\zeta_{t-1} + \theta_2\zeta_{t-2}$$

where ζ_t is (highly) conditionally heteroscedastic and $E(\zeta_t \zeta_{t-i}) = 0$ for all $i = 1, 2, 3, ...$

Using Moore and Roche's calibrated baseline parameters, the values of $\rho_\pi = 0.1$ and $\phi = 0.999$, and our simulated series for s, we can compute $(1 - \rho_\pi L)(1 - \phi L)\Delta s_t$ directly. We call this series Δs^f_t, where "^{f}" denotes filtered. Then we have

$$\Delta s^f_t = \zeta_t + \theta_1\zeta_{t-1} + \theta_2\zeta_{t-2} \tag{4.15}$$

This series is an MA(2) in a conditional heteroscedastic error ζ. We can use the Method of Moments to get good estimates of θ_1 and θ_2. Then denoting estimates by $\widehat{\ }$ we can compute the genuine innovation in the change in s as

$$\widehat{\zeta}_t = (1 + \theta_1 L + \theta_2 L^2)^{-1}\Delta s^f_t \tag{4.16}$$

As showed above, the change in spot rates, Δs_t, is an ARMA(2,2) and the filtered series for s, Δs^f_t, is an MA(2) in the conditional heteroscedastic error ζ, which we call the properties implied by the theoretical model[14] thereafter. We can calculate innovation $\widehat{\zeta}_t$ using the undetermined coefficient method. We now have a series of innovation $\widehat{\zeta}_t$ which we can apply ARCH class models' estimations to directly. We estimate the best fitting ARCH class model(s) to innovation $\widehat{\zeta}_t$. The model is able to explain conditional volatility in spot returns as long as innovation $\widehat{\zeta}_t$ has

[14] Moore and Roche's model in Moore and Roche (2006).

ARCH performance. It is emphasized clearly that the innovation $\widehat{\zeta}_t$ in the ARMA(2,2) process for spot returns is by definition conditionally heteroscedastic. The data we will simulate for spot returns and for the innovation in the spot returns process will definitely be conditionally heteroscedastic. We give the proof in the next section.

5 Solution and evaluation

We now turn our focus to presenting the baseline simulation results. First, we check if the simulations[15] in the baseline for the level of exchange rates (S) have the same time series properties as in Moore and Roche (2006). Second, we evaluate the efficiency of the simulations that capture the implied properties of the theoretical model. We also assess sensitivity to parameter changes.

We numerically solve the model. We employ the quarterly calibration parameter values assumed in the baseline framework in Moore and Roche (2006), where the parameters β and δ are chosen from the literature; γ and ϕ are chosen to make sure that the surplus consumption ratio ($\overline{X}$) is approximately 5% and that the value of local risk aversion is no more than 10; the parameters of endowment and money growth are chosen by using US data. Beside the parameterization in the baseline framework, in the sensitivity analysis, we set $\gamma = 0.7$, $\delta = -0.0025$, $\rho_\pi = 0$ or $\phi = 0.995$, respectively. The baseline's parameterization is displayed in Table 1.

Parameter	**Variable**	**Value**
Dsicount factor	β	0.99
Curvature of the utility function	γ	0.5
Parameter in steady state surplus consumption	δ	-0.005
AR(1) coefficient of log surplus consumption	ϕ	0.999
AR(1) coefficient of money growth	ρ_π	0.1
AR(1) coefficient of consumption growth	ρ_μ	0.00
Unconditional mean of money growth at steady state	$\overline{\pi}$	0.0136
Unconditional mean of consumption growth at steady state	$\overline{\mu}$	0.004725
Standard deviation of money growth	σ_u	0.00946
Standard deviation of consumption growth	σ_v	0.0075
Steady state value of surplus consumption ratio	$\overline{X}$	0.0506
Log steady state value of surplus consumption ratio	$\log(\overline{X})$	-2.9845
Max value of surplus consumption ratio	$X_{\max}$	0.0833
Max value of log surplus consumption ratio	$x_{\max}$	-2.4858
Local Relative Risk Aversion ($\preceq 10$)		9.88826

Notes: All parameters are quoted from Moore and Roche (2006).

Table 1: Model parameter values in baseline

Correct simulation techniques can guarantee to construct accurate exchange rates and spot returns. The procedure of simulation is executed as follows:

1. to generate a time series of consumption growth ($\triangle c$) using Equation (4.01), where we set initial consumption growth at its steady state value as given in Table 1;
2. to generate a time series of money growth ($\triangle m$) using Equation (4.02), where we set initial money growth at its steady state value as given in Table 1;
3. to accumulate the variables generated in step 1 and 2 so as to obtain the log level;
4. to generate a time series of log surplus consumption ratio ($x = \log X$) using Equations (4.04) and (4.05), where we set initial log surplus consumption ratio x at $\overline{x}$ and initial sensitivity function $\lambda(x)$ at $\lambda(\overline{x})$;
5. to construct a time series of exchange rates S using Equation (4.06);
6. to repeat steps 1-5 for each series that is constructed in the baseline framework and sensitivity analysis, respectively.

[15] We code in Matlab.

Before we investigate, examine and evaluate the theoretical model's implied properties, we carry out two simulation exercises. We start to simulate several time series (132 data points) that approximately have the properties of volatility and persistence of the exchange rates in the first-differenced data and the spot returns as in Moore and Roche's Table 9 and 10. We compare the statistics in the theoretical economy to those of the empirical data. The aim of the first exercise is to assess our simulation techniques by (more or less) replicating the results in Moore and Roche (2006). We also simulate several longer series (10000 data points) to capture Moore and Roche's model properties. The motivation of the latter is that Moore and Roche simulate only 132 observations and so the estimates of their model's properties are very imprecise. We are willing to deal with imprecision when we estimate the properties of the real world data where only 132 observations are available as in Moore and Roche (2006). However, we can generate longer series to reduce imprecision when estimating the model's properties. Hence the parameter values, variances, etc. estimated in the second exercise will be closer to the model's true parameters, variances, etc.

5.1 Simulation

Simulation 1

We simulated the model 1000 times generating log surplus consumption ratio ($x = \log X$), money growth ($\triangle m$) and consumption growth ($\triangle c$) for 132 observations[16]. After producing x, m and c we can construct exchange rates S.

We report the statistics of volatility and persistence for both the spot returns and the first-differenced exchange rates in Table 2. We find the approximately consistent results by comparing Moore and Roche's results to ours. The statistics of the first-differenced (FD for short afterwards) data are reported in Panel A of Table 2, where we filter the logarithm of the simulated exchange rates, $log(S)$, using the FD filter. First, for the property of volatility, in the baseline, the mean of volatility of the FD $log(S)$ in our case is 5.75, which is close to Moore and Roche's 6.36 and is much closer to the empirical value, 5.09, while the std. dev. of volatility in our case, 0.080, is less than Moore and Roche's 0.232. We also look at one other case apart from the baseline. In $\delta = -0.0025$ of the sensitivity analysis, the mean of volatility in our case is 5.51, which is similar to 5.38 in Moore and Roche (2006) and is much closer to 5.09 of the empirical data and even is better than our baseline's 5.75. At the same time, in $\delta = -0.0025$, the std. dev. of volatility in our case is 0.054, which is less than 0.143 in Moore and Roche (2006). We find that, in the sensitivity analysis, volatility rises if we increase the parameter value of γ and the absolute value of δ, respectively, or decrease the parameter values of ϕ and ρ_π, respectively, which is consistent with what is found in Moore and Roche (2006)[17].

Second, we find that the persistence of our simulated data is approximately consistent with that of the empirical and the theoretical data in Moore and Roche (2006). Our simulated data is a little bit negative autocorrelated while the autocorrelation parameters of the theoretical data in Moore and Roche (2006) are positive or negative. However, both the absolute values approximately tend to be around the value point at 0.02. We also find, but do not report, the consistent high persistence of the exchange rates filtered by using the Hodrick-Prescott filter as in Moore and Roche (2006). Our findings suggest an later application of GARCH class model to conditional volatility. In GARCH specifications, the autoregressive root which governs the persistence of volatility shocks is the sum of the ARCH parameter[18] (α) plus the GARCH parameter (β). When this root is very close to unity volatility shocks are quite persistent and die out rather slowly.

We report the properties of spot returns in Panel B of Table 2. We find the consistent results that are compared to those in Moore and Roche (2006). We also find that the statistics of spot returns in Panel B are the same as those of the first-differenced exchange rates in Panel A due to the same log first-order difference process.

In general, we (approximately) replicate the results of Moore and Roche (2006). The data

[16] We generate log surplus consumption ratio (x), money growth ($\triangle m$) and consumption growth ($\triangle c$) for 232 observations and discard the first 100 observations for each series in the baseline and sensitivity analysis, respectively.

[17] Moore and Roche (2006) report the same mean of volatility in the baseline and $\rho_\pi = 0$, which is still consistent with our results.

[18] See notes of the methodology in the working paper of forecasting for notations of GARCH class conditional volatility models.

Panel A
Properties of exchange rates in first-differenced data

	Moore and Roche (2006)						Liu (2007)				
	Empirical Data	**Simulated Data**									
		Baseline	γ=0.7	δ=-0.0025	ρ(π)=0	φ=0.995	Baseline	γ=0.7	δ=-0.0025	ρ(π)=0	φ=0.995
Volatility (%)											
Mean	5.09	6.36	7.42	5.38	6.36	8.48	5.75	7.59	5.51	6.52	8.86
Std.Dev	1.705	0.232	0.203	0.143	0.232	0.233	0.080	0.070	0.054	0.087	0.081
Persistence											
Mean	0.07	0.02	0.01	0.02	-0.02	-0.01	-0.04	-0.02	-0.02	-0.05	-0.03
Std. Dev	0.051	0.004	0.004	0.004	0.004	0.004	0.158	0.135	0.134	0.156	0.130

Panel B
Properties of spot returns

	Moore and Roche (2006)						Liu (2007)				
	Empirical Data	**Simulated Data**									
		Baseline	γ=0.7	δ=-0.0025	ρ(π)=0	φ=0.995	Baseline	γ=0.7	δ=-0.0025	ρ(π)=0	φ=0.995
Volatility (%)											
Mean	5.09	6.36	7.42	5.38	6.36	8.48	5.75	7.59	5.51	6.52	8.86
Std. Dev	1.705	0.232	0.203	0.143	0.232	0.233	0.080	0.070	0.054	0.087	0.081
Persistence											
Mean	0.07	0.02	0.01	0.02	-0.02	-0.01	-0.04	-0.02	-0.02	-0.05	-0.03
Std. Dev	0.051	0.004	0.004	0.004	0.004	0.004	0.158	0.135	0.134	0.156	0.130

Notes: Empirical data refers to nine real exchange rates (CAD/USD, GBP/USD, JPY/USD, CHF/USD, EUR/USD, CAD/EUR, GBP/EUR, JPY/EUR, and CHF/EUR) obtained from DataStream International in Moore and Roche (2006). See Moore and Roche's table 2 for notes to empirical data. Statistics of empirical data are calculated basing on the displayed numbers in Moore and Roche (2006)'s table 3 and 4. The mean of empirical data is the average of nine exchange rates. The Std. Dev of empirical data is the statistic of the series of nine exchange rates. The statistics of simulated data of Moore and Roche (2006) are quoted directly from Moore and Roche (2006)'s table 9 and 10. The statistics of our simulated data are given under the title of "Liu (2007)". The standard deviation (std dev for short) reported in theoretical economy are the averages from the 1000 simulations. Volatility is measured by the std dev and persistence is measured by the first-order autocorrelation coefficient.

Table 2: Statistics of empirical data vs. theoretical data

we simulated meet Moore and Roche's statistical criteria[19] and have the (approximately) same time series properties as in Moore and Roche (2006). Moore and Roche use the moment expressions in Equation (4.08)[20] to provide some intuition to their simulated results. The volatility of the fundamentals is able to explain the volatility in the nominal exchange rates.

