

On the Fundamental Theorems of General Equilibrium[†]

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0. *Introduction*

The three fundamental theorems of general equilibrium theory are the propositions that, under appropriate hypotheses, (i) a competitive equilibrium exists; (ii) a competitive equilibrium is Pareto efficient; and (iii) a Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments. Of these theorems, assertion (ii) (often called the First Welfare Theorem) is mathematically virtually trivial, whereas the existence and decentralization results are usually considered “deeper.”

In this paper, we will provide a simple generalization of the existence theorem for economies where Walras’ Law (which asserts that the value of excess demand is zero) need not be satisfied out of equilibrium. Assertion (iii) (the Second Welfare Theorem) is an almost immediate corollary of this generalization. Our approach makes it clear that, given the existence of equilibrium, the first and second welfare theorems are equally “trivial”; indeed, we show that they can be proved in very similar ways.

We begin in Section 1 by establishing equilibrium existence for a “generalized competitive” mechanism. In Section 2 we apply this result to a “fixed allocation” mechanism. We take up the welfare theorems in Section 3, and Section 4 summarizes our findings.

1. *Generalized Competitive Mechanisms*

Let the basic data of the economy be given by the specification of preferences $\{\succsim^h\}$, endowments $\{\omega^h\}$, and production sets $\{Y^f\}$, where consumers are indexed by $h = 1, \dots, H$, and firms by $f = 1, \dots, F$. For all h , consumer h ’s preference

ordering \succsim^h is defined¹ over \mathfrak{R}_+^ℓ , and his endowment ω^h belongs to \mathfrak{R}_+^ℓ , where ℓ is the number of commodities (the assumption that the consumption space is the positive orthant is more restrictive than necessary). For all f , firm f 's production set Y^f is a subset of \mathfrak{R}^ℓ .

A *generalized competitive mechanism* (GCM) is a rule that, for each vector of prices p (in the unit simplex) and each specification $\{y^f\}$ of production plans by firms (where $y^f \in Y^f$ for all f), assigns to each consumer h an income $I^h(p, \{y^f\})$. One example of a GCM is, of course, the ordinary competitive mechanism, in which consumer h is assigned income $p \cdot \omega^h + \sum_f \theta_f^h (p \cdot y^f)$, where θ_f^h is consumer h 's share in firm f 's profit $p \cdot y^f$, and $\sum_h \theta_f^h = 1$ for all f . Another example is the mechanism that, given some fixed allocation² $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ and prices p , gives consumer h income $p \cdot \tilde{x}^h$.

An *equilibrium* of a GCM is a price vector p and an allocation $\{\{x^h\}, \{y^f\}\}$ such that (i) for each h , x^h is preference-maximizing in \mathfrak{R}_+^ℓ , subject to the constraint $p \cdot x^h \leq I^h(p, \{y^f\})$; (ii) for each f , y^f is profit-maximizing in Y^f given prices p ; and (iii) $\sum_h x^h = \sum_f y^f + \sum_h \omega^h$.

¹ As usual, " $x \succsim^h y$ " means " x is weakly preferred to y by consumer h ."

² An *allocation* is a specification $\{x^h\}$ of consumers' consumption bundles together with a specification $\{y^f\}$ of firms' production plans. The allocation is *feasible* if $x^h \in \mathfrak{R}_+^\ell$ for all h , $y^f \in Y^f$ for all f , and $\sum_h x^h \leq \sum_f y^f + \sum_h \omega^h$. The allocation is *Pareto efficient* if it is feasible if there is no other feasible allocation $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ such that $\tilde{x}^h \succ^h x^h$ for h , with strict preference for some h .

Notice that, in the first example above, the value of excess demand,

$$p \cdot \left(\sum_h (x^h - \omega^h) - \sum_f y^f \right),$$

is zero, regardless of whether $(p, \{\{x^h\}, \{y^f\}\})$ constitutes an equilibrium, so long as consumers exhaust their income (i.e., $p \cdot x^h = I^h$). In other words, Walras' Law holds for the ordinary competitive mechanism when preferences are strictly monotonic. It is clear, however, that Walras' Law generally fails for the second example. Although assumptions guaranteeing that Walras' Law holds are usually invoked to prove existence theorems, we shall now show that a rather weaker condition will suffice. This observation will enable us to establish an existence theorem for the second example.

Given a GCM, let $Z(\cdot)$ be the corresponding excess demand correspondence.

