Integrated OU processes and non-Gaussian OU-based stochastic volatility models

Ole E. Barndorff-Nielsen
Centre for Mathematical Physics and Stochastics (MaPhySto), University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark.
oebn@imf.au.dk

Neil Shephard
Nuffield College, Oxford OX1 1NF, UK.
neil.shephard@nuf.ox.ac.uk

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Abstract

In this paper we study the detailed distributional properties of integrated non-Gaussian OU (intOU) processes. Both exact and approximate results are given. We emphasise the study of the tail behaviour of the intOU process. Our results have many potential applications in financial economics, for OU processes are used as models of instantaneous volatility in stochastic volatility (SV) models. In this case an intOU process can be regarded as a model of integrated volatility. Hence the tail behaviour of the intOU process will determine the tail behaviour of returns generated by SV models.

Keywords: Background driving Lévy process; Chronometer; Co-break; Econometrics; Integrated volatility; Kumulant function; Lévy density; Lévy process; Option pricing; OU processes; Stochastic volatility.

1 Introduction

In the stochastic volatility (SV) model for log-prices of stocks and for log exchange rates a basic Brownian motion is generalised to allow the volatility term to vary over time. Then the log-price $y^*(t)$ follows the solution to the stochastic differential equation (SDE),

$$\mathrm{d}y^*(t) = \{\mu + \beta\tau(t)\}\,\mathrm{d}t + \tau^{1/2}(t)\mathrm{d}w(t), \tag{1}$$

where $\tau(t)$, the *instantaneous* or *spot volatility*, is going to be assumed to (almost surely) have locally square integrable sample paths, while being stationary and stochastically independent of the standard Brownian motion $w(t)$. Over an interval of time of length $\Delta > 0$ returns are defined as

$$y_n = y^*(\Delta n) - y^*((n-1)\Delta), \qquad n = 1, 2, ... \tag{2}$$

which implies that whatever the model for τ, it follows that

$$y_n|\tau_n \sim N(\mu\Delta + \beta\tau_n, \tau_n), \tag{3}$$

where

$$\tau_n = \tau^*(n\Delta) - \tau^*\{(n-1)\Delta\}, \quad \text{and} \quad \tau^*(t) = \int_0^t \tau(u)\mathrm{d}u.$$

In econometrics $\tau^*(t)$ and τ_n are called *integrated volatility* and *actual volatility*, respectively. Both definitions play a central role in the probabilistic and statistical analysis of SV models. Of course $\tau^*(t)$ can be thought of as a generalised subordinator, or "chronometer", for Brownian motion with drift; more specifically, $y^*(t)$ is representable as $\mu t + b^\beta(\tau^*(t))$ where b^β denotes Brownian motion with drift β and is independent of τ^* (cf. Barndorff-Nielsen and Shephard (2001)). Reviews of the literature on SV models are given in Taylor (1994), Shephard (1996) and Ghysels, Harvey, and Renault (1996), while statistical and probabilistic aspects are studied in detail in Barndorff-Nielsen and Shephard (2001).

As a result of (3), SV models can deliver returns which are stationary, serially dependent so long as τ_n is dependent, while the marginal distribution of returns will be thicker tailed than normal due to the mixing over the random τ_n. However, when Δ is large the dependence is mild, while the distribution of returns is close to normality. The latter result holds so long as $\tau(u)$ is ergodic for then, as $t \to \infty$,

$$t^{-1}\tau^*(t) = t^{-1}\int_0^t \tau(u)\mathrm{d}u \overset{a.s.}{\to} \xi = \mathrm{E}(\tau(t)),$$

implying, for the SV model, that (e.g. Barndorff-Nielsen and Shephard (2001))

$$\Delta^{-1/2}\{y_n - \mu\Delta - \beta\tau^*(\Delta)\} \overset{\mathcal{L}}{\to} N(0,\xi)$$

as $\Delta \to \infty$. This result is called "aggregational Gaussianity."

Aggregational Gaussianity has been much discussed in the econometric literature (e.g. in the ARCH literature it goes back to Diebold (1988, pp. 12-16)). Here we use an exchange rate dataset kindly made available to us by Olsen and Associates to empirically verify this. It records every five minutes the most recent quote to appear on the Reuters screen from 1st December 1986 until 30th November 1996. This data, together with various adjustments we have made to it, is discussed extensively in our data Appendix. Here we focus on the Dollar/Deutsch-Mark series. We report in Figure 1 non-parametric estimates of the log-density of returns recorded over intervals of length five minutes ($\Delta = 1$, $T = 705,313$), 20 minutes ($\Delta = 4$, $T = 176,325$), 1 hour ($\Delta = 12$, $T = 58,775$), 6 hours ($\Delta = 72$, $T = 9,775$), one day ($\Delta = 288$, $T = 2,440$) and one week ($\Delta = 1440$, $T = 488$). Also plotted is the fit of the generalised hyperbolic (GH) distribution, which is a five parameter flexible mixture of normals model discussed by, for instance, Barndorff-Nielsen (1977), Barndorff-Nielsen (1997) and Eberlein and Prause (2000). Its parameters were chosen using maximum likelihood methods. The non-parametric estimator

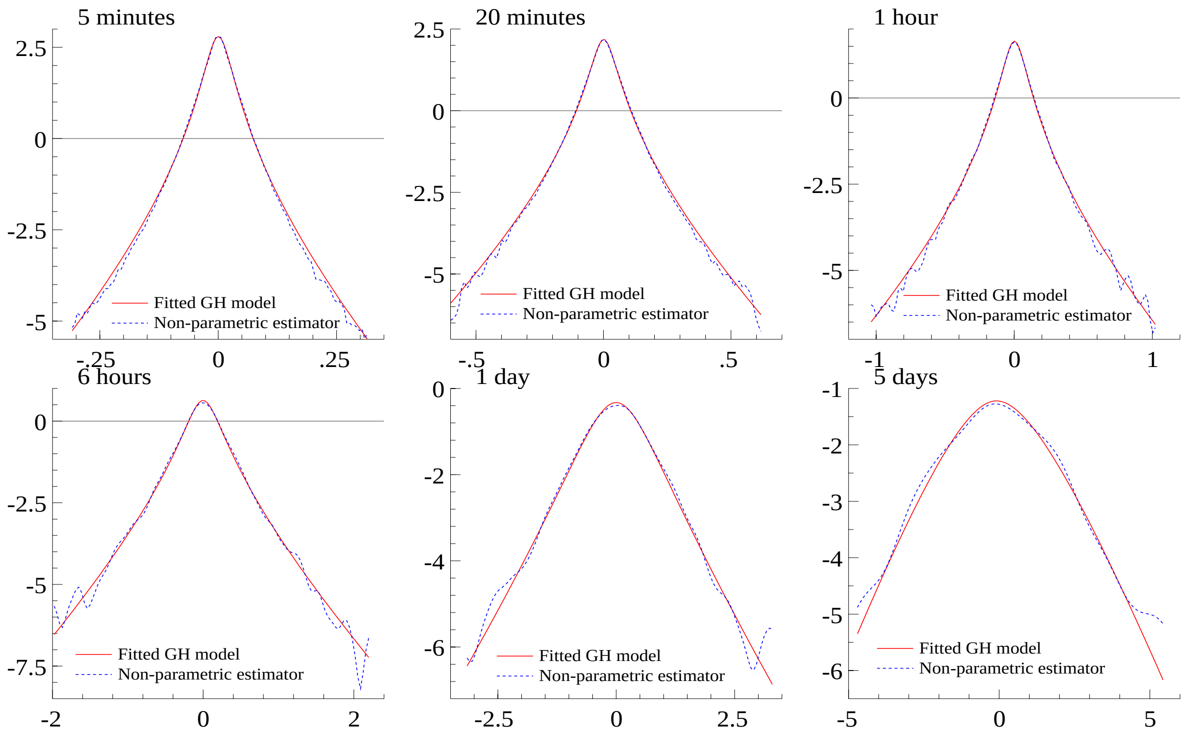

Figure 1: *Log-densities based on the five minute Olsen group data. Movements on the US Dollar against German DM from December 1986 to November 1996 over various intervals of time. Drawn log-densities are computed using a non-parametric estimator as well as the ML estimation of a generalised hyperbolic model. File name is* `boll_brown.in7`.

of the log-density is constructed by using the log of Gaussian kernel estimator coded in Applied Statistics Algorithm AS 176 by Bernard Silverman, which is available at `StatLib` and `NAG`, as well as in many statistical software environments such as `Ox` (Doornik (2001)). The bandwidth is chosen to be $1.06\widehat{\sigma}T^{-1/5}$, where T is the sample size and $\widehat{\sigma}$ is the empirical standard deviation of the returns (this is an optimal choice against a mean square error loss for Gaussian data).

