A continuous model of multilateral bargaining with random arrival times*

Attila Ambrus† and Shih En Lu‡

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Abstract

This paper proposes a continuous-time model framework of bargaining, which is analytically tractable even in complex situations like coalitional bargaining. The main ingredients of the model are: (i) players get to make offers according to a random arrival process; (ii) there is a deadline that ends negotiations. In the case of n-player group bargaining, there is a unique subgame-perfect Nash equilibrium, and the share of the surplus a player can expect is proportional to her arrival rate. In general coalitional bargaining, existence and uniqueness of Markov perfect equilibrium is established. In convex games, the set of limit payoffs as the deadline gets infinitely far away exactly corresponds to the core. The limit allocation selected from the core is determined by the relative arrival rates. As an application of the model, legislative bargaining with deadline is investigated.

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†Department of Economics, Harvard University, Cambridge, MA 02138, ambrus@fas.harvard.edu

‡Department of Economics, Harvard University, Cambridge, MA 02138, shihenlu@fas.harvard.edu

1 Introduction

The idea of explicitly modeling the dynamic aspects of bargaining goes back to Stahl (1972) and Rubinstein (1982).[1] They analyze a bargaining game between two players who take turns at making proposals. The key element of the model is impatience: players care not only about what share of the surplus they acquire, but they prefer to reach an agreement earlier rather than later. Remarkably, any nonzero degree of impatience leads to a unique subgame-perfect equilibrium prediction.

The Stahl-Rubinstein model was extended in many directions.[2] One line of research investigates the robustness of the predictions of dynamic bargaining models with respect the specification of the bargaining protocol. The general conclusion of these papers is that uniqueness of equilibrium is not robust to changing various aspects of the Rubinstein model. For early papers along this line, see Binmore *et al.* (1986) and Dekel (1990). Perry and Reny (1993) and Sákovics (1993) introduce a continuous-time framework, endogenize the order and timing of offers, and show that a continuum of divisions of the surplus can be supported in subgame-perfect Nash equilibrium.

The literature on multilateral group bargaining examines situations in which there are more than one parties involved, but all of them must agree in order to implement an agreement. A straightforward extension of Rubinstein's model yields a severe multiplicity of subgame-perfect equilibria if the number of players is at least three, even though there is a unique stationary equilibrium.[3] Coalitional bargaining investigates more complicated situations, when agreements are possible among subgroups of players, and the surpluses that different coalitions of players can split among each other can differ. In these games, a proposer has to choose both a coalition to approach and a division of the surplus that the coalition generates (for discrete-time models see Gul (1989), Chatterjee *et al.* (1993), Moldovanu and Winter (1995), Bloch (1996), Okada (1996), Ray and Vohra (1997) and (1999), Konishi and Ray (2003), Gomes (2005), and for a continuous-time model see Perry and Reny (1994)).

Although the importance of coalitional bargaining is recognized in the theoretical literature, its application is limited by the fact that models proposed thus far are hard to analyze, sensitive to the specification of the bargaining protocol, and - related to the former two issues - not amenable to obtaining general

[1] See also Shaked and Sutton (1984) for a simpler analysis of Rubinstein's game.

[2] A major extension of the dynamic bargaining model framework that we do not take up in this paper involves incorporating private information held by one or more of the bargaining parties. For early references on bargaining with asymmetric information, see Fudenberg and Tirole (1983), Sobel and Takahashi (1983), Cramton (1984), Fudenberg *et al.* (1985), Rubinstein (1985), and Gul *et al.* (1986). For a relatively recent survey of the topic, see Ausubel *et al.* (2002).

[3] See for example Osborne and Rubinstein (1990, p63). See also Krishna and Serrano (1996), who modify the game such that players can exit the game with partial agreements, and obtain a unique equilibrium.

predictions. For this reason, existing applications like Ray and Vohra (2001), Genicot and Ray (2003), and Aghion *et al.* (2007) do not simultaneously tackle coalition formation and the division of surplus within a coalition, but only focus on one of these issues (for example, by fixing how surplus is divided within a coalition).

In this paper, we aim to propose a model framework that is sufficiently analytically tractable to facilitate the analysis of complex bargaining situations like coalitional bargaining. We consider a continuous-time framework, in which players get random opportunities to approach others and make a proposal. In particular, we assume that the points in time at which a player gets the chance to make a proposal correspond to arrival times of a Poisson procedure, and that arrival times of different players are independent of each other. Players with higher arrival rates can in expectation propose more frequently. This might be either a consequence of institutional features, like certain members of a legislature (party leaders or other elected officials within the legislature) enjoying preferential treatment in initiating proposals, or of how much attention and resources a player can devote to the bargaining procedure at hand. Making an offer in a bargaining procedure might involve preparing a written contract proposal, getting approval from various actors (superior, board of trustees, or one's spouse), organizing a meeting/conference call with the parties to be approached, and communicating the offer. Players can be heterogeneous in how much time each of these steps requires.

Any player, when getting a chance to make a proposal, can approach any coalition of players that contains the proposer. The values of possible coalitions (how much surplus any group of players can generate by themselves) are exogenously given by a superadditive characteristic function. Once an offer is made, we assume that the approached parties react immediately, and all of them have to accept the proposal in order for an agreement to be reached. Once an agreement is reached (by some coalition), the game ends. If a proposal is rejected by any of the approached players, the game continues, and players wait for the next arrival time. Two highlighted special cases that fit into this general framework are n-player group bargaining, where only the grand coalition can generate positive surplus, and legislative bargaining along the lines of Baron and Ferejohn (1989), where any large enough coalition of players (in the case of simple majority, voting coalitions involving more than half of the players) can end the game by reaching an agreement. Although the framework we consider is flexible enough to facilitate the analysis of these special cases as well as many others, there are various extensions of the model that would make it more realistic in other situations, like introducing a (probabilistic or deterministic) lag in responding to an offer, allowing bargaining to continue even after an agreement is reached (with either only the players who were not part of previous agreements staying in the game, or all players staying in the game and potentially willing to renegotiate existing agreements), or allowing for external effects of agreements on the excluded players. These are issues that have been investigated in previous models of coalitional bargaining, and we plan to revisit them in future work, using the framework introduced in this paper.

One important component of our model is that we assume the existence of a deadline, after which the game ends even if no agreement was reached, and all players receive a payoff of zero.[4] Our motivation for proceeding this way is twofold. First, in many real-world bargaining situations, there are natural deadlines that end negotiations. If the NHL (National Hockey League) and the NHL Players' Association do not reach an agreement by a certain date, then the season needs to be canceled, as happened in 2004. If an organization owning the broadcasting rights to an event does not reach an agreement with a TV station on the terms of broadcast by the time the event takes place, and the event is only of interest if broadcast live, then the surplus is lost for both parties. For reaching an out-of-court settlement, the announcement of the verdict poses a final deadline. Finally, in legislative bargaining, the end of the legislature's mandate provides an upper bound on how long negotiations can last. Our second motivation is technical: having a deadline helps in deriving a unique prediction for the bargaining game. We pay highlighted attention to characterizing limit equilibrium payoffs as the deadline gets infinitely far away, and consider this exercise a way of deriving a prediction in bargaining games with no clear deadline for the end of negotiations. We also note that because of the continuous-time framework, there is no highlighted (and arguably unrealistic) "last period" in our model, in the sense that it is a 0-probability event that some player receives an arrival right at the deadline. Instead, as the deadline approaches, it simply becomes more and more likely that there will not be time for reaching an agreement before time expires.

Using the above framework, we first investigate n-player group bargaining. We show that there is a unique subgame-perfect Nash equilibrium, explicitly derive equilibrium payoffs, and show that the expected payoff of every player is monotonic in the length of the game, and in the limit as the deadline gets infinitely far away, players share the surplus in proportion to the arrival rates. Hence, in our framework, being able to propose more frequently increases a player's payoff, as opposed to what happens in the models of Perry and Reny (1993) and Sákovics (1993). In the latter models the longer a player's waiting time is, the more costly it is for her opponent to reject an offer, as the opponent would have to wait a long time before her counteroffer can feasibly be accepted. By contrast, in our model, approached players only accept an offer if they are offered at least their equilibrium continuation payoffs, and a player's continuation payoff increases in her arrival rate because of the higher probability that she receives the proposer surplus (the difference between the total pie and the sum of continuation values). The uniqueness of subgame-perfect equilibrium, which

[4]For deadline effects in discrete-time bargaining models, see Fershtman and Seidmann (1993), who examine bilateral bargaining with a particular commitment, and Norman (2002), who investigates legislative bargaining.

contrasts with infinite-horizon models of group bargaining with discounting, is due to the anchoring effect of the deadline.

In general coalitional bargaining, unlike in the special case of group bargaining, it is not only the arrival rates and the length of the game that determine expected payoffs in equilibrium, but also the values of different coalitions. For example, if the continuation payoff of a given player rises above her marginal contribution to the grand coalition as the deadline gets further away, the other players stop approaching her and instead propose to each other. In general, the characteristic function imposes constraints on the sum of payoffs of players in different coalitions, while arrival rates determine which payoff among those satisfying the constraints is achieved in equilibrium. The interplay of coalitional constraints and different frequencies of proposals potentially results in a complicated equilibrium path, which can be nonmonotonic. That is, a player's expected payoff can both increase or decrease over different ranges of time before the deadline. In general, a player that can propose very frequently finds it optimal to have an intermediate time horizon for negotiations.

In the general framework, we show both the existence and the uniqueness of Markov perfect equilibrium (where the payoff relevant state is the time remaining before the deadline). We are unaware of any uniqueness result in the literature that applies to such a large class of coalitional bargaining problems (Eraslan (2002) shows uniqueness of stationary equilibria in legislative bargaining games, which is a relatively simple special case of our general setup). While the proof of this result is involved, the rough intuition behind it is simple: In Markov perfect equilibrium all players, upon an arrival, approach one of the coalitions that maximize the difference between the value of the coalition and the sum of continuation values of other players in this coalition. In other words, players always approach the coalitions that are cheapest relative to their values. Suppose now that two Markov perfect equilibria generate different continuation value paths. Consider the earliest time such that continuation values always coincide in the two equilibria between this time and the deadline (such a time always exists because at the deadline all continuation values are zero in any equilibrium). The fact that continuation value paths differ before this time means that there is a coalition that gets relatively more expensive than another coalition in one equilibrium than in the other one. We show that this requires that close enough to the deadline, players in the first coalition are approached more frequently, relative to players in the other coalition, in one equilibrium versus the other one. However, we show that this contradicts the requirement that every player at every point of time should approach the relatively cheapest coalition, since that implies that the coalition becoming relatively more expensive should be approached relatively less frequently.

We also provide a characterization of possible limit equilibrium payoffs in games with convex characteristic functions. We show that there is an exact equivalence between possible limit payoffs with different arrival rates, and the core of the underlying cooperative game. In particular, any limit payoff of the game for any set of arrival rates has to be in the core, and any point of the core is the unique limit payoff of the game for some vector of arrival rates. We show by example that this core convergence result does not generally hold for nonconvex games with nonempty core. Our results are similar to some of the findings obtained in other models of coalitional bargaining, like Chatterjee et al. (1993) and Perry and Reny (1994). The difference is that through the arrival rates we provide an explanation that which point of the core is attained in a given specification of the game, and show that by varying arrival rates each point of the core can be attained as a limit equilibrium payoff.

We conclude the paper with an application of legislative bargaining with deadline. We show that expected payoffs are monotonic in arrival rates, and provide a simple algorithm that determines limit payoffs as the deadline gets infinitely far away. The payoffs of the q players (where q is the quota that is required to pass a bill) with the lowest arrival rates are always equalized if the time horizon is long enough, but the expected payoff of players with the highest arrival rates can remain bounded away from the former players even in the limit as the deadline gets infinitely far away. Just like in the general setup, a player who can propose sufficiently frequently finds it optimal to set an intermediate deadline for negotiations, as opposed to a very short or very long one.

2 The model

Consider a bargaining situation with set of players $N = \{1, 2, ..., n\}$ and characteristic function $V : 2^N \to \mathbb{R}_+$, where $V(C)$ for $C \subset N$ denotes the surplus that players in C can generate by themselves (without players in $N \backslash C$). We refer to elements of 2^N as *coalitions*. We assume that V is *superadditive*, i.e. that for any two coalitions C_1, C_2 such that $C_1 \cap C_2 = \varnothing$, the value of the coalitions satisfy $V(C_1) + V(C_2) \leq V(C_1 \cup C_2)$. Occasionally we will refer to the collection (N, V) as the underlying cooperative game behind the dynamic bargaining model investigated. The core of the underlying cooperative game is defined as: $\mathcal{C}(V) = \{x \in \mathbb{R}^n : \sum\limits_{i \in C} x_i \geq V(C) \ \forall \ C \subset N \text{ and } \sum\limits_{i \in N} x_i = V(N) \ \}$.

The dynamic bargaining game we investigate is defined as follows. The game is set in continuous time, starting at $-T < 0$. There is a Poisson arrival process associated with each player i, with arrival rate λ_i. The processes are independent from each other. For future reference, we define $\lambda \equiv \sum_{i=1}^n \lambda_i$. Whenever the process realizes for a player i, she can make an offer $x = (x_1, x_2, ..., x_n)$ to a coalition $C \subseteq N$ satisfying $i \in C$. The offer x must have the following characteristics:

1. $x_j \geq 0$ for all $1 \leq j \leq n$;
2. $\sum_{j=1}^n x_j \leq V(C)$.

Players in $C \backslash \{i\}$ immediately and sequentially accept or reject the offer (the order in which they do so turns out to be unimportant). If everyone accepts, the game ends and all players in N are paid their share according to x. For simplicity, we assume that players do not discount their payoffs.[5] If an offer is rejected by at least one of the respondents, it is taken off the table, and the game continues with the same Poisson arrival rates. If no offer has been accepted at time 0, the game ends, and all players receive payoff 0.

3 Group bargaining

In order to facilitate understanding of the model framework, we start the analysis with the simplest possible specification of the model: n-player group bargaining. Formally, in this section we assume that $V(N) = 1$, and that $V(C) = 0$ $\forall\ C \neq N$. Since only the grand coalition can generate value, the acceptance of every player is required for any outcome with nonzero payoffs.

As is well-known in the literature, if the number of players is at least 3, in an alternating-offer bargaining game with infinite horizon, any division of the surplus can be supported in subgame-perfect Nash equilibrium (SPNE), if players are patient enough.[6] In Section 8, we show that this conclusion remains valid in our continuous-time framework with random arrivals. In stark contrast to this, in the game with deadline, there is a unique SPNE for any vector of arrival rates. Below we show this result formally. However, since the role of the deadline is hidden in the proof of the theorem, we find it instructive to first provide an intuitive proof for the uniqueness of Markov perfect equilibrium (MPE) in a game with two players.[7] By Markov perfect equilibrium, we mean a subgame-perfect Nash equilibrium in which any proposer's offer depends only on the time remaining before the deadline (not the history of the game beforehand), and any acceptance decision depends only on the time remaining before the deadline, the current proposal on the table, and the acceptance/rejection decisions already made to the current proposal.

Suppose there are two players in the game, with arrival rates λ_1 and λ_2. Let $v_i(t)$ denote the expected payoff of player i if she gets an arrival at time t, and $w_i(t)$ denote the expected payoff of player i if player $j \neq i$ gets an arrival at t. Note that in any MPE, $v_i(0) = 1$ and $w_i(0) = 0$. We can then explicitly compute an MPE by assuming that $w_i(t) = \int_t^0 \left[\lambda_i e^{-\lambda(\tau-t)} v_i(\tau) + \sum_{j\neq i} \lambda_j e^{-\lambda(\tau-t)} w_i(\tau)\right] d\tau$ and $v_i(t) = w_i(t) + e^{\lambda t}$, where $e^{\lambda t}$ is the probability that no player gets an arrival after t. Note that continuation payoffs that satisfy the conditions above are consistent with equilibrium, since a player, whenever she gets an arrival, offers exactly the continuation value of the other player and keeps the rest of $V(N)$

[5] Incorporating discount factors in the model is straightforward and does not change the qualitative conclusions of the model. See the discussion in Subsection 8.1.

