Power and bipower variation with stochastic volatility and jumps

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Abstract

This paper shows that realised power variation and its extension we introduce here called realised bipower variation is somewhat robust to rare jumps. We show realised bipower variation estimates integrated variance in SV models — thus providing a model free and consistent alternative to realised variance. Its robustness property means that if we have an SV plus infrequent jumps process then the difference between realised variance and realised bipower variation estimates the quadratic variation of the jump component. This seems to be the first method which can divide up quadratic variation into its continuous and jump components. Various extensions are given. Proofs of special cases of these results are given. Detailed mathematical results will be reported elsewhere.
Keywords: Bipower variation; Integrated variance; Jump process; Power variation; Quadratic variation; Realised variance; Realised volatility; Semimartingale; Volatility.

Contents

1 Introduction

1.1 Motivation

The econometrics of financial volatility has made very significant recent process due to the harnessing of high frequency information through the use of realised variances and volatility. Here we discuss generalisations of objects of that type, studying sums of powers and products of powers of absolute returns. We will show that these quantities, called realised power variation and the new realised bipower variation we introduce here, are quite robust to rare jumps in the log-price process. In particular we demonstrate that it is possible, in theory, to untangle the presence of volatility and rare jumps by using power and bipower variation. Realised bipower variation also provides a new asymptotically unbiased, model free econometric estimator of integrated variance in stochastic volatility models. This estimator is robust to the presence of jumps. Hence, in theory, we can now decompose quadratic variation into the contribution from the continuous component of log-prices and the impact of jumps. We believe this is the first paper to do this without making strong parametric assumptions.

To start suppose $\hbar > 0$ is some fixed time period (e.g. a trading day or a calendar month) and that the log-price of an asset is written as $y^*(t)$ for $t \geq 0$. Then the i-th $\hbar$ "low frequency" return is

$$y_i = y^*\left(i\hbar\right) - y^*\left(\left(i-1\right)\hbar\right), \quad i = 1, 2, \ldots .$$

For concreteness we will often refer to the i-th period as the i-th day.

Suppose that additionally we record the prices at M equally spaced time points on the i-th day. Then we can define the "high frequency" returns as

$$y_{j,i} = y^*\left(\left(i-1\right)\hbar + \hbar j M^{-1}\right) - y^*\left(\left(i-1\right)\hbar + \hbar\left(j-1\right)M^{-1}\right), \quad j = 1, 2, ..., M. \tag{1}$$

Here $y_{j,i}$ is the j-th intra-$\hbar$ return for the i-th day (e.g. if $M = 288$, then this is the return for the j-th 5 minute period on the i-th day). As a result, for example, $y_i = \sum_{j=1}^{M} y_{j,i}$.

To illustrate this notation we will look at the first three days of the Olsen Dollar/DM high frequency series. It starts on 1st December 1986 and ignores weekend breaks. This series is constructed every five minutes by the Olsen group from bid and ask quotes which appeared on the Reuters screen (see Dacorogna, Gencay, Müller, Olsen, and Pictet (2001) for details). The log values of the raw data, having being preprocessed in a manner discussed in the Appendix of Barndorff-Nielsen and Shephard (2002a), is shown in the top left of Figure 1. We have transformed the data to start at zero at time zero. The series constitutes an empirical approximation, recorded every five minutes, to our y^* process. The top right part of the Figure shows the three daily returns $y_{1,i} = y_i$, i.e. $M = 1$, calculated off this series. These returns are shown as large

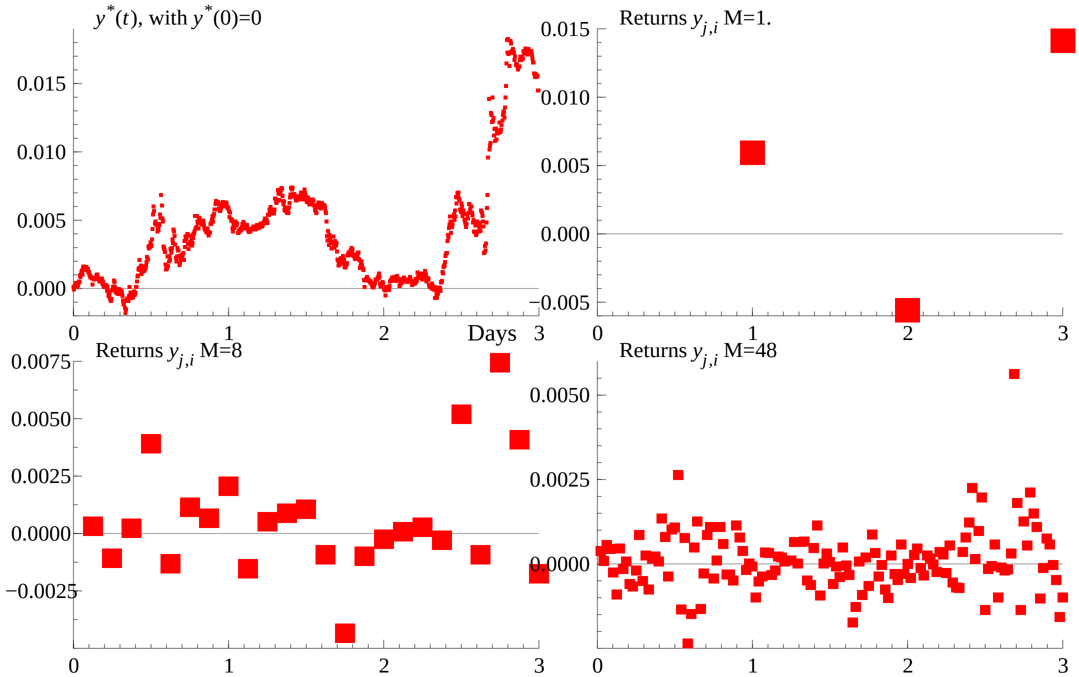

Figure 1: *DM/Dollar dataset from the Olsen and Associates database. Top left: raw* 5 *minute data. Top right: daily returns,* $M = 1$*. Bottom left:* 3 *hour returns,* $M = 8$*. Bottom right:* 30 *minute returns,* $M = 48$*.*

squares. The bottom left part shows $y_{j,i}$, for $j = 1, 2, ..., 8$ and $i = 1, 2, 3$. These are 3 hour returns. Finally, the bottom right graph shows the effect of taking $M = 48$, which corresponds to 30 minute returns.

We start out this paper by assessing how jumps in log-prices effect the volatility measure: *realised power variation*

$$\{y_M^*\}_i^{[r]} = \left(\frac{\hbar}{M}\right)^{1-r/2} \sum_{j=1}^{M} |y_{j,i}|^r, \quad r > 0$$

and, to a lesser extent, its unnormalised version[1]

$$[y_M^*]_i^{[r]} = \sum_{j=1}^{M} |y_{j,i}|^r.$$

[1]The similarly named p-variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$\sup_{\kappa} \sum |f(x_i) - f(x_{i-1})|^p,$$

where the supremum is taken over all subdivisions κ of $[a, b]$. If this function is finite then f is said to have bounded p-variation on $[a, b]$. The case of $p = 1$ gives the usual definition of bounded variation. This concept of p-variation has been studied recently in the probability literature. See the work of Lyons (1994), Mikosch and Norvaiša (2000) and Lyons and Qian (2002).

These quantities have recently been formalised by Barndorff-Nielsen and Shephard (2003a). Here we give two examples of this. When $r = 2$ this yields the traditional *realised variance*[2] or *realised quadratic variation*

$$\{y_M^*\}_i^{[2]} = [y_M^*]_i^{[2]} = \sum_{j=1}^{M} y_{j,i}^2. \tag{2}$$

This is linked to its square root version, the *realised volatility*[3] $\sqrt{\sum_{j=1}^{M} y_{j,i}^2}$. When $r = 1$ we produce $\{y_M^*\}_i^{[1]} = \sqrt{\frac{\hbar}{M}} \sum_{j=1}^{M} |y_{j,i}|$. Of course many other values of r are also interesting.

The most important practical contribution of this paper is the introduction of a new cross term estimator called *realised bipower variation*

$$\{y_M^*\}_i^{[r,s]} = \left\{ \left(\frac{\hbar}{M} \right)^{1-(r+s)/2} \right\} \sum_{j=1}^{M-1} |y_{j,i}|^r \, |y_{j+1,i}|^s, \quad r, s \geq 0. \tag{3}$$

In particular, for example, we will show that in the $r = s = 1$ case

$$\{y_M^*\}_i^{[1,1]} = \sum_{j=1}^{M-1} |y_{j,i}| \, |y_{j+1,i}| \tag{4}$$

will, up to a simple known multiple, converge to the same probability limit as realised variance when prices follow a SV process and that for (4) the limit does not change with added rare jumps. This provides

- a new way of making inference and predicting integrated variance, perhaps the single most important term in the econometrics of volatility,
- a simple way of measuring the impact of jumps on quadratic variation.

1.2 Outline of the paper

In Section 2 of this paper we review the well known probability limit of realised variance, which is built on the theory of semimartingales and the quadratic variation process. We specialise the theory down to a class of continuous sample path stochastic volatility semimartingales. For this class it is possible to extend the quadratic variation process to the power variation process, which allows us to derive the probability limit of realised power variation. Finally, we introduce the idea of bipower variation and study some of its properties.

[2] Realised variances have been used for a long time in financial economics — see, for example, Poterba and Summers (1986), Schwert (1989) and Dacorogna, Müller, Olsen, and Pictet (1998). Realised variance has been studied from a methodological viewpoint by Andersen, Bollerslev, Diebold, and Labys (2001), Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen and Shephard (2002a). See Andersen, Bollerslev, and Diebold (2003) and Barndorff-Nielsen and Shephard (2004, Ch. 7) for surveys of this area, including discussions of the related literature.

[3] Note that in econometrics sums of squared returns are sometimes called realised volatility.

