Management of a Capital Stock by Strotz's Naive Planner

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Abstract

A generalized version of the capital management problem posed in a classic paper by R. H. Strotz is analyzed for the case of the "naive" planner who fails to anticipate any impending change in his own preferences. By imposing progressively stronger restrictions on the primitives of the problem — namely, the planner's discounting function, his utility index function, and the investment technology — the path of the capital stock is characterized first implicitly as the solution to a differential equation and then explicitly via formulae that may or may not be expressible in closed form. Inasmuch as this procedure turns out to leave the discounting function essentially unrestricted, the theory can accommodate, in particular, decision makers who discount time according to the type of hyperbolic curve said to be suggested by psychological studies. Strategies for numerical computation of capital paths are discussed and are demonstrated in sample planning problems.

JEL classification codes: C60, D91, E21.

Keywords: consumption, computation, hyperbolic discounting, time preference.

0. OUTLINE

§1 discusses the rationale for and the objectives of this project.

§2 states the planner's problem, identifies the prerequisites for time-consistent behavior, and presents a schematic analysis of the general case in which consistency cannot be guaranteed.

§3 carries out this analysis and illustrates the resulting theory.

- **§3.1** shows how linearity of the investment technology allows us to characterize the path of the capital stock as the solution to an initial value problem.
- **§3.2** shows how we can solve this problem and determine the associated path when the utility index function has either an exponential or a basal specification.
- **§3.3** defines the stationary hyperbolic discounting function.
- **§3.4** applies our findings to simple "cake-eating" problems.

§4 comments on the feasibility of approximating capital paths numerically using either of two computational strategies.

§5 considers planning problems with credit constraints.

- **§5.1** demonstrates the difficulty of extending our analysis to cover such scenarios.
- **§5.2** obtains results for the restricted class of "savings-initiation" problems.

§6 offers two concluding comments.

1. INTRODUCTION

When Robert H. Strotz published his investigation [28] of "Myopia and Inconsistency in Dynamic Utility Maximization," he credited Allais, Hayek, and Samuelson with having already raised, or having at least [p. 165] "alluded to" the questions that he then went on to address. But despite this generosity, it is Strotz's paper that has achieved the status of the universally-cited founding document of the branch of economic theory that studies agents whose preferences change over time, and whose voluntary behavior at one moment they might themselves wish to thwart at another.

It may be that Strotz's contribution is still remembered so long after its appearance primarily because the framework in which he formulated the issue of time inconsistency remains, for the most part, that in which it is studied today. His paper considered a choice at (variable) time τ among feasible consumption plans C with the goal of maximizing a utility function of the form [p. 167]

$$\Phi_\tau = \int_0^T \lambda(t-\tau)u[C(t),t]\mathrm{d}t, \tag{1}$$

and with the statement of this problem he placed the theoretical innovation of changing preferences squarely within the context of what were to become two of its most important applications: the analyses of personal savings and national investment. Strotz showed that in this setting time consistency of the decision maker's behavior is equivalent to log-linearity of the discounting function (a specification of λ that he dubbed [p. 172] "the harmony case"), thereby also demonstrating what is needed to model a time-*inconsistent* decision maker. And he also introduced the important distinction between the cognitive assumptions of "sophistication" (i.e., awareness of any inconsistency) and "naivete" (the absence of such awareness), discussing each of these two possibilities in turn.[1]

If the first few papers to follow up on Strotz's essay (namely, those of Pollak [24], Peleg and Yaari [22], and Goldman [10]) dealt primarily with the sophisticated agent, it is only because his analysis of this case [pp. 173–175] was soon found to be defective (see [24, pp. 207–208]) and led to an extended and productive debate about the nature of strategic equilibrium in dynamic "intrapersonal" games (a question addressed more recently by Asheim [2]). Less easy to understand is why, when applied work incorporating time inconsistency finally began to appear — with notable contributions by Laibson [15, 16], Barro [3], Harris and Laibson [11, 12], and Krusell and Smith [14] — sophistication continued to be imposed with at most a cursory acknowledgement that there might be an alternative. This custom seems to have originated in [15], where Laibson declared [p. 451] that it had become "standard practice to formally model a consumer as a sequence of temporal selves making choices in a dynamic game (e.g., Pollak [1968], Peleg and Yaari [1973], and Goldman [1980])." But the papers cited here are, of course, precisely the three mentioned above in connection with the *purely theoretical* project of correcting the defective Strotzian analysis; these authors do not even take up the question of whether the sophisticated agent is an appropriate modelling device for any particular applied problem; and thus it appears that assuming sophistication has *become* standard practice simply by being *described* as such, and with less than adequate consideration of its relative merits vis-a-vis naivete in the economic contexts of interest.

In the context of savings behavior, Laibson does offer the argument that observed large holdings of illiquid assets constitute evidence that investors sometimes [15, p. 444] "prefer to constrain their own future choices"; and as examples of such "golden egg" investments he mentions real estate, business equity, durable goods, pensions, and other retirement (e.g., IRA, Keogh, and 401k) plans. There is undoubtedly some truth to this claim, though Laibson himself concedes there to be other reasons to purchase a home or a Honda than merely [p. 443] to remove money from one's bank account before one has

[1]This terminology is apparently due to Pollak [24].

a chance to spend it. And rather than as intrapersonal commitment devices designed by sophisticates to achieve self-control, it may be more realistic to interpret employer-matched savings and tax-advantaged retirement plans as incentive schemes designed by paternalistic employers and governments to manipulate the behavior of their employees and constituents.

Two excellent points of departure for thinking about these issues are the 1984 and 1991 Richard T. Ely lectures delivered, respectively, by Thomas Schelling [27] and George Akerlof [1]. The first deals exclusively with sophistication, the second with naivete, and the two together suffice to dismiss any claim that one or the other assumption is universally valid.[2] We shall not attempt here to summarize the conclusions of these lectures, which merit being read in their entirety. But it is worth pointing out that while the problem of intertemporal resource allocation is mentioned only in passing [p. 6] in the course of Schelling's engaging tour of sophisticated behavior patterns, it features prominently in Akerlof's [pp. 6–7] discussion of naive decision making.

Yet another source of insightful commentary on Strotz's cognitive dichotomy is the more recent work of O'Donoghue and Rabin [19, 20, 21] on the opposing phenomena of procrastination and preproperation. In the first of the cited papers, these authors take on the task of [19, p. 104] "explicitly comparing [the two] competing assumptions," and having done so warn [p. 119] that while "[p]eople clearly have some degree of sophistication, ... economists should be cautious when [working] solely with [this] assumption." At the same time, O'Donoghue and Rabin observe that presuming awareness of any changing preferences does seem entirely natural from the perspective of rational choice orthodoxy; indeed, another eminent economic theorist has suggested [personal communication] that conscious or unconscious imperatives of professional self-preservation have had as much to do with the traditional bias towards sophistication as any reasoned argument in its favor.

Our intention in this paper, however, is not to adjudicate territorial disputes between the two assumptions in question — this being more properly a task for empirical or experimental work informed by cognitive psychology. Rather, our aim is to provide an alternative to the methodology based on sophistication by investigating the pure theory of the naive agent; understanding, of course, that any satisfactory picture of the impact of time inconsistency on human decision making is likely to combine elements of both approaches.[3] Happily, filling the theoretical vacuum left by the neglect of the naivete assumption will require simply that we carry out the analysis of this case hinted at decades ago by Strotz himself, and for this reason the reader may wish to glance at the relevant section of [28, pp. 170–171].

[2]Schelling writes [p. 1, emphasis added] that "a person in evident possession of her faculties and knowing what she is talking about will rationally seek to prevent, to compel, or to alter her own later behavior — to restrict her own options in violation of *what she knows will be her preference* at the time the behavior is to take place." Akerlof, with other situations in mind, declares [p. 17, emphasis added] that a "modern view of behavior, based on twentieth-century anthropology, psychology, and sociology is that individuals have utilities that do change and, in addition, *they fail fully to foresee those changes* or even recognize that they have occurred."

[3]O'Donoghue and Rabin [21] have already taken a step in this direction by allowing for what they call "partial naivete."

More concretely, our goal is to determine, insofar as possible, the behavior of a naive planner faced with a generalized version of the Strotzian capital management problem mentioned above. At each moment, taking into account the level of the capital stock inherited from earlier moments, such an agent will identify that feasible consumption plan which is optimal with respect to his current preferences — a plan that we too can identify by solving the ordinary differential equation (ODE) that constitutes the relevant first-order condition for optimality. Failing to anticipate any impending change in his own outlook, the planner will use this currently-optimal plan to choose his instantaneous rate of consumption (or, equivalently, the instantaneous rate of change of his stock of capital). The resulting relationship between the level of the capital stock (appearing in the current resource constraint) and its time derivative (essentially the current choice variable) can then be captured in a second ODE, and it is by solving *this* equation that we can ascertain the actual ("historical") capital and consumption paths.