[6] See for example Osborne and Rubinstein (1990, p63).

[7] The arguments can be formalized and extended to provide a proof for uniqueness of MPE for any number of players. This is not included in the paper because Theorem 1 establishes a stronger result.

for herself. The solution of the resulting system of differential equations yields expected continuation payoff functions $W_i(t) = \frac{\lambda_i}{\lambda}(1 - e^{\lambda t})$ and proposer payoffs $V_i(t) = \frac{\lambda_i}{\lambda} + \frac{\lambda_j}{\lambda} e^{\lambda t}$ for $i = 1, 2$. Suppose now that equilibrium is not unique. Then there exists $t < 0$ such that either the supremum or the infimum of player i's MPE payoffs, when the other player receives an arrival at t, is not equal to $W_i(t)$. Assume the former (the argument for the latter is exactly symmetric). Let $\overline{w}_i(t)$ denote the supremum of player i's expected payoff in MPE when $j \neq i$ gets an arrival at t, and let $\overline{v}_i(t)$ denote the supremum of player i's expected payoff in MPE when she gets an arrival at t. Note that $\overline{w}_i(t)$ is bounded above by the expected payoff resulting from player i expecting the highest possible MPE payoff at any future arrival:

$$\overline{w}_i(t) \leq \int_t^0 \left[\lambda_i e^{-\lambda(\tau - t)} \overline{v}_i(\tau) + \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} \overline{w}_i(\tau) \right] d\tau \qquad (*)$$

Moreover, the history-independent nature of MPE implies that: (i) $\overline{w}_i(t)$ is equal to the supremum of player i's expected payoff in MPE if no player gets an arrival at t; and (ii) $\overline{v}_i(t) = \overline{w}_i(t) + e^{\lambda t}$, where $e^{\lambda t}$ is the probability that no player gets an arrival after t. Substituting the latter into (*) implies that, since $W_i(t) = \int_t^0 \left[\lambda_i e^{-\lambda(\tau - t)} V_i(\tau) + \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} W_i(\tau) \right] d\tau$, if $\overline{w}_i(t) - W_i(t) = \varepsilon > 0$, then there has to be a strictly positive mass of times τ later than t such that $\overline{w}_i(\tau) - W_i(\tau) \geq \varepsilon$. In particular, the latest time such that $\overline{w}_i(\tau) - W_i(\tau) \geq \varepsilon$ has to be larger than $t + \frac{1}{\lambda} \varepsilon e^{\lambda t}$, a term increasing in t. Repeated use of this argument then establishes that $\overline{w}_i(0) - W_i(0) \geq \varepsilon$, which leads to a contradiction since at 0, the continuation value of i if she does not get a proposal has to be 0 in all MPE.

We now formally prove the stronger result of the uniqueness of SPNE in n-player group bargaining games. The proofs of all formal results are in the Appendix.

Theorem 1: In any SPNE, the n-player group bargaining game ends at the first realization of the Poisson arrival process for any player. After any arrival, an offer is made to N and all players accept. SPNE payoff functions are unique, with player i receiving $\frac{\lambda_i}{\lambda} + (1 - \frac{\lambda_i}{\lambda}) e^{\lambda t}$ when she makes the offer at time t, and $\frac{\lambda_i}{\lambda}(1 - e^{\lambda t})$ when she is not the proposer.

Theorem 1 implies that the expected payoffs of players converge to the relative arrival rates as the deadline gets infinitely far away: the expected payoff of player i converges to $\frac{\lambda_i}{\lambda}$ as $T \to \infty$. Moreover, a player's expected payoff, both unconditionally and conditionally on getting an arrival, is monotonically increasing in her arrival rate, at all times. Figure 1 below depicts expected continuation payoffs of the game if $n = 3$, and arrival rates are $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, and $\lambda_3 = \frac{1}{6}$. Continuation values at the deadline are 0 for all players. Going back in time, continuation values start increasing at the rate corresponding to arrival rates, and converge to the relative arrival rates.

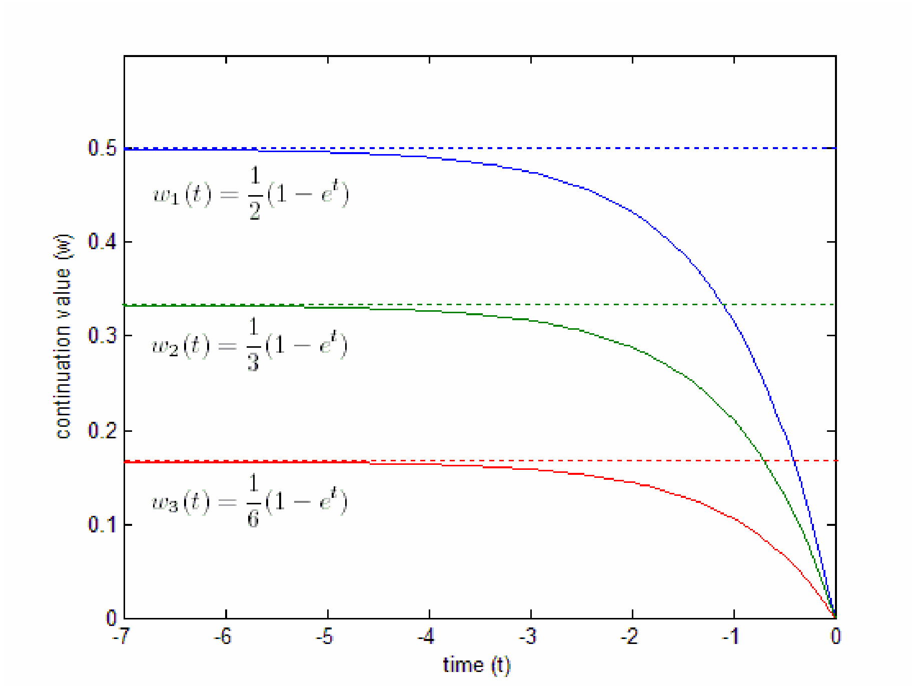

Figure 1.

Figure 2 depicts the payoff of players in the same game conditional on getting an arrival, at different points of time. These payoffs are interconnected with the expected continuation payoffs depicted on the previous graph in that any player getting an arrival needs to offer exactly the continuation payoff to the other two players. Therefore, any player close to the deadline can keep most of the pie to herself, and payoffs conditional on proposing are monotonically decreasing in time.

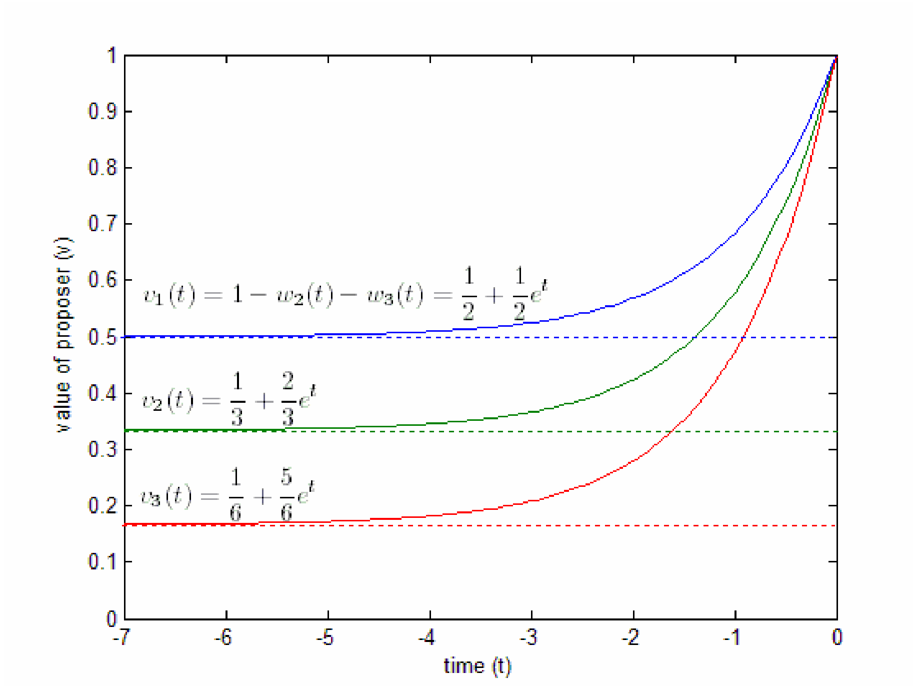

Figure 2.

The fact that in our model, a player's ability to make offers more frequently increases her expected payoff is in contrast with the predictions of the models in Perry and Reny (1993) and Sákovics (1993). In these models, a player that can only speak infrequently obtains a higher share of the surplus because she can credibly threaten to impose a higher time cost on other players, should her offer be rejected.

4 Examples of coalitional bargaining

In this section, we turn our attention to the general framework introduced in Section 2, where coalitions other than the grand coalition can also generate positive value. We present several examples that illustrate some of the additional features of equilibrium dynamics, relative to the simpler context of group bargaining. In all of the following examples, we assume that $n = 3$ and $V(N) = 1$. These games each have a unique MPE (a general feature of our framework, as shown in section 5), which we simply refer to as "equilibrium" below.

Example 1. First, consider a game in which $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $\lambda_2 = \frac{1}{6}$, $V(C) = \frac{2}{5}$ if $|C| = 2$, and $V(C) = 0$ if $|C| = 1$. This is the same game as the one in Figure 1, the only difference being that two-player coalitions can generate a positive value. However, in this game, the marginal contribution of any coalition to the grand coalition is never less than its share of arrivals. For example, the marginal contribution of any single player, $\frac{3}{5}$, is higher than her relative arrival rate. As a result, everyone always approaches the grand coalition, and continuation values evolve as in Figure 1.

Example 2. Consider next the game in which $\lambda_1 = \frac{2}{3}$, $\lambda_2 = \lambda_3 = \frac{1}{6}$, $V(C) = \frac{1}{2}$ if $|C| = 2$, and $V(C) = 0$ if $|C| = 1$. In this game, player 1's relative arrival rate, $\frac{\lambda_1}{\lambda} = \frac{2}{3}$, is greater than her contribution to the grand coalition. Hence, before a certain time, players 2 and 3 do not propose to her with probability 1. Instead, they randomize between approaching the grand coalition and coalition $\{2,3\}$ in a way that keeps player 1's continuation payoff constant at $\frac{1}{2}$ (which makes the other players indifferent between including and excluding her). The continuation payoffs of players 2 and 3 keep increasing as we get further away from the deadline (since they are always included in the coalition approached) and converge to $(\frac{1}{4}, \frac{1}{4})$, which results from sharing the value of coalition $\{2,3\}$ in proportion with the relative arrival rates. The situation is depicted in Figure 3.

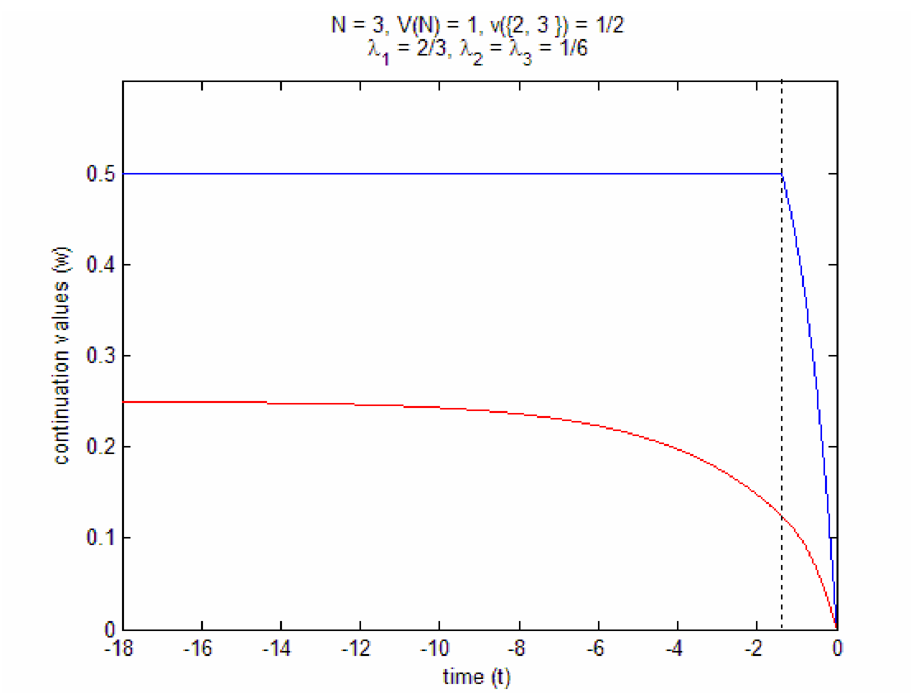

Figure 3.

Example 3. The next example shows that besides a player's marginal contribution to the grand coalition, her contribution to other coalitions can also impose a constraint on the limit payoff the player can expect in equilibrium. Let $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $V(\{1\}) = \frac{1}{2}$, $V(\{1,2\}) = \frac{5}{6}$, and let the value of all other coalitions other than the grand coalition be 0. As depicted on Figure 4, going back in time from the deadline, all players' payoffs start increasing at the same rate. When player 3's continuation payoff reaches $\frac{1}{6}$, her marginal contribution to the grand coalition, the other two players stop proposing her with probability 1, in a way that keeps player 3's continuation payoff constant at $\frac{1}{6}$. The other two players' continuation payoffs keep increasing until player 2's payoff reaches $\frac{1}{3}$, which is her marginal contribution to the value of coalition $\{1,2\}$. At this point, player 1 starts proposing with positive probability to the singleton coalition involving only herself (that is, excluding player 2), and player 2's continuation payoff is kept constant at $\frac{1}{3}$. Finally, player 1's payoff converges to $\frac{1}{2}$, the value she can generate by herself.

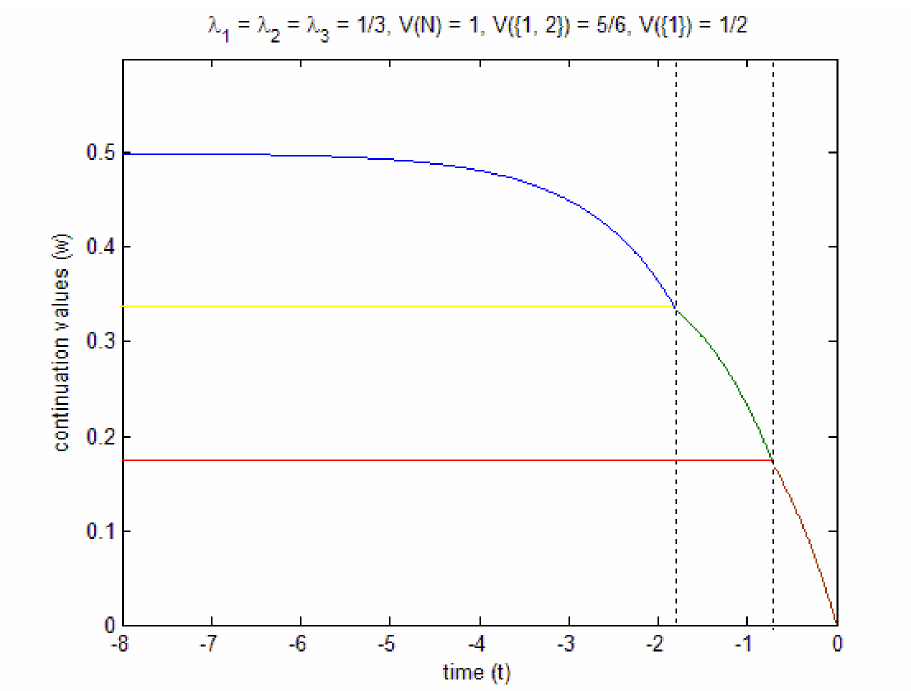

Figure 4.