Simulation 2

We simulated the model once for 10000 observations to investigate the model's implied properties: Δs_t is an ARMA(2,2); Δs_t^f is an MA(2); Δs_t and its innovation ζ are conditional heteroscedastic. The time series of spot returns is constructed by using the simulated 10000 quarterly exchange rates S. We investigate the time series of spot returns to the model's implied properties. Innovations of spot returns are subject to conditional volatility. In the baseline, all initial values and parameterization are the same as in Simulation 1 using Table 1. We repeat this exercise for each of the sensitivity variants to see the implications that this has on the conditional volatility of the model. The testing results are reported in Table 3-5.

$$\Delta s_t^f = c_0 + \theta_1 \zeta_{t-1} + \theta_2 \zeta_{t-2}$$

		Sensitivity analysis			
	Baseline	$\gamma = 0.7$	$\delta = -0.0025$	$\rho(\pi) = 0$	$\phi = 0.995$
Panel A: parameter estimates					
c_0	-8.44E-07	-1.61E-06	8.53E-07	4.76E-07	1.24E-06
	(-0.319)	(-0.443)	(0.395)	(0.194)	(0.347)
θ_1	-0.7852*	-0.8088*	-1.1354*	-0.7173*	-0.7851*
	(-84.638)	(-85.936)	(-114.597)	(-82.015)	(-84.469)
θ_2	-0.2112*	-0.1878*	0.1376*	-0.2792*	-0.2114*
	(-22.775)	(-19.940)	(13.884)	(-31.926)	(-22.743)
Panel B: adjusted R-squared ($\overline{R}^2$)					
Lag 1-2	0.50	0.51	0.57	0.45	0.50
Lag 3-10	0.011	0.002	0.006	0.011	0.012

Notes: In Panel A, t-ratios in parentheses; * denotes significance at the 1% level; in Panel B, adjusted R-squared is generated by regressing the filtered spot returns on the lags 1-2 and on the lags 3-10 of the conditional heteroscedastic errors, respectively.

Table 3: The filtered spot returns of an MA(2) process

Table 3 shows that Δs_t^f is an MA(2) in both the baseline and sensitivity analysis. The estimates of the constant, θ_1, and θ_2 in Equation (4.15) are displayed in Panel A of Table 3. All the coefficients of θ_1 and θ_2 are highly significant (at 1%) with the non-significant constants in both the baseline and sensitivity analysis. Furthermore, in order to prove that Δs_t^f is an MA(2) we examine if the regressions have the zero adjusted r-squares when regressing the filtered series Δs_t^f on the lags greater than 2. Panel B of Table 3 displays that the conditional heteroscedastic error ζ is able to explain approximately 50% variability in $\triangle s_t^f$ on the first two lags for autocorrelations while the adjusted r-squares are zero on those lags greater than 2 (from 3rd to 10th). The time series of the filtered spot returns Δs_t^f is indeed an MA(2).

In Table 4, we report that the time series of spot returns Δs_t is an ARMA(2,2) in both the baseline and sensitivity analysis. In Panel A of Table 4, almost all parameter estimates on the

[19] We thank Prof. Michael Moore and Dr. Maurice Roche for their kind help. We check our simulations by comparing their statistics with ours. We report the relevant statistics of the simulations: 1) the std. dev. of $log\ X_{home}$ ($log\ X_{foreign}$) is 30.27% (27.24%) while Moore and Roche's std. dev. of $logX$ is about 25%; 2) the std. dev. of the change in $log\ X_{home}$ ($log\ X_{foreign}$) is 7.63% (6.82%) while Moore and Roche's std. dev. of the change in $logX$ is be about 6.5%; 3) the std. dev. of (home and foreign) consumption growth is 0.75% while Moore and Roche's std. dev. of consumption growth is 0.75%. It is exogenous. Moore and Roche suggest that the std. dev. of the change in log surplus consumption is at least 10 time that of consumption growth; 4) the log of the surplus consumption ratio is always negative and the level of the surplus consumption ratio is always non-negative, which is consistent with Moore and Roche (2006) and Campbell and Cochrane (1999). The reason is "in the continuous-time limit, the x_t process never attains the region $x > x_{\max}$". So the log surplus consumption ratio is always negative since $x \prec x_{\max} = -2.4858$ as in Table 1.

[20] The expressions in Equation (4.08) are approximations based on Moore and Roche's equation (A14).

$$\Delta s_t = b_0 + b_1 \Delta s_{t-1} + b_2 \Delta s_{t-2} + b_3 \zeta_{t-1} + b_4 \zeta_{t-2}$$

		Sensitivity analysis			
	Baseline	$\gamma = 0.7$	$\delta = -0.0025$	$\rho(\pi) = 0$	$\phi = 0.995$
Panel A: parameter estimates					
b_0	1.88E-05	-0.0002	0.0006	-2.84E-05	-5.25E-05
	(0.03)	(-0.22)	(0.72)	(-0.04)	(-0.06)
b_1	-0.407**	0.236	0.310*	-0.526*	-0.856*
	(-2.34)	(0.47)	(42.27)	(-12.49)	(-16.02)
b_2	0.065*	-0.081	-0.977*	0.424*	-0.800*
	(2.62)	(-0.57)	(-135.41)	(10.09)	(-15.46)
b_3	0.426*	-0.241	-0.322*	0.619*	0.854*
	(2.45)	(-0.48)	(-35.999)	(13.88)	(14.80)
b_4	-0.092*	0.098	0.964*	-0.299*	0.760*
	(-3.54)	(0.68)	(108.94)	(-6.74)	(13.51)
Panel B: autocorrelation function - Ljung-Box Q-statistics					
Lag 5	21.92*	1.25	31.58*	39.29*	12.24*
Lag 6	23.10*	17.42*	31.84*	52.32*	27.26*
Lag 7	43.50*	34.13*	33.80*	79.02*	35.45*
Lag 8	94.06*	35.04*	34.34*	86.73*	35.63*
Lag 9	94.09*	35.64*	36.81*	87.00*	42.92*
Lag 10	96.27*	62.53*	38.01*	87.03*	45.35*
Lag 11	125.95*	63.84*	41.10*	108.99*	58.48*
Lag 12	128.56*	65.08*	41.10*	125.11*	60.56*
Lag 13	128.74*	65.16*	52.52*	126.48*	60.57*
Lag 14	131.16*	80.14*	54.17*	152.42*	69.87*
Lag 15	170.39*	84.86*	60.11*	156.55*	98.98*
Lag 16	172.84*	85.65*	60.25*	156.77*	102.52*
Lag 17	175.52*	85.66*	68.44*	168.16*	106.75*
Lag 18	178.91*	89.22*	97.16*	203.27*	106.77*
Lag 19	180.06*	90.18*	100.20*	210.43*	143.29*
Lag 20	203.46*	90.19*	105.38*	229.71*	144.66*
Lag 21	208.70*	108.18*	107.90*	231.47*	146.22*
Lag 22	211.01*	120.63*	129.35*	231.98*	171.55*
Lag 23	224.30*	124.49*	135.93*	272.40*	183.62*
Lag 24	250.72*	124.50*	136.31*	289.97*	183.70*
Lag 25	276.47*	127.92*	136.40*	317.54*	231.46*
Lag 26	282.12*	135.13*	136.53*	330.41*	232.76*
Lag 27	291.42*	154.12*	142.92*	332.55*	237.85*
Lag 28	306.40*	156.72*	163.95*	338.43*	247.68*
Lag 29	307.41*	158.20*	164.29*	339.59*	247.70*
Lag 30	308.27*	159.22*	168.17*	357.51*	261.98*
Lag 31	347.08*	159.97*	168.27*	386.10*	270.27*
Lag 32	379.88*	166.66*	169.24*	419.37*	287.26*
Lag 33	380.04*	170.14*	189.73*	420.40*	287.27*
Lag 34	425.18*	181.09*	190.07*	423.04*	350.75*
Lag 35	431.59*	181.20*	191.49*	453.34*	351.61*
Lag 36	431.62*	181.88*	192.77*	461.11*	355.35*

Notes: t-ratios in parentheses; * and ** denote significance at the 1% and 5% levels, respectively; Q-statistic probabilities are adjusted for 4 ARMA term(s).

Table 4: The theoretical spot returns of an ARMA(2,2) process

AR(1), AR(2), MA(1), and MA(2) terms of the ARMA(2,2) model for Δs_t, except those in $\gamma = 0.7$, are significant while all constants are not significant. We use the Ljung-Box test for autocorrelations and partial autocorrelations of the equation residuals. In Panel B of Table 4, the Ljung-Box Q-statistics and their p-values of the correlogram at each lag are highly significant up to 36 lags of the specified order. The Ljung-Box test statistic rejects the null hypothesis of no autocorrelation at the 1% significance level for almost all lags except the lag 5 in $\gamma = 0.7$. The time series of the spot returns Δs_t is indeed an ARMA(2,2).

5.2 Conditional heteroscedasticity

The identification of conditional heteroscedasticity is often based on testing whether squared or absolute returns are autocorrelated[21]. We use the ARCH-LM test[22] to test if the simulated spot return itself (without an ARMA process) and the simulated spot return of an ARMA(2,2) process from which the innovation ($\widehat{\zeta}_t$) in Equation (4.16) comes are conditionally heteroscedastic. In details, we regress the squared residuals of the simulated spot return itself and the squared innovations of the ARMA(2,2) spot returns on a constant and lagged squared residuals up to order 9, respectively. We quote the value and significance of the test. In Table 5, it is showed that both the F-statistic and χ^2-statistic of the test in Panel A and B are very significant (at 1%) in both the baseline and sensitivity analysis, respectively, suggesting the presence of ARCH in the simulated spot returns. The simulated data for spot returns and for the innovation in the spot returns process are definitely conditionally heteroscedastic.

ARCH-LM heteroscedasticity test		Sensitivity analysis			
	Baseline	$\gamma = 0.7$	$\delta = -0.0025$	$\rho(\pi) = 0$	$\phi = 0.995$
Panel A: the simulated spot returns					
F-statistic	375.289*	362.624*	416.187*	513.570*	298.724*
χ^2-statistic	2526.07*	2461.83*	2726.23*	3162.22*	2120.08*
Panel B: the simulated spot returns of an ARMA (2,2)					
F-statistic	372.800*	362.008*	424.955*	417.074*	293.176*
χ^2-statistic	2513.41*	2458.55*	2767.60*	2730.30*	2088.85*

Notes: * denotes significance at the 1% level.

Table 5: Testing conditional heteroscedasticity for the theoretical spot returns

We have numerically solved Moore and Roche's model (Equation (4.06)) and simulated the artificial data which have the same time series properties not only as those found in Moore and Roche (2006) but also as those implied by the theoretical model. We have also assessed the sensitivity of the results to parameter changes. The theoretical model is able to explain conditional volatility of exchange rates. We find ARCH effects in the simulated spot returns where the simulated data for spot return and for innovation in the spot return process are definitely conditionally heteroscedastic. We will treat the simulated data and its derivatives as though they were real world data and apply GARCH class models directly to their innovations[23] ($\widehat{\zeta}_t$) that are subject to conditional volatility. We are going to establish the exact dynamic form conditionally heteroscedasticity takes and see whether or not the dynamics match those from the actual monthly data in the rest parts of the paper.

[21] "A POWERFUL TEST FOR CONDITIONAL HETEROSCEDASTICITY FOR FINANCIAL TIME SERIES WITH HIGHLY PERSISTENT VOLATILITIES" by Rodríguez and Ruiz (2003).

[22] The ARCH test is a Lagrange multiplier (LM) test for autoregressive conditional heteroskedasticity (ARCH) in the residuals (Engle 1982). The test can also be thought of as a test for autocorrelation in the squared residuals. As well as testing the residuals of an estimated model, the ARCH test is frequently applied to raw returns data.

[23] Innovations $\widehat{\zeta}_t$ in Equation (4.16) are available after estimating Δs_t^f of an MA(2) in Equation (4.15) using the simulated S in Equation (4.06).

6 The form of conditional heteroscedasticity

We estimate the form of the conditional heteroscedasticity which is implied in the Moore and Roche's model. Generalised autoregressive conditionally heteroscedastic (GARCH) class models are used to capture conditional volatility. Specifically, we employ symmetric (ARCH, GARCH and GARCH-M[24]) and asymmetric (TARCH/GJR, EGARCH, PARCH and CGARCH) conditional volatility models for the best estimates. We present impulse response function (IRF) for the best GARCH models for the ARCH processes we estimate. We then draw the IRFs to establish the exact dynamic form the conditional heteroscedasticity takes. After comparing the IRFs, we show that the two IRFs from the simulated quarterly and from the empirical monthly data "look the same" with an approximately monotonic decreasing fashion. We conclude that the Lucas two-country monetary model with habit is capable of producing the same kind of ARCH features as we see in the real data. For simplicity, we call the Lucas two-country monetary model with habit in Moore and Roche (2006) as "the theoretical model", the GARCH class conditional volatility models as "the empirical model", the data simulated by using the theoretical model as "the theoretical data", and the spot USD/GBP exchange rates collected from Thomson Datastream as "the empirical data". Specifically, the empirical spot returns consist of the empirical monthly spot return[25] which is obtained from monthly averages of daily spot rates in that month and the empirical quarterly spot return which is obtained from quarterly point spot rates[26] in time.