The figures show some interesting features. At low levels of aggregation the “pine tree” feature of the log of the density of price changes in the exchange rate hold. This can be seen even at 6 hour returns. However, for daily data the log-density is a more linear in the tails. At the weekly level the tails seem heavier than linear and the quadratic approximation of the Gaussian seems to be closer to the mark. These observations also appear if we study the moments of the changes data. Table 1 shows the first four moments for the changes at different levels of aggregation. The big feature of the table is that as the level of aggregation increases so the kurtosis falls. At the weekly level the kurtosis is still above three, indeed this value is statistically significant, however it is not massively so. An interesting feature of the Table is the skewness statistics, which are all positive. However, these statistics do not really yield a

	Mean	Variance	Skewness	Kurtosis
5 minutes	-0.0000256	0.001847	0.146	44.2
20 minutes	-0.000102	0.006803	0.0628	27.6
1 hour	-0.000307	0.01929	0.263	21.3
6 hours	-0.00184	0.1162	0.0959	9.47
1 day	-0.00738	0.4903	0.00328	5.27
1 week	-0.0369	2.427	0.144	3.77

Table 1: *Raw mean, variance and standardised (by the standard deviation) third and fourth moments of the aggregated versions of the Olsen exchange rate data.*

consistent pattern which suggests the changes are mildly positively skewed but this is not a large feature of the series. Closely similar features to those discussed here are observed in other areas of study, particularly in turbulence, see for instance, Barndorff-Nielsen (1979).

As we saw above, if spot volatility is ergodic then SV models imply aggregational Gaussianity. In this paper we try to refine this result. We study the situtation where spot volatility follows a non-Gaussian Ornstein-Uhlenbeck (OU) process which is the solution to the stochastic differencial equation (SDE)

$$\mathrm{d}\tau(t) = -\lambda\tau(t)\mathrm{d}t + \mathrm{d}z(\lambda t), \quad \lambda > 0,$$

where $z(t)$ is a subordinator — that is a process with non-negative, independent and stationary increments (see, for example, Bertoin (1996) and Sato (1999)). Such models have been introduced in this context by Barndorff-Nielsen and Shephard (2001), while we call the corresponding $\tau^*(t)$ integrated OU or *intOU* processes. We will study the distribution of τ_n for these models with Δ fixed, which will imply the distribution of returns y_n. In particular we will derive the behaviour of the tails of actual volatility and so of returns from these SV models. This is of considerable practical importance and quite some interest in the recent literature. Andersen, Bollerslev, Diebold, and Ebens (2001) and Andersen, Bollerslev, Diebold, and Labys (2001a) have recently used realised volatility estimators of τ_n to claim that actual volatility is typically close to being lognormal for a wide range of Δ. This would imply returns are normal lognormal. Can such a claim hold if volatility is of OU type? Barndorff-Nielsen and Shephard (2001) have assumed $\tau(t)$ is distributed as an inverse Gaussian variable and then claimed that actual volatility is close to being inverse Gaussian for all Δ, implying returns would be normal inverse Gaussian. Can such claims be rationalised?

The outline of the paper is as follows. In section 2 we introduce the intOU processes, which are integrals of OU processes. The theory for these processes will be developed in the first case for general OU processes — not constraining ourselves to the non-negative case needed for volatility. The Section continues with a discussion of the properties of predictions from such models, as well as the behaviour of increments from intOU process. Section 3 gives more concrete results in the special case of non-negative OU processes. This Section will contain the answers to the above questions. Section 4 looks at superposition extensions of our basic models, while Section 5 concludes. Section 6 contains a discussion of the data used in this paper.

2 intOU processes

2.1 Basic model structure

This paper discusses analytic results on the distributional behaviour of the stochastic process $x^*(t)$ defined by

$$x^*(t) = \int_0^t x(s)\mathrm{d}s,$$

where $x(t)$ is a strictly stationary process on the real line which satisfies a SDE of the form

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(\lambda t).$$

Here the rate parameter λ is arbitrary positive and $z(t)$ is a homogeneous background driving Lévy process (BDLP) — that is it is a process with independent and stationary increments. The $x(t)$ process is said to be of OU type or an OU process (and is familiar in the Gaussian case where the Lévy process is Brownian motion). Correspondingly, we say that $x^*(t)$ is an intOU process. The OU process is representable (in law) as

$$x(t) = e^{-\lambda t}x(0) + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}z(\lambda s).$$

As indicated in the introduction, our main interest is where $x(t)$ is a purely non-negative process. In such cases we will often switch notation from $x(t)$ to $\tau(t)$ in order to make this clear.

Barndorff-Nielsen and Shephard (2001) have studied some of the stochastic properties of $x(t)$ and the reader is referred there for a discussion of the associated literature. They established the notation that if $x(t)$ is an OU process with a marginal law D, then we say $x(t)$ is a D-OU process. Further, if $z(1)$ has law D, then we say $x(t)$ is an OU-D process. Typical choices of D are the inverse Gaussian (IG) and gamma distributions.

A major feature of the intOU process $x^*(t)$ is that (see Barndorff-Nielsen (1998))

$$\begin{aligned} x^*(t) &= \lambda^{-1}\{z(\lambda t) - x(t) + x(0)\} \\ &= \lambda^{-1}(1-e^{-\lambda t})x(0) + \lambda^{-1}\int_0^t \left\{1 - e^{-\lambda(t-s)}\right\}\mathrm{d}z(\lambda s), \end{aligned} \tag{4}$$

which has a simple structure. An interesting characteristic of this expression is that $x^*(t)$ has continuous sample paths when $\lambda > 0$, while $z(\lambda t)$ and $x(t)$ have jumps (or breaks). Hence we have produced a process with a continuous sample path by taking linear combinations of two upward jumping processes. As a result $z(\lambda t)$ and $x(t)$ co-break (Clements and Hendry (1999, Ch. 9) introduced the concept of a co-break, where components of a multivariate series exhibit breaks but a linear combination of that series does not). Figure 2 shows this feature for a Γ-OU process, that is a process with a gamma distributed marginal law $\Gamma(\nu, \alpha)$ with probability density

$$\frac{\alpha^\lambda}{\Gamma(\nu)} x^{\nu-1} e^{-\alpha x}.$$

For the simulated process we plot $x^*(t)$ and $z(t\lambda)$ against t. The intOU process has no jumps, although the gradient of the process clearly changes over time. The BDLP has its familiar upward jumps. Further, in the case of a non-negative process, $\tau^*(t)$ has a lower bound made up

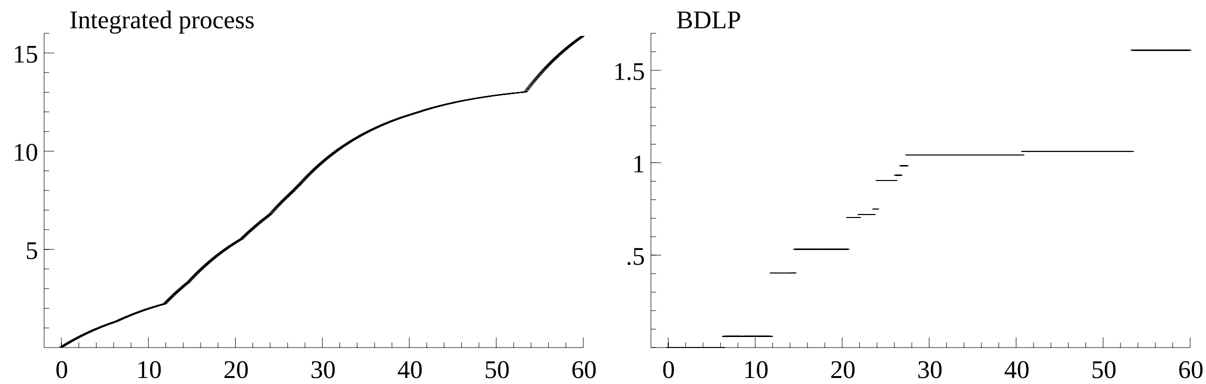

Figure 2: $\Gamma-$OU process with $\nu = 3$, $\alpha = 8.5$. Left: plot of $x^*(t)$ against t. Right: plot of $z(\lambda t)$, the BDLP, against t.

of $\lambda^{-1}(1 - e^{-\lambda t})\tau(0)$.