Example 4. Consider now a game with exactly the same characteristic function as in Example 2, but with arrival rates $\lambda_1 = \frac{4}{5}$, $\lambda_2 = \lambda_3 = \frac{1}{10}$. In this game, player 1 proposes so frequently relative to other players that even if the others exclude her from the offer for sure, her continuation payoff can still increase when going back in time, as Figure 5 illustrates. However, if we get sufficiently far away from the deadline, player 1's expected payoff falls back to $\frac{1}{2}$. This is because in order to keep player 1's continuation payoff above $\frac{1}{2}$ when she is excluded from other players' offers, she needs to receive a relatively high payoff as a proposer when she gets an arrival. Since the other two players' continuation payoffs are strictly increasing when going back in time, player 1's payoff conditional on being the proposer shrinks. As soon as player 1's payoff falls back to $\frac{1}{2}$, the other players start including her in the offer again with positive probability. Moreover, the probability that the grand coalition is proposed to, conditional on an arrival, converges to 1 as the deadline gets infinitely far away.

This example illustrates two interesting nonmonotonicities associated with multilateral bargaining with deadline. The first is that while close to the deadline, everyone approaches the grand coalition and therefore agreements are Pareto efficient, and taking the deadline to infinity implies convergence to efficiency, there can be an intermediate range of time horizons in which inefficient agreements are made with high probability.[8] The second one is that a player's expected payoff can be a nonmonotonic function of the time remaining before the deadline. Player 1 in the above example prefers the intermediate time horizon $T = 3$ to both the very short horizon $T = \frac{1}{2}$ (which implies too high probability of no agreement reached in time) and the very long one $T = 10$ (which implies that player 1 has to offer a high payoff to the players with slower arrival processes).

[8] In fact, it is easy to construct examples in which, during an intermediate time range, an inefficient agreement is proposed with probability 1, conditional on arrival by any player in this range.

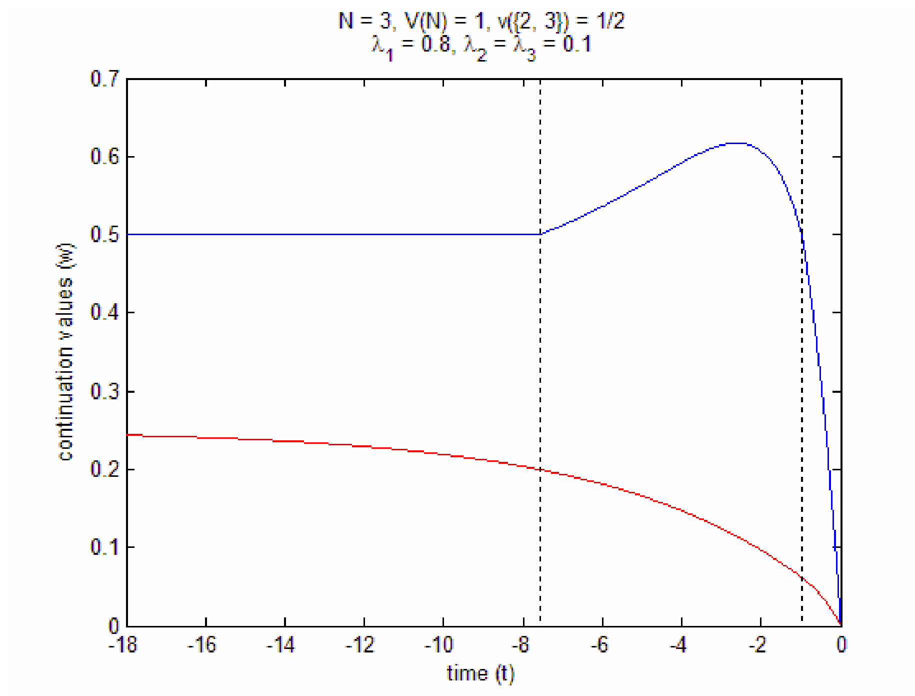

Figure 5.

Example 5. Next consider the classic example of one seller and two buyers, with the seller having only one object to sell. Let the seller be player 1. Here $V(N) = V(\{1,2\}) = V(\{1,3\}) = 1$, while the value of all other coalitions are 0. Assume $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$. Note that the marginal contribution of both buyers to the grand coalition is 0. As Figure 6 illustrates, this results in the continuation payoffs of the buyers first increasing, but at some point starting to decrease and eventually converging to 0. In the limit as the deadline gets far away, the seller obtains all the surplus. At any point of time, if a buyer gets an arrival, she proposes to the 2-player coalition involving herself and the seller. At the same time, the seller always approaches one of the two buyers with equal probability, which keeps the continuation values of buyers equal to each other (if one buyer has a higher continuation value than the other, then the seller would not approach her).

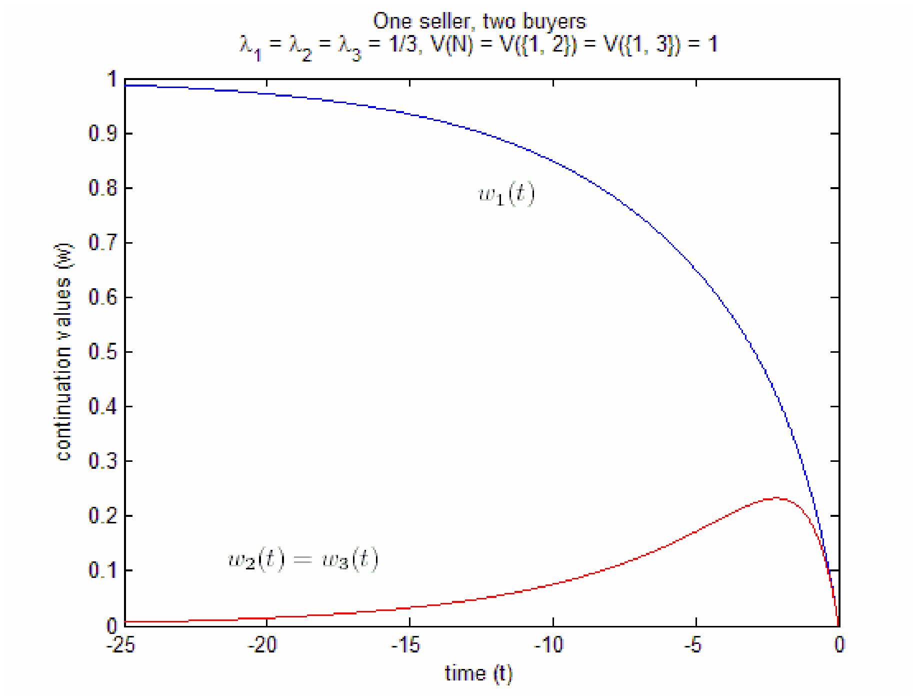

Figure 6.

Example 6. Our final example shows a situation in which the underlying cooperative game has an empty core. Let $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $V(\{1,2\}) = V(\{1,3\}) = V(\{2,3\}) = \frac{3}{4}$ and $V(\{1\}) = V(\{2\}) = V(\{3\}) = 0$. Here, each player's marginal contribution to the grand coalition is $\frac{1}{4}$. Hence, once continuation values reach $\frac{1}{4}$, all players switch to proposing to 2-player coalitions, in a way that keeps everyone's continuation payoff constant at this level. This implies that players make inefficient agreements even in the limit as the deadline gets far away.

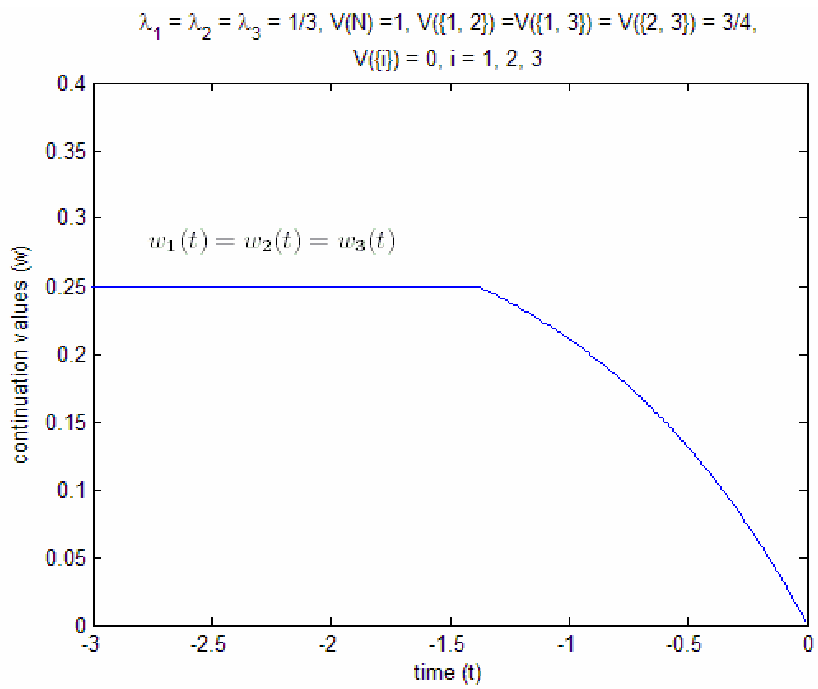

Figure 7.

5 Existence and uniqueness of Markov-perfect equilibrium

Below we show that in every game fitting into the general model framework introduced in Section 2, an MPE exists, and the continuation payoff functions of all players are uniquely determined in MPE. That is, while strategies in the general model might not be uniquely determined in MPE, they can only vary in a payoff-irrelevant way.[9] We are not aware of any similar uniqueness result in the literature that would apply to a general class of coalitional bargaining games. We also note that in discrete models with finite horizon, uniqueness of MPE payoffs does not hold in general, even in relatively simple bargaining games like legislative bargaining, since the way players break ties in some period can generate multiplicity in all preceding periods.[10] In our continuous framework,

[9] For example, if there are three players with equal arrival rates, and all players approach two-player coalitions in a way that every player is approached by others with the same probability, then it is payoff-irrelevant whether all players approach each of the other two players with probability 1/2-1/2, or whether player 1 always approaches player 2, player 2 always approaches player 3, and player 3 always approaches player 1.

[10] The resulting multiplicity can be quite severe: see Norman (2002) on this point.

tie-breaking ceases to be an issue. Whether SPNE are unique in our framework remains an open question.

We prove the existence of MPE by considering a sequence of discrete-time approximations of the continuous game in which the time lag between periods goes to zero. Each of the games along this sequence has a Markov perfect equilibrium that can be obtained by backward induction. Moreover, the resulting equilibrium continuation value functions are Lipschitz-continuous, with a uniform Lipschitz constant given by arrival rates and $V(N)$. Hence, by the Ascoli-Arzela theorem, there is a subsequence of the games such that the associated continuation payoffs uniformly converge to a limit function (which is Lipschitz-continuous with the same constant). The proof is concluded by constructing strategies that are optimal given the above limit continuation functions, and generate exactly the same continuation payoffs.

Let G stand for a generic game defined in Section 2.

Theorem 2: G has an MPE.

We proceed by showing a simple result that reveals an important feature of MPE in our model, and will be used in the subsequent uniqueness proof. It states that in MPE at any point of time any player, if getting an arrival, only approaches coalitions that maximize the difference between the value of the coalition and the sum of continuation payoffs of players other than her in the coalition. Intuitively, players only approach coalitions that are the cheapest to buy relative to the value they can generate.

Claim 1: In any MPE, at any $t \leq 0$ where $i \in N$ receives an arrival, she approaches a coalition $C \in \arg\max_{D \subset N} V(D) - \sum_{j \in D \setminus \{i\}} w_j(t)$ and offers exactly $w_j(t)$ to every $j \in C \setminus \{i\}$. Furthermore, the offer is accepted with probability 1.

Next we establish the uniqueness of MPE payoffs. The intuitive summary of the proof is as follows. Suppose that there are two Markov-perfect equilibria, A and B, with different continuation payoff functions. Suppose that, going backwards in time, continuation payoffs between the two equilibria first diverge at t.

In the first part of the proof, we show that close to t, for any player, continuation values depend primarily on the probability of being approached; that is, having a strictly higher continuation value in A than in B is generally associated with being approached strictly more often under A than under B. To see why this is the case, note that in general, holding arrival rates and future expected payoffs fixed, one expects both the probability of being approached and the share obtained when proposing to influence continuation values. However, close to t, the difference between the two equilibria for the former, a jump variable, is of a greater order of magnitude than for the latter, which is Lipschitz-continuous. The actual proof proceeds as follows: define $f_i(\tau)$ as the difference between player i's payoff in equilibrium A (w_i^A) and her payoff in equilibrium B (w_i^B), at time τ; let $g_i(\tau)$ be the analogous difference in the density of being proposed to by another player. To show that we can find τ arbitrarily close to t such that $\sum_{j\in N}\left[f_j(\tau)\int_\tau^t g_j(t')dt'\right] > 0$ (Lemma 2), we start by showing that $f_i(\tau) - \int_\tau^t e^{-\lambda(t'-\tau)} w_i^B(t') g_i(t')dt'$ is bounded by a quantity on the order of $(t-\tau)f_i$, when (roughly speaking) $|f_i|$ is large enough compared to $|f_j|$ for $j \neq i$. Lemma 1, a technical result, allows us to eliminate the $e^{-\lambda(t'-\tau)} w_i^B(t')$ term inside the integral. The short argument following Lemma 2 allows for the right endpoint of the interval to be different from t, as long as it is close enough to t. In particular, this allows us to pick an interval I where, for any pair of coalitions (C, C'), the sign of the change (as we move from A and B) in the difference of continuation values between the two coalitions ($f_{C,C'}$) remains the same for all times within I.

The second part of the proof starts with observing that optimality imposes that a coalition can take proposals away from another as we switch equilibria only if it has become relatively "cheaper". Denoting the change (from equilibrium A to equilibrium B) in the relative probability of the two coalitions being approached by $g_{C,C'}$, we have that $f_{C,C'}g_{C,C'} \leq 0$. Since within I, $f_{C,C'}$ retains the same sign, then so does $g_{C,C'}$, so that for any $\tau \in I$, $f_{C,C'}(\tau)\int_I g_{C,C'} \leq 0$. Thus, loosely speaking, for any coalition, having a strictly higher continuation value in A than in B is generally associated with being approached less often under A than under B. This observation is the opposite of the one made in the first part of the proof, but for coalitions instead of individual players. Finally, using simple calculations, we derive a contradiction with the result obtained in the first part.

Theorem 3: MPE payoff functions are unique.

6 Limit of payoffs when the time horizon increases

In this section, we investigate the relationship between the parameters of the game (characteristic function, arrival rates) and expected payoffs in MPE as the time horizon grows indefinitely. We note that this is equivalent to investigating the effect of increasing arrival rates with a fixed time horizon, while keeping relative arrival rates unchanged. We first show that if the underlying cooperative game has a nonempty core, then any point of the core can be obtained as the unique MPE (which also turns out to be the unique SPNE) of the game for some vector of arrival rates. Next, we establish a partial converse of this result and show that in convex games, for any vector of arrival rates, expected payoffs converge to a point of the core. Although the latter result extends to all three-player games with nonempty core, we show by example that it does not hold for all games with nonempty core.

