The Aggregate Weak Axiom in a Financial Economy through Dominant Substitution Effects

By John K.-H. Quah*

Abstract: Consider a two period financial economy with incomplete markets and with agents having von Neumann-Morgenstern utility functions. It is well known that when the economy's endowments are collinear, its excess demand function will obey the weak axiom when certain mild restrictions are imposed on agents' coefficient of relative risk aversion. This result is obtained through the application of a theorem on the law of demand (for individual demand) formulated independently by Milleron (1974), and Mitjuschin and Polterovich (1978). In this paper, we develop their arguments further and apply them to economies without collinear endowments. We identify conditions which guarantee that the economy's excess demand function obeys the weak axiom near an equilibrium price.

Keywords: income effects, substitution effects, monotonicity, law of demand, weak axiom.

JEL Classification Nos: D11, D50, D51, D52, G11

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*I am grateful to the Economic and Social Research Council for providing me with financial support under its Research Fellowship Scheme and to Thorsten Hens and Michael Jerison for stimulating comments. I would also like to thank Felicity Nyan for help with preparing the manuscript and much else besides.

1. Introduction

At least since the work of Sonnenschein, Debreu and Mantel, it is well known that the utility maximization hypothesis imposes little structure on the excess demand function of an exchange economy. If all agents have continuous and locally non-satiated preferences then the economy's aggregate excess demand will satisfy continuity and Walras' law, but beyond that, little can be said. More specifically, if an agent's *individual* excess demand function is derived from utility maximization, it will satisfy strong structural properties like the weak and strong axioms, but *aggregate* excess demand need not have those properties, even if all agents in the economy are utility-maximizing. This is unfortunate, because it is well known that an economy with an excess demand function which obeys the weak axiom is very well behaved. Such an economy will have a unique equilibrium price vector which is stable with respect to Walras' tatonnement and, furthermore, one can obtain intuitive comparative statics results when preferences or endowments are perturbed.[1] An economy with an excess demand function obeying the weak axiom need not admit a utility maximizing representative consumer, but such a structural property does go some way towards justifying this widely made assumption. For all these reasons and more, it is important to find plausible conditions on agents' preferences and/or endowments which will guarantee that the economy's excess demand function obeys the weak axiom or some other desirable structural property.

A significant literature has developed which addresses precisely this demand aggregation issue; a result which figures prominently in this literature is a theorem developed independently by Milleron (1974) and Mitjuschin and Polterovich (1978) (henceforth to be referred to as MMP). Let $u : R^l_{++} \to R$ be a utility function and let $f : R^l_{++} \times R_{++} \to R^l_{++}$ be the demand function it generates; we say that f obeys *monotonicity* or *the law of demand* if at any two distinct price vectors p and p' in R^l_{++}, and wealth $w > 0$,

$$(p - p')^T(f(p, w) - f(p', w)) < 0.$$

(Note that, crucially, while prices may vary, wealth is fixed at w.) It is well known that while this property is, in some precise sense, encouraged by substitution effects, the presence of income effects means that it does not follow from utility maximization alone. The contribution of MMP is to identify precisely those conditions on u which guarantee that substitution effects always dominate income effects, so that f is monotonic.[2, 3]

If u is the utility function defined over contingent consumption in l states of the world, it would be quite standard to assume that u has the expected utility form, i.e., $u(x) = \sum_{i=1}^{l} \pi_i \bar{u}(x^i)$ where $\pi_i > 0$ is the agent's subjective probability of state i occurring and $\bar{u} : R_{++} \to R$ is the Bernoulli utility function. In this case, one could show that the conditions of MMP are satisfied if the agent's coefficient of relative risk aversion does not vary by more than four, i.e.

$$\max_{r>0} \left(-\frac{r\bar{u}''(r)}{\bar{u}'(r)} \right) - \min_{r>0} \left(-\frac{r\bar{u}''(r)}{\bar{u}'(r)} \right) < 4.$$ [4]

The significance of this result for general equilibrium theory has to do with the fact that monotonicity is preserved by aggregation across agents, unlike the weak or strong axioms, which generally are not. Consider a market where all agents have monotonic demand functions (which can differ across agents). Assume that each agent has some wealth which is independent of price, and then consider a price change from p to p'. Each agent's demand will respond monotonically to this price change, and it is very clear that the aggregate demand of this market, $F : R^l_{++} \to R^l_{++}$, will also be monotonic, i.e., satisfy $(p - p')^T (F(p) - F(p')) < 0$ whenever $p \neq p'$.

Of course, there is a serious problem when one tries to apply this result in a general equilibrium model: with one significant exception, the wealth distribution in the economy will not be independent of price. The exception are exchange economies with collinear endowments, i.e., where all agents have endowments that are some (scalar) fraction of the aggregate endowment. In this case, a price change which preserves the value of aggregate endowment, will also preserve the wealth of every agent. If all agents have monotonic demand functions, then the economy's aggregate demand will also obey a restricted version of this property. Denoting the economy's aggregate demand by $\tilde{F}$ and the aggregate endowment by $\bar{\omega}$ in R^l_{++}, we have $(p-p')^T(\tilde{F}(p)-\tilde{F}(p')) < 0$ whenever $p \neq p'$ and $p^T\bar{\omega} = p'^T\bar{\omega}$. One can then use this to show that the economy's excess demand function, Z, where $Z(p) = \tilde{F}(p)-\bar{\omega}$, will also obey a similar version of monotonicity and hence the weak axiom. So by making an assumption which is stronger than the weak axiom at the individual level, one obtains the weak axiom in the aggregate.[5]

Our discussion so far has focused on exchange economies. We now turn to two-period financial economies with possibly incomplete markets. There are two reasons why MMP-type results are interesting in this context. Firstly, in this context, it is standard to assume that agents maximize expected utility; with these utility functions, the MMP approach leads to restrictions on the coefficients of relative risk aversion which can be easily interpreted. Secondly, the MMP approach can be extended to incomplete markets with relative ease. There is a natural analog to the complete markets result which can be obtained by a relatively modest tweaking of the assumptions and arguments made when markets are complete. The behavioral restrictions on the agents, i.e., on their coefficients of relative risk aversion remain unchanged; two assumptions must now be made on endowments - that they are collinear (as before) and that they are in the span of the asset space. Once these are in place, the economy's excess demand will obey the weak axiom.[6]

In the context of economies with collinear endowments, there are at least two papers which have sought to go beyond the claims made in the previous paragraph. While retaining the spanning assumption, Dana (1995) has extended the MMP results to infinite dimensional commodity spaces with possibly incomplete markets. In a finite dimensional context, the spanning requirement has been weakened in Bettzuge (1998), which shows that a joint restriction on the collinear endowments and the asset structure will suffice. In this paper we have retained both the finite dimensional and spanning assumptions and devoted our attention somewhere else.

We consider a financial economy with incomplete markets and focus on the *local properties of demand near a given equilibrium price.* Using MMP-type arguments, we identify conditions which guarantee that the economy's excess demand function for securities will obey the weak axiom near this equilibrium. Our local approach means that unlike other papers using the MMP approach (like Dana (1995) and Bettzuge (1998)), we are not able to address the issue of the global uniqueness of equilibrium. However, the local weak axiomatic structure of the excess demand function obtained through our conditions is still important because it guarantees the local stability of the equilibrium price and also the possibility of nice, i.e., intuitive, local comparative statics.

The major advantage of our approach is that we no longer require endowments to be collinear. We take as *given* the distribution of demand and endowments at the equilibrium price and ask what restrictions on agents' risk attitudes will guarantee that the economy's excess demand function for securities will obey the weak axiom. We recover, as a special case, the known result pertaining to collinear endowments; more generally, we show that the less collinear is the endowment distribution relative to the demand distribution (in some formal sense), the more stringent will be the conditions on agents' coefficient of relative risk aversion.

The paper is organized as follows. Section 2 contains a formal discussion of results pertaining to the monotonicity of individual demand and also of various concepts central to the MMP approach. The major result here is a 'translation theorem' that relates monotonicity conditions given in terms of direct utility functions with those given in terms of indirect utility functions. This is of great technical importance for us, since, following Quah (2000), the approach taken in this paper makes central use of the indirect utility function. Indeed, the conditions needed for our main results are first obtained as conditions on agents' indirect utility functions, and are only then translated into conditions on direct utility functions via the translation theorem. The main results of the paper, which pertain to the local weak axiom in financial economies, are presented in Section 3. This section also contains a comparison of the MMP approach to the aggregation problem that we have adopted in this paper, with the other major approach which focuses on conditions leading to 'increasing dispersion' (see Jerison (1999) for a survey and analysis of this approach).

2. The Monotonicity of Individual Demand for Securities [7]

We assume that there are two dates, 0 and 1. There is one good for consumption at date 0, and at date 1 there are l states of the world, with one good in each state. The economy has m securities or assets, with $m \leq l$. The $m \times l$ matrix D gives the payoffs of these securities at date 1, with the ijth entry being the payoff of the ith security in state j. We assume that the rank of D is m, so there are no redundant securities.