The fact that $z(\lambda t)$ and $x(t)$ co-break has deep implications for the use of this model. Suppose we focus for a moment on the case where $x^*(t)$ is a positive process. We can then use it as a chronometer of Brownian motion with drift, implying the resulting process $y^*(t)$ has continuous sample paths. This contrasts with the usual case of subordination in the probability literature where the Brownian motion plus drift is subordinated by a Lévy process, $z(t)$. In that case the resulting $y^*(t)$ process must have jumps.

Although $z(\lambda t)$ and $x(t)$ co-break, they do not co-integrate (Engle and Granger (1987) introduced the concept of a co-integration, where components of a multivariate series exhibit nonstationarity but linear combinations of that series do not). Instead, the long-run behaviour of $x^*(t)$ is dominated by $z(\lambda t)$. This is clear from rewriting (4) as

$$\lambda x^*(t) - z(\lambda t) = x(0) - x(t),$$

which means $x^*(t)$ and $z(\lambda t)$ (rather than $x(t)$ and $z(\lambda t)$) co-integrate. So roughly, for large λt, $\lambda x^*(t)$ will have the same distribution as $z(\lambda t)$ — the error in this approximation is a stationary process. The distribution of the error, for large t and $x(t)$ being a D-OU process, is approximately the difference of two independent random variables drawn from the distribution D.

2.2 The problem of prediction

In this section we will calculate the cumulants of $x^*(t)$, both unconditionally and conditionally on $x(0)$. The latter result is of fundamental importance in option pricing where analytically calculating the conditional cumulant function is enough to be able to compute European style options very rapidly. The former result will allow us to think about the unconditional distribution of returns.

The attractive feature of (4) is that the density of the future intOU process $x^*(t)|x(0)$ is determined by just

$$z(\lambda t) - x(t)|x(0).$$

As both $z(\lambda t)$ and $x(t)$ are linear we can see that this will be mathematically tractable. In particular (4) implies we have only to study the stochastic properties of the innovations for the intOU process

$$\lambda^{-1} \int_0^t \left\{1 - e^{-\lambda(t-s)}\right\} \mathrm{d}z(\lambda s) = \int_0^t \varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s) \tag{5}$$

$$\stackrel{\mathcal{L}}{=} \lambda^{-1} \int_0^{\lambda t} \left(1 - e^{-s}\right) \mathrm{d}z(s), \tag{6}$$

where

$$\varepsilon(t;\lambda) = \lambda^{-1}(1 - e^{-\lambda t}). \tag{7}$$

This allows us to compactly write

$$x^*(t) = \int_0^t \varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s) + \varepsilon(t;\lambda)x(0) \tag{8}$$

$$= \int_0^t \varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s) + \varepsilon(t;\lambda) \int_{-\infty}^0 e^s \mathrm{d}z(s), \tag{9}$$

the latter being entirely in terms of the BDLP z, here extended to be defined on the whole real line. In this section we will show how to compute the cumulants of the $x^*(t)$ process.

2.3 Notation

It will be helpful later to here define various pieces of notation which will be central to our results. First we note that

$$\varepsilon(t;\lambda) = t\varepsilon(1;\lambda t)$$

and

$$\varepsilon(t;\lambda) = \lambda^{-1}\{1 - \varepsilon(t;\lambda)_{/t}\} \tag{10}$$

or, equivalently,

$$\varepsilon(t;\lambda)_{/t} = 1 - \lambda\varepsilon(t;\lambda), \tag{11}$$

where / in subscript position indicates differentiation. Further, we shall use the following notation for Laplace and cumulant transforms of a random variate x:

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}$$

and

$$\bar{\mathrm{K}}\{\theta \ddagger x\} = \log \mathrm{E}\{e^{-\theta x}\},$$

where the latter notation is primarily used for positive variates x. Further, in the context of OU processes we write

$$\acute{\kappa}(\zeta) = \mathrm{C}\{\zeta \ddagger x(t)\} \quad \text{and} \quad \kappa(\zeta) = \mathrm{C}\{\zeta \ddagger z(1)\}, \tag{12}$$

$$\acute{k}(\theta) = \bar{\mathrm{K}}\{\theta \ddagger x(t)\} \quad \text{and} \quad k(\theta) = \bar{\mathrm{K}}\{\theta \ddagger z(1)\}, \tag{13}$$

and note that (see Barndorff-Nielsen (2000b) and Barndorff-Nielsen and Shephard (2001))

$$\acute{\kappa}(\zeta) = \int_0^\infty \kappa(e^{-s}\zeta)\mathrm{d}s \quad \text{and} \quad \kappa(\zeta) = \zeta\acute{\kappa}'(\zeta), \tag{14}$$

while

$$\acute{k}(\theta) = \int_0^\infty k(e^{-s}\theta)\mathrm{d}s \quad \text{and} \quad k(\theta) = \theta\acute{k}'(\theta). \tag{15}$$

It then follows that if we write the cumulants of $x(t)$ and $z(1)$ (when they exist) as, respectively, $\acute{\kappa}_m$ and κ_m $(m = 1, 2, ...)$ we have that

$$\kappa_m = m\acute{\kappa}_m, \quad \text{for} \quad m = 1, 2, ... \,.$$

A special case of this is that the means of $x(t)$ and $z(1)$ are identical, while the variance of the former is twice that of the latter. Finally we introduce the notation

$$\acute{k}^*(\theta) = \bar{\mathrm{K}}\{\theta \ddagger x^*(t)\},$$

for the integrated process.

Henceforth, for clarity, we shall refer to the quantities $\bar{\mathrm{K}}\{\theta \ddagger x\}$, $k(\theta)$, $\acute{k}(\theta)$ and $\acute{k}^*(\theta)$ as *kumulant* functions, to distinguish them from the other *cumulant* functions $\mathrm{C}\{\zeta \ddagger x\}, \kappa(\zeta)$ and $\acute{\kappa}(\zeta)$.

Some examples of the structure of such kumulants are given in Table 2. The only troublesome derivation is the OU-Γ case where we know that $k(\theta) = \nu \log(1 + \theta/\alpha)$ which implies

$$\begin{aligned}
\acute{k}(\theta) &= \nu \int_0^\infty \log\left(1 + \frac{\theta}{\alpha} e^{-s}\right) \mathrm{d}s \\
&= \nu \int_0^{\theta/\alpha} \frac{1}{t} \log(1+t)\, \mathrm{d}t \qquad \text{where} \qquad t = \frac{\theta}{\alpha} e^{-s} \\
&= \nu \sum_{j=1}^{\infty} (-1)^j \frac{(\theta/\alpha)^j}{j^2} \qquad \text{for} \qquad 0 \le \theta/\alpha < 1.
\end{aligned}$$

In this Table, $P(\psi)$ denotes a Poisson distribution with parameter ψ, $IG(\delta, \gamma)$ is an inverse Gaussian variable which has the density

$$\frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} \exp\left\{ -\frac{1}{2} \left(\delta^2 x^{-1} + \gamma^2 x \right) \right\}, \quad \gamma \ge 0, \quad \delta, x > 0, \tag{16}$$

and $TS(\kappa, \delta, \gamma)$ is the tempered stable. The tempered stable derives from the positive κ-stable law $S(\kappa, \delta)$ which has the cumulant transform $-(2\delta^2\theta)^\kappa$, $0 < \kappa < 1$ and density $p(x; \kappa, \delta)$. Then the density of the tempered stable is defined by

$$e^{(\delta\gamma)^{2\kappa}} p(x; \kappa, \delta) e^{-\frac{1}{2}\gamma^2 x}, \qquad \kappa \in (0,1), \delta > 0, \gamma \ge 0, \tag{17}$$

the $IG(\delta, \gamma)$ being the special case of $TS(\kappa, \delta, \gamma)$ determined by $\kappa = \frac{1}{2}$. The class of TS laws was introduced by Hougaard (1986), for applications to survival modelling. However, the same laws had been considered earlier, in an exponential family setting, by Tweedie (1984).