Our results on the relationship between limit points of MPE payoffs and the core resemble some existing results from the literature of coalitional bargaining. In Chatterjee et al. (1993), any limit of efficient stationary equilibria as the discount factor goes to 1 has to be in the core, and strict convexity of the characteristic function is sufficient for the existence of efficient equilibrium.[11] Perry and Reny (1994) show that in their continuous-time framework: (i) non-core allocations cannot be supported as stationary SPNE; and (ii) in totally balanced games, an allocation can be supported in SPNE if and only if it is in the core. For other related results in various coalitional bargaining situations, see Moldovanu and Winter (1995), Bloch (1996) and Evans (1997). Our results differ from all the above findings in that in our model, for every specification of the game, there is a unique MPE, but as we vary arrival rates, we can support any core allocation to be the unique limit MPE payoff of the game. Therefore, in convex games we establish an exact equivalence between the limit payoffs of the game and the core, and we provide a theory through the arrival rates that explains which point of the core becomes the limit payoff for a concrete specification of the game.

Throughout this section, we denote the core as $\mathcal{C}(V)$.

Theorem 4: For every $x \in \mathcal{C}(V)$, there exist arrival rates $\{\lambda_i\}_{i \in N}$ such that expected payoffs in SPNE are unique for any finite $T > 0$, and x is the limit of SPNE payoffs as $T \to \infty$.

The following definition and lemma are needed to prove Theorem 5, which is a partial converse of Theorem 4.

Definition: A bargaining game is convex if $V(C \cup A) - V(C) \geq V(C' \cup A) - V(C')$, whenever $C \supset C'$ and $C \cap A = C' \cap A = \varnothing$.

Claim 2: If V is convex, then for any $\varepsilon > 0$, there exists T^* such that in any MPE of a game with $T > T^*$, continuation values are such that $\sum_{i \in C} w_i(t) \geq V(C) - \varepsilon$, $\forall\ C \subset N$ and $t \leq -T^*$.

Claim 2 implies that in any limit MPE payoff of a convex game, the sum of payoffs for coalition members have to be at least as high as the value of the coalition. Since this holds for the grand coalition as well, the resulting allocation has to be in the core.

[11] However, the limit of efficient stationary equilibria is always one point in the core (the one that Lorenz-dominates all core allocations). Moreover, there can exist inefficient stationary equilibria too, the limit of which does not have to be in the core.

Theorem 5: If V is convex, then for any fixed vector of positive arrival rates $(\lambda_i)_{i\in N}$, and any x which can be obtained as a limit of Markov perfect equilibrium payoffs when taking T to infinity, it holds that $x \in \mathcal{C}(V)$.

The same result, namely that any limit payoff of the game has to be in the core, can also be established for all 3-player games with nonempty core.[12]

Theorem 6: If $N = 3$ and $\mathcal{C}(V) \neq \varnothing$, then for any fixed vector of positive arrival rates $\{\lambda_i\}_{i\in N}$, any limit allocation (w_1, w_2, w_3) of a MPE is in $\mathcal{C}(V)$.

We conclude the section by providing an example of a 4-player game with a nonempty core, in which payoffs converge to a point outside the core, as the deadline gets infinitely far away. In this example, the underlying game is not totally balanced; that is, although the game has a nonempty core, it is not true that all subgames have a nonempty core. We do not know whether the core convergence result can be extended to all totally balanced games, a superset of both convex games and 3-player games with nonempty core.

Example 7. Let $N = \{1, 2, 3, 4\}$, $V(N) = 1$, $V(\{1, 2, 3\}) = \frac{1}{2}$, $V(\{1, 2\}) = V(\{2, 3\}) = V(\{3, 1\}) = \frac{3}{8}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{15}$, $\lambda_4 = \frac{4}{5}$. The game is not totally balanced since the core in the subgame with players 1, 2 and 3 is empty. However, the complete game has a nonempty core; for example, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathcal{C}(V)$. It can be shown that in the limit as the deadline gets infinitely far away, expected payoffs converge to the inefficient allocation $(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.

7 Application: legislative bargaining with deadline

In this section, we investigate the special case of our model in which the value of every coalition of size at least K is 1, and the value of all other coalitions is 0. This corresponds to a legislative bargaining game in the tradition of Baron and Ferejohn (1989), in which $n > K$ legislators are voting on how to allocate a fixed surplus among each other, with a K-majority voting rule. In particular, $K = \lfloor \frac{n}{2} \rfloor + 1$ corresponds to a simple majority voting rule. The differences between this setup and the model proposed by Baron and Ferejohn are that: (i) instead of discrete periods with a proposer randomly selected in each period, time is continuous, and proposal times arrive randomly (hence the time between subsequent proposals is random); (ii) there is a deadline after which the surplus is lost. We regard the first difference to be technical, while the second one to be substantial. Having a deadline in legislative bargaining is a natural assumption in many cases: one possible final deadline is when the mandate of the legislature expires. Another example is coalitional government formation when the

[12] In fact, for any vector of arrival rates, it is fairly straightforward to characterize the limit payoffs for all 3-player games with nonempty core. But doing so is cumbersome because many different cases need to be considered.

constitution prescribes a new general election if a government is not successfully formed by a certain deadline.

Our first result establishes that if a player has a higher arrival rate (can propose more frequently in expectations) than another player, then the first player's continuation value is always weakly above the second one's.

Claim 3: If $\lambda_i \leq \lambda_{i'}$, then $w_i(t) \leq w_{i'}(t)\ \forall\ t \leq 0$.

Next we show that the limit of expected shares of the surplus as the time horizon goes to infinity can be obtained by a simple procedure. Without loss of generality assume that players are ordered such that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, and that $\lambda = 1$. Assume also that $\lambda_1 \neq \lambda_K$ (the arrival rates of the K players with the lowest arrival rates are not exactly the same). This assumption can be easily dispensed with without invalidating the results below, but it simplifies the exposition.

For every $j \in \{1, ..., n\}$ define $\underline{x}^j = (x_1^j, ..., x_n^j)$ as follows:

Let x be the solution to the equation $jx + (1 - (K-1)x) \sum\limits_{i=j+1}^{n} \lambda_i = 1$. Then for every $i \in \{1, ..., j\}$ let $x_i^j = x$, and for every $i \in \{j+1, ..., n\}$ let $x_i^j = (1 - (K-1)x)\lambda_i$. Note that by definition $x_{j'+1}^j \geq x_{j'}^j$ whenever $j' > j$.

Let j^* be the smallest $j \in \{1, ..., n\}$ such that $x_{j^*+1}^{j^*} \geq x_{j^*}^{j^*}$ (assume this holds trivially for $j^* = n$). It is easy to verify that $j \geq K$. To simplify notation, let $x_i^* = x_i^{j^*}\ \forall\ i \in \{1, ..., n\}$.

Theorem 7: As $T \to -\infty$, continuation payoffs converge to $(x_1^*, ..., x_n^*)$. There exists $T^* < 0$ such that $w_i(t) = w_{i'}(t)\ \forall\ i, i' \in \{1, ..., j^*\}$ and $t < T^*$.

The above theorem implies that if the deadline is sufficiently far away, then the continuation values of the j^* players (where $j^* \geq K$) with the lowest arrival rates are equal, and each of these players is approached with positive probability.[13] The continuation values of the rest of the players are above the previous level, and ordered according to the relative arrival rates.

For illustration, consider the following two three-player examples. Assume first that $\lambda_1 = 0.25$, $\lambda_2 = 0.3$, and $\lambda_3 = 0.45$. As Figure 8 shows, close to the deadline, $w_1 < w_2 < w_3$. However, since in this region player 3 approaches player 1 with probability 1 (and player 2 with probability 0), eventually the absolute value of the derivative of w_1 becomes greater than that of w_2, and at a certain time ($t^* \approx -0.78$), the continuation values of players 1 and 2 become the same. Before t^* both players 1 and 2 are approached by player 3 with positive probability, while player 3 is not approached by the other players. In this

[13] Not with equal probability though. Players with lower arrival rates are approached by others more frequently - this is what keeps the continuation values of the j^* players equal.

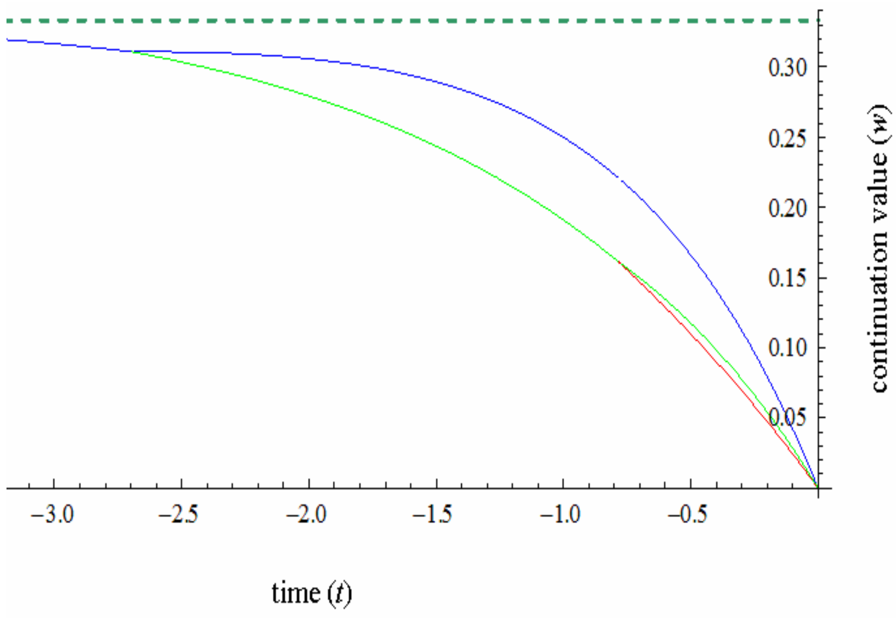

Figure 1:

game player 3's arrival rate is not high enough so that her continuation value permanently stays above the other players' given that she is not approached by them. At a certain time ($t^{**} \approx -2.7$) the continuation values of players 1 and 2 catch up with the continuation value of player 3. Before t^{**} all players' continuation values are equal and converge to $\frac{1}{3}$ as $t \to -\infty$.

Figure 8.

Assume next that $\lambda_1 = 0.15$, $\lambda_2 = 0.25$, and $\lambda_3 = 0.6$. As the Figure 9 shows, again close to the deadline, $w_1 < w_2 < w_3$, and there is a point ($t^* \approx -1.71$) such that w_1 becomes equal to w_2. However, in this example, despite the fact that player 3 is never approached by any of the other players, her continuation value always stays above the continuation values of the other players, because her relative arrival rate is high enough. If the deadline gets infinitely far away, w_3 converges to approximately 0.426, while w_1 and w_2 converge to approximately 0.287.

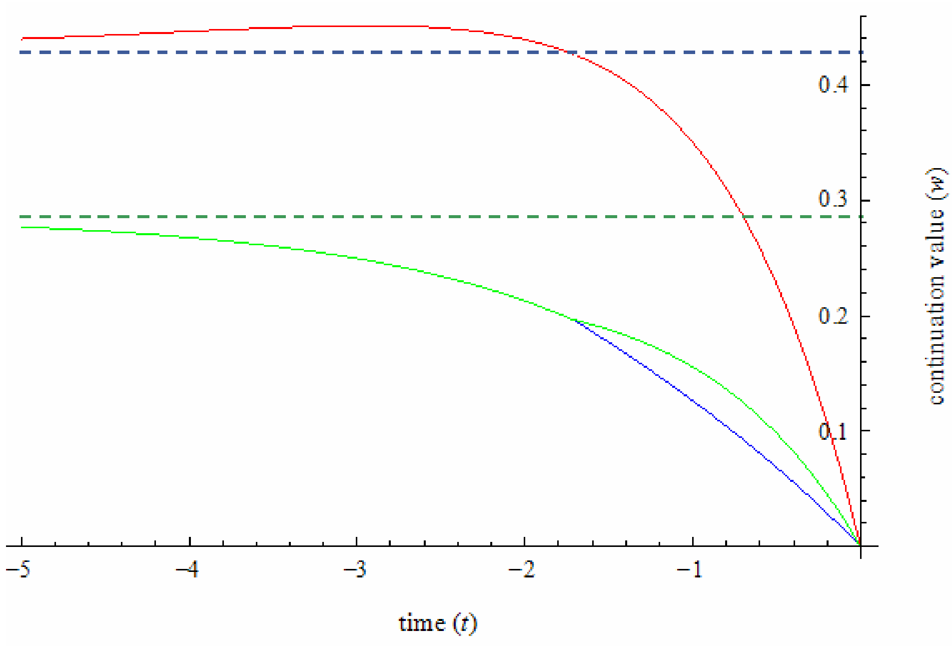

Figure 9.

Both of the above examples show that the difference between expected payoffs of a player who can propose more frequently and a player who can propose less frequently is non-monotonic over time. Going back in time starting from $t = 0$, the difference between player 2's continuation value and player 1's continuation value is first increasing, then decreasing. Moreover, the second example reveals that the continuation value of a player itself might not be monotonic. Going back in time, player 3's continuation value first increases strictly above its limit value, and converges there from above. Hence, for a player who can propose frequently, the deadline resulting in the highest expected payoff is an intermediate one. If the deadline is too close, the probability of not reaching an agreement (because of no arrival) is too high, while if the deadline is too far out, then the player has to offer a relatively high share of the surplus to players with lower arrival rates.

8 Discussion: extensions

Our model framework can be extended in many different directions. Some of these are mentioned in the introduction of the paper, like considering asymmetric information, or allowing for the game to continue after an agreement is reached by some coalitions. We leave these investigations to future research.

However, other interesting extensions can be incorporated into the model in a straightforward manner. Below we discuss these.

8.1 Discounting

The presence of a deadline and random arrival times imply that our model yields sharp predictions without introducing discounting. However, discounting can be added easily. To see this, consider the case of group bargaining, as in Section 3, but assume now a constant discount rate r common to all agents. Payoffs then become $w_i(t) = \frac{\lambda_i}{\lambda+r}(1 - e^{(\lambda+r)t})$. As before, players' expected payoffs are proportional to their arrival rates, and $w_i'(0)$ (more precisely, the left-hand derivative of w_i at 0) is the same as in the case without discounting for each player i. Now, however, at the limit as the deadline gets infinitely far away, payoffs converge to $\frac{\lambda_i}{\lambda+r}$ instead of $\frac{\lambda_i}{\lambda}$. This means that the sum of limit expected payoffs is less than $V(N)$ and decreasing in r. In the general coalitional bargaining framework, a model with constant common discount rate r can be rewritten as a model with no discount rate and an auxiliary player 0 that has arrival rate r, and in case of arrival, approaches the singleton coalition $\{0\}$ (which can be guaranteed by setting $V(\{0\}) = V(N)$).

8.2 Gradually disappearing pies

Our model assumes that the surplus that any coalition can generate stays constant until a certain point of time (the deadline) and then discontinuously drops to zero. Although there are many situations in which there is such a highlighted point of time that makes subsequent agreements infeasible, in other cases it is more realistic to assume that the surpluses that players can generate start decreasing at some point, but only go to zero gradually. For example, agreeing upon broadcasting the games of a sports season yields diminishing payoffs once the season started, but if there are still remaining games in the season, a fraction of the original surplus can still be attained.

Some of our results can be extended to this framework. For example, the case of group bargaining remains tractable when $V(N)$ is time-dependent, even without assuming specific functional forms. Indeed, if $V(N)(t)$ is continuous and nonincreasing, and there is some time t^* at which $V(N)$ becomes zero, our argument for the uniqueness of SPNE payoffs applies with minor modifications. The uniqueness of the solution to the system of differential equations still follows from the Picard-Lindelof theorem. Continuation payoff functions are then $w_i(t) = \lambda_i \int_t^\infty e^{-\lambda(\tau-t)} V(N)(\tau) d\tau$, so payoffs remain proportional to arrival rates at all times, and since the grand coalition always forms at an arrival, the sum of expected payoffs across all players is simply the expected size of the pie at the next arrival (0 after t^*). Even if we do not assume that there is a time t^* as above, but instead only that $V(N)(t)$ is nonincreasing and $\lim_{t\to\infty} V(N)(t) = 0$, the informal argument presented for the uniqueness of MPE in group bargaining can be formalized and shown to apply to this context. In particular, one can show that if there are two MPE yielding payoffs to a player that differ by $\varepsilon > 0$

at some point of time, then there has to be a sequence of times going to infinity such that everywhere along this sequence, expected payoffs of i differ by at least ε. This leads to a contradiction, since after some point the whole pie is worth less than ε. Furthermore, the same continuation payoffs apply in this case as in the previous one.