We assume that the commodity space is R^{l+1}_{++}; for a typical bundle $x = (x^0, x^1, x^2, ..., x^l)$, the first entry x^0 represents consumption at date 0, while contingent consumption at date 1 is represented by the vector $x^{-0} = (x^1, x^2, ..., x^l)$. A function $u : R^{l+1}_{++} \rightarrow R$ is a *regular utility function* if it has the following properties: it is C^2, its partial derivatives are strictly positive, it is differentiably strictly quasi-concave, and the sets $\mathcal{C}_{\bar{x}} = \{x \in R^{l+1}_{++} : u(x) \geq u(\bar{x})\}$ are closed in R^{l+1} for any $\bar{x}$ in R^{l+1}_{++}. We call a preference (more formally, a preorder) on R^{l+1}_{++} *regular* if it is representable by a regular utility function.

We denote the set of arbitrage free security prices by Q', which we know from standard theory satisfies $Q' = \{q' \in R^m : q = Dp \text{ for some } p \in R^l_{++}\}$. In other words, Q' is the set of security price vectors implied by strictly positive state price vectors. It follows that Q' is an open and convex cone in R^m. The contingent consumption implied by a portfolio of securities θ in R^m is $D^T\theta$; we assume that there exists some portfolio θ such that $D^T\theta \gg 0$. Consider an agent who, at date 0, has a regular preference $\succeq$ (represented by some regular utility function u) over R^{l+1}_{++}. This agent chooses a portfolio of securities which maximizes his utility, subject to the constraint implied by his wealth and the prevailing prices. We denote his wealth by $w > 0$; the prices he faces is represented by some vector $q = (q^0, q^1, ..., q^m)$ in the set $Q = R_{++} \times Q'$; so q^0 is the price of the date 0 good, and $q^{-0} = (q^1, q^2, ..., q^m)$ is the vector of security prices. The agent is constrained to choose from his budget set; at wealth w and price q, the set is

$$B(q,w) = \{x \in R^l_{++} : x^0 \leq (w - \theta^T q^{-0})/q^0 \text{ and } x^{-0} \leq D^T \theta \text{ for some } \theta \in R^m\}.$$

Since we assume that there is θ such that $D^T\theta \gg 0$, this set is nonempty for all (q,w) in $Q \times R_{++}$.

The regularity of u guarantees that there is a unique solution x_* to the problem of maximizing $u(x)$ subject to x in $B(q,w)$; since D is of rank m, the portfolio choice θ_* required to achieve x_* is also unique. We define the function $g : Q \times R_{++} \to R^{m+1}$ by $g(q,w)^0 = x^0_*$ and $g(q,w)^{-0} = \theta_*$. We shall refer to $g(q,w)$ as the *demand of the agent at* (q,w), with the first entry representing his choice of consumption level at date 0 and the other entries representing his chosen portfolio of securities. The regularity of u guarantees that g obeys the budget identity, i.e., $q^T g(q,w) = w$, and is C^1.

The function g is said to obey *monotonicity* or *the law of demand* if for any (q,w) and (q',w) in $Q \times R_{++}$, with $q \neq q'$, we have $(q-q')^T(g(q,w) - g(q',w)) < 0$. In particular, this property guarantees that when the price of security increases, its demand will fall. A sufficient (and effectively necessary) condition for monotonicity is that $\partial_q g(q,w)$ is negative definite at all (q,w) in $Q \times R_{++}$. By the Slutsky decomposition we can write $\partial_q g(q,w)$ as the difference between the substitution and income effect matrices; so long as g is generated by utility maximization, the former is always negative semidefinite, but the presence of the income effect matrix means that $\partial_q g(q,w)$ is not generally negative definite. So additional conditions on the utility function are needed to guarantee that substitution effects dominate income effects. We will discuss this next.

At each x in R^{l+1}_{++}, let $U(x)$ be the collection of regular and concave utility functions which represent $\succeq$ is some open and convex neighborhood of x. That $\succeq$ is representable by a regular utility function is true by definition; less trivially, it is also true that local representations which are *both* concave and regular exist, so that $U(x)$ is always nonempty.[8] For each $\hat{u}$ in $U(x)$, we can compute

$$(1) \qquad \psi_{\hat{u}}(x) \equiv -\frac{x^T \partial^2 \hat{u}(x) x}{\partial \hat{u}(x) x}.$$

Following Quah (2003), we define the *direct MMP coefficient* (or, for short, MMP coefficient) of $\succeq$ at x by

$$\psi_{\succeq}(x) = \inf_{\hat{u} \in U(x)} \psi_{\hat{u}}(x).$$

Since $\hat{u}$ is concave, $\psi_{\hat{u}}(x) \geq 0$, so $\psi_{\succeq}(x) \geq 0$ for all x. It is possible for $\psi_{\succeq} \equiv 0$; indeed, if $\succeq$ is homothetic, it must be representable by a concave and 1-homogeneous utility function $\hat{u}$ and one could check that $\psi_{\hat{u}}(x) = 0$ for all x. The next result identifies a condition on the MMP coefficient which is sufficient to guarantee that g obeys monotonicity. It is a fairly straightforward adaptation, to a financial setting with incomplete markets, of the original monotonicity theorem due to Milleron (1974) and Mitjuschin and Polterovich (1978).

Note that the matrix $(m+1) \times (l+1)$ matrix $\bar{D} = (\bar{d}_{ij})_{0 \leq i,j \leq l}$ referred to in the next proposition has $\bar{d}_{00} = 1$, $\bar{d}_{0j} = \bar{d}_{i0} = 0$ for all i and j, and $\bar{d}_{ij} = d_{ij}$ for all $i, j \geq 1$. We introduce it so that the consumption implied by $g(q,w)$, which is $g(q,w)^0$ at date 0 and contingent consumption of $D^T g(q,w)^{-0})$ at date 1 can be more succinctly written as $\bar{D}^T g(q,w)$.

Proposition 2.1 (The MMP monotonicity theorem): *Suppose that the demand function $g : Q \times R_{++} \to R_{++} \times R^m$ is generated by a regular preference $\succeq$ on R^{l+1}_{++}.*

(i) If at some (q, w), we have $\psi_{\succeq}(\bar{D}^T g(q, w)) < 4$, then there exists an open neighborhood N around q in which g is monotonic, i.e., $(q' - q'')^T (g(q', w) - g(q'', w)) < 0$ whenever q' and q'' are in N and $q' \neq q''$.

(ii) If $\psi_{\succeq}(x) < 4$ for all x in R^{l+1}_{++}, then g is a monotonic demand function.

As we had pointed out earlier, a homothetic preference will have an MMP coefficient which is identically zero, so we conclude that such a preference will generate a monotonic demand function. Indeed when the preference is homothetic, it is not difficult to prove the monotonicity of g directly, using the fact that g is now linear in w (see Mas-Colell et al (1995)). So we can view Proposition 2.1 as a far reaching generalization of the simple observation that homothetic preferences give rise to monotonic demand functions.

The condition $\psi_{\succeq} < 4$ is not just sufficient for monotonicity - there is also a sense in which it is necessary for monotonicity. In one form or another, this fact is well known (see, for example, Mas-Colell (1991) and Quah (2003)). Quah (2003) also provides an economic interpretation of $\psi_{\succeq}$ which is valid for any regular preference $\succeq$, but the special attractiveness of the MMP coefficient in the context of financial decision making has to do with the next result we present.

Suppose that $\succeq$ is representable by an additive utility function u, i.e., $u(x) = \sum_{i=0}^{l} u_i(x^i)$, where $u_i' > 0$ and $u_i'' < 0$ for $i = 0, 1, 2, ..., l$. In this case, one could impose a very useful bound on $\psi_{\succeq}$. Define the function $B_u : R^{l+1}_{++} \to R$ by

$$B_u(x) \quad = \quad \max_{0\leq i\leq l}\left(-\frac{x^i u_i''(x^i)}{u_i'(x^i)}\right) - \min_{1\leq i\leq l}\left(-\frac{x^i u_i''(x^i)}{u_i'(x^i)}\right); \tag{2}$$

the next result can be found in Quah (2003).

PROPOSITION 2.2: *Suppose u is a regular and additive utility function defined on R^{l+1}_{++}, and let $\succeq$ be the preference over R^l_{++} that it represents. Then for any x in R^l_{++}, $\psi_{\succeq}(x) \leq B_u(x)$.*

In the context of financial decision making, the assumption that u is additive is standard and has sound axiomatic foundations. Indeed, it is commonplace to formulate u as

$$u(x) \quad = \quad \bar{u}(x^0) + \delta\left[\sum_{i=1}^{l}\pi_i \bar{u}(x^i)\right] \tag{3}$$

where δ represents the discount rate and $\pi_i > 0$ the subjective probability of state i occurring, so $\sum_{i=1}^{l}\pi_i = 1$. In this case, $B_u(x)$ is uniformly bounded by the variation in the agent's coefficient of relative risk aversion, i.e., for all x in R^{l+1}_{++}, $B_u(x) \leq V_{\bar{u}}$, where

$$V_{\bar{u}} \quad = \quad \max_{r>0}\left(-\frac{r\bar{u}''(r)}{\bar{u}'(r)}\right) - \min_{r>0}\left(-\frac{r\bar{u}''(r)}{\bar{u}'(r)}\right). \tag{4}$$

Proposition 2.2 is very useful: combining it with Proposition 2.1, we see that a regular and additive utility function u will generate a monotonic demand function if $B_u(x) < 4$ for all x. More generally, imagine any theorem which includes a condition of the form '$\psi_{\succeq}(x) < M$'; if $\succeq$ is representable by a regular and additive utility function u, then the claim of the theorem is still true if the condition '$\psi_{\succeq}(x) < M$' is replaced by the condition '$B_u(x) < M$.' Whether or not the latter condition is strong depends on the size of M, but at least the economic interpretation on the condition, in terms of the agent's coefficient of relative risk aversion, is completely straightforward.