6.1 Estimation of GARCH models

We estimate the best fit GARCH model(s) for the time series of the theoretical (simulated) spot return Δs_t and for the time series of the empirical (real) monthly and quarterly USD/GBP spot returns, respectively, in terms of significance of coefficients, asymmetric effects and persistent shocks as well as the relationship of return with risk. We then do the same for the innovation $\widehat{\zeta_t}$ in the time series representation for Δs_t that is an ARMA(2,2) and the residuals in the MA(1) process for the empirical monthly spot return and in the ARMA(2,3) process for the empirical quarterly spot return, respectively. The first exercise is a misspecification if the Moore and Roche's model is "true (i.e. under the null of Moore and Roche (2006)). But the reason for considering the FOREX spot return itself in the first exercise is simply because this is exactly what empirical researchers tend to do. Also we could argue that the persistence properties in the FOREX spot return at the quarterly frequency are rather weak so that modelling ARCH effects in Δs itself rather than the innovation in the time series model for Δs is a minor misspecification. This exercise also ties in closely with the results in the previous forecasting paper in which we find the best fit forecasting model for the spot return rather than its innovation.

6.1.1 Theoretical estimation

We found the presence of ARCH effects in the theoretical spot return Δs_t using Engle (1982) test as we did in Section 5. It suggests that the GARCH class models are appropriate for the theoretical data. In Table 6 and 7, we report value and significance of the estimates of the GARCH(1,1)[27] class conditional volatility models for the theoretical data in baseline.

[24] With the same conditional variance equation as in the GARCH model, the GARCH-in-mean (GARCH-M for short) model has a different conditional mean equation where the conditional variance of asset returns enters into the conditional mean equation, for example, $y_t = c + bx_t + dh_t + \tau_t$ where $\tau_t \sim N(0, h_t)$, which says that the return is partly determined by its risk. The GARCH-M model is often used in financial applications where the expected return on an asset is related to the expected asset risk. The estimated coefficient on the expected risk is a measure of the risk-return tradeoff. In empirical applications, the conditional variance term, h_t^2, appears directly in the conditional mean equation, rather than in square root form h_t (p480, Brooks (2002)). See the working paper of forecasting for the details of the other GARCH class models.

[25] We use the same empirical monthly data set as those employed in the working paper for forecasts. The time series of the empirical monthly spot return is constructed by using the daily spot USD/GBP exchange rate, where the average of daily prices in that month (quarter) as the proxy of monthly (quarterly) price. The empirical monthly data spans the period from 1973 to 2005, which is the same as in Moore and Roche (2006) in terms of the spot GBP/USD exchange rate. See the forecasting working paper for the details.

[26] The empirical quarterly point spot rates of USD/GBP in time cover the sample period from 1973:1 to 2005:4, which is the same as in Moore and Roche (2006) in terms of the spot GBP/USD exchange rate.

[27] We found that the GARCH(1,1) class models have better estimating performances than the GARCH(p,q) models with higher orders ($1 < p \leqq 9$, $1 < q \leqq 9$) when we estimated the GARCH(9,9) models then removed insignificant lags one at a time (re-estimating each time).

Mean Equation

$$\Delta s_t = c + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + dh_t + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 \qquad \text{ARCH(1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)-M}$$

$$\ln(h_t^2) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln(h_{t-1}^2) + \eta\frac{\tau_{t-1}}{h_{t-1}} \qquad \text{EGARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1} \text{ where } I_t = 1 \text{ if } \tau_t < 0 \text{ and 0 otherwise} \qquad \text{TARCH(1,1)}$$

$$h_t^{\vartheta} = \omega + \alpha\left(|\tau_{t-1}| - \upsilon\tau_{t-1}\right)^{\vartheta} + \beta h_{t-1}^{\vartheta} \qquad \text{PARCH(1,1)}$$

$$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right) \qquad \text{CGARCH(1,1)}$$

Estimating model	c	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
ARCH(1)	0.00049	-	0.0000912*	15.077*	-	-	-	-	-	-	-	-
	(0.74)	-	(4.29)	(7.15)	-	-	-	-	-	-	-	-
GARCH(1,1)	0.00016*	-	0.0000027*	0.578*	0.420*	-	-	-	-	-	-	-
	(3.46)	-	(7.99)	(15.80)	(11.48)	-	-	-	-	-	-	-
GARCH(1,1)-M	0.00048*	-0.045	0.0000023*	0.447*	0.551*	-	-	-	-	-	-	-
	(5.23)	(-1.37)	(10.09)	(23.59)	(29.05)	-	-	-	-	-	-	-
EGARCH(1,1)	0.000006	-	-6.054*	1.017***	0.164	-0.016	-	-	-	-	-	-
	(0.01)	-	(-3.31)	(1.95)	(0.64)	(-0.17)	-	-	-	-	-	-
TARCH(1,1)	0.00025*	-	0.0000029*	1.213*	0.398*	-	0.193	-	-	-	-	-
	(4.80)	-	(675.74)	(11.76)	(21.99)	-	(1.18)	-	-	-	-	-
PARCH(1,1)	0.00029*	-	0.005**	0.537*	0.605*	-	-	-0.019	0.644*	-	-	-
	(6.08)	-	(2.39)	(12.27)	(27.96)	-	-	(-0.38)	(7.47)	-	-	-
CGARCH(1,1)	0.00026*	-	0.002	-0.038	0.514**	-	-	-	-	0.999*	0.445*	0.030
	(5.67)	-	(0.04)	(-1.03)	(2.28)	-	-	-	-	(32.32)	(11.82)	(0.76)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.

Table 6: Estimates of GARCH class conditional volatility models for the theoretical quarterly spot return itself in baseline

Mean Equation

$$\Delta s_t = c + c_1\Delta s_{t-1} + c_2\Delta s_{t-2} + c_3\tau_{t-1} + c_4\tau_{t-2} + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + c_1\Delta s_{t-1} + c_2\Delta s_{t-2} + c_3\tau_{t-1} + c_4\tau_{t-2} + dh_t + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 \qquad \text{ARCH(1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)-M}$$

$$\ln(h_t^2) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln(h_{t-1}^2) + \eta\frac{\tau_{t-1}}{h_{t-1}} \qquad \text{EGARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1} \text{ where } I_t = 1 \text{ if } \tau_t < 0 \text{ and 0 otherwise} \qquad \text{TARCH(1,1)}$$

$$h_t^{\vartheta} = \omega + \alpha\left(|\tau_{t-1}| - \upsilon\tau_{t-1}\right)^{\vartheta} + \beta h_{t-1}^{\vartheta} \qquad \text{PARCH(1,1)}$$

$$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right) \qquad \text{CGARCH(1,1)}$$

Estimating model	c	c_1	c_2	c_3	c_4	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
ARMA(2,2)-ARCH(1)	-0.00026	-0.825*	0.015	0.903*	0.032	-	0.0000117*	38.372*	-	-	-	-	-	-	-	-
	(-1.07)	(-2.81)	(0.16)	(2.93)	(0.21)	-	(244.33)	(3.43)	-	-	-	-	-	-	-	-
ARMA(2,2)-GARCH(1,1)	0.00017*	1.203*	-0.218*	-1.401*	0.412*	-	0.0000025*	0.463*	0.535*	-	-	-	-	-	-	-
	(5.30)	(21.25)	(-4.19)	(-24.66)	(7.77)	-	(10.08)	(22.49)	(25.92)	-	-	-	-	-	-	-
ARMA(2,2)-GARCH(1,1)-M	-0.00039	1.382*	-0.535*	-1.319*	0.531*	-0.049	0.0000037*	0.711*	0.287*	-	-	-	-	-	-	-
	(-0.29)	(12.66)	(-6.30)	(-13.23)	(7.21)	(-0.63)	(5.89)	(7.99)	(3.22)	-	-	-	-	-	-	-
ARMA(2,2)-EGARCH(1,1)	0.00073*	0.998*	-0.103***	-1.121*	0.218*	-	-1.099*	0.948*	0.943*	-0.049*	-	-	-	-	-	-
	(11.57)	(11.13)	(-1.66)	(-12.84)	(3.53)	-	(-20.56)	(38.36)	(182.28)	(-3.43)	-	-	-	-	-	-
ARMA(2,2)-TARCH(1,1)	0.00019*	-0.303	0.068**	0.173	-0.171**	-	0.0000028*	1.171*	0.401*	-	0.186	-	-	-	-	-
	(6.45)	(-0.80)	(2.31)	(0.45)	(-2.12)	-	(9.93)	(12.96)	(22.13)	-	(1.31)	-	-	-	-	-
ARMA(2,2)-PARCH(1,1)	-0.0000048	-0.735*	0.112*	0.730*	-0.116*	-	0.004*	0.526*	0.617*	-	-	-0.026	0.744*	-	-	-
	(-0.13)	(-8.29)	(3.14)	(8.85)	(-2.72)	-	(2.79)	(13.41)	(29.04)	-	-	(-0.55)	(10.70)	-	-	-
ARMA(2,2)-CGARCH(1,1)	-0.004*	0.889*	0.115	-0.869*	-0.093	-	0.000068*	0.097*	0.623*	-	-	-	-	0.957*	0.152*	0.234*
	(-6.22)	(12.13)	(1.57)	(-12.18)	(-1.34)	-	(49467.35)	(5.40)	(32.80)	-	-	-	-	(256.20)	(18.48)	(9.30)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.

Table 7: Estimates of GARCH class conditional volatility models for the theoretical quarterly spot return of an ARMA(2,2) in baseline

We report the results of estimating GARCH class models for the simulated Δs_t series itself in baseline in Table 6. For all specifications of the GARCH class models we employ, the coefficients on the lagged squared error (ARCH) term in the conditional variance equation are statistically significant except the non-significant ARCH term in the CGARCH(1,1) model while the coefficients on the lagged conditional variance (GARCH) term are statistically significant except the one in the EGARCH(1,1) model. The asymmetry terms (η, μ, υ and γ) in the EGARCH, TARCH, PARCH and CGARCH models respectively are not significant. The sum of the ARCH and GARCH coefficients for the GARCH, GARCH-M, EGARCH and PARCH models respectively is (approximately) close to unity, which implies that shocks to conditional variance will be (highly) persistent. This can be found by using the models to forecast future values of the conditional variance for the real USD/GBP spot returns as in the previous paper. A large sum of these coefficients (e.g. the TARCH model) will imply that a large positive or a large negative return will lead future forecasts of the variance to be high for a protracted period. The variance intercept terms (ω) are significant except the non-significant one in the CGARCH model, where the variance intercept terms (ω) in the ARCH, GARCH, GARCH-M and TARCH models are very small, while the coefficients on the significant GARCH terms are larger ($\gtrapprox 0.4$). The conditional standard deviation term that is introduced into the mean equation of the GARCH-M model is not significant, which suggests that the property that higher market-wide risk would lead to higher returns is not available. We find but do not report the results of additional ARCH effects up to the order 9 in the residuals after the model estimates. The presence of additional ARCH is in the residuals of the estimated models except the PARCH and CGARCH models.

Next, we report the estimating results of the GARCH class models for the innovation for Δs_t that is an ARMA(2,2) in baseline in Table 7. For all cases, the coefficient estimates on the variance intercept, ARCH and GARCH terms in the conditional variance equation are highly statistically significant (at 1%). Also, the coefficient estimates on the asymmetry terms (η and γ) in the EGARCH and CGARCH models are highly statistically significant (at 1%), which suggests, as expected, that negative shocks imply a higher next period conditional variance than positive shocks of the same magnitude. The persistence of volatility shocks is found for the GARCH, GARCH-M, and PARCH models due to the sum of the ARCH and GARCH coefficients close to unity which is compared to a large sum ($\gtrapprox 1.6$) of these coefficients for the EGARCH and TARCH models and a small sum (≈ 0.7) for the CGARCH model. We find the presence of the additional ARCH effects in the residuals for the EGARCH and CGARCH models. Again, the conditional standard deviation term in the mean equation of the GARCH-M model is not significant.

It is noted that EGARCH is the best fit model to additional ARCH effects due to the presence of ARCH effects in the residuals for both the simulated Δs_t and its innovation. EGARCH and CGARCH are the best fit models to asymmetry effects because of the significant asymmetric terms in the conditional variance equation as shown in Table 7. PARCH is the best fit asymmetric model to persistent shocks in terms of the sum of the ARCH and GARCH coefficients close to unity as found in both Table 6 and 7. Taking account of the properties of conditional volatility that is not only conditional but also asymmetric, the asymmetric CGARCH, EGARCH and PARCH conditional volatility models are the best fit estimating models for the simulated data. Further details of the selection process are not reported.

