

Local existence for the non-resistive MHD equations in Besov spaces

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Abstract

In this paper we prove the existence of solutions to the viscous, non-resistive magnetohydrodynamics (MHD) equations on the whole of \mathbb{R}^n , $n = 2, 3$, for divergence-free initial data in certain Besov spaces, namely $\mathbf{u}_0 \in B_{2,1}^{n/2-1}$ and $\mathbf{B}_0 \in B_{2,1}^{n/2}$. The a priori estimates include the term $\int_0^t \|\mathbf{u}(s)\|_{H^{n/2}}^2 ds$ on the right-hand side, which thus requires an auxiliary bound in $H^{n/2-1}$. In 2D, this is simply achieved using the standard energy inequality; but in 3D an auxiliary estimate in $H^{1/2}$ is required, which we prove using the splitting method of Calderón (*Trans. Amer. Math. Soc.* **318**(1), 179–200, 1990). By contrast, our proof that such solutions are unique only applies to the 3D case.

Keywords: Besov spaces, magnetohydrodynamics, MHD

1. Introduction

In this paper we prove local-in-time existence of weak solutions to the non-resistive magnetohydrodynamics (MHD) equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p_* = (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (1.1a)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u}, \quad (1.1b)$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0, \quad (1.1c)$$

on the whole of \mathbb{R}^n for $n = 2, 3$, with divergence-free initial data in Besov spaces as follows:

$$\mathbf{u}_0 \in B_{2,1}^{n/2-1}(\mathbb{R}^n) \quad \text{and} \quad \mathbf{B}_0 \in B_{2,1}^{n/2}(\mathbb{R}^n).$$

In particular, we prove the following theorem.

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Theorem 1.1. *Let $n = 2, 3$. For $\mathbf{u}_0 \in B_{2,1}^{n/2-1}(\mathbb{R}^n)$ and $\mathbf{B}_0 \in B_{2,1}^{n/2}(\mathbb{R}^n)$ with $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{B}_0 = 0$, there exists a time $T_* = T_*(\nu, \mathbf{u}_0, \|\mathbf{B}_0\|_{B_{2,1}^{n/2}}) > 0$ such that the equations (1.1) have at least one weak solution (\mathbf{u}, \mathbf{B}) , with*

$$\begin{aligned} \mathbf{u} &\in L^\infty([0, T_*]; B_{2,1}^{n/2-1}(\mathbb{R}^n)) \cap L^1(0, T_*; B_{2,1}^{n/2+1}(\mathbb{R}^n)), \\ \mathbf{B} &\in L^\infty([0, T_*]; B_{2,1}^{n/2}(\mathbb{R}^n)). \end{aligned}$$

This result is the natural generalisation of the main result of Fefferman et al. (2014), in which local-in-time existence of strong solutions to (1.1) was proved on the whole of \mathbb{R}^n with $n = 2, 3$, with divergence-free initial data $\mathbf{u}_0, \mathbf{B}_0 \in H^s(\mathbb{R}^n)$, for $s > n/2$. This depended upon a commutator estimate, a partial generalisation of that of Kato and Ponce (1988), which does not hold for $s = n/2$.

In this paper, we work instead in the space $B_{2,1}^{n/2}$, which is the natural replacement for the space $H^{n/2}$: it is the largest Besov space which still embeds in L^∞ (unlike $H^{n/2}$). Thanks to the properties of the heat equation in Besov spaces, we require one fewer derivative for the initial data \mathbf{u}_0 , requiring only that $\mathbf{u}_0 \in B_{2,1}^{n/2-1}(\mathbb{R}^n)$; but with no diffusion term in the \mathbf{B} equation we still require $\mathbf{B}_0 \in B_{2,1}^{n/2}(\mathbb{R}^n)$.

This paper, like Fefferman et al. (2014), builds on a number of previous results for the non-resistive MHD equations, including Jiu and Niu (2006), Fan and Ozawa (2009) and Zhou and Fan (2011). Moreover, for the fully ideal MHD equations (with no diffusion in either equation), Miao and Yuan (2006) proved existence and uniqueness of solutions to fully ideal MHD in the Besov space $B_{p,1}^{1+n/p}(\mathbb{R}^n)$.

Nonetheless, the results for the non-resistive equations are still much weaker than those for the fully diffusive MHD equations, in which the term $-\eta\Delta\mathbf{B}$ appears in (1.1b): in 2D one has global existence and uniqueness of weak solutions, and in 3D one has local existence of weak solutions, much like the Navier–Stokes equations; these results go back to Duvaut and Lions (1972) and Sermange and Temam (1983). A detailed discussion of previous work on the subject can be found in the introduction to Fefferman et al. (2014).

The rest of the paper is structured as follows:

- In Section 2, we recall some of the theory of Besov spaces used throughout the paper.
- In Section 3, we prove two of the key a priori estimates necessary in the proof of Theorem 1.1: these two estimates apply equally in both 2D and 3D.
- In Section 4, we prove additional estimates on the term $\int_0^T \|\mathbf{u}(t)\|_{H^{n/2}}^2 dt$, which appears on the right-hand side of the estimate for the \mathbf{u} equation proved in Section 3, in order to close up the a priori estimates. Different arguments are required in 2D and 3D.
 - In 2D, this is easily taken care of using the energy inequality (see Section 4.1).

– In 3D, this needs a careful argument, based on the splitting method of Calderón (1990), to yield an $H^{1/2}$ estimate for the Navier–Stokes equations (see Section 4.2).

- In Section 5, with the necessary estimates completed, the rest of the proof of Theorem 1.1 is outlined.
- In Section 6 we prove that, in 3D, the solution whose existence is asserted by Theorem 1.1 is unique.

Surprisingly, the proof of uniqueness in 2D is more difficult and it has only been resolved recently by Wan (2015). Furthermore, note that we require the initial data to have finite energy, taking \mathbf{u}_0 and \mathbf{B}_0 in inhomogeneous Besov spaces rather than their homogeneous counterparts. For further discussion on both these issues, see the conclusion (Section 7).

2. Besov Spaces

Here we recall some of the standard theory of Besov spaces which we will use throughout the paper; we use, as far as possible, the same notation as Bahouri, Chemin and Danchin (2011), and refer the reader to Chapter 2 therein for proofs and many more details that we must omit.

2.1. Definitions

For the purposes of this section, given a function ϕ and $j \in \mathbb{Z}$ we denote by ϕ_j the dilation

$$\phi_j(\xi) = \phi(2^{-j}\xi).$$

Let \mathcal{C} be the annulus $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$. There exist radial functions $\chi \in C_c^\infty(B(0, 4/3))$ and $\varphi \in C_c^\infty(\mathcal{C})$ both taking values in $[0, 1]$ such that

$$\text{for all } \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi_j(\xi) = 1, \quad (2.1a)$$

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad (2.1b)$$

$$\text{if } |j - j'| \geq 2, \text{ then } \quad \text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \emptyset, \quad (2.1c)$$

$$\text{if } j \geq 1, \text{ then } \quad \text{supp } \chi \cap \text{supp } \varphi_j = \emptyset; \quad (2.1d)$$

the set $\tilde{\mathcal{C}} := B(0, 2/3) + \mathcal{C}$ is an annulus, and

$$\text{if } |j - j'| \geq 5, \text{ then } \quad 2^{j'} \tilde{\mathcal{C}} \cap 2^j \mathcal{C} = \emptyset. \quad (2.1e)$$

Furthermore, we have

$$\text{for all } \xi \in \mathbb{R}^n, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi_j^2(\xi) \leq 1, \quad (2.1f)$$

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi_j^2(\xi) \leq 1. \quad (2.1g)$$

Denote by

$$\mathcal{F}[u](\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} u(x) dx,$$

the Fourier transform of u , and let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. Given a measurable function σ defined on \mathbb{R}^n with at most polynomial growth at infinity, we define the Fourier multiplier operator M_σ by $M_\sigma u := \mathcal{F}^{-1}(\sigma \hat{u})$.

For $j \in \mathbb{Z}$, the *inhomogeneous dyadic blocks* Δ_j are defined as follows:

$$\begin{aligned} \text{if } j \leq -2, \quad & \Delta_j u = 0, \\ \Delta_{-1} u &= M_\chi u = \int_{\mathbb{R}^n} \tilde{h}(y) u(x-y) dy, \\ \text{if } j \geq 0, \quad & \Delta_j u = M_{\varphi_j} u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y) dy. \end{aligned}$$

The inhomogeneous low-frequency cut-off operator S_j is defined by

$$S_j u := \sum_{j' \leq j-1} \Delta_{j'} u.$$

For $j \in \mathbb{Z}$, the *homogeneous dyadic blocks* $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator \dot{S}_j are defined as follows:

$$\begin{aligned} \dot{\Delta}_j u &= M_{\varphi_j} u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y) dy, \\ \dot{S}_j u &= M_{\chi_j} u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y) u(x-y) dy. \end{aligned}$$

Formally, we can write the following *Littlewood–Paley decompositions*:

$$\text{Id} = \sum_{j \in \mathbb{Z}} \Delta_j \quad \text{and} \quad \text{Id} = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j.$$

In the inhomogeneous case, the decomposition makes sense in $\mathcal{S}'(\mathbb{R}^n)$: if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, then $u = \lim_{j \rightarrow \infty} S_j u$ in $\mathcal{S}'(\mathbb{R}^n)$. Unfortunately, the homogeneous case is a little more involved. We denote by $\mathcal{S}'_h(\mathbb{R}^n)$ the space of tempered distributions such that

$$\lim_{\lambda \rightarrow \infty} \|M_{\theta(\lambda \cdot)} u\|_{L^\infty} = 0 \quad \text{for any } \theta \in C_c^\infty(\mathbb{R}^n).$$

Then the homogeneous decomposition makes sense in $\mathcal{S}'_h(\mathbb{R}^n)$: if $u \in \mathcal{S}'_h(\mathbb{R}^n)$, then $u = \lim_{j \rightarrow \infty} \dot{S}_j u$ in $\mathcal{S}'_h(\mathbb{R}^n)$. Moreover, using the homogeneous decomposition, it is straightforward to show that

$$\dot{S}_j u = \sum_{j' \leq j-1} \dot{\Delta}_{j'} u.$$

Given a real number s and two numbers $p, r \in [1, \infty]$, the *homogeneous Besov space* $B_{p,r}^s(\mathbb{R}^n)$ consists of those distributions u in $\mathcal{S}'_h(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}_{p,r}^s} := \left(\sum_{j \in \mathbb{Z}} 2^{rjs} \|\dot{\Delta}_j u\|_{L^p}^r \right)^{1/r} < \infty$$

if $r < \infty$, and

$$\|u\|_{\dot{B}_{p,\infty}^s} := \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p} < \infty$$

if $r = \infty$. This is a normed space, and its norm is independent of the choice of function φ used to define the blocks $\dot{\Delta}_j$. Note that a distribution $u \in \mathcal{S}'_h(\mathbb{R}^n)$ belongs to $\dot{B}_{p,r}^s(\mathbb{R}^n)$ if, and only if, there exists a constant C and a non-negative sequence $(d_j)_{j \in \mathbb{Z}}$ such that

$$\text{for all } j \in \mathbb{Z}, \quad \|\dot{\Delta}_j u\|_{L^p} \leq C d_j 2^{-js} \quad \text{and} \quad \|(d_j)\|_{\ell^r} = 1. \quad (2.2)$$

It follows immediately from (2.1g) that the seminorms $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}_{2,2}^s}$ are equivalent, and hence that $\dot{H}^s \subset \dot{B}_{2,2}^s$ and that both spaces coincide for $s < n/2$.