In Section 3 we find the probability limit of realised power variation in the case where we add a compound Poisson process to the SV process. We see that sometimes the limit is not changed. We extend this analysis to realised bipower variation and see that the robustness result holds so long as the joint powers sum is less than or equal to two. This result is the main contribution of this paper for it means, in particular, that realised bipower variation can be setup so that it consistently estimates the integrated variance even in the presence of jumps.

In Section 4 we conduct a simulation study of a SV process plus jump process, demonstrating that the theory seems to yield useful predictions. In Section 5 we apply the theory to some empirical data and in Section 6 we indicate various possible extensions of our work. Section 7 concludes.

2 Basic development and ideas

2.1 Realised variance and quadratic variation

The probability limit of realised variance (2) is known under the assumption that y^* is a semimartingale ($\mathcal{SM}$) using the theory of quadratic variation (e.g. Jacod and Shiryaev (1987, p. 55)). Recall if $y^* \in \mathcal{SM}$, then we can write

$$y^*(t) = \alpha^*(t) + m^*(t), \tag{5}$$

where α^*, a drift term, has *locally bounded variation* paths and m^* is a *local martingale*. One of the most important aspects of semimartingales is the *quadratic variation* (QV) process. This is defined as

$$[y^*](t) = \text{p}-\lim_{M\to\infty} \sum_{j=1}^{M} \{y^*(t_j) - y^*(t_{j-1})\}^2, \tag{6}$$

for any sequence of partitions $t_0 = 0 < t_1 < ... < t_M = t$ so long as $\sup_j\{t_j - t_{j-1}\} \to 0$ for $M \to \infty$. This implies that if $y^* \in \mathcal{SM}$ then

$$[y^*_M]^{[2]}_i = \sum_{j=1}^{M} y^2_{j,i} \xrightarrow{p} [y^*](\hbar i) - [y^*](\hbar(i-1)) = [y^*]_i,$$

meaning realised variance consistently estimates increments of QV.

In general

$$[y^*](t) = [y^{*c}](t) + \sum_{0\le s\le t} \{\Delta y^*(s)\}^2, \tag{7}$$

where y^{*c} is the continuous local martingale component of y^* and $\Delta y^*(t) = y^*(t) - y^*(t-)$ is the jump at time t. If α^* is continuous then we obtain the simplification

$$[y^*](t) = [m^*](t) = [m^{*c}](t) + \sum_{0\le s\le t} \{\Delta m^*(s)\}^2.$$

2.2 Stochastic volatility

To extend the results on quadratic variation we will need more assumptions. We start by stating two.

- (a) m^* is a stochastic volatility (SV) process[4]

$$m^*(t) = \int_0^t \sigma(u)\mathrm{d}w(u), \tag{8}$$

  where w is standard Brownian motion, $\sigma(t) > 0$, the *spot volatility* process, is càdlàg and locally bounded away from zero and the integrated variance process

$$\sigma^{2*}(t) = \int_0^t \sigma^2(u)\mathrm{d}u, \tag{9}$$

  satisfies $\sigma^{2*}(t) < \infty$ for all $t < \infty$.

- (b) the mean process α^* is continuous.

We call semimartingales satisfying (a) and (b) members of the *continuous SV semimartingales* class (denoted $\mathcal{SVSM}^c$). Clearly $\mathcal{SVSM}^c \subset \mathcal{SM}^c \subset \mathcal{SM}$, where $\mathcal{SM}^c$ is the class of continuous semimartingales.

Importantly, if $y^* \in \mathcal{SVSM}^c$ then

$$[y^*](t) = \sigma^{2*}(t). \tag{10}$$

If $D_-\{g(x)\}$ denotes the left derivative of a function g, $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\{g(x) - g(x-\varepsilon)\}$, then

$$D_-\{[y^*](t)\} = \sigma^2(t-), \quad \text{or} \quad \frac{\partial [y^*](t)}{\partial t} = \sigma^2(t)$$

under the additional assumption that σ is continuous. Finally, for all $y^* \in \mathcal{SVSM}^c$

$$\{y_M^*\}_i^{[2]} = [y_M^*]_i^{[2]} = \sum_{j=1}^M y_{j,i}^2 \xrightarrow{p} \sigma_i^2, \quad \text{where} \quad \sigma_i^2 = \sigma^{2*}(\hbar i) - \sigma^{2*}(\hbar(i-1)).$$

We call σ_i^2 and $\sqrt{\sigma_i^2}$ the actual variance and actual volatility, respectively, of the SV component over the i-th interval. All these results on SV processes are very well known. They hold irrespective of the relationship between α^*, σ and w.

[4]The literature of SV models is discussed by, for example, Harvey, Ruiz, and Shephard (1994), Taylor (1994), Ghysels, Harvey, and Renault (1996), Shephard (1996), Kim, Shephard, and Chib (1998) and Barndorff-Nielsen and Shephard (2004, Ch. 4 and 5).

2.3 Power variation process

The quadratic variation process $[y^*]$ was generalised to the r-th order *power variation* process ($r > 0$) by Barndorff-Nielsen and Shephard (2003a). It is defined, when it exists, as

$$\{y^*\}^{[r]}(t) = p\text{-}\lim_{\delta \downarrow 0} \delta^{1-r/2} \sum_{j=1}^{\lfloor t/\delta \rfloor} |y_j(t)|^r ,$$

where over an interval of length $\delta > 0$ the equally spaced j-th return is

$$y_j = y_j(t) = y^*(j\delta) - y^*((j-1)\delta).$$

Here, for any real number a, $\lfloor a \rfloor$ denotes the largest integer less than or equal to a.

The normalisation $\delta^{1-r/2}$ is key in power variation.

1. When $r = 2$ the normalisation is one and so disappears.

2. When $r > 2$ the normalisation goes off to infinity as $\delta \downarrow 0$.

3. When $r < 2$ the normalisation goes to zero as $\delta \downarrow 0$.

The key property of power variation for $\mathcal{SVSM}^c$ processes is given as follows.

Theorem 1 *If $y^* \in \mathcal{SVSM}^c$ and additionally (σ, α^*) are independent of w, then*

$$\{y^*\}^{[r]}(t) = \mu_r \int_0^t \sigma^r(s)\mathrm{d}s,$$

for $r > 0$ where $\mu_r = 2^{r/2}\Gamma\left(\frac{1}{2}(r+1)\right)/\Gamma\left(\frac{1}{2}\right)$.

□

Proof. See Barndorff-Nielsen and Shephard (2003a).

□

Remark 1 *It is so far unclear to us as to the substantial impact of relaxing the independence condition on (σ, α^*) and w. Note, no additional assumptions are made on the (σ, α^*) process beyond those stated in (a) and (b). Also note that $\mu_r = \mathrm{E}|u|^r$ where $u \sim N(0,1)$.*

This implies

$$\left(D_- \left[\mu_r^{-1} \{y^*\}^{[r]}(t)\right]\right)^{1/r} = \sigma(t-), \quad \text{or} \quad \left(\frac{\partial \mu_r^{-1} \{y^*\}^{[r]}(t)}{\partial t}\right)^{1/r} = \sigma(t)$$

if σ is continuous.

The probability limit of realised power variation follows immediately from the definition of the power variation process. Thus under the conditions of Theorem 1, as $M \to \infty$,

$$\begin{array}{rcl} \{y_M^*\}_i^{[r]} & \xrightarrow{p} & \{y^*\}^{[r]}(\hbar i) - \{y^*\}^{[r]}(\hbar(i-1)) \\ & = & \mu_r \int_{\hbar(i-1)}^{\hbar i} \sigma^r(s)\mathrm{d}s. \end{array} \tag{11}$$

Barndorff-Nielsen and Shephard (2003a) have extended the above convergence in probability to a distribution theory for $\{y^*\}^{[r]}(t) - \mu_r \int_0^t \sigma^r(s)\mathrm{d}s$. See further Section 6.4. Andreou and Ghysels (2003) have used power variations to test for changes in the level of volatility in financial markets.

2.4 Bipower variation process

In the context of multivariate covariation Barndorff-Nielsen and Shephard (2002b) sometimes found it helpful to study cross terms of the type

$$\delta^{-1} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} y_j^2 y_{j+1}^2 \xrightarrow{p} \int_0^t \sigma^4(u)\mathrm{d}u.$$

Here we provide an extension of this. First we define the *bipower variation process*

$$\{y^*\}^{[r,s]}(t) = p\text{-}\lim_{\delta \downarrow 0} \delta^{1-(r+s)/2} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j|^r |y_{j+1}|^s, \quad r, s \geq 0$$

provided this exists. We will define a more general k-th order power variation process in Section 6.2. Clearly the value of $r+s$ is crucial here, as it determines the normalisation by δ. Importantly

$$\{y^*\}^{[r,0]}(t) = \{y^*\}^{[0,r]}(t) = \{y^*\}^{[r]}(t).$$

Hence the bipower variation process includes as a special case the power variation process.

If $y^* \in \mathcal{SVSM}^c$ and additionally (σ, α^*) are independent of w, then we would expect that

$$\{y^*\}^{[r,s]}(t) = \mu_r \mu_s \int_0^t \sigma^{r+s}(u)\mathrm{d}u,$$

for $r + s > 0$ where $\mu_r = 2^{r/2}\Gamma\left(\frac{1}{2}(r+1)\right)/\Gamma\left(\frac{1}{2}\right)$. We will discuss a proof of this general result elsewhere. Again it is unclear as to the importance of the assumed independence assumption. Overall, this result means that it provides a second way of accessing integrated power volatility, extending previous work on realised power variation. For now we prove a special case.

Theorem 2 *Suppose $y^* \in \mathcal{SVSM}^c$ and additionally σ is independent of w and $\alpha^* = 0$. Then, for any $r > 0$, we have*

$$\{y^*\}^{[r,r]}(t) = \mu_r^2 \int_0^t \sigma^{2r}(s)\mathrm{d}s. \tag{12}$$

□

Proof. It is convenient to introduce the notation

$$\{y_\delta^*\}^{[r,r]}(t) = \delta^{1-r} \sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^r.$$

Here, and elsewhere, we use δ rather than M as lower index; recall that $\delta M = t$. The value of the bipower process $\{y^*\}^{[r,r]}$ at time t is the probability limit, as $\delta \downarrow 0$, of $\{y_\delta^*\}^{[r,r]}(t)$. In determining this limit it causes no loss of generality to assume that t/δ is an integer M, say.

