Retrading in Market Games *

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Abstract

When agents are not price takers, they typically cannot obtain an efficient reallocation of resources in one round of trade. This paper presents a non-cooperative model of imperfect competition where agents can retrade allocations, consistent with the Edgeworth's idea of recontracting. We show that there are allocations on the Pareto frontier that can be approximated arbitrarily closely when trade is myopic, i.e., when agents play a static Nash equilibrium at every round of retrading. We then show that the converging sequence of allocations generated by myopic retrading can also be supported along some retrade-proof Subgame Perfect Equilibrium path when traders anticipate future rounds of retrading.

Keywords: Market Games, Retrading, Myopic versus Far-sighted Behavior, Retrade Proofness.

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1 Introduction

In Edgeworth (1881), we find the following definition: "A final settlement is a settlement which cannot be varied by recontract within the field of competition". In this definition of a final settlement, the emphasis is on outcomes that are immune to recontracting. When individuals interact cooperatively, outcomes immune to recontracting are defined to lie in the core of an exchange economy (Debreu and Scarf (1963)). In contrast, our emphasis is on a non-cooperative formulation of recontracting in a general equilibrium model characterized by imperfect competition. When the outcomes of trade are inefficient, traders must be allowed to reopen markets. The allocations from the previous round of trade are the initial endowments in any new round of trade, while the rules of exchange remain constant. This generates an iterative process of retrading in which traders are able to reopen markets before they consume. We focus on the issue of whether retrading will allow traders to approximate allocations on the Pareto frontier.

The non-cooperative game of exchange we use is the Shapley-Shubik (1977) market game, where the rules of exchange allow all traders to influence prices by sending quantity signals. With a finite number of traders, Dubey and Rogawski (1990) have shown, under some mild regularity assumptions on preferences, that the Nash equilibrium outcomes of the market game are Pareto optimal if and only if the initial endowments of traders are Pareto optimal as well. This result allows us to study the incentives traders have to reopen markets before they consume their final allocations even in trading environments characterized by complete information.

In our model, traders can reopen markets a finite or infinite number of times before they consume. We think of the number of times traders can reopen markets as away of capturing the frequency with which they can retrade. At each round of trade all commodities are exchanged at trading posts except for the numeraire commodity, in which bids for all other commodities have to be made. For each non-numeraire commodity, traders can submit bids for the commodity and make offers of a quantity of the commodity, at the relevant trading post. In any new round of trade, the endowments of individuals are their final allocations from the previous round of trade. Using these endowments, individuals now make bids and offers in the trading posts and obtain allocations determined by the same price formation rule and allocation rule. The cost of reopening trading posts in any new round of trade is measured by a common discount factor for all traders.

We study the outcomes of myopic retrading as well as far-sighted retrading. A path of myopic retrading only requires that each period allocation be a Nash equilibrium outcome given the final allocation of the previous period. With far-sighted retrading, traders anticipate that there will be retrading in future time-periods.

With myopic retrading, we show that there are allocations on the Pareto frontier that can be approximated arbitrarily closely along some equilibrium path of retrading, as the discount factor is close enough to perfect patience and the number of allowable retrading periods is large enough. We construct an example in which there is a unique path of myopic retrading, which approximates the Pareto frontier. The same sequence of allocations that approximates a Pareto optimal allocation under myopic retrading can be sustained by a Subgame Perfect Equilibrium profile under far-sighted retrading. The approximation result with far sighted retrading is shown under two different information scenarios. In the first, each trader uses anonymous strategies where current bids and offers are conditioned only on the allocation obtained from the preceeding round of trade and on the aggergate bids and offers in the preceeding round of trade. With this restriction, deviations from the equilibrium path of play are punished by no trade. This is unsatisfactory as now (off the equilibrium path of play) traders may have an incentive to reopen trading posts. We, then show that when strategies are required to be retrade proof, both on and off the equilibrium path of play, if all traders are able to observe the identity of the deviating trader, the approximation result still holds.

However, we also show that, along any equilibrium path of finite retrading, with or without far-sighted behavior, no allocation on the Pareto frontier can be attained even when the cost of reopening trading posts is negligible.

We are also able to demonstrate that any Subgame Perfect Equilibrium that sustains a sequence of allocations that converges to some allocation on the Pareto frontier must have the property that it must look increasingly similar to the sequence of allocations generated by myopic retrading. Moreover, the set of allocations supported by Subgame Perfect Equilibrium profiles is shown to expand as the cost of reopening trading posts falls. This weak monotonicity result holds with finite as well as infinite horizon.

All the results just described are first proved under the simplifying assumption that traders can consume commodities (all tradeable) only after having stopped trading. However, we show that all results extend to the more general class of games where traders can decide to consume part of their current endowment at any time, while remaining on the market with the rest.

Our model of retrading can also be derived as reduced form of a model where the tradeable goods are actually assets. The goods that agents consume can be simply viewed as derived from the flow yields of the currently owned stock of assets. With this interpretation in mind, our model of retrading can be thought of as providing a rationale for resale markets where assets (more generally, durable goods) are traded. Moreover, in this case the issue of consumption becomes irrelevant, since the assets owned by each individual at any given time cannot be consumed, they can only be kept or traded.

Finally, although we focus on the possibility of eventually reaching an efficient allocation of resources (or assets) through retrading, we point out that a new type of market failure also arises in market games with retrading: there are "bad" Subgame Perfect Equilibria where traders delay trade only because the other traders do the same.

The rest of the paper is organized as follows. The next subsection compares our retrading model and our results with the related literature. The next section presents the economy and the basic models of non-cooperative trade that we study. Section 3 gives a simple example, in which the unique equilibrium path of retrading converges to the competitive equilibrium. Section 4 characterizes the equilibria of the benchmark retrading model with myopic players. Section 5 characterizes the Subgame Perfect Equilibria and retrade-proof equilibria of the market game with far-sighted retrading. Section 6 contains the extensions to the case where each trader can always choose between consuming and trading any subset of her own commodities and to the case where the tradeable goods are assets. Some more technical material is relegated to the appendix.

1.1 Related literature

The model we study as well as the results we obtain are different from the body of related work that studies dynamic noncooperative games of exchange.

Gale (1986a, 1986b, 1987) and McLennan and Sonnenschein (1991) look at a model where traders are repeatedly pair-wise matched and bargain over the trades that they make with each other. With a continuum of traders, complete information, and endogenous replacement, there is a stationary equilibrium which converges to the competitive equilibrium as the discount factor converges to one. The following are the main differences from our model: (1) We have a finite number of traders; (2) Gale's traders make direct transfers to each other in pairs, which are independent of the transfers made within other matched pairs at each round of trade (in contrast, in our model, trade is anonymous and each commodity is traded at a common price); (3) Once a pair agree to trade, they exit and are replaced by identical copies. In this sense, in contrast to our model, the same set of traders *never really agree to retrade* with each other along the equilibrium path of play. In that framework, retrading refers to the fact that any type has a positive probability of being repeatedly matched with any given other type of trader. Moreover, in order to obtain convergence to efficient allocations, Gale needs traders to be far-sighted. In contrast, we are able to obtain convergence when traders are myopic.

Dubey, Sahi and Shubik (1993) is closer to our paper, as they also study retrading in market games. However, they have a model with a continuum of agents.[1] Moreover, They do not allow for discounting of future consumption. They show that if equilibria in the one-shot market game fail to coincide with competitive equilibria due to the endowment constraints in the numeraire commodity binding for non-negligible subsets of traders, competitive equilibria can nevertheless be approx-

[1]A model of retrading with a continuum of agents corresponds to Edgeworth's notion of recontracting in a field of perfect competition. In contrast, our model studies recontracting in a field of imperfect competition.

imated arbitrarily when traders are allowed to reopen trading posts before they consume their final allocations. In our model, with a finite number of agents, the Nash equilibria of the market game are Pareto inefficient even when endowment constraints in the numeraire commodity don't bind for any individual trader.

The process of myopic retrading that we study in Section 4 shares with the iterative processes studied by Dreze and de la Vallee Poussin (1971), Malinvaud (1972) and Allen, Dutta and Polemarchakis (1999), the property that reallocations can be Pareto improving at each step.

Peck and Shell (1990)[2] study a model of a market game where traders can make arbitrarily large short sales, so that net trades are small relative to gross trades. Using this model they show that, at equilibrium, no individual action has a big effect on market prices, and therefore equilibrium allocations approximate competitive equilibrium allocations. Introducing the possibility of arbitrarily large short sales requires traders in their model to satisfy a budget constraint. They postulate some form of outside enforcement of the budget constraint via a bankruptcy rule. With these features, allowing for short sales has similar effects on imperfect competition as allowing for retrading (as they point out in footnote 6).

2 The Economy

We study trade in pure exchange economies with a finite set of commodities L (indexed by l), a finite set of individuals I (indexed by i). Each individual's consumption set is $\Re^L_+$, and his endowment is denoted by $w^i \in \Re^L_{++}$. The utility function is $u^i : \Re^L_+ \rightarrow \Re$. A pure exchange economy is $E = \{L, (u^i, w^i) : i \in I\}$. An allocation $x = (x^1, ..., x^I)$ such that $x^i \in \Re^L_+$ for all $i \in I$ is feasible if, in addition, $\sum_{i \in I} x^i = \sum_{i \in I} w^i$. A feasible allocation x is Pareto optimal if there is no other feasible allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all $i \in I$ with $u^i(y^i) > u^i(x^i)$ for some $i \in I$. Throughout the paper, we keep the total endowments of each commodity fixed. Let P denote the set of Pareto optimal allocations and let IR denote the set of individually rational allocations x such that $u^i(x) \geq u^i(w)$ for all $i \in I$. Let F de-

[2]For a related liquidity based approximation result see also Okuno and Schmeidler (1986).

note the set of feasible allocations, i.e., $F \equiv \{x \in \Re_{+}^{LI} : \sum_{i \in I} x_l^i = \sum_{i \in I} w_l^i, l = 1, ..., L\}$. Throughout the paper, we make the following assumption on the fundamentals of the exchange economy:

Assumption 1 *For each $i \in I$, u^i is strictly monotone, strictly-concave, element of C^r, $r \geq LI$, and the closure of the indifference curves through w^i are contained in $\Re_{++}^L$ and remain bounded away from the boundary of the consumption set.*

2.1 The one-shot market game

In this section we describe the Shapley-Shubik (1977) market game of non-cooperative exchange. Each trader makes bids and offers of commodities at trading posts where commodities are exchanged; all bids are denoted in some numeraire commodity, which we set to be commodity 1. Traders are allowed to make offers in all the other commodities $2, ..., L$, each one traded on one of $L - 1$ trading posts. A strategic action for a trader i is a vector $s^i = (b_2^i, ..., b_L^i, q_2^i, ..., q_L^i)$ where b_l^i denotes the bid for commodity l while q_l^i denotes the offer of commodity l, $l = 2, ..., L$. The corresponding set of strategic actions for each trader i is $S^i(w^i) = \{(b_2^i, ..., b_L^i, q_2^i, ..., q_L^i)$ such that $b_l^i \geq 0, \sum_{i \in I} b_l^i \leq w_1^i$, $0 \leq q_l^i \leq w_l^i$, $l = 2, ..., L\}$. All bids and offers have to be non-negative and the offer of a commodity made by a trader cannot *exceed* his endowment of that commodity. For each action profile $s = (s^1, ..., s^I)$, at the trading post for commodity l the aggregate bid is $B_l = \sum_{i \in I} b_l^i$ and the aggregate offer is $Q_l = \sum_{i \in I} q_l^i$. The corresponding price is $\pi_l(s) = \frac{B_l}{Q_l}$ if $B_l > 0$ and $Q_l > 0$; $\pi_l(s) = 0$ otherwise. For each trader i, the allocation rule determines commodity holdings as follows: If $\pi_l(s) \neq 0$, $x_1^i(s) = w_1^i - \sum_{l=2}^{L} b_l^i + \sum_{l=2}^{L} q_l^i \pi_l(s)$ and $x_l^i(s) = w_l^i - q_l^i + \frac{b_l^i}{\pi_l(s)}$, $l = 2, ..., L$. If $\pi_l = 0$, $x_l^i(s) = w_l^i$, for all $i \in I$. Let $v^i(s^i, s_{-i})$ be the payoff associated with s. A Nash equilibrium profile is s^* such that $v^i(s^{*i}, s_{-i}^*) \geq v^i(s^i, s_{-i}^*)$, for all $s^i \in S^i(w^i)$ and $i \in I$. A Nash equilibrium profile s^* such that $b_l^{i*} > 0, q_l^{i*} > 0$ for all $l = 2, ..., L$ and $i \in I$ is a non-trivial Nash equilibrium. A Nash equilibrium profile s^* such that $b_l^{i*} > 0$, $0 < \sum_{l=2}^{L} b_l^i < w_1^i$, $0 < q_l^{i*} < w_l^i$ for all $l = 2, ..., L$ and $i \in I$ is an *interior Nash equilibrium.* Let $N(w)$ denote the set of interior Nash equilibrium allocations of the market game.