Starting with a completely nonparametric capital management problem, we shall carry out the analysis sketched above by adopting at each stage the weakest set of functional-form assumptions that will allow us to proceed. This method will lead naturally to a sequence of characterizations of the planner's behavior of ever-increasing strength: first, in Section 2.3, as the output of a mathematical algorithm presented schematically; second, in Section 3.1, as the solution to an initial value problem; third, in Sections 3.2 and 5.2, via explicit formulae for the historical capital path; and fourth, in Section A.1, via formulae expressible in closed form. Together with the brief discussion of computational feasibility to be found in Section 4, these characterizations are intended to delineate a sort of "analytical possibilities frontier" — to acquaint the reader with approximate rates of substitution between generality and tractability in the environment under investigation. And in addition to providing a basis for theoretical work seeking to extend this frontier, it is hoped that the results reported here will prove useful to applied economists wishing to incorporate time-inconsistent preferences into models of, for example, financial markets or the macroeconomy.

2. NAIVE CAPITAL MANAGEMENT

2.1. The planner's problem

The *calendar date* t will advance as a continuous variable from the initial date 0 through a fixed planning horizon $T > 0$. At calendar date t, the *scheduling date* s will range over the interval $[t, T]$, allowing the planner to consider both his present and future behavior. Each possible pattern of (anticipated future) consumption will be encoded in a *consumption schedule* $x : [t, T] \to \Re$, and the planner's date-t preferences over candidate schedules will be presumed to admit a utility representation of the form

$$U^t := \int_t^T \delta(s, t) u(x(s), s) \mathrm{d}s \tag{2}$$

(cf. Equation 1) for some strictly positive-valued *discounting function* δ and some *index function* u that is both strictly increasing and strictly concave in its first (consumption) argument.[4]

Note that the representation in Equation 2 imposes additive separability of the total utility into increments contributed by the various scheduling dates, as well as multiplicative separability of the increment contributed by date s into a *discount factor* $\delta(s,t)$ independent of the rate of consumption and a *utility index* $u(x(s),s)$ independent of the calendar date.[5] The *discount rate* applied at calendar date t to scheduling date s can be calculated via the relation

$$\rho(s,t) := -\delta_1(s,t)/\delta(s,t) \tag{3}$$

(where the subscript denotes differentiation with respect to the indicated argument). And the convenient normalization $\delta(t,t) = 1$ then leads to the inverse relation

$$\delta(s,t) = \exp \int_t^s [-\rho(v,t)]\, \mathrm{d}v. \tag{4}$$

Given a consumption schedule x, the associated *capital schedule* k will evolve according to the differential equation

$$k_1(s) = f(k(s),s) - x(s), \tag{5}$$

where the *technology* f supplying the anticipated return on accumulated capital will be assumed to be both weakly increasing and weakly concave in its first (capital stock) argument.[6] Using Equation 5 to change variables, we can express the planner's date-t objective function as

$$U^t = \int_t^T \delta(s,t) u(f(k(s),s) - k_1(s), s) \mathrm{d}s, \tag{6}$$

to be maximized now by choice of the capital path k.

Our goal is to determine the *history* h of the capital stock over the domain $[0,T]$. At date t, this history will constrain the planner's choice of k to among those satisfying

$$k(t) = \begin{cases} h(0) = K & \text{for } t = 0, \\ h(t) & \text{for } t > 0; \end{cases} \tag{7}$$

and we shall impose also the terminal condition

$$k(T) = 0. \tag{8}$$

(See Figure 1.) Hence, in summary, the planner's problem at calendar date t is to select a capital path k that maximizes the objective function in Equation 6 subject to the constraints in Equations 7–8.

[4] Similar representations were proposed by Samuelson [26] and axiomatized by Koopmans [13] and Fishburn and Rubinstein [7]. Frederick et al. [8] provide an extensive review of theoretical, experimental, and observational studies relating to this functional form.

[5] Thus, as Strotz puts it [p. 168], the planner can express his enthusiasm for consuming from his stock of champagne on the date of his birth by assigning a high value to the utility index u(two glasses of champagne per diem, planner's birth date), but this enthusiasm can neither increase nor decrease as the date approaches.

[6] Note that, as implied by Equation 5, capital is consumable ("like rabbits," according to Phelps and Pollak [23, p. 187]).

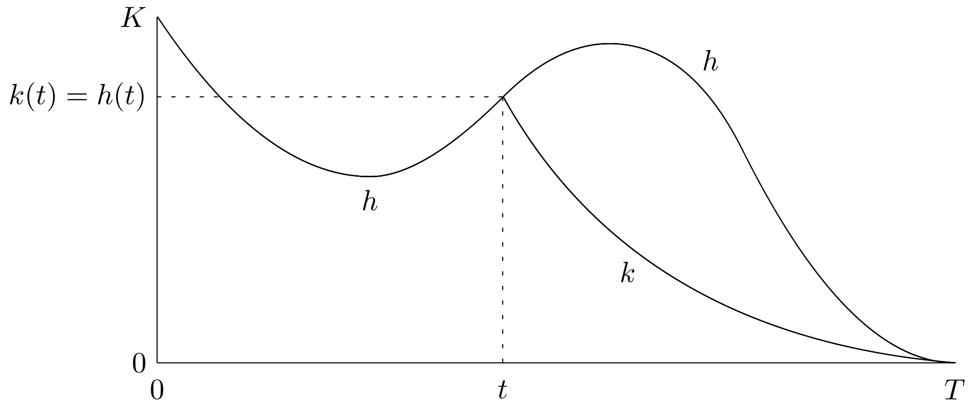

Figure 1: Equations 7–8. A candidate capital path k considered at calendar date t must connect the points $\langle t, h(t)\rangle$ and $\langle T, 0\rangle$.

2.2. The question of consistency

Strotz inquires [p. 171]: "Under what circumstances will an individual who continuously re-evaluates his planned course of consumption confirm his earlier choices and follow out the consumption plan originally selected?" Or, in our terminology: Under what conditions will the solutions to the planner's problem at different calendar dates coincide?

The answer to this question (noted already by Burness [5]) is that coincidental plans will arise if and only if there exists a function σ satisfying

$$\delta(s,t) = \sigma(s)/\sigma(t); \tag{9}$$

that is, if and only if the discounting function is multiplicatively *separable*.[7] To see that this property is necessary (its sufficiency is immediate), observe that any capital schedule k^t rendering U^t stationary must satisfy the Euler equation

$$\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)f_1(k^t(s),s) + \frac{\mathrm{d}\left[\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)\right]}{\mathrm{d}s} = 0 \tag{10}$$

at each $s \in [t,T]$.[8] Equation 10 has first integral

$$\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)\exp\int_s^T \left[-f_1(k^t(v),v)\right]\mathrm{d}v = \tau(t), \tag{11}$$

where the constant $\tau(t)$ can be interpreted as the reciprocal of the shadow price of utility in terms of date T consumption. And if the schedule k^0 remains optimal at each calendar date $t \in (0,T)$ then we can let

$$\sigma(s) = \left[u_1(f(k^0(s),s) - k_1^0(s),s)\exp\int_s^T \left[-f_1(k^0(v),v)\right]\mathrm{d}v\right]^{-1}, \tag{12}$$

[7] Separability is equivalent to the *calendar invariance* condition that, for each $0 \le t < \bar{t} \le s \le T$ and $\epsilon > 0$, we have $\delta(s+\epsilon,t)/\delta(s,t) = \delta(s+\epsilon,\bar{t})/\delta(s,\bar{t})$.

[8] See [6, pp. 28–36], [9, pp. 14–18], or [18, pp. 179–183]. Equation 10 is in fact necessary and sufficient for the optimality of k^t since $G(k(s),k_1(s),s) = \delta(s,t)u(f(k(s),s) - k_1(s),s)$ is concave in $\langle k(s),k_1(s)\rangle$. This concavity follows, in turn, from the inequalities $G_{11} = \delta\left[u_{11}f_1^2 + u_1 f_{11}\right] \le 0$, $G_{22} = \delta u_{11} \le 0$, and $G_{11}G_{22} - G_{12}G_{21} = \delta^2 u_1 u_{11} f_{11} \ge 0$. (See [6, pp. 81–91].)

whereupon Equation 9 holds since $\tau(t) = \delta(t,t)/\sigma(t) = 1/\sigma(t)$.

With regard to the representation in Equation 2, it is common to assume (as does Strotz) that the discounting function is *stationary*; i.e., that there exists a function σ such that

$$\delta(s,t) = \sigma(s-t).^{9} \tag{13}$$

Under this assumption, the planner's point of view undergoes a rigid translation with the advance of the calendar date, and in a sense the future never actually arrives at the present.[10]

A third possible assumption is that of *log-linearity*, a property that δ is said to exhibit whenever it admits a function σ such that

$$\delta(s,t) = \exp\left[-\sigma(t)[s-t]\right].^{11} \tag{14}$$

Such a function, when it exists, supplies the common discount rate applied by the planner at a given calendar date to all future scheduling dates.