Model	$k(\theta) = \log E\{e^{-\theta z(1)}\}$	$\acute{k}(\theta) = \log E\{e^{-\theta x(t)}\}$
OU-$\Gamma(\nu, \alpha)$	$-\nu \log(1 + \theta\alpha^{-1})$	$\nu \sum_{j=1}^{\infty} (-1)^j (\theta/\alpha)^j j^{-2}$
OU-IG(δ, γ)	$\delta\gamma - \delta\gamma(1 + 2\gamma^{-2}\theta)^{1/2}$	Not known
OU-P(ψ)	$-\psi\left(1 - e^{-\theta}\right)$	$-\psi\left\{E_1(\theta) + \log\theta + \gamma\right\}$
OU-TS(κ, δ, γ)	$(\delta\gamma)^{2\kappa} - \delta^{2\kappa}(\gamma^2 + 2\theta)^\kappa$	Not known
$\Gamma(\nu, \alpha)$-OU	$-\nu\theta\left(\alpha + \theta\right)^{-1}$	$-\nu \log\left(1 + \theta\alpha^{-1}\right)$
IG(δ, γ)-OU	$-\theta\delta\gamma^{-1}\left(1 + 2\theta\gamma^{-2}\right)^{-1/2}$	$\delta\gamma - \delta\gamma\left(1 + 2\theta\gamma^{-2}\right)^{1/2}$
TS(κ, δ, γ)-OU	$-2\delta^{2\kappa}\kappa\theta(\gamma^2 + 2\theta)^{\kappa-1}$	$(\delta\gamma)^{2\kappa} - \delta^{2\kappa}(\gamma^2 + 2\theta)^\kappa$

Table 2: *Kumulant functions for common models. In the OU-P case, γ is Euler's constant and $E_1(x) = \int_x^\infty y^{-1} e^{-y} dy$, the exponential integral.*

2.4 Conditional first two moments

Recalling we wrote the first two cumulants of $z(1)$ as κ_1 (which also equals $\mathrm{E}(x(t))$ and κ_2 (which equals $2\mathrm{Var}(x(t))$, then

$$\begin{aligned}
\mathrm{E}\left\{ \lambda^{-1} \int_0^{\lambda t} \left(1 - e^{-s}\right) \mathrm{d}z(s) \right\} &= \lambda^{-1}\kappa_1 \int_0^{\lambda t} \left(1 - e^{-s}\right) \mathrm{d}s \\
&= \lambda^{-1}\kappa_1 \left(\lambda t - 1 + e^{-\lambda t}\right).
\end{aligned}$$

The implication is that

$$\begin{aligned} \mathrm{E}\left\{x^*(t)|x(0)\right\} &= \lambda^{-1}(1-e^{-\lambda t})x(0)+\lambda^{-1}\kappa_1\left(\lambda t-1+e^{-\lambda t}\right) && (18)\\ &= \varepsilon(t;\lambda)\{x(0)-\kappa_1\}+\kappa_1 t && (19)\end{aligned}$$

which implies, of course, $\mathrm{E}\left\{x^*(t)\right\}=\kappa_1 t$. The corresponding result for the conditional variance is

$$\begin{aligned} \mathrm{Var}\left\{x^*(t)|x(0)\right\} &= \lambda^{-2}\kappa_2\int_0^{\lambda t}(1-e^{-s})^2\mathrm{d}s\\ &= \lambda^{-2}\kappa_2\left(\lambda t-2+2e^{-\lambda t}+\frac{1}{2}-\frac{1}{2}e^{-2\lambda t}\right), && (20)\end{aligned}$$

while

$$\mathrm{Var}\left\{x^*(t)\right\}=\kappa_2\lambda^{-2}\left\{e^{-\lambda t}-1+\lambda t\right\}.$$

2.5 Cumulant functions for $x^*(t)|x(0)$ and $x^*(t)$

One of the main advantages of the OU process is that we are able to derive the conditional cumulant function of $x^*(t)|x(0)$. From (8) it follows that

$$\mathrm{C}\{\zeta\ddagger x^*(t)|x(0)\}=\mathrm{C}\left\{\zeta\ddagger\int_0^t\varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s)\right\}+i\zeta\varepsilon(t;\lambda)x(0), \qquad (21)$$

where

$$\begin{aligned} \mathrm{C}\left\{\zeta\ddagger\int_0^t\varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s)\right\} &= \mathrm{C}\left\{\zeta\ddagger\lambda^{-1}\int_0^{\lambda t}(1-e^{-\lambda t+u})\mathrm{d}z(u)\right\}\\ &= \int_0^{\lambda t}\mathrm{C}\{\zeta\lambda^{-1}(1-e^{-\lambda t+u})\ddagger z(1)\}\mathrm{d}u\\ &= \lambda\int_0^t\mathrm{C}\{\zeta\varepsilon(t-s;\lambda)\ddagger z(1)\}\mathrm{d}s\\ &= \lambda\int_0^t\kappa(\zeta\varepsilon(t-s;\lambda))\mathrm{d}s\\ &= \lambda\int_0^t\kappa(\zeta\varepsilon(s;\lambda))\mathrm{d}s. && (22)\end{aligned}$$

The result for the conditional cumulant function allows us to easily calculate the unconditional cumulant function. From (8) and (22) it follows that

$$\begin{aligned} \mathrm{C}\{\zeta\ddagger x^*(t)\} &= \mathrm{C}\left\{\zeta\ddagger\int_0^t\varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s)\right\}+\acute{\kappa}(\zeta\varepsilon(t;\lambda))\\ &= \lambda\int_0^t\kappa(\zeta\varepsilon(s;\lambda))\mathrm{d}s+\acute{\kappa}(\zeta\varepsilon(t;\lambda)). && (23)\end{aligned}$$

It is sometimes helpful to express

$$\int_0^t\kappa(\zeta\varepsilon(s;\lambda))\mathrm{d}s=\int_0^{\varepsilon(t;\lambda)}(1-\lambda r)^{-1}\kappa(\zeta r)\mathrm{d}r, \qquad (24)$$

which implies (21) and (23) become

$$\mathrm{C}\{\zeta \ddagger x^*(t)|x(0)\} = \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1}\kappa(\zeta r)\mathrm{d}r + i\zeta\varepsilon(t;\lambda)x(0)$$

and

$$\mathrm{C}\{\zeta \ddagger x^*(t)\} = \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1}\kappa(\zeta r)\mathrm{d}r + \acute{\kappa}(i\zeta\varepsilon(t;\lambda)).$$

2.6 Cumulant functionals

At a more abstract level it is sometimes helpful to have generic results for the cumulant functions of the x^* process. A convenient way of doing this is via

$$f \bullet x^* = \int_0^\infty f(t)\mathrm{d}x^*(t),$$

where f denotes an "arbitrary" function. We find that

$$\begin{aligned}
f \bullet x^* &= \int_0^\infty f(t)x(t)\mathrm{d}t \\
&= \int_0^\infty f(t)\left\{e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}z(\lambda s) + e^{-\lambda t}x(0)\right\}\mathrm{d}t \\
&= \int_0^\infty f(t)e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}z(\lambda s)\mathrm{d}t + \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t x(0) \\
&= \int_0^\infty \int_s^\infty f(t)e^{-\lambda(t-s)}\mathrm{d}t\mathrm{d}z(\lambda s) + \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t x(0) \\
&= \int_0^\infty \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\mathrm{d}z(\lambda s) + \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t x(0) \\
&= \int_0^\infty \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\mathrm{d}z(\lambda s) + \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t x(0)
\end{aligned}$$

and hence

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*|x(0)\} = \lambda \int_0^\infty \kappa\left(\zeta \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s + i\zeta \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t x(0).$$

The corresponding unconditional cumulant functional is

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*\} = \lambda \int_0^\infty \kappa\left(\zeta \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s + \acute{\kappa}(\zeta \int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t). \tag{25}$$

We note, in passing, that using (14), (25) may be given the alternative forms

$$\begin{aligned}
\mathrm{C}\{\zeta \ddagger f \bullet x^*\} &= \lambda \int_{-\infty}^\infty \kappa\left(\zeta \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s & (26)\\
&= \zeta \int_0^\infty f(s)\acute{\kappa}'\left(\zeta \int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s, & (27)
\end{aligned}$$

see Barndorff-Nielsen (2000b).

Formula (23) is recovered from (25) by choosing $f(s) = \mathbf{1}_{[0,t]}(s)$. The joint cumulant function of $x^*(s)$ and $x^*(t)$, for $0 < s < t$, may be obtained by letting $f = \phi\mathbf{1}_{[0,s]} + \psi\mathbf{1}_{[0,t]}$, etc.

3 Positive processes

3.1 Background

From now on we suppose that the OU process is positive and, correspondingly, we switch notation from x and x^* to τ and τ^*. We wish to investigate the nature of the intOU process τ^* somewhat more closely as this is of main concern in connection with the models introduced in Barndorff-Nielsen and Shephard (2001).