In the one-shot market game with variable offers, observe that the trivial Nash equilibrium where $b_l^{*i} = q_l^{*i} = 0$ for all l and $i \in I$, always exists and yields the initial endowments as the final allocation. When $w \in P \cap \Re_{++}^{LI}$, there is also an interior Nash equilibrium at which individuals consume their initial endowments guaranteeing that $N(w) \cap P \neq \phi$, since w would be an element of such an intersection (see, for instance, Dubey and Rogawski (1990)). What happens when $w \notin P$? Consider the following three properties:

- **(P1)** *(Static inefficiency)* If $w \notin P$, then $N(w) \cap P = \phi$.
- **(P2)** *(Weak gains from trade)* If $w \notin P$, there exists $x \in N(w)$ such that $u^i(x^i) \geq u^i(w^i)$ for all $i \in I$, with $u^i(x^i) > u^i(w^i)$ for some $i \in I$.
- **(P3)** *(Strong gains from trade)* If $w \notin P$, there exists $x \in N(w)$ such that $u^i(x^i) > u^i(w^i)$ for all $i \in I$.

(P1) requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is no interior Nash equilibrium allocation that is also Pareto optimal. **(P2)** requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is nevertheless some interior Nash equilibrium allocation that makes at east one trader better-off relative to his endowments. **(P3)** requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is some interior Nash equilibrium allocation that makes all traders better-off relative to their endowments.

Although when we state results in later sections we directly assume that one or all of **(P1),(P2),(P3)** characterize $N(w)$ whenever $w \notin P$, it is worth pointing out that when preferences and endowments satisfy Assumption 1, Dubey and Rogawski (1990) show that **(P1)**, **(P2), (P3)** characterize $N(w)$ whenever $w \notin P$ (see also Peck, Shell and Spear (1992) for simillar results in a related market game). Further, Dubey and Rogawski (1990) also imply that if $w \in N(w)$, then $w \in P$.

We conclude this section with a result that characterizes the set of interior Nash equilibria. Consider two different endowment vectors w and w' with the same set of feasible allocations (i.e., the aggregate amount of each commodity is the same at w and w') but there is some individual $\bar{\imath}$ who is better off at w' relative to w. In the following proposition (see Shapley (1976) for a simillar argument in the case of two commodities and two individuals) we show that there exists an interior Nash equilibrium with endowments w' at which individual $\bar{\imath}$ is better off than at a different interior Nash equilibrium with endowments w. Remark that the result stated below (and proved in the appendix) is a direct proof that $N(w)$ is non-empty whenever $w \notin P$.[3] It also shows that $N(w)$ is characterized by **(P3)** whenever $w \notin P$.

Proposition 1 *Suppose preferences and endowments of individuals satisfy Assumption 1. Consider two endowment vectors* $w \gg 0$ *and* $w' \gg 0$, $w, w' \notin P$, $\sum_{i=1}^{I} w_l^i = \sum_{i=1}^{I} w_l'^i$, $l = 1, ..., L$ *such that* $u^{\bar{\imath}}(w^{\bar{\imath}}) < u^{\bar{\imath}}(w'^{\bar{\imath}})$ *for some* $\bar{\imath} \in I$. *Then, there exists* $x \in N(w)$ *and* $x' \in N(w')$ *such that* $u^{\bar{\imath}}(x^{\bar{\imath}}) < u^{\bar{\imath}}(x'^{\bar{\imath}})$ *for the same individual* $\bar{\imath}$.

Proof. See appendix. **QED.**

Proposition 1, together with Dubey and Rogawski (1990) result that if $w \in P \cap \Re_{++}^{LI}$ then $w \in N(w)$, guarantees the non-emptiness of $N(w)$ for exchange economies that satisfy Assumption 1.

2.2 The market game with retrading

From the results discussed in the preceeding section, it follows that the gains from trade are never exhausted after a one-period exchange. Therefore there are always *incentives to retrade.* In this section we describe an exchange mechanism that takes into account these incentives. Trading posts can reopen over a sequence of finite or infinite periods, $t = 0, 1, ..., T$. At each t an action for trader i is a vector s_t^i. The corresponding set of strategic actions at t for each trader i is $S_t^i(x_{t-1}^i)$, since the endowments for the traders at time t are the allocations obtained from trading in the previous period. Start from $s_{-1}^i = (0, ..., 0)$ for all $i \in I$ and $x_{l,-1}^i = w_l^i$, for all $l = 1, ..., L$ and for all $i \in I$. For each strategic action profile s_t, in the trading post for commodity l, the aggregate bid is $B_{l,t}$, the aggregate offer is $Q_{l,t}$, with the corresponding price $\pi_{l,t}(s_t)$, defined as in the static game. For each trader i, the allocations $x_t^i(s_t)$ are also defined as before. Along a sequence of action

[3] Dubey and Shubik (1978) show, under weaker assumptions, that the set of non-trivial Nash equilibria is non-empty.

profiles $s = \{s_0, ..., s_t, ...\}$, we say that player i *stops trading* after period $\tilde{T}^i$ iff $b^i_{l,t'} = q^i_{l,t'} = 0$ for all $t' \geq \tilde{T}^i$, $l = 2, ..., L$ and $b^i_{l,t'} \neq 0$, $q^i_{l,t'} \neq 0$ for all $t' < \tilde{T}^i$, $l = 2, ..., L$. Even though traders can stop trading at different times, it is convenient not to complicate notation by explicitly keeping track of traders who drop out. We can do so without loss of generality as the bids and offers of a trader can be zero at any round of trade and hence a trader i who stops trading at some period $\tilde{T}^i$ can be counted as a market player who makes zero bids and offers in all periods including and subsequent to $\tilde{T}^i$.

In what follows we shall consider two models of retrading, labeled as **myopic** and **far-sighted**.

Case 1 (**Myopic retrading**): When retrading is *myopic*, at each new round of retrading traders behave in a very simple way: at each new round of retrading, they choose a vector of bids and offers that constitutes a static Nash equilibrium to the final allocation obtained from the previous round of trade. In the notation developed before, at each t, the strategy profile chosen, s_t, satisfies the condition that $x_t(s_t) \in N(x_{t-1})$. Traders consume when they stop trading.[4] As the utility function of each trader is continuous and the set of feasible allocations compact, we remark that even when $\tilde{T}^i = \infty$, the payoff to any player i remains well-defined. Myopic traders can be seen as traders who do not expect that trading posts can be reopened, so they play their best responses as if the current trading round were the last. Consistent with this, we will study myopic retrading without discounting, even though the results extend to the case where discounting occurs.

Case 2 (**Far-sighted retrading**): When retrading is *far-sighted*, all traders know that future play will, in general, be conditioned on the outcomes of the current round of trade. Here, as before, we assume that an individual trader consumes only when she has stopped trading. However, now we endow each trader i with a common discount factor δ. When T is finite, δ lies in $[0, 1]$. When $T = \infty$, δ lies in $[0, 1)$.[5] Trader i's payoff, once she has stopped trading in period $\tilde{T}^i$, is $\delta^{\tilde{T}^i} u^i(x^i_{\tilde{T}^i})$. A history of play at period t is $h_t = \{s_0, ..., s_{t-1}\}$. The corresponding set of histories

[4] Note that the assumption that each trader consumes when he stops trading is only made for simplicity, and it is not crucial for the results, as discussed in Section 6.

[5] We interpret δ as a measure of the cost of reopening trading posts in any new round of trade.

is denoted by H_t. A pure strategy for trader i is a sequence $\sigma^i = \{\sigma_0^i, ..., \sigma_t^i, ...\}$ with $\sigma_t^i : H_t \rightarrow S_t^i$ for all t. Denote by $\sigma^i|h_t$ the restriction of σ^i to the subgame from period t after history h_t. A pure strategy profile $\sigma = (\sigma^1, ..., \sigma^I)$ is a Subgame Perfect Equilibrium (SPE henceforth) if, for every h_t, the restriction $\sigma^i|h_t$ for all traders $i \in I$ is a Nash equilibrium in the subgame from period t. Let $\tilde{X}(\delta, w, T)$ denote the set of SPE allocations of the market game with far-sighted retrading.

3 An example

In this section, we analyze retrading in an example. There are two commodities and two individuals, with quasi-linear utility functions $u^k(x) = x_1^k + f^k(x_2^k)$, $k = i, j$. We assume that $f^k(.)$ is strictly monotone, strictly concave, twice-continuously differentiable and satisfies the boundary condition that $\lim_{x_2 \rightarrow 0} \partial f^k(x_2) = \infty$ for $k = i, j$. Further, for simplicity, we choose the units in which commodities are measured so that $\sum_k w_2^k = 1$. We focus on retrading in the "sell-all" market game. The "sell-all" version of the Shapley-Shubik market game is obviously simpler than the variable offers version: At each time t where trader i is still active, his offer is assumed to equal $x_{2,t-1}^i$, which is the endowment of commodity 2 inherited from the trades of the previous period. Other than for this simplification, the strategies, aggregate variables, and the allocation rules are identical to the more general variable offers model described before.[6] In this case there is a unique Nash equilibrium with trade in the one-shot market game.[7] This means that finitely repeated trade would not add anything, whereas we now show that finite retrading leads the traders towards the competitive allocation *even* if they are myopic.

It will be convenient to refer to $w_2^i = \alpha_0^i \in (0, 1)$ as individual i's initial share of commodity 2 and α_t^i as individual i's share at the end of round $t-1$ of retrading.

[6] One additional difference would be in the precise definition of what it means to stop trading in the sell-all market game. We avoid the formal definitions since they are not relevant for this example, but the intuitive feature of any such definition is that traders must bid the exact amounts that give them back the endowments obtained with their last real trade.

[7] The existence of an equilibrium with trade follows from Dubey (1980), Remark 2. Further, using Remark 5 in Dubey and Rogowski (1990), it also follows that if $w \notin P$ then $N_{sellall}(w)$ satisfies **(P1)**, **(P2)**, **(P3)**.

For the moment, we simply assume that at any round of trade, all traders have enough of the numeraire commodity to ensure existence of an interior one-shot Nash equilibrium in any one round of trade[8].

Using the allocation rule, we obtain that at any round of retrading t, $t = 0, 1, ...,$ if the current profile of actions is $s_t = (b^j_{2,t}, b^i_{2,t})$, player $i's$ objective function at time t is

$$x^i_{1,t} - b^i_t + \alpha^i_t B_t + f^i\left(\frac{b^i_t}{B_t}\right)$$

where $B_t = b^i_{2,t} + b^j_{2,t}$. Using the fact that the ratio $\frac{b^i_t}{B_t} = \alpha^i_{t+1}$, if the current profile of actions s_t is an interior Nash equilibrium, we can rewrite the first-order conditions of traders to obtain the dynamical system that characterizes the evolution of the sequence of allocations generated by myopic retrading:

$$\frac{\partial f^i\left(\alpha^i_{t+1}\right)\left(1 - \alpha^i_{t+1}\right)}{\partial f^j\left(1 - \alpha^i_{t+1}\right)\alpha^i_{t+1}} = \frac{\left(1 - \alpha^i_t\right)}{\alpha^i_t}$$

Evidently, a stationary point of the preceeding map is an interior allocation on the Pareto frontier. Moreover, as both individuals have quasi-linear utility functions, the allocations of commodity 2 is uniquely determined at an interior Pareto optimum. Let $\bar{\alpha}^i$ denote individual $i's$ share of commodity 2 at the interior Pareto optimum. Suppose $\alpha^i_0 < \bar{\alpha}^i$. Then, as $f^k(.)$ is strictly concave, we must have that $\frac{\partial f^i\left(\alpha^i_0\right)}{\partial f^j\left(1-\alpha^i_0\right)} > 1$. Moreover, $\frac{\left(1-\alpha^i_0\right)}{\alpha^i_0} > \frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$. For all $t > 0$ such that $\frac{\partial f^i\left(\alpha^i_t\right)}{\partial f^j\left(1-\alpha^i_t\right)} > 1$, $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} < \frac{\left(1-\alpha^i_t\right)}{\alpha^i_t} < \frac{\left(1-\alpha^i_{t-1}\right)}{\alpha^i_{t-1}}$. If there exists $\hat{t}$ such that $\frac{\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\hat{t}}\right)} < 1$, $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} > \frac{\left(1-\alpha^i_{\hat{t}}\right)}{\alpha^i_{\hat{t}}}$ and as long as for all $t > \hat{t}$, $\frac{\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\hat{t}}\right)} < 1$, we must have that $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} > \frac{\left(1-\alpha^i_t\right)}{\alpha^i_t} > \frac{\left(1-\alpha^i_{t-1}\right)}{\alpha^i_{t-1}}$.

[8]By rewriting the first -order conditions for individual i at an interior Nash equilibrium, at period t we obtain the equation $b^i_t = g^i(\alpha^i_t, \alpha^i_{t+1})$ where $g^i(\alpha^i_t, \alpha^i_{t+1}) = \alpha^i_{t+1}\partial f^i(\alpha^i_{t+1})\frac{1-\alpha^i_{t+1}}{1-\alpha^i_t}$. We require that at period t, $g^i(\alpha^i_t, \alpha^i_{t+1}) \leq x^i_{1,t}$ for $i = 1, 2$. Starting from $t = 1$ and applying the above equality and inequality recursively, we obtain that at each $T' \leq T$, the required inequality is $w^i_1 \geq K^i_{T'}$ where $K^i_{T'} = \sum_{t=1}^{T'-1} \frac{(1-\alpha^i_{t+1})g^i(\alpha^i_t,\alpha^i_{t+1})}{\alpha^i_t g^j(\alpha^i_t,\alpha^i_{t+1})} + g^i(\alpha^i_{T'}, \alpha^i_{T'+1}) + T' - 1$, $i \neq j$, $i, j = 1, 2$. Without loss of generality, as $\lim_{x_2 \to 0} \partial f^k(x_2) = \infty$ for $k = i, j$, at each t, α^i_t lies in a compact set bounded away from 0 and 1. If follows that $\max_{T' \leq T} K^i_{T'}$ $i = 1, 2$, is finite and therefore, if $w^i_1 \geq \max_{T' \leq T} K^i_{T'}$, $i = 1, 2$, an interior Nash equilibrium exists at each t.