We also define the *inhomogeneous Besov space* $B_{p,r}^s(\mathbb{R}^n)$ as the space of those distributions u in $\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{B_{p,r}^s} := \left(\sum_{j \in \mathbb{Z}} 2^{rjs} \|\Delta_j u\|_{L^p}^r \right)^{1/r} < \infty$$

if $r < \infty$, and

$$\|u\|_{B_{p,\infty}^s} := \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p} < \infty$$

if $r = \infty$. It is straightforward to show that $B_{p,r}^s = \dot{B}_{p,r}^s \cap L^p$, and that $B_{p,r}^s$ is always a Banach space. For that reason, we focus mainly on homogeneous Besov spaces; most of the following results have inhomogeneous versions, which can be found in Sections 2.7 and 2.8 of Bahouri et al. (2011).

2.2. Embeddings

Much like the Sobolev embeddings, Besov spaces embed in certain L^p spaces with the correct exponents. We quote the two embeddings we will use most frequently.

Proposition 2.1 (Proposition 2.20 in Bahouri et al. (2011)). *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. For any real number s , we have the continuous embedding*

$$\dot{B}_{p_1,r_1}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{p_2,r_2}^{s-n(1/p_1-1/p_2)}(\mathbb{R}^n).$$

Proposition 2.2 (Proposition 2.39 in Bahouri et al. (2011)). *For $1 \leq p \leq q \leq \infty$, we have the continuous embedding*

$$\dot{B}_{p,1}^{n/p-n/q}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

Note that the homogeneous Besov space $\dot{B}_{p,r}^s(\mathbb{R}^n)$ is a Banach space if, and only if, either $s < n/p$, or $s = n/p$ and $r = 1$ (in contrast to its inhomogeneous counterpart). Indeed, it is the case $\dot{B}_{p,1}^{n/p}$ that most interests us, especially when $p = 2$, for three reasons: it is a Banach space, it embeds continuously in $L^\infty(\mathbb{R}^n)$ by Proposition 2.2, and it is a Banach algebra. The last fact follows from Bony's paraproduct decomposition, which we outline now.

2.3. Homogeneous Paradifferential Calculus

Let u and v be tempered distributions in $\mathcal{S}'_h(\mathbb{R}^n)$. We have

$$u = \sum_{j' \in \mathbb{Z}} \dot{\Delta}_{j'} u \quad \text{and} \quad v = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j v,$$

so, at least formally,

$$uv = \sum_{j, j' \in \mathbb{Z}} \dot{\Delta}_{j'} u \dot{\Delta}_j v.$$

One of the key techniques of paradifferential calculus is to break the above sum into three parts, as follows: define

$$\dot{T}_u v := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v,$$

and

$$\dot{R}(u, v) := \sum_{|k-j| \leq 1} \dot{\Delta}_k u \dot{\Delta}_j v.$$

At least formally, the following *Bony decomposition* holds true:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v).$$

We now state two standard estimates on \dot{T} and \dot{R} that we will use in proving our a priori estimates in Section 3.

Lemma 2.3 (Theorem 2.47 from Bahouri et al. (2011)). *Let $s \in \mathbb{R}$ and $t < 0$. There exists a constant $C = C(s, t)$ such that for any $p, r_1, r_2 \in [1, \infty]$, $u \in \dot{B}_{p, r_1}^t$ and $v \in \dot{B}_{p, r_2}^s$,*

$$\|\dot{T}_u v\|_{\dot{B}_{p, r}^{s+t}} \leq C \|u\|_{\dot{B}_{\infty, r_1}^t} \|v\|_{\dot{B}_{p, r_2}^s}$$

with $\frac{1}{r} = \min \left\{ 1, \frac{1}{r_1} + \frac{1}{r_2} \right\}$.

Lemma 2.4 (Theorem 2.52 from Bahouri et al. (2011)). *Let $s_1, s_2 \in \mathbb{R}$ such that $s_1 + s_2 > 0$. There exists a constant $C = C(s_1, s_2)$ such that, for any $p_1, p_2, r_1, r_2 \in [1, \infty]$, $u \in \dot{B}_{p_1, r_1}^{s_1}$ and $v \in \dot{B}_{p_2, r_2}^{s_2}$,*

$$\|\dot{R}(u, v)\|_{\dot{B}_{p, r}^{s_1+s_2}} \leq C \|u\|_{\dot{B}_{p_1, r_1}^{s_1}} \|v\|_{\dot{B}_{p_2, r_2}^{s_2}}$$

provided that

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

From Lemmas 2.3 and 2.4 it is straightforward to prove that, if $s > 0$ and $p, r \in [1, \infty]$ such that either $s < n/p$, or $s = n/p$ and $r = 1$, then there is a constant C depending only on s and the dimension n such that

$$\|uv\|_{\dot{B}_{p, r}^s} \leq C \left(\|u\|_{L^\infty} \|v\|_{\dot{B}_{p, r}^s} + \|u\|_{\dot{B}_{p, r}^s} \|v\|_{L^\infty} \right).$$

In particular, $L^\infty \cap \dot{B}_{p, r}^s$ is a Banach algebra. Moreover, as $\dot{B}_{p, 1}^{n/p}$ embeds continuously in L^∞ (by Proposition 2.2), we see that $\dot{B}_{p, 1}^{n/p}$ is an algebra and

$$\|uv\|_{\dot{B}_{p, 1}^{n/p}} \leq c \|u\|_{\dot{B}_{p, 1}^{n/p}} \|v\|_{\dot{B}_{p, 1}^{n/p}}. \quad (2.3)$$

3. A Priori Estimates

We first prove the two main a priori estimates that we will use in the existence proof: to streamline the presentation we prove the estimates formally for \mathbf{u} and \mathbf{B} which solve equations (1.1).

Proposition 3.1. *If (\mathbf{u}, \mathbf{B}) solve equations (1.1) on $[0, T]$, then there is a constant c_1 such that, for all $t \in [0, T]$,*

$$\|\mathbf{B}(t)\|_{\dot{B}_{2,1}^{n/2}} \leq \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{n/2}} \exp\left(c_1 \int_0^t \|\nabla \mathbf{u}(s)\|_{\dot{B}_{2,1}^{n/2}} ds\right).$$

Before embarking on the proof, we state a lemma we require, which is a particular case of Lemma 2.100 from Bahouri et al. (2011).

Lemma 3.2. *Let $-1 - n/2 < \sigma < 1 + n/2$ and $1 \leq r \leq \infty$. Let \mathbf{v} be a divergence-free vector field on \mathbb{R}^n , and set $Q_j := [(\mathbf{v} \cdot \nabla), \dot{\Delta}_j]f$. There exists a constant $C = C(\sigma, n)$, such that*

$$\left\| (2^{j\sigma} \|Q_j\|_{L^2})_j \right\|_{\ell^r} \leq C \|\nabla \mathbf{v}\|_{\dot{B}_{2,\infty}^{n/2} \cap L^\infty} \|f\|_{\dot{B}_{2,r}^\sigma}.$$

Proof of Proposition 3.1. Given $j \in \mathbb{Z}$, apply the homogeneous Littlewood–Paley operator $\dot{\Delta}_j$ (see Section 2.1) to the equation (1.1b) for \mathbf{B} to obtain

$$\frac{\partial}{\partial t} \dot{\Delta}_j \mathbf{B} + \dot{\Delta}_j [(\mathbf{u} \cdot \nabla) \mathbf{B}] = \dot{\Delta}_j [(\mathbf{B} \cdot \nabla) \mathbf{u}].$$

As $\dot{B}_{2,1}^{n/2}$ is an algebra (see equation (2.3)), we have

$$\|(\mathbf{B} \cdot \nabla) \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \leq \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}.$$

By (2.2), we may write

$$\|\dot{\Delta}_j [(\mathbf{B} \cdot \nabla) \mathbf{u}]\|_{L^2} \leq C d_j(t) 2^{-jn/2} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}$$

where $d_j(t)$ denotes a sequence in $\ell^1(\mathbb{Z})$ whose sum is 1.