There are other ways of formulating sufficient conditions for monotonicity besides using the MMP coefficient. In particular, conditions could be stated in terms of the indirect preference. For any (p, w) in R^{l+2}_{++}, the regularity of $\succeq$ guarantees that there is a unique bundle $\hat{x}$ in R^{l+1}_{++} which satisfies the following conditions: $p^T\hat{x} \leq w$ and $\hat{x} \succeq x$ for all x satisfying $p^T x \leq w$. In other words, $\hat{x}$ is the demand at (p, w) in the classical sense, i.e., it is the demand at (p, w) when markets are complete. We denote this by $f(p, w)$ and will refer to f as the *classical demand function.* The indirect preference *induced* (or *generated*) by $\succeq$ refers to the preorder $\succeq'$ defined on (p, w) in R^{l+1}_{++}, such that $(p, w) \succeq' (p', w')$ whenever $f(p, w) \succeq f(p', w')$.

If u is a regular utility function representing $\succeq$, then the induced indirect preference $\succeq'$ is representable by the indirect utility function $v : R^{l+2}_{++} \rightarrow R$, where $v(p, w) = u(f(p, w))$. It is well known that the regularity of u will guarantee that v is a *regular indirect utility function*; by this we mean that it has the following properties: it is homogeneous of degree zero, it is C^2, its partial derivative with respect to the price of any good is strictly negative, and it is differentiably strictly quasiconvex in prices (see Mas-Colell (1985)). We call an indirect preference on R^{l+2}_{++} *regular* if it admits a regular indirect utility function. It follows that a regular preference must generate a regular indirect preference.

We denote by $V(p, w)$ the set of indirect utility functions which are *both* regular and convex in prices and which represent $\succeq'$ in an open and convex neighborhood of (p, w). It is known that for all (p, w) in R^{l+2}_{++}, the set $V(p, w)$ is nonempty.[9] For each $\hat{v}$ in $V(p, w)$, we may construct

$$(5) \qquad \phi_{\hat{v}}(p, w) \quad = \quad -\frac{p^T \partial_p^2 \hat{v}(p, w) p}{\partial_p \hat{v}(p, w) p};$$

the indirect MMP coefficient of $\succeq'$ at (p, w), denoted by $\phi_{\succeq'}(p, w)$, is defined as

$$(6) \qquad \phi_{\succeq'}(p, w) \quad = \inf_{\hat{v} \in V(p,w)} \phi_{\hat{v}}(p, w).$$

Note that $\phi_{\succeq'}$ is always nonnegative. The next result gives us the relationship between $\phi_{\succeq'}$ and $\psi_{\succeq}$.

THEOREM 2.3 (The translation theorem): *Let $\succeq'$ be the indirect preference generated by the regular (direct) preference $\succeq$ on R^{l+1}_{++}. Then $\psi_{\succeq}(f(p, w)) = \phi_{\succeq'}(p, w)$ for all (p, w) in R^{l+2}_{++}.*

This result is very useful since it allows us to translate conditions stated in terms of $\psi_{\succeq}$ into conditions stated in terms of $\phi_{\succeq'}$ and vice versa. So for example, we conclude from this theorem and Proposition 2.1 that the demand function g generated by $\succeq$ will be monotonic if the indirect preference $\succeq'$ obeys $\phi_{\succeq'}(p, w) < 4$ for all (p, w) in R^{l+2}_{++}. In fact, the principal use of Theorem 2.3 in this paper is in the opposite direction. All the results of the next two sections are stated in terms of conditions on $\psi_{\succeq}$—we do this because direct preferences are usually thought of as more familiar than indirect preferences and also (crucially) because $\psi_{\succeq}$ is bounded by B_u, which has a very straightforward interpretation—but an examination of the proofs will reveal that the conditions in our results were originally imposed on $\phi_{\succeq'}$ and translated into conditions on $\psi_{\succeq}$ only at the final step, using Theorem 2.3.

3. THE LOCAL WEAK AXIOM FOR MARKET EXCESS DEMAND

In this section we will examine the structure of demand near an equilibrium in a financial economy, which we will denote as $\mathcal{F}$. As in Section 2, we assume that there are two dates, 0 and 1. At date 1 there are l states of the world, with one good in each state. The economy has m securities, with $m \leq l$. The $m \times l$ matrix D gives the payoffs of these securities and the rank of D is m. Note that the familiar complete markets exchange economy, which we shall refer to as a *classical exchange economy* can be thought of as a financial economy with $m = l$ and the payoff matrix $D = I$.

The agents in $\mathcal{F}$ are drawn from a compact metric space of types, A. The distribution of types in $\mathcal{F}$ is given by the Borel probability measure μ on A. To each type a is associated an endowment ω_a in R^{m+1}, so ω_a^0 represents type a's endowment of the date 0 good while ω_a^{-0} in R^m is type a's endowment of securities. We assume that the map from a to ω_a is continuous. We also assume that $\omega_a^0 \geq 0$ and $D^T \omega_a^{-0} \geq 0$, with either strictly positive. To each a is also associated a C^1 demand function $g_a : Q \times R_{++} \to R^{m+1}$ which is generated by a regular preference $\succeq_a$. We assume that the maps from (a, q, w) to $g_a(q, w)$ and $\partial_q g_a(q, w)$ are continuous.

The agent in $\mathcal{F}$ derives his wealth from his endowment; we denote type a's demand, as a function of the security price vector by $\tilde{g}_a$, i.e., $\tilde{g}_a(q) = g_a(q, q^T \omega_a)$. Note that since q is in Q, there is $p \gg 0$ such that $q^{-0} = Dp$, so that, given our assumptions on ω_a, the agent's wealth at price q,

$$q^T \omega_a = q^0 \omega_a^0 + p^T (D^T \omega_a^{-0}) > 0.$$

The mean demand function of $\mathcal{F}$ is $G : Q \to R^{m+1}$, given by $G(q) = \int_A \tilde{g}_a(q) d\mu$; this is well-defined and C^1, with $\partial_q G(q) = \int_A \partial_q \tilde{g}_a(q) d\mu$. The excess demand at price q in Q is defined as $\zeta(q) = G(q) - \bar{\omega}$, where $\bar{\omega} = \int_{a \in A} \omega_a d\mu$ is the economy's mean endowment. The function ζ is C^1, homogeneous of degree zero, and satisfies Walras' Law, i.e., $q^T \zeta(q) = 0$ at all q in Q. We assume that $\mathcal{F}$ has an equilibrium price at $\bar{q}$, i.e., $\zeta(\bar{q}) = 0$.

We wish to identify the conditions under which ζ obeys the *weak axiom locally at* $\bar{q}$; formally, this property requires that there be a neighborhood of $\bar{q}$ such that $(q-\bar{q})^T\zeta(q) < 0$ whenever q is in that neighborhood and $\bar{q}$ and q are not collinear. In particular, this implies that a small rise in the price of i above $\bar{q}^i$ leads to excess supply, and a small fall in the price of i below $\bar{q}^i$ leads to excess demand. A sufficient condition for the local weak axiom is to require ζ to obey the *differentiable weak axiom at* $\bar{q}$, by which we mean that $\partial_q\zeta(\bar{q})$ is negative definite on the set $\bar{q}^{\perp} = \{z \in R^{m+1} : z^T\bar{q} = 0\}$, i.e., $z^T\partial_q\zeta(\bar{q})z < 0$ for all $z \neq 0$ in $\bar{q}^{\perp}$.

Our goal is to formulate a condition which guarantees the local weak axiom at $\bar{q}$ in terms of the distribution of demand and endowments at that price and some bound on agents' MMP coefficients. Put another way, we first observe that at any given equilibrium price, there are many ways demand and endowments can be distributed. For any *given* distribution, we wish to determine the restrictions (if any) on the local behavior of demand, as measured by the MMP coefficients, which will guarantee that excess demand obeys the local weak axiom. Potentially, the type of distributional information on demand and endowments needed for the formulation of a sensible theorem could be complicated, or at least complicated to state, but happily, it turns out that all the distributional information required can be captured by a few properly constructed covariance matrices.