Suppose there exists $\tilde{t} > \hat{t}$ such that $\frac{\partial f^i\left(\alpha_{\tilde{t}}^i\right)}{\partial f^j\left(1-\alpha_{\tilde{t}}^i\right)} > 1$. Consider the ratio

$$\frac{\partial f^i\left(\alpha_{\tilde{t}}^i\right)...\partial f^i\left(\alpha_{\hat{t}}^i\right)}{\partial f^j\left(1-\alpha_{\tilde{t}}^i\right)...\partial f^j\left(1-\alpha_{\hat{t}}^i\right)}.$$

Notice that if the above ratio is equal to one we must be on the Pareto frontier. On the other hand the above ratio must be strictly greater than one as otherwise $\frac{1-\alpha_{\tilde{t}}^i}{\alpha_{\tilde{t}}^i} > \frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$ a contradiction. Therefore, $\frac{\partial f^i\left(\alpha_{\tilde{t}}^i\right)...\partial f^i\left(\alpha_{\hat{t}}^i\right)}{\partial f^j\left(1-\alpha_{\tilde{t}}^i\right)...\partial f^j\left(1-\alpha_{\hat{t}}^i\right)} > 1$, which implies that $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} < \frac{1-\alpha_{\tilde{t}}^i}{\alpha_{\tilde{t}}^i} < \frac{\left(1-\alpha_{\hat{t}-1}^i\right)}{\alpha_{\hat{t}-1}^i}$. By repeating the above argument from $\tilde{t}$, it follows that the ratio $\frac{\left(1-\alpha_t^i\right)}{\alpha_t^i}$ converges to $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$ and therefore, α_t^i to $\bar{\alpha}^i$. A symmetric argument establishes convergence when $\alpha_0^i > \bar{\alpha}^i$. An immediate consequence is that the sequence of allocations generated by myopic retrading must converge to the Pareto frontier. Moreover, note that from the equations that determine final allocations, we also obtain that individuals consumption of commodity 1 is identical to that at the competitive equilibrium.

What about far-sighted retrading? We show that the sequence of allocations generated by myopic retrading can be supported as SPE outcomes when T is very large but finite. Consider the sequence of allocations $y_1, ..., y_t, ...$, with $y_0 = w$, associated with myopic retrading. Note that $y_t = N_f(y_{t-1})$,[9] $t = 1, ...$, with the associated sequence of payoffs $u(y_1), ..., u(y_t), ...$ in utility space $\Re^2$. Consider the following strategy profile $\tilde{\sigma}$. For $t \leq \underline{T} + 1$, play $\tilde{s}_t$ such that $y_t^i = x^i(\tilde{s}_t)$ (and $\tilde{u}_t^i = u^i(x^i(\tilde{s}_t)))$ as long as $\tilde{h}_t = \{\tilde{s}_0, ..., \tilde{s}_{t-1}\}$; otherwise, if there has been a deviation, play s_t' such that $x(s_t') = N_f(x_{t-1})$, for all $t \leq T$. We need to show that $\tilde{\sigma}$ is a SPE. By construction, after any deviation, both players continue to choose bids according to one-shot Nash equilibria. As all sequences of allocations generated by one-shot Nash equilibria converge to the same allocation for both commodities, no player has an incentive to deviate when T is large.

[9]The subscript f refers to "fixed" offers, since in this example offers are not strategic.

4 Myopic retrading

We start the general analysis with myopic retrading. We show that, starting from an arbitrary configuration of initial endowments, traders are able to converge to some allocation in the Pareto set. Nevertheless, convergence cannot take place in a finite number of rounds of myopic retrading. We state the results only for the market game with variable offers described in Section 2.2, but we note that the same results also obtain for the "sell-all" market game (as one could guess from the previous section).

The following definition identifies the sequences of allocations that are consistent with myopic retrading.

Definition of Myopic Retrading: A sequence of allocations $\{x_t\}$, $t = 1, ...$ is generated by myopic retrading if and only if it satisfies the inclusion $x_t \in N(x_{t-1})$, for all $t \geq 1$.[10] An allocation y is stationary with myopic retrading if and only if $y = N(y)$.

Some notation is needed before proving the main result of this section. For any allocation y, let $u(y) = (u^1(y^1), ..., u^I(y^I))$. For any $K \subset \Re^{LI}$, let $u(K) = \{u(y) : y \in K\}$. Observe that $u(K) \subset \Re^I$, for all K. Let $\|.\|$ denote the Euclidian norm. Then, we define the distance between a vector y and a set K as $d(u(y), u(K)) \equiv \inf_{\hat{u} \in u(K)} \|u(y) - \hat{u}\|$.

Proposition 2 *Consider $w \in \Re^{LI}_{++}$, suppose that $N(w)$ satisfies **(P1)-(P2)** whenever $w \notin P$. Then, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...$, $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$.*

Proof. If $w \in P$, then $w = N(w)$ and we are done. Therefore assume that $w \notin P$. Consider the sequence of sets $N_1, ..., N_t, ...$, with $y_0 = w$, and $N_t = \{x : x \in N(y),\ for\ some\ y \in N_{t-1}\}$, $t = 1, ...$, with the associated sequence of sets $u(N_1), ..., u(N_t), ...$ in utility space $\Re^I$. By **(P2)** there exists a set of sequences, denoted by $\{\tilde{U}_t\}$, where each sequence in this set, $\tilde{u}_t$, $t = 0, 1, ...$, is such that

[10] x_0 is obviously the initial endowment.

$\tilde{u}_t \in u(N_t)$ for all t and $\tilde{u}_{t+1} > \tilde{u}_t$ at each t. Denote by $\{\tilde{Y}_t\}$ (and, respectively, by $\tilde{y}_t,\ t = 0, 1, ...$ its generic element) the associated set of sequences of allocations generated by myopic retrading and satisfying **P2** (all starting from $y_0 = w$). Note that for each $i \in I$, any sequence $\tilde{u}_t^i,\ t = 0, 1, ...,$ in $\{\tilde{U}_t^i\}$ is bounded above, as the utility of each individual is continuous and the set of feasible allocations is compact. Let $\bar{u}^i$ denote the supremum of the sequence $\tilde{u}_t^i$. As every increasing sequence converges to the supremum, it follows that the sequence $\tilde{u}_t$ converges to $\bar{u} = (\bar{u}^1, ..., \bar{u}^I)$, the component-wise supremum of $\tilde{u}_t = (\tilde{u}_t^1, ..., \tilde{u}_t^I),\ t = 0, 1, ...$ Moreover, it also follows that the associated sequence of allocations $\tilde{y}_t,\ t = 0, 1, ...$ converges to some allocation $\bar{y}$ such that $u(\bar{y}) = \bar{u}$. By considering every sequence of utilities in $\{\tilde{U}_t\}$ and the corresponding sequences in $\{\tilde{Y}_t\}$, we obtain a set of allocations $\bar{Y}_w$ which consists of the limit allocations of each of those sequences of allocations $\tilde{y}_t,\ t = 0, 1, ...$ If we show that there exists some sequence of allocations $\tilde{y}_t,\ t = 0, 1, ...$ in $\{\tilde{Y}_t\}$ that converges to $\bar{y} \in \bar{Y}_w$ such that $\bar{y}$ is stationary under myopic retrading, we are done: in fact, as Dubey and Rogawski (1990) have shown that if $w \in N(w)$ then $w \in P$, if $\bar{y} = N(\bar{y})$, $\bar{y} \in P$.

To this end, define the binary relation $\prec_w$ on F as follows: Given two feasible allocations x and y, $y \prec x$ if $x \in N_t$ and $y \in N_{t'}$, $t' < t$, for some N_t, $N_{t'}$ in the sequence of sets $N_1, ..., N_t, ...$ (with $y_0 = w$) and either (a) $x \in N(y)$ and $u^i(x) \geq u^i(y)$ for all $i \in I$ and $u^i(x) > u^i(y)$ for some $i \in I$ or (b) x is the limit of a sequence of allocations $\{x_t\}$, $t = 1, ...$ with $x_t \in N(x_{t-1})$, where $x_0 = y$. Remark that $\prec_w$ is transitive: if $y \prec_w x$ and $x \prec_w z$, then by (b) $y \prec_w z$. Therefore, $\prec_w$ is a partial order on F (page 13, Kelley (1955)). Consider any sequence of allocations $\tilde{y}_t,\ t = 0, 1, ...$ in $\{\tilde{Y}_t\}$ (i.e., in the set of sequences generated by myopic retrading satisfying **(P2)**). Note that either $\tilde{y}_t \prec_w \tilde{y}_{t'}$ or $\tilde{y}_{t'} \prec_w \tilde{y}_t$ for all $t \neq t'$; moreover, if $\tilde{y}_t \prec_w \tilde{y}_{t'}$ and $\tilde{y}_{t'} \prec_w \tilde{y}_t$, $\tilde{y}_t = \tilde{y}_{t'}$. Therefore, $\prec_w$ is a linear ordering (page 14, Kelley (1955)) and hence any sequence in $\{\tilde{Y}_t\}$ is a chain given $\prec_w$ (page 15, Kelley (1955)). By Kuratowski's lemma (page 33, Kelley (1955)), each chain in a partially ordered set is contained in a maximal chain. Moreover, any chain in F under $\prec_w$, and hence even the maximal chain, is a (weak) subset of some sequence of allocations in $\{\tilde{Y}_t\}$. We have already shown that every sequence in $\{\tilde{Y}_t\}$ converges to some allocation $\bar{y}$. Focusing on the maximal chain, $\tilde{y}_t \prec_w \bar{y}$ for all t, and hence $\bar{y}$ is an upper bound (page 13, Kelley (1955)) of the chain consisting of the sequence of allocations $\tilde{y}_t,\ t = 0, 1, ...$ under $\prec_w$. Given that F is partially ordered under $\prec_w$ and every linearly ordered subset has an upper bound, Zorn's lemma (page 33, Kelley (1955)) guarantees that F has a maximal element under $\prec_w$. By definition, a maximal element cannot preceed any other element of F. It follows that there is some sequence of allocations in $\{\tilde{Y}_t\}$ that contains a maximal point. From this, it follows that, as the maximal element of a sequence of allocations must be its upper bound and hence its limit allocation, there is some $\bar{y} \in \bar{Y}_w$ such that $\bar{y} = N(\bar{y})$, and therefore $\bar{y} \in P$. **QED.**

The above proposition demonstrates that starting from an arbitrary Pareto suboptimal vector of initial endowments, there is some sequence of allocations, generated by myopic retrading, that converges to an allocation that is stationary under myopic retrading, and hence to some allocation on the Pareto set. Note that each profile of actions along the myopic retrading sequence constitutes a static Nash equilibrium to the allocation inherited from the preceeding round of trade. By **(P2),** for every configuration of Pareto suboptimal endowments, there is a static Nash equilibrium at which allocations are such that every trader is at least as well-off and some trader(s) strictly better-off relative to their initial endowments. This implies that the sequence of utility profiles associated with the sequence of allocations generated by myopic retrading is an increasing sequence. But then, along each dimension, corresponding to a specific individual, this sequence of utilities must converge to its supremum, which in turn determines the limit of the sequence of allocations generated by myopic retrading. To complete the proof, it suffices to show that some limit allocation must be stationary under myopic retrading. To this end, we define a binary relation (on the set of feasible allocations) which is a suborder of pareto dominance. Each sequence of allocations generated by myopic retrading satisfying **(P2)** is a linearly ordered chain under this binary relation, and under this binary relation, any linearly ordered chain is a subset of some sequence of allocations in the set of all sequences of allocations generated by myopic retrading satisfying **(P2).** By Kuratowski's lemma, this set must contain a maximally ordered chain under this binary relation. As each sequence of allocations generated by myopic retrading has an upper bound (as the set of feasible allocations is compact), by Zorn's lemma some sequence of allocations in the set of all sequences of allocations generated by myopic retrading satisfying **(P2)** has a maximal element. If a sequence of allocations generated by myopic retrading satisfying **(P2)** has a maximal element, then the maximal element must be its upper bound and therefore its limit allocation. But then this limit allocation is stationary under myopic retrading and hence a Pareto optimal allocation.[11,12]

Evidently, the preceeding proposition goes through with the stronger requirement that $N(w)$ satisfies **(P3)** whenever $w \notin P$. The result (as well as the next one) can also be extended to $\delta < 1$. The reason for dealing only with $\delta = 1$ in the propositions of this section is that this is the case where the assumptions of myopic retrading make the most sense intuitively: in fact, a myopic player who does not discount the future can be assumed to believe he will consume right away. So myopia here means that traders can't predict that after trading they will change their mind, trading again instead of consuming.

Remark 1 At each stage of myopic retrading, the final allocation from the preceeding round of trade defines the distribution of endowments for a "new" economy. As the sequence of allocations converge to some allocation on the Pareto frontier, in the limit we obtain an economy with Pareto optimal endowments. As no trade is the only outcome at the competitive equilibrium of an economy with Pareto optimal endowments, in this sense the converging sequence of allocations associated with myopic retrading converges to competitive equilibria of the limit economy as well.