For the term $(\mathbf{u} \cdot \nabla) \mathbf{B}$, we use Bony's paraproduct decomposition:

$$(\mathbf{u} \cdot \nabla) \mathbf{B}_\ell = \sum_{k=1}^n [\dot{T}_{\mathbf{u}_k} \partial_k \mathbf{B}_\ell + \dot{T}_{\partial_k \mathbf{B}_\ell} \mathbf{u}_k + \dot{R}(\mathbf{u}_k, \partial_k \mathbf{B}_\ell)].$$

Consider the second term $\dot{T}_{\partial_k \mathbf{B}_\ell} \mathbf{u}_k$: by Lemma 2.3 we have

$$\begin{aligned} \|\dot{T}_{\partial_k \mathbf{B}_\ell} \mathbf{u}_k\|_{\dot{B}_{2,1}^{n/2}} &\leq c \sum_{k=1}^n \|\partial_k \mathbf{B}_\ell\|_{\dot{B}_{\infty,\infty}^{-1}} \|\mathbf{u}_k\|_{\dot{B}_{2,1}^{n/2+1}} \\ &\leq c \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}, \end{aligned}$$

where we have used that $\dot{B}_{2,1}^{n/2} \hookrightarrow \dot{B}_{\infty,\infty}^0$ (by Proposition 2.1). For the third term $\dot{R}(\mathbf{u}_k, \partial_k \mathbf{B}_\ell)$, we apply Lemma 2.4:

$$\begin{aligned} \|\dot{R}(\mathbf{u}_k, \partial_k \mathbf{B}_\ell)\|_{\dot{B}_{2,1}^{n/2}} &\leq c \sum_{k=1}^n \|\mathbf{u}_k\|_{\dot{B}_{2,1}^{n/2+1}} \|\partial_k \mathbf{B}_\ell\|_{\dot{B}_{\infty,\infty}^{-1}} \\ &\leq c \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}, \end{aligned}$$

as above. Using (2.2), we obtain

$$\begin{aligned} \sum_{k=1}^n \|\dot{\Delta}_j \dot{T}_{\partial_k \mathbf{B}_\ell} \mathbf{u}_k\|_{L^2} &\leq cd_j(t) 2^{-jn/2} \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}, \\ \sum_{k=1}^n \|\dot{\Delta}_j \dot{R}(\mathbf{u}_k, \partial_k \mathbf{B}_\ell)\|_{L^2} &\leq cd_j(t) 2^{-jn/2} \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}. \end{aligned}$$

For the term $\dot{T}_{\mathbf{u}_k} \partial_k \mathbf{B}_\ell$, let us write

$$\begin{aligned} \sum_{k=1}^n \dot{\Delta}_j \dot{T}_{\mathbf{u}_k} \partial_k \mathbf{B}_\ell &= \sum_{j' \in \mathbb{Z}} \sum_{k=1}^n \dot{\Delta}_j \left(\dot{S}_{j'-1} \mathbf{u}_k \partial_k \dot{\Delta}_{j'} \mathbf{B}_\ell \right) \\ &= \sum_{k=1}^n \dot{S}_{j-1} \mathbf{u}_k \partial_k \dot{\Delta}_j \mathbf{B}_\ell \\ &\quad + \sum_{j' \in \mathbb{Z}} \sum_{k=1}^n (\dot{S}_{j'-1} \mathbf{u}_k - \dot{S}_{j-1} \mathbf{u}_k) \partial_k \dot{\Delta}_j \dot{\Delta}_{j'} \mathbf{B}_\ell \\ &\quad + \sum_{j' \in \mathbb{Z}} \sum_{k=1}^n [\dot{\Delta}_j, \dot{S}_{j'-1} \mathbf{u}_k \partial_k] \left(\dot{\Delta}_{j'} \mathbf{B}_\ell \right) \\ &=: (\dot{S}_{j-1} \mathbf{u} \cdot \nabla) \dot{\Delta}_j \mathbf{B}_\ell + P_j + Q_j. \end{aligned}$$

For P_j , by (2.1c) we have

$$\begin{aligned} P_j &:= \sum_{|j-j'| \leq 1} \sum_{k=1}^n (\dot{S}_{j'-1} \mathbf{u}_k - \dot{S}_{j-1} \mathbf{u}_k) \dot{\Delta}_j \dot{\Delta}_{j'} \partial_k \mathbf{B}_\ell \\ &= \sum_{k=1}^n (\dot{\Delta}_{j-1} \mathbf{u}_k) (\dot{\Delta}_j \dot{\Delta}_{j+1} \partial_k \mathbf{B}_\ell) - \sum_{k=1}^n (\dot{\Delta}_{j-2} \mathbf{u}_k) (\dot{\Delta}_j \dot{\Delta}_{j-1} \partial_k \mathbf{B}_\ell), \end{aligned}$$

so as $\|\dot{\Delta}_j \partial_k \mathbf{B}\|_{L^2} \simeq 2^j \|\dot{\Delta}_j \mathbf{B}\|_{L^2}$ we have

$$\begin{aligned} 2^{jn/2} \|P_j\|_{L^2} &\leq c \left(4 \cdot 2^{j-1} \|\dot{\Delta}_{j-1} \mathbf{u}\|_{L^\infty} 2^{jn/2} \|\dot{\Delta}_j \mathbf{B}_\ell\|_{L^2} \right. \\ &\quad \left. + 2 \cdot 2^{j-2} \|\dot{\Delta}_{j-2} \mathbf{u}\|_{L^\infty} 2^{jn/2} \|\dot{\Delta}_j \mathbf{B}_\ell\|_{L^2} \right) \\ &\leq cd_j(t) \|\mathbf{u}\|_{\dot{B}_{\infty, \infty}^1} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \\ &\leq cd_j(t) \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}. \end{aligned}$$

For Q_j , we apply Lemma 3.2: note that

$$Q_j := \sum_{j' \in \mathbb{Z}} [\dot{\Delta}_j, \dot{S}_{j'-1}(\mathbf{u} \cdot \nabla)] \left(\dot{\Delta}_{j'} \mathbf{B}_\ell \right)$$

so

$$\begin{aligned} \left\| \left(2^{jn/2} \|Q_j\|_{L^2} \right)_j \right\|_{\ell^1} &\leq c \|\nabla \mathbf{u}\|_{\dot{B}_{2, \infty}^{n/2} \cap L^\infty} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \\ &\leq c \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{n/2}} \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \end{aligned}$$

since $\dot{B}_{2,1}^{n/2}$ embeds continuously in both $\dot{B}_{2,\infty}^{n/2}$ (by Proposition 2.1) and L^∞ (by Proposition 2.2). So by (2.2),

$$\|Q_j\|_{L^2} \leq cd_j(t)2^{-jn/2}\|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}\|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}.$$

By combining all the above estimates, we obtain

$$\frac{\partial}{\partial t}\dot{\Delta}_j\mathbf{B} + (\dot{S}_{j-1}\mathbf{u} \cdot \nabla)\dot{\Delta}_j\mathbf{B} = F_j(t), \quad (3.1)$$

where

$$\|F_j(t)\|_{L^2} \leq cd_j(t)2^{-jn/2}\|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}\|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}.$$

Taking the inner product of (3.1) with $\dot{\Delta}_j\mathbf{B}$ and using the fact that \mathbf{u} (and hence $\dot{S}_{j-1}\mathbf{u}$) is divergence-free, we obtain

$$2^{jn/2}\frac{d}{dt}\|\dot{\Delta}_j\mathbf{B}\|_{L^2} \leq 2cd_j(t)\|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}\|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}$$

so summing in j yields

$$\frac{d}{dt}\|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}} \leq c\|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{n/2}}\|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}$$

and the result follows by Gronwall's inequality. \square

Our second estimate, for the \mathbf{u} equation alone, is stated for a general forcing term \mathbf{f} .

Proposition 3.3. *Let $\mathbf{f} \in L^1(0, T; \dot{B}_{2,1}^{n/2-1}(\mathbb{R}^n))$. Suppose \mathbf{u} solves*

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad (3.2a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3.2b)$$

on the time interval $[0, T]$. Then there is a constant c_2 such that, for all $t \in [0, T]$,

$$\begin{aligned} & \|\mathbf{u}(t)\|_{\dot{B}_{2,1}^{n/2-1}} + \nu \int_0^t \|\nabla\mathbf{u}(s)\|_{\dot{B}_{2,1}^{n/2}} ds \\ & \leq \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}} + c_2 \int_0^t \|\mathbf{u}(s)\|_{H^{n/2}}^2 ds + c_2 \int_0^t \|\mathbf{f}(s)\|_{\dot{B}_{2,1}^{n/2-1}} ds. \end{aligned}$$

Note that in the particular case $\mathbf{f} = (\mathbf{B} \cdot \nabla)\mathbf{B} = \nabla \cdot (\mathbf{B} \otimes \mathbf{B})$, we have

$$\|\mathbf{f}\|_{\dot{B}_{2,1}^{n/2-1}} = \|\nabla \cdot (\mathbf{B} \otimes \mathbf{B})\|_{\dot{B}_{2,1}^{n/2-1}} \leq \|\mathbf{B}\|_{\dot{B}_{2,1}^{n/2}}^2 \quad (3.3)$$

since $\dot{B}_{2,1}^{n/2}$ is an algebra.

In the proof we will need the following inequality:

$$\sum_{j \in \mathbb{Z}} 2^{j(n-2)} \langle \dot{\Delta}_j[(\mathbf{u} \cdot \nabla)\mathbf{u}], \dot{\Delta}_j\mathbf{u} \rangle \leq c\|\mathbf{u}\|_{H^{n/2}}^2\|\mathbf{u}\|_{\dot{B}_{2,1}^{n/2-1}}. \quad (3.4)$$

This result can be easily obtained by an elementary modification of Lemma 1.1 from Chemin (1992), which in particular shows that for a divergence-free vector field \mathbf{u} we have

$$\langle \Lambda^{n/2-1}[(\mathbf{u} \cdot \nabla)\mathbf{u}], \Lambda^{n/2-1}\mathbf{u} \rangle \leq c\|\mathbf{u}\|_{H^{n/2}}^2\|\mathbf{u}\|_{\dot{H}^{n/2-1}}. \quad (3.5)$$

Proof of Proposition 3.3. Applying the Littlewood–Paley operator $\dot{\Delta}_j$ to equation (3.2) yields

$$\frac{\partial}{\partial t} \dot{\Delta}_j \mathbf{u} + \dot{\Delta}_j [(\mathbf{u} \cdot \nabla) \mathbf{u}] - \nu \Delta \dot{\Delta}_j \mathbf{u} + \nabla \dot{\Delta}_j p = \dot{\Delta}_j \mathbf{f}.$$

Taking the inner product with $\dot{\Delta}_j \mathbf{u}$ yields

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_j \mathbf{u}\|_{L^2}^2 + c\nu 2^{2j} \|\dot{\Delta}_j \mathbf{u}\|_{L^2}^2 \leq \left| \langle \dot{\Delta}_j [(\mathbf{u} \cdot \nabla) \mathbf{u}], \dot{\Delta}_j \mathbf{u} \rangle \right| + \left| \langle \dot{\Delta}_j \mathbf{f}, \dot{\Delta}_j \mathbf{u} \rangle \right|.$$

Now, using estimate (3.4), by (2.2) (and dividing by $2^{j(n-2)}$) we obtain

$$\left| \langle \dot{\Delta}_j [(\mathbf{u} \cdot \nabla) \mathbf{u}], \dot{\Delta}_j \mathbf{u} \rangle \right| \leq cd_j(t) 2^{-j(n/2-1)} \|\mathbf{u}\|_{H^{n/2}}^2 \|\dot{\Delta}_j \mathbf{u}\|_{L^2}.$$

Hence

$$\begin{aligned} \frac{d}{dt} \|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2}^2 + c\nu 2^{2j} \|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2}^2 \\ \leq cd_j(t) 2^{-j(n/2-1)} \left(\|\mathbf{u}(t)\|_{H^{n/2}}^2 + \|\mathbf{f}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \right) \|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2}. \end{aligned}$$