8.3 Infinite horizon with constant pies and discounting

Without deadline, our model yields very similar results to a discrete-time model in which a proposer is selected randomly at every period (with perhaps a positive probability of no proposer selected). For example, in the group bargaining case with infinite horizon and constant discount rate r, one can show that if r is low enough, then any allocation of the surplus can be supported in SPNE.[14] A further similarity with the discrete model is that there is only one stationary SPNE in discounted infinite-horizon group bargaining, which is characterized by:

$$w_i = \int_0^\infty [\lambda_i e^{-(\lambda+r)\tau}(1 - \sum_{j \in N \setminus \{i\}} w_j) + \sum_{j \in N \setminus \{i\}} \lambda_j e^{-(\lambda+r)\tau} w_i] d\tau.$$

The solution of this system is $w_i = \frac{\lambda_i}{r+\lambda}$. This implies that as $r \to 0$, $w_i \to \frac{\lambda_i}{\lambda}$. That is, the limit of stationary equilibrium payoffs of the infinite-horizon model as the discount rate goes to 0 is equal to the limit of SPNE payoffs of the finite-horizon model as the horizon goes to infinity. In fact, the same conclusion is true for the legislative bargaining games analyzed in Section 7. It is straightforward to verify that the limit payoffs in Theorem 7 are exactly the limit stationary SPNE payoffs of the infinite horizon version of our game, as the discount rate goes to 0. This is in contrast with the findings of of Norman (2002) for discrete time legislative bargaining games.[15] It is an open question what class of coalitional bargaining problems this result can be extended to.

9 Appendix

Proof of Theorem 1: Let $\overline{v_i}(t)$ and $\underline{v_i}(t)$ be the supremum and the infimum, over all SPNEs and all histories preceding t, of player i's share when she makes an offer at time t. Let $\overline{w_i}(t)$ and $\underline{w_i}(t)$ be the supremum and the infimum of

[14] In particular, the type of construction in p63 of Osborne and Rubinstein (originally by Shaked) supports even the most extreme allocation in which one player gets all the surplus.

[15] Norman shows that for generic specifications of the legislative bargaining game with finite horizon there is a unique SPNE, but players' continuation values over time in equilibrium are typically changing in a non-monotonic way, even far away from the deadline. The paper argues that this implies that play in longer and longer finite-horizon versions of the game does not approximate play in stationary SPNE of the infinite horizon game, although it is not addressed formally whether limit payoffs as the horizon goes to infinity converge to the stationary SPNE payoffs. See also Snyder et al. for an investigation of stationary SPNE payoffs in discrete time legislative bargaining games.

player i's share when player $j \neq i$ is making an offer, over all SPNEs, histories and $j \neq i$.

Let $\lambda = \sum_{i=1}^{n} \lambda_i$. Note that the density of i getting the next arrival of any player, at a time x units from now, is $\lambda_i e^{-\lambda x}$.

First, note that $\overline{v_i}(t) + \sum_{j \neq i} \underline{w_j}(t) = 1$, since this will be true in an SPNE where i offers everyone $\underline{w_j}(t)$ and takes the rest, and where, if any such offer by i is rejected, we move to a SPNE giving a continuation value of $\underline{w_j}(t)$ to the first rejector.

Consider the following profile: when any player $k \neq i$ makes an offer, the offer to player i must be $\overline{w_i}(t)$, and the offer to all $j \neq i, k$ is at least $\underline{w_j}(t)$. If k offers less to any player, the offer is rejected by that player; if player $j \neq i, k$ is the rejector, we move to an SPNE giving player k an expected payoff of $\underline{w_k}(t)$, and if player i is the rejector, we move to an SPNE giving player i an expected payoff of $\overline{w_i}(t)$. If k makes the correct offer and player j is the first rejecting the offer, then we move to an equilibrium giving $\underline{w_j}(t)$ to j. When i makes an offer, she gives herself $\overline{v_i}(t)$ and gives $\underline{w_j}(t)$ to all $j \neq i$, as specified above.

To show that the exhibited profile is an SPNE, we need to verify that it indeed exists, i.e. that offers are feasible. Note that player k's offer is feasible if $\overline{w_i}(t) + \sum_{j \neq i} \underline{w_j}(t) \leq 1$. But this must be true since the sum of all continuation values in any SPNE must be less than 1. As established above, player i's offer is feasible. We also need to check that players' actions are optimal. The only case where this is not trivial is that when k makes an offer, she may prefer to make one that is rejected by i. However, this will not be the case in an interval close to 0 where the probability of any future arrival $\leq \frac{1}{n}$, since then $\overline{w_k}(t) \leq \frac{1}{n} = 1 - \frac{n-1}{n} \leq 1 - \sum_{i \neq k} \overline{w_i}(t)$, so k will want the offer to be accepted. Denote this interval $[s, 0]$ (so $s = \frac{1}{\lambda} \ln(\frac{n-1}{n})$).

The above profile is of course the best possible one for i, so on $[s, 0]$ we have:

$$\begin{aligned}
\overline{w_i}(t) &= \int_t^0 \left[\lambda_i e^{-\lambda(\tau - t)} \overline{v_i}(\tau) + \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} \overline{w_i}(\tau) \right] d\tau \\
&= \int_t^0 \left[\lambda_i e^{-\lambda(\tau - t)} (1 - \sum_{j \neq i} \underline{w_j}(\tau)) + \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} \overline{w_i}(\tau) \right] d\tau
\end{aligned}$$

Since $\overline{w_i}(t)$ is the integral of a continuous function, its derivative exists, so:

$$\begin{aligned}
\overline{w_i}'(t) &= -\lambda_i(1-\sum_{j\neq i}\underline{w_j}(t)) - \sum_{j\neq i}\lambda_j\overline{w_i}(t) \\
&\quad +\lambda\int_t^0\left[\lambda_i e^{-\lambda(\tau-t)}(1-\sum_{j\neq i}\underline{w_j}(\tau)) + \sum_{j\neq i}\lambda_j e^{-\lambda(\tau-t)}\overline{w_i}(\tau)\right]d\tau \\
&= -\lambda_i(1-\sum_{j\neq i}\underline{w_j}(t)) - \sum_{j\neq i}\lambda_j\overline{w_i}(t) + \lambda\overline{w_i}(t) \\
&= \lambda_i(\overline{w_i}(t) - 1 + \sum_{j\neq i}\underline{w_j}(t))
\end{aligned}$$

Similarly, we note that $\underline{v_i}(t)+\sum_{j\neq i}\overline{w_j}(t) = 1$ on $[s,0]$, since this occurs when i offers everyone $\overline{w_j}(t)$ and takes the rest, and where, if i gives any less than $\overline{w_j}(t)$ to a player, we move to a SPNE giving a continuation value of $\overline{w_j}(t)$ to the first rejector. On $[s,0]$, $\overline{w_j}(t)$ and the probability of a future arrival are close to 0, so it will be optimal for i to make such an offer. By a similar argument as above, we can show that:

$$\underline{w_i}'(t) = \lambda_i(\underline{w_i}(t) - 1 + \sum_{j\neq i}\overline{w_j}(t))$$

Thus on a nontrivial interval $[s,0]$, we have a system of $2n$ differential equations continuous in t, and Lipschitz continuous in $2n$ unknown functions with initial values $\overline{w_i}(0) = \underline{w_i}(0) = 0$. By the Picard-Lindelof theorem, this initial value problem has a unique solution. It is easy to check that the following functions constitute the solution:

$$\overline{w_i}(t) = \underline{w_i}(t) = \frac{\lambda_i}{\lambda}(1-e^{\lambda t}) \equiv w_i(t)$$

The above argument can be iterated for $[2s,s]$ since the game ending at s with payoffs $w_i(s)$ is simply a scaled version of the original game, and so on. QED

Proof of Theorem 2: For any $k \in \mathbb{Z}_+$, let G^k denote the following discrete time game:

There are k periods. In each period there is an arrival with probability $1-e^{-\lambda\frac{T}{k}}$. The relative probabilities of which player gets the arrival (if there is one) are $\lambda_1, ..., \lambda_N$.

A simple argument shows that in every SPNE of G^k, in period k every player approaches N and offers 0 to everyone in $N\backslash\{i\}$.

For each of these games, there exists an SPNE in pure strategies. Such a profile can be constructed as follows:

Denote the coalition player i approaches in period m by $C_i^k(m)$. Let $w_i^k(m)$ denote the continuation payoff of i if no agreement was made in $1, ..., m-1$.

For every $i \in N$, let $w_i^k(k+1) = 0$.

Suppose now that $C_i^k(m)$ and $w_i^k(m)$ are defined for every $m > k'$, where $k' \in \{1, ..., k\}$. For any $i \in N$, define $C_i^k(k')$ such that $C_i^k(k') \in \underset{C \subset N : i \in C}{\arg\max} V(C) - \sum_{j \in C \setminus \{i\}} w_j^k(k'+1)$ (if there are multiple C like that, then select any of them). For every $i, j \in N$, define $I_{ij}^{k,k'}$ the indicator function that takes value 1 if $j \in C_i^k(k')$, and 0 otherwise. Finally, define

$$w_i^k(k') = \frac{\lambda_i}{\lambda} e^{-\lambda \frac{T}{K}} [V(C_i^k(k')) - \sum_{j \in C \setminus \{i\}} w_j^k(k'+1)]$$

$$+ \sum_{j \in N \setminus \{i\}} \frac{\lambda_j}{\lambda} e^{-\lambda \frac{T}{K}} I_{ji}^{k,k'} w_i^k(k'+1) + (1 - e^{-\lambda \frac{T}{K}}) w_i^k(k'+1).$$

The above procedure iteratively defines $C_i^k(m)$ and $w_i^k(m)$ $\forall$ $i \in N$ and $m \in \{1, ..., k\}$.

It is easy to see that the following strategies constitute an SPNE: player i at period m approaches $C_i^k(m)$ and offers $w_j^k(m+1)$ to every $j \in C_i^k(m) \setminus \{i\}$. If player i is approached at period m by any other player, then she accepts the offer if it is at least $w_j^k(m+1)$, and rejects it otherwise.

Denote the above-defined strategy profile in G^k by s^k.

Based on s^k, for every $i \in N$, define strategy $\widehat{s}_i^k$ of i in G, the following way:

If i gets an arrival at $t = -\frac{(k-m-\alpha)T}{k}$ for $\alpha \in (0, 1]$ and $m \in \{0, ..., k-1\}$ (or $\alpha = m = 0$ for $t = -T$), she approaches $C_i^k(m+1)$ and offers $w_j^k(m+2)$ (as defined in G^k) to every $j \in C_i^k(m+1) \setminus \{i\}$. If player i is approached at $t = -\frac{(k-m-\alpha)T}{k}$ for $\alpha \in (0, 1]$ and $m \in \{0, ..., k-1\}$ (or $\alpha = m = 0$) by any player, then she accepts the offer if and only if it is at least $w_i^k(m+2)$. (For ease of exposition, we will omit the $\alpha = m = 0$ from the argument below.)

Define the expected continuation value of player i at time t according to the above profile in G by $\widehat{w}_i^k(t)$. First, note that given $\widehat{s}_{-i}^k$, strategy $\widehat{s}_i^k$ specifies an optimal action for i if she has an arrival, at every $t \in [-T, 0]$. Next, we bound the suboptimality of $\widehat{s}_i^k$ when i considers an offer. Observe that by construction, at any time $t = -\frac{(k-m)T}{k}$ for $m \in \{0, ..., k-1\}$, $\widehat{w}_i^k(t) = w_i^k(m+1)$. Moreover, for $t = -\frac{(k-m-\alpha)T}{k}$, $\widehat{w}_i^k(t) \to w_i^k(m+2)$ as $\alpha \nearrow 1$. Given that $\widehat{s}^k$ is Markovian, the optimal action for i in G if she is approached by any other player at $t = -\frac{(k-m-\alpha)T}{k}$ for $\alpha \in (0, 1]$ and $m \in \{0, ..., k-1\}$ is, independently of payoff-irrelevant history, to accept the offer if it is at least $\widehat{w}_i^k(t)$, and reject it otherwise. Instead, strategy $\widehat{s}_i^k$ specifies that i accepts the offer if and only if it is at least $w_j^k(m+2)$; hence, after some histories, $\widehat{s}_i^k$ specifies a suboptimal action for i. However, since $\widehat{w}_i^k(t)$ is between $w_j^k(m+1)$ and $w_j^k(m+2)$, the difference between the expected payoff resulting from following $\widehat{s}_i^k$ versus choosing the optimal action at t is bounded by $\left| w_j^k(m+1) - w_j^k(m+2) \right|$. Given that the probability of an arrival between $t = -\frac{(k-m)T}{k}$ and $t = -\frac{(k-m-1)T}{k}$ is $1 - e^{-\lambda\frac{k}{T}}$, $\left|w_j^k(m+1) - w_j^k(m+2)\right| \leq V(N)(1 - e^{-\lambda\frac{k}{T}})$. This means that for any $\varepsilon > 0$, there is a $k^\varepsilon \in Z^+$ such that for any $k > k^\varepsilon$, $\widehat{s}^k$ specifies an ε-perfect equilibrium of G (which is also Markovian, by construction).

Consider now continuation payoff functions $\widehat{w}_i^k()$ for $i \in N$. Define $\widehat{t}^k(\tau) = \left\lceil (T+\tau)\frac{k}{T} \right\rceil$. By construction,

$$\widehat{w}_i^k(t) = \int_t^0 [\lambda_i e^{-\lambda(\tau - t)} \left(V[C_i^k(\widehat{t}^k(\tau))] - \sum_{j \in C_i^k(\widehat{t}^k(\tau)) \setminus \{i\}} w_j^k(\widehat{t}^k(\tau)+1) \right)$$
$$+ \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} w_i^k(\widehat{t}^k(\tau)+1)] d\tau.$$

It is easy to see that for every $i \in N$ and $k \in \mathbb{Z}_+$, $\widehat{w}_i^k()$ is Lipschitz-continuous with Lipschitz constant $\lambda V(N)$. Moreover, all $\widehat{w}_i^k()$ are uniformly bounded by 0 below and $V(N)$ above. Therefore, by the Ascoli-Arzela theorem (see Royden (1988), p169), the sequence of functions $(\widehat{w}_1^k(), ..., \widehat{w}_N^k())$ has a subsequence that converges uniformly to functions $(\widehat{w}_1^*(), ..., \widehat{w}_N^*())$, as $k \to \infty$. Moreover, these functions are also Lipschitz-continuous with the same constant. Without loss of generality, assume that the original sequence is convergent.

Take now any $t \in [-T, 0]$, and any $C \subset N$ such that for every $k' \in \mathbb{Z}_+$, there exists $k > k'$ such that $C_i^k(\widehat{t}^k(t)) = C$ (that is, C is approached by i at t according to $\widehat{s}_i^k$). Recall that for every $\varepsilon > 0$, there exists $k^\varepsilon \in \mathbb{Z}_+$ such that for every $k > k^\varepsilon$, $\widehat{s}^k$ is an ε-perfect profile, and $(\widehat{w}_1^k(), ..., \widehat{w}_N^k()) \to (\widehat{w}_1^*(), ..., \widehat{w}_N^*())$. Thus, if continuation payoffs at t are given by $\widehat{w}_1^*(t), ..., \widehat{w}_N^*(t)$, then it is optimal for i to approach C and offer $\widehat{w}_j^*(t)$ to everyone in $C \backslash \{i\}$, and accepting this offer is optimal for players in $C \backslash \{i\}$.