We also look at the cases apart from the baseline and assess sensitivity to parameter changes. We find but do not report the results of ARCH properties of the significant coefficients, asymmetry effects, persistent shocks, additional ARCH, and non-significant conditional variance terms in the mean equation of the GARCH-M models in the sensitivity analysis. It is found consistent with what we find in the baseline. As well as we take a look at, but do not report, the results of estimates and additional ARCH for the other models[28] and for steps in model selection process. For other cases, we find not only consistent results but also more information compared to those disclosed in Table 6 and 7. For example, the conditional variance term introduced into the mean equation for the GARCH-M model has a positive sign and is highly significant (at 1%). This suggests that higher market-wide risk, proxied by the conditional variance, will lead to higher returns. Thus the parameter (d) of the conditional variance in the mean equation of the GARCH-M model can be interpreted as a risk premium. The theoretical model is able to capture the relationship between return and risk where return is partly determined by risk.

[28] We find but do not report the best fit GARCH estimates for the simulated filtered series Δs_t^f in Equation (4.15) and the simulated genuine innovation $\widehat{\zeta}_t$ in Equation (4.16). Our aim is to maximally capture ARCH basing on the model properties even its implied properties.

We summarize the findings in both the baseline and sensitivity analysis as well as the results for the other relevant models we are interested in. Basing on the analysis and summary, the main conclusions are given as follows:

- GARCH class conditional volatility models are appropriate for the theoretical quarterly FOREX data in which the predictable properties of conditional volatility are found.
- The presence of additional ARCH effects is in the residuals of the estimated standard GARCH class models.
- In symmetric conditional volatility models, GARCH(1,1) is the best fit estimating model to conditional volatility for the theoretical spot return and its innovation.
- In asymmetric conditional volatility models, CGARCH, EGARCH and PARCH are the best fit estimating models to conditional volatility for the theoretical spot return and its innovation in terms of the properties of asymmetry, additional ARCH, and persistent volatility shock, respectively.
- In the sensitivity analysis, more significant results on ARCH effects, asymmetric effects and persistent volatility shocks to conditional volatility are found than those disclosed in the baseline. $\gamma = 0.7$ has the superior performance to others in the sensitivity analysis.
- The theoretical model can generate realistic conditional volatility even asymmetry conditional volatility.

As stated above, the consistent estimating results suggest that the asymmetric CGARCH, EGARCH and PARCH conditional volatility models are the best fit GARCH models to conditional volatility for the theoretical data.

6.1.2 Empirical estimation

We turn to focus on the actual data. As known, the GARCH class conditional volatility models are appropriate for the empirical monthly spot return[29] of the USD/GBP exchange rate. We report the model estimates for the monthly USD/GBP spot return itself in Table 8. For all specifications, the coefficients on the lagged squared error (ARCH) term in the conditional variance equation are highly statistically significant (at 1%) except the non-significant ARCH term in the CGARCH(1,1) model. All coefficients on the lagged conditional variance (GARCH) term are highly statistically significant (at 1%). The asymmetry term (γ) in the conditional variance equation of the CGARCH model is very significant with a positive sign, suggesting that negative shocks imply a higher next period conditional variance than positive shocks of the same magnitude. The sum of the coefficients on the ARCH and GARCH terms of the PARCH model compared with that of these two coefficients of the other models is approximately close to unity (≈ 0.92), which implies that shocks to conditional variance will be persistent. The variance intercept terms (ω) are significant except the non-significant one in the PARCH model, where the variance intercept terms (ω) in ARCH, GARCH, GARCH-M and TARCH are very small. It is found that the conditional variance term in the mean equation of the GARCH-M model is not significant. The presence of additional ARCH is in the residuals of the estimated ARCH model while there is no additional ARCH in the residuals for other estimated GARCH class models[30].

Moreover, we report the estimating results of the GARCH class models for the innovation of the monthly USD/GBP spot return that is of an MA(1) process[31] in Table 9.

[29] In the working paper for forecasting, we find that the time series of the monthly USD/GBP spot return is stationary using a unit root test (ADF test). We compute the Engle (1982) test for ARCH effects to make sure that the GARCH-type models are appropriate for the data. We find the highly significant (at 1%) F-statistic and LM-statistic of the test by regressing the squared residuals on a constant and 9 lags. The presence of ARCH is in the residuals for the monthly USD/GBP return.

[30] See the relevant parts in the forecasting working paper for the further details.

[31] In the working paper for forecasting, we find that the time series of the monthly USD/GBP spot return is an MA(1) using Schwarz's (1978) Bayesian information criterion (SBIC) that is recommended by Diebold (2001). We also report the estimates of the ARMA model where the MA(1) term is highly significant (at 1%) and the results of the Ljung-Box test. See the forecasting working paper for the further details.

Mean Equation

$$\Delta s_t = c + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + dh_t^2 + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$h_t^2 = \omega + \alpha\tau_{t-1}^2$	ARCH(1)
$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2$	GARCH(1,1)
$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2$	GARCH(1,1)-M
$\ln(h_t^2) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln(h_{t-1}^2) + \eta\frac{\tau_{t-1}}{h_{t-1}}$	EGARCH(1,1)
$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1}$ where $I_t = 1$ if $\tau_t < 0$ and 0 otherwise	TARCH(1,1)
$h_t^\vartheta = \omega + \alpha\left(\lvert\tau_{t-1}\rvert - \upsilon\tau_{t-1}\right)^\vartheta + \beta h_{t-1}^\vartheta$	PARCH(1,1)
$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right)$	CGARCH(1,1)

Estimating model	c	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
ARCH(1)	-0.001	-	0.00042*	0.322*	-	-	-	-	-	-	-	-
	(-1.19)	-	(9.46)	(3.03)	-	-	-	-	-	-	-	-
GARCH(1,1)	-0.00057	-	0.00009**	0.259*	0.609*	-	-	-	-	-	-	-
	(-0.62)	-	(2.48)	(3.26)	(5.74)	-	-	-	-	-	-	-
GARCH(1,1)-M	0.00089	-2.609	0.00009**	0.257*	0.607*	-	-	-	-	-	-	-
	(0.48)	(-0.79)	(2.46)	(3.19)	(5.58)	-	-	-	-	-	-	-
EGARCH(1,1)	-0.00040	-	-1.384**	0.392*	0.857*	-0.015	-	-	-	-	-	-
	(-0.38)	-	(-2.49)	(4.31)	(12.44)	(-0.31)	-	-	-	-	-	-
TARCH(1,1)	-0.00063	-	0.00009**	0.215*	0.611*	-	0.065	-	-	-	-	-
	(-0.60)	-	(2.48)	(2.72)	(5.66)	-	(0.64)	-	-	-	-	-
PARCH(1,1)	-0.00023	-	0.007	0.190*	0.732*	-	-	0.005	0.773***	-	-	-
	(-0.22)	-	(0.70)	(3.36)	(8.55)	-	-	(0.03)	(1.94)	-	-	-
CGARCH(1,1)	0.00081	-	0.00060*	-0.072	0.701*	-	-	-	-	0.938*	0.089	0.293*
	(0.92)	-	(3.92)	(-0.75)	(6.44)	-	-	-	-	(19.18)	(1.16)	(2.81)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.

The time series of the empirical monthly spot return itself is obtained from monthly averages of daily spot rates in that month.

Table 8: Estimates of GARCH class conditional volatility models for the empirical monthly spot return of USD/GBP itself

Mean Equation

$$\Delta s_t = c + c_1\tau_{t-1} + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + c_1\tau_{t-1} + dh_t^2 + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 \qquad \text{ARCH(1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)-M}$$

$$\ln(h_t^2) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln(h_{t-1}^2) + \eta\frac{\tau_{t-1}}{h_{t-1}} \qquad \text{EGARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1} \text{ where } I_t = 1 \text{ if } \tau_t < 0 \text{ and } 0 \text{ otherwise} \qquad \text{TARCH(1,1)}$$

$$h_t^\vartheta = \omega + \alpha\left(|\tau_{t-1}| - \upsilon\tau_{t-1}\right)^\vartheta + \beta h_{t-1}^\vartheta \qquad \text{PARCH(1,1)}$$

$$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right) \qquad \text{CGARCH(1,1)}$$

Estimating model	c	c_1	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
MA(1)-ARCH(1)	-0.00082	0.410*	-	0.00042*	0.187**	-	-	-	-	-	-	-	-
	(-0.53)	(7.76)	-	(9.07)	(2.04)	-	-	-	-	-	-	-	-
MA(1)-GARCH(1,1)	-0.00078	0.396*	-	0.00006***	0.139*	0.754*	-	-	-	-	-	-	-
	(-0.55)	(8.43)	-	(1.90)	(2.73)	(8.31)	-	-	-	-	-	-	-
MA(1)-GARCH(1,1)-M	0.00033	0.396*	-2.674	0.00006***	0.140*	0.752*	-	-	-	-	-	-	-
	(0.12)	(8.41)	(-0.43)	(1.88)	(2.70)	(8.13)	-	-	-	-	-	-	-
MA(1)-EGARCH(1,1)	-0.00048	0.405*	-	-12.511*	0.226***	-0.622*	-0.002	-	-	-	-	-	-
	(-0.30)	(8.48)	-	(-7.98)	(1.75)	(-3.01)	(-0.03)	-	-	-	-	-	-
MA(1)-TARCH(1,1)	-0.00084	0.396*	-	0.00006***	0.130**	0.754*	-	0.014	-	-	-	-	-
	(-0.57)	(8.38)	-	(1.93)	(2.45)	(8.53)	-	(0.19)	-	-	-	-	-
MA(1)-PARCH(1,1)	-0.00090	0.390*	-	0.00069	0.142*	0.772*	-	-	-0.003	1.344	-	-	-
	(-0.61)	(8.39)	-	(0.32)	(3.16)	(8.77)	-	-	(-0.02)	(1.58)	-	-	-
MA(1)-CGARCH(1,1)	-0.00067	0.385*	-	0.00053*	-0.030	-0.345	-	-	-	-	0.912*	0.107*	0.139
	(-0.48)	(8.11)	-	(4.91)	(-0.40)	(-0.57)	-	-	-	-	(17.02)	(2.80)	(1.08)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.
The time series of the empirical monthly MA(1) spot return is obtained from monthly averages of daily spot rates in that month.

Table 9: Estimates of GARCH class conditional volatility models for the empirical monthly MA(1) spot return of USD/GBP

For all specifications in Table 9, the coefficients on the ARCH, and GARCH terms in the conditional variance equation are statistically significant except neither in the CGARCH(1,1) model. Particularly, all GARCH coefficient estimates except the one in the CGARCH model are highly statistically significant (at 1%). None of the asymmetry terms in the asymmetric models is significant. Also in most cases, the variance intercept terms (ω) are significant except the non-significant intercept term in the PARCH model. The variance intercept terms in ARCH, GARCH, GARCH-M and TARCH are very small. Again, the PARCH model is able to capture persistent volatility shocks due to the sum of its ARCH and GARCH terms approximately close to unity (≈ 0.91) and the conditional variance term in the mean equation of the GARCH-M model is not significant. We find the presence of ARCH effects up to the order 9 in the residuals of the estimated ARCH and EGARCH models using the Engle (1982) test.

So far, we estimated the best fit GARCH models for the spot returns of the theoretical and empirical data and then their innovations that are in the time series representations for the corresponding spot returns of an ARMA process. We compare the results in Table 6-9. On the one hand, the main results in Table 6 and 8 for the spot return itself (without the ARMA process) are that 1) in both Table 6 and 8 only the CGARCH model has the non-significant ARCH coefficient estimates; 2) only the EGARCH model in Table 6 has the non-significant GARCH coefficient estimate; 3) only the CGARCH model in Table 8 has the significant asymmetric coefficient estimate in the variance equation. The CGARCH and EGARCH models in Table 8 have better estimating performances than theirs in Table 6 in terms of significance of the coefficients on the GARCH and asymmetry terms. The estimating results in Table 8 for the empirical spot return itself where the coefficients on the important terms in the variance equation are significant are superior to those in Table 6 for the theoretical Δs_t itself. On the other hand, the main differences of estimating results between Table 7 and 9 for the innovations of the ARMA spot returns are that, in Table 9, the CGARCH model has the nonsignificant coefficient estimates on the ARCH, GARCH and asymmetric terms and the EGARCH model has the insignificant asymmetric term, while the coefficients on these terms are highly statistically significant in Table 7. The CGARCH and EGARCH models in Table 7 have better estimating performances than theirs in Table 9 due to the significant ARCH, GARCH and asymmetric terms in the variance equation. The estimating results in Table 7 for the innovation in the theoretical Δs_t of an ARMA(2,2) process are superior to those in Table 9 for the residual in the MA(1) process for the empirical monthly spot return. It is found in common in Table 6-9 that the PARCH model is the only asymmetric conditional volatility model that captures persistent shock to conditional variance, and the conditional variance term in the mean equation of the GARCH-M model is globally insignificant, and the variance intercept terms in ARCH, GARCH, GARCH-M and TARCH compared to those in CGARCH, EGARCH and PARCH are very small.