The latter two assumptions on the discounting function are together sufficient for intertemporal consistency; indeed, any two of separability, stationarity, and log-linearity jointly imply the third. Thus standard, "exponential" discounting (which satisfies all three assumptions) is the unique stationary specification of δ that leads to time-consistent planning.

2.3. Schematic analysis of the general case

When the discounting function is inseparable and hence intertemporal consistency fails, the observed behavior will generally have the "unpleasant feature," pointed out by Blackorby et al. [4, p. 239], that "*ex post* [it] makes no sense from any point of view." The reason for this unpleasantness is that while at each calendar date t our naive planner will be following a capital schedule k^t that appears for the moment to be optimal, he will adhere to this plan only for an instant and will soon find that he prefers a new "optimal" schedule $k^{t+\mathrm{d}t}$. Accordingly, although the choice of k^t is made with each future scheduling date taken into consideration, this choice affects the history h only by determining the instantaneous rate

$$h_1(t) = k_1^t(t) \tag{15}$$

of capital accumulation at calendar date t. (See Figure 2.)

Using Equations 7 and 15 to calculate

$$\tau(t) = u_1(f(h(t),t) - h_1(t),t)\exp\int_t^T \left[-f_1(k^t(v),v)\right] \mathrm{d}v \tag{16}$$

[9] Stationarity is equivalent to the *absolute invariance* condition that, for each $0 \leq t < \bar{t} \leq T$ and $\epsilon > 0$, we have $\delta(t+\epsilon,t) = \delta(\bar{t}+\epsilon,\bar{t})$.

[10] Samuelson [26, p. 160] writes that "as the individual moves along in time there is a sort of perspective phenomenon in that his view of the future in relation to his instantaneous time position remains invariant, rather than his evaluation of any particular year." (See also the commentary, cited by Strotz [p. 170], of Rosenstein-Rodan [25].)

[11] Log-linearity is equivalent to the *scheduling invariance* condition that, for each $0 \leq t \leq s < \bar{s} \leq T$ and $\epsilon > 0$, we have $\delta(s+\epsilon,t)/\delta(s,t) = \delta(\bar{s}+\epsilon,t)/\delta(\bar{s},t)$.

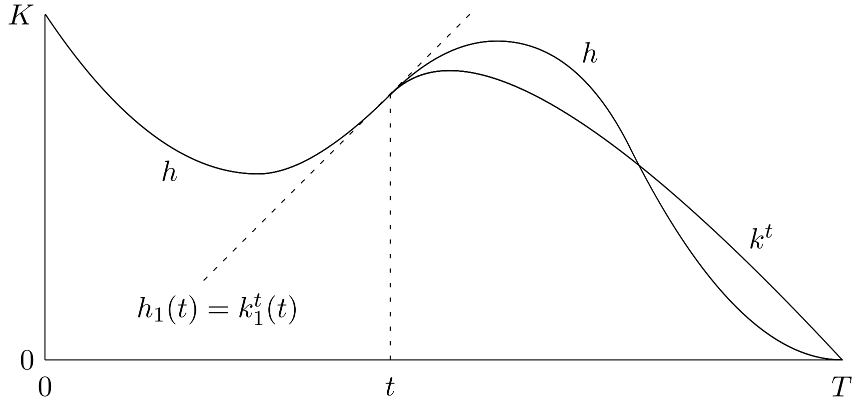

Figure 2: Equation 15. At calendar date t, the history h must be tangent to the optimal capital schedule k^t.

allows us to rewrite Equation 11 as

$$\underbrace{\delta(s,t)u_1(f(k^t(s),s) - k^t_1(s),s)}_{\text{marginal utility of consumption}} \underbrace{\exp\int_t^s f_1(k^t(v),v)\mathrm{d}v}_{\text{price deflator}} = u_1(f(h(t),t) - h_1(t),t), \tag{17}$$

a first-order ODE in k^t that in combination with the terminal condition

$$k^t(T) = 0 \tag{18}$$

can in principle be solved to yield an expression of the form

$$k^t(s) = \Xi(s,t,h(t),h_1(t)) \tag{19}$$

valid for $s \in [t,T]$. And finally, setting $s = t$ in Equation 19 leads to the relation

$$h(t) = \Xi(t,t,h(t),h_1(t)), \tag{20}$$

a first-order ODE in h that in combination with the initial condition

$$h(0) = K \tag{21}$$

can (again, in principle) be solved to yield the historical path of the capital stock.

3. ANALYTICAL RESULTS

3.1. Linear technologies

Section 2.3 provides a recipe for computing the history h from the primitives of the planner's problem; namely, the discounting function δ, the index function u, and the technology f. As we have seen, this computation requires the solution, in sequence, of two first-order ODEs, the first (Equation 17) determining k^t as a function of the scheduling date and the second (Equation 20) determining h as a function of the calendar date. The immediate difficulty in carrying out these tasks is that the unknown path k^t in

Equation 17 affects both the marginal utility of consumption and the price deflator, and may therefore be in a nonlinear relationship with its derivative k_1^t. Linearity of this ODE (a property sufficient but perhaps not necessary for analytical solvability) can be assured, however, if the technology is itself linear; that is, if

$$f(z,s) = \alpha(s)z + \beta(s) \tag{22}$$

for given functions α and β supplying, respectively, an "interest rate" and a flow of "exogenous income." In this case the price deflator

$$\pi(s,t) := \exp \int_t^s \alpha(v) \mathrm{d}v \tag{23}$$

is independent of the optimal capital path, and since $u_1(\cdot, s)$ is invertible (being strictly decreasing) we can put Equation 17 into the standard form

$$k_1^t(s) - \alpha(s)k^t(s) = \beta(s) - u_1(\cdot, s)^{\mathrm{inv}} \left[\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(s,t)\pi(s,t)} \right], \tag{24}$$

with integrating factor

$$\exp \int_0^s [-\alpha(v)]\, \mathrm{d}v = \pi(s,0)^{-1}. \tag{25}$$

Solving the terminal value problem formed by Equations 18 and 24 then leads to the formula

$$k^t(s) = \int_s^T \left[u_1(\cdot, v)^{\mathrm{inv}} \left[\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\pi(v,t)} \right] - \beta(v) \right] \frac{\mathrm{d}v}{\pi(v,s)} \tag{26}$$

for k^t over the domain $[t, T]$, and hence to the expression

$$h(t) = \int_t^T \left[u_1(\cdot, v)^{\mathrm{inv}} \left[\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\pi(v,t)} \right] - \beta(v) \right] \frac{\mathrm{d}v}{\pi(v,t)} \tag{27}$$

for the relation in Equation 20.

3.2. Tractable index functions

Exponential utility. We have seen that the assumption of a linear technology reduces the problem of determining the history of the capital stock to that of solving the initial value problem formed by Equations 21 and 27. Again we shall focus on parameterizations for which the latter is a *linear* ODE, and one of two such cases arises when, for some strictly positive-valued mapping γ, the planner's index function has the exponential specification

$$u(z,s) = -\gamma(s)\exp[-z/\gamma(s)], \tag{28}$$

with associated marginal utility

$$u_1(z,s) = \exp[-z/\gamma(s)] \tag{29}$$

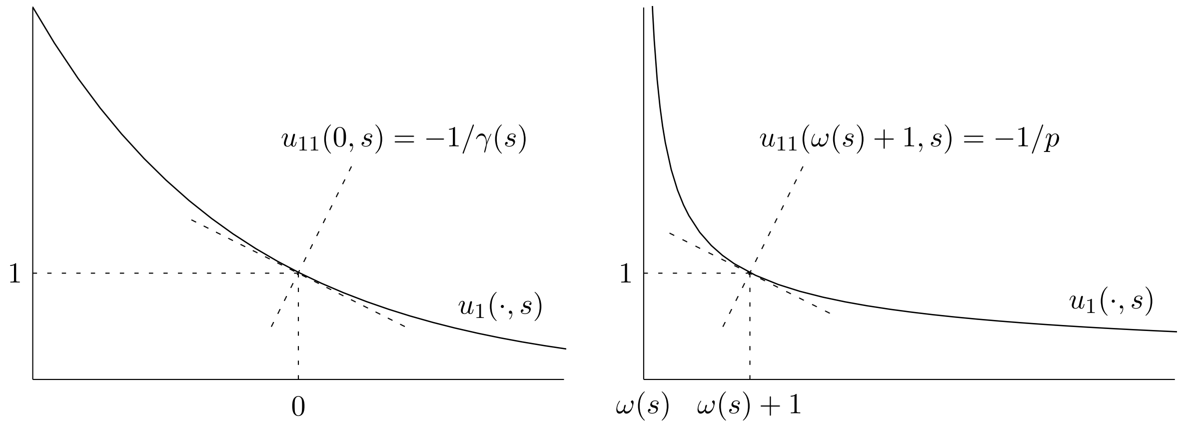

Figure 3: Marginal utility curves for the exponential (left panel) and basal (right panel) index specifications. Note that each curve is strictly decreasing and asymptotes to zero. While exponential utility permits any rate of consumption at scheduling date s, basal utility requires a rate of at least $\omega(s)$. The parameters $\gamma(s)$ and p equal, respectively, the slopes of the lines normal to the marginal utility curve at 0 for the exponential case and at $\omega(s)+1$ for the basal case.