For any $t \in [-T, 0]$, let $n^k(t) = [-T, 0] \cap [t - \frac{1}{k}, t + \frac{1}{k}]$, so that $\bigcap_{k=1,2,...} n^k(t) = \{t\}$. For any $i \in N$, let $p_i^k(t) \in \Delta(2^N)$ be defined such that $p_i^k(t)(C) = \frac{L(\{x \in n^k(t) : \hat{s}_i^k(x) = C\})}{|n^k(t)|}$, where $L()$ stands for the Lebesgue measure. Note that the set $\{x \in n^k(t) : \hat{s}_i^k(t) = C\}$ is measurable by construction. Since $\Delta(2^N)$ is compact, the sequence $(p_i^k(t))_{k=1,2,...}$ has a convergent subsequence. Take any convergent subsequence and denote the limit by $p_i^*(t)$.

Now define strategies s_i^* for all $i \in N$ as follows: for any $t \in [-T, 0]$, if i has an arrival at t, then she approaches $C \subset N$ with probability $p_i^*(t)(C)$ and offers $\widehat{w}_j^*(t)$ to every $j \in C \backslash \{i\}$; if i is approached at t by any other player, then she accepts the offer if and only if it is at least $\widehat{w}_i^*(t)$. Observe that if $p_i^*(t)(C) > 0$, then there is $k > k'$ such that $p_i^k(t)(C) > 0 \; \forall \; k' \in \mathbb{Z}_+$. Since for every $\varepsilon > 0$, there is $k^\varepsilon \in \mathbb{Z}_+$ such that for every $k > k^\varepsilon$, $\widehat{s}^k$ is an ε-perfect profile, $\bigcap_{k=1,2,...} n^k(t) = \{t\}$, $\widehat{w}_1^k()$ goes to $\widehat{w}_1^*()$ uniformly as $k \to \infty$, and $\widehat{w}_1^*()$ is continuous, it has to be that approaching C and offering $\widehat{w}_j^*(t)$ to every $j \in C \backslash \{i\}$ is the optimal action for i if other players play according to s_{-i}^*. Moreover, if continuation values are indeed given by $\widehat{w}_i^*(t) \; \forall \; i \in N$, then it is optimal for i to accept an offer if and only if it is at least $\widehat{w}_i^*(t)$.

All that remains to be shown is that s^* generates continuation payoffs $\widehat{w}_i^*(t)$ for every $i \in N$ and $t \in [-T, 0]$. To see this, first note that since the functions $\widehat{w}_1^k(), ..., \widehat{w}_N^k()$ are Lipschitz-continuous for every $k \in \mathbb{Z}_+$, they are almost everywhere differentiable. The same holds for $\widehat{w}_1^*(), ..., \widehat{w}_N^*()$. Hence, for almost every $t \in [-T, 0]$, both $\widehat{w}_1^*(), ..., \widehat{w}_N^*()$ and $\widehat{w}_1^k(), ..., \widehat{w}_N^k()$ for all $k \in \mathbb{Z}_+$ are differentiable at t. Consider any such t. Since $\widehat{s}^k$ generates continuation payoffs $\widehat{w}_1^k(), ..., \widehat{w}_N^k()$, the construction of s^* implies that the derivative at t of the continuation payoff of any $i \in N$ generated by s^* is equal to the derivative of $\widehat{w}_i^*()$ at t. Since this holds for almost all $t \in [-T, 0]$, and the continuation payoff of $i \in N$ at $t = 0$ generated by any strategy is equal to $\widehat{w}_i^*(0) = 0$, the profile s^* indeed generates continuation payoffs $\widehat{w}_1^*(), ..., \widehat{w}_N^*()$.

This establishes that s^* is an SPNE. By construction, it is Markovian. QED

Proof of Claim 1: Note that $\sum_{j \in N} w_j(t) \leq 1 - e^{\lambda t}$, where $e^{\lambda t} > 0$ is the probability that no one has the chance to make an offer during $[t, 0]$. Furthermore, in any MPE, if $C \subset N$ is approached by i at t, and every $j \in N \backslash \{i\}$ is offered strictly more than $w_j(t)$, then the offer has to be accepted by everyone with probability 1. Therefore, player i can guarantee a payoff strictly larger than $w_i(t)$ by approaching N and offering $w_j(t) + \varepsilon$ to every $j \in N \backslash \{i\}$ for small enough $\varepsilon > 0$. On the other hand, a rejected offer results in continuation payoff $w_i(t)$ for i. Next, note that approaching a coalition C and offering strictly less than $w_j(t)$ to some $j \in C$ results in rejection of the offer with probability 1, and is therefore not optimal. Approaching a coalition C and offering $w_j(t) + \varepsilon$ for $\varepsilon > 0$ to some $j \in C$ is also suboptimal, because offering instead $w_j(t) + \varepsilon / N$ to every $j \in C \backslash \{i\}$ results in acceptance of the offer with probability 1 and strictly higher payoff. Therefore, whatever coalition C is approached, player i has to offer exactly $w_j(t)$ to every $j \in C \backslash \{i\}$. It cannot be that this offer is accepted with probability less than 1, since then player i could strictly improve her payoff by offering slightly more than $w_j(t)$ to every $j \in C \backslash \{i\}$, since that offer would be accepted with probability 1. Finally, it cannot be that $C \notin \arg\max_{D \subset N} V(D) - \sum_{j \in D \backslash \{i\}} w_j$, since then approaching some $C' \in \arg\max_{D \subset N} V(D) - \sum_{j \in D \backslash \{i\}} w_j$ instead, and offering slightly more than $w_j(t)$ to every $j \in C' \backslash \{i\}$ would result in a strictly higher payoff. QED

Proof of Theorem 3: The proof requires the introduction of some extra notation. Let $v_i(t) = \max_{C \ni i} \left\{ v(C) - \sum_{j \in C \backslash \{i\}} w_j(t) \right\}$. Also, let $p_{ij}(t)$ be the probability of j receiving an offer at time t given that i gets an arrival at that time, and let $p_{iC}(t)$ be the probability of C (and only C) receiving an offer at time t given that i gets an arrival at that time.

We proceed by contradiction. Suppose two MPEs, A and B, of the same bargaining game with characteristic function V and arrival rates $(\lambda_1, ..., \lambda_n)$ do not have the same continuation value functions.

First note that if $\lambda_i = 0$, the only possible MPE continuation value for i is 0 at all times. The game is then equivalent to an alternative game with players $N \backslash \{i\}$ and characteristic function $V'(C) = V(C \cup \{i\}), \forall C \subset N \backslash \{i\}$. So we assume without loss of generality that $\lambda_i > 0, \forall i \in N$.

Let $t = \min\{\tau | f_i(t') = 0, \forall t' \in [\tau, 0], \forall i \in N\}$. Note that $t < 0$ since there must be some nontrivial interval where proposing to a coalition of value $V(N)$ is strictly optimal for everyone. When the only such coalition is N, MPE payoffs are clearly unique within this interval; the same can be shown if multiple coalitions have value $V(N)$.[16] Thus we have $w_i(t) > 0$.

Define $f_i(\tau) = w_i^A(\tau) - w_i^B(\tau)$, and note that $f_i(\tau)$ is Lipschitz continuous.

For $\tau < t$, let $F_i(\tau) = \max_{t' \in [\tau, t]} |f_i(t')|$, so $F_i(\tau)$ is Lipschitz continuous and nonincreasing.

Let $I(\tau) = \arg\max_i F_i(\tau)$, and let $S = \{t' | F_{I(t')}(t') = \left| f_{I(t')}(t') \right| \text{ and } t' < t\}$. Note that for all $\tau < t$, $S \cap [\tau, t) \neq \varnothing$.

Define $g_j(\tau) = \displaystyle\sum_{i \neq j} \lambda_i p_{ij}^A(\tau) - \sum_{i \neq j} \lambda_i p_{ij}^B(\tau)$. Note that since we are considering equilibria, continuation value functions exist, which implies that $\int_\tau^t g_j(t')dt'$ exists for all $\tau < t$.

Lemma 1: Let $g(.)$ be an integrable function taking values between $-K$ and K, and let $h(.) \geq 0$ be Lipschitz continuous with Lipschitz bound L. Then $\left| \int_\tau^t g(t')h(t')dt' - h(t) \int_\tau^t g(t')dt' \right| < 2L(t - \tau) \max_{\tau' \in [\tau, t]} \left| \int_{\tau'}^t g(t')dt' \right|$.

Proof: Suppose $\int_{t_1}^{t_2} g(t')dt' = 0$ and $\left| \int_\tau^{t_2} g(t')dt' \right| \leq C$ for all $\tau \in [t_1, t_2]$. Then $\left| \int_{t_1}^{t_2} g(t')h(t')dt' \right| < LC(t_2 - t_1)$, as the maximum corresponds to the case where $h(\tau)$ follows a Lipschitz bound, $g(\tau) = K$ for $\tau \in [\max\{t_2 - \frac{C}{K}, \frac{t_1+t_2}{2}\}, t_2]$, $g(\tau) = -K$ for $\tau \in [t_1, \min\{t_1 + \frac{C}{K}, \frac{t_1+t_2}{2}\}]$, and $g(\tau) = 0$ for $\tau \in (\min\{t_1 + \frac{C}{K}, \frac{t_1+t_2}{2}\}, \max\{t_2 - \frac{C}{K}, \frac{t_1+t_2}{2}\})$.

Now partition $[\tau, t]$ into measurable sets A and B, where $\int_S g(t')dt' = 0$ for any connected set $S \subset A$ and $\int_U g(t')dt'$ has the same sign as $\int_\tau^t g(t')dt'$ for any connected set $U \subset B$. Then we have $\left| \int_\tau^t g(t')h(t')dt' - h(t) \int_\tau^t g(t')dt' \right| \leq \left| \int_A g(t')h(t')dt' \right| + \left| \int_B g(t')h(t')dt' - h(t) \int_B g(t')dt' \right| < LC(t-\tau) + L(t-\tau) \left| \int_\tau^t g(t')dt' \right|$, where $C = \max_{\tau' \in [\tau, t]} \left| \int_{\tau'}^t g(t')dt' \right|$. QED

Lemma 2: $\forall \varepsilon > 0$, $\exists \tau \in [t - \varepsilon, t)$ such that $\sum_{j \in N} \left[f_j(\tau) \int_\tau^t g_j(t')dt' \right] > 0$.

[16] Note that at $t = 0$, the left derivative of continuation value functions must exist since $w_i(0) = 0$ (so the probability of being proposed to given an arrival, which may be discontinuous, does not affect the rate of change of w_i). In fact, we have $w_i'(0) = -\lambda_i V(N)$. So when there are coalitions $C_1, ..., C_m \neq N$ with $i \in C_j$ and $V(C_j) = V(N)$ for $j = 1, ..., m$, it must be true that in a neighborhood of 0, i only proposes to C_k with positive probability if $\sum_{l \in C_k} \lambda_l \leq \sum_{i \in C_j} \lambda_l$ for $j = 1, ..., m$. When there are multiple such coalitions, if feasible, they will be proposed to such that their continuation values are equalized; if equalization cannot be achieved, those having the choice between many such coalitions will propose to the cheapest one.

Proof: Note that:

$$\begin{aligned} f_j(\tau) &= \int\limits_{\tau}^{t} e^{-\lambda(t'-\tau)} \left[\begin{array}{c} \lambda_j(v_j^A(t') - v_j^B(t')) + (w_j^A(t') - w_j^B(t')) \left(\sum\limits_{i \neq j} \lambda_i p_{ij}^A(t') \right) \\ + w_j^B(t') \left(\sum\limits_{i \neq j} \lambda_i p_{ij}^A(t') - \sum\limits_{i \neq j} \lambda_i p_{ij}^B(t') \right) \end{array} \right] dt' \\ &= \lambda_j \int_{\tau}^{t} e^{-\lambda(t'-\tau)} \left[v_j^A(t') - v_j^B(t') \right] dt' + \int_{\tau}^{t} e^{-\lambda(t'-\tau)} f_j(t') \left(\sum_{i \neq j} \lambda_i p_{ij}^A(t') \right) dt' \\ &\quad + \int_{\tau}^{t} e^{-\lambda(t'-\tau)} w_j^B(t') g_j(t') dt' \end{aligned}$$

So we have:

$$\begin{aligned} \left| f_j(\tau) - \int_{\tau}^{t} e^{-\lambda(t'-\tau)} w_j^B(t') g_j(t') dt' \right| &\leq \lambda(t-\tau) \sum_{i \in N \setminus \{j\}} F_i(\tau) + \lambda(t-\tau) F_j(\tau) \\ &= \lambda(t-\tau) \sum_{i \in N} F_i(\tau) \end{aligned}$$

Also, by Lemma 1,

$$\begin{array}{c} \left| \int_{\tau}^{t} g_j(t') dt' - \frac{e^{\lambda(t-\tau)}}{w_j^B(t)} \int_{\tau}^{t} e^{-\lambda(t'-\tau)} w_j^B(t') g_j(t') dt' \right| \\ < 2L_j(t-\tau) \max_{\tau' \in [\tau, t]} \left| \int_{\tau'}^{t} e^{-\lambda(t'-\tau)} w_j^B(t') g_j(t') dt' \right| \\ \leq 2L_j(t-\tau) \left(F_j(\tau) + \lambda(t-\tau) \sum_{i \in N} F_i(\tau) \right) \end{array}$$

where L_j, the Lipschitz constant for $\frac{e^{\lambda(t-\tau)}}{w_j^B(t)}$, is finite when $w_j^B(t) \neq 0$. Combining this with the above yields:

$$\begin{array}{c} \left| f_j(\tau) - \frac{w_j^B(t)}{e^{\lambda(t-\tau)}} \int_{\tau}^{t} g_j(t') dt' \right| \\ < \lambda(t-\tau) \sum_{i \in N} F_i(\tau) + 2 \frac{w_j^B(t)}{e^{\lambda(t-\tau)}} L_j(t-\tau) \left(F_j(\tau) + \lambda(t-\tau) \sum_{i \in N} F_i(\tau) \right) \\ \leq (\lambda + 2L_j w_j^B(t)) \sum_{i \in N} F_i(\tau)(t-\tau) + 2\lambda L_j w_j^B(t) \sum_{i \in N} F_i(\tau)(t-\tau)^2 \\ \equiv \sum_{i \in N} F_i(\tau)(t-\tau)(k_j + q_j(t-\tau)) \end{array}$$

For $(t-\tau)$ small enough, $q_j(t-\tau)$ is negligible, so we omit it below. Thus:

$$f_j(\tau) \frac{w_j^B(t)}{e^{\lambda(t-\tau)}} \int_{\tau}^{t} g_j(t') dt' \geq [f_j(\tau)]^2 - k_j(t-\tau) \sum_{i \in N} F_i(\tau) \left| f_j(\tau) \right|$$

We have that for all $\tau \in S$:

$$f_j(\tau) \int_{\tau}^{t} g_j(t') dt' \geq \frac{1}{w_j^B(t)} \left([f_j(\tau)]^2 - k_j(t-\tau) n \left| f_{I(\tau)}(\tau) \right| \left| f_j(\tau) \right| \right)$$

So $\forall\ \tau \in (t - \frac{\sqrt{\min_j \{1, w_j^B(t)\}}}{2n^2 \max_j k_j}, t) \cap S$:

$$
\begin{aligned}
f_j(\tau)\int_\tau^t g_j(t')dt' &> \frac{1}{w_j^B(t)}\left([f_j(\tau)]^2 - |f_j(\tau)|\frac{\left|f_{I(\tau)}(\tau)\right|}{2n}\sqrt{w_j^B(t)}\right) \\
&\geq -\frac{\left|f_{I(\tau)}(\tau)\right|^2}{16n^2}
\end{aligned}
$$

and so $f_{I(\tau)}(\tau)\int_\tau^t g_{I(\tau)}(t')dt' \geq \frac{\left|f_{I(\tau)}(\tau)\right|^2}{2}$.