We also note that the results in Table 7 are highly similar to those in Table 8 while the results in Table 6 are highly similar to those in Table 9. Specifically, for both Table 7 and 8, all coefficients on the GARCH term are highly statistically significant and asymmetric effects are present where the asymmetric coefficients in the CGARCH model in both tables and in the EGARCH model in Table 7 are highly statistically significant while none of asymmetric terms is significant in both Table 6 and 9. The GARCH coefficients in the EGARCH model in Table 6 and in the CGARCH model in Table 9 are insignificant. It suggests that the estimates of conditional volatility for the innovation in the theoretical quarterly spot return of an ARMA(2,2) process provide highly consistent performance as those for the empirical monthly spot return itself without an ARMA process do. At this moment, both cases make the maximal capture of estimating information of conditional volatility in either theoretical or empirical frames. This, we think, could be one reason why empirical researchers tend to consider the FOREX spot return itself not its innovation as mentioned at the section's beginning. It also answers why we estimated and forecasted volatility for the FOREX spot return rather than its innovation previously.

We estimate but do not report the best fit models for the empirical daily and quarterly spot returns of the USD/GBP exchange rate, where the empirical quarterly spot return is obtained from quarterly averages of daily spot rates in that quarter as the proxy of quarterly prices. We find that the empirical averaging data at a higher (e.g. daily/monthly) frequency is able to provide highly similar properties of conditional volatility to those implied in the theoretical time point data at a low (e.g. quarterly) frequency. In the theoretical framework, modelling ARCH effects in the innovation that is in the time series presentation for the theoretical spot return is able to explain realistic conditional volatility even if the theoretical data is at a low frequency. In the empirical framework, realistic conditional volatility could be well captured by using empirical high(er) frequency data for spot return due to "noise" or "imperfection" in the real world. "Noise" mentioned here could be crises, noisy traders, momentum, psychology, central bank intervention and macroeconomic variables etc. in daily life. "Noise" makes empirical data at the same frequency as theoretical data is imprecise so as to lose or change partial information. Hence, information obtained by using the empirical high frequency data is able to match that implied by using the theoretical low frequency data. This is why we find the consistent results as in Table 7 and 8. It is also a reasonable solution that using empirical mixed data with a diverse (e.g. from low to high) frequency captures conditional volatility as theoretical data implies.

After comparing the empirical monthly averaging data to the theoretical quarterly time point data, we also look at the estimating results for the empirical quarterly spot return that is constructed by using quarterly USD/GBP point spot rates in time. Using the unit root test and Engle (1982) test, we find that the GARCH class models are appropriate for the time series of the quarterly spot return that is stationary. The estimating results for the empirical quarterly point data are (more or less) similar to the results as found consistent for the theoretical quarterly point data and the empirical monthly averaging data.

We report the model estimates for the quarterly USD/GBP spot return itself in Table 10. The coefficients on the lagged squared error (ARCH) term in the conditional variance equation of the GARCH-M, EGARCH and CGARCH models are statistically significant while other ARCH coefficient estimates are not significant. For all specifications, the coefficients on the lagged conditional variance (GARCH) term are highly statistically significant (at 1%) except the non-significant GARCH term in the CGARCH(1,1) model. The conditional variance term that appears in the mean equation of the GARCH-M model has a negative sign and is significant at the 10% level, which suggests that higher market-wide risk, proxied by the conditional variance, will lead to lower returns. All estimating results mentioned above are distinguished from those in Table 6 and 8. The asymmetry term in the conditional variance equation of the CGARCH model is very significant with a positive sign, which is the same as that in Table 8. The variance intercept terms in the GARCH, TARCH and PARCH are not significant, which is slightly similar to that in Table 8 where only the variance intercept term in the PARCH model is not significant. It is common that the variance intercept terms in GARCH, GARCH-M and TARCH are very small and only the PARCH model has the sum of the ARCH and GARCH coefficients close to unity to capture persistent shocks to conditional variance as in Table 6 and 8. Generally, the estimating results for the empirical quarterly USD/GBP spot return itself in Table 10 have slightly more similarities to those in Table 8 for the empirical monthly USD/GBP spot return itself than those in Table 6 for the theoretical quarterly spot return itself.

We report the estimating results of the GARCH class models for the innovation of the quarterly USD/GBP spot return that is of an ARMA(2,3) process[32] in Table 11. In Table 11, the coefficients on the ARCH, and GARCH terms in the conditional variance equation are statistically significant except both the non-significant ARCH and GARCH terms in the CGARCH(1,1) model and the non-significant ARCH term in the ARCH model. Particularly, all GARCH coefficient estimates except the one in the CGARCH model are highly statistically significant (at 1%), which is the same as in Table 9. The asymmetry terms (η, μ and γ) in the asymmetric EGARCH, TARCH and CGARCH models are significant, which is highly similar to the asymmetric estimates in Table 7. The EGARCH, PARCH and TARCH models capture persistent volatility shocks due to the sum of their ARCH and GARCH terms approximately close to unity, respectively. The conditional variance term in the mean equation of the GARCH-M model has a negative sign and is significant at the 5% level, suggesting that higher market-wide risk will lead to lower returns. The variance intercept terms only in the ARCH, GARCH-M and CGARCH models are significant. The variance intercept terms in the GARCH, GARCH-M and TARCH are globally small as found in Table 6-11. Generally, the estimating results for the empirical quarterly USD/GBP ARMA(2,3) spot return in Table 11 have slightly more similarities to those in Table 7 for the theoretical quarterly ARMA(2,2) spot return than those in Table 9 for the empirical monthly USD/GBP MA(1) spot return.

[32]We find that the time series of the quarterly USD/GBP spot return constructed from quarterly point spot rates in time is an ARMA(2,3) using Schwarz's (1978) Bayesian information criterion (SBIC) that is recommended by Diebold (2001).

Mean Equation

$$\Delta s_t = c + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + dh_t^2 + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 \qquad \text{ARCH(1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)-M}$$

$$\ln(h_t^2) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln(h_{t-1}^2) + \eta\frac{\tau_{t-1}}{h_{t-1}} \qquad \text{EGARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1} \text{ where } I_t = 1 \text{ if } \tau_t < 0 \text{ and 0 otherwise} \qquad \text{TARCH(1,1)}$$

$$h_t^\vartheta = \omega + \alpha\left(|\tau_{t-1}| - \upsilon\tau_{t-1}\right)^\vartheta + \beta h_{t-1}^\vartheta \qquad \text{PARCH(1,1)}$$

$$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right) \qquad \text{CGARCH(1,1)}$$

Estimating model	c	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
ARCH(1)	-0.00009	-	0.002*	0.153	-	-	-	-	-	-	-	-
	(-0.02)	-	(5.56)	(1.23)	-	-	-	-	-	-	-	-
GARCH(1,1)	-0.002	-	0.00016	0.073	0.867*	-	-	-	-	-	-	-
	(-0.36)	-	(0.86)	(1.39)	(7.88)	-	-	-	-	-	-	-
GARCH(1,1)-M	0.039***	-15.74***	0.00067***	0.153***	0.591*	-	-	-	-	-	-	-
	(1.66)	(-1.74)	(1.78)	(1.83)	(3.23)	-	-	-	-	-	-	-
EGARCH(1,1)	-0.007	-	-11.68*	0.189**	-0.968*	-0.006	-	-	-	-	-	-
	(-1.55)	-	(-41.75)	(2.27)	(-25.43)	(-0.07)	-	-	-	-	-	-
TARCH(1,1)	-0.001	-	0.00015	0.111	0.873*	-	-0.078	-	-	-	-	-
	(-0.25)	-	(0.94)	(1.36)	(9.33)	-	(-0.94)	-	-	-	-	-
PARCH(1,1)	-0.003	-	0.007	0.065	0.916*	-	-	-0.534	0.523	-	-	-
	(-0.87)	-	(0.23)	(0.67)	(7.66)	-	-	(-0.59)	(0.31)	-	-	-
CGARCH(1,1)	-0.00028	-	0.002*	-0.180*	0.223	-	-	-	-	0.927*	0.063	0.494*
	(-0.09)	-	(3.68)	(-2.59)	(0.84)	-	-	-	-	(9.47)	(0.86)	(3.08)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.
The time series of the empirical quarterly spot return itself is obtained from quarterly point spot rates in time.

Table 10: Estimates of GARCH class conditional volatility models for the empirical quarterly spot return of USD/GBP itself

Mean Equation

$$\Delta s_t = c + c_1\Delta s_{t-1} + c_2\Delta s_{t-2} + c_3\tau_{t-1} + c_4\tau_{t-2} + c_5\tau_{t-3} + \tau_t \quad \tau_t \backsim N(0, h_t^2)$$

(where $\Delta s_t = c + c_1\Delta s_{t-1} + c_2\Delta s_{t-2} + c_3\tau_{t-1} + c_4\tau_{t-2} + c_5\tau_{t-3} + dh_t^2 + \tau_t \quad \tau_t \sim N(0, h_t^2)$ to GARCH(1,1)-M)

Variance Equation

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 \qquad \text{ARCH(1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \text{GARCH(1,1)-M}$$

$$\ln\left(h_t^2\right) = \omega + \alpha\left[\frac{\tau_{t-1}}{h_{t-1}} - \sqrt{\frac{2}{\pi}}\right]\tau_{t-1}^2 + \beta\ln\left(h_{t-1}^2\right) + \eta\frac{\tau_{t-1}}{h_{t-1}} \qquad \text{EGARCH(1,1)}$$

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 + \mu\tau_{t-1}^2 I_{t-1} \text{ where } I_t = 1 \text{ if } \tau_t < 0 \text{ and } 0 \text{ otherwise} \qquad \text{TARCH(1,1)}$$

$$h_t^\vartheta = \omega + \alpha\left(|\tau_{t-1}| - \upsilon\tau_{t-1}\right)^\vartheta + \beta h_{t-1}^\vartheta \qquad \text{PARCH(1,1)}$$

$$h_t^2 = \omega + \rho\left(q_{t-1} - \omega\right) + \varphi\left(\tau_{t-1}^2 - h_{t-1}^2\right) + \alpha\left(\tau_{t-1}^2 - q_{t-1}\right) + \gamma\left(\tau_{t-1}^2 - q_{t-1}\right)D_{t-1} + \beta\left(h_{t-1}^2 - q_{t-1}\right) \qquad \text{CGARCH(1,1)}$$

Estimating model	c	c_1	c_2	c_3	c_4	c_5	d	ω	α	β	η	μ	υ	ϑ	ρ	φ	γ
ARMA(2,3)-	-0.001	-0.284*	-0.959*	0.520*	1.023*	0.300*	-	0.002*	0.159	-	-	-	-	-	-	-	-
ARCH(1)	(-0.23)	(-14.44)	(-44.04)	(5.31)	(38.04)	(3.08)	-	(6.23)	(1.23)	-	-	-	-	-	-	-	-
ARMA(2,3)-	-0.002	-0.283*	-0.954*	0.508*	1.023*	0.286*	-	0.00046	0.153***	0.675*	-	-	-	-	-	-	-
GARCH(1,1)	(-0.37)	(-14.84)	(-47.17)	(4.77)	(42.82)	(2.78)	-	(1.47)	(1.89)	(3.90)	-	-	-	-	-	-	-
ARMA(2,3)-	0.137**	-0.277*	-0.957*	0.532*	1.033*	0.313*	-58.345**	0.00064*	0.066*	0.689*	-	-	-	-	-	-	-
GARCH(1,1)-M	(2.17)	(-18.59)	(-57.34)	(6.54)	(51.46)	(3.91)	(-2.38)	(2.58)	(2.86)	(6.69)	-	-	-	-	-	-	-
ARMA(2,3)-	-0.00061	-0.282*	-0.965*	0.525*	1.031*	0.300*	-	-0.769	0.189***	0.899*	0.127***	-	-	-	-	-	-
EGARCH(1,1)	(-0.13)	(-15.66)	(-53.87)	(5.49)	(44.15)	(3.33)	-	(-1.36)	(1.88)	(10.50)	(1.88)	-	-	-	-	-	-
ARMA(2,3)-	-0.00095	-0.285*	-0.965*	0.513*	1.029*	0.275*	-	0.00020	0.188***	0.829*	-	-0.198***	-	-	-	-	-
TARCH(1,1)	(-0.20)	(-16.81)	(-59.85)	(5.64)	(37.95)	(3.29)	-	(1.09)	(1.80)	(7.09)	-	(-1.79)	-	-	-	-	-
ARMA(2,3)-	0.00008	-0.284*	-0.960*	0.539*	1.032*	0.319*	-	0.015	0.109***	0.816*	-	-	-0.634	0.615	-	-	-
PARCH(1,1)	(0.02)	(-15.94)	(-52.30)	(6.30)	(45.82)	(4.04)	-	(0.34)	(1.91)	(6.8)	-	-	(-1.57)	(0.65)	-	-	-
ARMA(2,3)-	-0.00282	-0.213**	-0.746*	0.319*	0.745*	0.278*	-	0.002*	-0.367	0.354	-	-	-	-	0.671**	0.251	0.378**
CGARCH(1,1)	(-0.64)	(-2.00)	(-6.58)	(2.57)	(5.92)	(3.21)	-	(4.45)	(-1.37)	(0.62)	-	-	-	-	(2.32)	(0.91)	(1.99)

Notes: z-statistics in parentheses; *, ** and *** denote significance at the 1%, 5% and 10% levels, respectively.
The time series of the empirical quarterly ARMA(2,3) spot return is obtained from quarterly point spot rates in time.