(see Figure 3) and elasticity of intertemporal substitution

$$\varepsilon(z,s) := \frac{-u_1(z,s)}{z u_{11}(z,s)} = \gamma(s)/z. \tag{30}$$

In this case Equation 27 evaluates to

$$h(t) = [\alpha(t)h(t) + \beta(t) - h_1(t)] \int_t^T \frac{\gamma(v)\mathrm{d}v}{\gamma(t)\pi(v,t)} \cdots$$
$$\cdots + \int_t^T \frac{[\gamma(v)\log[\delta(v,t)\pi(v,t)] - \beta(v)]\,\mathrm{d}v}{\pi(v,t)} \tag{31}$$

and with the definition

$$\Theta(t) := \left[\int_t^T \frac{\gamma(v)\mathrm{d}v}{\gamma(t)\pi(v,t)}\right]^{-1} \tag{32}$$

can be put into the standard form

$$h_1(t) + [\Theta(t) - \alpha(t)]\,h(t) = \beta(t) + \Theta(t)\int_t^T \frac{[\gamma(v)\log[\delta(v,t)\pi(v,t)] - \beta(v)]\,\mathrm{d}v}{\pi(v,t)}, \tag{33}$$

with integrating factor

$$\exp\int_0^t [\Theta(v) - \alpha(v)]\,\mathrm{d}v = \pi(t,0)^{-1}\exp\int_0^t \Theta(v)\mathrm{d}v. \tag{34}$$

The initial value problem in question is then solved by the capital history

$$h(t) = K\pi(t,0)\exp\int_0^t [-\Theta(w)]\,\mathrm{d}w + \int_0^t \pi(t,v)\exp\int_v^t [-\Theta(w)]\,\mathrm{d}w \cdots$$
$$\cdots \times \left[\beta(v) + \Theta(v)\int_v^T \frac{[\gamma(w)\log[\delta(w,v)\pi(w,v)] - \beta(w)]\,\mathrm{d}w}{\pi(w,v)}\right]\mathrm{d}v. \tag{35}$$

Basal utility. The second case in which Equation 27 is linear arises when, for some real-valued mapping ω and some parameter $p > 0$, the planner's index function has the basal specification

$$u(z,s) = \begin{cases} [1-1/p]^{-1}[z-\omega(s)]^{1-1/p} & \text{for } p \neq 1, \\ \log[z-\omega(s)] & \text{for } p = 1; \end{cases} \tag{36}$$

with associated marginal utility

$$u_1(z,s) = [z-\omega(s)]^{-1/p} \tag{37}$$

(again see Figure 3) and elasticity of intertemporal substitution

$$\varepsilon(z,s) = p\,[1-\omega(s)/z]\,. \tag{38}$$

In this case Equation 27 evaluates to

$$h(t) = [\alpha(t)h(t) + \beta(t) - \omega(t) - h_1(t)] \int_t^T \frac{[\delta(v,t)\pi(v,t)]^p \,\mathrm{d}v}{\pi(v,t)} \cdots$$
$$\cdots - \int_t^T \frac{[\beta(v) - \omega(v)]\,\mathrm{d}v}{\pi(v,t)} \tag{39}$$

and with the definition

$$\Psi(t) := \left[\int_t^T \delta(v,t)^p \pi(v,t)^{p-1} \mathrm{d}v\right]^{-1} \tag{40}$$

can be put into the standard form

$$h_1(t) + [\Psi(t) - \alpha(t)]\,h(t) = \beta(t) - \omega(t) - \Psi(t) \int_t^T \frac{[\beta(v) - \omega(v)]\,\mathrm{d}v}{\pi(v,t)}, \tag{41}$$

with integrating factor

$$\exp \int_0^t [\Psi(v) - \alpha(v)]\,\mathrm{d}v = \pi(t,0)^{-1} \exp \int_0^t \Psi(v)\mathrm{d}v. \tag{42}$$

The initial value problem formed by Equations 21 and 41 is then solved by the capital history

$$h(t) = K\pi(t,0) \exp \int_0^t [-\Psi(w)]\,\mathrm{d}w + \int_0^t \pi(t,v) \exp \int_v^t [-\Psi(w)]\,\mathrm{d}w \cdots$$
$$\cdots \times \left[\beta(v) - \omega(v) - \Psi(v) \int_v^T \frac{[\beta(w) - \omega(w)]\,\mathrm{d}w}{\pi(w,v)}\right] \mathrm{d}v. \tag{43}$$

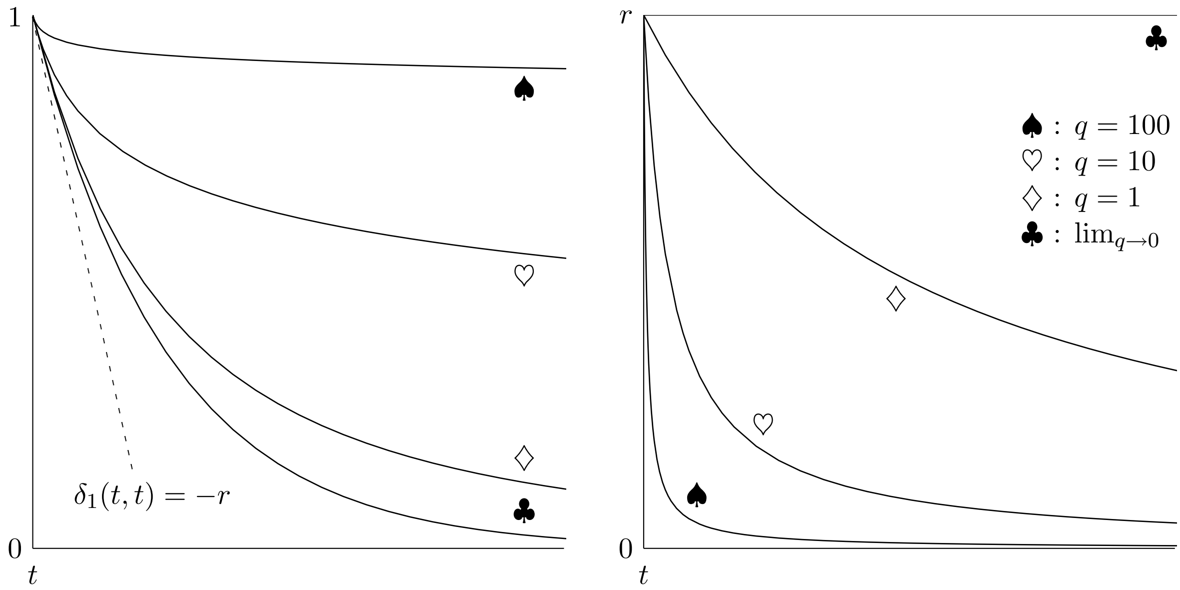

Figure 4: Hyperbolic discount factor (left panel) and discount rate (right panel) curves for fixed r and variable q. (Cf. [17, p. 581].) Since $\lim_{q\to 0} \delta(s,t) = e^{-r[s-t]}$, the parameter q controls the degree of distortion relative to exponential discounting with discount rate r.

3.3. Hyperbolic discounting

Equations 35 and 43 provide explicit formulae for the historical path of a capital stock managed by Strotz's naive planner under the assumptions of a linear technology and either exponential or basal utility. It should be noted that these results impose no conditions whatsoever on the discounting function (which need not even be stationary), and that in this respect the naive planning model provides a convenient laboratory for examining nonstandard functional forms for δ.

One notable nonstandard form is the stationary hyperbolic specification

$$\delta(s,t) = [1 + q\,[s-t]]^{-r/q} \tag{44}$$

advocated by Loewenstein and Prelec [17], among others. (See Figure 4.) When $q, r > 0$, the associated discount rate function

$$\rho(s,t) = r\,[1 + q\,[s-t]]^{-1} \tag{45}$$

is decreasing in the scheduling date — a phenomenon often generated in discrete-time models using the "quasi-hyperbolic" (or [14] "quasi-geometric") specification originally proposed by Phelps and Pollak [23]. And writing $r = \rho(t,t)$ and $q = -\rho_1(t,t)/\rho(t,t)$ suggests interpreting these parameters of the hyperbolic form as short-term discount rates of, respectively, first and second order.

3.4. Cake-eating problems

Within the class of planning problems defined in Section 2.1, the simplest are those in which $f(z,s) = 0$ (a trivially linear technology). Here the planner's task amounts to scheduling over the interval $[0,T]$ his consumption of a non-perishable "cake" of size K.