Dividing through by $\|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2}$ and multiplying by $e^{c\nu 2^{2j}t}$ yields

$$\begin{aligned} \frac{d}{dt} \left(e^{c\nu 2^{2j}t} \|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2} \right) \\ \leq ce^{c\nu 2^{2j}t} d_j(t) 2^{-j(n/2-1)} \left(\|\mathbf{u}(t)\|_{H^{n/2}}^2 + \|\mathbf{f}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \right). \end{aligned}$$

Integrating in time from 0 to t yields

$$\begin{aligned} \|\dot{\Delta}_j \mathbf{u}(t)\|_{L^2} \leq \|\dot{\Delta}_j \mathbf{u}_0\|_{L^2} e^{-c\nu 2^{2j}t} \\ + c 2^{-j(n/2-1)} \int_0^t d_j(s) e^{-c\nu 2^{2j}(t-s)} \left(\|\mathbf{u}(s)\|_{H^{n/2}}^2 + \|\mathbf{f}(s)\|_{\dot{B}_{2,1}^{n/2-1}} \right) ds. \end{aligned} \quad (3.6)$$

As $e^{-c\nu 2^{2j}t} \leq 1$ for all t , multiplying (3.6) by $2^{j(n/2-1)}$ and summing in j yields

$$\|\mathbf{u}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \leq \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}} + c \int_0^t \left(\|\mathbf{u}(s)\|_{H^{n/2}}^2 + \|\mathbf{f}(s)\|_{\dot{B}_{2,1}^{n/2-1}} \right) ds.$$

Taking the L^∞ norm over $t \in [0, T]$ yields

$$\|\mathbf{u}\|_{L^\infty(0, T; \dot{B}_{2,1}^{n/2-1})} \leq \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}} + c \int_0^T \left(\|\mathbf{u}(t)\|_{H^{n/2}}^2 + \|\mathbf{f}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \right) dt.$$

Multiplying (3.6) by $\nu 2^{2j(n/2+1)}$ and then taking the L^1 norm over $t \in [0, T]$ yields

$$\begin{aligned} \nu 2^{j(n/2)} \|\dot{\Delta}_j \nabla \mathbf{u}\|_{L^1(0, T; L^2)} \leq 2^{j(n/2-1)} \|\dot{\Delta}_j \mathbf{u}_0\|_{L^2} \int_0^T \nu 2^{2j} e^{-c\nu 2^{2j}t} dt \\ + c \int_0^T \int_0^t d_j(s) \nu 2^{2j} e^{-c\nu 2^{2j}(t-s)} \left(\|\mathbf{u}(s)\|_{H^{n/2}}^2 + \|\mathbf{f}(s)\|_{\dot{B}_{2,1}^{n/2-1}} \right) ds dt. \end{aligned}$$

Using Young's inequality for convolutions and the fact that

$$\int_0^T c\nu 2^{2j} e^{-c\nu 2^{2j} t} dt = 1 - e^{-c\nu 2^{2j} T} \leq 1$$

yields

$$\begin{aligned} \nu 2^{j(n/2)} \|\dot{\Delta}_j \nabla \mathbf{u}\|_{L^1(0,T;L^2)} &\leq c 2^{j(n/2-1)} \|\dot{\Delta}_j \mathbf{u}_0\|_{L^2} \\ &+ c \int_0^T d_j(t) \left(\|\mathbf{u}(t)\|_{H^{n/2}}^2 + \|\mathbf{f}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \right) dt. \end{aligned}$$

Summation in j and the Monotone Convergence Theorem yields

$$\nu \|\nabla \mathbf{u}\|_{L^1(0,T;\dot{B}_{2,1}^{n/2})} \leq \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}} + c \int_0^T \left(\|\mathbf{u}(t)\|_{H^{n/2}}^2 + \|\mathbf{f}(t)\|_{\dot{B}_{2,1}^{n/2-1}} \right) dt.$$

This completes the proof. \square

4. Uniform Bounds in 2D and 3D

To turn our a priori estimates into a rigorous proof, we consider a Fourier truncation of the equations (1.1). We define the Fourier truncation \mathcal{S}_R as follows:

$$\widehat{\mathcal{S}_R f}(\xi) = \mathbb{1}_{B_R}(\xi) \hat{f}(\xi),$$

where B_R denotes the ball of radius R centered at the origin. Note that

$$\begin{aligned} \|\mathcal{S}_R f - f\|_{H^s}^2 &= \int_{(B_R)^c} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \\ &= \int_{(B_R)^c} \frac{1}{(1 + |\xi|^2)^k} (1 + |\xi|^2)^{s+k} |\hat{f}(\xi)|^2 d\xi \\ &\leq \frac{1}{(1 + R^2)^k} \int_{(B_R)^c} (1 + |\xi|^2)^{s+k} |\hat{f}(\xi)|^2 d\xi \\ &\leq \frac{C}{R^{2k}} \|f\|_{H^{s+k}}^2. \end{aligned}$$

Hence

$$\|\mathcal{S}_R f - f\|_{H^s} \leq C(1/R)^k \|f\|_{H^{s+k}}, \quad (4.1)$$

$$\|\mathcal{S}_R f - \mathcal{S}_{R'} f\|_{H^s} \leq C \max\{(1/R)^k, (1/R')^k\} \|f\|_{H^{s+k}}. \quad (4.2)$$

We consider the truncated MHD equations on the whole of \mathbb{R}^n :

$$\frac{\partial \mathbf{u}^R}{\partial t} - \nu \Delta \mathbf{u}^R + \nabla p_*^R = \mathcal{S}_R[(\mathbf{B}^R \cdot \nabla) \mathbf{B}^R] - \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla) \mathbf{u}^R], \quad (4.3a)$$

$$\frac{\partial \mathbf{B}^R}{\partial t} = \mathcal{S}_R[(\mathbf{B}^R \cdot \nabla) \mathbf{u}^R] - \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla) \mathbf{B}^R], \quad (4.3b)$$

$$\nabla \cdot \mathbf{u}^R = \nabla \cdot \mathbf{B}^R = 0, \quad (4.3c)$$

with initial data $\mathcal{S}_R \mathbf{u}_0, \mathcal{S}_R \mathbf{B}_0$. By taking the truncated initial data as we have, we ensure that $\mathbf{u}^R, \mathbf{B}^R$ lie in the space

$$V_R := \{f \in L^2(\mathbb{R}^n) : \hat{f} \text{ is supported in } B_R\},$$

as the truncations are invariant under the flow of the equations. The Fourier truncations act like mollifiers, smoothing the equation; in particular, on the space V_R it is easy to show that

$$F(\mathbf{u}^R, \mathbf{B}^R) := \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla)\mathbf{B}^R]$$

is Lipschitz in \mathbf{u}^R and \mathbf{B}^R . Hence, by Picard's theorem for infinite-dimensional ODEs (see Theorem 3.1 in Majda and Bertozzi (2002), for example), there exists a solution $(\mathbf{u}^R, \mathbf{B}^R)$ in V_R to (4.3) for some time interval $[0, T(R)]$.

The solution will exist as long as the relevant norms of \mathbf{u}^R and \mathbf{B}^R remain finite. Repeating the a priori estimates from Proposition 3.1 we obtain

$$\|\mathbf{B}^R(t)\|_{\dot{B}_{2,1}^{n/2}} \leq \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{n/2}} \exp\left(c_1 \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{n/2}} ds\right),$$

where the constant c_1 is independent of R . Repeating Proposition 3.3 for the equation

$$\frac{\partial \mathbf{u}^R}{\partial t} + \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla)\mathbf{u}^R] - \nu \Delta \mathbf{u}^R + \nabla p_*^R = \mathbf{f}^R, \quad (4.4a)$$

$$\nabla \cdot \mathbf{u}^R = 0. \quad (4.4b)$$

yields

$$\begin{aligned} & \|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^{n/2-1}} + \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{n/2}} ds \\ & \leq \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}} + c_2 \int_0^t \|\mathbf{u}^R(s)\|_{H^{n/2}}^2 ds + c_2 \int_0^t \|\mathbf{f}^R(s)\|_{\dot{B}_{2,1}^{n/2-1}} ds, \end{aligned}$$

where the constant c_2 is independent of R .

Turning these estimates into uniform bounds on \mathbf{u}^R and \mathbf{B}^R which are independent of R depends on the dimension, so we consider the 2D and 3D cases separately. However, in both cases we will make use of the following standard energy estimate:

$$\begin{aligned} & \sup_{t \in [0, T]} \|\mathbf{u}^R(t)\|_{L^2}^2 + \sup_{t \in [0, T]} \|\mathbf{B}^R(t)\|_{L^2}^2 + \nu \int_0^T \|\nabla \mathbf{u}^R(s)\|_{L^2}^2 ds \\ & \leq 2(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2) \end{aligned} \quad (4.5)$$

for any $T > 0$, which can be obtained by taking the inner product of (1.1a) with \mathbf{u}^R , the inner product of (1.1b) with \mathbf{B}^R , and adding.

4.1. Uniform Bounds in Two Dimensions

In 2D, the term $\int_0^t \|\mathbf{u}^R(s)\|_{H^{n/2}}^2 ds$ is simply $\int_0^t \|\mathbf{u}^R(s)\|_{H^1}^2 ds$. Using the standard energy estimate (4.5) we may bound this as follows:

$$\begin{aligned} & \int_0^t \|\mathbf{u}^R(s)\|_{H^1}^2 ds \leq \int_0^t \|\mathbf{u}^R(s)\|_{L^2}^2 ds + \int_0^t \|\nabla \mathbf{u}^R(s)\|_{L^2}^2 ds \\ & \leq 2\left(t + \frac{1}{\nu}\right) (\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2). \end{aligned} \quad (4.6)$$

Using this, we show that \mathbf{u}^R and \mathbf{B}^R are uniformly bounded.