Table 11: Estimates of GARCH class conditional volatility models for the empirical quarterly ARMA(2,3) spot return of USD/GBP

Spot return itself	Theoretical quarterly spot return		Empirical monthly spot return						Empirical quarterly spot return					
	α_{th}	β_{th}	α_{em}	t_α	*rejection*	β_{em}	t_β	*rejection*	α_{em}	t_α	*rejection*	β_{em}	t_β	*rejection*
ARCH(1)	15.077	-	0.322	-138.81	*yes*	-	-	-	0.153	-119.90	*yes*	-	-	-
	(2.11)	(-)	(0.11)			(-)			(0.12)			(-)		
GARCH(1,1)	0.578	0.420	0.259	-4.03	*yes*	0.609	1.77	*no*	0.073	-9.71	*yes*	0.867	4.06	*yes*
	(0.04)	(0.04)	(0.08)			(0.11)			(0.05)			(0.11)		
GARCH(1,1)-M	0.447	0.551	0.257	-2.36	*yes*	0.607	0.51	*no*	0.153	-3.52	*yes*	0.591	0.21	*no*
	(0.02)	(0.02)	(0.08)			(0.11)			(0.08)			(0.18)		
EGARCH(1,1)	1.017	0.164	0.392	-6.87	*yes*	0.857	10.05	*yes*	0.189	-9.95	*yes*	-0.968	-29.76	*yes*
	(0.52)	(0.26)	(0.09)			(0.07)			(0.08)			(0.04)		
TARCH(1,1)	1.213	0.398	0.215	-12.60	*yes*	0.611	1.98	*no*	0.111	-13.52	*yes*	0.873	5.07	*yes*
	(0.10)	(0.02)	(0.08)			(0.11)			(0.08)			(0.09)		
PARCH(1,1)	0.537	0.605	0.190	-6.13	*yes*	0.732	1.48	*no*	0.065	-4.91	*yes*	0.916	2.60	*yes*
	(0.04)	(0.02)	(0.06)			(0.09)			(0.10)			(0.12)		
CGARCH(1,1)	-0.038	0.514	-0.072	-0.36	*no*	0.701	1.72	*no*	-0.180	-2.05	*yes*	0.223	-1.09	*no*
	(0.04)	(0.23)	(0.10)			(0.11)			(0.07)			(0.27)		

Notes: The t-tests are H_0: $\alpha = \alpha_{th}$, H_1: $\alpha \neq \alpha_{th}$ and H_0: $\beta = \beta_{th}$, H_1: $\beta \neq \beta_{th}$. The test statistics are $t_\alpha = (\alpha_{em} - \alpha_{th})/SE(\alpha_{em})$, $t_\beta = (\beta_{em} - \beta_{th})/SE(\beta_{em})$, respectively. α is the coefficient estimate on the ARCH term. β is the coefficient estimate on the GARCH term. The subscript "$_{th}$" refers to the theoretical data. The subscript "$_{em}$" refers to the empirical data. The time series of the empirical monthly spot return is obtained from monthly averages of daily spot rates in that month. The time series of the empirical quarterly spot return is obtained from quarterly point spot rates in time. Given the t-ratios and critical values (2 for a 5% test), the null hypotheses are rejected. "-" indicates N/A. The standard errors in parentheses are placed below the coefficient estimates.

Table 12: Testing significant differences of the empirical estimates from the theoretical estimates for the spot return itself

Spot return of the ARMA process	Theoretical quarterly ARMA(2,2) spot return		Empirical monthly MA(1) spot return						Empirical quarterly ARMA(2,3) spot return					
	α_{th}	β_{th}	α_{em}	t_α	*rejection*	β_{em}	t_β	*rejection*	α_{em}	t_α	*rejection*	β_{em}	t_β	*rejection*
ARCH(1)	38.372	-	0.187	-415.28	*yes*	-	-	-	0.159	-293.44	*yes*	-	-	-
	(11.17)	-	(0.09)			(-)			(0.13)			(-)		
GARCH(1,1)	0.463	0.535	0.139	-6.38	*yes*	0.754	2.41	*yes*	0.153	-3.86	*yes*	0.675	0.80	*no*
	(0.02)	(0.02)	(0.05)			(0.09)			(0.08)			(0.17)		
GARCH(1,1)-M	0.711	0.287	0.140	-11.06	*yes*	0.752	5.02	*yes*	0.066	-27.95	*yes*	0.689	3.90	*yes*
	(0.09)	(0.09)	(0.05)			(0.09)			(0.02)			(0.10)		
EGARCH(1,1)	0.948	0.943	0.226	-5.59	*yes*	-0.622	-7.57	*yes*	0.189	-7.55	*yes*	0.899	-0.50	*no*
	(0.02)	(0.01)	(0.13)			(0.21)			(0.10)			(0.09)		
TARCH(1,1)	1.171	0.401	0.130	-19.56	*yes*	0.754	4.00	*yes*	0.188	-9.45	*yes*	0.829	3.67	*yes*
	(0.09)	(0.02)	(0.05)			(0.09)			(0.10)			(0.12)		
PARCH(1,1)	0.526	0.617	0.142	-8.53	*yes*	0.772	1.76	*no*	0.109	-7.29	*yes*	0.816	1.66	*no*
	(0.04)	(0.02)	(0.04)			(0.09)			(0.06)			(0.12)		
CGARCH(1,1)	0.097	0.623	-0.030	-1.68	*no*	-0.345	-1.59	*no*	-0.367	-1.73	*no*	0.354	-0.47	*no*
	(0.02)	(0.02)	(0.08)			(0.61)			(0.27)			(0.57)		

Notes: The t-tests are H_0: $\alpha = \alpha_{th}$, H_1: $\alpha \neq \alpha_{th}$ and H_0: $\beta = \beta_{th}$, H_1: $\beta \neq \beta_{th}$. The test statistics are $t_\alpha = (\alpha_{em} - \alpha_{th})/SE(\alpha_{em})$, $t_\beta = (\beta_{em} - \beta_{th})/SE(\beta_{em})$, respectively. α is the coefficient estimate on the ARCH term. β is the coefficient estimate on the GARCH term. The subscript "$_{th}$" refers to the theoretical data. The subscript "$_{em}$" refers to the empirical data. The time series of the empirical monthly spot return is obtained from monthly averages of daily spot rates in that month. The time series of the empirical quarterly spot return is obtained from quarterly point spot rates in time. Given the t-ratios and critical values (2 for a 5% test), the null hypotheses are rejected. "-" indicates N/A. The standard errors in parentheses are placed below the coefficient estimates.

Table 13: Testing significant differences of the empirical estimates from the theoretical estimates for the ARMA spot return

6.1.3 Testing significance of estimates

In order to check and make sense of the estimates exposed previously, the final thing we are going to do is to test significant differences of the empirical estimates from the theoretical estimates. Treating the theoretical estimates as fixed numbers, we take the empirical estimates of the GARCH parameters and their standard errors and use t-test to test each of the GARCH parameter's significance from that estimated from the theoretical data. The null hypothesis is that the population GARCH parameters (α, β) are the theoretical estimates against the 5% two-sided alternative. We quote the test statistics and report the results in Table 12 and 13.

In Table 12, we report the results of testing estimation significance for the spot return itself. Given the t-ratios of the estimated ARCH and GARCH coefficients and 5% two-sided critical values[33], on the one hand, for the empirical monthly spot return itself in the middle part (from the 4th to 9th columns) of Table 12, only the CGARCH model has the non-significant ARCH and GARCH parameters while the rest models have the significant ARCH coefficients and only the EGARCH model has the significant GARCH estimate. The CGARCH model has the best estimating performance because the estimated values of both the ARCH and GARCH coefficients are indistinguishable statistically from the theoretical estimated values of these two parameters. The EGARCH model has both the statistically distinguishable ARCH and GARCH parameters compared to those of the theoretical estimates. On the other hand, for the empirical quarterly spot return itself in the right hand side (from the 10th to 15th columns) of Table 12, all models have the significant ARCH coefficients. The GARCH-M and CGARCH model have the insignificant GARCH parameters while the other models have the significant GARCH estimates. The CGARCH model is one of the best estimating models with less significant differences. Summarily, in Table 12, the GARCH class models for the empirical monthly spot return of averaging data have at least the same (even better) estimating performance as (than) theirs for the empirical quarterly spot return of point data in time. The empirical monthly spot return has less estimation differences of significance to those of the theoretical quarterly spot return than the empirical quarterly spot return.

In Table 13, we report the results of testing estimation significance for the spot return of the ARMA process. Given the t-ratios of the estimated ARCH and GARCH coefficients and 5% two-sided critical values, on the one hand, for the empirical monthly MA(1) spot return in the middle part (from the 4th to 9th columns) of Table 13, all models except CGARCH have the significant ARCH coefficients while all models except CGARCH and PARCH have the significant GARCH parameters. Only the CGARCH model has the non-significant ARCH and GARCH estimates, which is the same as in Table 12 for the empirical monthly spot return itself. The CGARCH model has the best estimating performance because the estimated values of the ARCH and GARCH coefficients are indistinguishable statistically from the theoretical estimated parameter values. The PARCH model is the second best estimating model due to the indistinguishable GARCH coefficient. On the other hand, for the empirical quarterly ARMA(2,3) spot return in the right hand side (from the 10th to 15th columns) of Table 13, all models except CGARCH have the significant ARCH coefficients while the GARCH, CGARCH, EGARCH and PARCH models have the non-significant GARCH parameters. Again, the CGARCH model is the only model that has the statistically indistinguishable ARCH and GARCH parameters from those theoretical estimated values. Generally, in Table 13, the GARCH class models for the empirical quarterly ARMA(2,3) spot return of point data in time have at least the same[34] (even better) estimating performance as (than) theirs for the empirical monthly MA(1) spot return of averaging data. The the empirical quarterly ARMA(2,3) spot return has less estimation differences of significance to those of the theoretical quarterly ARMA(2,2) spot return than the empirical monthly MA(1) spot return. The CGARCH, PARCH and EGARCH models are the top three best estimating models.

Overall, taking account of the theoretical and empirical estimates on significance of coefficients (especially those on the GARCH terms), asymmetric effects, additional ARCH, and persistent volatility shocks and the results of testing estimation significant differences in Table 6-13, it is found that the best fit estimating models for both the theoretical and empirical data are the asymmetric CGARCH, EGARCH, and PARCH conditional volatility models, which is consistent

[33] For the degrees of freedom greater than around 25, the 5% two-sided critical value is approximately ± 2. As a rule of thumb, the null hypothesis would be rejected if the t-statistic exceeds 2 in absolute value.

[34] For the empirical quarterly spot return, the ARCH, GARCH-M, CGARCH, PARCH and TARCH models have the same estimating performances and the GARCH and EGARCH models have the better performances of the ARCH and GARCH parameters compared to theirs for the empirical monthly spot return.

with what we found in our working paper[35] for forecasts. The theoretical model can produce the same kind of ARCH estimates as we see in the real data.