Thus, $\sum_{j\in N} f_j(\tau)\int_\tau^t g_j(t')dt' > \frac{7\left|f_{I(\tau)}(\tau)\right|^2}{16}$. The result follows since for all $\tau < t$, $S \cap [\tau, t) \neq \varnothing$. QED

Now let $f_{C,C'}(\tau) \equiv f_C(\tau) - f_{C'}(\tau) \equiv \sum_{i\in C} f_i(\tau) - \sum_{i\in C'} f_i(\tau)$, and $Z_{t'}(\tau) \equiv \sum_{j\in N}\left[f_j(t')\int_\tau^t g_j(s)ds\right]$. So we know that $\forall\varepsilon > 0$, $\exists t' \in [t-\varepsilon, t)$ such that $Z_{t'}(t') > 0$.

Since $Z_{t'}(\tau)$ and $f_{C,C'}(\tau)$ are continuous in τ, and $Z_{t'}(t) = 0$, we have that for all $\varepsilon > 0$, there must exist a nontrivial interval $[t', t''] \in [t-\varepsilon, t]$ where $Z_{t'}(\tau)$ is strictly decreasing, and $f_{C,C'}(\tau)$ do not change sign, for all $(C, C') \in 2^N \times 2^N$. Then $0 < Z_{t'}(t') - Z_{t'}(t'') = \sum_{j\in N}\left[f_j(t')\int_{t'}^{t''} g_j(s)ds\right]$.

In the rest of the proof, we will show that $\sum_{j\in N}\left[f_j(t')\int_{t'}^{t''} g_j(s)ds\right] \leq 0$, which gives us the desired contradiction.

Let $O : 2^N\backslash\varnothing \to \{1, 2, ..., 2^n - 1\}$. We will use this ordering of coalitions to define shifts of proposals from a coalition to another in an intuitive (but cumbersome) way. Define $g_{iC}(\tau) = p_{iC}^A(\tau) - p_{iC}^B(\tau)$. Let the sequences (ordered according to O) $A_i^0(t) = \{C \in 2^N\backslash\varnothing | g_{iC}(\tau) > 0\}$ and $B_i^0(t) = \{C \in 2^N\backslash\varnothing | g_{iC}(\tau) < 0\}$. Let $g_{iC}^0(\tau) = |g_{iC}(\tau)|$. Denote the k^{th} element of $A_i^0(t)$ as $A_i^0(t)_k$, and similarly for the other sequence. If $g_{iA_i^k(t)_1}^k(\tau) > g_{iB_i^k(t)_1}^k(\tau)$, let $A_i^{k+1}(t) = A_i^k(t)$, and $B_i^{k+1}(t)$ be such that $B_i^{k+1}(t)_m = B_i^k(t)_{m+1}$ (so $B_i^{k+1}(t)$ is one element shorter than $B_i^k(t)$; this will be referred to as "shifting $B_i^k(t)$"); if $g_{iA_i^k(t)_1}^k(\tau) < g_{iB_i^k(t)_1}^k(\tau)$, shift $A_i^k(t)$, but leave $B_i^k(t)$ unchanged; if $g_{iA_i^k(t)_1}^k(\tau) = g_{iB_i^k(t)_1}^k(\tau)$, shift both sequences. Then define:

$$
g_{iC}^{k+1}(\tau) = \left\{\begin{matrix} g_{iC}^k(\tau) - g_{iB_i^k(t)_1}^k(\tau) \text{ if } C = A_i^k(t)_1 \text{ and } g_{iA_i^k(t)_1}^k(\tau) > g_{iB_i^k(t)_1}^k(\tau) \\ g_{iC}^k(\tau) - g_{iA_i^k(t)_1}^k(\tau) \text{ if } C = B_i^k(t)_1 \text{ and } g_{iA_i^k(t)_1}^k(\tau) < g_{iB_i^k(t)_1}^k(\tau) \\ g_{iC}^k(\tau) \text{ otherwise} \end{matrix}\right\}
$$

Now define:

$$
g_{iC,C'}(\tau) = \left\{\begin{matrix} \min\{g_{iC}^k(\tau), g_{iC'}^k(\tau)\} \text{ if } C = A_i^k(t)_1 \text{ and } C' = B_i^k(t)_1 \text{ for some } k \\ -\min\{g_{iC}^k(\tau), g_{iC'}^k(\tau)\} \text{ if } C = B_i^k(t)_1 \text{ and } C' = A_i^k(t)_1 \text{ for some } k \\ 0 \text{ otherwise} \end{matrix}\right\}
$$

Finally, let $g_{C,C'}(\tau) = \sum_{i\in N}\lambda_i g_{iC,C'}(\tau)$. Observe that $g_{C,C'}(\tau)$ has a simple interpretation: it measures the frequency of proposals gained by coalition C

from C' in equilibrium A relative to equilibrium B. It is easy to verify that $\sum_{C'\in 2^N} g_{C,C'}(\tau) = -\sum_{C'\in 2^N} g_{C',C}(\tau) \equiv g_C(\tau)$, and $g_i(\tau) = \sum_{C\ni i} g_C(\tau)$.

By optimality, it is clear that $f_{C,C'}(\tau)g_{C,C'}(\tau) \le 0$, for all $C, C' \in 2^N$ and $\tau < 0$. Since $f_{C,C'}(\tau)$ maintain their sign in $[t', t'']$, we have: $f_{C,C'}(t')\int_{t'}^{t''} g_{C,C'}(s)ds \le 0$. Now note that:

$$\begin{aligned}
\sum_{(C,C')\in 2^N\times 2^N} f_{C,C'}(t')\int_{t'}^{t''} g_{C,C'}(s)ds = & \sum_{(C,C')\in 2^N\times 2^N} f_C(t')\int_{t'}^{t''} g_{C,C'}(s)ds \\
& -\sum_{(C,C')\in 2^N\times 2^N} f_{C'}(t')\int_{t'}^{t''} g_{C,C'}(s)ds \\
= \sum_{C\in 2^N} f_C(t')\int_{t'}^{t''} g_C(s)ds + \sum_{C'\in 2^N} f_{C'}(t')\int_{t'}^{t''} g_{C'}(s)ds & \\
= 2\sum_{C\in 2^N}\left(\sum_{i\in C} f_i(t')\int_{t'}^{t''} g_C(s)ds\right) & \\
= 2\sum_{i\in N} f_i(t')\left(\sum_{C\ni i}\int_{t'}^{t''} g_C(s)ds\right) & \\
= 2\sum_{i\in N} f_i(t')\int_{t'}^{t''} g_i(s)ds &
\end{aligned}$$

Thus, $\sum_{i\in N} f_i(t')\int_{t'}^{t''} g_i(s)ds \le 0$, which completes the contradiction. QED

Proof of Theorem 4: The statement holds vacuously if $\mathcal{C}(V) = \varnothing$, so we assume $\mathcal{C}(V) \ne \varnothing$. Normalize payoffs with $V(N) = 1$. Let $x \in \mathcal{C}(V)$, and set $\lambda_i = x_i$. For any $T > 0$, consider continuation value functions $w_i(t) = \lambda_i(1 - e^t), \forall\, t \in [-T, 0]$, $i \in N$, and specify strategies as follows:

For every $i \in N$, if player i gets the chance to make an offer at $t \in [-T, 0]$, she approaches the grand coalition and offers exactly $w_j(t)$ to every $j \in N\backslash\{i\}$. If player i gets approached at t, then independently of who approached her and what coalition was approached, she accepts the offer if and only if she is offered at least $w_i(t)$.

We will show that the strategies specified above comprise a subgame perfect equilibrium, in which expected payoffs are given by $x_i(1 - e^t)$ for every $i \in N$. First, note that if no offer gets accepted at t, and afterwards everyone plays according to the prescribed strategies, then the expected continuation payoff of player j is:

$$\int_t^0 [\lambda_j e^{-\lambda(\tau-t)}(1 - \sum_{k\ne j}\lambda_k(1-e^\tau)) + \left(\sum_{k\ne j}\lambda_k\right) e^{-\lambda(\tau-t)}\lambda_j(1-e^\tau)]d\tau = w_j(t).$$

In particular, the expected payoff of player j at the beginning of the game is $w_j(-T) = x_j(1 - e^{-T})$.

Second, note that given other players' strategies, the best offer player i can give to the grand coalition is the one specified above, and since $1 - \sum_{j\ne i} w_j(t) = w_i(t) + e^t > w_i(t)$, it yields a higher payoff than giving an unacceptable offer.

Next, note that $w_j(t) \leq x_j \; \forall \; t \in [-T, 0]$ and $j \in N$. Since $x \in \mathcal{C}(V)$, this implies that $\sum_{j \in C} w_j \leq V(N) - V(N \backslash C)$ for any $C \subset N$. Given others' strategies, this means that there is no $C \subset N$ such that player i could give an acceptable offer to coalition $N \backslash C$ and get a strictly higher payoff than what she obtains when following the strategy prescribed above.

We conclude that no player can profitably deviate, given the above profile, at any point where it is her turn to make an offer.

If player $i \in N$ is approached by another player at t, then rejecting the offer results in continuation payoff $w_i(t)$, which means that it is optimal to reject the offer when it is not above $w_i(t)$, and it is optimal to accept the offer when it is not below $w_i(t)$. Hence, the strategy prescribed above is optimal for i.

Thus, there exists a SPNE such that expected payoffs are given by $x_i(1-e^T)$ for every $i \in N$.

Showing that any subgame perfect equilibrium yields the same expected payoffs as the one specified above is similar to showing uniqueness of subgame perfect equilibrium payoffs in the N-player group bargaining game.

Finally, note that as $T \to \infty$, the expected SPNE payoff of player i goes to x_i. QED

Proof of Claim 2: First, note that for any $i \in N$ and any $t \leq 0$, $w_i(t) < V(\{i\})$ implies that $V(C \cup \{i\}) - w_i(t) > V(C)$ for any $i \notin C$. This and Lemma 0 imply that at any time such that $w_i(t) < V(\{i\})$ in a Markov perfect equilibrium, any player $j \in N$ will include player i in the approached coalition at an arrival, and offer her exactly $w_i(t)$. Furthermore, note that if player i has the chance to make an offer at t, then she can guarantee a payoff of at least $V(\{i\})$ by approaching herself. This implies that $w_i(t)$ is bounded below by $\int_t^0 [\lambda_i e^{-\lambda(\tau - t)} V(\{i\}) + \sum_{j \neq i} \lambda_j e^{-\lambda(\tau - t)} w_i(\tau)] d\tau$, which implies that $w_i(t) \geq V(\{i\})(1 - e^{\lambda_i t})$ in every MPE. Therefore, if $T_1(\varepsilon) = \min_{i \in N} \frac{1}{\lambda_i} \ln \frac{\varepsilon}{V(\{i\})}$, then for any $t \leq T_1(\varepsilon)$ and $i \in N$, $w_i(t) \geq V(\{i\}) - \varepsilon$, for every $\varepsilon > 0$.

Assume now that for some $K \in \{1, ..., n-1\}$, there exists a finite $T_K(\varepsilon)$ for any $\varepsilon > 0$ such that for every $C \subset N$ with $|C| \leq K$, it holds that $\sum_{i \in C} w_i(t) \geq V(C) - \varepsilon$, $\forall \; t \leq T_K(\varepsilon)$. Below we show that this implies that for any $\varepsilon > 0$, there exists a finite $T_{K+1}(\varepsilon)$ such that for every $C \subset N$ with $|C| \leq K+1$, it holds that $\sum_{i \in C} w_i(t) \geq V(C) - \varepsilon, \forall \; t \leq T_{K+1}(\varepsilon)$. Fix any $\varepsilon > 0$ and any C with $|C| = K+1$. >From the induction assumption, $\sum_{i \in C'} w_i(t) \geq V(C') - \varepsilon/2$, $\forall \; t \leq T_K(\varepsilon/2)$ and $C' \subsetneqq C$. Consider now any $t \leq T_K(\varepsilon/2)$ and assume that $\sum_{i \in C} w_i(t) < V(C) - \varepsilon$. Suppose that there is $i \in N$ such that i does not approach everyone in C with probability 1 at t. Let D be such that there is a positive probability that D is approached at t by i, and $C \nsubseteq D$. Since $t \leq T_K(\varepsilon/2)$, $\sum_{i \in C \cap D} w_i(t) \geq V(C \cap D) - \varepsilon/2$. Then $\sum_{i \in C} w_i(t) < V(C) - \varepsilon$

implies $\sum\limits_{i \in C \setminus D} w_i(t) < V(C) - V(C \cap D) - \varepsilon/2$. Convexity of V then implies $\sum\limits_{i \in C \setminus D} w_i(t) < V(D \cup C) - V(D)$. By Lemma 0, i could strictly improve her payoff by approaching $D \cup C$ instead of D, a contradiction. Therefore, for any $C \subset N$ for which $|C| \leq K+1$, $\sum\limits_{i \in C} w_i(t) < V(C) - \varepsilon$ and $t \leq T_K(\varepsilon/2)$ imply that everyone in C is approached by every player at t with probability 1. Therefore, for $t \leq T_K(\varepsilon/2)$, $\sum\limits_{i \in C} w_i(t)$ is bounded below by $\min(V(C) - \varepsilon, k(t))$, where $k(t) = \int\limits_t^{T_K(\varepsilon/2)} \left[\left(\sum\limits_{i \in C} \lambda_i \right) e^{-\lambda(\tau - t)} V(C) + \left(\sum\limits_{j \notin C} \lambda_j \right) e^{-\lambda(\tau - t)} w_i(\tau) \right] d\tau$. Thus, there exists $T^C_{K+1}(\varepsilon)$ such that $\sum\limits_{i \in C} w_i(t) \geq V(C) - \varepsilon$, $\forall\ t \leq T^C_{K+1}(\varepsilon)$. Then for $T_{K+1}(\varepsilon) = \min \left\{ \min\limits_{C:|C|=K+1} T^C_{K+1}(\varepsilon), T_K(\varepsilon) \right\}$, for every $C \subset N$ with $|C| \leq K+1$, it holds that $\sum\limits_{i \in C} w_i(t) \geq V(C) - \varepsilon$, $\forall\ t \leq T_{K+1}(\varepsilon)$.

The claim follows by induction. QED

Proof of Theorem 5: Fix a convex V, and let $T^1, T^2, ... \longrightarrow \infty$ and $t^1, t^2, ... \longrightarrow -\infty$ such that $t^k \geq -T^k\ \forall\ k \in \mathbb{Z}_+$. For every $k \in \mathbb{Z}_+$, let $\underline{w}^k(t^k)$ be the continuation payoff vector at t^k of an MPE of the bargaining game with time horizon T^k, and assume $\underline{w}^k(t^k) \to \underline{w}^*$. By Claim 2, $\sum\limits_{i \in C} w_i^* \geq V(C)\ \forall$ $C \subset N$. In particular, $\sum\limits_{i \in N} w_i^* \geq V(N)$. Also, since $\sum\limits_{i \in N} w_i^k(t^k) \leq V(N)\ \forall$ $k \in \mathbb{Z}_+$, $\sum\limits_{i \in N} w_i^* \leq V(N)$. Combining the above yields $\underline{w}^* \in \mathcal{C}(V)$. QED

Proof of Theorem 6: First we establish the following lemma:

Lemma 3: In a MPE, suppose $\exists T^*$ s.t. $\forall t < T^*$, player i is proposed to with probability 1 when any player $j \in N \setminus \{i\}$ makes an offer, and that $\lambda_j > 0, \forall j$. Then if the core is nonempty and $\lim_{t \to -\infty} w_j(t) \equiv w_j$ exists for each $j \in N$, the limit allocation $(w_1, w_2, ..., w_n)$ is in the core.