Let $\sigma_j > 0$ be defined by

$$\sigma_j^2 = \sigma^{2*}(j\delta) - \sigma^{2*}((j-1)\delta).$$

We then have

$$\{y_\delta^*\}^{[r,r]}(t) \overset{law}{=} \delta^{1-r} \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r |u_j|^r |u_{j+1}|^r$$

where the u_j are i.i.d. standard normal. An application of Barndorff-Nielsen and Shephard (2003b, Corollary 4.3) to the sum on the right hand side shows that

$$\delta^{1-r} \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r (|u_j|^r |u_{j+1}|^r - \mu_r^2) \xrightarrow{p} 0$$

for $\delta \downarrow 0$.

Since (cf. Barndorff-Nielsen and Shephard (2003a))

$$\delta^{1-r} \sum_{j=1}^{M} \sigma_j^{2r} \xrightarrow{p} \sigma^{2r*}(t) \tag{13}$$

it only remains to prove that

$$\delta^{1-r} \left\{ \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r - \sum_{j=1}^{M} \sigma_j^{2r} \right\} \xrightarrow{p} 0. \tag{14}$$

In fact we have the stronger result that

$$\delta^{1-r} \left\{ \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r - \sum_{j=1}^{M} \sigma_j^{2r} \right\} = O(\delta) \tag{15}$$

To establish this we do the following rewrite

$$\begin{aligned} \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r - \sum_{j=1}^{M} \sigma_j^{2r} &= \sum_{j=1}^{M-1} (\sigma_j^r \sigma_{j+1}^r - \sigma_j^{2r}) - \sigma_M^{2r} \\ &= \sum_{j=1}^{M-1} \sigma_j^r (\sigma_{j+1}^r - \sigma_j^r) - \sigma_M^{2r} \end{aligned}$$

$$= \sum_{j=1}^{M-1} \frac{\sigma_j^r}{\sigma_{j+1}^r + \sigma_j^r} (\sigma_{j+1}^{2r} - \sigma_j^{2r}) - \sigma_M^{2r}$$

$$= \frac{1}{2} \sum_{j=1}^{M-1} \left(\frac{2\sigma_j^r}{\sigma_{j+1}^r + \sigma_j^r} - 1 \right) (\sigma_{j+1}^{2r} - \sigma_j^{2r}) - \frac{1}{2}(\sigma_1^{2r} + \sigma_M^{2r})$$

$$= -\frac{1}{2} \sum_{j=1}^{M-1} \frac{\sigma_{j+1}^r - \sigma_j^r}{\sigma_{j+1}^r + \sigma_j^r} (\sigma_{j+1}^{2r} - \sigma_j^{2r}) - \frac{1}{2}(\sigma_1^{2r} + \sigma_M^{2r})$$

$$= -\frac{1}{2} \sum_{j=1}^{M-1} \frac{(\sigma_{j+1}^{2r} - \sigma_j^{2r})^2}{(\sigma_{j+1}^r + \sigma_j^r)^2} - \frac{1}{2}(\sigma_1^{2r} + \sigma_M^{2r}).$$

Further, letting $\psi_j = \delta^{-1/2}\sigma_j$ we find

$$\delta^{1-r} \left\{ \sum_{j=1}^{M-1} \sigma_j^r \sigma_{j+1}^r - \sum_{j=1}^{M} \sigma_j^{2r} \right\} = -\frac{1}{2}\delta \left\{ \sum_{j=1}^{M-1} \frac{(\psi_{j+1}^{2r} - \psi_j^{2r})^2}{(\psi_{j+1}^r + \psi_j^r)^2} + \psi_1^{2r} + \psi_M^{2r} \right\}.$$

By condition (a) we have, in particular, that

$$0 < \inf_{0 \le s \le t} \sigma(s) \le \inf \psi_j \le \sup \psi_j \le \sup_{0 \le s \le t} \sigma(s) < \infty$$

uniformly in δ. From this it is immediate that $\psi_1^{2r} + \psi_M^{2r}$ is uniformly bounded from above. Furthermore,

$$\sum_{j=1}^{M-1} \frac{(\psi_{j+1}^{2r} - \psi_j^{2r})^2}{(\psi_{j+1}^r + \psi_j^r)^2} \le \frac{1}{4} \frac{1}{\inf_{0 \le s \le t} \sigma^r(s)} \sum_{j=1}^{M-1} (\psi_{j+1}^{2r} - \psi_j^{2r})^2$$

where $\inf_{0 \le s \le t} \sigma^r(s) > 0$ and

$$\sum_{j=1}^{M-1} (\psi_{j+1}^{2r} - \psi_j^{2r})^2 \xrightarrow{p} [\sigma^{2r}](t)$$

which implies (15) and hence (12).

□

By setting $r = 1$ we have a new estimator of an important quantity in financial econometrics, integrated variance.

Finally, we have the simple result in the $\mathcal{SVSM}^c$ case that

$$[y_\delta^*]^{[2]}(t) - \mu_1^{-2} \{y_\delta^*\}^{[1,1]}(t) \xrightarrow{p} 0,$$

so that

$$[y^*] = \mu_1^{-2} \{y^*\}^{[1,1]}.$$

We will see in the next Section that when there are jumps this term will not have a zero probability limit rather it will converge to a positive, but finite quantity. It will become clear that this difference can be used to test for the presence of jumps.

We now introduce an additional regularity condition.

Condition (V′). The volatility process σ has the property

$$\lim_{\delta \downarrow 0} \delta^{1/2} \sum_{j=1}^{M} |\sigma^r(\eta_j) - \sigma^r(\xi_j)| = 0$$

for some $r > 0$ (equivalently for every $r > 0$) and for any $\xi_j = \xi_j(\delta)$ and $\eta_j = \eta_j(\delta)$ such that

$$0 \leq \xi_1 \leq \eta_1 \leq \delta \leq \xi_2 \leq \eta_2 \leq 2\delta \leq \cdots \leq \xi_M \leq \eta_M \leq t.$$

□

Using this condition we may strengthen Theorem 2 to

Theorem 3 *Let the situation be as in Theorem 2 and suppose moreover that* σ *satisfies condition* **(V′)**. *Then*

$$\delta^{1-r} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j|^r |y_{j+1}|^r = \mu_r^2 \int_0^t \sigma^{2r}(s) \mathrm{d}s + o_p(\delta^{1/2})$$

□

Proof By the proof of Theorem 2, particularly the conclusion (15), it suffices to note that when conditions (a) and **(V′)** hold, then together they imply that

$$\delta^{1-r} \sum_{j=1}^{M} \sigma_j^{2r} - \sigma^{2r*}(t) = o_p(\delta^{1/2}).$$

This latter relation is in fact the content of Barndorff-Nielsen and Shephard (2003b, Lemma 2).

□

Theorem 3 will be essential in deriving distributional limit results for realised power variation. See Section 6.4 below.

3 SV process plus large but rare jumps

3.1 Rare jumps and quadratic variation

Consider the log-price process

$$y^*(t) = y^{(1)*}(t) + y^{(2)*}(t), \tag{16}$$

with $y^{(1)*} \in \mathcal{SVSM}^c$ and

$$y^{(2)*}(t) = \sum_{i=1}^{N(t)} c_i, \tag{17}$$

where N is a counting process such that $N(t) < \infty$ (for all $t > 0$) and $\{c_i\}$ are a collection of non-zero random variables. In the special case where N is a homogeneous Poisson process and c_i is i.i.d. from some distribution D then $y^{(2)*}$ is a compound Poisson process (written $\mathcal{CPP}$). Such jump process models have often been used as components of price processes (e.g. Merton (1976), Andersen, Benzoni, and Lund (2002), Johannes, Polson, and Stroud (2002) and Chernov, Gallant, Ghysels, and Tauchen (2002)).

The QV for this jump plus $\mathcal{SVSM}^c$ process is well known (e.g. see the discussion in Andersen, Bollerslev, Diebold, and Labys (2001)) and is reported below in the following theorem.

Proposition 4 *Suppose $y^{(1)*} \in \mathcal{SVSM}^c$ and $y^{(2)*}$ is given in (17), then*

$$\begin{aligned} [y^*](t) &= \sigma^{2*}(t) + \sum_{i=1}^{N(t)} c_i^2 \\ &= \left[y^{(1)*}\right]^{[2]}(t) + \left[y^{(2)*}\right]^{[2]}(t). \end{aligned}$$

□

Proof. Trivial from (7) and (10).

□

3.2 Rare jumps and power variation

We can generalise the above discussion to deal with power variation. This captures the main theoretical effect we are after in this paper.