6.2 Impulse response function

We establish the dynamic form and analyze the dynamic properties of conditional volatility we estimate by examining the impulse response function (IRF). We have an interest to know how does a unit innovation to conditional volatility affect it, now and in future? An impulse response function measures the effect of a transitory shock to current volatility h_0^2 on future volatilities h_t^2. To achieve this, we read off the coefficients in the moving average representation of the process. For example, We consider the GARCH(1,1) model and its conditional variance process is defined by

$$h_t^2 = \omega + \alpha\tau_{t-1}^2 + \beta h_{t-1}^2 \qquad \tau_t \backsim N(0, h_t^2)$$

Subtracting 1 from each of the time subscripts, an infinite number of successive substitutions of the conditional variance would yield

$$h_t^2 = \omega(1 + \beta + \beta^2 + ...) + \alpha\tau_{t-1}^2 \left(1 + \beta L + \beta^2 L^2 + ...\right) + \beta^\infty h_0^2$$

The expression on the RHS is simply a constant, and as the number of observations tends to infinity, β^∞ will tend to zero. Hence, equivalently, the volatility equation can be written as

$$h_t^2 = \varpi + Z_1\tau_{t-1}^2 + Z_2\tau_{t-2}^2 + Z_3\tau_{t-3}^2 + ...Z_t\tau_0^2$$

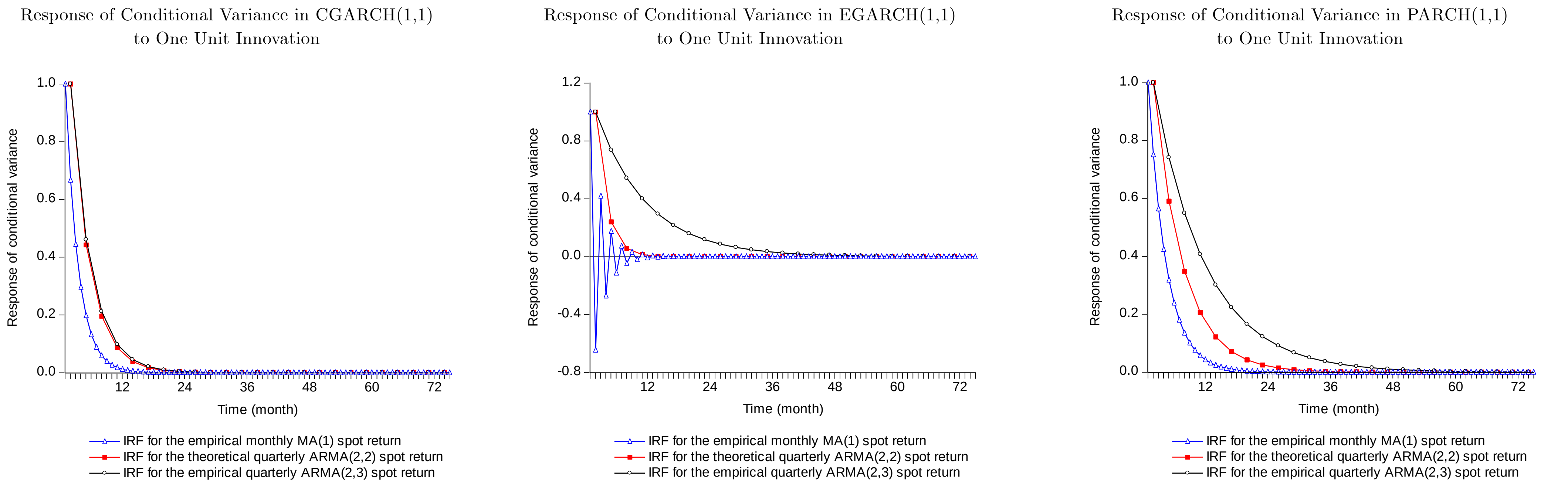
where ϖ is the constant term, $Z_1 = a$, $Z_2 = \alpha\beta$, $Z_3 = \alpha\beta^2$, $Z_4 = \alpha\beta^3$, ..., $Z_t = \alpha\beta^{(t-1)}$. The full set of impulse-response coefficients, $\{Z_1, Z_2...Z_t\}$, tracks the complete dynamic response of h_t^2 to the shock. In other words, the autoregressive model is written as a moving average.

Using the IRF we can convert the conditional variance (h_t^2) in the variance equation of our best fit estimating CGARCH(1,1), EGARCH(1,1) and PARCH(1,1) models for the theoretical (simulated) ARMA(2,2) spot return (Δs_t) into their infinite moving average forms, respectively. The IRF is then simply the graph of Z_i against $i = 1, 2, 3....$ We repeat this process for the best estimating models for the conditional variance of the empirical monthly USD/GBP spot return that is an MA(1) and for the conditional variance of the empirical quarterly USD/GBP spot return that is an ARMA(2,3). We start the IRFs for the theoretical and empirical data at the same point. Three IRFs start from the same initial response (e.g. one unit shock) and then any differences between the dynamic patterns of the three can be seen clearly. We draw and compare the impulse response functions (IRFs) for the conditional variance (h_t^2) of the three ARCH processes we estimated. It is noted that we compress the scale so the monthly IRFs from the empirical data are synchronized with the quarterly IRFs from the theoretical data. For the latter, we need the IRFs using the theoretical model's parameters in the baseline as in Table 1 to get the theoretical time series of the ARMA(2,2) spot return we simulated.

In Figure 1, we display the impulse responses of conditional variance of the best fit estimating models at time t to one unit shock in variance at time 0 for the theoretical and the empirical ARMA spot returns, respectively. First, we consider the sign of the responses. For the empirical monthly IRFs from the the empirical monthly MA(1) spot return (blue lines), the line graphs on the left and right hand sides (LHS and RHS) show that a shock to the conditional variance in the CGARCH and PARCH models respectively always has a positive impact on the future conditional volatility since the impulse responses are positive. The line graph in the middle position shows that a shock to the conditional variance in the EGARCH model has a positive (negative) impact at the odd (even) time points on the future conditional volatility since the impulse response is positive (negative) at that time. For the theoretical quarterly IRFs from the theoretical quarterly ARMA(2,2) spot return (red lines), for all three cases, the conditional volatility always has a positive response to shock at all time points until the effect of the shock dies out. The empirical quarterly IRFs from the empirical quarterly ARMA(2,3) spot return (black lines) have positive signs for all three cases, which is the same as the theoretical quarterly IRFs have. The empirical quarterly IRFs seem to be having more similarities to the theoretical quarterly IRFs than the empirical monthly IRFs in terms of the sign of the responses.

[35] In the working paper of forecasting, although we conclude that no single model dominates for forecasts, the CGARCH, EGARCH and PARCH models have locally the best forecasting performances compared to those of other symmetric and asymmetric GARCH class models.

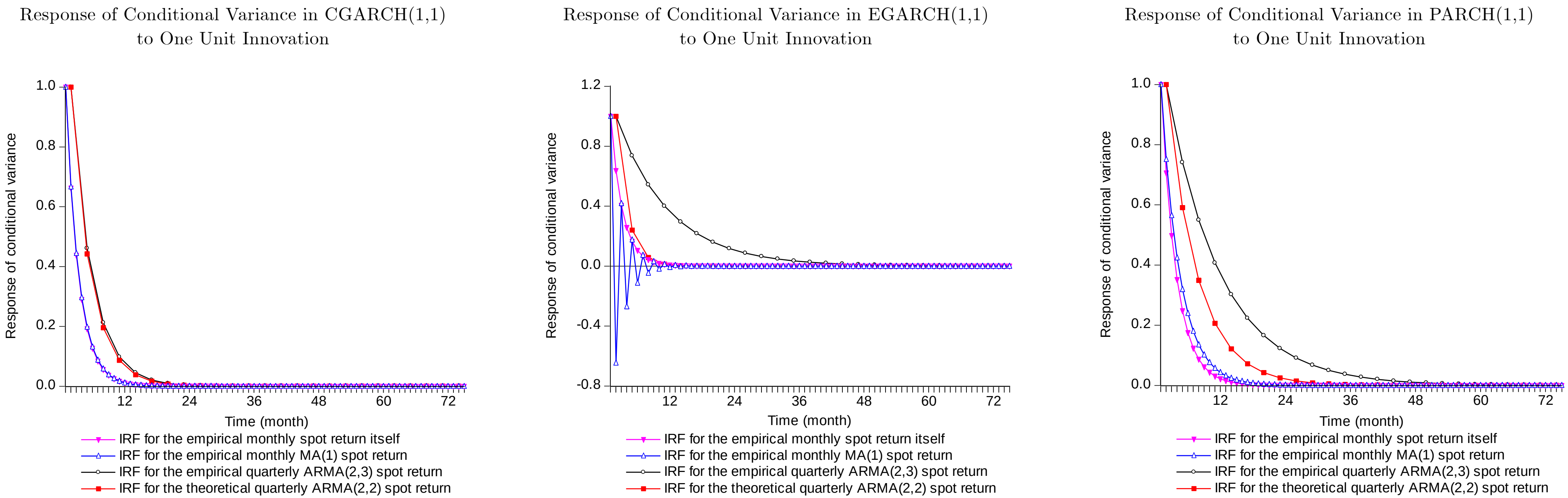
Innovations in conditional variance of white noise errors



Notes: The time series of the empirical monthly MA(1) spot return is obtained from monthly averages of daily spot rates in that month; the time series of the empirical quarterly ARMA(2,3) spot return is obtained from quarterly point spot rates in time.

Figure 1: Impulse response functions for one unit innovation in conditional variance from the empirical and the theoretical ARMA spot returns

Innovations in conditional variance of white noise errors



Notes: The time series of the empirical monthly spot return itself is obtained from monthly averages of daily spot rates in that month; the time series of the empirical monthly MA(1) spot return is obtained from monthly averages of daily spot rates in that month; the time series of the empirical quarterly ARMA(2,3) spot return is obtained from quarterly point spot rates in time.

Figure 2: Impulse response functions for one unit innovation in conditional variance from the empirical and the theoretical spot returns

Second, we deliberate on the magnitude of the responses and the time periods the effect of the shock takes to die out. In Figure 1, the monthly impulse responses are small while the quarterly impulse responses are big. For the empirical monthly IRFs from the empirical monthly MA(1) spot return (blue lines), for all three cases, the effect of the shock takes approximately $1.5-2$ year to die out. In details, the effect of the shock to the conditional variance in the CGARCH, EGARCH and PARCH models dies down toward zero (≈ 0.001) after taking approximately 17, 15, and 23 months, respectively. In contrast, for the theoretical quarterly IRFs from the theoretical quarterly ARMA(2,2) spot return (red lines), the effect of the shock to the conditional variance in the CGARCH, EGARCH and PARCH models takes 24, 15, and 39 months respectively to die out, while, for the empirical quarterly IRFs from the empirical quarterly ARMA(2,3) spot return (black lines), the effect of the shock to the conditional variance in the CGARCH, EGARCH and PARCH models takes 27, 66, and 66 months respectively to die down toward zero (≈ 0.001). For the CGARCH model, the empirical quarterly IRF has the highly similar magnitudes of responses and time periods for dying out as the theoretical quarterly IRF has so that both IRFs look almost same (overlapping). For the EGARCH and PARCH models, the empirical monthly IRFs have more similarities to the theoretical quarterly IRFs than the empirical quarterly IRFs. The quarterly IRFs take longer time than what the monthly IRFs take to die down, where the empirical quarterly IRFs with the biggest magnitudes take longest time. All IRFs from the empirical and the theoretical ARMA spot returns in Figure 1 show a clear trend of dying out, where the effect of the shock to the conditional variance in the EGARCH model takes the shortest time while the one to the PARCH model takes the longest time, in short, $EGARCH < CGARCH < PARCH$.

Third, we consider the dynamic features of the responses. For the empirical monthly IRFs from the empirical monthly MA(1) spot return (blue lines), the impulse responses of the conditional variance in the CGARCH and PARCH models on the LHS and RHS respectively in Figure 1 monotonically decrease up to 72 months until the effect of the shock dies out. Specifically, the IRF to the CGARCH model tends to more intensely decline than what the IRF to the PARCH model does within the first 12 months particularly on the time interval $[4, 12]$, while the impulse responses of the conditional variance in the EGARCH model in the middle part fluctuate around zero with a gradually (from strong to weak) falling trend as the effect of the shock dies out. All theoretical quarterly IRFs from the theoretical quarterly ARMA(2,2) spot return (red lines) decay in a monotonic decreasing fashion. It is found in common that both the empirical monthly and the theoretical quarterly IRFs to the CGARCH and PARCH models drop slowly with a relatively smooth trend while both the empirical monthly and the theoretical quarterly IRFs to the EGARCH model descends quickly with a comparably steep trend. It is different from the dynamic movements of the empirical monthly and the theoretical quarterly IRFs that the empirical quarterly IRFs from the empirical quarterly ARMA(2,3) spot return (black lines) have the more intense decline to the CGARCH model than theirs to the EGARCH and PARCH models in which the empirical quarterly IRFs are quite smooth with a generally falling trend. In Figure 1, the shock has a diminishing impact on future conditional volatility, and all of the empirical and theoretical IRFs show a (approximately) monotonic decreasing fashion until they finally die out.

It is emphasized that we are interested in the IRF's dynamics not size of typical shock (e.g. variance of shocks) "hitting" the variances. The empirical and theoretical IRFs would differ anyway because we compare monthly average data against quarterly point data in time. For the empirical and theoretical data, the dynamic patterns of the CGARCH and PARCH models respectively always look close on the IRFs than those to the EGARCH model, which, as we expect, is consistent with the results of testing significance of the data estimates disclosed previously where the CGARCH and PARCH parameters of the empirical data estimates are not significantly different from the ones of theoretical data.