Firstly, by re-scaling $\bar{q}$ if necessary, we can assume that $\bar{q}^T\bar{\omega} = 1$. In other words, we have normalized the equilibrium price vector so that the mean wealth is 1. We define a new probability measure $\hat{\mu}$ on A: for any measurable subset S of A, define $\hat{\mu}(S) = \int_S \bar{q}^T\omega_a d\mu$. The effect of $\hat{\mu}$ is to 're-weigh' agents according to their contribution to average wealth at the equilibrium price $\bar{q}$. For each a, we define $\hat{g}_a(\bar{q})$ by $\hat{g}_a(\bar{q}) = \tilde{g}_a(\bar{q})/\bar{q}^T\omega_a$. So $\hat{g}_a(\bar{q})$ is just the projection of $\tilde{g}_a(\bar{q})$ onto the *mean endowment budget plane*, $B = \{x \in R^{m+1} : \bar{q}^T x = \bar{q}^T\bar{\omega}\}$. Similarly we define $\hat{\omega}_a = \omega_a/\bar{q}^T\omega_a$, the projection of ω_a onto B. In Figure 1, projected demand bundles are depicted by squares, and projected endowments by circles. The distributional information we require is captured by the covariance matrices $\text{Cov}(\hat{g}(\bar{q}), \hat{g}(\bar{q}))$, $\text{Cov}(\hat{\omega}, \hat{\omega})$ and $\text{Cov}(\hat{g}(\bar{q}), \hat{\omega})$, where all of them are computed with the probability measure $\hat{\mu}$.

To specify the local *behavior* of demand (as opposed to the *position* of each demand bundle) we impose a condition on the agents' MMP coefficients. Recall that the consumption (in R^{l+1}_{++}) implied by $\tilde{g}_a(\bar{q})$ is $\bar{D}^T\tilde{g}_a(\bar{q})$; at that bundle, type a has an MMP coefficient of $\psi_{\succeq_a}(\bar{D}^T\tilde{g}_a(\bar{q}))$. We denote

$$\bar{\psi}(\bar{q}) = \sup_{a \in A} \psi_{\succeq_a}(\bar{D}^T\tilde{g}_a(\bar{q}))$$

and will refer to $\bar{\psi}(\bar{q})$ as the *MMP bound*. The next result, which is the main theorem of this paper, gives a condition for the local weak axiom at $\bar{q}$ which involves the MMP bound and the distribution of endowments and demand, as captured by the covariance matrices.

THEOREM 3.1: *Suppose that the economy $\mathcal{F}$ has a normalized equilibrium price at $\bar{q}$. Then ζ obeys the differentiable weak axiom at $\bar{q}$ whenever the matrix*

$$L(\bar{q}) = -4\left[\text{Cov}(\hat{g}, \hat{g}) - \text{Cov}(\hat{g}, \hat{\omega})\right] + \bar{\psi}(\bar{q})\text{Cov}(\hat{g} - \hat{\omega}, \hat{g} - \hat{\omega}) \tag{7}$$

is negative definite on the plane $\bar{q}^{\perp}$.

(Note that the argument $\bar{q}$ has been dropped from $\hat{g}$ to save space. Note also that, if we so wish, we can write $\text{Cov}(\hat{g} - \hat{\omega}, \hat{g} - \hat{\omega})$ as $\text{Cov}(\hat{g}, \hat{g}) + \text{Cov}(\hat{\omega}, \hat{\omega}) - 2\text{Cov}(\hat{g}, \hat{\omega})$.)

Theorem 3.1 gives sufficient conditions for ζ to satisfy the weak axiom at $\bar{q}$ in terms of the distribution of projected demand and endowments (as measured by their covariance matrices) and the local behavior of demand as measured by the MMP bound $\bar{\psi}(\bar{q})$. We argue that an examination of these conditions will show them to be reasonably mild, so that there is indeed a sound foundation for assuming the local weak axiom at an equilibrium price.

Note firstly that the theorem contains the known result for collinear endowments as a special case. If ω_a are collinear for all a, $\hat{\omega}_a$ is identical for all a, so that $L(\bar{q}) = \mathrm{Cov}(\hat{g}, \hat{g})(\bar{\psi}(\bar{q}) - 4)$. Since the matrix $\mathrm{Cov}(\hat{g}, \hat{g})$ is always positive semidefinite, the matrix $L(\bar{q})$ will be negative semidefinite if $\bar{\psi}(\bar{q}) < 4$ and it will be negative definite on the plane orthogonal to $\bar{q}$ if $\mathrm{Cov}(\hat{g}, \hat{g})$ is positive definite on that plane. More generally, the restriction implied by the negative definiteness of $L(\bar{q})$ can be usefully broken into two parts. Since $\mathrm{Cov}(\hat{g} - \hat{\omega}, \hat{g} - \hat{\omega})$ is positive semidefinite and $\bar{\psi}(\bar{q}) \geq 0$, $L(\bar{q})$ can be negative definite only if

(A) $\mathrm{Cov}(\hat{g}, \hat{g}) - \mathrm{Cov}(\hat{g}, \hat{\omega})$ *is positive definite on* $\bar{q}^{\perp}$.

Provided (A) is satisfied, $L(\bar{q})$ will be negative definite on $\bar{q}^{\perp}$ if

(B)

$$\bar{\psi}(\bar{q}) < \min_{\{z \in R^m :\, z \neq 0 \text{ and } z \perp \bar{q}\}} \frac{\mathrm{Var}(z^T \hat{g}, z^T \hat{g}) - \mathrm{Cov}(z^T \hat{g}, z^T \hat{\omega})}{\mathrm{Var}\left(z^T(\hat{g} - \hat{\omega}), z^T(\hat{g} - \hat{\omega})\right)}.$$

Condition (A) guarantees that the right hand side of this inequality is strictly positive, so there will always be some nonempty range of values for $\bar{\psi}(\bar{q})$ which satisfies (B). In short, it is clear that $L(\bar{q})$ is negative definite on $\bar{q}^{\perp}$ if and only if conditions (A) and (B) are satisfied, so by Theorem 3.1, *conditions (A) and (B) imply that* ζ *obeys the local weak axiom at* $\bar{q}$. We now examine, in turn, the significance of conditions (A) and (B).

How restrictive is condition (A)? It is reasonable to say that it is a big improvement over assuming collinear endowments; whether distributions of demand and endowments actually satisfy the property is an empirical issue, but (unlike collinear endowments) it is not prima facie implausible. The type of equilibrium situation which creates problems for the aggregate weak axiom occurs when those agents who are relatively well endowed with a particular commodity also tend to consume more of that commodity. It stands to reason that this will cause problems: in this case, a rise in the price of the commodity will raise the wealth of the greatest consumers of that commodity, which potentially negates the tendency to substitute away from it. Condition (A) does not completely exclude the possibility of positive correlation between demand and endowments, but it does require that this phenomenon be, in a specific sense, smaller than the variance of demand.

Is condition (A) necessary? More specifically, we can ask the following: for *any* particular distribution of demand and endowments at equilibrium, is there some restriction on $\bar{\psi}(\bar{q})$ which guarantees the local weak axiom? The answer to the second question is 'no', so the answer to the first is 'yes.' To understand this, first note that the MMP coefficient for a homothetic preference is everywhere zero. The tightest possible restriction on $\bar{\psi}(\bar{q})$ that one could impose is to require it to be zero, but even then *all* homothetic preferences will be admissible. So if it were the case that for any distribution of demand and endowments, some restriction on agents' MMP coefficients is sufficient to guarantee the weak axiom, we will in effect be saying that so long as preferences are homothetic, the local weak axiom holds, never mind how demand and endowments are distributed at equilibrium. Such a result will run up against the indeterminacy theorem of Hens (2001), which says that the excess demand function of a financial economy need not have any structure (and in particular need not obey the weak axiom) even if all agents have homothetic preferences. (Hens' result is a generalization to financial economies of the well known indeterminacy theorem of Mantel (1976), which only applies to classical exchange economies.[10]) In short, when preferences are homothetic, condition (B) is automatically satisfied so condition (A) is sufficient to guarantee the local weak axiom. However, when condition (A) is violated, the Mantel-Hens indeterminacy theorem tells us that local violations of the weak axiom can indeed occur even if all preferences are homothetic.

We now turn to condition (B). When endowments are collinear, the bound on $\bar{\psi}(\bar{q})$ is 4, which, at least when interpreted as a restriction on the variation of the coefficient of relative risk aversion, seems very permissive. One would expect this restriction to be more stringent when endowments are non-collinear. To have some sense of the magnitudes involved, it is useful to have some way measuring the dispersion of demand relative to that of endowments.

We assume that $\mathrm{Var}(\hat{g}, \hat{g})$ is positive definite on $\bar{q}^{\perp}$. This matrix is always positive semidefinite, so to assume that it is positive definite is a very modest extension. With this assumption, we know there must exist a nonnegative number θ such that $\theta\, Var(\hat{g}, \hat{g}) - Var(\hat{\omega}, \hat{\omega})$ is positive semidefinite on $\bar{q}^{\perp}$. Similarly, there must be nonnegative numbers K_1 and K_2 such that $K_2\, Var(\hat{g}, \hat{g}) - Cov(\hat{g}, \hat{\omega})$ and $K_1\, Var(\hat{g}, \hat{g}) + Cov(\hat{g}, \hat{\omega})$ are positive semidefinite matrices. We assume that θ, K_1 and K_2 are chosen to be the smallest nonnegative numbers for which the conditions are satisfied. Clearly, a large θ will mean that the variance of demand is small relative to the variance of endowment; K_1 and K_2 can be similarly interpreted. Intuitively, we would expect that the larger are these coefficients the more stringent will be the conditions on the MMP bound needed to guarantee the local weak axiom. This is borne out by the following corollary which gives a list of conditions guaranteeing monotonicity. We intersperse each condition with our comments. Note that distributional conditions imposed in all four cases of the corollary satisfy condition (A).[11]

COROLLARY 3.2: *Suppose that economy $\mathcal{F}$ has a normalized equilibrium price at $\bar{q}$ and that* $\mathrm{Cov}(\hat{g}, \hat{g})$ *is positive definite on $\bar{q}^{\perp}$. Then ζ satisfies the differentiable local weak at $\bar{q}$, if any of the following situations hold.*

(i) $\theta < 1$ and $\bar{\psi}(\bar{q}) \leq 2$.