That is, for any prices p , $Z(p) = \left\{ z \mid z = \sum_h (x^h - \omega^h) - \sum_f y^f, \text{ and } x^h \text{ and } y^f \text{ are such} \right.$

that

$$x^h \text{ maximizes } z^h \text{ subject to } p \cdot x^h \leq I^h$$

and

$$y^f \text{ maximizes firm } f \text{'s profit in } Y^f \text{ given prices } p \}.$$

The following is our basic existence result:

Lemma (Existence): Given a GCM, suppose that $Z(\cdot)$ is well defined, upper hemi-continuous, and convex- and compact-valued. Suppose that if p is such that $p_i = 0$ for some i , then for all $z \in Z(p)$, $z_i > 0$. Finally, suppose that, for all p and all $z \in Z(p)$,

either (a) $z = 0$ or (b) there exist i and j such that $z_i > 0$ and $z_j \leq 0$. (Note that this last hypothesis constitutes a weakening of Walras' Law). Then, there exists an equilibrium.³

Proof: Because $Z(\cdot)$ is upper hemi-continuous and, for any p and i , $z_i > 0$ if $z \in Z(p)$ and $p_i = 0$, there exists $\delta > 0$ such that, for all p , all i and all $z \in Z(p)$, $z_i > 0$ if $p_i < \delta$.

Hence, upper hemi-continuity implies that there exists $K > 0$ such that for all p , all

$z \in Z(p)$, and all j , $z_j + Kp_j > 0$. Define the correspondence

$$H(p) = \left\{ h \mid h = \frac{z + Kp}{\sum_i (z_i + Kp_i)}, z \in Z(p) \right\}.$$

The correspondence $H(\cdot)$ takes the unit simplex to itself. It is upper hemi-continuous and convex-and compact-valued because $Z(\cdot)$ is. Therefore, by the Kakutani fixed point lemma, there exists \bar{p} such that $\bar{p} \in H(\bar{p})$. Choose $\bar{z} \in Z(\bar{p})$ such that

$$\bar{p} = \frac{\bar{z} + K\bar{p}}{\sum_i (z_i + K\bar{p}_i)}.$$

Then, for all j ,

$$(*) \quad \bar{p}_j = \frac{\bar{z}_j + K\bar{p}_j}{\sum_i (\bar{z}_i + K\bar{p}_i)}.$$

If $\bar{p}_j = 0$, for some j then, by hypothesis, $\bar{z}_j > 0$, which contradicts (*). Hence, $\bar{p}_j > 0$

for all j . Thus, if $\sum_i \bar{z}_i > 0$, then from (*), $\bar{z}_j > 0$ for all j , contradicting our weakened

Walras' Law. Similarly, if $\sum_i \bar{z}_i < 0$, then (*) implies that $\bar{z}_j < 0$ for all j , also a

³ Varian (1981) makes a related observation for excess demand functions.

contradiction. Therefore, $\sum_i \bar{z}_i = 0$, and so, from (*), $\bar{z}_j = 0$ for all j , i.e., \bar{p} and the allocation corresponding to \bar{z} constitute an equilibrium.

Q.E.D.

2. *Application of the Existence Lemma*

Theorem 1 (Existence of equilibrium at a Pareto efficient allocation): Let the allocation $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ be Pareto efficient, and suppose that, for all h , all components of \tilde{x}^h are strictly positive.⁴ Suppose that, for all h and p , $I^h(p, \{y^f\}) = p \cdot \tilde{x}^h$. Assume that preferences are convex, continuous, and strictly monotone, and that production sets are convex, closed, and bounded.⁵ Then an equilibrium exists.

Proof: Because the aggregate feasible set $\left(\sum_h \omega^h + \sum_f Y^f\right) \cap \mathfrak{R}_+^\ell$ is bounded, we can choose M big enough so that if each consumer h is limited to the truncated consumption set $\mathfrak{R}_+^\ell \cap [-M, M]^\ell$, then any allocation $\{\{x^h\}, \{y^f\}\}$ for which, for some h x^h is on the truncation boundary must be infeasible. Let $Z(\cdot)$ be the excess demand correspondence for the truncated consumption sets. It is well-defined, upper hemi-continuous, and compact-valued because agents' (i.e., consumers' and firms') objectives are continuous,

⁴ This hypothesis can be relaxed using standard methods, as in Debreu (1959).

⁵ The boundedness assumption can be dropped if we impose certain conditions on the aggregate production set $\sum_f Y^f$; see Debreu (1959).

their choice sets are closed and bounded, and each \tilde{x}^h is strictly positive. It is convex-valued because agents' objectives and their choice sets are convex.

Consider p such that $p_i = 0$ for some i . Choose $z \in Z(p)$ and write

$$z = \sum_h (x^h - \omega^h) - \sum_f y^f. \text{ Because preferences are strictly monotone, } x_i^h = M \text{ for all } h.$$

Hence, from our choice of M , $z_i > 0$.