Theorem 5 *Let $y^* = y^{(1)*} + y^{(2)*}$ with $y^{(1)*}$ and $y^{(2)*}$ being independent. Suppose $y^{(1)*} \in \mathcal{SVSM}^c$, additionally (σ, α^*) are independent of w and $y^{(2)*}$ is given in (17), then*

$$\mu_r^{-1}\{y^*\}^{[r]}(t) \xrightarrow{p} \begin{cases} \int_0^t \sigma^r(u)\mathrm{d}u, & r \in (0,2) \\ [y^*](t), & r = 2 \\ \infty, & r > 2. \end{cases}$$

□

Remark 2 *The same setting has been independently studied recently by Woerner (2002) who reaches the same conclusion but for r restricted to $(1, \infty)$ and using a different technique. See also Lépingle (1976) for an earlier investigation.*

Proof. It is clear that as $\delta \downarrow 0$ then generally

$$\sum_{j=1}^{\lfloor t/\delta \rfloor} \left| y_j^{(2)}(t) \right|^r \xrightarrow{p} \sum_{i=1}^{N(t)} |c_i|^r ,$$

where

$$y_j^{(2)}(t) = y^{(2)*}(j\delta) - y^{(2)*}((j-1)\,\delta).$$

It follows from this that when we also normalise the sum as $\delta \downarrow 0$

$$\delta^{\beta} \sum_{j=1}^{\lfloor t/\delta \rfloor} \left| y_j^{(2)}(t) \right|^r \xrightarrow{p} \begin{cases} 0 & \beta > 0,\ r \in (0,2) \\ [y^{(2)*}](t) & \beta = 0,\ r = 2 \\ \infty & \beta < 0,\ r > 2. \end{cases}$$

Hence the power variation of $y^{(2)*}$ is either zero, the quadratic variation or infinity, depending upon the value of r. Furthermore

$$\sum_{j=1}^{\lfloor t/\delta \rfloor} |y_j(t)|^r = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left| y_j^{(1)}(t) \right|^r + O_p(N(t)),$$

as there are only $N(t)$ contributions to $y^{(2)*}$. As a result we have that

$$\mu_r^{-1} \delta^{\beta} \sum_{j=1}^{\lfloor t/\delta \rfloor} |y_j(t)|^r \xrightarrow{p} \begin{cases} \int_0^t \sigma^r(u) \mathrm{d}u, & \beta = 1 - r/2,\ r \in (0,2) \\ [y^*](t), & \beta = 0,\ r = 2 \\ \infty, & \beta < 0,\ r > 2. \end{cases}$$

This delivers the required result by definition of $\{y^*\}^{[r]}(t)$.

□

Theorem 5 implies that:

1. When $r \in (0,2)$ the probability limit of realised power variation is unaffected by the presence of jumps.

2. When $r > 2$ the probability limit is determined by the jump component and so scaling means this goes off to infinity.

3. When $r = 2$ both components contribute.

The result is not very sensitive for it would hold for any jump process as long as there are a finite number of jumps in any finite period of time. In particular, then, the jumps can be serially dependent.

The above implies that

$$\sigma(t-) = \left(D_- \left[\mu_r^{-1} \{y^*\}^{[r]}(t) \right] \right)^{1/r},$$

when $r < 2$. Hence it is possible, in theory, to estimate the spot volatility in the presence of rare jumps.

3.3 Rare jumps and bipower variation

The logic of the proof of Theorem 5 carries over to the bipower variation process. In particular we deliver the stimulating result on bipower given in the following theorem.

Theorem 6 *Let the setting be as in Theorem 5. Then*

$$\mu_r^{-1}\mu_s^{-1}\left\{y^*\right\}^{[r,s]}(t) = \begin{cases} \int_0^t \sigma^{r+s}(u)\mathrm{d}u, & r+s \in (0,2) \\ \int_0^t \sigma^2(u)\mathrm{d}u, & r+s=2,\ r,s>0 \\ \infty, & r+s>2 \end{cases} .$$

□

Proof. Exactly the same argument as for Theorem 5 produces this result due to the finite number of jumps so long as $r+s<2$. When $r+s=2$ the scaling no longer has any impact. However, defining

$$\delta_{j,j+1} = \begin{cases} 1, & y_j^{(2)} = y_{j+1}^{(2)} = 0 \\ 1, & y_j^{(2)} \neq 0, y_{j+1}^{(2)} \neq 0 \\ 0, & \text{elsewhere} \end{cases} ,$$

we have an indicator which is one either if there are no jumps or the jumps are contiguous. Then

$$\sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s = \sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s \delta_{j,j+1} + \sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s (1-\delta_{j,j+1}).$$

Now

$$\sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s (1-\delta_{j,j+1}) = O(N(t)),$$

as the terms in the sum are zero unless there is a jump and there are $N(t)$ jumps. Further, when there is a jump the corresponding contiguous (and so non-jump) return goes to zero as $\delta \downarrow 0$. Thus we have that

$$\sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s (1-\delta_{j,j+1}) \xrightarrow{p} 0.$$

Further, the probability of having any contiguous jumping returns goes to zero as $\delta \downarrow 0$ so

$$\sum_{j=1}^{\lfloor t/\delta \rfloor -1} |y_j|^r |y_{j+1}|^s \delta_{j,j+1} - \sum_{j=1}^{\lfloor t/\delta \rfloor -1} \left|y_j^{(1)}\right|^r \left|y_{j+1}^{(1)}\right|^s \xrightarrow{p} 0.$$

This produces the desired result. The result for $r+s>2$ is immediate.

□

Remark 3 *This result has a special case, of a great deal of applied interest. When $r+s=2$ then so long as $r,s>0$*

$$\mu_r^{-1}\mu_s^{-1}\left\{y^*\right\}^{[r,s]}(t) = \int_0^t \sigma^2(u)\mathrm{d}u,$$

integrated variance. Hence the integrated variance can be consistently estimated in the presence of rare but large jumps. It also immediately implies that

$$[y^*]^{[2]}(t) - \mu_r^{-1}\mu_s^{-1}\{y^*\}^{[r,s]}(t) = \sum_{i=1}^{N(t)} c_i^2,$$

the jump contribution to the QV. We believe this is the first analysis which has been able to feasibly decompose the quadratic variation into the contributions from the continuous and jump components of the log-price.

4 Some simulations of SV plus large rare jumps

4.1 Basic simulation

In this Section we will illustrate some of these results by simulating an SV plus jump process. We start with a discussion of the spot volatility process. We use a Feller or Cox, Ingersoll, and Ross (1985) square root process for σ^2

$$\mathrm{d}\sigma^2(t) = -\lambda\left\{\sigma^2(t) - \xi\right\}\mathrm{d}t + \omega\sigma(t)\mathrm{d}b(\lambda t), \quad \xi \geq \omega^2/2, \quad \lambda > 0, \tag{18}$$

where b is a standard Brownian motion process. The square root process has a marginal distribution

$$\sigma^2(t) \sim \Gamma(2\omega^{-2}\xi, 2\omega^{-2}) = \Gamma(\nu, a), \quad \nu \geq 1,$$

with a mean of $\xi = \nu/a$, a variance of $\omega^2 = \nu/a^2$ and

$$\mathrm{Cor}\left\{\sigma^2(t), \sigma^2(t+s)\right\} = \exp(-\lambda|s|).$$

In the context of SV models this is often called the Heston (1993) model. We take $\alpha^* = 0$ and so

$$y^*(t) = \int_0^t \sigma(u)\mathrm{d}w(u) + \sum_{i=1}^{N(t)} c_i.$$

We rule out leverage (e.g. Black (1976) and Nelson (1991)) by assuming $\mathrm{Cor}\{b(\lambda t), w(t)\} = 0$. We take $\hbar = 1$, $\lambda = 0.01$, $\nu = 0.1$ and $a = 0.2$. We randomly scatter 10 jumps in a time interval of 50 days, while the jumps are $NID(0, 0.64\nu/a)$. The latter means that when there is a jump, the jump has the same variance as that expected over a 64% of a day period of trading when there is no jump. Thus these are large but quite rare jumps.

We report results based on $M = 12$, $M = 72$ and $M = 288$, which are typical practical values for these types of methods. The first row of Figure 2 corresponds to $M = 12$, the second has $M = 72$ and the third $M = 288$. Figures 2(a), (c) and (e) show the sample path of the y^* and N processes recorded at the resolution of M. Notice we have not used a standard time series plot here, we plot the processes using dots so that the jumps have the potential of being seen. As M increases the sample path of the discretised process reveals the jump in the process.

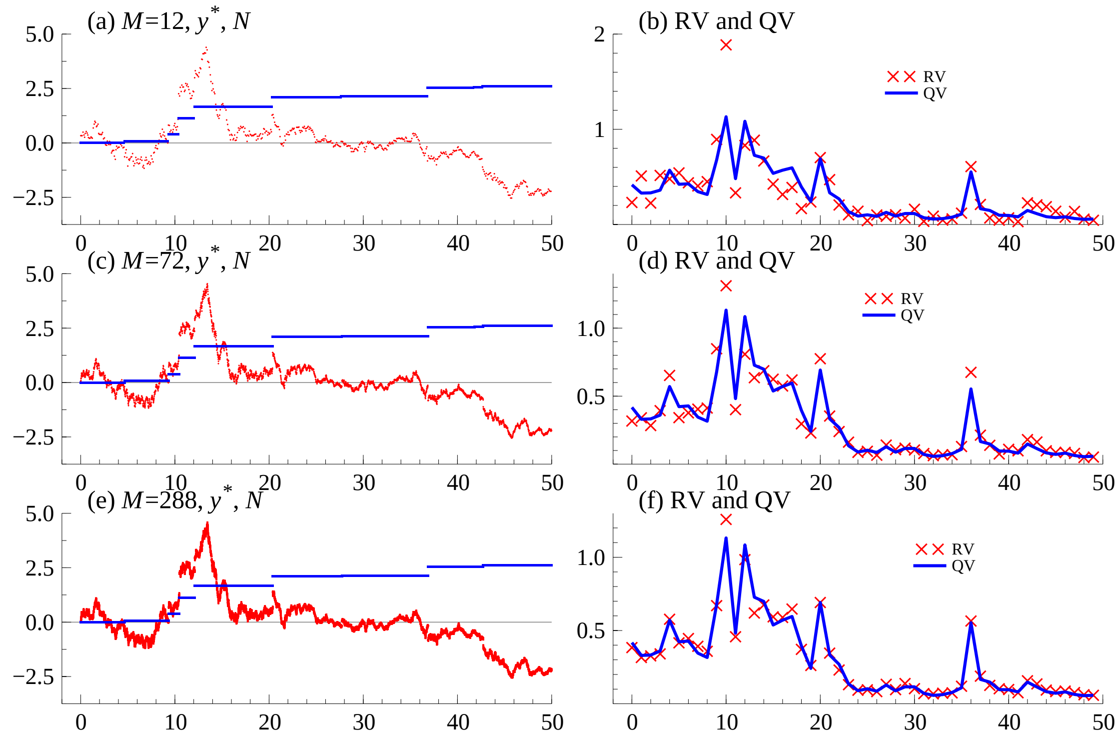

Figure 2: *Simulation from a jump plus diffusion based SV model using a square root variance processes. (a), (c) and (e) show $y^*(t)$ and $0.5N(t)$. The latter shows the position of the jumps. (b), (d) and (f) show σ_i^2 and $[y_M^*]_i^{[2]}$. Code is available at:* `jump.ox`.