Proof: $\forall t < T^*$, we know that $w_i(t) \geq \int_t^{T^*} e^{-\lambda(\tau - t)} [\lambda_i (V(N) - \sum_{j \neq i} w_j(\tau)) + (\lambda - \lambda_i) w_i(\tau)] d\tau$ because by Lemma 0, player i can always get $V(N) - \sum_{j \neq i} w_j(\tau)$ by proposing to the grand coalition, and $j \neq i$ will always offer $w_i(t)$ to player i. Let T^{**} be such that $|w_j(t) - \lim_{t \to -\infty} w_j(t)| < \frac{\varepsilon \lambda_i}{2 \lambda n}, \forall t < T^{**}$ and $j \in N$, and let $\widetilde{T} = \min\{T^*, T^{**}\}$. Then $\forall t < \widetilde{T}$, we have $w_i(t) \geq \int_t^{\widetilde{T}} e^{-\lambda(\tau - t)} [\lambda_i (V(N) - \sum_{j \neq i} w_j - \frac{\varepsilon \lambda_i (n-1)}{2 \lambda n}) + (\lambda - \lambda_i)(w_i(t) - \frac{\varepsilon \lambda_i}{\lambda n})] d\tau = \frac{1}{\lambda}(1 - e^{-\lambda(T - t)}) [\lambda_i (V(N) - \sum_{j \neq i} w_j - \frac{\varepsilon \lambda_i (n-1)}{2 \lambda n}) + (\lambda - \lambda_i)(w_i(t) - \frac{\varepsilon \lambda_i}{\lambda n})]$. This implies $\frac{\lambda - (1 - e^{-\lambda(\tau - t)})(\lambda - \lambda_i)}{\lambda_i (1 - e^{-\lambda(\tau - t)})} w_i(t) > V(N) - \sum_{j \neq i} w_j - \varepsilon$. Taking the limit as $t \to -\infty$ gives $w_i \geq V(N) - \sum_{j \neq i} w_j - \varepsilon$, or $\sum_{j \in N} w_j \geq V(N) - \varepsilon$. Letting $\varepsilon \to 0$ yields $\sum_{j \in N} w_j \geq V(N)$. Of course, we also know that $\sum_{j \in N} w_j \leq V(N)$, so $\sum_{j \in N} w_j = V(N)$.

Let $\{D_i\}_{i=1}^m$ be the set of all coalitions satisfying $V(D_i) = V(N)$. Let $D = \cap_{i=1}^m D_i$. Thus if $C \not\supseteq D$, $V(C) < V(N)$. Let $p_C(t)$ be the probability that if someone makes an offer at time t, it is made to coalition C, and let $p_j(t)$ be the probability that j is part of a coalition receiving an offer. We have $V(N) = \sum_{j \in N} w_j = \lim_{t \to -\infty} \int_t^0 e^{-\lambda(\tau - t)} \left(\sum_{C \subseteq N} p_C(\tau) V(C) \right) d\tau = \sum_{C \subseteq N} V(C) \left(\lim_{t \to -\infty} \int_t^0 e^{-\lambda(\tau - t)} p_C(\tau) d\tau \right)$. It follows that:

$$\sum_{i=1}^m \left(\lim_{t \to -\infty} \int_t^0 e^{-\lambda(\tau - t)} p_{D_i}(\tau) d\tau \right) = 1$$

so $\lim_{t \to -\infty} \int_t^0 e^{-\lambda(\tau - t)} p_j(\tau) d\tau = 1, \forall j \in D$.

Suppose the limit allocation is not in the core. Then $\exists C \subsetneqq N$ such that $\sum_{j \in C} w_j < V(C)$. Thus $\exists T^N$ such that $\forall t < T^N$, each member $k \in C$ receives at least $w_k + \varepsilon$ when she proposes because she could propose to C. This yields a contradiction when $C \cap D \neq \varnothing$ by an argument similar to that in the first paragraph of the proof, using the fact that a player $j \in C \cap D$ will have $\lim_{t \to -\infty} \int_t^0 e^{-\lambda(\tau - t)} p_j(\tau) d\tau = 1$ (her limit payoff would have to exceed w_j). If $C \cap D = \varnothing$, then by superadditivity, since $V(D) = V(N) \geq V(C \cup D)$, it follows that $V(C) = 0$, so it is impossible that $\sum_{j \in C} w_j < V(C)$. QED

Proof of theorem: Let $f(C) \equiv V(C) - \sum_{j \in C} w_j$, and $R = \max_{C \subseteq N} f(C)$. If $R \leq 0$, then we are done, so assume $R > 0$. Let $S = \arg\max_{C \subseteq N} f(C) = \{S_i\}_{i=1}^m$. Note that $S_i \cap S_j \neq \varnothing, \forall 1 \leq i \neq j \leq m$, since otherwise $f(S_i \cup S_j) \geq f(S_i) + f(S_j) > f(S_i), f(S_j)$. Also note that $\exists T^*$ s.t. $\forall t < -T^*$, if j proposes to a coalition C with positive probability, then $C \cap S_i \neq \varnothing, \forall 1 \leq i \leq m$, since otherwise j would do strictly better by proposing to $C \cup S_i$.

Clearly, if $|S_i| = 1$ for some i, then we are done by Lemma 3, as there is a time before which everyone proposes to the player in S_i with probability 1.

Now suppose $|S_1| = 3$. If $|S| = 1, 2$ or 3, then it is easy to see that we are done again by Lemma 3. Thus we need to study case A: $S = \{\{1,2,3\}, \{1,2\}, \{2,3\}, \{3,1\}\}$.

Finally suppose $|S_i| = 2, \forall\ i$. If $|S| = 2$, we are done. So we need to consider case B: $S = \{\{1,2\}, \{2,3\}, \{3,1\}\}$, and without loss of generality, case C: $S = \{\{1,2\}\}$.

Suppose that $\{1,2\} \in S$. This means that $V(\{1,2\}) - w_1 - w_2 \geq V(N) - w_1 - w_2 - w_3$, so $w_3 \geq V(N) - V(\{1,2\})$. If $\{2,3\} \in S$, $w_1 \geq V(N) - V(\{2,3\})$, so $w_1 + w_3 \geq 2V(N) - V(\{1,2\}) - V(\{2,3\}) \geq V(\{3,1\})$, with the latter inequality due to the nonemptyness of $\mathcal{C}(V)$. But in both cases A and B, $\{3,1\} \in S$, implying $R = V(\{3,1\}) - w_1 - w_3 \leq 0$, a contradiction.

Now consider case C. If $V(\{1,3\}) - w_1 - w_3 \neq V(\{2,3\}) - w_2 - w_3$, then $\exists T^*$ s.t. $\forall t < -T^*$, player 3 proposes to either player 1 or player 2 with probability 1, in which case we are done. Thus $V(\{1,3\}) - w_1 = V(\{2,3\}) - w_2 \geq V(N) - w_1 - w_2$. Then $w_1 \geq V(N) - V(\{2,3\})$ and $w_2 \geq V(N) - V(\{1,3\})$, so we get a contradiction as above.

Having exhausted all cases, we conclude that $(w_1, w_2, w_3) \in \mathcal{C}(V)$. QED

Proof of Claim 3: Claim 1 implies that at any t in MPE, any player j

proposes to the cheapest coalition of size K that includes her. Suppose $w_i(t) > w_{i'}(t)$. Then if any player $j \in N\backslash\{i, i'\}$ approaches i with positive probability, she must approach i' with probability 1. Moreover, if i' approaches i with positive probability, then i approaches i' with probability 1. Since $\lambda_i \leq \lambda_{i'}$, the probability of being included in a proposal at time t is greater for i' than for i.

Also, $w_i(t) > w_{i'}(t)$ implies that $v_{i'}(t) - w_{i'}(t) \geq v_i(t) - w_i(t)$, so $\lambda_{i'}(v_{i'}(t) - w_{i'}(t)) \geq \lambda_i(v_i(t) - w_i(t))$.

Combining the above two facts, we obtain that the left-hand derivative of $w_{i'}$ at t is weakly smaller (i.e. more negative or less positive) than the left-hand derivative of w_i whenever $w_i(t) > w_{i'}(t)$. Since $w_i(0) = w_{i'}(0)$, such t cannot exist. QED

Proof of Theorem 7: First we establish the following lemma:

Lemma 4: There exists a time $t' \leq 0$ such that $w_i(t') = w_{i'}(t')\ \forall\ i, i' \in \{1, ..., K\}$.

Proof: Note that if there is no t' as in the lemma, then at any time, player 1 is approached by every player with probability 1 at all times. Let $\widehat{w}_j(t) = \min\limits_{C \subset N\backslash\{j\}:\ |C| = K-1} \sum\limits_{i \in C} w_i(t) + w_j(t)$. Since player 1 always has the lowest continuation value by Claim 1, $\widehat{w}_1(t) = \min\limits_{C \subset N:\ |C| = K} \sum\limits_{i \in C} w_i(t) \leq \frac{K}{n}$. Thus we have $w_1'(t) = -\lambda_1(1 - \widehat{w}_1(t)) \leq -\lambda_1 \frac{n-K}{n}$, so that $w_1(t) \geq -\lambda_1 \frac{n-K}{n} t$ for all $t < 0$, which is clearly impossible for t far enough from 0. Therefore, there must be a time t' as stated in the lemma. QED

Proof of Theorem: Let $t^1 = \max\limits_{t<0}\{t | w_i(t) = w_{i'}(t)\ \forall\ i, i' \in \{1, ..., K\}\}$. By Lemma 4 and the continuity of continuation values, t^1 is well-defined. Define j^1 such that $w_i(t^1) = w_{i'}(t^1)\ \forall\ i, i' \in \{1, ..., j^1\}$, and $w_{j^1}(t^1) < w_{j^1+1}(t^1)$ (if $j^1 < N$). Define $w^*(t^1) \equiv w_1(t^1) = ... = w_{j^1}(t^1)$. Note that since $j^1 \geq K$, every player at t^1 only approaches players in $\{1, ..., j^1\}$, approaches exactly $K - 1$ of them, and offers $w^*(t^1)$ to each of them. Consider now the auxiliary continuation payoff path $w^a = (w_1^a, ..., w_{j^1}^a)$, which is the unique solution of the differential equation $\frac{\partial w_i^a(t)}{\partial t} = -\frac{1}{j^1}[w^a(K-1) \sum\limits_{i > j^1} \lambda_i + \sum\limits_{i \leq j^1} \lambda_i] + \lambda w^a$ with terminal condition $w_i^a(0) = 0\ \forall\ i \in \{1, ..., j^1\}$. It corresponds to a payoff path resulting from all players at any time approaching players in $\{1, ..., j^1\}$ in a way that keeps the payoff of all players in $\{1, ..., j^1\}$ the same. This path is not necessarily feasible in the sense that in order to keep payoffs within $\{1, ..., j^1\}$ the same, player 1 may need to be approached more frequently than if everyone approached her with probability 1, in order to keep her on par with player j^1. At the same time, along this path, the continuation payoff of j^1 might need to grow more slowly than if all players approached her with 0 probability. That is, along the auxiliary path we ignore the constraint that at any point of time, the probability with which a player approaches another one has to be between 0 and 1. It is easy to see that $w_1^a(t) = ... = w_{j^1}^a(t)$ is continuous and strictly monotonically decreasing in t, and it converges to a limit higher than $w^*(t^1)$ as $t \to -\infty$. Moreover, as t decreases, the frequency at which player 1 (resp.

player j^1) needs to be approached in order to keep her at the same continuation payoff as other players in $\{1, ..., j^1\}$ is decreasing (resp. increasing), since the payoff of any proposer at t is $1 - (K-1)w_1^a(t)$, which is strictly increasing in t (implying that players who propose more frequently have a greater advantage later in the game).

Note that for all $\varepsilon > 0$, there must be some interval within $(t_1, t_1 + \varepsilon)$ along which $w_{j^1}(t) - w_1(t)$ increases in MPE. All the proposal probabilities are of course feasible in MPE, so at t_1, there must exist a set of feasible proposal probabilities keeping w_{j^1} and w_1 (and thus also $w_2, ..., w_{j^1-1}$) the same.

Consider now t^a such that $w_1^a(t^a) = ... = w_{j^1}^a(t^a) = w^*(t^1)$. Since $w_1(t^1) = ... = w_{j^1}(t^1) = w^*(t^1)$, along the auxiliary path at t^a (and after) all players have to be approached with feasible probabilities by every player. As noted earlier, maintaining $w_1^a = ... = w_{j^1}^a$ becomes progressively easier as we move back in time, so the auxiliary path employs feasible probabilities to the left of t^a. This means that there is an MPE of the original game for which there is an interval on the left of t^1 such that in this interval, all players propose only to players in $\{1, ..., j^1\}$ in a way that keeps these players' continuation payoffs equal throughout the interval. Moreover, over this interval continuation values $w_1(t) = ... = w_{j^1}(t)$ are strictly decreasing. By Theorem ... all MPE have the same continuation value functions over the interval.

Suppose first that $w_1(t) = ... = w_{j^1}(t)$ is strictly below $w_{j^1+1}(t)$ $\forall$ $t < t^1$. This means that players $\{j^1 + 1, ..., N\}$ are never approached in MPE. Since $w_1(t) = ... = w_{j^1}(t)$ is monotonic in t and bounded within the interval $[0, 1]$, it converges as $t \to -\infty$. Since at any $t < t^1$ a proposer has to offer $(K-1)w_1(t)$ to other players, the fact that players in $\{j^1 + 1, ..., n\}$ are never approached by any player and that $w_1(t)$ converges as $t \to -\infty$ together imply that $w_i(t)$ converges for all $i \in \{j^1 + 1, ..., n\}$, too. Note that if there is an arrival at any time t, independently of the proposal the value of the approached coalition is 1, and the proposal is accepted by probability 1. Hence, as $t \to -\infty$, $\sum\limits_{i \in N} w_i \to 1$. The above results imply that the limit of continuation values as $t \to -\infty$ is equal to $\underline{x}^{j^1}$, as defined in the procedure above. The definition of j^* together with Claim 1 then imply $j^1 \geq j^*$. It cannot be that $j^1 > j^*$ and $\underline{x}^{j^1} \neq \underline{x}^{j^*}$, since along an auxiliary path where all players at all times approach only players in $\{1, ..., j^*\}$ in a way that keeps the latter players' payoffs equal, the payoff of any player $i \in \{1, ..., j^*\}$ has to converge to a limit that is weakly greater than her limit MPE payoff $\underline{x}_i^{j^1}$ (since they are being proposed to at least as often as in the MPE at all times), and by definition $\underline{x}_i^{j^1} \geq \underline{x}_i^{j^*}$.

Suppose next that at some time $t < t^1$, $w_1(t) = ... = w_{j^1}(t) = w_{j^1+1}(t)$. Let $t^2 = \max\limits_{t<t^1}\{t | w_{j^1}(t) = w_{j^1+1}(t)\}$ and let j^2 such that $j^2 = \max\limits_{i \in N}\{i | w_{j^1}(t) = w_i(t)\}$. Analogous arguments as above establish that there is an interval on the left of t^2 on which $w_1 = ... = w_{j^2}$, and $w_1(t) = ... = w_{j^2}(t)$ is strictly decreasing in t.

Since n is finite, continuing the same argument establishes that there are $t^k < ... < t^1 \leq 0$ (for some $k \geq 1$) and $j^k > ... > j^1 \geq K$ such that on (∞, t^k), $w_1 = ... = w_{j^k}$, $w_1(t) = ... = w_{j^k}(t)$ is strictly decreasing in t, and that $w_i \to \underline{x}_i^*$

as $t \to -\infty$. QED

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