Remark: Note that $\theta < 1$ if and only if demand is more dispersed than endowments in the sense of having a bigger variance, i.e., $\mathrm{Var}(\hat{g}, \hat{g}) - \mathrm{Var}(\hat{\omega}, \hat{\omega})$ is positive definite on $\bar{q}^{\perp}$. So (i) can be re-phrased as saying that *the local weak axiom at $\bar{q}$ is guaranteed if demand is more dispersed than endowments and the MMP bound is less than 2.* This provides a very clean generalization of the known result that an MMP bound of 4 guarantees the local weak axiom when endowments are collinear.

(ii) $K_1 = K_2 = 0$ and $\bar{\psi}(\bar{q}) < 4/(1+\theta)$.

Remark: We consider here a highly stylised (thought not completely unrealistic) scenario in which the covariance of the endowment and demand distributions is zero. (Of course, it is sufficient for this that endowments and demand are independently distributed.) This brings into sharp relief the impact of θ on the MMP bound. Since $4/(1+\theta)$ is decreasing in θ, the greater is the dispersion of endowment relative to demand, the more stringent is the condition on the MMP bound needed for the weak axiom. When $\theta = 0$ it equals 4 (as expected), when $\theta = 1$, it equals 2 (in agreement with (i)) and tends to zero as θ goes to infinity. (Note that the MMP bound cannot, by definition, fall below zero.)

(iii) $\theta < 1$ *and* $\bar{\psi}(\bar{q}) < 2(2 + 2K_1)/(1 + \theta + 2K_1)$.

Remark: This is just a more refined version of (i), which uses the precise values of θ and K_1 to give a more permissive MMP bound: note that the bound on the MMP bound is now *larger* than 2.

(iv) $\theta \geq 1$, $K_2 < 1$, *and* $\bar{\psi}(\bar{q}) < 4(1 - K_2)/[(\theta - 1) + 2(1 - K_2)]$.

Remark: This concerns the case where demand is less dispersed than endowments (so $\theta \geq 1$) and where the covariance between demand and endowments is not too great (in the sense that $K_2 < 1$). The condition on the MMP bound (it is now lower than 2) is tighter than in (i) and (iii), as one would expect. The condition becomes more stringent as K_2 or θ increases, and in fact it tends to zero as θ tends to infinity or K_2 tends to 1 from below.

Suppose that each for each type a in the economy, $\succeq_a$ is representable by a utility function of the form

$$u_a(x) = \bar{u}_a(x^0) + \delta_a \left[\sum_{i=1}^{l} \pi_{ai} \bar{u}_a(x^i) \right]$$

where δ_a represents type a's discount rate and π_{ai} is a's subjective probability of state i occurring, so $\sum_{i=1}^{l} \pi_{ai} = 1$. As we had pointed out in the Section 2, type a's MMP coefficient is then uniformly bounded by the variation in the coefficient of relative risk aversion; formally, $\psi_{\succeq_a} \leq V_{\bar{u}_a}$, with the latter as defined by (4). Thus Theorem 3.1 and Corollary 3.2 will still be true if $\sup_{a \in A} V_{\bar{u}_a}$ were to replace $\bar{\psi}(\bar{q})$. In other words, it suffices to impose bounds on the variation in the coefficient of relative risk aversion for all agent types in the economy. For example, Corollary 3.2(i) will say that when demand is more dispersed than endowments, the local weak axiom holds if, for each agent type, the variation in the agent's coefficient of relative risk aversion is smaller than 2. It is worth pointing out the obvious here: we are not imposing a bound on the variation of the coefficient of relative risk aversion *across* agents, but rather a bound on the variation for *each* agent. So one agent type can have a coefficient of relative risk aversion between 5 and 7, another between 15 and 17, etc.

Relationship with the increasing dispersion approach to the aggregation problem

Finally, we wish to relate the approach to the aggregation problem adopted in this paper with the other approach commonly used to deal with this issue. The approach adopted in this paper - via the MMP coefficient - has sometimes been referred to as the "dominating substitution effects" approach (see Mas-Colell (1991)). This is an apt description, since by controlling an agent's MMP coefficient we are controlling the degree to which his substitution effects dominate his income effects. The contribution of this paper is to determine, in an economy with non-collinear endowments, a precise bound on agents' MMP coefficients which is sufficient to guarantee that, following any small price change from the equilibrium price, the substitution effects which arise across all agents dominate, on average, the corresponding income effects.

There is another approach to the aggregation problem which focuses, not on getting substitution effects to dominate income effects, but on getting average income effects to be well-behaved. To put it in differentiable terms, since the substitution effect matrix of each agent in the economy is always negative semidefinite, a sufficient condition for the excess demand function to satisfy the differentiable weak axiom at some equilibrium price is for the average of income effect matrices at that price to be positive definite. As pointed out in Jerison (1999), what is needed for this property to hold is *increasing dispersion*: in exchange economies this means that if all agents were to receive a little more income (while holding prices fixed), the distribution of their excess demand will have a greater variance than before. Thus in this approach, either directly or indirectly, conditions are imposed on the collective behavior of agents' income expansion paths.

We wish to demonstrate that the MMP approach is distinct from the increasing dispersion approach to the aggregation problem. To do this, we show with an example that a bound on the MMP coefficient of an agent imposes no restriction on the direction of his income expansion path, so that any positive bound on the MMP coefficients for all agents in the economy cannot imply the increasing dispersion property.

Consider the utility function $u(x) = \sum_{i=0}^{l} k^i (x^i - b^i)^\theta$ where $k = (k^0, k^1, ..., k^l) \gg 0$ and $b = (b^0, b^1, ..., b^l) \gg 0$. Assume that markets are complete and that $D = I$. Suppose that at some price $\hat{p}$ and income 1, the demand is $\hat{x}$, where $\hat{p}^T \hat{x} = 1$. So long as $\hat{x} \gg b$, this can always be arranged: by the first order conditions, we need only choose k^i to satisfy $k^i \theta (\hat{x}^i - b^i)^{\theta - 1} = \lambda \hat{p}^i$ where λ is the Langrange multiplier. With k chosen as such, locally at $\hat{x}$, the income expansion path is in the direction of $\hat{x} - b$, which could be any positive direction since b can take on any values provided $\hat{x} \gg b$. Turning to the MMP coefficient,

$$\begin{aligned} \psi_u(\hat{x}) &= \frac{\sum \theta (1-\theta) k^i (\hat{x}^i)^2 (\hat{x}^i - b^i)^{\theta - 2}}{\sum \theta \hat{x}^i k^i (\hat{x}^i - b^i)^{\theta - 1}} \\ &= \frac{\sum (1-\theta)(\hat{x}^i)^2 (\hat{x}^i - b^i)^{-1} \hat{p}^i}{\sum \hat{p}^i \hat{x}^i} \end{aligned}$$

$$= \quad (1-\theta)\sum \frac{(\hat{x}^i)^2\hat{p}^i}{(\hat{x}^i - b^i)},$$

which will be arbitrarily close to zero when θ is sufficiently close to 1. Geometrically, as θ increases to one and $\psi_u(\hat{x})$ approaches zero, the indifference surface around $\hat{x}$ flattens.

So we see that a bound on this agent's MMP coefficient, however close to zero, is compatible with an income expansion path in any positive direction. This means that one can easily construct a classical exchange economy where, at an equilibrium price, agents have income expansion paths which violate the dispersion property, and yet the economy has an MMP bound which is small enough to ensure that its excess demand function satisfies the differentiable local weak axiom.

Note that the opposite is also true. Consider a classical exchange economy with collinear endowments. The MMP approach says that an MMP bound of 4 will guarantee monotonicity for individual demand and hence the aggregate weak axiom for excess demand. In this approach, agents need not have the same preference, but the preference of every agent must satisfy the bound on the MMP coefficient. The increasing dispersion approach gives alternative conditions for the weak axiom in such an economy. An early and influential paper using this approach is Hildenbrand (1983). The paper assumes that all agents share the same demand function generated by some regular preference. It shows that when endowments are collinear and has a distribution represented by a downward sloping density function, then the average of income effects will be positive semidefinite. Such an economy will have an *average* (or mean) excess demand function which obeys the weak axiom, even though the preference need not generate a monotonic *individual* demand function and the

MMP coefficient of the preference need not be bounded above by 4 or any other number.