Thus, to apply our Lemma, it remains to verify that $Z(\cdot)$ satisfies our weak

Walras' Law. Suppose, for given p and $\bar{z} \in Z(p)$, that $\bar{z} \neq 0$. Write

$$\bar{z} = \sum_h (\bar{x}^h - \omega^h) - \sum_f \bar{y}^f. \text{ By definition of the GCM, } p \cdot \bar{x}^h \leq p \cdot \tilde{x}^h \text{ for all } h. \text{ From profit}$$

maximization $p \cdot \bar{y}^f \geq p \cdot \tilde{y}^f$ for all f . Therefore, $p \cdot \bar{z} \leq p \cdot \left(\sum_h (\tilde{x}^h - \omega^h) - \sum_f \tilde{y}^f \right) = 0$,

where the last equation holds because $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is Pareto efficient and preferences are

strictly monotone. Thus, there exists good j such that $\bar{z}_j \leq 0$. If $\bar{z} \leq 0$, then the

allocation $\{\{\bar{x}^h\}, \{\bar{y}^f\}\}$ is feasible. Because consumer h can afford \tilde{x}^h , $\bar{x}^h \succeq^h \tilde{x}^h$. Thus

$\{\{\bar{x}^h\}, \{\bar{y}^f\}\}$ is Pareto efficient. But since $\bar{z} \neq 0$, there exists j such that $\bar{z}_j < 0$, i.e.,

there exists a Pareto efficient allocation with excess supply, a contradiction of strictly

monotone preferences. Thus, there exists i such that $\bar{z}_i > 0$ and so all the hypotheses of

the Lemma hold. Hence, there exists an equilibrium $\hat{p}, \{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ when consumers are

confined to their truncated consumption sets. Because the allocation is feasible, no \hat{x}^h

can lie on the truncation boundary. Suppose, for consumer h , there exists $\hat{\hat{x}}^h$ outside his

truncated consumption set such that he strictly prefers \hat{x}^h to \tilde{x}^h and $\hat{p} \cdot \hat{x} \leq \hat{p} \cdot \tilde{x}^h$. But then strict monotonicity and preference convexity imply that any strict convex combination of \hat{x}^h and \tilde{x}^h is strictly preferred to \tilde{x}^h , which implies that there exists a consumption bundle in the truncated consumption set strictly preferred to \tilde{x}^h , a contradiction. We conclude that, for all h , \hat{x}^h globally maximizes consumer h 's preferences subject to his budget constraint. That is, $\hat{p}, \{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ is a full-fledged equilibrium.

Q.E.D.

3. *The Welfare Theorems*

The first welfare theorem asserts that a competitive equilibrium is Pareto efficient.

The natural generalization to our framework is the following

Theorem 2: (First Welfare Theorem): If preferences are strictly monotone, then any equilibrium of a GCM is Pareto efficient.

Proof: Suppose that prices \hat{p} and allocation $\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ constitute an equilibrium of a GCM. Suppose, contrary to the Theorem, there exists a feasible allocation $\{\{x^h\}, \{y^f\}\}$ that Pareto dominates $\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$. By definition of equilibrium and from strictly monotone preferences, we have

$$(1) \quad \sum_h x^h = \sum_h \omega^h + \sum_f y^f \quad \text{and} \quad \sum_h \hat{x}^h = \sum_h \omega^h + \sum_f \hat{y}^f$$

and

$$(2) \quad \hat{p} \cdot x^h \geq \hat{p} \cdot \hat{x}^h \quad \text{for all } h,$$

with strict inequality for those consumers h who strictly prefer x^h to \hat{x}^h . Thus, summing (2) across consumers, we obtain

$$(3) \quad \sum_h \hat{p} \cdot x^h > \sum_h \hat{p} \cdot \hat{x}^h .$$

From profit maximization, we have $\hat{p} \cdot y^f \leq \hat{p} \cdot \hat{y}^f$ for all f , and so

$$(4) \quad \sum_f \hat{p} \cdot y^f \leq \sum_f \hat{p} \cdot \hat{y}^f .$$

Subtracting (4) from (3) and also subtracting endowments, we obtain

$$(5) \quad \hat{p} \cdot \left(\sum_h (\hat{x}^h - \omega^h) - \sum_f \hat{y}^f \right) < \hat{p} \cdot \left(\sum_h (x^h - \omega^h) - y^f \right),$$

which contradicts (1). Thus, the equilibrium allocation is Pareto efficient after all.

Q.E.D.