Theorem 4.1. *Let $n = 2$, and let $(\mathbf{u}^R, \mathbf{B}^R)$ be the solution to (4.3). There is a time $T_* = T_*(\nu, \|\mathbf{u}_0\|_{B_{2,1}^0}, \|\mathbf{B}_0\|_{B_{2,1}^1}) > 0$ such that*

$$\begin{aligned} \mathbf{u}^R & \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap L^1(0, T_*; \dot{B}_{2,1}^2(\mathbb{R}^2)), \\ \mathbf{B}^R & \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^1(\mathbb{R}^2)). \end{aligned}$$

Proof. Let

$$\begin{aligned} M_1 &= \|\mathbf{u}_0\|_{\dot{B}_{2,1}^0} + \frac{2c_2}{\nu} (\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2), \\ M_2 &= 2c_2 (\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2). \end{aligned}$$

Substituting from equation (4.6) into Proposition 3.3, we obtain

$$\|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^0} + \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^1} ds \leq M_1 + M_2 t + c_2 \int_0^t \|\mathbf{f}^R(s)\|_{\dot{B}_{2,1}^0} ds.$$

Using (3.3) and substituting in from Proposition 3.1, we obtain

$$\begin{aligned} \|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^0} + \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^1} ds & \\ & \leq M_1 + M_2 t + \int_0^t \|\mathbf{B}_0\|_{\dot{B}_{2,1}^1}^2 c_2 \exp\left(2c_1 \int_0^\tau \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^1} ds\right) d\tau. \\ & \leq M_1 + M_2 t + M_3 t \exp\left(2c_1 \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^1} ds\right), \end{aligned}$$

where $M_3 = c_1 \|\mathbf{B}_0\|_{\dot{B}_{2,1}^1}^2$. Let

$$\begin{aligned} X_R(t) &= \|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^0}, \\ Y_R(t) &= \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^1} ds. \end{aligned}$$

Then we can rewrite the last inequality as

$$X_R(t) + Y_R(t) \leq M_1 + M_2 t + M_3 t \exp(2c_1 Y_R(t)/\nu). \quad (4.7)$$

Set

$$T_* = \min \left\{ \frac{M_1}{M_2}, \frac{M_1}{M_3} \exp(-6c_1 M_1/\nu) \right\}.$$

It remains to show that $X_R(t) + Y_R(t) \leq 3M_1$ for all $t \in [0, T_*]$ and all $R > 0$. To that end, note that $Y_R(t)$ is continuous and $Y_R(0) = 0$. Now, suppose $t < T_*$ and $Y_R(t) \leq 3M_1$; then

$$\begin{aligned} Y_R(t) & \leq M_1 + M_2 t + M_3 t \exp(2c_1 Y_R(t)/\nu) \\ & < M_1 + M_1 + M_1 \exp\left(\frac{2c_1}{\nu} [Y_R(t) - 3M_1]\right) \\ & \leq 3M_1. \end{aligned}$$

This means that $Y_R(t)$ can never equal $3M_1$ on the interval $[0, T_*)$; so $Y_R(t) < 3M_1$ for all $t \in [0, T_*)$. The result follows from inequality (4.7) and Proposition 3.1. \square

Before moving onto the 3D case, it is worth noting that in 2D the existence time T_* depends only on the norm $\|\mathbf{u}_0\|_{B_{2,1}^0}$ rather than \mathbf{u}_0 itself.

4.2. Uniform Bounds in Three Dimensions

In 3D, we take initial data $\mathbf{u}_0 \in B_{2,1}^{1/2}(\mathbb{R}^3)$ and $\mathbf{B}_0 \in B_{2,1}^{3/2}(\mathbb{R}^3)$. Instead of being able to use the energy inequality, we require the following auxiliary estimate to bound $\int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds$.

Proposition 4.2. *Let $n = 3$. There exist constants c_3 and c_4 and a time $T_1 = T_1(\nu, \mathbf{u}_0)$ such that, if $T \leq T_1$, $R > 0$ and*

$$\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \leq \frac{\nu}{2(c_3 c_4)^{1/4}} =: C_*, \quad (4.8)$$

the solution $(\mathbf{u}^R, \mathbf{B}^R)$ of (4.3) satisfies

$$\begin{aligned} \int_0^T \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds &\leq \frac{1}{\nu} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^2 + \frac{8c_3}{\nu^3} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^4 \\ &\quad + \frac{3}{2\nu} \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^2 + \frac{4c_3}{\nu^3} \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^4. \end{aligned} \quad (4.9)$$

Note carefully that the estimate (4.9) is conditional on assumption (4.8) holding: once we have proved the proposition, we will require a further lemma to ensure that there is a time such that assumption (4.8) holds, and thus avoid a circular argument.

Proof of Proposition 4.2. The proof is based on the proof of Theorem 1 in Marín-Rubio, Robinson and Sadowski (2013), which in turn is based on the proof of Theorem 3.4 in Chemin et al. (2006); the original idea of splitting the equation is due to Calderón (1990).

First, let us consider the Stokes equation with initial data \mathbf{u}_0 :

$$\frac{\partial \mathbf{h}}{\partial t} - \nu \Delta \mathbf{h} + \nabla p_{\mathbf{h}} = 0, \quad (4.10a)$$

$$\nabla \cdot \mathbf{h} = 0, \quad (4.10b)$$

$$\mathbf{h}(0) = \mathbf{u}_0, \quad (4.10c)$$

Thanks to the properties of the Stokes equation and of Fourier truncations, the solution of the equation

$$\frac{\partial \mathbf{h}^R}{\partial t} - \nu \Delta \mathbf{h}^R + \nabla p_{\mathbf{h}^R} = 0, \quad (4.11a)$$

$$\nabla \cdot \mathbf{h}^R = 0, \quad (4.11b)$$

$$\mathbf{h}^R(0) = \mathcal{S}_R \mathbf{u}_0, \quad (4.11c)$$

is given by $\mathbf{h}^R = \mathcal{S}_R \mathbf{h}$.

Let us decompose $\mathbf{u}^R = \mathbf{h}^R + \mathbf{v}^R + \mathbf{w}^R$, where \mathbf{v}^R and \mathbf{w}^R satisfy

$$\frac{\partial \mathbf{v}^R}{\partial t} - \nu \Delta \mathbf{v}^R + \nabla p_{\mathbf{v}^R} = \mathcal{S}_R [(\mathbf{B}^R \cdot \nabla) \mathbf{B}^R], \quad (4.12a)$$

$$\nabla \cdot \mathbf{v}^R = 0, \quad (4.12b)$$

$$\mathbf{v}^R(0) = 0, \quad (4.12c)$$

and

$$\frac{\partial \mathbf{w}^R}{\partial t} - \nu \Delta \mathbf{w}^R + \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla) \mathbf{u}^R] + \nabla p_{\mathbf{w}}^R = 0, \quad (4.13a)$$

$$\nabla \cdot \mathbf{w}^R = 0, \quad (4.13b)$$

$$\mathbf{w}^R(0) = 0, \quad (4.13c)$$

respectively.

Applying $\Lambda^{1/2}$ to (4.13a) and taking the inner product with $\Lambda^{1/2} \mathbf{w}^R$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{w}^R\|_{\dot{H}^{1/2}}^2 + \nu \|\mathbf{w}^R\|_{\dot{H}^{3/2}}^2 &= \langle \Lambda^{1/2} \mathcal{S}_R[(\mathbf{u}^R \cdot \nabla) \mathbf{u}^R], \Lambda^{1/2} \mathbf{w}^R \rangle \\ &= \langle (\mathbf{u}^R \cdot \nabla) \mathbf{u}^R, \Lambda \mathbf{w}^R \rangle \\ &\leq \|\mathbf{u}^R\|_{L^6} \|\nabla \mathbf{u}^R\|_{L^2} \|\Lambda \mathbf{w}^R\|_{L^3} \\ &\leq c \|\mathbf{u}^R\|_{\dot{H}^1}^2 \|\mathbf{w}^R\|_{\dot{H}^{3/2}} \\ &\leq c (\|\mathbf{h}^R\|_{\dot{H}^1}^2 + \|\mathbf{v}^R\|_{\dot{H}^1}^2 + \|\mathbf{w}^R\|_{\dot{H}^1}^2) \|\mathbf{w}^R\|_{\dot{H}^{3/2}} \\ &\leq c \|\mathbf{h}^R\|_{\dot{H}^1}^2 \|\mathbf{w}^R\|_{\dot{H}^{3/2}} + c \|\mathbf{v}^R\|_{\dot{H}^1}^2 \|\mathbf{w}^R\|_{\dot{H}^{3/2}} \\ &\quad + c \|\mathbf{w}^R\|_{\dot{H}^{1/2}} \|\mathbf{w}^R\|_{\dot{H}^{3/2}}^2 \end{aligned}$$

by interpolation. Using Young's inequality, we obtain

$$\frac{d}{dt} \|\mathbf{w}^R\|_{\dot{H}^{1/2}}^2 + \nu \|\mathbf{w}^R\|_{\dot{H}^{3/2}}^2 \leq \frac{c_3}{\nu} \|\mathbf{h}^R\|_{\dot{H}^1}^4 + \frac{c_3}{\nu} \|\mathbf{v}^R\|_{\dot{H}^1}^4 + c_4 \|\mathbf{w}^R\|_{\dot{H}^{1/2}} \|\mathbf{w}^R\|_{\dot{H}^{3/2}}^2.$$

For any $T > t > 0$, integrating in time over $[0, t]$ yields

$$\begin{aligned} &\|\mathbf{w}^R(t)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^t \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ &\leq \frac{c_3}{\nu} \int_0^t \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds + \frac{c_3}{\nu} \int_0^t \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds \\ &\quad + c_4 \int_0^t \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}} \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ &\leq \frac{c_3}{\nu} \int_0^T \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds + \frac{c_3}{\nu} \int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds \\ &\quad + \frac{1}{2} \sup_{s \in [0, T]} \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 + \frac{c_4}{2} \left(\int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \right)^2, \end{aligned}$$

so taking the supremum on the left-hand side over $t \in [0, T]$ yields

$$\begin{aligned} &\sup_{s \in [0, T]} \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ &\leq \frac{4c_3}{\nu} \int_0^T \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds + \frac{4c_3}{\nu} \int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds \\ &\quad + 2c_4 \left(\int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \right)^2. \end{aligned} \quad (4.14)$$

Set

$$T(R) := \sup \left\{ T \geq 0 : \int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \leq \frac{\nu}{2c_4} \right\}$$

so that for all $T \in [0, T(R)]$ we have

$$\begin{aligned} & \sup_{s \in [0, T]} \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ & \leq \frac{4c_3}{\nu} \int_0^T \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds + \frac{4c_3}{\nu} \int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds. \end{aligned} \quad (4.15)$$

We now seek a bound on the right-hand side: indeed, if we can find a time T_0 such that

$$\frac{4c_3}{\nu} \int_0^{T_0} \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds + \frac{4c_3}{\nu} \int_0^{T_0} \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds < \frac{\nu^2}{2c_4}, \quad (4.16)$$

then $T(R) \geq T_0$. To see this, we proceed along the same lines as in the proof of Theorem 4.1: first note that

$$\int_0^{T(R)} \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds = \frac{\nu}{2c_4}$$

by continuity; but if $T(R) < T_0$ then (4.14) and (4.16) would imply that

$$\int_0^{T(R)} \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds < \frac{\nu}{2c_4},$$

which is a contradiction, and thus we must have $T(R) \geq T_0$.