So the MMP approach employed in this paper, and the increasing dispersion approach are two distinct approaches to the aggregation problem. Of course, this is not the same as saying that the (loosely speaking) 'regularizing' effects of *both* approaches could not be operating in the same setting to give some weak axiom-type structure to aggregate demand or excess demand. Indeed, in the context of classical exchange economies with collinear endowments, Quah (2000) has shown how both approaches may work in combination. In the context of economies with non-collinear endowments, it remains to be seen that a hybrid approach to the aggregation problem will lead to interesting results.

Appendix

The proof of Proposition 2.1 relies on the following lemma, which relates g with the classical demand function f. Though it is not always stated in the form presented here, the result is well known so we will skip the proof (see, for example, Magill and Quinzii (1996)).

Lemma A: *Let $\succeq$ be a regular preference generating the demand function g. There is a C^1 function $P : Q \to R^{l+1}_{++}$ such that $\bar{D}^T g(q,w) = f(P(q),w)$ and $\bar{D}P(q) = q$*

Proof of Proposition 2.1: The original MMP monotonicity result was set in a complete markets context, i.e., they identify conditions which guarantee the monotonicity of the classical demand function f. This proposition is just an extension of the MMP result to the function g. Our proof will rely on the original MMP result and also on Lemma A, with the latter allowing us to 'move' between g and f.

(i) By Lemma A, $\bar{D}^T g(q,w) = f(P(q),w)$. It is known that when $\psi_{\succeq}(\bar{D}^T g(q,w)) =$

$\psi_{\succeq}(f(P(q),w)) < 4$, there is an open and convex neighborhood N of $P(q)$ such that for any distinct p' and p'' in N, we have $(p'-p'')^T(f(p',w)-f(p'',w)) < 0$ (see Mas-Colell (1991) and Quah (2003)). Lemma A tells us that P is C^1, so in particular it is continuous and there is an open and convex neighborhood M of q such that whenever q' is in M, $P(q')$ is in N. So if q' and q'' are distinct prices in M, we have $(P(q')-P(q''))^T(f(P(q'),w)-f(P(q''),w)) < 0$. Substituting $\bar{D}^T g(q',w) = f(P(q'),w)$ and $\bar{D}^T g(q'',w) = f(P(q''),w)$ into this inequality tells us that $(q'-q'')^T(g(q',w)-g(q'',w)) < 0$.

(ii) By Lemma A, we have

$$(q'-q'')^T(g(q',w)-g(q'',w)) = (P(q')-P(q''))^T(f(P(q'),w)-f(P(q''),w)).$$

The latter will be less than zero if q' and q'' (and hence $P(q')$ and $P(q'')$) are distinct and $\psi_{\succeq}(x) < 4$ for all x (see Mas-Colell (1991) and Quah (2003)). *QED*

The proof of Theorem 2.3 requires the following lemma.

LEMMA B: *Let u be a regular utility function that generates a regular indirect utility function v. (i) If u is locally concave at $x^* = f(p^*,1)$, then $\phi_v(p^*,1) \leq 2$. Conversely, if $\phi_v(p^*,1) < 2$, then u is locally concave at $x^* = f(p^*,1)$. (ii) If v is locally convex in prices at $(p^*,1)$, then $\psi_u(f(p^*,1)) \leq 2$. Conversely, if $\psi_u(f(p^*,1)) < 2$, then v is locally convex in prices at $x^* = f(p^*,1)$.*

Proof: See and adapt the proof of Proposition 2.4 in Quah (2000).

PROOF OF THEOREM 2.3: Without loss of generality, we may assume that $w = 1$. We will show that at $p = p^*$, we have $\psi_{\succeq'}(f(p^*,1)) \geq \phi_{\succeq}(p^*,1)$. The proof of the other direction is analogous. Let u be a locally concave and regular representation of $\succeq'$. Using a linear transformation on u if necessary, we could guarantee that $\partial_x u(x^*)x^* = 1$, where

$x^* = f(p^*, 1)$. The function $\tilde{u} = h \circ u$, where h is increasing, satisfies

$$(8) \qquad \psi_{\tilde{u}}(x) \quad = \quad -\frac{h''(u(x))}{h'(u(x))}(\partial_x u(x)x) - \frac{x^T \partial_x^2 u(x) x}{\partial_x u(x) x}.$$

If $\psi_u(x^*) = M$, then by choosing h such that $h^{''}/h^{'} = M' - 2$, with M' greater than M, we find that $\psi_{\tilde{u}}(x^*) < 2$. By Lemma B(ii), $\tilde{v} = h \circ v$, the indirect utility generated by $\tilde{u}$ is convex in prices in a neighborhood of $(p^*, 1)$. Note that $-\partial_p v(p^*, 1)p^* = v_w(p^*, 1) = \partial_x u(x^*)x^* = 1$, and that $\phi_v(p^*, 1) \leq 2$, the latter following from the concavity of u (by Lemma B(i)). Therefore

$$\phi_{\tilde{v}}(p^*, 1) \quad = \quad -\left[\frac{h''(v(p^*, 1))}{h'(v(p^*, 1))}\right] \partial_p v(p^*, 1)p^* + \phi_v(p^*, 1) \leq M'.$$

Since M' could be chosen to be arbitrarily close to M, we obtain $\psi_{\succeq'}(f(p^*, 1)) \geq \phi_{\succeq}(p^*, 1)$.

QED

The proof of Theorem 3.1 is quite elaborate, so it is best that we break it up into a few more manageable lemmas. Let $v : R^{l+2}_{++} \to R$ be an indirect utility function. We define

$$\epsilon_v(p, w) = \frac{w v_{ww}(p, w)}{v_w(p, w)}.$$

Since v_w is the marginal utility of wealth, ϵ_v is the *wealth elasticity of the marginal utility of wealth.* The relationship between $\phi_v(p, w)$ and $\epsilon_v(p, w)$ is given in the next lemma. (This result can also be found in Quah (2000) but we reproduce it here for completeness.)

Lemma C: *For any regular indirect utility function $v : R^{l+2}_{++} \to R$,*

$$(9) \qquad \phi_v(p, w) \quad = \quad 2 + \epsilon_v(p, w).$$

Proof: Since v is zero-homogeneous, the denominator of $\phi_v(p,w)$, which is $\partial_p v(p,w)p$ (see (5)), equals $-v_w(p,w)$ (by Euler's identity). The numerator of the $\phi_v(p,w)$ formula, which is $-p^T\partial_p^2 v(p,w)p$, can also be re-written in terms of v_w and v_{ww}. Since $\partial v/\partial p^i$ is (-1)-homogeneous, Euler's identity tells us that

$$\sum_{j=0}^{l} \frac{\partial^2 v}{\partial p^j \partial p^i} p^j = -\frac{\partial v}{\partial p^i} - w\frac{\partial^2 v}{\partial w \partial p^i}.$$

(Note that we have omitted the arguments to save space.) So

$$p^T\partial_p^2 v(p,w)p = \sum_{i=0}^{l} -p^i\frac{\partial v}{\partial p^i} - \sum_{i=0}^{l} wp^i\frac{\partial^2 v}{\partial w \partial p^i}.$$

Using Euler's identity again, we see that the first sum on the right equals wv_w and the second sum equals $wv_w + w^2 v_{ww}$. So we obtain (9). QED

The next lemma is the central technical result of this paper because it links type a's indirect MMP coefficient, $\phi_{\succeq'_a}$, with $\partial_q \tilde{g}_a$. The proof makes crucial use of Roy's identity.

Lemma D: *Suppose that there is an open and convex neighborhood N_a of $(q, q^T\omega_a)$ in which g_a is generated by a regular indirect preference $\succeq'_a$. Then for any $z \in R^{m+1}$,*

$$\begin{aligned} 4z^T\partial_q\tilde{g}_a(q)z \;\; &\leq \;\; -4\left[\frac{(z^T\tilde{g}_a(q))^2}{q^T\omega_a} - \frac{(z^T\omega_a)(z^T\tilde{g}_a(q))}{q^T\omega_a}\right] \\ &\quad +\phi_{\succeq'_a}\left(P(q), q^T\omega_a\right)\frac{\left[z^T(\tilde{g}_a(q)-\omega_a)\right]^2}{q^T\omega_a}. \end{aligned} \tag{10}$$

Proof: To simplify notation, we will drop the subscript a. Since $\tilde{g}(q) = g(q, q^T\omega)$, we have

$$\partial_q\tilde{g}(q) \;\; = \;\; \partial_q g(q, q^T\omega) + \partial_w g(q, q^T\omega)\omega^T \tag{11}$$

Our first objective is to show that for any z in R^{m+1}, and denoting the vector $\partial_q P(q)z$ in R^{l+1} by $\hat{z}$, we have

$$(12) \qquad z^T \partial_q \tilde{g}(q) z = \hat{z}^T \partial_p f(P(q), q^T\omega)\hat{z} + \left[\hat{z}^T \partial_w f(P(q), q^T\omega)\right]\left[\hat{z}^T(\bar{D}^T\omega)\right].$$

By Lemma A, $q = \bar{D}P(q)$; differentiating this with respect to q we obtain $I = \bar{D}\partial_q P(q)$. So the matrix $\partial_q P(q)$ is a right inverse of $\bar{D}$. Differentiating the identity $\bar{D}^T g(q, w) = f(P(q), w)$ with respect to q we obtain

$$(13) \qquad \bar{D}^T \partial_q g(q, w) = \partial_p f(P(q), w)\partial_q P(q).$$