Figures 2(b), (d) and (f) show the sample paths of the daily realised variances, while we also draw the corresponding daily increments in the quadratic variation, which is obviously the sum of the integrated variance and the sums of squared jumps. We see that the RV becomes more accurate with M, as we expect from theory. However, we can also see that it is quite an inaccurate estimator.

Figures 3(a), (c) and (e) show the more innovative results. It displays our estimator of the integrated variance, using the theory of realised bipower variation. In particular we are using $r = s = 1$ in this work. This is contrasted with the actual increment of the integrated variance. We can see that when $M = 12$ this is a poor estimator and is, in particular, influenced by the large jumps which appear in the sample. However for moderate M the statistic seems to be quite informative, while when $M = 288$ the estimator is reasonably accurate.

Figures 3(b), (d) and (f) shows the difference between the realised variance and the realised bipower variation. The theory suggests this is a consistent estimator of the quadratic variation of the jump component. For small M it is very inaccurate, but for moderate and large M the

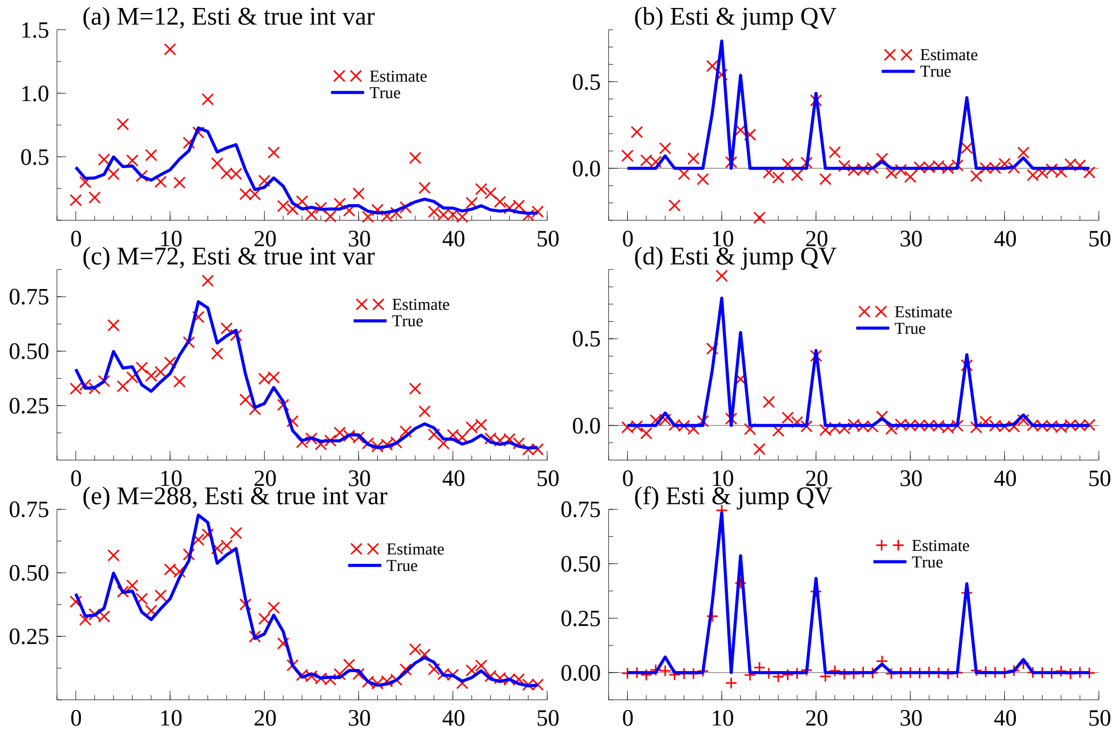

Figure 3: *Simulation from a jump plus diffusion based SV model, estimating the integrated variance and the QV of the jump components. Rows correspond to M=12, M=72 and M=288 respectively. (a), (c) and (e) plot the daily integrated variance* σ_i^2 *and the consistent estimator* $\mu_1^{-2}\left\{y_M^*\right\}_i^{[1,1]}$. *(b), (d) and (f) plot the jump QV and consistent estimator. Code is available at:* `jump.ox`.

estimator works rather well.

We back up these results by repeating the above analysis but using $5,000$ days, having jumps of the same size as before but now on average every 20 days. We then focus on the distribution of the estimation errors. In particular we look in detail at

$$[y^*]_i - [y_M^*]_i^{[2]},$$

the realised variance error,

$$\left[y^{(1)*}\right]_i - \mu_1^{-2}\left\{y_M^*\right\}_i^{[1,1]} = \sigma_i^2 - \mu_1^{-2}\left\{y_M^*\right\}_i^{[1,1]},$$

the realised bipower variation error and, defining

$$\widehat{\left[y_M^{(2)*}\right]_i^{[2]}} = [y_M^*]_i^{[2]} - \mu_1^{-2}\left\{y_M^*\right\}_i^{[1,1]},$$

then

$$\left[y^{(2)*}\right]_i - \widehat{\left[y_M^{(2)*}\right]_i^{[2]}} = \left[[y^*]_i - \sigma_i^2\right] - \left[[y_M^*]_i^{[2]} - \mu_1^{-2}\left\{y_M^*\right\}_i^{[1,1]}\right],$$

	$[y^*]_i - [y_M^*]_i^{[2]}$			$\left[y^{(1)*}\right]_i - \mu_1^{-2}\{y_M^*\}_i^{[1,1]}$			$\left[y^{(2)*}\right]_i - \widehat{\left[y_M^{(2)*}\right]_i^{[2]}}$		
	mean	.025	.975	mean	.025	.975	mean	0.025	0.975
M=12	-.00658	-.413	.330	-.00120	-.386	.351	.00404	-.173	.211
M=48	-.00817	-.302	.221	-.00316	-.284	.237	.00441	-.133	.160
M=72	-.0103	-.235	.117	-.00317	-.174	.139	.00225	-.075	.087
M=144	-.0102	-.197	.0884	-.00220	-.117	.101	.00137	-.050	.064
M=288	-.00983	-.151	.0625	-.00140	-.075	.073	.00098	-.034	.045
M=576	-.00963	-.130	.0452	-.00155	-.062	.050	.00133	-.024	.034

Table 1: Finite sample behaviour of the estimators of the quadratic variation of prices, integrated variance, and quadratic variation of the jump process. Reported are the errors. So we calculate the mean value and the 2.5% and 97.5% quantiles of the sampling distribution. Code: quasi_RV.ox

the realised jump error. The results are given in Table 1. This records the mean error, to see if the estimators are unbiased, as well as the 2.5% and 97.5% quantiles.

For the realised variance error there is a small bias for small M, which quickly disappears as M increases. The 95% range of the error falls quite quickly, although the interval is still substantial when $M = 576$. An interesting feature is that the errors are roughly symmetric in this analysis.

For the realised bipower variation error the results are rather similar to the RV case. The estimator is roughly unbiased for moderate M, but the error is substantial for large M. Interesting the 95% interval is larger for this error than for the corresponding RV case.

For the realised jump error, the estimator is roughly unbiased even for moderate values of M and the 95% intervals are roughly symmetric and smaller than for the other estimators.

4.2 No jump case

Interestingly, when we repeat the analysis, removing all jumps and comparing the performance of $[y_M^*]_i^{[2]}$ and $\mu_1^{-2}\{y_M^*\}_i^{[1,1]}$ as estimators of σ_i^2 we obtained the results given in Table 2, which indicate that the RV is slightly preferable in terms of accuracy.

4.3 Improving the finite sample behaviour

It is clear that

$$\left[y^{(2)*}\right]_i = [y^*]_i - \sigma_i^2 = \sum_{i=N\{\hbar(i-1)\}}^{N(\hbar i)} c_j^2 \geq 0,$$

while the estimator

$$\widehat{\left[y_M^{(2)*}\right]_i^{[2]}} = [y_M^*]_i^{[2]} - \mu_1^{-2}\{y_M^*\}_i^{[1,1]},$$

	$[y^*]_i - [y^*_M]^{[2]}_i$			$\sigma^2_i - \mu_1^{-2}\{y^*_M\}^{[1,1]}_i$		
	mean	.025	.975	mean	.025	.975
M=12	.00271	-.331	.339	.00260	-.367	.361
M=48	.00140	-.229	.226	-.000438	-.276	.237
M=72	-.000898	-.140	.119	-.00115	-.163	.143
M=144	-.000901	-.107	.0911	-.000691	-.113	.102
M=288	-.000615	-.0677	.0654	-.000398	-.0729	.0740
M=576	-.000327	-.0466	.0468	-.000809	-.0575	.0509

Table 2: Finite sample behaviour of the estimators of the quadratic variation of prices, integrated variance, and quadratic variation of the jump process. Reported are the errors. So we calculate the mean value and the 2.5% and 97.5% quantiles of the sampling distribution. Code: quasi_RV.ox

can be negative. Hence it would be sensible to sometimes replace it by the consistent estimator

$$\max\left([y^*_M]^{[2]}_i - \mu_1^{-2}\{y^*_M\}^{[1,1]}_i, 0\right).$$

This would suggest the estimator of the actual variance of the SV component

$$\min\left\{[y^*_M]^{[2]}_i, \mu_1^{-2}\{y^*_M\}^{[1,1]}_i\right\}.$$

Repeating the above results we produce the analysis of errors given in Table 3. We see that the finite sample behaviour of the integrated variance terms does not improve, but for the quadratic variation of the jump component the improvement is quite considerable. Importantly, the latter term now has a very short right hand tail, which means we rarely overstate the presence of jumps by a large amount. Of course this estimator has the disadvantage of being biased.