Remark 2 Proposition 2 holds even when we limit attention to sequences of Pareto undominated Nash equilibrium allocations in the paths of myopic retrading. In

[11]For a similar argument see Allen, Dutta and Polemarchakis (1999) (see observation 8 page 16) who study a convergent iterative process where at each step individuals exchange assets to share risks inherent in the multiplicity of competitive equilibria.

[12]Note that it is not possible to show that all limit allocations are stationary points under myopic retrading, because some sequences of allocations generated by myopic retrading satisfying (**P2**) may have a limit allocation where the Nash equilibrium correspondence fails to be continuous.

other words, consider the sequences of sets $\tilde{N}_1, ..., \tilde{N}_t, ...,$ where $\tilde{N}_t$ are the Pareto undominated allocations in N_t, $t = 1, ...$ Remark that as the set of interior Nash equilibrium allocations N_t is a closed subset of the set of feasible allocations, it is also compact, and therefore $\tilde{N}_t$ is non-empty.[13]

Although Proposition 2 demonstrates that traders will obtain allocations in the vicinity of the Pareto set, it still leaves open the question of whether traders are able to converge to an allocation *on* the Pareto frontier after a *finite* number of rounds of myopic retrading.

Proposition 3 *If $w \notin P$ and $N(w)$ satisfies **(P1)**, there is no $T < \infty$, and no sequence of allocations $\{y_t\}$, $t = 1, ...,$ $y_t \in N(y_{t-1})$, with $y_0 = w$ and $t = 0, ..., T$, such that $y_T \in P$.*

Proof. If $y_T \in P$, then $y_T = N(y_T)$. Moreover, as $w \notin P$, there must be some $T' < T$ such that the allocation obtained at $T' - 1$, $y_{T'-1}$, is not in P, while for all $t \geq T'$ $y_t \in P$. Then we must have that $y_{T'} \in N(y_{T'-1}) \cap P$, a contradiction. **QED.**

The intuition behind this result is simple. If trade concludes after some finite length of time, at some finite stage in the game it must be the case that while the traders' inherited allocation from the previous period is Pareto suboptimal, the final allocation they obtain after reopening trading posts is both (a) a Pareto optimal allocation, and (b) satisfies the inequalities for a Nash equilibrium allocation for the one-shot market game with the traders' inherited allocation as the endowment. But by **(P1)**, with Pareto suboptimal endowments no Nash equilibrium allocation of the one-shot market game can ever be Pareto optimal. This guarantees that no

[13]In market games like the one in Peck, Shell and Spear (1992), the presence of a budget constraint and the bankruptcy rules to insure feasibility imply that the equilibrium of the sell-all model is also an equilibrium of the more general variable offers model, and under some conditions (see for instance Goenka, Kelly and Spear (1998) such an equilibrium is the "best" Nash equilibrium in N_t. However, in the Shapley-Shubik model that we use it is not true that the equilibrium of the sell-all model is always an equilibrium of the more general variable offers model, and even when it is there is no reason why it should be better than any other equilibrium. Hence, since in the Shapley Shubik model there is no salient type of equilibrium that always belongs to $\tilde{N}_t$, the analysis does not benefit from limiting attention to the undominated Nash equilibria.

allocation *on* the Pareto set will be attained by traders after a *finite* number of rounds of myopic retrading. Without discounting, this implies that trading posts will *always* be reopened. This makes the assumption of myopic traders hard to swallow, but the next section shows that not only the results above extend to far-sighted behavior, but also that far-sighted behavior becomes indistinguishable from myopic behavior over the process of retrading.

5 Far-sighted retrading

Let us now allow traders to be far-sighted. The approximation result of Proposition 2 is confirmed, and we show that the set of SPE paths "converges" to the set of myopic retrading paths as traders keep retrading. Far-sighted retrading can lead the economy to Pareto improvements "faster" than myopic retrading, but doesn't have to. We will see that it is possible to have a new kind of market failure with far-sighted retrading: traders may delay trade along a SPE path merely because they expect other traders to do the same. We will also see that anonymous strategies, i.e., strategies that can only be functions of prices and aggregate demand for each commodity, are sufficient to guarantee the existence of SPE profiles such that the final allocation converges to the Pareto set. On the other hand, the appealing property of renegotiation proofness, appropriately defined for our context, can only be proved by allowing for individual punishment strategies.

The next proposition and its corollary provide a negative result, in line with Proposition 3.

Proposition 4 *If $w \notin P$, and $N(w)$ satisfies* ***(P1)****, $\tilde{X}(\delta, w, T) \cap P = \phi$, for all $\delta \in [0,1]$, and all $T < \infty$.*

Proof. Let $\bar{T} \leq T$ be the first period at which an allocation $x_{\bar{T}} \in P$ is obtained along some SPE path. Given that trade cannot take place after reaching the Pareto set, it must be the case that traders stop trading at some $\bar{T} \leq T$, i.e., $b^i_{l,t} = q^i_{l,t} = 0\ \forall i, \forall l, \forall t > \bar{T}$. Moreover, since $\bar{T}$ is the first period where p is reached, $x_{\bar{T}-1} \notin P$. As $x_{\bar{T}-1} \notin P$, by **(P1),** $N(x_{\bar{T}-1}) \cap P = \phi$. This is a contradiction, since, at the last round of trade, any SPE profile requires the final allocation to be in the set of Nash equilibrium allocations with respect to the inherited allocation. **QED.**

Corollary 1 *If $w \notin P$, and $N(w)$ satisfies **(P1)**, $\tilde{X}(\delta, w, \infty) \cap P = \phi$, for all $\delta \in [0, 1)$.*

Proof. When $\delta \in [0, 1)$, any trader gets a payoff of zero if he trades indefinitely. Therefore, along any SPE path, all traders will stop trading after some finite length of time, implying that there exists a $\bar{T} < \infty$ such that $b^i_{l,t} = q^i_{l,t} = 0$ for all $t \geq \bar{T}$, $l = 2, ..., L$. Trade stops before $\bar{T}' = \inf_T\{T : b^i_{l,t} = q^i_{l,t} = 0 \ for \ all \ t \geq T, \ l = 2, ..., L, \ i \in I\}$. Then, the proof immediately follows from Proposition 4. **QED.**

Even far-sighted traders cannot obtain allocations on the Pareto set. As trade always concludes after some finite length of time, at some finite stage in the game, it must be the case that both the traders' inherited allocation from the previous period and the final one are Pareto suboptimal, otherwise there would be a contradiction with **(P1)**. We now extend the approximation result obtained under myopic retrading to this world of far-sighted players.

Proposition 5 *If $N(w)$ satisfies **(P1)-(P3)** whenever $w \notin P \cap \Re^{LI}_{++}$, then, for every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}(\delta, w, T)$ such that $d(u(y), u(P)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$.*

Proof. Using Proposition 2, for δ close to 1 we obtain that whenever $w \notin P \cap \Re^{LI}_{++}$, if $N(w)$ satisfies **(P1)-(P3)** whenever $w \notin P$, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...$, $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$. Now construct the following strategy profile $\tilde{\sigma}$. For $t \leq \underline{T} + 1$, play $\tilde{s}_t$ such that $y^i_t = x^i(\tilde{s}_t)$ (and $\tilde{u}^i_t = u^i(x^i(\tilde{s}_t)))$ as long as $\tilde{h}_t = \{\tilde{s}_0, ..., \tilde{s}_{t-1}\}$; otherwise, if there has been a deviation, play $b^i_{\bar{t}} = q^i_{\bar{t}} = 0$, $i \in I$, for all $\bar{t} > t$. Finally, when $t > \underline{T} + 1$, play $b^i_t = q^i_t = 0$. To complete the proof, we need to show that $\tilde{\sigma}$ is a SPE. By construction, observe that no player has an incentive to deviate after $\underline{T} + 1$ or in any subgame following a deviation from the SPE path. It remains to check that no player has an incentive to deviate at any $t \leq \underline{T} + 1$. Indeed, consider player i who deviates at t choosing some action $s_t'^i$. As $b_{t'}^i = q_{t'}^i = 0$, $i \in I$, for all $t' > t$, denote i's maximum payoff from such a deviation by $\delta^{t+1} v^i(s_t'^i, \tilde{s}_{-i,t})$, where $x^i(s_t'^i, \tilde{s}_{-i,t})$ is the resulting allocation for i when i chooses $s_t'^i$ while all other players choose according to $\tilde{\sigma}$. On the other hand, his payoff from continuing to choose according to $\tilde{\sigma}$ is $\delta^{T^i(\tilde{\sigma})} u^i(y)$. As $y_t \in N(y_{t-1})$, we must have $v^i(s_t'^i, \tilde{s}_{-i,t})$ less than or equal to $u^i(y_t^i) < u^i(y^i_{T^i(\tilde{\sigma})})$. Consider $\delta^{T^i(\tilde{\sigma})} u^i(y^i_{T^i(\tilde{\sigma})}) - \delta^{t+1} v^i(s_t'^i, \tilde{s}_{-i,t}) = \delta^{t+1}[\delta^{T^i(\tilde{\sigma})-t-1} u^i(y^i_{T^i(\tilde{\sigma})}) - v^i(s_t'^i, \tilde{s}_{-i,t})]$. Let δ_i^{t+1} be such that $[\delta^{T^i(\tilde{\sigma})-t-1} u^i(y^i_{T^i(\tilde{\sigma})}) - v^i(s_t'^i, \tilde{s}_{-i,t})] = 0$. Set $\underline{\delta} = \inf_{i, 0 \leq t \leq T^i(\tilde{\sigma})-1} \delta_i^{t+1}$. It follows that for all $\delta \in [\underline{\delta}, 1)$, $\tilde{\sigma}$ is a SPE strategy profile. **QED.**

The proof of Proposition 5 shows that the sequence of allocations generated by myopic retrading that converges to the Pareto set can be supported as SPE outcome with far-sighted retrading when traders are sufficiently patient. To understand the main idea of the proof, consider first the case where $\delta = 1$. The strategy profile is constructed so that traders continue to choose the bids and offers that implement the sequence of allocations generated by myopic trade. If a trader deviates at some round of trade, in all subsequent rounds of trade all traders make null bids and offers at the trading post for each commodity, thus ensuring that no trade is the outcome. In the no trade phase, no individual trader has an incentive to deviate. This is because the bids and offers at any round of trade constitute a static Nash equilibrium to the final allocation from the previous round of trade, which implies no individual trader can gain by deviating, as a deviation will be followed by no trade in all subsequent rounds. Under **(P3)**, all traders *strictly* gain in utility along some sequence of allocations generated by myopic retrading. This implies that if traders are sufficiently patient, they will prefer to retrade over consuming their current allocation. Remark that the the above argument would also go through if at each t, along the equilibrium path of play, traders were required to choose strategies according to some undominated myopic Nash equilibrium.

The following corollary shows that Proposition 5 goes through even when strategies are conditioned only on a subset of the entire history of play namely, the aggregate bids and offers at each trading post in the preceeding round of trade. Denote by σ_M an *anonymous* strategy profile where each player i conditions his choice of bids and offers in period t, (b_t^i, q_t^i), only on $B_{l,t-1}$ and $Q_{l,t-1}$, $l = 2, ..., L$ (and therefore on the preceeding period's market price vector $\pi_{t-1}(s_{t-1})$), and on her own individual allocation $x^i_{t-1}(s_{t-1})$. Let $\tilde{X}_M(\delta, w, T)$ the set of SPE allocations for strategy profiles in Σ_M.

Corollary 2 *For every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}_M(\delta, w, T)$ such that $d(u(y), u(P \cap IR)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$.*

Proof. It is sufficient to observe that the sequence of allocations along the SPE path, $y_0, ..., y_t, ...$, used in the proof of Proposition 4 can also be supported by a strategy profile $\tilde{\sigma}_M$ specified as follows. For $t \leq \underline{T}$, play $\tilde{s}_t$ as long as $B_{t-1} = \tilde{B}_{t-1}$ and $Q_{t-1} = \tilde{Q}_{t-1}$; otherwise, if there has been a deviation, play $b^i_{\bar{t}} = q^i_{\bar{t}} = 0$, $i \in I$, for all $\bar{t} > t$. Finally, when $t > \underline{T}$, play $b^i_t = q^i_t = 0$. It is immediate that $\sigma_M \in \Sigma_M$ is also a SPE strategy profile. **QED.**

Although anonymous strategy profiles minimize the amount of information used to sustain a sequence of myopic Nash equilibria along the equilibrium path of a SPE strategy profile, Proposition 5 is problematic because deviations from the equilibrium path of play are punished by no trade, but if the allocation reached is suboptimal no trade is not retrade-proof. This leads us to define retrade-proof strategy profiles. Consider a sequence of feasible allocations $x_t, t = 0, 1, ...$ such that there exists some finite time period T such that $x_{t'} = x_T$ for all $t' > T$. For a given value of δ, we say the sequence of allocations is retrade proof if there is no $x \in N(x_T)$ such that $u^i(x_T) < \delta u^i(x)$ for all $i \in I$. Under the strategy profile σ, let $T(\sigma)$ be the set of all time periods at which trade stops under the strategy profile σ both on or off the equilibrium path of play. Under the strategy profile σ, let $X(\sigma)$ be the set of sequences of allocations generated by σ both on and off the equilibrium path of play. It follows that for each sequence of allocations x_t, $t = 0, 1, ...$ in $X(\sigma)$ there exists $\hat{T} \in T(\sigma)$ such that $x_{t'} = x_{\hat{T}}$ for all $t' > \hat{T}$. Then, σ is retrade-proof if every sequence of allocations in $X(\sigma)$ is retrade-proof. Let $\tilde{X}^R(\delta, w, T)$ denote the set of allocations supported by retrade proof SPE.