If we adopt stationary hyperbolic discounting, then imposing exponential utility with constant intertemporal substitution parameter $\gamma(s) = c > 0$ leads via Equation 35 to the capital history

$$\begin{aligned} h(t) &= K[1-t/T] + \int_0^t \frac{T-t}{[T-v]^2} \int_v^T [-cr/q] \log\left[1+q\left[w-v\right]\right] \mathrm{d}w\mathrm{d}v \\ &= [T-t]\left[\frac{K}{T} - \frac{cr}{q}\int_{qT}^{q[T-t]} \frac{[\bar{v} - [1+\bar{v}]\log[1+\bar{v}]]\,\mathrm{d}\bar{v}}{\bar{v}^2}\right] \\ &= [T-t]\left[\frac{K}{T} - \frac{cr}{q}\left[\frac{[1+\bar{v}]\log[1+\bar{v}]}{\bar{v}} + \mathrm{Li}^2[-\bar{v}]\right]_{\bar{v}=qT}^{q[T-t]}\right], \end{aligned} \tag{46}$$

where the polylogarithm operator

$$\mathrm{Li}^n z := \sum_{i=1}^{\infty} z^i i^{-n} \tag{47}$$

— ordinarily treated as a known function — generalizes the natural logarithm in the sense that $\log z = \mathrm{Li}^1[1-1/z]$. Alternatively, imposing basal utility with $\omega(s) = 0$ (and $p \neq q/r$) leads via Equation 43 to the history

$$h(t) = K \exp \int_0^t \frac{[q-pr]\mathrm{d}w}{1-[1+q[T-w]]^{1-pr/q}}, \tag{48}$$

which resists further simplification. Note that as $q \searrow 0$ and hyperbolic discounting tends to exponential, these results tend to the textbook cake-eating formulae

$$h(t) = [T-t]\,[K/T - crt/2] \tag{49}$$

for the exponential and

$$h(t) = K\left[\frac{1-e^{pr[T-t]}}{1-e^{prT}}\right] \tag{50}$$

for the basal case (cf. [24, p. 206], [18, p. 182], and [29, p. 547 ff.]).

The above calculations demonstrate that even in the simplest of capital management problems, evaluating the integrals in Equations 35 and 43 to obtain closed-form histories may require the use of non-elementary functions or may be altogether infeasible. And as shown in Section A.1, our experience here turns out to be typical: While closed-form paths can be obtained for a broad class of problems under exponential utility, the same cannot be said of basal utility.

4. COMPUTATIONAL STRATEGIES

4.1. Numerical integration

Whether or not the formulae in Equations 35 and 43 can be expressed in closed form, the corresponding linear/exponential and linear/basal capital histories can be approximated with a high degree of precision by computing the relevant integrals numerically. This

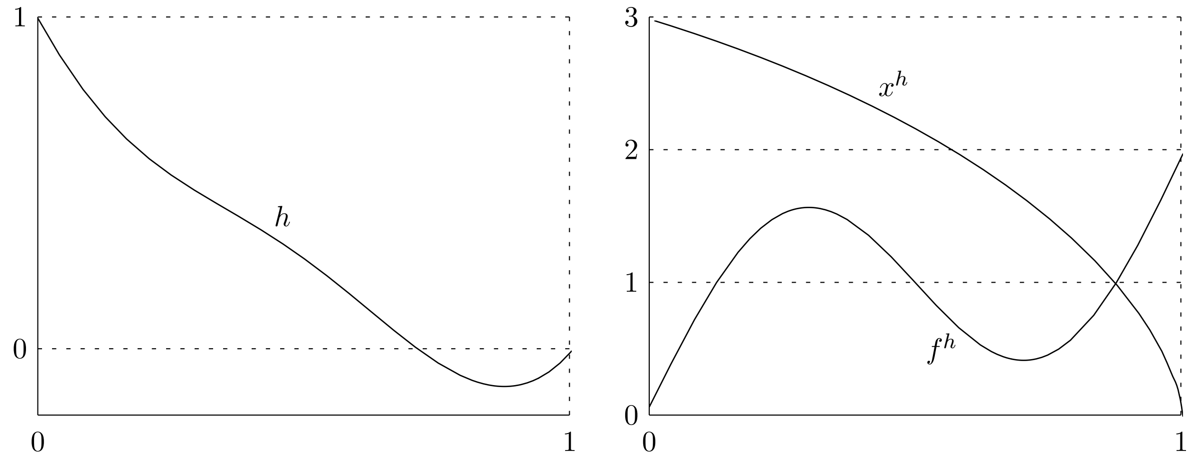

Figure 5: Capital history (left panel) and consumption and income histories (right panel) for a planning problem with a linear technology, basal utility, and bi-level discounting; computed by numerical integration.

strategy is easily implemented in *Mathematica* using the built-in function `NIntegrate`, though the successive integrals being nested (i.e., involving dummy variables as limits of integration) rather than multiple does keep the exercise from being entirely trivial. The latter complication can be dealt with by making use of `InterpolatingFunction` objects to record intermediate approximations (e.g., to the functions Θ and Ψ). And as these interpolations are carried out on a finer and finer temporal grid, the overall approximation error can be presumed to become arbitrarily small.

To demonstrate the capabilities of the numerical integration strategy, Figure 5 plots the capital history h, the consumption history

$$x^h(t) := f(h(t), t) - h_1(t), \tag{51}$$

and the income history

$$f^h(t) := f(h(t), t) \tag{52}$$

for the capital management problem with parameters $T = K = 1$, linear technology

$$f(z, s) = z/10 + 2s + \sin[2\,\mathrm{pi}\, s], \tag{53}$$

basal index function $u(z, s) = 2z^{1/2}$, and (nonstationary) bi-level discounting function

$$\delta(s, t) = \begin{cases} 1 & \text{for } s \le [T + t]/2, \\ 1/2 & \text{for } s > [T + t]/2. \end{cases} \tag{54}$$

Here the planner's consumption — which decreases at an increasing rate from 3 to 0 over the interval $[0, 1]$ — is funded at first from his initial endowment $h(0) = 1$ and his small investment income $\alpha(t)h(t) = h(t)/10$, later with substantial contributions from his exogenous income $\beta(t) = 2t + \sin[2\,\mathrm{pi}\, t]$, and towards the end using a loan to reconcile decreasing outflows with increasing inflows.[12]

[12]Note that the initial endowment is measured in units of capital, while income is measured in units of capital per unit time.

4.2. *Numerical optimization*

When the technology f is nonlinear or the index function u is neither exponential nor basal, we can still attempt to approximate the capital history h by employing the method of finite differences (see, e.g., [9, p. 4]) together with numerical optimization. Dividing the interval $[0, T]$ into $m+1$ parts of equal length $\Delta := T/[m+1]$, this strategy uses the objective function

$$\hat{U}^t = \sum_{s=t+\Delta}^{T} \delta(s,t)u\left(f(k(s),s) - [k(s) - k(s-\Delta)]/\Delta, s\right)\Delta \qquad [s \text{ increment} = \Delta] \tag{55}$$

to set up a version of the planner's problem at date $t \propto \Delta$ with finitely many choice variables.[13] This discretized problem can then be fed into *Mathematica*'s built-in function `NMaximize`, and the resulting solution vector $\hat{k}^t$ used to assign $\hat{h}(t+\Delta) = \hat{k}^t(t+\Delta)$. When the discretization parameter m is sufficiently large, we can hope that an interpolation to the vector $\hat{h}$ of estimates constructed recursively in this fashion will approximate the unknown history h with negligible error.

While numerical optimization is (at least potentially) widely applicable, its drawbacks are many: Firstly, solving m maximization problems (the first in m variables, the second in $m-1$ variables, and so on) begins to require a substantial amount of either time or computing power as the parameter m, and hence the accuracy of the method, increases. Secondly, the recursive nature of the calculation presumably causes the approximation error to grow as the calendar date advances. And finally, we have no guarantee that `NMaximize` will always return correct solutions, or even any solutions at all. (Indeed, problems involving nonlinear technologies seem to present a particular challenge in this respect.)

As an application of the numerical optimization strategy, Figure 6 plots the histories h, x^h, and f^h for the capital management problem with parameters $T = 1$ and $K = 0$, linear technology

$$f(z,s) = z/2 + 3/2 - \sin[\mathrm{pi}\, s], \tag{56}$$

arctangential index function $u(z,s) = \arctan[2z]$, and hyperbolic discounting function $\delta(s,t) = [1+s-t]^{-1}$. Here the planner transfers inflows from good times to bad by at first saving and then borrowing, and in doing so achieves a consumption history that is quite smooth in comparison with his sinusoidal exogenous income.

5. CREDIT CONSTRAINTS

5.1. *Discussion of the general case*

The planner's problem defined in Section 2.1 imposes the boundary conditions listed in Equations 7–8, but does not constrain the choice of capital path k on the interior of the interval $[t, T]$ relevant at calendar date t. More specifically, this formulation endows the planner with an infinite capacity to borrow that we may in some circumstances wish to

[13] Since $k(t)$ and $k(T)$ are fixed by boundary conditions, the problem in question involves precisely $[T-t]/\Delta - 1$ such variables.