	IntVar error			Jump error		
	mean	.025	.975	mean	.025	.975
M=12	.0210	-.278	.401	-.0181	-.173	.0271
M=48	.0144	-.202	.273	-.0131	-.133	.0220
M=72	.0064	-.121	.149	-.0073	-.075	.0162
M=144	.0041	-.099	.110	-.0050	-.050	.0146
M=288	.0029	-.059	.082	-.0033	-.034	.0080
M=576	.0020	-.044	.057	-.0022	-.024	.0066

Table 3: Improved finite sample behaviour of the estimators of the integrated variance and quadratic variation of the jump process. Reported are the errors. So we calculate the mean value and the 2.5% and 97.5% quantiles of the sampling distribution. Code: quasi_RV.ox

5 Initial empirical work

To illustrate some of the new theory we return to the dataset discussed in the introduction. We now work with the full dataset from 1st December 1986 until 30th November 1996. All calculations will be based on $M = 288$, that is we will employ 5 minute returns, calculating aggregate volatility statistics for each day within the sample.

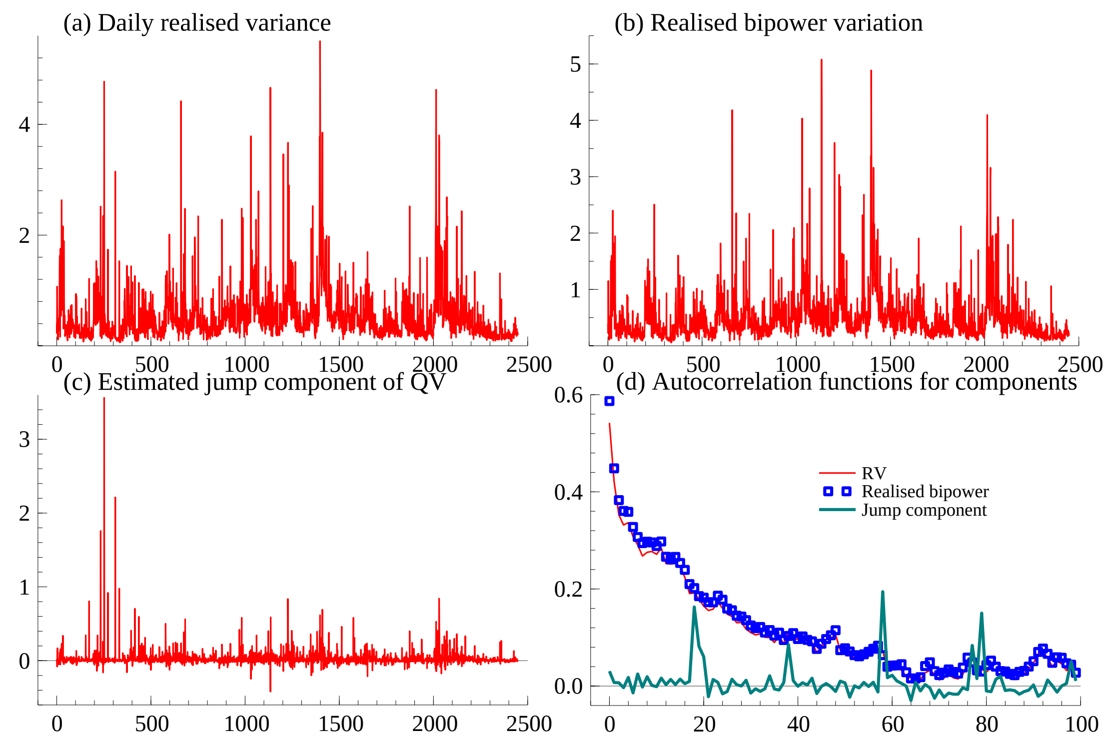

Figure 4: *DM/Dollar daily volatility measurements based on M=288. Code is available at:* `jump.ox`.

Figure 4(a) shows the time series of the daily realised variance, while 4(b) shows the corresponding realised bipower variation. The difference between these two estimates, which is our estimate of the daily increments of QV of the jump component, is given in Figure 4(c). Finally, 4(d) shows the correlograms of the realised variances, the realised bipower variations and the estimated jump components of QV.

The main features of Figure 4 is that the estimate of the integrated variance is the dominating component of the realised variance. Some of the estimates of the jump QV are quite large, especially at the start of the period. However, these jumps tend not to be very serially dependent. The serial dependence in the estimated integrated variances is larger than the RVs. This is interesting, however the increase in dependence is quite modest.

During this period, the average value of the jump component to daily QV is 0.0310. Average realised variance is 0.528, hence the size of the jump component is quite modest for the dataset.

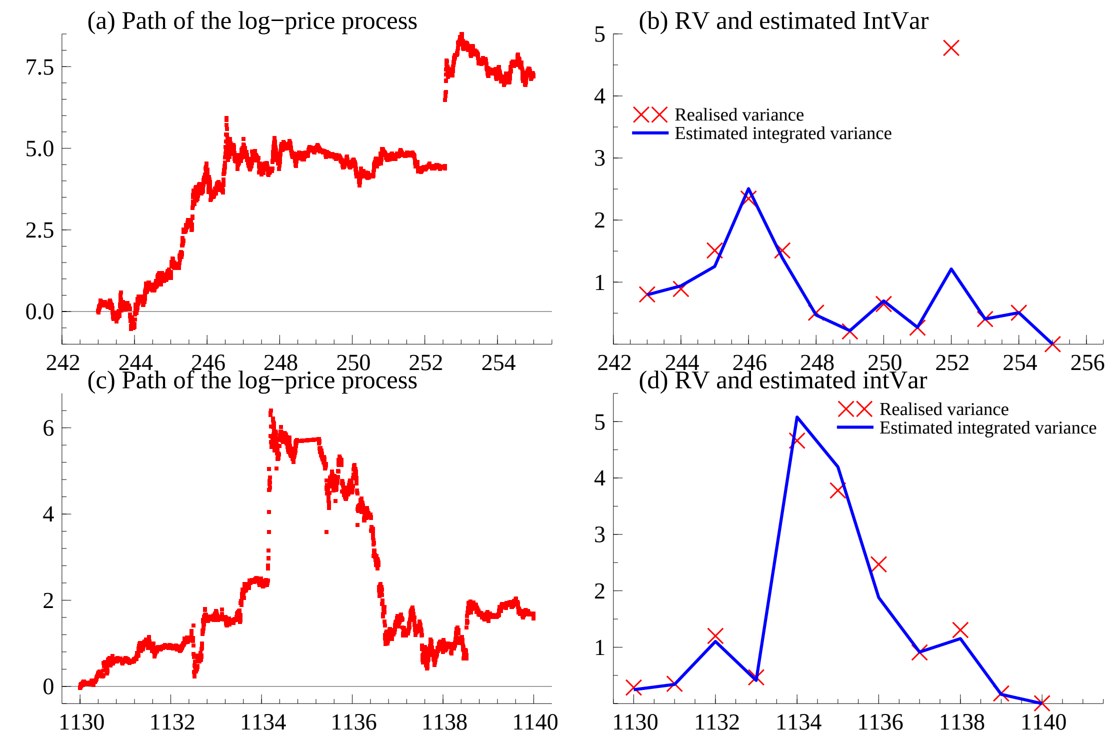

Figure 5: *Two examples of small stretches of data with large realised variances. The log-prices are recorded every 5 minutes in (a) and (c). (a) has a large step change in the prices, which one may view as a jump. (c) has a number of large positive returns which are in sequence. Code is available at:* `jump_RV.ox`.

Figure 5 focuses on two time-periods with large realised variances. In both cases we look at 10 day periods of time, plotting in Figures 5(a) and (c) the log-price process every 5 minutes. Throughout in Figures 5(a) and (c) we plot the prices using dots, rather than the more standard time-series lines. In Figure 5(a) there is a large uptick in the price, with a movement of nearly one in a five minute period. That day had a large realised variance, shown in 5(b), but a much smaller estimate of the integrated variance. Hence the statistics are attributing a large component of the realised variance to the jump.

This effect contracts with 5(c) where there is a three unit increase in the log-price, but this happens over around an hour long period with many positive returns. We can see in Figure 5(d) that the corresponding realised variance is very high, but so is the estimated integrated variance. Hence in this case the statistics have not flagged up a jump in the price, even though prices were moving rapidly in one direction.

6 Extensions and discussion

6.1 Robust estimation of integrated covariance

A vital concept in financial econometrics is covariance. We can formally base this on the concept of quadratic covariation. We set this up using a bivariate semimartingale x^* and y^*. Then the quadratic covariation between x^* and y^* is

$$[y^*, x^*](t) = \text{p}-\lim_{M\to\infty} \sum_{j=1}^{M} \{y^*(t_j) - y^*(t_{j-1})\}\{x^*(t_j) - x^*(t_{j-1})\}. \tag{19}$$

Note that using this notation

$$[y^*, y^*](t) = [y^*](t),$$

the QV of the y^* process. Quadratic covariation has been recently studied in the econometric literature by, for example, Barndorff-Nielsen and Shephard (2002b) and Andersen, Bollerslev, Diebold, and Labys (2003).

In the context of multivariate $\mathcal{SVSM}^c$ models (Barndorff-Nielsen and Shephard (2002b)) then we write

$$\begin{pmatrix} y^*(t) \\ x^*(t) \end{pmatrix} = \alpha^*(t) + m^*(t), \quad \text{where} \quad m^*(t) = \int_0^t \Theta(u)\mathrm{d}w(u),$$

where α^* has continuous locally bounded variation paths, while the elements of Θ are assumed càdlàg and w is a vector of independent standard Brownian motions. We write $\Sigma(t) = \Theta(t)\Theta(t)'$, and we need to make the additional assumption that $\int_0^t \Sigma_{k,l}(u)\mathrm{d}u < \infty$, $k, l = 1, 2$, to ensure that m^* is a local martingale. This setup has the important property that

$$[y^*, x^*](t) = \int_0^t \Sigma_{1,2}(u)du,$$

the integrated covariance of the price process. Andersen, Bollerslev, Diebold, and Labys (2003) has emphasised the importance of estimating this type of term. They discuss the use of realised covariation to carry this out. A distribution theory for this estimator under the above type of model structure is given in Barndorff-Nielsen and Shephard (2002b).