First, let us find a bound for the \mathbf{h} term. Applying $\Lambda^{1/2}$ to (4.10a) and taking the inner product with $\Lambda^{1/2}\mathbf{h}$ yields

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{h}\|_{\dot{H}^{1/2}}^2 + \nu \|\mathbf{h}\|_{\dot{H}^{3/2}}^2 \leq 0.$$

For any $T > t > 0$, integrating in time over $[0, t]$ yields

$$\frac{1}{2} \|\mathbf{h}(t)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^t \|\mathbf{h}(s)\|_{\dot{H}^{3/2}}^2 ds \leq \frac{1}{2} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^2,$$

and thus

$$\sup_{s \in [0, T]} \|\mathbf{h}(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\mathbf{h}(s)\|_{\dot{H}^{3/2}}^2 ds \leq 2\|\mathbf{u}_0\|_{\dot{H}^{1/2}}^2. \quad (4.17)$$

By interpolation,

$$\int_0^T \|\mathbf{h}(s)\|_{\dot{H}^1}^4 ds \leq \frac{2}{\nu} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^4. \quad (4.18)$$

Hence $\|\mathbf{h}(s)\|_{\dot{H}^1}^4$ is integrable on $[0, T]$, and thus we may choose T_1 such that

$$\int_0^{T_1} \|\mathbf{h}(s)\|_{\dot{H}^1}^4 ds < \frac{\nu^3}{16c_3c_4}.$$

By the properties of the Stokes equation, this implies that

$$\int_0^{T_1} \|\mathbf{h}^R(s)\|_{\dot{H}^1}^4 ds < \frac{\nu^3}{16c_3c_4} \quad (4.19)$$

for all $R > 0$. Note that, unlike the 2D case, T_1 really depends on the whole of \mathbf{u}_0 .

Secondly, let us find a bound for the \mathbf{v} term. Applying $\Lambda^{1/2}$ to (4.12a) and taking the inner product with $\Lambda^{1/2}\mathbf{v}^R$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}^R\|_{\dot{H}^{1/2}}^2 + \nu \|\mathbf{v}^R\|_{\dot{H}^{3/2}}^2 &\leq \|(\mathbf{B}^R \cdot \nabla) \mathbf{B}^R\|_{\dot{H}^{1/2}} \|\mathbf{v}^R\|_{\dot{H}^{1/2}} \\ &\leq \|\mathbf{B}^R\|_{\dot{B}_{2,1}^{3/2}}^2 \|\mathbf{v}^R\|_{\dot{H}^{1/2}} \end{aligned}$$

by (3.3). Dropping the second term on the left-hand side yields

$$\frac{d}{dt} \|\mathbf{v}^R\|_{\dot{H}^{1/2}} \leq \|\mathbf{B}^R\|_{\dot{B}_{2,1}^{3/2}}^2.$$

For any $T > t > 0$, integrating in time over $[0, t]$ and taking the supremum over $t \in [0, T]$ yields

$$\sup_{s \in [0, T]} \|\mathbf{v}^R(s)\|_{\dot{H}^{1/2}} \leq \int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds.$$

This implies that

$$\|\mathbf{v}^R(t)\|_{\dot{H}^{3/2}}^2 \leq \frac{1}{\nu} \|\mathbf{B}^R(t)\|_{\dot{B}_{2,1}^{3/2}}^2 \int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds,$$

so that

$$\sup_{s \in [0, T]} \|\mathbf{v}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^{3/2}}^2 ds \leq 3 \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^2. \quad (4.20)$$

Hence by interpolation,

$$\int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds \leq \frac{1}{\nu} \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^4. \quad (4.21)$$

Now, let $T \leq T_1$ be any time such that assumption (4.8) holds. Then we obtain

$$\int_0^T \|\mathbf{v}^R(s)\|_{\dot{H}^1}^4 ds \leq \frac{\nu^3}{16c_3c_4}. \quad (4.22)$$

Combining (4.19) and (4.22) yields (4.16) with $T_0 = T$, and hence $T(R) \geq T$ for all such T ; in particular, $T(R) \geq T_1$.

Moreover, (4.15) holds on the interval $[0, T]$, and substituting (4.18) and (4.21) into (4.15) yields

$$\begin{aligned} \sup_{s \in [0, T]} \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 + \nu \int_0^T \|\mathbf{w}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ \leq \frac{8c_3}{\nu^2} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^4 + \frac{4c_3}{\nu^2} \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^4. \end{aligned} \quad (4.23)$$

Hence, using (4.17), (4.20) and (4.23), we obtain

$$\begin{aligned}
& \sup_{s \in [0, T]} \|\mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds \\
& \leq \sup_{s \in [0, T]} \|\mathbf{h}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \mathbf{h}^R(s)\|_{\dot{H}^{1/2}}^2 ds \\
& \quad + \sup_{s \in [0, T]} \|\mathbf{v}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \mathbf{v}^R(s)\|_{\dot{H}^{1/2}}^2 ds \\
& \quad + \sup_{s \in [0, T]} \|\mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 + 2\nu \int_0^T \|\nabla \mathbf{w}^R(s)\|_{\dot{H}^{1/2}}^2 ds \\
& \leq 2\|\mathbf{u}_0\|_{\dot{H}^{1/2}}^2 + \frac{16c_3}{\nu^2} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^4 \\
& \quad + 3 \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^2 + \frac{8c_3}{\nu^2} \left(\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^4.
\end{aligned}$$

This completes the proof. \square

Proposition 4.2 appears to show that the existence time for the \mathbf{u} equation depends on the existence time for the \mathbf{B} equation; but it is clear from Proposition 3.1 that the existence time for the \mathbf{B} equation ought to depend on the existence time for the \mathbf{u} equation. In order to circumvent this seemingly circular argument, we now show that there is some (short) time interval such that (4.8) holds for all $R > 0$.

Lemma 4.3. *There is a time $T_2 = T_2(\nu, \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}, \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}) > 0$ such that*

$$\int_0^T \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \leq \frac{\nu}{2(c_3 c_4)^{1/4}} =: C_*$$

for all $T \leq \min\{T_1, T_2\}$ and all $R > 0$.

Proof. Define

$$Z_R(t) := \int_0^t \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds.$$

Using the estimate on $\|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}$ from Proposition 3.1, we obtain

$$Z_R(t) \leq t \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2 \exp \left(2c_1 \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds \right).$$

Using the estimate on $\int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds$ from Proposition 3.3, we obtain

$$\begin{aligned}
Z_R(t) \leq t \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2 \exp \left(\frac{2c_1}{\nu} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}} + \frac{2c_1 c_2}{\nu} \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds \right. \\
\left. + \frac{2c_1 c_2}{\nu} Z_R(t) \right). \quad (4.24)
\end{aligned}$$

Recall from (4.8) that $C_* := \frac{\nu}{2(c_3 c_4)^{1/4}}$. Let

$$T_2 := \frac{C_*}{\|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2} \exp \left(-\frac{2c_1}{\nu} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}} - \frac{2c_1 c_2}{\nu^2} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^2 - \frac{8c_1 c_2 c_3}{\nu^4} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^4 \right. \\ \left. - \frac{3c_1 c_2}{\nu^2} C_* - \frac{2c_1 c_2}{\nu} C_*^2 - \frac{4c_1 c_2 c_3}{\nu^4} C_*^4 \right).$$

Suppose $t < \min\{T_1, T_2\}$ and $Z_R(t) \leq C_*$. Then using Proposition 4.2 to estimate the term $\int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds$, from (4.24) we obtain

$$Z_R(t) \leq t \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2 \exp \left(\frac{2c_1}{\nu} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}} + \frac{2c_1 c_2}{\nu^2} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^2 + \frac{16c_1 c_2 c_3}{\nu^4} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^4 \right. \\ \left. + \frac{3c_1 c_2}{\nu^2} Z_R(t) + \frac{2c_1 c_2}{\nu} [Z_R(t)]^2 + \frac{8c_1 c_2 c_3}{\nu^4} [Z_R(t)]^4 \right) \\ \leq t \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2 \exp \left(\frac{2c_1}{\nu} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}} + \frac{2c_1 c_2}{\nu^2} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^2 + \frac{16c_1 c_2 c_3}{\nu^4} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^4 \right. \\ \left. + \frac{3c_1 c_2}{\nu^2} C_* + \frac{2c_1 c_2}{\nu} C_*^2 + \frac{8c_1 c_2 c_3}{\nu^4} C_*^4 \right) \\ < C_*.$$

As $Z_R(t)$ is continuous and $Z_R(0) = 0$, this means that $Z_R(t)$ can never equal C_* as long as $0 \leq t < \min\{T_1, T_2\}$, and hence $Z_R(t) < C_*$ for all $0 \leq t < \min\{T_1, T_2\}$. \square

Combining the energy estimate (4.5) with Proposition 4.2 and Lemma 4.3, we obtain the following bound on $\int_0^t \|\mathbf{u}^R(s)\|_{\dot{H}^{3/2}}^2 ds$:

$$\int_0^t \|\mathbf{u}^R(s)\|_{\dot{H}^{3/2}}^2 ds \\ \leq \int_0^t \|\mathbf{u}^R(s)\|_{L^2}^2 ds + \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{H}^{1/2}}^2 ds \\ \leq 2t(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2) + \frac{1}{\nu} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^2 + \frac{8c_3}{\nu^3} \|\mathbf{u}_0\|_{\dot{H}^{1/2}}^4 \\ + \frac{3}{2\nu} \left(\int_0^t \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^2 + \frac{4c_3}{\nu^3} \left(\int_0^t \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds \right)^4 \quad (4.25)$$

for all $0 \leq t \leq \min\{T_1, T_2\}$.

We can now proceed analogously to the 2D case and show that \mathbf{u}^R and \mathbf{B}^R are uniformly bounded in the corresponding Besov spaces, although the algebra is slightly more involved.