We now extend the approximation result obtained with anonymous strategy profiles to the case where we require retrade-proof SPE. Fix some $T > 0$.

Proposition 6 *If $N(w)$ satisfies **(P1)-(P3)** whenever $w \notin P \cap \Re^{LI}_{++}$, then, for every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}^R(\delta, w, T)$ such that $d(u(y), u(P)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$.*

Proof. See appendix. **QED.**

Suppose all traders are sufficiently patient and that there are at least three active traders on each side of a trading post. Consider a SPE strategy profile where deviations off the equilibrium path of play are not punished by no trade. What prevents a trader from deviating from the equilibrium path of play profile of bids and offers? By definition, the current profile of bids and offers is a one-shot Nash equilibrium. It follows that a trader will deviate if she anticipates that in the continuation subgame, there is an equilibrium which supports an allocation where she is better off relative to the allocation obtained as the limit of the sequence of allocations along the SPE path of play. To prevent such a contingency from occuring, the other traders punish the deviating trader by coordinating at each subsequent round of trade on a interior myopic Nash equlibrium at which the deviating trader is worse-off. Now, each individual trader strategies is conditioned on both the aggregate bids and offers and on the identity of the deviating trader. Proposition 1 guarantees the existence of such a strategy profile. When traders use such a strategy profile, following a deviation, bids and offers in the continuation subgame are conditioned on the identity of the deviator. Moreover, given δ, off the equilibrium path of play, by constructrion such a strategy profile allows traders to reopen trading posts as long as there are incentives to retrade.

Are there other SPE strategy profiles, which do not require players to implement allocations generated by myopic retrading but which, nevertheless, supports a sequence of allocations that approximate the Pareto frontier? While the answer is generally yes, the next proposition shows that any SPE strategy profile must, after some length of time, begin to look like a strategy profile that implements allocations generated by myopic retrading. Formally, for $T = \infty$, for any SPE strategy profile σ, let $y_1(\sigma), ..., y_t(\sigma), ..., y_{T_\sigma}(\sigma)$ (where $y_t = x(s_t(\sigma))$) denote the allocations generated along the equilibrium path of play associated with σ and T_σ denotes the last period *with trade* under σ. For $\tilde{T} < T_\sigma$, let $y_1(\sigma), ..., y_t(\sigma), ..., y_{\tilde{T}}(\sigma)$ denote a $\tilde{T}$ truncation of T_σ. We say that a SPE strategy profile σ approximates the Pareto frontier if for $\varepsilon > 0$ there exists $\tilde{T} \leq T_\sigma$ and $y_{\tilde{T}}(\sigma)$ such that $d(u(y_{\tilde{T}}(\sigma)), u(P)) < \varepsilon$. Moreover, for any $\epsilon > 0$, and $w \in R^{LI}_{++}$, let $N_\epsilon(w)$ denote the set of non-trivial $\epsilon-$Nash equilibrium allocations.[14]

Proposition 7 *For any SPE strategy profile σ that approximates the Pareto frontier, for every ϵ, there is a $\tilde{T} < T_\sigma - 1$ such that for each $t > \tilde{T}$, $y_t(\sigma) \in N_{\epsilon,t}(y_{t-1}(\sigma))$.*

Proof. Consider the sequence of strategies along the equilibrium path of play of σ, $s_1(\sigma), ..., s_t(\sigma), ...s_{T_\sigma}(\sigma)$. At any t such that $y_t(\sigma) \notin N_\epsilon(y_{t-1}(\sigma))$, there is some player i whose maximum payoff from a deviation, denoted by $v^i(s^i_t, s_{-i,t}(\sigma))$, where $x^i(s^i_t, \tilde{s}_{-i,t}(\sigma))$ is the resulting allocation for i when she chooses s^i_t while all other players choose according to σ, is such that $v^i(s^i_t, s_{-i,t}(\sigma)) - u^i(y^i_t(\sigma)) > 0$. By choosing $b^i_{t'} = q^i_{t'} = 0$, for all $t' > t$, player i can obtain a payoff $\delta^{t+1} v^i(s^i_t, s_{-i,t}(\sigma))$. As σ is SPE, it follows that $\delta^{t+1} v^i(s^i_t, s_{-i,t}(\sigma)) \leq \delta^{\tilde{T}+1} u^i(y^i_{t'}(\sigma))$ for all $\delta \in [\hat{\delta}, 1]$ and $t' > t$ and therefore, $u^i(y^i_{t'}(\sigma)) > v^i(s^i_t, s_{-i,t}(\sigma)) > u^i(y^i_t(\sigma))$. As σ approximates the Pareto frontier, for every $\epsilon > 0$, there exists $\tilde{T}$ such that if $t > \tilde{T}$ and $t' > t$, $u^i(y^i_{t'}(\sigma)) - u^i(y^i_t(\sigma)) < \epsilon$ and therefore, $v^i(s^i_t, s_{-i,t}(\sigma)) - u^i(y^i_t(\sigma)) < \epsilon$ which implies that $y_{t+1}(\sigma) \in N_{\epsilon_t}(y_t(\sigma))$. **QED.**

When at some t players do not choose bids and offers according to myopic retrading, along they obtain an allocation $y_t \notin N(y_{t-1})$ (where y_{t-1} is the allocation obtained from $t-1$). This implies that there must be some individual i who would have incentive to deviate from the SPE strategy profile at t and then choose $b^i_{t'} = q^i_{t'} = 0$ for all $t' > t$. Therefore, if $\{y_t : t \geq 0\}$ is generated along some SPE path of play, it must be the case that the gain in utility for i in the continuation game along the SPE path of play from $t+1$ outweighs the gain in utility from deviating at t. As we approach the Pareto frontier along a SPE, an individual's gain in the continuation game along the SPE path becomes smaller, and so must the gain in utility by deviating from the equilibrium path of play. Remark that a similar result goes through for when T is large but finite (simply substitute T for T_σ throughout).

[14]A non-trivial $\epsilon-$Nash equilibrium allocation x satisfies the condition that there is a profile s' with $x^i = x^i(s')$ such that for all $i \in I$, $u^i(x^i(s')) \geq u^i(x^i(s^i, s'_{-i})) - \epsilon$ for all $s^i \in S^i(w)$.

It is intuitive that far-sighted retrading can lead the economy within a given neighborhood of the Pareto frontier faster than myopic retrading. To see this, consider $\delta < 1$ and T such that $d(u(y_T), U(P \cap IR)) < \epsilon$, where y_T is the allocation obtained through the myopic retrading path $y_0, ..., y_t, ..., y_T$. The same path can be sustained as a SPE path of far-sighted retrading, hence far-sighted traders can always do at least as well as myopic traders. They can also do strictly better: Suppose that $T > 3$; then it is possible to construct a SPE profile where at the first round of trade the obtained allocation is directly y_{T-1} as long as y_{T-1} can be attained by some combination of bids and offers with the initial endowments w. The threat of no trade in the last round can make deviations from this profile not attractive, if δ is high enough and the game satisfies **P3**. Does this mean that far-sighted retrading *always* leads to greater gains in efficiency? The answer is no, and the following remark shows that there are also SPE of the game that make traders worse off than with just one round of trade.[15] A *new* type of *market failure* can also arise: there are SPE where traders will delay trade only because all other traders do the same.

Remark 3 By Proposition 1, we know that there always exists a static Nash equilibrium in the one-shot market game where all traders gain relative to the no-trade equilibrium. Denote the bid-offer profile that constitutes a Nash equilibrium with trade $s^* = (b^*, q^*)$. Now suppose that traders are allowed to retrade in an extra round of trade. Consider the following strategy profile $\tilde{\sigma}$: (1) for all $i \in I$, play $s^i_{0,l} = (b^i_{0,l}, q^i_{0,l}) = (0, 0)$ for all $l = 2, ..., L$; (2) if $s^i_{0,l} = (b^i_{0,l}, q^i_{0,l}) = (0, 0)$ for all $l = 2, ..., L$, and all $i \in I$, play $s = (b^*, q^*)$ next round; otherwise, play $b^i_1 = q^i_1 = 0$, for all $i \in I$. Then, for each $\delta \in (\frac{u^{\bar{\imath}}(w^{\bar{\imath}})}{u^{\bar{\imath}}(x^{\bar{\imath}}(s^*))}, 1]$, where $\bar{\imath} = \arg\max_{i \in I} \left\{ \frac{u^i(w^i)}{u^i(x^i(s^*))} \right\}$, $\tilde{\sigma}$ is a SPE. However, observe that for $\delta \in (\frac{u^{\bar{\imath}}(w^{\bar{\imath}})}{u^{\bar{\imath}}(x^{\bar{\imath}}(s^*))}, 1)$, at $\tilde{\sigma}$, all traders obtain payoffs which are Pareto dominated by their payoffs corresponding to the static Nash equilibrium.

The next result shows that the set of SPE allocations with far-sighted retrading expands as δ becomes larger.

[15] Note also that the Pareto set can be approximated only in terms of final allocation, whereas discounting makes the convergence process itself "inefficiently long" in terms of utility.

Proposition 8 *Consider $\delta', \delta'' \in [0,1]$ such that $\delta' \leq \delta''$. For each $T < \infty$, then, $\tilde{X}(\delta', w, T) \subseteq \tilde{X}(\delta'', w, T)$.*

Any allocation that satisfies the inequalities that characterize the sequence of allocations along the SPE path for a specific δ must continue to do so as δ becomes larger. The proof is in the Appendix, and the result can be easily extended to the case where traders can retrade infinitely often.

6 Discussion on consumption and asset trading

Throughout the paper we have made the simplifying assumption that consumption by trader i may occur only after he has stopped trading. A natural question to ask is whether therefore our results hold when individual traders can decide otherwise, i.e., when they can opt to consume part of their current endowment instead of using it all for trading purposes. We will divide the analysis of this issue in two parts. First, we give a direct answer to this question, keeping the assumption that all tradeable goods are also consumable. The second part of the analysis makes an argument that in fact one of the best interpretations of our model ought to be the case where the tradeable goods on the trading posts are assets, which are long-lived, yield consumption indirectly, but are not directly consumable themselves. In the second case, of course, the consumption issue becomes irrelevant.

Let us start by keeping the assumption that all tradeable are consumables. Clearly, when the discount factor is equal to one it cannot make a difference, and the SPE profiles that approximate points on the Pareto set remain SPE profiles even when consumption is in principle allowed at any time. On the other hand, when the discount factor is in the open interval $(0,1)$, individuals will typically have an incentive to consume (part of) their endowments even before leaving the market. An important observation is that along a SPE path, the bids and offers typically do not exhaust the endowments at any round of trade. In other words, considering a sequence of actions $s_0, ..., s_t, ...$ that constitutes a SPE path of far-sighted retrading, it is typically the case that $q_t^i < x_{t-1}^i$ at all times.[16] Therefore, individuals can

[16]Recall that we are looking at environments in which no trader has a shortage of tradeable

consume a small fraction of their current endowments at each new round of trade and not necessarily affect the SPE profile of retrading. Hence, it is obvious that allowing traders to consume whenever they want the final utilities must be higher. But what matters here is that if δ is high enough the same path $s_0, ..., s_t, ...$ can remain an equilibrium path of retrading. The equilibrium path used in the approximation results is such that everybody is made better off by each successive round of trade, and hence, for δ high enough, the utility difference can always compensate for the longer wait to consume. A deviation to consume current endowments that affects the feasibility of bids and offers at the current or subsequent rounds of trade will not be profitable.

Even though it should be clear from the above discussion that our assumption of "consumption at the end" is irrelevant for the main results, it is also worth noting that this consumption issue would not even be raised if the trading posts were just markets for assets. Let $x = (x_1, ..., x_L)$ be reinterpreted as an allocation of assets. For any x^i, let $y^i = (y^i_1, ..., y^i_M)$ be the associated allocation of commodities.[17] Let $v^i(y)$ represent trader $i's$ preferences over the commodity bundle y. Traders are endowed with assets but not commodities. A feasible allocation of assets generates a feasible allocation of commodities. An allocation of assets is Pareto optimal if and only if the associated allocation of commodities is Pareto optimal. Traders trade assets $2, ..., L$ using asset $l = 1$ as numeraire. For simplicity, we assume that traders cannot trade commodities directly. They can only trade commodities indirectly, by trading assets. The retrading process, both myopic and far-sighted, is as in the previous sections. The difference is that now at each round of trade, if x^i_t is trader $i's$ current allocation of assets, then y^i_t is trader $i's$ current commodity bundle, which he consumes to obtain a current utility of $v^i(y^i_t)$. Then, trader $i's$ total utility from retrading will be $\sum_{t=0}^{T} \delta^t v^i(y^i_t)$.

With this specification, all our previous results apply by appropriately rephrasing the propositions and proofs. After all players have stopped trading, the final

goods.