If we pre-multiply $\bar{D}^T \partial_q g(q, w)$ by $\hat{z}^T = (\partial_q P(q)z)^T$ and post-multiply it by z, we obtain $z\partial_q g(q, w)z$ so we conclude that

$$(14) \qquad z^T \partial_q g(q, w) z = (\partial_q P(q)z)^T \partial_p f(P(q), w)(\partial_q P(q)z) = \hat{z}^T \partial_p f(P(q), w)\hat{z}.$$

Since g is zero-homogeneous, $\partial_q g(q, w)q = -w\partial_w g(q, w)$. Similarly, because the map from (q, w) to $\bar{D}^T g(q, w) = f(P(q), w)$ is zero-homogeneous, $\partial_p f(P(q), w)\partial_q P(q)q = -w\partial_w f(P(q), w)$. Post-multiplying (13) by q, we obtain $\bar{D}^T \partial_w g(q, w) = \partial_w f(P(q), w)$. Since $\partial_q P(q)^T$ is the left inverse of $\bar{D}^T$, pre-multiplying this equation by $z^T \partial_q P(q)^T$ gives us

$$(15) \qquad z^T \partial_w g(P(q), w) = (\partial_q P(q)z)^T \partial_w f(P(q), w) = \hat{z}^T \partial_w f(P(q), w).$$

Equations (11), (14) and (15), with $w = q^T\omega$ will give us (12).

Suppose that g is generated by the direct preference $\succeq$, which has $\succeq'$ as its associated indirect preference. Let v be an indirect utility function which represents $\succeq'$ in some open neighborhood of $(p, p^T\theta)$, We will now show that for any vector $\hat{z}$ in R^{l+1},

$$(16) \qquad 4\hat{z}^T \partial_p f(p, \theta^T p)\hat{z} + 4\left(\hat{z}^T \partial_w f(p, \theta^T p)\right)\left(\hat{z}^T\theta\right)$$

$$\leq \quad -4\left[\frac{(\hat{z}^T f(p,\theta^T p))^2}{p^T\theta} - \frac{(\hat{z}^T\theta)(\hat{z}^T f(p,\theta^T p))}{p^T\theta}\right] + \phi_v(p, p^T\theta)\frac{[\hat{z}^T(f(p,p^T\theta)-\theta)]^2}{p^T\theta}.$$

By making the substitutions $\hat{z} = \partial_q P(q)z$, $p = P(q)$, $\bar{D}P(q) = q$, $\theta = \bar{D}^T\omega$, and $f(P(q),(\bar{D}^T\omega)^T P(q)) = \bar{D}g(q, q^T\omega)$ in (16), and replacing the left hand side by $4z^T\tilde{g}(q)z$ (which we can do by (12)), we obtain

$$4z^T\partial_q\tilde{g}(q)z \quad \leq \quad -4\left[\frac{(z^T\tilde{g}(q))^2}{q^T\omega} - \frac{(z^T\omega)(z^T\tilde{g}(q))}{q^T\omega}\right] + \phi_v(P(q), q^T\omega)\frac{\left[z^T(\tilde{g}(q)-\omega)\right]^2}{q^T\omega}.$$

By the definition of ϕ, we may replace $\phi_v(P(q), q^T\omega)$, with $\phi_{\succeq'_a}(P(q), q^T\omega)$ and so obtain (10).

It remains for us to proof (16). By Roy's identity, $f(p,w) = -\partial_p v(p,w)^T/v_w(p,w)$. Differentiating this with respect to p and using the fact that $v_w(p,w) = -\partial_p v(p,1)p/w$, we obtain

$$\partial_p f \quad = \quad -\frac{\partial_p^2 v}{v_w} - \frac{(\partial_p v)^T p^T \partial_p^2 v}{w v_w^2} - \frac{(\partial_p v)^T(\partial_p v)}{w v_w^2}. \tag{17}$$

(Note that we have omitted the arguments to save space.) Differentiating Roy's identity with respect to w gives us

$$\partial_w f \quad = \quad \frac{v_{ww}(\partial_p v)^T}{v_w^2} + \frac{(\partial_p v)^T}{w v_w} + \frac{\partial_p^2 v\; p}{w v_w} \tag{18}$$

Combining (17) and (18) and using Roy's identity again, we see that

$$\begin{aligned} &\hat{z}^T\partial_p f\hat{z} + (\hat{z}^T\theta)(\partial_w f^T\hat{z}) \\ = \quad & -\frac{\hat{z}^T\partial_p^2 v\hat{z}}{v_w} + \left[\hat{z}^T(f+\theta)\right]\frac{p^T\partial_p^2 v\;\hat{z}}{w v_w} - \frac{(\hat{z}^T f)^2}{w} + \left(-\frac{1}{w} - \frac{v_{ww}}{v_w}\right)(\hat{z}^T f)(\hat{z}^T\theta) \end{aligned}$$

Using Lemma C and denoting $s = z^T(f+\theta)/w$, the right hand of this equation could be re-written (by 'completing squares') as

$$-\frac{1}{v_w}\left(\hat{z} - \frac{s}{2}p\right)^T\partial_p^2 v\left(\hat{z} - \frac{s}{2}p\right) + \frac{ws^2}{4}\phi - \frac{(\hat{z}^T f)^2}{w} + \left(-\frac{1}{w} - \frac{(\phi-2)}{w}\right)(\hat{z}^T f)(\hat{z}^T\theta),$$

which is clearly less than

$$\frac{ws^2}{4}\phi - \frac{(\hat{z}^T f)^2}{w} + \left(-\frac{1}{w} - \frac{\phi - 2}{w}\right)(\hat{z}^T f)(\hat{z}^T \theta).$$

Substituting $p^T\theta$ for w, re-substituting $\hat{z}^T(f+\theta)/w$ for s, and re-arranging the terms in this expression, we obtain (16). QED

Proof of Theorem 3.1: Note that by Theorem 2.3, we may replace $\phi_{\succeq'_a}(P(q), q^T\omega_a)$ in (10) with $\psi_{\succeq_a}(\bar{D}^T\tilde{g}(q))$. Furthermore, at the equilibrium price $\bar{q}$, we may replace $\psi_{\succeq_a}(\bar{D}^T\tilde{g}(\bar{q}))$ with $\bar{\psi}(\bar{q})$ since the latter is greater than the former by definition. Making this replacement and then aggregating the inequality over A, we obtain

$$\begin{aligned} 4z^T\partial_q\zeta(\bar{q})z &= 4z^T\partial_q G(\bar{q})z \\ &\leq -4\left[\int_A \frac{(z^T\tilde{g}_a(\bar{q}))^2}{\bar{q}^T\omega_a}d\mu - \int_A \frac{(z^T\omega_a)(z^T\tilde{g}_a(\bar{q}))}{\bar{q}^T\omega_a}d\mu\right] + \bar{\psi}(\bar{q})\int_A \frac{\left[z^T(\tilde{g}_a(\bar{q})-\omega_a)\right]^2}{\bar{q}^T\omega_a}d\mu. \end{aligned} \tag{19}$$

Recall that the equilibrium price $\bar{q}$ was normalized to satisfy $\bar{q}^T\bar{\omega} = 1$. Recall also the definitions of $\hat{\mu}$, $\hat{\omega}_a$ and $\hat{g}_a$ in Section 3. The first integral on the right hand side of (19)

$$\int_A \frac{(z^T\tilde{g}_a(\bar{q}))^2}{\bar{q}^T\omega_a}d\mu = \int_A (z^T\hat{g}_a(\bar{q}))^2 d\hat{\mu};$$

a similar transformation can be made to the other integrals on the right hand side of (19). This gives us

$$\begin{aligned} 4z^T\partial_q\zeta(\bar{q})z \leq & -4\left[\int_A (z^T\hat{g}_a(\bar{q}))^2 d\hat{\mu} - \int_A (z^T\hat{\omega}_a)(z^T\hat{g}_a(\bar{q}))d\hat{\mu}\right] \\ & +\bar{\psi}(\bar{q})\int_A \left[z^T(\hat{g}_a(q)-\hat{\omega}_a)\right]^2 d\hat{\mu}. \end{aligned} \tag{20}$$

Since $\bar{q}$ is the equilibrium price,

$$\int_A z^T\hat{g}_a(\bar{q})d\hat{\mu} = \int_A z^T g_a(\bar{q})d\mu = \int_A z^T\omega_a d\mu = \int_A z^T\hat{\omega}_a d\hat{\mu}.$$

It follows that the right hand side of (20) equals

$$-4\left[\mathrm{Var}(z^T\hat{g}, z^T\hat{g}) - \mathrm{Cov}(z^T\hat{g}, z^T\hat{\omega})\right] + \bar{\psi}(\bar{q})\mathrm{Var}(z^T(\hat{g}-\hat{\omega}), z^T(\hat{g}-\hat{\omega})),$$

which is $z^T L(\bar{q})z$. It follows that if $L(\bar{q})$ is negative definite on $\bar{q}^{\perp}$, so is $\partial_q\zeta(\bar{q})$. QED

Proof of Corollary 3.2: By Theorem 3.1, we need only show that the conditions in (i) to (iv) all lead to $L(\bar{q})$ being negative definite on $\bar{q}^{\perp}$. For (i), we re-write

$$(21) \qquad L(\bar{q}) \;=\; -2\left[\mathrm{Cov}(\hat{g},\hat{g}) - \mathrm{Cov}(\hat{\omega},\hat{\omega})\right] + (\bar{\psi}(\bar{q})-2)\mathrm{Cov}(\hat{g}-\hat{\omega}, \hat{g}-\hat{\omega}).$$

If $\theta < 1$, $\mathrm{Cov}(\hat{g},\hat{g}) - \mathrm{Cov}(\hat{\omega},\hat{\omega})$ is positive definite on $\bar{q}^{\perp}$ while $\mathrm{Cov}(\hat{g}-\hat{\omega},\hat{g}-\hat{\omega})$ is positive semidefinite, so $L(\bar{q})$ is negative definite on $\bar{q}^{\perp}$ if $\bar{\psi}(\bar{q}) \leq 2$.