Theorem 4.4. *Let $n = 3$, and let $(\mathbf{u}^R, \mathbf{B}^R)$ be the solution to (4.3). There is a time $T_* = T_*(\nu, \mathbf{u}_0, \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}) > 0$ such that*

$$\mathbf{u}^R \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^{1/2}(\mathbb{R}^3)) \cap L^1(0, T_*; \dot{B}_{2,1}^{5/2}(\mathbb{R}^3)), \\ \mathbf{B}^R \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^{3/2}(\mathbb{R}^3)).$$

Proof. Let

$$\begin{aligned} M_1 &= \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}} + \frac{c_2}{\nu} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^2 + \frac{8c_2c_3}{\nu^3} \|\mathbf{u}_0\|_{\dot{B}_{2,1}^{1/2}}^4 \\ M_2 &= 2c_2(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^2}^2). \end{aligned}$$

Substituting from equation (4.25) into Proposition 3.3, when $t \leq \min\{T_1, T_2\}$ we obtain

$$\begin{aligned} \|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^{1/2}} + \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds \\ \leq M_1 + M_2 t + c_2 Z_R(t) + \frac{3c_2}{2\nu} (Z_R(t))^2 + \frac{4c_2c_3}{\nu^3} (Z_R(t))^4, \end{aligned}$$

where $Z_R(t) := \int_0^t \|\mathbf{B}^R(s)\|_{\dot{B}_{2,1}^{3/2}}^2 ds$ as above. Letting $M_3 = \|\mathbf{B}_0\|_{\dot{B}_{2,1}^{3/2}}^2$, Proposition 3.1 yields

$$\begin{aligned} Z_R(t) &\leq M_3 \int_0^t \exp\left(2c_1 \int_0^\tau \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds\right) d\tau \\ &\leq M_3 t \exp\left(2c_1 \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds\right). \end{aligned}$$

Setting

$$\begin{aligned} X_R(t) &= \|\mathbf{u}^R(t)\|_{\dot{B}_{2,1}^{1/2}}, \\ Y_R(t) &= \nu \int_0^t \|\nabla \mathbf{u}^R(s)\|_{\dot{B}_{2,1}^{3/2}} ds, \end{aligned}$$

yields

$$\begin{aligned} X_R(t) + Y_R(t) &\leq M_1 + M_2 t + c_2 M_3 t \exp(2c_1 Y_R(t)/\nu) \\ &\quad + \frac{3c_2}{2\nu} M_3^2 t^2 \exp(4c_1 Y_R(t)/\nu) + \frac{4c_2c_3}{\nu^3} M_3^4 t^4 \exp(8c_1 Y_R(t)/\nu). \end{aligned} \tag{4.26}$$

Let

$$M_4 = (2 + c_2)M_1 + \frac{3c_2}{2\nu} M_1^2 + \frac{4c_2c_3}{\nu^3} M_1^4,$$

and set

$$T_* = \min\left\{T_1, T_2, \frac{M_1}{M_2}, \frac{M_1}{M_3} \exp(-2c_1 M_4/\nu)\right\}.$$

It suffices to show that $X_R(t) + Y_R(t) \leq M_4$ for all $t \in [0, T_*)$ and all $R > 0$. To see this, note that $Y_R(t)$ is continuous and $Y_R(0) = 0$. Now, suppose $t < T_*$ and $Y_R(t) \leq M_4$; then

$$\begin{aligned} Y_R(t) &\leq M_1 + M_2 t + c_2 M_3 t \exp(2c_1 Y_R(t)/\nu) \\ &\quad + \frac{3c_2}{2\nu} M_3^2 t^2 \exp(4c_1 Y_R(t)/\nu) + \frac{4c_2c_3}{\nu^3} M_3^4 t \exp(8c_1 Y_R(t)/\nu) \\ &< M_1 + M_1 + c_2 M_1 \exp\left(\frac{2c_1}{\nu} [Y_R(t) - M_4]\right) \\ &\quad + \frac{3c_2}{2\nu} M_1^2 \exp\left(\frac{4c_1}{\nu} [Y_R(t) - M_4]\right) + \frac{4c_2c_3}{\nu^3} M_1^4 \exp\left(\frac{8c_1}{\nu} [Y_R(t) - M_4]\right) \\ &\leq M_4. \end{aligned}$$

This means that $Y_R(t)$ can never equal M_4 on the interval $[0, T_*)$, hence $Y_R(t) < M_4$ for all $t \in [0, T_*)$. The result follows from inequality (4.26) and Proposition 3.1. \square

Notice that, in the 3D case, T_1 (and hence T_*) depends on \mathbf{u}_0 itself, and not just on the norm $\|\mathbf{u}_0\|_{B_{2,1}^{1/2}}$.

5. Existence Proof

In summary, in either the 2D or the 3D case, there is some time T_* such that

$$\mathbf{u}^R \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^{n/2-1}(\mathbb{R}^n)) \cap L^1(0, T_*; \dot{B}_{2,1}^{n/2+1}(\mathbb{R}^n)), \quad (5.1a)$$

$$\mathbf{B}^R \text{ is uniformly bounded in } L^\infty(0, T_*; \dot{B}_{2,1}^{n/2}(\mathbb{R}^n)). \quad (5.1b)$$

Having obtained these uniform bounds, in this section we outline the proof of Theorem 1.1, using broadly the same method as in Section 4.2 of McCormick, Robinson and Rodrigo (2014) to show the existence of a weak solution.

Let us first note that since the initial data is taken in inhomogeneous Besov spaces, the standard energy estimate (4.5) implies that \mathbf{u}^R and \mathbf{B}^R are uniformly bounded in $L^\infty(0, T; L^2(\mathbb{R}^n))$ for any $T > 0$, and hence the uniform bounds (5.1) imply that

$$\mathbf{u}^R \text{ is uniformly bounded in } L^\infty(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)) \cap L^1(0, T_*; B_{2,1}^{n/2+1}(\mathbb{R}^n)), \quad (5.2a)$$

$$\mathbf{B}^R \text{ is uniformly bounded in } L^\infty(0, T_*; B_{2,1}^{n/2}(\mathbb{R}^n)). \quad (5.2b)$$

5.1. Bounds on the Time Derivatives

We first obtain uniform bounds on the time derivatives $\frac{\partial \mathbf{u}^R}{\partial t}$ and $\frac{\partial \mathbf{B}^R}{\partial t}$. By first applying the Leray projector Π to the equations, we may eliminate the pressure term in (4.3) and consider the equations

$$\frac{\partial \mathbf{u}^R}{\partial t} - \nu \Pi \Delta \mathbf{u}^R = \mathcal{S}_R \Pi[(\mathbf{B}^R \cdot \nabla) \mathbf{B}^R] - \mathcal{S}_R \Pi[(\mathbf{u}^R \cdot \nabla) \mathbf{u}^R], \quad (5.3a)$$

$$\frac{\partial \mathbf{B}^R}{\partial t} = \mathcal{S}_R \Pi[(\mathbf{B}^R \cdot \nabla) \mathbf{u}^R] - \mathcal{S}_R \Pi[(\mathbf{u}^R \cdot \nabla) \mathbf{B}^R]. \quad (5.3b)$$

Taking the $\dot{B}_{2,1}^{n/2-1}$ norm of both sides of (5.3a) yields

$$\begin{aligned} \left\| \frac{\partial \mathbf{u}^R}{\partial t} \right\|_{\dot{B}_{2,1}^{n/2-1}} &\leq \nu \left\| \Delta \mathbf{u}^R \right\|_{\dot{B}_{2,1}^{n/2-1}} \\ &\quad + \left\| (\mathbf{B}^R \cdot \nabla) \mathbf{B}^R \right\|_{\dot{B}_{2,1}^{n/2-1}} + \left\| (\mathbf{u}^R \cdot \nabla) \mathbf{u}^R \right\|_{\dot{B}_{2,1}^{n/2-1}} \\ &\leq \nu \underbrace{\left\| \mathbf{u}^R \right\|_{\dot{B}_{2,1}^{n/2+1}}}_{\in L^1(0, T_*)} + \underbrace{\left\| \mathbf{B}^R \right\|_{\dot{B}_{2,1}^{n/2}}^2}_{\in L^\infty(0, T_*)} + \underbrace{\left\| \mathbf{u}^R \right\|_{\dot{B}_{2,1}^{n/2}}^2}_{\in L^1(0, T_*)} \end{aligned}$$

where we have used the fact that, by interpolation, the uniform bounds (5.1) imply that \mathbf{u}^R is uniformly bounded in $L^2(0, T_*; \dot{B}_{2,1}^{n/2}(\mathbb{R}^n))$. Similarly, taking the $\dot{B}_{2,1}^{n/2-1}$ norm of both sides of (5.3b) yields

$$\left\| \frac{\partial \mathbf{B}^R}{\partial t} \right\|_{\dot{B}_{2,1}^{n/2-1}} \leq 2 \underbrace{\|\mathbf{B}^R\|_{\dot{B}_{2,1}^{n/2}}}_{\in L^\infty(0, T_*)} \cdot \underbrace{\|\mathbf{u}^R\|_{\dot{B}_{2,1}^{n/2}}}_{\in L^2(0, T_*)}.$$

Hence

$$\frac{\partial \mathbf{u}^R}{\partial t} \text{ is uniformly bounded in } L^1(0, T_*; \dot{B}_{2,1}^{n/2-1}(\mathbb{R}^n)), \quad (5.4a)$$

$$\frac{\partial \mathbf{B}^R}{\partial t} \text{ is uniformly bounded in } L^2(0, T_*; \dot{B}_{2,1}^{n/2-1}(\mathbb{R}^n)). \quad (5.4b)$$

Repeating these bounds using the inhomogeneous norms and (5.2) implies that

$$\frac{\partial \mathbf{u}^R}{\partial t} \text{ is uniformly bounded in } L^1(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)), \quad (5.5a)$$

$$\frac{\partial \mathbf{B}^R}{\partial t} \text{ is uniformly bounded in } L^2(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)). \quad (5.5b)$$

5.2. Strong Convergence

Using the uniform bounds (5.2) and (5.5), one may use the Banach–Alaoglu theorem to extract a weakly-* convergent subsequence such that

$$\begin{aligned} \mathbf{u}^{R_m} &\overset{*}{\rightharpoonup} \mathbf{u} && \text{in } L^\infty(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)) \cap L^1(0, T_*; B_{2,1}^{n/2+1}(\mathbb{R}^n)), \\ \mathbf{B}^{R_m} &\overset{*}{\rightharpoonup} \mathbf{B} && \text{in } L^\infty(0, T_*; B_{2,1}^{n/2}(\mathbb{R}^n)), \\ \frac{\partial \mathbf{u}^{R_m}}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial \mathbf{u}}{\partial t} && \text{in } L^1(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)), \\ \frac{\partial \mathbf{B}^{R_m}}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial \mathbf{B}}{\partial t} && \text{in } L^2(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)). \end{aligned}$$

We now show that (\mathbf{u}, \mathbf{B}) is a *weak* solution of the equations. By embedding the Besov spaces $B_{2,1}^s$ in the corresponding Sobolev spaces H^s , and using a variant of the Aubin–Lions compactness lemma (see Proposition 2.7 in Chemin et al. (2006)), there exists a subsequence of $(\mathbf{u}^{R_m}, \mathbf{B}^{R_m})$ that converges strongly in $L^2(0, T; H^s(K))$ for any $s \in (\frac{n}{2} - 1, \frac{n}{2})$ and any compact subset $K \subset \mathbb{R}^n$; and thus they also converge strongly in $L^2(0, T; L^2(K))$, and hence the limit satisfies

$$\mathbf{u}, \mathbf{B} \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; V(\mathbb{R}^n)).$$

This local strong convergence allows us to pass to the limit in the nonlinear terms: an argument similar to Proposition 4.5 in McCormick et al. (2014) will show that (after passing to a subsequence)

$$\mathcal{S}_{R_m}[(\mathbf{u}^{R_m} \cdot \nabla) \mathbf{B}^{R_m}] \overset{*}{\rightharpoonup} (\mathbf{u} \cdot \nabla) \mathbf{B}$$

(and so on) in $L^2(0, T; V^*(\mathbb{R}^n))$ (see §2.2.4 of Chemin et al. (2006) for full details). Thus (\mathbf{u}, \mathbf{B}) is indeed a weak solution of (1.1).