[17] As a metaphor, think of the allocation of assets as being allocation of trees, and the vector y would be the corresponding allocation of fruits. People consume fruits, not trees, but trade trees only in this interpretation of the model.

allocation of assets will keep giving the same consumption bundle to all traders thereafter every period. If we extended the model to allow for stochastic yields of assets, then asset trading could continue forever, since every shock on the productivity of assets may change the incentives (or needs) of traders to readjust their asset portfolio. Issues related to uncertainty and/or asymmetric information are however beyond the objective of this paper.

7 Conclusion

The main result of this paper has been to show that allowing retrading in markets where the one-shot allocations are inefficient allows traders to approximate allocations on the Pareto frontier arbitrarily closely.

This "approximation" result, however, needs to be qualified on the following grounds: (1) allocations on the Pareto frontier are never attained in finite time by retrading; (2) getting to an allocation close to the Pareto frontier may take several rounds of retrading and therefore, when traders discount future consumption heavily, in payoff space traders may still be far away from the Pareto frontier of utilities; (3) there is a huge multiplicity of equilibria with retrading, and therefore not all Subgame Perfect Equilibrium allocations with retrading are close to the Pareto frontier; (4) in other contexts (see for instance Jehiel and Moldovanu (1999)), where there are externalities in consumption and traders use trading mechanisms which allow some subset of traders to be excluded from the market, retrading may not approximate allocations on the Pareto frontier.

Beside the issue of efficiency, this paper has also demonstrated some interesting "behavioral" properties of retrading processes. In particular, we have shown that the set of equilibrium paths of retrading that converge to the Pareto frontier when agents are forward looking shrinks towards the converging path of myopic retrading. We have also shown by example that convergence holds even when there is a unique Nash equilibrium in the one-shot game, i.e., in a context where finitely repeated trade could not have efficient equilibrium outcomes. The properties of retrading that we have studied seem therefore to be quite general, and independent on the assumptions made on the rationality of traders.

8 Appendix

8.1 Proof of Proposition 1

In order to prove Proposition 1, it is convenient to associate to the original exchange economy and market game, a pseudo exchange economy and market game. In the original exchange economy, there are L commodities and individual $i's$ is characterised by his consumption set $\Re^L_+$, endowments $w \in \Re^L_{++}$ and utility function $u : \Re^L_+ \to \Re$. In the pseudo-exchange economy, commodity 1 is replaced by $L-1$ copies of itself i.e. commodity 1 is replaced by commodities labelled by the pair $(1, l)$, $l = 2, ..., L$ implying that there are $2(L-1)$ commodities. A commodity bundle in the original excahnge economy is $x \in \Re^L$. A commodity bundle in the pseudo exchange economy is $\hat{x} = (\hat{x}_{1,2}, ..., \hat{x}_{1,L}, \hat{x}_2, ..., \hat{x}_L) \in \Re^{2(L-1)}$ with $\sum_{l=2}^{L} \hat{x}_{1,l} = x_1$ and $\hat{x}_l = x_l$, $l = 2, ..., L$. The consumption set of individual i in the pseudo-exchange economy is $\Re^{2(L-1)}_+$. We abuse notation slightly to denote individual's preferences over the new set of commodities by the utility function $u^i(\sum_{l=2}^{L} \hat{x}_{1,l}, \hat{x}_2, ..., \hat{x}_L) = u^i(x_1, x_2, ..., x_L)$ where $\sum_{l=2}^{L} \hat{x}_{1,l} = x_1$ and $\hat{x}_l = x_l$, $l = 2, ..., L$ with endowments $\hat{w}^i = (\hat{w}^i_{1,2}, ..., \hat{w}^i_{1,L}, \hat{w}^i_2, ..., \hat{w}^i_L)$ where $\hat{w}^i_{1,l} > 0$, $l = 2, ..., L$, $\sum_{l=2}^{L} \hat{w}^i_{1,l} = w^i_1$ and $\hat{w}^i_l = w^i_l$, $l = 2, ..., L$. The allocation $(\hat{x}_{1,2}, ..., \hat{x}_{1,L}, \hat{x}_2, ..., \hat{x}_L)$ is feasible in the pseudo exchange economy if and only if $\sum_{i=1}^{I} \sum_{l=2}^{L} \hat{x}^i_{1,l} = \sum_{i=1}^{I} \sum_{l=2}^{L} w^i_{1,l}$ and $\sum_{i=1}^{I} \hat{x}^i_l = \sum_{i=1}^{I} \hat{w}^i_l$, $l = 2, ..., L$. Remark that if the allocation $(\hat{x}_{1,2}, ..., \hat{x}_{1,L}, \hat{x}_2, ..., \hat{x}_L)$ is feasible in the pseudo exchange economy, the allocation $(\sum_{l=2}^{L} \hat{x}_{1,l} = x_1, x_2, ..., x_L)$ where $\hat{x}_l = x_l$, $l = 2, ..., L$ is also feasible in the original excahnge economy. Moreover, if the allocation $(x_1, x_2, ..., x_L)$ is feasible in the original exchange economy, the allocation $(\hat{x}_{1,2}, ..., \hat{x}_{1,L}, \hat{x}_2, ..., \hat{x}_L)$ where $\sum_{l=2}^{L} \hat{x}_{1,l} = x_1$ and $\hat{x}_l = x_l$, $l = 2, ..., L$ is feasible in the pseudo-exchange economy. Therefore, the set of feasible utility profiles in the pseudo exchange economy coincides with the set of feasible utility profiles in the original exchange economy. Consider the allocation x in the original economy where $u(x) \gg u(w)$. Consider the allocation $(\hat{x}_{1,2}, ..., \hat{x}_{1,L}, x_2, ..., x_L)$ in the pseudo exchange economy where $\sum_{l=2}^{L} \hat{x}_{1,l} = x_1$ and $\hat{x}_l = x_l$, $l = 2, ..., L$. It follows that $u^i(\sum_{l=2}^{L} \hat{x}_{1,l}, \hat{x}^i_2, ..., \hat{x}^i_L) =$

$u^i(x^i_1, x^i_2, ..., x^i_L) > u^i(w^i_1, w^i_2, ..., w^i_L) = u^i(\sum_{l=2}^{L} \hat{w}^i_{1,l}, \hat{w}^i_2, ..., \hat{w}^i_L)$. Conversely, if $\hat{x}$ is an allocation in the pseudo exchange economy such that $u(\hat{x}) \gg u(\hat{w})$, then the allocation $x = (\sum_{l=2}^{L} \hat{x}_{1,l} = x_1, x_2, ..., x_L)$ $\hat{x}_l = x_l$, $l = 2, ..., L$ in the original exchange economy is such that $u(x) \gg u(w)$. Next, we turn to specification of the pseudo market game. In the pseudomarket game, bids for commodity l are denoted in commodity $(1, l)$. A strategy for player i is therefore $(\hat{b}^i, \hat{q}^i) = (\hat{b}^i_2, ..., \hat{b}^i_L, \hat{q}^i_2, ..., \hat{q}^i_L)$ such that $0 \leq \hat{b}^i_l \leq \hat{w}^i_{1,l}$ and $0 \leq \hat{q}^i_l \leq \hat{w}^i_l$, $l = 2, ..., L$. The price formation rules and allocation rules remain unchanged. At a strategy profile where $\pi_l > 0$, $l = 2, ..., L$, the payoff function for individual i is

$$u^i\left(\sum_{l=2}^{L}\left(\hat{w}^i_{1,l} - \hat{b}^i_{1,l} + \pi_l \hat{q}^i_l\right), \hat{w}^i_2 - \hat{q}^i_2 + \frac{\hat{b}^i_2}{\pi_2}, ..., \hat{w}^i_L - \hat{q}^i_L + \frac{\hat{b}^i_L}{\pi_L}\right)$$

Remark that an interior Nash equilibrium of the pseudo market game where $0 < \hat{b}^i_l < \hat{w}^i_{1,l}$ and $0 < \hat{q}^i_l < \hat{w}^i_l$, $l = 2, ..., L$, $i = 1, ..I$ which yields allocation $\hat{x}$ is also an interior Nash equilibrium of the original market game, with $b^i_l = \hat{b}^i_l$, and $q^i_l = \hat{q}^i_l$, $l = 2, ..., L$, which yields allocation x where $x = (\sum_{l=2}^{L} \hat{x}_{1,l} = x_1, \hat{x}_2, ..., \hat{x}_L)$. To see why, note that the first-order conditions that characterise an interior Nash equilibrium in the pseudo market game as they coincide with the first -order conditions that characterise an interior Nash equilibrium. Moreover, it also follows that by appropiately choosing $(\hat{w}^i_{1,2}, ..., \hat{w}^i_{1,L})$ for all $i = 1, ..., I$, any interior Nash equilibrium of the original market game is also an interior Nash equilibrium of the pseudo market game.

Lemma 1 *Suppose $w \notin P$ in the original exchange economy. Then, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$, there is some feasible allocation x with $\|x - w\| < \varepsilon$ and $u(x) \gg u(w)$ such that $x \in N(w)$ in the original market game.*

Proof. For the purposes of the proof, it is convenient to work with excess demands. Let $z^i_l = x^i_l - w^i_l$. Consider the set

$$\bar{Z} \equiv \left\{z \in \Re^L : u^i(z + w^i) \geq u^i(w^i),\ for\ all\ i \in I\right\}.$$

As $u^i(.)$ is concave, $\bar{Z}$ is convex (intersection of convex upper contour sets). Remark that 0 is on the boundary of $\bar{Z}$.[18] It follows, by applying the supporting hyperplane theorem, there exists a vector $\tilde{\pi} = (\tilde{\pi}_1, ..., \tilde{\pi}_L) \neq 0$ such that $\tilde{\pi}\bar{z} \geq 0$ for all $\bar{z} \in \bar{Z}$. Consider the hyperplane $\left\{z \in \Re^L : \tilde{\pi}z = 0\right\}$. Remark that $\left\{z \in \Re^L : \tilde{\pi}z = 0\right\} \cap \bar{Z} = \{0\}$. Next, note that $\tilde{\pi}_l > 0$ for all $l = 1, ..., L$.[19] As the indifference sets of individuals through 0 are smooth, then the boundary of the set $\tilde{Z} \equiv \left\{z \in \Re^L : u^i(z + w^i) = u^i(w^i),\ for\ all\ i \in I\right\}$ is not differentiable at 0. Therefore, there exists a continuum of hyperplanes $\left\{z \in \Re^L : \tilde{\pi}z = 0\right\}$ such that $\left\{z \in \Re^L : \tilde{\pi}z = 0\right\} \cap \bar{Z} = \{0\}$, $\tilde{\pi}_l > 0$ for all $l = 1, ..., L$. We can choose $\tilde{\pi} \gg 0$ such that for each i $\partial_x u^i(w) \neq \tilde{\pi}$,[20] i.e., such that $\left\{z \in \Re^L : u^i(z + w^i) > u^i(w^i)\right\} \cap \left\{z \in \Re^L : \tilde{\pi}z = 0\right\} \neq \phi$. Without loss of generality, by an appropriate normalization, we can set $(\tilde{\pi}_1, ..., \tilde{\pi}_L) = (1, \pi_2, ..., \pi_L) \gg 0$ where $\pi_l = \frac{\tilde{\pi}_l}{\tilde{\pi}_1}$. In the remainder of the proof we work with the vector $\pi = (1, \pi_2, ..., \pi_L) \gg 0$. As $w \notin P$, there exists a partition of the set of individuals into two non-empty sets I' and I'' such that for all $i' \in I'$, $\left(1, \frac{\partial_{x_2} u^{i'}(w^{i'})}{\partial_{x_1} u^{i'}(w^{i'})}, ..., \frac{\partial_{x_L} u^{i'}(w^{i'})}{\partial_{x_1} u^{i'}(w^{i'})}\right) < (1, \pi_2, ..., \pi_L)$ while for each $i'' \in I''$, $\left(1, \frac{\partial_{x_2} u^{i''}(w^{i''})}{\partial_{x_1} u^{i''}(w^{i''})}, ..., \frac{\partial_{x_L} u^{i''}(w^{i''})}{\partial_{x_1} u^{i''}(w^{i''})}\right) > (1, \pi_2, ..., \pi_L)$. It folllows that there exists $\bar{\epsilon}' > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}'$, the set

$$Z(\epsilon) \equiv \left\{z : \left\|z^i\right\| < \varepsilon, \pi z^i = 0\ and\ u^i(z^i + w^i) > u^i(w^i)\ \forall i,\ \sum_i z_l^i = 0\ \forall l\right\}$$

is non empty. Take any $z \in Z(\bar{\epsilon'})$ and consider a pseudo-exchange economy where $\hat{z}_{1,l}^i = -\pi_l z_l^i$, $\hat{z}_l^i = z_l^i$ for all $i \in I$ and all $l = 2, ..., L$. Note that $\sum_{i=1}^I \sum_{l=1}^L \hat{z}_{1,l}^i = 0$, $\sum_i \hat{z}_l^i = 0$, $l = 2, ..., L$ and

$$\pi_l = -\frac{\hat{z}_{1,l}^i}{\hat{z}_l^i}\ l = 2, ..., L$$

whenever $\hat{z}_{1,l}^i \neq 0$ and $\hat{z}_l^i \neq 0$, $l = 2, ..., L$. Now consider the equations

$$\frac{\partial_{x_l} u^i(z^i + w^i)}{\partial_{x_1} u^i(z^i + w^i)} = (\pi_l)^2 \frac{\hat{Q}_{l,-i}}{\hat{B}_{l,-i}}, i = 1, ..., I,\ l = 2, ..., L$$

[18] Note that even if players have different endowments in the commodity space, in the excess demand space they all have the same status quo at 0.