3.2 Lévy densities

In order to study the tail behaviour of $\tau^*(t)$ we will study the tail behaviour of the Lévy density in detail. We recall that knowledge of the Lévy density is enough to produce the characteristic function via the Lévy-Khintchine formula, which obviously characterises the density of τ^*. However, the connection between the probability density of τ^* and the associated Lévy density is more intimate than this indicates, particular when we are interested in explicitly studying the tail behaviour of the density of $\tau^*(t)$. This is important, for the right hand tail will determine the an essential part of the behaviour of returns when the intOU models are used for integrated volatility in stochastic volatility models. More specifically, for the infinitely divisible laws there are many useful relations and points of similarity between the probability measures and probability densities of the laws on the hand and their associated Lévy measures and Lévy densities on the other.[1] See Sato (1999, Corollary 25.8, Theorems 28.4, 53.6, 53.8) and Bingham, Goldie, and Teugels (1989, p. 341) and references given there; cf. also Embrechts and Goldie (1981), Sato and Steutel (1998) and Barndorff-Nielsen (2000a).

By (23), the kumulant function of τ^* is

$$\bar{\mathrm{K}}\{\theta \ddagger \tau^*(t)\} = \lambda \int_0^t k(\theta\varepsilon(s;\lambda))\mathrm{d}s + \acute{k}(\theta\varepsilon(t;\lambda)). \tag{28}$$

Let u and $\acute{u}$ denote the Lévy densities of $z(1)$ and $\tau(t)$, respectively. They are related by

$$u(x) = -\acute{u}(x) - x\acute{u}'(x)$$

and

$$\acute{u}(x) = x^{-1}U^+(x),$$

where

$$U^+(x) = \int_x^\infty u(y)\mathrm{d}y$$

[1]There are also very intriguing differences, so that simple-minded guessing about similarities will often not work.

cf. (Barndorff-Nielsen and Shephard (2001). From (28) we may determine an expression for the Lévy density $v(y;t;\lambda)$ of $\tau^*(t)$. We have

$$\begin{aligned}\bar{\mathrm{K}}\{\theta \ddagger \tau^*(t)\} &= -\lambda \int_0^t \int_0^\infty (1-e^{-\theta\varepsilon(s;\lambda)x})u(x)\mathrm{d}x\mathrm{d}s \\ &\quad - \int_0^\infty (1-e^{-\theta\varepsilon(t;\lambda)x})\acute{u}(x)\mathrm{d}x\end{aligned}$$

and by the substitutions $r=\varepsilon(s;\lambda)$ and $y=rx$, and using (11), this gives

$$\bar{\mathrm{K}}\{\theta \ddagger \tau^*(t)\} = -\int_0^\infty (1-e^{-\theta y})v(y;t;\lambda)\mathrm{d}y,$$

where

$$v(y;t;\lambda) = \lambda \int_0^{\varepsilon(t;\lambda)} \{r(1-\lambda r)\}^{-1} u(r^{-1}y)\mathrm{d}r + \varepsilon(t;\lambda)^{-1}\acute{u}(\varepsilon(t;\lambda)^{-1}y). \tag{29}$$

Recalling that $\acute{u}(x)=x^{-1}U^+(x)$, the latter expression may be written as

$$v(y;t;\lambda) = \lambda \int_0^{\varepsilon(t;\lambda)} \{r(1-\lambda r)\}^{-1} u(r^{-1}y)\mathrm{d}r + y^{-1}U^+(\varepsilon(t;\lambda)^{-1}y). \tag{30}$$

Letting $\delta(t;\lambda)=\varepsilon(t;\lambda)^{-1}$ and using the substitution $q=r^{-1}$ we then obtain

$$\begin{aligned}v(y;t;\lambda) &= \lambda \int_{\delta(t;\lambda)}^\infty (q-\lambda)^{-1}u(qy)\mathrm{d}q + y^{-1}U^+(\varepsilon(t;\lambda)^{-1}y) & (31)\\ &= \lambda \int_0^\infty (w+\delta(t;\lambda)-\lambda)^{-1}u((w+\delta(t;\lambda))y)\mathrm{d}w + y^{-1}U^+(\delta(t;\lambda)y). & (32)\end{aligned}$$

Letting

$$\bar{u}(x)=xu(x) \quad \text{and} \quad \bar{v}(y;t;\lambda)=yv(y;t;\lambda) \tag{33}$$

the latter relation may be reexpressed as

$$\bar{v}(y;t;\lambda) = \lambda \int_0^\infty \{(w+\delta(t;\lambda)-\lambda)(w+\delta(t;\lambda))\}^{-1}\bar{u}((w+\delta(t;\lambda))y)\mathrm{d}w + U^+(\delta(t;\lambda)y) \tag{34}$$

Thus, in particular, if $u(x)\sim x^{-1-a}$ for an $a\in(0,1)$ and $x\downarrow 0$ then

$$v(y;t;\lambda) \sim \left\{\lambda \int_{\delta(t;\lambda)}^\infty q^{-1-a}(q-\lambda)^{-1}\mathrm{d}q + a^{-1}\varepsilon(t;\lambda)^a\right\} y^{-1-a}$$

for $y\downarrow 0$.

More detailed calculations can be carried out in particular cases, as for the TS and Γ settings that we discuss next.

Example 3.1 OU-TS case. In this model u is of the TS form

$$u(x)=x^{-1-a}e^{-x},$$

with $0<a<1$. Hence

$$
\begin{aligned}
U^+(x) &= \int_x^\infty y^{-1-a}e^{-y}\mathrm{d}y \\
&= a^{-1}x^{-a}e^{-x} - a^{-1}\int_x^\infty y^{-a}e^{-y}\mathrm{d}y \qquad (35) \\
&= \Gamma(-a,x) \qquad (36)
\end{aligned}
$$

and

$$
\acute{u}(x) = x^{-1}\Gamma(-a,x). \qquad (37)
$$

Here

$$
\Gamma(\alpha,x) = \int_x^\infty \xi^{\alpha-1}e^{-\xi}\mathrm{d}\xi
$$

is the incomplete gamma function. We recall that

$$
\Gamma(\alpha+1,x) = \alpha\Gamma(\alpha,x) + x^\alpha e^{-x} \qquad (38)
$$

and, for $\alpha > 0$ and $x \to \infty$,

$$
\Gamma(\alpha,x) \sim x^{\alpha-1}e^{-x}\left\{1 + (\alpha-1)x^{-1} + (\alpha-1)(\alpha-2)x^{-2} + \cdots\right\} \qquad (39)
$$

(cf. Abramowitz and Stegun (1970, formula 6.5.32)). This implies, in particular, that

$$
\Gamma(-a,x) \sim x^{-a-1}e^{-x} \qquad (40)
$$

for $x \to \infty$.

It follows, from (37) and (39), that

$$
\acute{u}(x) \sim a^{-1}x^{-1-a} \quad \text{for} \quad x \downarrow 0,
$$

while

$$
\acute{u}(x) \sim x^{-5/2}e^{-x} \quad \text{for} \quad x \to \infty.
$$

Combining (32) and (36) we moreover find

$$
\begin{aligned}
v(y;t;\lambda) &= \lambda y^{-1-a}e^{-\delta(t;\lambda)y}\int_0^\infty (w+\delta(t;\lambda))^{-1-a}(w+\delta(t;\lambda)-\lambda)^{-1}e^{-wy}\mathrm{d}w \\
&\quad + y^{-1}\Gamma(-a,\delta(t;\lambda)y). \qquad (41)
\end{aligned}
$$

Here, for $y \to \infty$

$$
\int_0^\infty (w+\delta(t;\lambda))^{-1-a}(w+\delta(t;\lambda)-\lambda)^{-1}e^{-wy}\mathrm{d}w \sim \varepsilon(t;\lambda)^{2+a}e^{\lambda t}y^{-1}, \qquad (42)
$$

while, by (40),

$$
y^{-1}\Gamma(-a,\delta(t;\lambda)y) \sim \varepsilon(t;\lambda)^{1+a}y^{-2-a}e^{-\delta(t;\lambda)y}.
$$

Thus, all in all, for $y \to \infty$

$$v(y;t;\lambda) \sim c_\infty(t;\lambda,a) y^{-2-a} e^{-\delta(t;\lambda)y} \tag{43}$$

and for $y \downarrow 0$, by (29) and (35),

$$v(y;t;\lambda) \sim c_0(t;\lambda,a) y^{-1-a}, \tag{44}$$

where

$$c_\infty(t;\lambda,a) = e^{\lambda t} \varepsilon(t;\lambda)^{1+a}$$
$$c_0(t;\lambda,a) = \lambda \int_0^{\varepsilon(t;\lambda)} r^a (1-\lambda r)^{-1} \mathrm{d}r + a^{-1} \varepsilon(t;\lambda)^a.$$