6. Uniqueness

We now prove a uniqueness result in 3D.

Proposition 6.1. *Let $(\mathbf{u}_j, \mathbf{B}_j)$, $j = 1, 2$, be two solutions of (1.1) with the same initial conditions $\mathbf{u}_j(0) = \mathbf{u}_0$, $\mathbf{B}_j(0) = \mathbf{B}_0$, such that*

$$\begin{aligned} \mathbf{u}_j &\in L^\infty(0, T_*; B_{2,1}^{1/2}(\mathbb{R}^3)) \cap L^1(0, T_*; B_{2,1}^{5/2}(\mathbb{R}^3)), \\ \mathbf{B}_j &\in L^\infty(0, T_*; B_{2,1}^{3/2}(\mathbb{R}^3)). \end{aligned}$$

Then $(\mathbf{u}_1, \mathbf{B}_1) = (\mathbf{u}_2, \mathbf{B}_2)$ as functions in $L^\infty(0, T; L^2(\mathbb{R}^3))$.

Proof. Take the equations for $(\mathbf{u}_1, \mathbf{B}_1)$ and $(\mathbf{u}_2, \mathbf{B}_2)$ and subtract: writing $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{z} = \mathbf{B}_1 - \mathbf{B}_2$ and $q = p_1 - p_2$, we obtain

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}_1 \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{u}_2 - \nu \Delta \mathbf{w} + \nabla q = (\mathbf{B}_1 \cdot \nabla) \mathbf{z} + (\mathbf{z} \cdot \nabla) \mathbf{B}_2, \quad (6.1a)$$

$$\frac{\partial \mathbf{z}}{\partial t} + (\mathbf{u}_1 \cdot \nabla) \mathbf{z} + (\mathbf{w} \cdot \nabla) \mathbf{B}_2 = (\mathbf{B}_1 \cdot \nabla) \mathbf{w} + (\mathbf{z} \cdot \nabla) \mathbf{u}_2. \quad (6.1b)$$

Taking the inner product of (6.1a) with \mathbf{w} and (6.1b) with \mathbf{z} , and adding, yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\mathbf{w}\|_{L^2}^2 + \|\mathbf{z}\|_{L^2}^2) + \nu \|\nabla \mathbf{w}\|_{L^2}^2 \\ &= \langle (\mathbf{z} \cdot \nabla) \mathbf{B}_2, \mathbf{w} \rangle - \langle (\mathbf{w} \cdot \nabla) \mathbf{u}_2, \mathbf{w} \rangle + \langle (\mathbf{z} \cdot \nabla) \mathbf{u}_2, \mathbf{z} \rangle - \langle (\mathbf{w} \cdot \nabla) \mathbf{B}_2, \mathbf{z} \rangle \\ &\leq \|\mathbf{z}\|_{L^2} \|\nabla \mathbf{w}\|_{L^2} \|\mathbf{B}_2\|_{L^\infty} + \|\mathbf{w}\|_{L^2} \|\nabla \mathbf{w}\|_{L^2} \|\mathbf{u}_2\|_{L^\infty} \\ &\quad + \|\mathbf{z}\|_{L^2}^2 \|\nabla \mathbf{u}_2\|_{L^\infty} + \|\mathbf{w}\|_{L^6} \|\nabla \mathbf{B}_2\|_{L^3} \|\mathbf{z}\|_{L^2} \\ &\leq (\|\mathbf{w}\|_{L^2} + \|\mathbf{z}\|_{L^2}) \|\nabla \mathbf{w}\|_{L^2} \left(\|\mathbf{u}_2\|_{\dot{B}_{2,1}^{3/2}} + \|\mathbf{B}_2\|_{\dot{B}_{2,1}^{3/2}} \right) + \|\mathbf{z}\|_{L^2}^2 \|\nabla \mathbf{u}_2\|_{\dot{B}_{2,1}^{3/2}}, \end{aligned}$$

so by Young's inequality

$$\begin{aligned} &\frac{d}{dt} (\|\mathbf{w}\|_{L^2}^2 + \|\mathbf{z}\|_{L^2}^2) + \nu \|\nabla \mathbf{w}\|_{L^2}^2 \\ &\leq \frac{c}{\nu} \left(\|\mathbf{u}_2\|_{\dot{B}_{2,1}^{3/2}}^2 + \|\mathbf{B}_2\|_{\dot{B}_{2,1}^{3/2}}^2 + \|\nabla \mathbf{u}_2\|_{\dot{B}_{2,1}^{3/2}} \right) (\|\mathbf{w}\|_{L^2}^2 + \|\mathbf{z}\|_{L^2}^2) \end{aligned}$$

and uniqueness follows by Gronwall's inequality. \square

Note, however, that this argument does not apply in 2D. This is because the term $\langle (\mathbf{w} \cdot \nabla) \mathbf{B}_2, \mathbf{z} \rangle$ cannot be estimated in the same way: in 3D we used the inequality

$$|\langle (\mathbf{w} \cdot \nabla) \mathbf{B}_2, \mathbf{z} \rangle| \leq \|\mathbf{w}\|_{L^6} \|\nabla \mathbf{B}_2\|_{L^3} \|\mathbf{z}\|_{L^2} \leq \|\nabla \mathbf{w}\|_{L^2} \|\mathbf{B}_2\|_{\dot{B}_{2,1}^{3/2}} \|\mathbf{z}\|_{L^2},$$

but in 2D the best we can do is

$$|\langle (\mathbf{w} \cdot \nabla) \mathbf{B}_2, \mathbf{z} \rangle| \leq \|\mathbf{w}\|_{L^\infty} \|\nabla \mathbf{B}_2\|_{L^2} \|\mathbf{z}\|_{L^2} \leq \|\mathbf{w}\|_{L^\infty} \|\mathbf{B}_2\|_{\dot{B}_{2,1}^1} \|\mathbf{z}\|_{L^2},$$

since the embedding $H^1 \hookrightarrow L^\infty$ fails to hold in 2D. While we could use the embedding $\dot{B}_{2,1}^1 \hookrightarrow L^\infty$, that would not allow us to absorb the term into the $\|\nabla \mathbf{w}\|_{L^2}$ term on the left-hand side.

This leaves us in the odd situation where we can prove uniqueness in 3D, but not in 2D; the recent paper of Wan (2015) uses a much more sophisticated argument involving mixed space-time Besov spaces to resolve uniqueness in 2D. More importantly, however, the argument above shows that a proof along the lines of Fefferman et al. (2014) would not necessarily work, since the uniqueness proof is just a simpler version of the proof that the truncated solutions $(\mathbf{u}^R, \mathbf{B}^R)$ are Cauchy in $L^\infty(0, T; L^2(\mathbb{R}^n))$.

7. Conclusion

With initial data $\mathbf{u}_0 \in B_{2,1}^{n/2-1}(\mathbb{R}^n)$ and $\mathbf{B}_0 \in B_{2,1}^{n/2}(\mathbb{R}^n)$ for $n = 2, 3$, we have proved the existence of a solution (\mathbf{u}, \mathbf{B}) satisfying

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T_*; B_{2,1}^{n/2-1}(\mathbb{R}^n)) \cap L^1(0, T_*; B_{2,1}^{n/2+1}(\mathbb{R}^n)), \\ \mathbf{B} &\in L^\infty(0, T_*; B_{2,1}^{n/2}(\mathbb{R}^n)). \end{aligned}$$

It is clear, however, that there is considerable scope for further work in a number of directions. While the a priori estimates in Section 3 depend only on the norms of the initial data in the corresponding *homogeneous* Besov spaces, that is $\|\mathbf{u}_0\|_{\dot{B}_{2,1}^{n/2-1}}$ and $\|\mathbf{B}_0\|_{\dot{B}_{2,1}^{n/2}}$, in 3D the use of the commutator estimate (3.4) forces the use of inhomogeneous spaces.

It is thus natural to ask whether all three norms on the right-hand side of (3.4) or (3.5) could be taken in homogeneous spaces: if such a generalisation could be proved, then in 3D the a priori estimates could be closed up while assuming only that $\mathbf{u}_0 \in \dot{B}_{2,1}^{1/2}$ and $\mathbf{B}_0 \in \dot{B}_{2,1}^{3/2}$ (though further work would be required to obtain a bona fide solution, as the method of Section 5 would no longer apply).

A partial generalisation of (3.5) is proved in McCormick et al. (2015): it is shown that

$$|\langle \Lambda^s[(\mathbf{u} \cdot \nabla)\mathbf{u}], \Lambda^s \mathbf{u} \rangle| \leq c \|\mathbf{u}\|_{\dot{H}^{s_1}} \|\mathbf{u}\|_{\dot{H}^{s_2}} \|\mathbf{u}\|_{\dot{H}^s},$$

provided that $s \geq 1$ and $s_1, s_2 > 0$ such that

$$1 \leq s_1 < \frac{n}{2} + 1 \quad \text{and} \quad s_1 + s_2 = s + \frac{n}{2} + 1.$$

Unfortunately the case we would want to apply requires $s = n/2 - 1$, which does not satisfy $s \geq 1$ in 2D or 3D.

Finally, the methods developed here allowed us to prove uniqueness in 3D, but not in 2D. Uniqueness in the 2D case has recently been shown by Wan (2015), using mixed space-time Besov spaces to resolve this issue.

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