[19] Indeed, suppose to the contrary that $\tilde{\pi}_l \leq 0$ for some l. Then, there exists $z' \in \left\{z \in \Re^L : \tilde{\pi}z = 0\right\}$ such that $w^i + z' \geq w^i$ for all $i \in I$ and therefore, as the utility functions are strongly monotone, $u^i(z' + w^i) > u^i(w^i)$ for all $i \in I$, a contradiction.

[20] Recall that the utility functions are twice continuously differentiable and smooth at 0 by Assumption 1.

$$\hat{z}_l^i = -\hat{q}_l^i + \left(\frac{\hat{b}_l^i}{\pi_l}\right), i = 1, ..., I,\ l = 2, ..., L.$$

As $\sum_i \hat{z}_l^i = 0,\ l = 2, ..., L,$

$$\pi_l = \frac{\hat{B}_l}{\hat{Q}_l}\ l = 2, ..., L.$$

Solving these equations we obtain that

$$\hat{B}_{l,-i} = \frac{-(\pi_l)^2 \left(\sum_{j\neq i} \hat{z}_l^j\right)}{\left(\frac{\partial_{x_l} u^i(z^i+w^i)}{\partial_{x_1} u^i(z^i+w^i)}\right) - \pi_l}, i = 1, ..., I,\ l = 2, ..., L,$$

$$\hat{Q}_{l,-i} = -\sum_{j\neq i} \hat{z}_l^j + \left(\frac{\hat{B}_{l,-i}}{\pi_l}\right), i = 1, ..., I,\ l = 2, ..., L.$$

After some manipulation, remark that

$$\left(\frac{\partial_{x_l} u^i(z^i + w^i)}{\partial_{x_1} u^i(z^i + w^i)}\right) - \pi_l = (\pi_l)^2 \frac{\hat{Q}_{l,-i}}{\hat{B}_{l,-i}} - \pi_l,\ l = 2, ..., L$$
$$= \pi_l \left(\frac{\pi_l \hat{Q}_{l,-i}}{\hat{B}_{l,-i}} - 1\right)\ l = 2, ..., L.$$

$\sum_{j\neq i} \hat{z}_l^j = \frac{\hat{B}_{l,-i}}{\pi_l}\left(1 - \frac{\pi_l \hat{Q}_{l,-i}}{\hat{B}_{l,-i}}\right)$ and therefore, $-\sum_{j\neq i} \hat{z}_l^j$ and $\pi_l - \left(\frac{\partial_{x_l} u^i(z^i+w^i)}{\partial_{x_1} u^i(z^i+w^i)}\right)$ have the same sign. Therefore, $\hat{B}_{l,-i} > 0,\ l = 2, ..., L.$ Consider now the excess demand vectors on the supporting hyperplane in $Z(\bar{\epsilon'})$ and consider the associated set of excess demand vectors in the pseudo exchange economy

$$\hat{Z}(\bar{\epsilon'}) = \left\{\hat{z} : \hat{z}_{1,l}^i = -\pi_l z_l^i, \hat{z}_l^i = z_l^i, l = 2, ..., L, i \in I, z \in Z(\bar{\epsilon'})\right\}.$$

As w is not Pareto optimal and $u(x) \gg u(w)$ in the original exchange economy, we must have that $\pi_l - \frac{\partial_{x_l} u^i(w^i)}{\partial_{x_1} u^i(w^i)} \neq 0$. Therefore,

$$\lim_{\hat{z}\to 0, \hat{z}\in\hat{Z}} \hat{B}_{l,-i} = \lim_{\hat{z}\to 0, \hat{z}\in\hat{Z}} \hat{Q}_{l,-i} = 0,\ i = 1, ..., I,\ l = 2, ..., L.$$

Let $\underline{\hat{w}}_{1,l,-i} = \min_{j\neq i}\left\{\hat{w}_{1,l}^j\right\} > 0$ and $\underline{\hat{w}}_{l,-i} = \min_{j\neq i}\left\{\hat{w}_l^j\right\} > 0,\ l = 2, ..., L.$ It follows that for each $i \in I$, there exists $\hat{z}_{1,l}^i \neq 0,\ \hat{z}_l^i \neq 0,\ l = 2, ..., L$ and $\pi_l = -\frac{\hat{z}_{1,l}^i}{\hat{z}_l^i}$ such that $0 < \hat{B}_{l,-i} < (I-1)\underline{\hat{w}}_{1,l,-i}$, $0 < \hat{Q}_{l,-i} < (I-1)\underline{\hat{w}}_{l,-i}$, and therefore $0 < \hat{b}_l^i < \hat{w}_{1,l}^i$ and $0 < \hat{q}_l^i < \hat{w}_l^i$, $i = 1, ..., I$, $l = 2, ..., L$, which yields $\hat{x}_{1,l}^i = \hat{w}_{1,l}^i + \hat{z}_{1,l}^i$, $\hat{x}_l^i = \hat{w}_l^i + \hat{z}_l^i$

for all $i = 1, ..., I$, $l = 2, ..., L$ as an interior Nash equilibrium allocation in the pseudo market game. It follows that $(\sum_{l=2}^{L} \hat{x}_{1,l} = x_1, x_2, ..., x_L)$, $x_l = \hat{x}_l$, $l = 2, ..., L$, is an interior Nash equilibrium allocation in the original market game such that $u(x) \gg u(w)$. In other words, there exists $\bar{\epsilon} \leq \bar{\epsilon'}$ such that $\forall \epsilon \in (0, \bar{\epsilon}]$ $\exists$ $x \in N(w)$, $\|x - w\| < \epsilon$, such that $u(x) \gg u(w)$. **QED.**

By the preceeding lemma, there exists $x' \in N(w')$ such that $u^{\bar{\imath}}(w^{\bar{\imath}}) < u^{\bar{\imath}}(w'^{\bar{\imath}}) < u^{\bar{\imath}}(x'^{\bar{\imath}})$ for the same individual $\bar{\imath}$. As the utility of each individual is continuous, there exists $\hat{\varepsilon} > 0$ such that for all feasible allocations x with $\|x - w\| < \hat{\varepsilon}$, $u^{\bar{\imath}}(x^{\bar{\imath}}) < u^{\bar{\imath}}(x'^{\bar{\imath}})$. But, then, by the preceeding lemma, there exists $\bar{\varepsilon}$ such that for all $\varepsilon \leq \bar{\varepsilon}$, there exists $x \in N(w)$ such that $\|x - w\| < \varepsilon$ and $u(x) \gg u(w)$. Let $\varepsilon' = \min\{\hat{\varepsilon}, \bar{\varepsilon}\}$. It immediately follows that there exists $x \in N(w)$ such that $\|x - w\| < \varepsilon'$, $u(x) \gg u(w)$ and $u^{\bar{\imath}}(x^{\bar{\imath}}) < u^{\bar{\imath}}(x'^{\bar{\imath}})$. **QED.**

8.2 Convergence with myopic retrading: an alternative proof

Let $\tilde{U}$ be the set of strictly monotone, strictly concave, C^r, $r \geq LI$, utility functions endowed with the topology of uniform convergence on compacts (see Mas-Colell (1985) for a definition). Let U^i be the subset of utility functions in $\tilde{U}$ which have the property that all $u \in U^i$ are finite in the corresponding norm. Then, as Dubey and Rogowski (1990) note, by Theorem 10.2 in Abraham and Robbins (1967), U^i is an open subset of a Banach set. Let $U = U^1 \times ... \times U^I$.

Proposition 2': Consider u in the countable intersection of a collection of open and dense subsets of U and $w \in \Re^{LI}_{++}$, suppose that $N(w)$ satisfies **(P1)-(P2)** whenever $w \notin P$. Then, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...,$ $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$.

Proof. By Propositions 3 and 4, remark 5 and section 5.1 in Dubey and Rogawski (1990), for every $w \in \Re^{LI}_{++}$, there is an open and dense subset of U so that for each $\bar{u}$ in this subset, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* (see page 295, Dubey and Rogowski (1990)). As the domain of η^* is compact, by the openness of transversal intersections (page

43, Mas-Colell (1985)), it follows that there is an $\varepsilon > 0$ such that for all w' with $\|w' - w\| < \varepsilon$, for each $\bar{u}$ in the open and dense subset of U associated with w, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^*. Consider a countable set of allocations, contained in $\Re^{LI}_{++}$, which is dense in F. Such a set exists. Let F_k denote the projection of F onto the k-th coordinate of $\Re^{LI}_{++}$. Remark that the set of rational numbers in F_k is a dense subset of F_k. Take the LI product of the subset of rational numbers contained in each F_k. Denote this set by Υ. Then, Υ is a countable set which is dense in F. For each allocation in $w \in \Upsilon$, there exists an open and dense subset of U such that (a) each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* and (b) there is an $\varepsilon > 0$ such that for all w' with $\|w' - w\| < \varepsilon$, for each $\bar{u}$ in the open and dense subset of U associated with w, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^*. It follows that by taking the countable intersection of the open and dense subsets of U corresponding to some $w \in \Upsilon$, we obtain, by the Baire property (page 10, Mas-Colell (1985)), a non-empty dense subset of U such that each u in this set and every $y \in F \cap \Re^{LI}_{++}$, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* and further, η^* is also transverse to every submanifold of N^*: therefore, η^* satisfies the definition of transverse stability. Fix u in this dense subset of U. Let $w \notin P$. Consider the sequence of sets $N_1, ..., N_t, ...$, with $y_0 = w$, and $N_t = \{x : x \in N(y),\ for\ some\ y \in N_{t-1}\}$, $t = 1, ...$, with the associated sequence of sets $u(N_1), ..., u(N_t), ...$ in utility space $\Re^I$. By **(P2)**, we can extract a sequence $\tilde{u}_t$, $t = 0, 1, ...$ such that $\tilde{u}_t \in u(N_t)$ and $\tilde{u}_{t+1} > \tilde{u}_t$, at each t, with $y_0, ..., y_t, ...$ the associated sequence of allocations. Note that for each $i \in I$, the sequence $\tilde{u}^i_t$, $t = 0, 1, ...$ is bounded above, as the utility of each individual is continuous and the set of feasible allocations is compact. Let $\bar{u}^i$ denote the supremum of the sequence $\tilde{u}^i_t$, $t = 0, 1, ...$ As every increasing sequence converges to the supremum, it follows that the sequence $\tilde{u}_t$, $t = 0, 1, ...$, converges to $\bar{u} = (\bar{u}^1, ..., \bar{u}^I)$, the component-wise supremum of $\tilde{u}_t = (\tilde{u}_t^1, ..., \tilde{u}_t^I)$, $t = 0, 1, ...$ Moreover, by passing to subsequence if necessary, without loss of generality, we may assume that the associated sequence of allocations $y_t, t = 0, 1, ...$ converges to some allocation $\bar{y}$ such that $u(\bar{y}) = \bar{u}$. By considering every sequence of utilities and the corresponding sequence of allocations generated by myopic retrading which satisfy **(P2)**, we obtain a set of allocations $\bar{Y}$ which consists of the limit allocations of each sequence of allocations $y_t, t = 0, 1, ...$ We have to show that for every $\varepsilon > 0$, there is some $\bar{y} \in \bar{Y}$ such that $d(u(\bar{y}), u(P \cap IR)) < \varepsilon$. Then, for every $\varepsilon > 0$ there will be some $T > 0$ and some sequence of allocations generated by myopic retrading $y_t, t = 0, 1, ...$ which converges to $\bar{y}$ such that (a) $d(u(\bar{y}), u(P \cap IR)) < \frac{\varepsilon}{2}$ and (b) for all $t > T$, $d(u(y_t), u(P \cap IR)) < \varepsilon$. Suppose to the contrary, that $\min_{y \in cl.(\bar{Y})} d(u(y), u(P \cap IR)) > 0$ with $\bar{y}' \in \arg\min_{y \in cl.(\bar{Y})} d(u(y), u(P \cap IR) cl(\bar{Y})$. Then, by **(P2)**, there exists an allocation $\hat{y} \in N(\bar{y}')$ and $i \in I$ such that $u^i(\hat{y}) > u^i(\bar{y}'^i)$. It follows that there exists $\varepsilon > 0$ such that for all $\|y - \bar{y}'\| < \tilde{\varepsilon}$, there exists $\tilde{y} \in N(y)$ such that $u^i(\tilde{y}^i) > u^i(\bar{y}'^i)$. Moreover, for every $\varepsilon > 0$, there is some $\bar{y} \in \bar{Y}$ such that $d(u(\bar{y}), u(\bar{y}'))$. Therefore, for every $\varepsilon > 0$, there exists a sequence of allocations $y'_t, t = 0, 1, ...$ generated by myopic retrading so that there is a $\tilde{T}$ such that for all $t > \tilde{T}$, $\|y'_t - \bar{y}'\| < \tilde{\varepsilon}$, there exists $y''_t \in N(y'_t)$, $u^i(y''^i_t) > u^i(\bar{y}'^i)$. Consider the sequence $y''_0, ..., y''_t, ...$ where for $t \leq \tilde{T},$, $y''_t = y'_t$, for $t = \tilde{T} + 1$, $y''_t \in N(y'_{t-1})$ such that $u^i(y''^i_t) > u^i(\bar{y}'^i)$ and for $t > \tilde{T} + 1$, $y''_t \in N(y''_{t-1})$ and $u^i(y''^i_t) > u^i(y''^i_{t-1})$. Let $\tilde{u}''_t$, $t = 0, 1, ...$ be the associated sequence of allocations. Remark that $\bar{u}'$ is no longer the component-wise supremum of $\tilde{u}''_t$, $t = 0, 1, ...$ Therefore, the sequence $\tilde{u}''_t$, $t = 0, 1, ...$ must converge to $\bar{u}'' = (\bar{u}''^1, ..., \bar{u}''^I)$, the component-wise supremum of $\tilde{u}''_t = (\tilde{u}''_t, ..., \tilde{u}''^I_t)$, $t = 0, 1, ...$ and the associated sequence of allocations $y''_0, ..., y''_t, ...$ must converge to some $\bar{y}''$ such that $u(\bar{y}'') = \bar{u}''$ such that $d(\bar{u}'', u(P \cap IR)) < d(u(\bar{y}'), u(P \cap IR))$, a contradiction. **QED.**

8.3 Proof of Proposition 6

We construct a strategy profile σ that has the following phases:

Phase (1): Players choose $\tilde{s}_t$ at each t that supports the sequence of allocations generated by myopic (interior) Nash retrading which converges to the Pareto frontier along the equilibrium path of play namely, $\{\tilde{y}_t\}$, $t = 0, 1, \ldots$. Such a sequence of allocations exists by Propisition 2 and is, by construction, retrade-proof. be the corresponding sequence of allocations. At each time t, if the history of play is as in phase(1), continue choosing $\tilde{s}_t$.