The proof of Theorem 2 will be recognized as identical to that usually given for the first welfare theorem. We have included it here primarily for comparison with the proof of the decentralization theorem:

Theorem 3: (Decentralization of a Pareto efficient allocation): Assume that preferences are strictly monotone. Suppose that allocation $\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ is Pareto efficient. Consider the GCM in which, given prices p , consumer h receives income $I^h = p \cdot \tilde{x}^h$. Then, if an equilibrium of this GCM exists, $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is an equilibrium allocation.

Proof: Suppose that \hat{p} is an equilibrium price vector and $\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ is a

corresponding equilibrium allocation for the GCM described. From Theorem 2,

$\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ is Pareto efficient. Therefore, because $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is also Pareto efficient

and consumer h can afford \tilde{x}^h , he must be indifferent between \tilde{x}^h and \hat{x}^h , and so, from

preference monotonicity, $\hat{p} \cdot \hat{x}^h = \hat{p} \cdot \tilde{x}^h$. Because firms are profit maximizing,

$\hat{p} \cdot \hat{y}^f \geq \hat{p} \cdot \tilde{y}^f$ for all f . If, for some firm f , the inequality is strict, then

$\hat{p} \cdot \sum_f \hat{y}^f > \hat{p} \cdot \sum_f \tilde{y}^f$, and so

$$(6) \quad \hat{p} \cdot \left(\sum_h \hat{x}^h - \sum_f \hat{y}^f \right) < \hat{p} \cdot \left(\sum_h \tilde{x}^h - \sum_f \tilde{y}^f \right).$$

But, since $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ and $\{\{\hat{x}^h\}, \{\hat{y}^f\}\}$ are both Pareto efficient and preferences are

strictly monotone, $\sum_h \tilde{x}^h - \sum_f \tilde{y}^f = \sum_h \hat{x}^h - \sum_f \hat{y}^f = \sum_h \omega^h$, contradicting (6). Thus,

$\hat{p} \cdot \hat{y}^f = \hat{p} \cdot \tilde{y}^f$ for all f . Since consumers and firms both are indifferent between the

hatted and the tildaed allocations, we conclude that $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is itself an equilibrium

allocation.

Q.E.D.

The proof of Theorem 3, like that of Theorem 2, is a simple revealed preference argument: given that existential problems can be ignored, agents stay at the pre-trade allocation unless they can make themselves better off. But if the pre-trade allocation is Pareto efficient, improvement is impossible. It is worth emphasizing that Theorem 3 requires *no convexity* assumptions. The theorem illustrates that convexity in decentralization theorems is needed only to show that equilibrium exists; it is not required to show that the equilibrium occurs at the Pareto efficient allocation. Indeed, it follows directly that if a Pareto efficient allocation cannot be supported as an equilibrium, then starting at this allocation, *no* equilibrium can exist.

Finally, if existence can be guaranteed without the use of convexity—as in large nonatomic economies—Theorem 3 ensures that Pareto efficient allocations can be decentralized.

The usual second welfare theorem follows immediately from Theorems 1 and 3:

Theorem 4: (Second Welfare Theorem): Suppose that preferences and production sets satisfy the hypotheses of Theorem 1. Then if $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is Pareto efficient and \tilde{x}^h is strictly positive for all h , there exist prices \hat{p} and balanced transfers $\{T^h\}$ (i.e., summing to zero) such that $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is an equilibrium allocation with respect to the mechanism that, for each p , gives consumer h the income $p \cdot \omega^h + \sum_f \theta_f^h p \cdot \tilde{y}^f + T^h$.

Proof: Under the hypotheses, Theorem 1 implies the existence of an equilibrium of the GCM in which, for each p , consumer h is assigned income $p \cdot \tilde{x}^h$. Theorem 3 then implies that $\{\{\tilde{x}^h\}, \{\tilde{y}^f\}\}$ is such an equilibrium together some price vector \hat{p} . To complete the argument, set $T^h = \hat{p} \cdot \tilde{x}^h - (\hat{p} \cdot \omega^h + \sum_f \theta_f^h \hat{p} \cdot \tilde{y}^f)$.

Q.E.D.

4. *Concluding Remarks*

This paper has reconsidered the principal theorems of general equilibrium theory.

We have attempted to show that

1) There are interesting general equilibrium models in which Walras' Law fails to hold out of equilibrium. However, these models may satisfy a weakened version of Walras' Law.

- 2) To prove the existence of an equilibrium, Walras' Law in its strong form can be replaced by a weakened version.
- 3) If existence is taken for granted, the second welfare theorem—and not just the first theorem—follows from a simple revealed preference analysis. The usual statement of the second welfare theorem involves an existential statement that is the reason behind its mathematical “difficulty.” Separation of the theorem into two parts—the existence part invoking the weakened Walras' Law—makes clear that the standard convexity conditions may be needed for existence but not for that part of the theorem that constitutes its real substance.

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