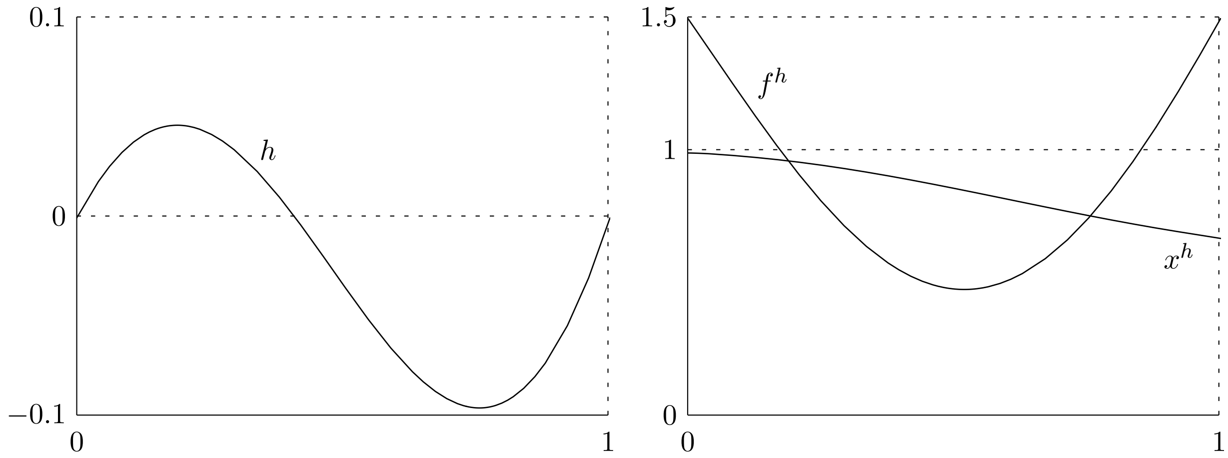

Figure 6: Capital history (left panel) and consumption and income histories (right panel) for a planning problem with a linear technology, arctangential utility, and hyperbolic discounting; computed by numerical optimization.

restrict via the requirement that

$$k(s) \geq g(s) \tag{57}$$

for each $s \in [t, T]$ and a given *credit limit* g satisfying $g(T) = 0$. Introducing such a constraint obliges us to substitute for U^t (see Equation 6) the Lagrangian maximand

$$L^t = \int_t^T [\delta(s,t)u(f(k(s),s) - k_1(s),s) + \mu(s)\,[k(s) - g(s)]]\,\mathrm{d}s; \tag{58}$$

and the corresponding Euler equation can be integrated to yield

$$\delta(s,t)u_1(f(k^t(s),s) - k_1^t(s),s)\exp\int_t^s f_1(k^t(w),w)\mathrm{d}w = u_1(f(h(t),t) - h_1(t),t)\cdots$$
$$\cdots - \int_t^s \mu^t(v)\exp\int_t^v f_1(k^t(w),w)\mathrm{d}w\mathrm{d}v \tag{59}$$

(cf. Equation 17), where the weakly positive-valued mapping μ^t returns the time-varying multiplier on the credit constraint at the L^t-extremum. Together with the complementary-slackness condition

$$\mu^t(s)\left[k^t(s) - g(s)\right] = 0, \tag{60}$$

Equation 59 characterizes the constrained-optimal schedule k^t for an arbitrary calendar date t. And with knowledge of these paths, we can in principle compute the history h of the capital stock just as in the unconstrained problem (see Section 2.3).

In the case of a linear technology (see Section 3.1), Equation 59 again takes the form of a linear ODE in k^t that together with the terminal condition in Equation 18 can be solved for the path

$$k^t(s) = \int_s^T \left[u_1(\cdot,v)^{\mathrm{inv}}\left[\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t),t)}{\delta(v,t)\pi(v,t)} - \int_t^v \frac{\mu^t(w)\mathrm{d}w}{\delta(v,t)\pi(v,w)}\right]\cdots\right.$$
$$\left.\cdots - \beta(v)\right]\frac{\mathrm{d}v}{\pi(v,s)} \tag{61}$$

(cf. Equation 26). But if we proceed to use this formula to eliminate k^t from Equation 60, we shall be left with an extremely complex integral equation in μ^t. And even if it were somehow possible to obtain from this equation an expression of the form

$$\mu^t(s) = \Upsilon(s, t, h(t), h_1(t)), \tag{62}$$

the resulting generalization

$$h(t) = \int_t^T \left[u_1(\cdot, v)^{\mathrm{inv}} \left[\frac{u_1(\alpha(t)h(t) + \beta(t) - h_1(t), t)}{\delta(v,t)\pi(v,t)} - \int_t^v \frac{\Upsilon(w, t, h(t), h_1(t))\mathrm{d}w}{\delta(v,t)\pi(v,w)} \right] \cdots \right.$$
$$\left. \cdots - \beta(v) \right] \frac{\mathrm{d}v}{\pi(v,t)} \tag{63}$$

of Equation 27 would no longer be linear under either exponential or basal utility. It follows — perhaps unsurprisingly — that we cannot solve the credit-constrained version of the planner's problem by analytical methods with the same degree of generality as we can the unconstrained version, and that our techniques will have to be tailored to the particular constrained problems we wish to study.

5.2. Savings-initiation problems

Rather than attempting to characterize behavior in the general credit-constrained capital management problem, let us restrict attention to a simpler class of environments. These are the *savings-initiation* problems satisfying $g(0) = K$ and having the property that for any $\bar{t} < T$ with $h(\bar{t}) > g(\bar{t})$, we have that $k^t(s) > g(s)$ for all $\bar{t} \leq t \leq s < T$. In other words, once the planner finds (at date $\bar{t}$) that his credit constraint is nonbinding, then from this moment on he always plans (at date t) to maintain positive "savings" up until the horizon.[14]

Since in a savings-initiation problem the inequalities $\bar{t} < t < T$ and $h(\bar{t}) > g(\bar{t})$ imply that $h(t) = k^t(t) > g(t)$, there exists (at least when $h - g$ is continuous) a date $t^\star$ up to and including which the credit constraint *always* binds and after which this constraint *never* binds except at date T. Moreover, for any $t \in (t^\star, T)$ the above property implies that $k^t(s) > g(s)$ and hence (by Equation 60) that $\mu^t(s) = 0$ for each $s \in (t, T)$. At $t \geq t^\star$, therefore, Equation 63 reverts to Equation 27 in the linear technology case, and so the latter ODE holds on the interval $[t^\star, T]$. But of course we know also that

$$h(t^\star) = g(t^\star), \tag{64}$$

which sets up an initial value problem solved under exponential utility by the path

$$h(t) = g(t^\star)\pi(t, t^\star) \exp \int_{t^\star}^t [-\Theta(w)]\,\mathrm{d}w + \int_{t^\star}^t \pi(t, v) \exp \int_v^t [-\Theta(w)]\,\mathrm{d}w \cdots$$
$$\cdots \times \left[\beta(v) + \Theta(v) \int_v^T \frac{[\gamma(w) \log [\delta(w, v)\pi(w, v)] - \beta(w)]\,\mathrm{d}w}{\pi(w, v)} \right] \mathrm{d}v \tag{65}$$

[14]Note that this is a property involving not just the primitives of the model but also the derived objects h and k^t. For this reason, we cannot be entirely sure whether or not a particular problem falls into the savings-initiation category prior to analyzing it.

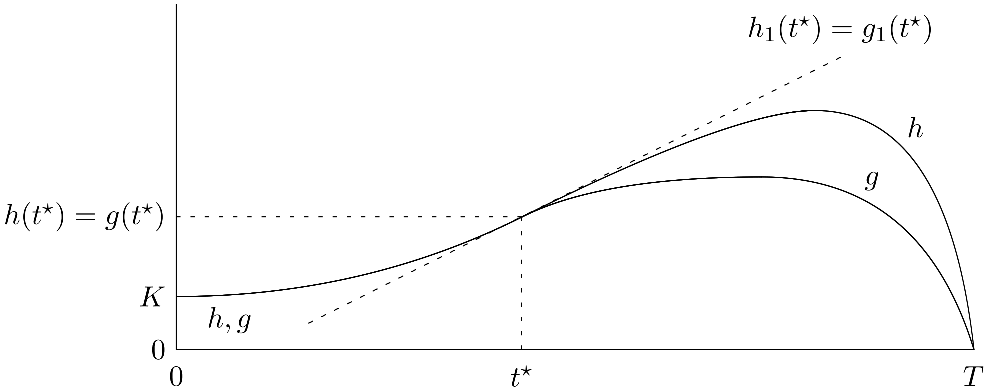

Figure 7: Equations 64 and 68. In a savings-initiation problem, the history h must be both equal and tangent to the credit limit g at date $t^\star$.