Phase (2): In Phase (2), some player $i(1)$ has deviated from the sequence of bids and offers in Phase (1). Consider a deviation from Phase (1) at some $t(1) < T$ by player $i(1)$ to some $s'^{i(1)}_{t(1)} \neq \tilde{s}^{i(1)}_{t(1)}$. As $\tilde{s}_{t(1)}$ is a myopic (interior) Nash equilibrium at $t(1)$, $u^{i(1)}(x^{i(1)}(s'^{i(1)}_{t(1)}, \tilde{s}_{-i(1),t(1)})) < u^{i(1)}(x^{i(1)}(\tilde{s}_{t(1)}))$. Let $\tilde{y}^{(2),j}_{t(1)} = x^j(s'^{i(1)}_{t(1)}, \tilde{s}_{-i(1),t(1)})$, $j = 1, \ldots, I$. Let $\tilde{y}^{(2)}_{t(1)} = (\tilde{y}^{(2),j} : j \in I)$. Remark that $\tilde{y}^{(2)}_{t(1)} \gg 0$ as at $t(1)$ all traders $i \neq i(1)$ are choosing strictly positive bids and offers. Therefore, by Proposition 1, there exists a sequence of allocations $\left\{\tilde{y}^{(2)}_t\right\}$ starting from $\tilde{y}^{(2)}_{t(1)}$, $t = t(1), t(1)+1, \ldots, T$, such that (i) $\tilde{y}^{(2)}_t \in N(\tilde{y}^{(2)}_{t-1})$ for all $t > t(1)$, (ii) $u^{i(1)}(\tilde{y}^{(2),i(1)}_t) < u^{i(1)}\left(\tilde{y}^{i(1)}_t\right)$ for all $t \geq t(1)$. Fix the sequence of allocations generated by σ in a phase(2) subgame to be $\left\{\tilde{y}^{(2)}_t\right\}$.

...

Phase (K): Consider a deviation from Phase (K-1) at some $t(K-1) > t(K-2)$ and $t(K-1) < T$ by player $i(K-1)$ to some $s'^{i(K-1)}_{t(K-1)} \neq \tilde{s}^{i(K-1)}_{t(K-1)}$. As $\tilde{s}_{t(K)}$ is a myopic (interior) Nash equilibrium at $t(K-1)$, $u^{i(K-1)}(x^{i(K-1)}(s'^{i(K-1)}_{t(K-1)}, \tilde{s}_{-i(K-1),t(K-1)})) < u^{i(K-1)}(x^{i(K-1)}(\tilde{s}_{t(K-1)}))$. Let $\tilde{y}^{(K),j}_{t(K-1)} = x^j(s'^{i(K-1)}_{t(K-1)}, \tilde{s}_{-i(K),t(K-1)})$, $j = 1, \ldots, I$. Let $\tilde{y}^{(K)}_{t(K-1)} = (\tilde{y}^{(K),j}_{t(K-1)} : j \in I)$. Remark that $\tilde{y}^{(K)}_{t(K-1)} \gg 0$ as at $t(K-1)$ all traders $i \neq i(K-1)$ are choosing strictly positive bids and offers. Therefore, by proposition 1, there exists a sequence of allocations $\left\{\tilde{y}^{(K)}_t\right\}$ starting from $\tilde{y}^{(K)}_{t(K-1)}$, $t = t(K-1), t(K-1)+1, \ldots, T$, such that (i) $\tilde{y}^{(K)}_t \in N(\tilde{y}^{(K)}_{t-1})$ for all $t > t(K-1)$, (ii) $u^{i(K-1)}(\tilde{y}^{(K),i(K-1)}_t) < u^{i(K-1)}\left(\tilde{y}^{i(K-1)}_t\right)$ for all $t \geq t(K-1)$. Fix the sequence of allocations generated by σ in a phase (K) subgame to be $\left\{\tilde{y}^{(K)}_t\right\}$.

...

Phase (T): In Phase (T), some player $i(T-1)$ has deviated from the sequence of bids and offers in Phase (T-1). Consider a deviation from Phase (T-1) at $T-1$ by player $i(T-1)$ to some $s'^{i(T-1)}_{T-1} \neq \tilde{s}^{i(T-1)}_{T-1}$. As $\tilde{s}_{T-1}$ is a myopic (interior) Nash equilibrium at $T-1$, $u^{i(T-1)}(x^{i(T-1)}(s'^{i(T-1)}_{T-1}, \tilde{s}_{-i(T-1),T-1})) < u^{i(T-1)}(x^{i(T-1)}(\tilde{s}_{T-1}))$.

Let $\tilde{y}^{(T),j}_{T-1} = x^j(s'^{i(T-1)}_{T-1}, \tilde{s}_{-i(T-1),T-1})$, $j = 1, ..., I$. Let $\tilde{y}^{(T)}_{T-1} = (\tilde{y}^{(T),j}_{T-1} : j \in I)$. Remark that $\tilde{y}^{(T)}_{T-1} \gg 0$ as at $T-1$ all traders $i \neq i(T-1)$ are choosing are choosing strictly positive bids and offers. Therefore, by proposition 1, there exists an allocation $\tilde{y}^{(T)}_t$ starting from $\tilde{y}^{(T)}_{T-1}$, $t = T-1, T$, such that (i) $\tilde{y}^{(T)}_T \in N(\tilde{y}^{(T)}_{T-1})$, (ii) $u^{i(K-1)}(\tilde{y}^{T,i(T-1)}_T) < u^{i(T-1)}\left(\tilde{y}^{(T-1),i(T-1)}_T\right)$.

Remark that the sequence of bids and offers chosen at each phase of the strategy profile σ will depend on the identity of the deviator from the path of play specified in the preceeding phase. The existence of such a strategy profile σ is guaranteed by Proposition 1. Consider the sequence of allocations in phase(1) $\{\tilde{y}_t\}$, $t = 0, 1, ..., T$. In proposition 5, we have already shown that for every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}(1)$ such that (i) $d(u(\tilde{y}_T), u(P)) < \varepsilon$ for all $\delta \in [\underline{\delta}(1), 1]$, $T \geq \underline{T}$ and (ii) no player i will have a unilateral incentive to stop trading at some $t < \underline{T}$ (iii) $\{\tilde{y}_t\}$, $t = 0, 1, ..., \underline{T}$ is retrade proof. By a simillar argument, it also follows that for each phase(K), K>1, K<T the sequence of allocations $\tilde{y}_0$,..., $\tilde{y}_{t(1)-1}$, $\tilde{y}^{(2)}_{t(1)}$, ..., $\tilde{y}^{(2)}_{t(2)-1}$,..., $\tilde{y}^{(K-1)}_{t(K-1)-1}$, $\tilde{y}^{(K)}_{t(K-1)}$,..., $\tilde{y}^{(K)}_{\underline{T}}$ there is a $\underline{\delta}(K)$ such that (i) no player i will have a unilateral incentive to stop trading at some $t < \underline{T}$ (iii) the sequence of allocations $\tilde{y}_0$,..., $\tilde{y}_{t(1)-1}$, $\tilde{y}^{(2)}_{t(1)}$, ..., $\tilde{y}^{(2)}_{t(2)-1}$,..., $\tilde{y}^{(K-1)}_{t(K-1)-1}$, $\tilde{y}^{(K)}_{t(K-1)}$,..., $\tilde{y}^{(K)}_{\underline{T}}$ is retrade proof. Fix $\underline{\delta} = \min_K \{\underline{\delta}(1), ..., \underline{\delta}(K), ..., \underline{\delta}(T)\}$. Remark that $\underline{\delta} < 1$. It follows that for all δ, $\underline{\delta} < \delta < 1$, σ satisfies the unimprovability by one-shot deviations and is, therefore, subgame perfect. be the corresponding sequence of allocations. But, then, for every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}^R(\delta, w, T)$ such that $d(u(y), u(P)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$. **QED.**

8.4 Proof of Proposition 8

We show that if $y' \in \tilde{X}(\delta', w, T)$, then $y' \in \tilde{X}(\delta'', w, T)$. In order to show this, the following lemma comes handy. Consider the strategy profile $\hat{\sigma}(\delta', y')$ which is identical to a SPE $\sigma(\delta', y')$ on the equilibrium path but differs off the equilibrium path in that, after any deviation from the equilibrium path of play at some time $t < T(\sigma(\delta', y'))$, $b^i_{t'} = q^i_{t'} = 0$, for all $t' > t$. Let $\hat{\Sigma}$ denote the corresponding set of strategies.

Lemma 2 *For any $T < \infty$, for all $\delta \in [0,1]$, $y' \in \tilde{X}(\delta', w, T)$ if and only if there is a $\hat{\sigma}(\delta', y') \in \hat{\Sigma}$ that supports y'.*

Proof. When $\delta = 0$, all traders stop trading at $t = 0$, implying that $x_0 \in N(w)$. It follows that any SPE strategy profile must be an element of $\hat{\Sigma}$. Suppose $\delta \in (0,1]$. If there is a $\hat{\sigma}(\delta', y') \in \hat{\Sigma}$ that supports y', by definition $y' \in \tilde{X}(\delta', w, T)$. Next, suppose that $\sigma(\delta', y')$ is a SPE strategy profile that yields $y' \in \tilde{X}(\delta', w, T)$. Then, $\hat{\sigma}(\delta', y')$ is also a SPE strategy that yields $y' \in \tilde{X}(\delta', w, T)$. By construction, observe that no player has an incentive to deviate after $T(\sigma(\delta', y')) + 1$ or after observing a deviation from the equilibrium path of play. Therefore, suppose player i deviates at t choosing some action $s_t'^i$. As $b_{\bar{t}}^i = q_{\bar{t}}^i = 0$, $i \in I$, for all $\bar{t} > t$, denote i's maximum payoff from such a deviation by $u_t^{d,i}(\delta') = (\delta')^{t+1} u^i(x^i(s_t'^i, s_{-i,t}(\sigma(\delta', y')))$ where $x^i(s_t'^i, s_{-i,t}(\sigma(\delta', y')))$ be the resulting allocation for i when he chooses $s_t'^i$ while all other players choose according to $\hat{\sigma}(\delta', y')$. Observe that as $\sigma(\delta', y')$ is itself a SPE, it must be the case that i's maximum payoff from deviating from the equilibrium path of play under the strategy profile $\sigma(\delta', y')$ cannot be less than $u_t^{d,i}(\delta')$. Therefore, if player i has no incentive to deviate from the equilibrium path of play under $\sigma(\delta', y')$, she cannot have an incentive to deviate from the equilibrium path of play under $\hat{\sigma}(\delta', y')$. **QED.**

Given Lemma 2, we can assume w.l.o.g that any $y' \in \tilde{X}(\delta', w, T)$ is supported by a SPE profile $\hat{\sigma}(\delta', y')$. We need to show that $\tilde{\sigma}(\delta', y')$ remains a SPE strategy profile when $\delta = \delta''$. For each i, let $\tilde{T}_i$ denote the final period when $s_t^i(\tilde{\sigma}(\delta', y')) \neq 0$. Then we must have, at each $t \leq \tilde{T}_i$, $u^i(x^i(s_t(\tilde{\sigma}(\delta', y'))) \leq (\delta')^{\tilde{T}_i - t} u^i(x^i(s_{\tilde{T}_i}(\tilde{\sigma}(\delta', y')))$ and for all $t' > t$, $t' \leq \tilde{T}_i$, $u^i(x^i(s_{i,t}', s_{-i,t}(\tilde{\sigma}(\delta', y'))) \leq (\delta')^{t'-t} u^i(x^i(s_{t'}(\tilde{\sigma}(\delta', y')))$. Finally, note that as $\delta'' > \delta'$, the above inequalities continue to hold when δ' is replaced by δ''. **QED.**

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