An important property of quadratic variation is the so called polarisation result that

$$[y^* + x^*](t) = [y^*](t) + [x^*](t) + 2[y^*, x^*](t),$$

which means that

$$[y^*, x^*](t) = \frac{1}{2}\left\{[y^* + x^*](t) - [y^*](t) - [x^*](t)\right\}.$$

Hence we can use realised bipower variation to consistently estimate $[y^*, x^*](t)$ or increments of it by estimating each of the individual terms. Further, straightforwardly this estimator is robust to jumps and we now have tools for assessing if the dependence between price processes is effected by the presence of jumps. We will explore this issue elsewhere in some detail.

6.2 Generalisation to multipower variation

We defined the bipower variation as

$$p-\lim_{\delta\downarrow 0}\delta^{1-(r+s)/2}\sum_{j=1}^{\lfloor t/\delta\rfloor-1}\left|y_j(t)\right|^r\left|y_{j+1}(t)\right|^s,\quad r,s\geq 0.$$

It is clear we can generalise this object by multiplying together a finite number of absolute returns raised to some non-negative power. In general we call this idea multipower variation. In particular the tripower variation process is

$$\{y^*\}^{[r,s,u]}(t)=p-\lim_{\delta\downarrow 0}\delta^{1-(r+s+u)/2}\sum_{j=1}^{\lfloor t/\delta\rfloor-2}\left|y_j(t)\right|^r\left|y_{j+1}(t)\right|^s\left|y_{j+2}(t)\right|^u,\quad r,s,u>0,$$

while the quadpower variation process is

$$\{y^*\}^{[r,s,u,v]}(t)=p-\lim_{\delta\downarrow 0}\delta^{1-(r+s+u+v)/2}\sum_{j=1}^{\lfloor t/\delta\rfloor-3}\left|y_j(t)\right|^r\left|y_{j+1}(t)\right|^s\left|y_{j+2}(t)\right|^u\left|y_{j+3}(t)\right|^v,\quad r,s,u,v>0.$$

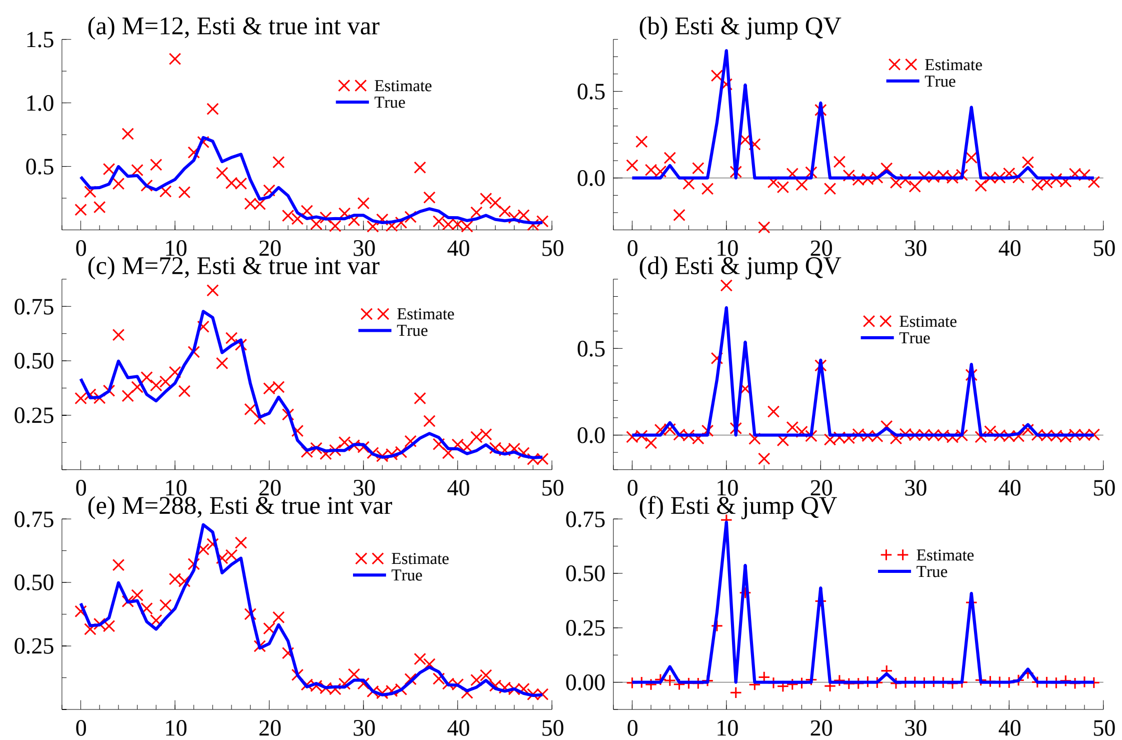

Figure 6: *Estimation based on tripower variation. Simulation from a jump plus diffusion based SV model. Rows correspond to $M=12$, $M=72$ and $M=288$ respectively. (a), (c) and (e) plot the daily integrated variance σ_i^2 and the consistent estimator $\mu_{2/3}^{-3}\{y_M^*\}_i^{[2/3,2/3,2/3]}$. (b), (d) and (f) plot the jump QV and consistent estimator. Code is available at:* `jump_RV.ox`.

Example 7 *An example of this is where we take $r = s = u = 2/3$, then*

$$\mu_{2/3}^{-3} \{y^*\}^{[2/3,2/3,2/3]}(t) = \int_0^t \sigma^2(u)\mathrm{d}u,$$

another new estimator of integrated variance. Figure 6 repeats the experiments reported in Figure 3 but replaces bipower with tripower. In particular the realised tripower terms we plot have

$$\mu_{2/3}^{-3} \sum_{j=1}^{M-2} |y_{j,i}|^{2/3} |y_{j+1,i}|^{2/3} |y_{j+2,i}|^{2/3}.$$

Likewise

$$\mu_{1/2}^{-4} \{y^*\}^{[1/2,1/2,1/2,1/2]}(t) = \int_0^t \sigma^2(u)\mathrm{d}u.$$

This result is interesting as it is estimating integrated variance based on square roots of absolute returns, which is quite a robust item. Figure 7 again repeats the previous experiments, but now based on this quadpower concept. The realised version of the quadpower is

$$\mu_{1/2}^{-4} \sum_{j=1}^{M-3} \sqrt{|y_{j,i}y_{j+1,i}y_{j+2,i}y_{j+3,i}|}.$$

We see from Figure 7 that we do not produce very different answers. An important feature of this graph is one estimate of the QV of the jump component which is highly negative. This makes little sense, as we know the jump component has to be non-negative. We discussed this issue in the context of bipower in Section 4.3. If we use the same strategy here then we produce much more sensible results. They are plotted in Figure 8. This suggests that the estimators are reasonably accurate even for quite small samples.

6.3 Some infinite activity Lévy processes

In our analysis we have shown that the realised power variation of a compound Poisson process converges in probability to zero when the power is less than two. An interesting extension of this result is to Lévy processes with infinite numbers of jumps in finite time intervals. Then the question is whether bipower variation is robust to these types of jumps?

Recall Lévy processes are processes with independent and stationary increments — see, for example, the review in Barndorff-Nielsen and Shephard (2004, Ch. 2 and 3). Examples of this type of process are the normal gamma (often called the variance gamma process) process and the normal inverse Gaussian process. These are due to Madan and Seneta (1990) and Madan, Carr, and Chang (1998) in the former case and Barndorff-Nielsen (1997) and Barndorff-Nielsen (1998) in the latter case. Here we explore this problem, without giving a complete solution.

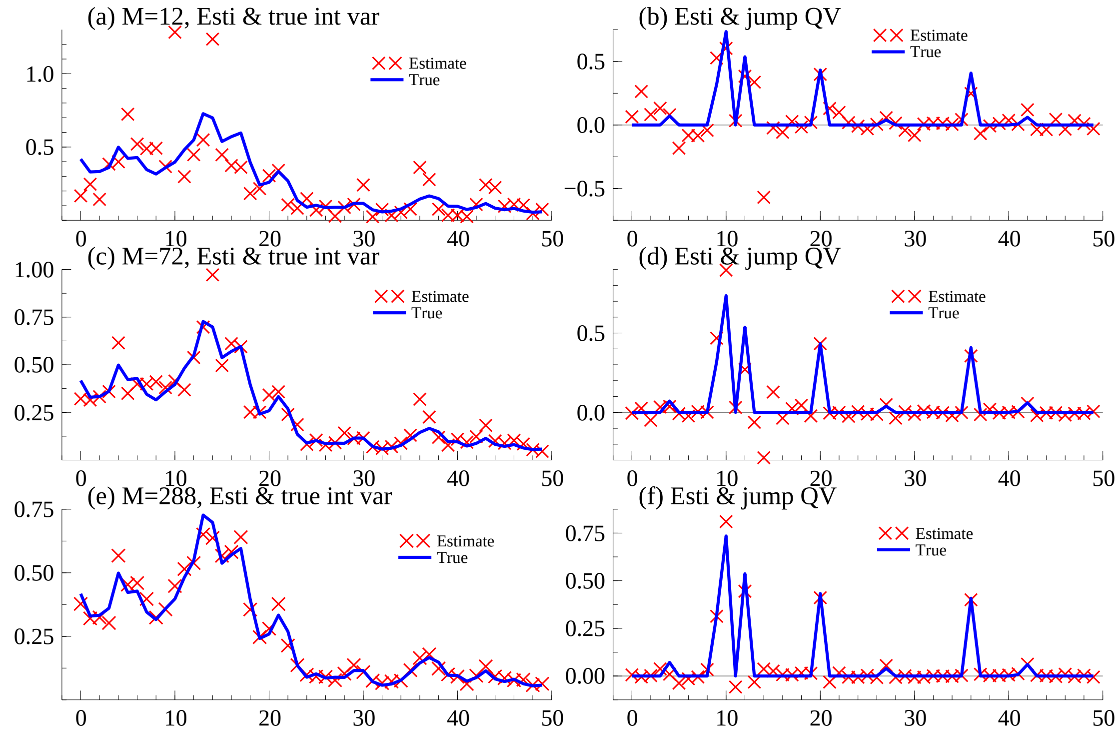

Figure 7: *Estimation based on quadpower variation. Simulation from a jump plus diffusion based SV model. Rows correspond to* $M = 12$*,* $M = 72$ *and* $M = 288$ *respectively. (a), (c) and (e) plot the daily integrated variance* σ_i^2 *and the consistent estimator* $\mu_{1/2}^{-4}\{y_M^*\}_i^{[1/2,1/2,1/2,1/2]}$*. (b), (d) and (f) plot the jump QV and consistent estimator. Code is available at:* `jump_RV.ox`.