As reported previously, the results of the ARCH estimates for the empirical monthly spot return itself (without an ARMA process) have more similarities to those for the innovation in the theoretical quarterly spot return of an ARMA(2,2) process. We plot the IRFs for our best fit estimating (CGARCH, EGARCH and PARCH) models for the conditional variance of the empirical monthly spot return itself due to its superior ARCH estimates. In Figure 2, we compare these IRFs with the IRFs implied by the empirical and theoretical ARCH processes from the ARMA spot returns as in Figure 1.

In Figure 2, for the empirical monthly IRFs from the empirical monthly spot return itself (mauve lines), the impulse responses are positive and smaller than both the empirical and theoretical quarterly impulse responses (red and black lines), while they are not greater ($\leqq$) than the absolute values of the IRFs' magnitudes from the empirical monthly MA(1) spot return (blue lines) at all time points for all model cases. The effect of the shock to the conditional variances in the CGARCH, EGARCH and PARCH models takes approximately 16, 15, and 19 months respectively to die out. The shock has a diminishing impact on future conditional volatility, and the impulse responses of the conditional volatility to shock tend to have a monotonic decreasing fashion, which is consistent with the findings in Figure 1. It is seen clearly that, for all cases, the IRFs in mauve monotonically decrease at a similar steep slope as the IRFs from the theoretical ARMA(2,2) spot return do, which is better than the IRFs in blue (from the empirical monthly MA(1) spot return) though the IRFs in blue are superior to those in black (from the empirical quarterly ARMA(2,3) spot return). It suggests that the dynamic forms of conditional heteroscedasticity for the empirical monthly spot return itself without an ARMA process have more similarities so as to look more consistent with those for the innovation in the theoretical quarterly ARMA(2,2) spot return. This, we think, could be another justification to answer why empirical researchers consider the FOREX spot return itself not its innovation.

We also look at the dynamic movements of the impulse responses to the shock for other type empirical data in the quarterly frequency where two time series of the empirical quarterly spot return itself are constructed from the empirical quarterly averages of daily USD/GBP spot rates in the quarter and from the empirical quarterly USD/GBP point spot rates in time, respectively; and one time series of the empirical quarterly ARMA(2,1) spot return is constructed from the quarterly averages of daily USD/GBP spot rates in that quarter. We find, but do not report, that, as the consistent result, all IRFs tend to have the monotonic decreasing fashion and the empirical monthly spot return itself constructed from the averaging data has the most similar ARCH dynamics to those of the theoretical quarterly data.

Overall, as showed in both Figure 1 and 2, the empirical monthly IRFs "look the same" to the theoretical quarterly IRFs for an approximately monotonic decreasing fashion as time goes. Shock has a weaken impact on future conditional volatility until it dies out. The IRFs for variance for the theoretical and empirical data do have similar dynamics though they have different orders of magnitude in which the theoretical IRFs are bigger and take longer time to die out. As we mentioned before, in the paper, it is dynamics we are interested in not size of typical shock "hitting" the variances. The magnitudes would differ anyway because the monthly average data is comparing against the quarterly point data in time. In details, the empirical data uses the average exchange rates whereas the theoretical data is a one-point-in-time observation. Variances of averages are always lower because averaging a series smooths it out. It may be this that is giving the bigger responses to the theoretical model. At the same time, another possible explanation might be the "imperfect" market in reality. In the FOREX markets, many endemic and exotic factors coexist simultaneously. Some of them counteract each other, or some earlier shocks (e.g. old news) are getting to be of less or no effect on the future conditional volatility, while these factors are not considered to live in the pure theoretical framework. For example, investors usually pay more attention to the recent release or announcement of financial news and the market could be influenced mainly by the late central bank intervention. The theoretical model can generate the same kind of dynamic features to ARCH as we see in the actual data.

7 Conclusion

In the paper, we attempt to give a theoretical underpinning to the well established empirical stylized fact that asset returns in general and the spot FOREX returns in particular display predictable volatility characteristics. After investigating Moore and Roche's habit version of Lucas to conditional volatility, we find that the Lucas two-country, two-good, two-money economy model with habit can generate realistic conditional volatility in spot FOREX return. Specifically, we research the Lucas two-country monetary model with habit in Moore and Roche (2006) and find the model's implied property of the ARMA(2,2) spot return. We numerically solve the model and test that the theoretical ARMA(2,2) spot return and its innovation in the spot return process are definitely conditionally heteroscedastic. We estimate the best fit GARCH models for the theoretical and empirical data and then establish the dynamic form for the conditional heteroscedasticity we estimate from the best fit GARCH models. Using the impulse response functions (IRFs) we show that the baseline theoretical data has "ARCH" properties in the quarterly frequency that match well the "ARCH" properties of the empirical monthly estimations.

The IRFs for the ARCH processes we estimate "look the same" with similar impulse responses to one unit shock in conditional variance of white noise errors. The impulse responses of conditional variance to shock tend to monotonically decrease until the effect of the shock dies out. On the other hand, concerning the highly consistent performance of the ARCH estimates and dynamic features between the empirical monthly spot return itself and the innovation in the theoretical quarterly ARMA(2,2) spot return, we answer why empirical researchers tend to consider the spot FOREX return itself rather than its innovation as we did previously in the working paper of forecasting. The Lucas two-country monetary model with habit is capable of producing the same kind of ARCH features as we see in the real data.

As one of the theoretical asset pricing models, the habit persistence model is able not only to explain persistent volatility on asset returns that is unconditional but also to generate volatility clustering in FOREX returns even its asymmetric property. Campbell and Cochrane (1999) is the earlier one that explains the dynamic behavior of asset prices using a consumption-based asset pricing model with an external habit. Moore and Roche (2006) extend the theoretical model with habit in Campbell and Cochrane (1999) to a two-country monetary economy to solve many FOREX puzzles simultaneously including mimicking the FOREX unconditional volatility. McQueen and Vorkink (2004) is an important paper to apply the theoretical asset pricing model to the issue of volatility clustering. McQueen and Vorkink (2004) develop a preference-based equilibrium asset pricing model that derives utility from both consumption and financial wealth to endogenously explain conditional volatility in US stock data. In McQueen and Vorkink (2004), a unique mental scorecard that records wealth changes and affects investors' level of risk aversion induces wealth-varying sensitivity to news causing subsequent stock volatility.

In the paper, we use the theoretical Lucas two-economy representative-agent model in Moore and Roche (2006), which combines the external habit in Campbell and Cochrane (1999) into a monetary framework, to explain conditional volatility in spot FOREX returns. According to the success of the capacity of both the theoretical asset pricing models in Moore and Roche (2006) and McQueen and Vorkink (2004) explaining volatility clustering, we summarize their main features as follows: 1) Moore and Roche (2006) derive utility from surplus consumption while McQueen and Vorkink (2004) derive utility from consumption and financial wealth changes; 2) Moore and Roche (2006) use an external scorecard of surplus consumption ratio while McQueen and Vorkink (2004) use an internal scorecard of prior investment performance; 3) Moore and Roche (2006) numerically solved the model using the quarterly calibrated parameters while McQueen and Vorkink (2004) numerically solved the model using the monthly calibrated parameters; 4) Moore and Roche (2006) mimic unconditional volatility in FOREX changes while Vorkink and McQueen (2004) explain volatility clustering in asset returns; 5) for both cases, we are moving away from a utility function (U) that can be written as the sum of discounted one-period utility (u) in "current" consumption.

In our opinion, both utility specifications in Moore and Roche (2006) and McQueen and Vorkink (2004) are at the heart of generating ARCH effects, which is overall consistent with what Cochrane (2001) suggests, "risk aversion depends on the level of consumption or wealth relative to some trend or the recent past". In other words, either surplus consumption utility in Moore and Roche (2006) or prior investment utility in McQueen and Vorkink (2004) could be the reason behind volatility clustering found in empirical facts. However, we think that the utility function in McQueen and Vorkink (2004) is strange. It includes wealth (changes) directly in utility implying that consumers care about wealth directly. But in economics we always think of wealth as an instrument that leads to utility via its ability to buy consumption rather than the object itself. To take an extreme case would we be happy including a utility that was a function of Treasury Bill holdings? Of course there is a precedent – money has been included in the utility function in macroeconomics. But this is more of a device rather than couched in a belief that money itself (rather than consumption) gives you utility. By contrast only consumption appears in a habit utility. It is true however that the habit term collects together past consumptions perhaps in a way that wealth could collect together future consumptions. But at least it is directly in terms of consumption whereas with wealth we would need to convert to consumption via e.g. current and future interest rates and maybe current and future (consumption) price levels. Hence, it is unreasonable to assume that people care about their changes in wealth separately in addition to the consumption stream that is affected by what the changes in wealth bring.

Finally, the habit persistence model is an industry standard now in macroeconomics and finance. Special attention is given to the role of habit persistence in explaining the equity premium puzzle, additional asset-pricing puzzles such as the risk-free-rate puzzle and the forecastability of excess returns (see, for example, Campbell and Cochrane, 1999), many exchange rate puzzles such as disconnect, forward bias, and Meese-Rogoff puzzles including mimicking unconditional volatilities of exchange rates and spot returns etc. (see, for example, Moore and Roche, 2006), observed business-cycle fluctuations and inflation dynamics, and in generating a theory of counter-cyclical markups of prices over marginal costs. The paper gives a study of the ability of habit persistence to account for conditional volatility in spot FOREX returns.

References

[1] Abel, A., 1990, "Asset prices under habit formation and catching-up-with-the-Joneses", American Economic Review, 80, 38–42.

[2] Boldrin, M., Christiano, L. and Fisher, J., 2001, "Habit persistence, asset returns, and the business cycle", American Economic Review, 91, 149–66.

[3] Campbell, John Y. and Cochrane, John H., 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", Journal of Political Economy, 107, no. 2, 205-251.

[4] Campbell, John Y., Lo, Andrew W. and Mackinlay, A.Craig, The Econometrics of Financial Markets, Princeton University Press, 1997.

[5] Christiano, L., Eichenbaum, M. and Evans, C., 2005, "Nominal rigidities and the dynamic effects of a shock to monetary policy", Journal of Political Economy, 113, 1–45.

[6] Cochrane, John H., Asset Pricing, Princeton University Press, 2001.

[7] Constantinides, G., 1990, "Habit formation: a resolution of the equity premium puzzle", Journal of Political Economy, 98, 519–43.

[8] Duesenberry, James.S., "Income, Savings and the Theory of Consumer Behavior", Harvard University Press, Cambridge, MA, 1949.

[9] LeBaron, B., "Agent-based Computational Finance", the preliminary draft for The Handbook of Computational Economics, vol. II„ edited by K. L. Judd and L. Tesfatsion, Brandeis University, 2005.

[10] Liu, Jingyi, Chapter 3, "Three Essays in Financial Economics", PhD Dissertation, University of Edinburgh, 2008.

[11] Lucas, Robert E., 1982, "Interest Rates and Currency Prices in a Two-country World." Journal of Monetary Economics, 10, 335-360.

[12] Mehra, R. and Prescott, E., 1985, "The equity premium: a puzzle", Journal of Monetary Economics, 15, 145–61.

[13] Moore, Michael J. and Roche, Maurice J., 2002, "Less of a Puzzle: a New Look at the Forward Forex Market." Journal. of International. Economics, 58, 387-411.

[14] Moore, Michael J. and Roche, Maurice J., "A neo-classical explanation of nominal exchange rate volatility" in Exchange Rate Economics: Where do we stand? edited by Paul de Grauwe, MIT Press, 2005.

[15] Moore, Michael J. and Roche, Maurice J, 2006, "Solving Exchange Rate Puzzles with neither Sticky Prices nor Trade Costs", working paper, Queen's University Belfast.

[16] McQueen, G. and Vorkink, K., 2004, "Whence garch? A preference-based explanation for conditional volatility", Review of Financial Studies, 17, 915-949.

[17] Engle, Robert.F. and Ng, V., "Measuring and testing the impact of news on volatility", 1993, Journal of Finance, 48, 1749–1778.

[18] Pagan, A.R., Schwert, G.W., (1990), "Altemative models for conditional stock volatility", Journal of Econometrics, 3, 267-290.

[19] Ravn, M., Schmitt-Grohé, S. and Uribe, M., 2006, "Deep habits", Review of Economic Studies, 73, 1–24.

[20] Sundaresan, S. M., 1989, "Intertemporally dependent preferences and the volatility of consumption and wealth", Review of Financial Studies, 2, 73–89.

[21] Taylor, Stephen J., Asset Price Dynamics, Volatility, and Prediction, Princeton University Press, 2005.

[22] Uribe, M., 2002, "The price-consumption puzzle of currency pegs", Journal of Monetary Economics, 49, 533–69.