(cf. Equation 35) and under basal utility by the path

$$h(t) = g(t^\star)\pi(t,t^\star)\exp\int_{t^\star}^{t}[-\Psi(w)]\,\mathrm{d}w + \int_{t^\star}^{t}\pi(t,v)\exp\int_{v}^{t}[-\Psi(w)]\,\mathrm{d}w\cdots$$
$$\cdots\times\left[\beta(v)-\omega(v)-\Psi(v)\int_{v}^{T}\frac{[\beta(w)-\omega(w)]\,\mathrm{d}w}{\pi(w,v)}\right]\mathrm{d}v \quad (66)$$

(cf. Equation 43). Together with the fact that, by our choice of $t^\star$, we have $h(t) = g(t)$ for each $t < t^\star$, Equations 65–66 supply full linear/exponential and linear/basal capital histories — though there remains the task of locating $t^\star$ itself.

In order to characterize $t^\star$, let us collect what we know about this "savings-initiation date." First of all, from Equation 27 we have the relationship

$$h(t^\star) = \int_{t^\star}^{T}\left[u_1(\cdot,v)^{\mathrm{inv}}\left[\frac{u_1(\alpha(t^\star)h(t^\star)+\beta(t^\star)-h_1(t^\star),t^\star)}{\delta(v,t^\star)\pi(v,t^\star)}\right]-\beta(v)\right]\frac{\mathrm{d}v}{\pi(v,t^\star)}, \quad (67)$$

now to be considered an algebraic equation in $t^\star$ rather than an ODE. We have also Equation 64, which allows us to replace $h(t^\star)$ with $g(t^\star)$ above. And furthermore, our many tacit smoothness assumptions ensure that the instantaneous rate $k_1^t(t)$ of capital accumulation is continuous in t; this continuity implies that

$$h_1(t^\star) = k_1^{t^\star}(t^\star) = \lim_{t\nearrow t^\star} k_1^t(t) = \lim_{t\nearrow t^\star} g_1(t) = g_1(t^\star) \quad (68)$$

(see Figure 7) by Equation 15 and the definition of $t^\star$; and replacing $h_1(t^\star)$ with $g_1(t^\star)$ above then yields an implicit formula

$$g(t^\star) = \int_{t^\star}^{T}\left[u_1(\cdot,v)^{\mathrm{inv}}\left[\frac{u_1(\alpha(t^\star)g(t^\star)+\beta(t^\star)-g_1(t^\star),t^\star)}{\delta(v,t^\star)\pi(v,t^\star)}\right]-\beta(v)\right]\frac{\mathrm{d}v}{\pi(v,t^\star)} \quad (69)$$

for $t^\star$ expressed entirely in terms of the primitives of the planner's problem. Note finally that this characterization reduces in the case of exponential utility to

$$g_1(t^\star)+[\Theta(t^\star)-\alpha(t^\star)]\,g(t^\star) = \beta(t^\star)+\Theta(t^\star)\int_{t^\star}^{T}\frac{[\gamma(v)\log[\delta(v,t^\star)\pi(v,t^\star)]-\beta(v)]\,\mathrm{d}v}{\pi(v,t^\star)} \quad (70)$$

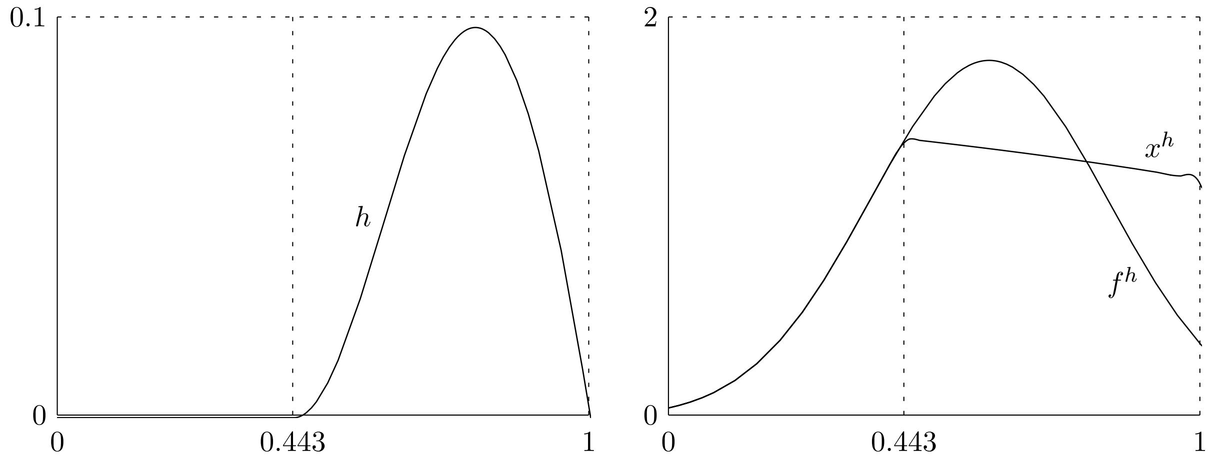

Figure 8: Capital history (left panel) and consumption and income histories (right panel) for a credit-constrained planning problem with a linear technology, exponential utility, and hyperbolic discounting; computed by numerical integration.

(cf. Equation 33) and in the case of basal utility to

$$g_1(t^\star) + [\Psi(t^\star) - \alpha(t^\star)]\, g(t^\star) = \beta(t^\star) - \omega(t^\star) - \Psi(t^\star) \int_{t^\star}^{T} \frac{[\beta(v) - \omega(v)]\, \mathrm{d}v}{\pi(v, t^\star)} \tag{71}$$

(cf. Equation 41).

Like Equations 35 and 43, Equations 65/70 and 66/71 can be used to approximate, respectively, linear/exponential and linear/basal capital histories by means of numerical integration (see Section 4.1). To demonstrate this capability, Figure 8 plots the paths h, x^h, and f^h for the capital management problem with credit limit $g(s) = 0$, parameters $T = 1$ and $K = 0$, linear technology

$$f(z, s) = z/10 + [10/\mathrm{pi}]^{1/2} e^{-10[s-3/5]^2}, \tag{72}$$

exponential index function $u(z, s) = -3\exp[-z/3]$, and hyperbolic discounting function $\delta(s, t) = [1 + s - t]^{-1/4}$. Here the credit constraint binds and the planner consumes his entire income up until the savings-initiation date $t^\star \cong 0.443$, at which point he begins to build up a reserve of capital from which he will later withdraw.

6. CONCLUDING COMMENTS

1. In a discussion of "The Uncertainties of Numerical Mathematics," Wolfram [30, pp. 952–953] advises that "[i]n many calculations, it is . . . worthwhile to go as far as you can symbolically, and then resort to numerical methods only at the very end [in order to avoid] the problems that can arise in purely numerical computations." This has been our method in studying the Strotzian capital management scenario, and reducing our exposure to the vagaries described by Wolfram has certainly been one of our motives for proceeding analytically to the maximum extent possible. But in addition to their usefulness in allowing us to test our computational machinery on safe ground, the analytical results obtained also possess an intrinsic value to the

extent that they reveal qualitative features of the planner's behavior. For example, it is apparent from Equation 35 (and obvious from Equations 74–76) that in the case of exponential utility the functions β and δ exert independent influences on the capital history h, and that this specification of the index function therefore rules out certain interaction effects of whose absence we might like to be aware.

2. In circumstances where we have developed explicit formulae for the history of the capital stock (see Sections 3.2, 3.4, 5.2, and A.1), we are already well equipped to investigate how the shape of the path h is affected by manipulating parameters such as the interest rate a (see Section A.1), the intertemporal substitutability measure p under basal utility (see Section 3.2), or the degree of distortion q under hyperbolic discounting (see Section 3.3). The reader may also wonder, however, about the effect on h of manipulating "cognitive parameters" — or, more crudely, of interchanging the assumptions of naivete and sophistication.

   While a determined attempt to investigate cognitive comparative statics of this sort is well beyond the scope of the present paper, it is worth pointing out what is known about the special case of a cake-eating problem with basal utility: Firstly, Pollak [24] shows that when $p = 1$ (i.e., when the index function is logarithmic) and $\omega(s) = 0$, the naive and sophisticated planners exhibit exactly the same behavior. And secondly, O'Donoghue and Rabin [19, p. 119, fn 24] claim that when $p \neq 1$, the sophisticated agent saves more or less than the naive agent according to whether $p < 1$ or $p > 1$.