Let z denote a Lévy process. It is characterised by the Lévy-Khintchine representation which says that we can write

$$\log \mathrm{E} \exp \{i\zeta z(1)\} = ai\zeta - \frac{1}{2}\sigma^2\zeta^2 - \int_{\mathbf{R}} \left\{1 - e^{i\zeta x} + i\zeta x \mathbf{1}_B(x)\right\} W(\mathrm{d}x), \tag{20}$$

where $a \in R$, $\sigma \geq 0$, $B = [-1, 1]$ and the Lévy measure W must satisfy

$$\int_{\mathbf{R}} \min\{1, x^2\} W(\mathrm{d}x) < \infty \tag{21}$$

and W has no atom at 0.

In this exposition we will assume the drift a and the volatility σ are zero, so z is a pure jump process. The Lévy measure controls the jumps in the process. If $\int_{\mathbf{R}} W(\mathrm{d}x) < \infty$ then the process is a compound Poisson process and so has a finite number of jumps. If this does not hold but $\int_{\mathbf{R}} \min\{1, |x|\} W(\mathrm{d}x) < \infty$ then we say the process is of B-activity. It has an infinite number of jumps in any finite time interval. However, the process has enough stability that we can decompose it into

$$z(t) = z_+(t) - z_-(t),$$

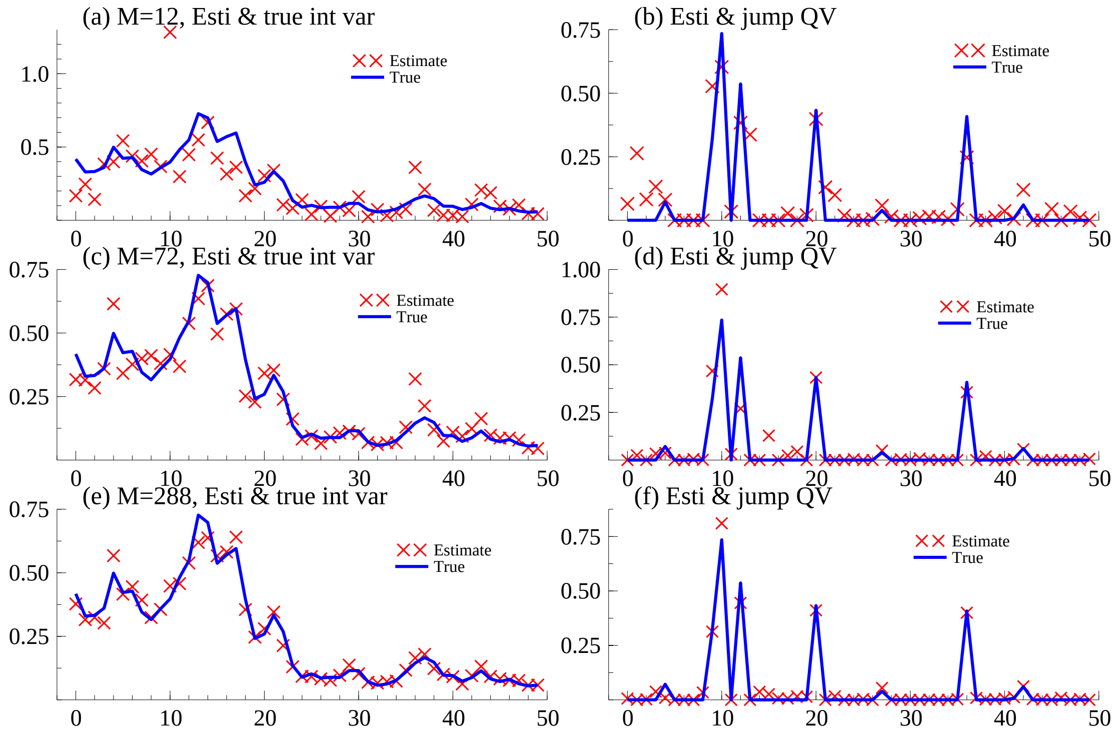

Figure 8: *Estimation based on the truncated version of quadpower variation, which is enforced to improve the finite sample behaviour of the estimator. Simulation from a jump plus diffusion based SV model. Rows correspond to* $M = 12$, $M = 72$ *and* $M = 288$ *respectively. (a), (c) and (e) plot the daily integrated variance* σ_i^2 *and the consistent estimator. (b), (d) and (f) plot the jump QV and consistent estimator. Code is available at:* `jump_RV.ox`.

where z_+ and z_- are independent subordinators. Recall a subordinator is a Lévy process with non-negative increments. Under the more general condition (21) this decomposition into positive and negative jumps is not possible.

We believe that the results of Section 3 extend to cases where the jump process is of the Lévy B-activity type and a discussion of this will be given elsewhere.

6.4 Asymptotic distribution

An important question is whether we can extend the convergence in probability result we saw for realised bipower variation to convergence in distribution? This type of result has been developed for realised variance by Barndorff-Nielsen and Shephard (2002a) and Barndorff-Nielsen and Shephard (2002b). It has been extended considerably in their work on realised power variation by Barndorff-Nielsen and Shephard (2003a) and Barndorff-Nielsen and Shephard (2003b). We intend to discuss this question formally elsewhere. Clearly the development of a distribution theory will allow us to test, robustly, for jumps in the log-price process. Here we state a particular result.

To include the possibility of the presence of a drift (or risk premium) α^* in the distributional theory we need to invoke the following condition.

Condition (M). The mean process α^* satisfies (pathwise)

$$\overline{\lim_{\delta \downarrow 0}} \max_{1 \leq j \leq M} \delta^{-1} |\alpha^*(j\delta) - \alpha^*((j-1)\delta)| < \infty.$$

□

This condition is implied by Lipschitz continuity and itself implies continuity of α^*.

Theorem 8 *Suppose that α^* and σ are independent of w and satisfy conditions (a), (b),* **(V′)** *and* **(M)**. *Then $y^* \in \mathcal{SVSM}^c$ and for any $r > 0$ we have, for $\delta \downarrow 0$ and*

$$\{y_\delta^*\}^{[r,r]}(t) = \delta^{1-r} \sum_{j=1}^{\lfloor t/\delta \rfloor - 1} |y_j|^r |y_{j+1}|^r,$$

that

$$\frac{\{y_\delta^*\}^{[r,r]}(t) - \mu_r^2 \int_0^t \sigma^{2r}(s) \mathrm{d}s}{\delta^{1/2} \mu_{2r}^{-1} \left[\nu_{2r2r} \{y_\delta^*\}^{[2r,2r]}(t) \right]^{1/2}} \stackrel{law}{\to} N(0,1)$$

where

$$\nu_{rr} = \mathrm{Var}\left(|u|^r |u'|^r \right)$$

u, u' being independent standard normal.

□

The proof runs along the lines of the discussion in Section 2 and of the main Theorem in Barndorff-Nielsen and Shephard (2003a); the details will be given elsewhere.

6.5 Other related work on jumps

Here we briefly discuss some related econometric work which has studied jump type models.

There is a considerable literature on estimating continuous time parametric models. A particular focus has been on the class of SV plus jumps model. Johannes, Polson, and Stroud (2002) provide filtering methods for parameterised stochastic differential equations which exhibit finite activity jumps. Their approach can be generalised to deal with infinite activity processes for it relies on the very flexible auxiliary particle filters introduced by Pitt and Shephard (1999). Andersen, Benzoni, and Lund (2002) and Chernov, Gallant, Ghysels, and Tauchen (2002) have used EMM methods to estimate and test some of these models. Maheu and McCurdy (2003) have constructed a jump/GARCH discrete time model to daily data which attempts to unscramble jumps and volatility changes.

There is a literature on discrete time parametric models of SV plus jumps. Chib, Nardari, and Shephard (2002) use simulation based likelihood inference which allows one to test for jumps in a standard manner. Other papers on this topic are pointed out in that paper's references.

Ait-Sahalia (2002) has recently asked the generic question of whether a Markov process exhibits jumps? He developed a theory for this based upon transition densities and then applied this to financial data.

Finally, and closest to this paper, Barndorff-Nielsen and Shephard (2003c) have studied the second order properties of realised variance under the assumption that the local martingale component of prices is a time-changed Lévy process. Such models will have jumps in the price process except in the Brownian motion special case. For time-changed Lévy processes the realised variance is an inconsistent estimator of the increments of the time-change. They used their results to suggest simple quasi-likelihood estimators of these types of models.

7 Conclusion

In this paper we have studied the recently formalised concept of realised power variation in the context of SV models where there can be occasional jumps. We have shown that sometimes the probability limit of these quantities are unaffected by rare jumps.

Realised power variation has inspired us to introduce realised bipower variation. This shares some robustness property, but can also be setup to estimate integrated power volatility in SV models. In a range of cases this produces an estimator of integrated variance in the presence of jumps. To our knowledge this is the first radical alternative to the commonly used realised variance estimator of this quantity. Importantly when we add jumps to the SV model the probability limit of the bipower estimator does not change, which means we can combine realised variance with realised bipower variation to estimate the quadratic variation of the jump component. We think our empirical work is the first time researchers have used this type of robust, model free estimators of jumps in financial markets.

Various extensions of our work have been developed. We have outlined multivariate versions of the methods, a distribution theory and an assessment of the robustness to the α^* process. We think our paper may open up a number of interesting research avenues for econometricians.

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