□

Example 3.2 *TS*-OU case. In this case

$$\acute{u}(x) = x^{-1-a} e^{-x}, \tag{45}$$

$(0 < a < 1)$. Thus we have

$$\begin{aligned} u(x) &= -\acute{u}(x) - x\acute{u}'(x) \\ &= a x^{-1-a} e^{-x} + x^{-a} e^{-x} \end{aligned}$$

and, again with $\delta(t;\lambda) = \varepsilon(t;\lambda)^{-1}$,

$$\delta(t;\lambda) \acute{u}(\delta(t;\lambda) y) = \varepsilon(t;\lambda)^a y^{-1-a} \exp\{-\delta(t;\lambda) y\}.$$

Consequently

$$\begin{aligned} v(y;t;\lambda) &= a\lambda y^{-1-a} \int_0^\infty (w+\delta(t;\lambda))^{-1-a} (w+\delta(t;\lambda)-\lambda)^{-1} e^{-wy} \mathrm{d}w e^{-\delta(t;\lambda)y} \\ &\quad + \lambda y^{-a} \int_0^\infty (w+\delta(t;\lambda))^{-a} (w+\delta(t;\lambda)-\lambda)^{-1} e^{-wy} \mathrm{d}w e^{-\delta(t;\lambda)y} \\ &\quad + \varepsilon(t;\lambda)^a y^{-1-a} \exp\{-\delta(t;\lambda) y\}. \end{aligned} \tag{46}$$

It follows from (42) that for $y \to \infty$

$$v(y;t;\lambda) \sim c_\infty(t;\lambda,a) y^{-1-a} e^{-\delta(t;\lambda)y}, \tag{47}$$

whereas for $y \downarrow 0$

$$v(y;t;\lambda) \sim c_0(t;\lambda,a) y^{-1-a} \tag{48}$$

and here

$$c_\infty(t;\lambda,a) = e^{\lambda t} \varepsilon(t;\lambda)^a$$

$$c_0(t;\lambda,a) = a\lambda\int_0^\infty (w+\delta(t;\lambda))^{-1-a}(w+\delta(t;\lambda)-\lambda)^{-1}\mathrm{d}w + \varepsilon(t;\lambda)^a.$$

Thus, in particular, for the IG-OU process τ, for any $t > 0$ the Lévy density of $\tau^*(t)$ has asymptotically the same upper and lower tail behaviour as for IG laws, so that the law of $\tau^*(t)$ is close to being IG. This follows as the Lévy density of an $IG(\delta,\gamma)$ is

$$\frac{\delta}{\sqrt{2\pi}}x^{-3/2}\exp\left(-\frac{\gamma^2}{2}x\right),$$

which implies $\acute{u}(x)$ in (45) results from the IG special case $IG(\sqrt{2\pi},\sqrt{2})$. The implication is that the distribution of $\tau^*(t)$ is close to an $IG(\sqrt{2\pi}c_\infty, \sqrt{2\delta(t,\lambda)})$ in the upper tail and to $IG(\sqrt{2\pi}c_0(t;\lambda,a),0)$ in the lower tail. A similar conclusion holds in general for the TS-OU processes. □

Example 3.3 OU-Γ case. For this model u is of the form

$$u(x) = x^{-1}e^{-x}$$

and proceeding as in Example 3.1 we find

$$U^+(x) = E_1(x),$$

where $E_1(x)$ is the exponential integral. Further

$$\begin{aligned} v(y;t;\lambda) &= \lambda y^{-1}e^{-\delta(t;\lambda)y}\int_0^\infty \{(w+\delta(t;\lambda))(w+\delta(t;\lambda)-\lambda)\}^{-1}e^{-wy}\mathrm{d}w \\ &\quad + y^{-1}E_1(\delta(t;\lambda)y). \end{aligned} \tag{49}$$

Since

$$E_1(x) \sim \begin{cases} x^{-1}e^{-x} & \text{for} \quad x\downarrow 0 \\ -\log x & \text{for} \quad x\to\infty \end{cases}, \tag{50}$$

(see, for example, Abramowitz and Stegun (1970, pp. 229 and 231)) it follows that

$$v(y;t;\lambda) \sim \begin{cases} e^{\lambda t}y^{-1}e^{-\delta(t;\lambda)y} & \text{for} \quad y\to\infty \\ y^{-1}\log y^{-1} & \text{for} \quad y\downarrow 0. \end{cases}$$

□

Example 3.4 Γ-OU case. With

$$\acute{u}(x) = x^{-1}e^{-x}$$

we have

$$u(x) = U^+(x) = e^{-x}$$

and

$$v(y;t;\lambda) = \lambda \int_{\delta(t;\lambda)}^{\infty} (q-\lambda)^{-1} e^{-qy} \mathrm{d}q + y^{-1} \exp\{-\delta(t;\lambda)y\}.$$

Here, by (50),

$$\begin{aligned} \int_{\delta(t;\lambda)}^{\infty} (q-\lambda)^{-1} e^{-qy} \mathrm{d}q &= e^{-\lambda y} \int_{\delta(t;\lambda)-\lambda}^{\infty} s^{-1} e^{-s} \mathrm{d}q \\ &= e^{-\lambda y} E_1((\delta(t;\lambda)-\lambda)y) \\ &\sim \begin{cases} e^{\lambda t} e^{-\delta(t;\lambda)y} & \text{for} \quad y \to \infty \\ e^{-\lambda y} \log y^{-1} & \text{for} \quad y \downarrow 0 \end{cases} \end{aligned}$$

implying

$$v(y;t;\lambda) \sim \begin{cases} e^{\lambda t} y^{-1} e^{-\delta(t;\lambda)y} & \text{for} \quad y \to \infty \\ \\ y^{-1} & \text{for} \quad y \downarrow 0. \end{cases}$$

Thus the conclusion is similar to that for the TS-OU processes. □

Example 3.5 LN-OU case. The lognormal (LN) distribution is known to be selfdecomposable, see Bondesson (1982, p. 18, Theorem 3.1.1, p. 48: Notes). However, an explicit expression for the Lévy density of LN is not known. Bondesson (2000) dicusses this open problem and notes that, based among other things on results in Embrechts, Goldie, and Veraverbeke (1979), the Lévy density u corresponding to the standard lognormal $LN(0,1)$ must, in all likelihood, satisfy

$$\bar{u}(x) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\log^2 x} \tag{51}$$

for $x \to \infty$. Based on this assumption we find for the intOU process, derived from the LN-OU process with standard LN marginal, that for $y \to \infty$

$$\bar{v}(y;t;\lambda) \sim \frac{1}{\sqrt{2\pi}} (\log y)^{-1} e^{-\frac{1}{2}\log^2(\delta(t;\lambda)y)} = (\log y)^{-1} \bar{u}(\delta(t;\lambda)y) \tag{52}$$

In particular, then, the tails of the marginal distributions of the int(LN-OU) process do not behave as do LN laws.

The verifiication of (52) is as follows. By the substitution $r = \log(w + \delta(t;\lambda))$, for the integral in (34) we obtain, as $y \to \infty$,

$$\begin{aligned} \int_{\log\delta(t;\lambda)}^{\infty} (e^r - \lambda)^{-1} \bar{u}(ye^r) \mathrm{d}r &\sim \frac{1}{\sqrt{2\pi}} \int_{\log\delta(t;\lambda)}^{\infty} e^{-r} e^{-\frac{1}{2}(r+\log y)^2} \mathrm{d}r \\ &\sim \frac{\sqrt{e}}{\sqrt{2\pi}} e^{\log y} \int_{\log\delta(t;\lambda)}^{\infty} e^{-\frac{1}{2}(r+\log y+1)^2} \mathrm{d}r \\ &= \sqrt{e} e^{\log y} \{1 - \Phi(\log(\delta(t;\lambda)y) + 1)\} \\ &\sim \frac{\sqrt{e}}{\sqrt{2\pi}} e^{\log y} (\log y)^{-1} e^{-\frac{1}{2}(\log y + \log\delta(t;\lambda)+1)^2} \\ &= \frac{\varepsilon(t;\lambda)}{\sqrt{2\pi}} e^{-\frac{1}{2}\log^2\delta(t;\lambda)} (\log y)^{-1} e^{-\frac{1}{2}\log^2 y - (\log(\delta(t;\lambda)+1)\log y} \\ &= o((\log y)^{-1} e^{-\log^2 y}) \end{aligned} \tag{53}$$