A. APPENDIX

A.1. Prospects for closed-form histories

Assume a linear technology and let $\alpha(s) = a \geq 0$, imposing a constant interest rate. Assume exponential utility and let $\gamma(s) = c > 0$, imposing constant intertemporal substitutability. Then

$$\Theta(t) = \left[\int_t^T e^{-a[v-t]}\mathrm{d}v\right]^{-1} = \begin{cases} [T-t]^{-1} & \text{for } a = 0, \\ a\left[1 - e^{-a[T-t]}\right]^{-1} & \text{for } a \neq 0; \end{cases} \tag{73}$$

and some straightforward calculations allow us to write the capital history in Equation 35 as

$$h(t) = B(t) + D(t) + \begin{cases} [1 - t/T]\, K & \text{for } a = 0, \\ c\,[t - T] + \dfrac{\left[1 - e^{-a[T-t]}\right][cT + K]}{1 - e^{-aT}} & \text{for } a \neq 0; \end{cases} \tag{74}$$

where the paths B and D are defined by

$$B(t) := \begin{cases} \displaystyle\int_0^t \frac{T-t}{T-v}\left[\beta(v) - \frac{1}{T-v}\int_v^T \beta(w)\mathrm{d}w\right]\mathrm{d}v & \text{for } a = 0, \\ \displaystyle\int_0^t \frac{1 - e^{-a[T-t]}}{1 - e^{-a[T-v]}}\left[\beta(v) - \frac{a}{1 - e^{-a[T-v]}}\int_v^T \beta(w) e^{-a[w-v]}\mathrm{d}w\right]\mathrm{d}v & \text{for } a \neq 0; \end{cases} \tag{75}$$

and

$$D(t) := \begin{cases} \displaystyle\int_0^t \frac{T-t}{[T-v]^2} \int_v^T c\log\delta(w,v)\mathrm{d}w\mathrm{d}v & \text{for } a=0, \\ \displaystyle\int_0^t \frac{a\left[1-e^{-a[T-t]}\right]}{\left[1-e^{-a[T-v]}\right]^2} \int_v^T ce^{-a[w-v]}\log\delta(w,v)\mathrm{d}w\mathrm{d}v & \text{for } a\neq 0; \end{cases} \tag{76}$$

and determined, respectively, by the functions β and δ.

Now suppose that the planner's exogenous income has the polynomial specification

$$\beta(s) = \sum_{n=0}^{\infty} b(n)s^n, \tag{77}$$

bearing in mind that (according to the Weierstrass Approximation Theorem) the polynomials are dense in the (Banach) space of continuous functions on $[0,T]$ under the maximum norm. When $a=0$ the necessary computations are then straightforward, and when $a\neq 0$ they can be carried out easily enough with the aid of Equations 83–84 from Section A.2 together with the First Fundamental Theorem of Calculus. Thus we obtain the path

$$B(t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{b(n)t\left[t^n-T^n\right]}{n+1} & \text{for } a=0, \\ \displaystyle\sum_{n=0}^{\infty} \frac{b(n)\left[\left[1-e^{-a[T-t]}\right]\Gamma^{n+1}\left[0,at\right]+\left[1-e^{at}\right]\Gamma^{n+1}\left[at,aT\right]\right]}{a^{n+1}\left[1-e^{-aT}\right]} & \text{for } a\neq 0; \end{cases} \tag{78}$$

where the gamma function

$$\Gamma^{n+1}\left[y,z\right] := \int_y^z v^n e^{-v}\mathrm{d}v \tag{79}$$

generalizes the factorial operator in the sense that $n! = \Gamma^{n+1}\left[0,\infty\right]$ for each integer $n\geq 0$. And likewise, when the discounting function has the stationary log-polynomial specification

$$\delta(s,t) = \exp\sum_{n=1}^{\infty} d(n)[s-t]^n \tag{80}$$

with polynomial discount rate function

$$\rho(s,t) = \sum_{n=1}^{\infty}[-nd(n)][s-t]^{n-1}, \tag{81}$$

Equations 83 and 85 from Section A.2 aid us in computing the path

$$D(t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{d(n)c[T-t]\left[T^n-[T-t]^n\right]}{n[n+1]} & \text{for } a=0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{d(n)c\left[1-e^{-a[T-t]}\right]n!}{a^{n+1}} \left[\frac{1}{1-e^{-a[T-v]}} - \log\left[1-e^{-a[T-v]}\right]\cdots\right. & \\ \displaystyle\left.\cdots + av - \sum_{i=0}^{n} \frac{\Gamma^{n-i+1}\left[a[T-v],\infty\right]\mathrm{Li}^i e^{-a[T-v]}}{e^{-a[T-v]}\left[n-i\right]!}\right]_{v=0}^{t} & \text{for } a\neq 0. \end{cases} \tag{82}$$

Thus, given any capital management problem with linear f, constant α, and continuous β; with exponential u and constant γ; and with stationary, continuous δ; Equations 74, 78, and 82 can be used to approximate with arbitrary precision the associated history h.

Can we in a similar fashion obtain closed-form capital histories for the case of basal utility, given only modest assumptions about the technology and the discounting function? The answer, unfortunately, is that we cannot — or at least that it is not obvious that we can. Close scrutiny of the (omitted) calculations leading to Equations 78 and 82 reveals our method here to be heavily reliant on the log-linearity in the dummy variable of the integrand in Equation 32. For the corresponding integrand in Equation 40 to have this property, it is necessary both that α be constant, as before, *and that δ itself be log-linear.* But together with stationarity, the latter assumption simply returns us to the rather uninteresting case of exponential discounting (see Section 2.2).

A.2. Indefinite integrals

The following indefinite integrals are used in Section A.1:

$$\int [v-w]^n e^{-a[v-w]} \mathrm{d}v = -a^{-n-1}\Gamma^{n+1}\left[a[v-w],\infty\right]. \tag{83}$$

$$\int \frac{\left[[av]^n\left[1-e^{-a[T-v]}\right] - e^{av}\Gamma^{n+1}\left[av,aT\right]\right]\mathrm{d}v}{\left[1-e^{-a[T-v]}\right]^2} = \frac{e^{aT}\Gamma^{n+1}\left[aT,\infty\right] - e^{av}\Gamma^{n+1}\left[av,\infty\right]}{a\left[1-e^{-a[T-v]}\right]}. \tag{84}$$

$$\int \frac{\Gamma^{n+1}\left[0,a[T-v]\right]\mathrm{d}v}{\left[1-e^{-a[T-v]}\right]^2} = \frac{n!}{a}\left[\frac{1}{1-e^{-a[T-v]}} - \log\left[1-e^{-a[T-v]}\right] \cdots \right.$$
$$\left. \cdots + av - \sum_{i=0}^{n}\frac{\Gamma^{n-i+1}\left[a[T-v],\infty\right]\mathrm{Li}^i e^{-a[T-v]}}{e^{-a[T-v]}\left[n-i\right]!}\right]. \tag{85}$$

Equations 83–84 can be verified routinely using the relation $\Gamma_1^{n+1}\left[z,\infty\right] = -z^n e^{-z}$. To verify Equation 85 (valid for integer $n \geq 0$), first confirm using the relation $\mathrm{Li}_1^{n+1}z = z^{-1}\mathrm{Li}^n z$ that the v-derivative of the right-hand-side of the equality takes the form

$$\frac{n!}{\left[1-e^{-a[T-v]}\right]^2} - \sum_{i=0}^{n}\frac{n!}{[n-i]!}\left[\left[a[T-v]\right]^{n-i}\mathrm{Li}^i e^{-a[T-v]} \cdots\right.$$
$$\left. \cdots - \Gamma^{n-i+1}\left[a[T-v],\infty\right]e^{a[T-v]}\left[\mathrm{Li}^i e^{-a[T-v]} - \mathrm{Li}^{i-1}e^{-a[T-v]}\right]\right]. \tag{86}$$

Using the identity $\Gamma^{n+1}\left[z,\infty\right] = \sum_{i=0}^{n}\frac{n!}{i!}\left[z^i e^{-z}\right]$ (valid for integer $n \geq 0$), this expression can be written in the expanded form

$$\frac{n!}{\left[1-e^{-a[T-v]}\right]^2} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}\frac{n!}{j!}\left[a[T-v]\right]^j\mathrm{Li}^i e^{-a[T-v]} \cdots$$
$$\cdots - \left[\sum_{j=0}^{n}\frac{n!}{j!}\left[a[T-v]\right]^j\mathrm{Li}^{-1}e^{-a[T-v]} + \sum_{i=1}^{n}\sum_{j=0}^{n-i}\frac{n!}{j!}\left[a[T-v]\right]^j\mathrm{Li}^{i-1}e^{-a[T-v]}\right]. \tag{87}$$

Since the difference of double sums is zero and since $\mathrm{Li}^{-1} z = z\,[1-z]^{-2}$, the v-derivative in question then equals

$$\frac{n! - \sum_{j=0}^{n} \frac{n!}{j!}\,[a[T-v]]^j\, e^{-a[T-v]}}{\left[1-e^{-a[T-v]}\right]^2} = \frac{\Gamma^{n+1}\,[0,\infty] - \Gamma^{n+1}\,[a[T-v],\infty]}{\left[1-e^{-a[T-v]}\right]^2} = \frac{\Gamma^{n+1}\,[0,a[T-v]]}{\left[1-e^{-a[T-v]}\right]^2}. \tag{88}$$

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