For (ii) to (iv), it useful to re-write

$$(22) \qquad L(\bar{q}) \;=\; (-4+\bar{\psi}(\bar{q}))\,\mathrm{Cov}(\hat{g},\hat{g}) - 2(\bar{\psi}(\bar{q})-2)\mathrm{Cov}(\hat{g},\hat{\omega}) + \bar{\psi}(\bar{q})\mathrm{Cov}(\hat{\omega},\hat{\omega}).$$

If $K_1 = K_2 = 0$, for $z \neq 0$ in $\bar{q}^{\perp}$,

$$z^T L(\bar{q})z \leq (-4+\bar{\psi}(\bar{q}))\mathrm{Var}(z^T\hat{g}, z^T\hat{g}) + \bar{\psi}(\bar{q})\theta\mathrm{Var}(z^T\hat{g}, z^T\hat{g}).$$

The right hand side is negative if $\bar{\psi}(\bar{q}) < 4/(1+\theta)$. So we have shown (ii). The proofs for (iii) and (iv) are similar. QED

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Footnotes

1. For a textbook reference to all the claims in this paragraph see Mas-Colell et al (1995). On comparative statics, some recent results which make use of a weak axiomatic structure on the excess demand function can be found in Nachbar (2002, 2004) and Quah (2003); Quah (2003) also has a discussion of comparative statics in a financial economy.

2. Milleron's paper is in French and never published, while the paper of Mitjuschin and Polterovich is in Russian. English language versions of their theorem can be found in Mas-Colell (1991) and Quah (2000, 2003), amongst other places. These English language versions repeat (or adapt) the proof used by Mitjuschin and Polterovich rather than the proof of Milleron.

3. For an alternative characterization of monotonicity using the normalized gradient function see Kannai (1989).

4. The usual (and perhaps better known) application of the MMP monotonicity theorem says that an *upper bound* of 4 on the coefficient of relative risk aversion is sufficient to guarantee monotonicity (see, for example, Mas-Colell (1991), Dana (1995) and Bettzuge (1998)). The subtler result here is from Quah (2003).

5. For the claims in this paragraph, see Mas-Colell (1991) or Mas-Colell et al (1995).

6. When markets are complete and agents maximize expected utility, there is a result which says that market excess demand obeys gross substitutability (and hence all the other nice properties which flow from it (see Mas-Colell et al (1995)) if all agents have coefficients of relative risk aversion which are bounded above by one (see Mas-Colell (1991) or Hens and Loffler (1995)). Note that this is an upper (and quite stringent) bound on the coefficient of

relative risk aversion, unlike the MMP condition which is a bound on the coefficient's variation across income. Nonetheless, this result is very nice because it requires no substantial assumptions on agents' endowments. Unfortunately, this result does not extend easily to incomplete markets. (See also Hens and Pilgrim (2002) for an extensive discussion of gross substitutability and related concepts in financial economies.)

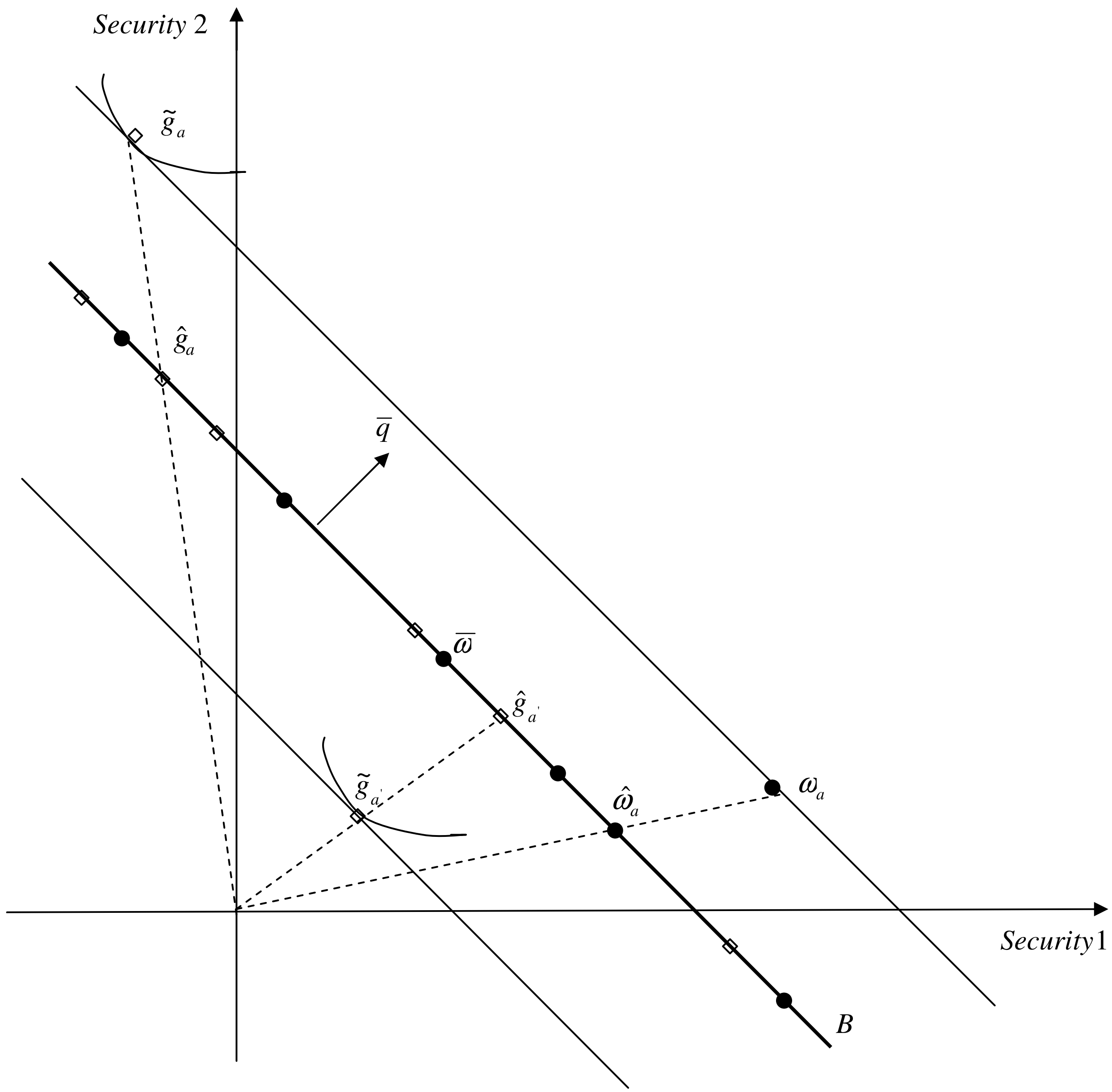
7. The formulation of a financial economy in this section and the next is broadly along standard lines. See Magill and Quinzii (1996) for a textbook introduction.

8. This claim is rather misleading only because the truth is considerably stronger (see Mas-Colell (1985)).

9. Once again, see Mas-Colell (1985). The results there pertain to the existence of concave utility representations for regular (direct) preferences, but it is clear they also apply, mutatis mutandi, to the representation of regular indirect preferences by indirect utility functions which are convex in prices.

10. The rationalizing economy constructed in Hens' indeterminacy theorem have agents with homothetic preferences and also, as in the model considered here, endowments which are in the asset span.

11. This is explicit in (ii) and (iv) where it is assumed that $K_2 < 1$. In (i) and (iii), we assume that $\mathrm{Cov}(\hat{g}, \hat{g}) - \mathrm{Cov}(\hat{\omega}, \hat{\omega})$ is positive definite on $\bar{q}^{\perp}$, which implies (A); we can see this by expanding the positive semidefinite matrix $\mathrm{Cov}\hat{g} - \hat{\omega}, \hat{g} - \hat{\omega})$.



Security 2
$\tilde{g}_a$
$\hat{g}_a$
$\overline{q}$
$\overline{\omega}$
$\hat{g}_{a'}$
$\tilde{g}_{a'}$
$\hat{\omega}_a$
ω_a
Security 1
B

FIGURE 1