Furthermore, (51) implies

$$U^{+}(x) \sim \frac{1}{\sqrt{2\pi}}(\log y)^{-1} e^{-\frac{1}{2}\log^2 y} \tag{54}$$

and combining (34), (53) and (54) we find (52). □

3.3 IG-OU case

We use the expression we have already seen, that

$$\begin{aligned}\overline{\mathrm{K}}\left\{\theta \ddagger \tau^*(t)|\tau(0)\right\} &= -\theta\varepsilon(t;\lambda)\tau(0) + \lambda\int_0^t k(\theta\varepsilon(s;\lambda))\mathrm{d}s \\ &= -\theta\varepsilon(t;\lambda)\tau(0) + \int_0^{1-e^{-\lambda t}} (1-u)^{-1}k(\lambda^{-1}\theta u)\mathrm{d}u, \end{aligned} \tag{55}$$

recalling that the kumulant function $k(\theta) = \log \mathrm{E}\left[\exp\left\{-\theta z(1)\right\}\right]$. Determining the expression for $\overline{\mathrm{K}}\left\{\theta \ddagger \tau^*(t)|\tau(0)\right\}$ in particular cases has been carried out in the context of option pricing based on OU volatility by Barndorff-Nielsen and Shephard (2001) and subsequently Nicolato and Venardos (2000) and Tompkins and Hubalek (2000). Here we discuss only the IG-OU case which was independently derived by Nicolato and Venardos (2000) and Tompkins and Hubalek (2000). From Table 2 we have that

$$k(\theta) = -\frac{\theta\delta}{\gamma}\left(1+2\theta\gamma^{-2}\right)^{-1/2}.$$

Then

$$\int_0^{1-e^{-\lambda t}} (1-u)^{-1}k(\lambda^{-1}\theta u)\mathrm{d}u = -\frac{\delta\theta}{\gamma\lambda}\int_0^{1-e^{-\lambda t}} (1-u)^{-1}u\left(1+\varkappa u\right)^{-1/2}\mathrm{d}u, \tag{56}$$

where $\varkappa = 2\gamma^{-2}\lambda^{-1}\theta$. Now

$$\int (1-u)^{-1}u\left(1+\varkappa u\right)^{-1/2}\mathrm{d}u = -\frac{2\sqrt{1+\varkappa u}}{\varkappa} + \frac{2\mathrm{arctanh}\left\{\frac{\sqrt{1+\varkappa u}}{\sqrt{1+\varkappa}}\right\}}{\sqrt{1+\varkappa}}.$$

Having derived $\overline{\mathrm{K}}\left\{\theta \ddagger \tau^*(t)|\tau(0)\right\}$ it is straightforward to calculate $\mathrm{C}\left\{\zeta \ddagger \tau^*(t)\right\}$ via (23). This cumulant function can be inverted to give the exact density of $\tau^*(t)$. Recall that for a random variable Y the distribution function can be obtained via the characteristic function using the Fourier inversion:

$$\Pr(\tau^*(t) > y) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \mathrm{Im}\exp\{-i\zeta y + \mathrm{C}\left\{\zeta \ddagger \tau^*(t)\right\}\zeta^{-1}\mathrm{d}\zeta.$$

Here we use this result to compute the density of $\tau^*(t)$ for the IG-OU case, written $p^*(x;\delta,\lambda)$. This is given, for three different choices of λ, in Figure 3. Together with this we have also plotted a right hand tail approximation $IG(\sqrt{2\delta(t,\lambda)}\mathrm{E}\left(\tau^*(t)\right), \sqrt{2\delta(t,\lambda)})$, which makes the mean of the process correct as well as the right hand tail. We also plot these densities on the log scale. We can see from (16), that the tail approximation works very well for small values of λ and less well when λ is large.

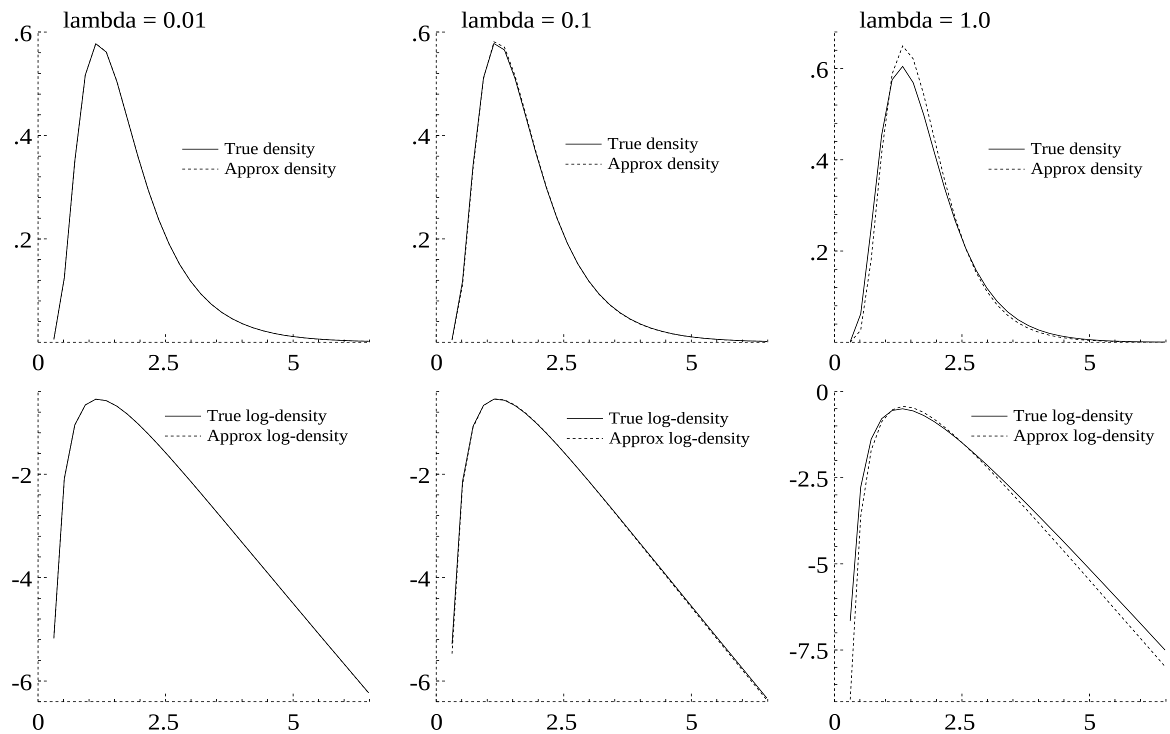

Figure 3: *The density* $p(x)$ *(top graphs) and* $\log p(x)$ *(bottom graphs) of* $\tau^*(1)$*, where the OU process* $\tau(t)$ *is distributed as* $IG(\sqrt{2\pi}, \sqrt{2})$*. The dotted line is the upper tail approximation* $IG(\sqrt{2\delta(t,\lambda)}\mathrm{E}\left(\tau^*(1)\right), \sqrt{2\delta(t,\lambda)})$*. The left hand graphs have* $\lambda = 0.01$, the middle 0.1 and *the right hand graphs have* $\lambda = 1$.

4 Superposition of two intOU processes

In practical application it is often helpful to allow for more flexible dynamic structures. A simple and mathematically tractable way of doing this is by adding together two (or more) independent OU processes, see Barndorff-Nielsen and Shephard (2001).

It is helpful in thinking about this issue to work with the IG-OU case, with

$$\tau(t) = \sum_{j=1}^{2} \tau_j(t), \qquad \text{where} \qquad \tau_j(t) \sim IG(\delta w_j, \gamma)\text{-OU},$$

where the weights $\{w_j\}$ are strictly positive and sum to one, while the corresponding damping values are $\{\lambda_j\}$. Again, the tail behaviour of $\tau^*(t)$ will be as for the IG laws.

5 Conclusion

In this paper we have carefully studied some of the properties of integrated OU processes. The main focus has been on studying cumulant functions of $x^*(t)$ unconditionally and conditionally on $x(0)$. The results have important implications for their use in, for example, option pricing models. Our main analytic conclusion is that if $x(t)$ is TS-OU or Γ-OU then while $x^*(t)$ is not distributed exactly as TS its tails do have this behaviour. A special case of this analysis are the important inverse Gaussian based models. Further, this type of result carries over to the Γ-OU process. These results are potentially important for it means that stochastic volatility models built out of OU processes with gamma or inverse Gaussian marginals will have tails which behave like normal gamma or normal inverse Gaussian distributions.

6 Acknowledgment

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6.1 Data appendix

The Olsen group have kindly made available to us a dataset which records every five minutes the most recent quote to appear on the Reuters screen from 1st December 1986 until 30th November 1996. When prices are missing they have interpolated them. Details of this processing is given in Dacorogna, Gencay, Muller, Olsen, and Pictet (2001). The same dataset was analysed by Andersen, Bollerslev, Diebold, and Labys (2001b). We follow the extensive work of Torben Andersen and Tim Bollerslev on this dataset, who remove much of the times when the market is basically closed. This includes almost all of the weekend, while they have taken out most US holidays. The result is what we will regard as a single time series of length 705,313 observations. Although many of the breaks in the series have been removed, sometimes there are sequences of very small price changes caused by, for example, unmodelled non-US holidays or data feed breakdowns. We deal with this by adding a Brownian bridge simulation to sequences of data where at each time point the absolute change in a five minute period is below 0.01%. That is, when this happens, we interpolate prices stochastically, adding a Brownian bridge with a standard deviation of 0.01 for each time period. By using a bridge process we are not effecting the long run trajectory of prices. Code to carry out this interpolation is in `mult_data.ox`. It is illustrated in Figure 4, which shows the first 500 observations in the Dollar/DM series together with another series on the Yen/Dollar. Later stretches of the data have less breaks in them, however this graph illustrates the effects of our intervention. Clearly our approach is ad hoc.

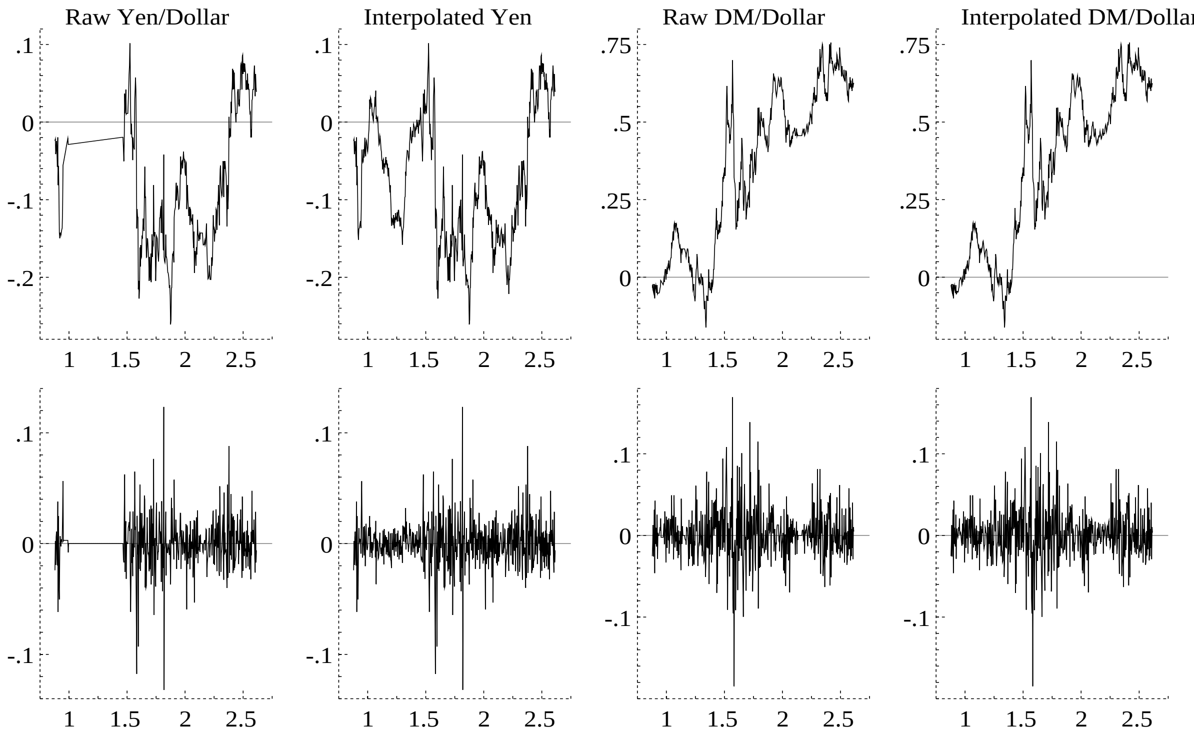

Figure 4: *Top line of graphs are the raw and interpolated data using a Brownian bridge interpolator. Bottom line of graphs is the corresponding returns. The x-axes are marked off in days.*

However, a proper statistical modelling of these breaks is very complicated due to their many causes and the fact that our dataset is enormous. And a more refined approach is unlikely to change the conclusions.

References

Abramowitz, M. and I. A. Stegun (1970). *Handbook of Mathematical Functions*. New York: Dover Publications Inc.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and H. Ebens (2001). The distribution of realised stock return volatility. *Journal of Financial Economics*. Forthcoming.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2001b). The distribution of exchange rate volatility. *Journal of the American Statistical Association 96*, 42–55.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2001a). Exchange rate returns standardised by realised volatility are (nearly) Gaussian. *Multinational Finance Journal*. Forthcoming.

Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London, Series A 353*, 401–419.

Barndorff-Nielsen, O. E. (1979). Models for non-Gaussian variation, with applications to turbulence. *Proceedings of the Royal Society. London. Series A. Mathematical and Physical Sciences 368*, 501–520.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics 24*, 1–14.

Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics 2*, 41–68.

Barndorff-Nielsen, O. E. (2000a). Probability densities and Lévy densities. Research report 18, MaPhySto, Aarhus University. Submitted.

Barndorff-Nielsen, O. E. (2000b). Superposition of Ornstein-Uhlenbeck type processes. *Theory of Probability and Its Applications*. Forthcoming.

Barndorff-Nielsen, O. E. and N. Shephard (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics (with discussion). *Journal of the Royal Statistical Society, Series B 63*, 167–241.

Bertoin, J. (1996). *Lévy Processes*. Cambridge: Cambridge University Press.

Bingham, N. H., C. M. Goldie, and J. L. Teugels (1989). *Regular Variation*. Cambridge: Cambridge University Press.

Bondesson, L. (1982). On simulation from infinitely divisible distributions. *Advances in Applied Probability 14*, 855–869.

Bondesson, L. (2000). On the Lévy measure of the lognormal and the logcauchy distributions. Research Report No. 10, Dept. Mathematical Statistics, Umeå University.

Clements, M. P. and D. F. Hendry (1999). *Forecasting Non-stationary Economic Time Series: The Zeuthen Lectures on Economic Forecasting*. Cambridge, Mass.: MIT Press.

Dacorogna, M. M., R. Gencay, U. A. Muller, R. B. Olsen, and O. V. Pictet (2001). *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

Diebold, F. X. (1988). *Empirical Modelling of Exchange Rate Dynamics*. New York: Springer-Verlag.

Doornik, J. A. (2001). *Ox: Object Oriented Matrix Programming, 3.0*. London: Timberlake Consultants Press.

Eberlein, E. and K. Prause (2000). The generalized hyperbolic model: financial derivatives and risk measures. FDM Preprint 56, University of Freiburg. To appear in Prooceedings of the Year 2000 Congress of the European Mathematical Society.

Embrechts, P. and C. M. Goldie (1981). Comparing the tail of an infinitely divisible distribution with integrals of its lévy measure. *Annals of Probability 9*, 468–481.

Embrechts, P., C. M. Goldie, and N. Veraverbeke (1979). Subexponentiality and infinite divisibility. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 49*, 335–347.

Engle, R. F. and C. W. J. Granger (1987). Co-integration and error correction: representation, estimation and testing. *Econometrica 55*, 251–276.

Ghysels, E., A. C. Harvey, and E. Renault (1996). Stochastic volatility. In C. R. Rao and G. S. Maddala (Eds.), *Statistical Methods in Finance*, pp. 119–191. Amsterdam: North-Holland.

Hougaard, P. (1986). Survival models for heterogeneous populations derived from stable distributions. *Biometrika 73*, 387–396.

Nicolato, E. and E. Venardos (2000). Derivative pricing in Barndorff-Nielsen and Shephard's OU type stochastic volatility models. Unpublished paper: Dept. of Mathematics Sciences, Aarhus University.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge: Cambridge University Press.

Sato, K. and F. W. Steutel (1998). Note on the continuation of infinitely divisible distributions and canonical measures. *Statistics 31*, 347–357.

Shephard, N. (1996). Statistical aspects of ARCH and stochastic volatility. In D. R. Cox, D. V. Hinkley, and O. E. Barndorff-Nielsen (Eds.), *Time Series Models in Econometrics, Finance and Other Fields*, pp. 1–67. London: Chapman & Hall.

Taylor, S. J. (1994). Modelling stochastic volatility. *Mathematical Finance 4*, 183–204.

Tompkins, R. and F. Hubalek (2000). On closed form solutions for pricing options with jumping volatility. Unpublished paper: Technical University, Vienna.

Tweedie, M. (1984). An index which distinguishes between some important exponential families. In J. Ghosh and J. Roy (Eds.), *Statistics: Applications and New Directions: Proc. Indian Statistical Institute Golden Jubilee International Conference*, pp. 579–604.