Option Valuation under Stochastic Volatility
With Mathematica Code

Published by: Finance Press, Newport Beach, California, USA
Cover design by: Brian Burton Design

For the latest available updates and corrections to this book and publisher contact information, visit the Internet sites:
http:// www.financepress.com or http://members.home.net/financepress

Trademark notice: *Mathematica* is a trademark of Wolfram Research, Inc., which is not associated with the author or publisher of this book.

International Standard Book Number 0-9676372-0-1
Library of Congress Card Number: 99-91935

Printed in the United States of America
Printed on acid-free paper

2 The Fundamental Transform

In this chapter we introduce a transform-based approach to solving the option valuation PDE that we developed in Chapter 1. The method is based on a generalized Fourier transform. A particular function, which we call the fundamental transform, plays an important role throughout the book. While the idea of a transform-based approach is not new, previous applications have tended to be model-specific. Not only are our results more general, but they encompass the situation when option prices, relative to a numeraire, are not martingales, but only strictly local martingales.

1 Assumptions

In Chapter 1, we developed a PDE for valuing options under stochastic volatility at (**1**.4.10). Now we specialize to time-homogeneous volatility processes of the form $dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t$. In other words, the volatility changes in time only through the Brownian noise and level-dependent coefficients; but there is no explicit time dependence.

Indeed, most models of the *actual* volatility process that are proposed by researchers are time-homogeneous. In particular, both GARCH-style models and their continuous-time limits are time-homogeneous. And, as we show later in Chapter 7, the time-homogeneity property can be preserved after risk adjustment. Briefly, this can be achieved with a power utility function using an infinite consumption horizon or a pure investor model with a distant planning horizon.

We take as constant both the dividend yield on the underlying security and the short-term interest rate. This too can be made consistent with a risk adjustment model. Finally, we make a smoothness assumption that we use in later chapters. In summary, we employ in this chapter and throughout much of the book the basic model given by:

Assumption 1. The martingale pricing process $\tilde{P}$ has the general form

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\tilde{B}_t \\ dV_t = \tilde{b}(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}, \tag{1.1}$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are correlated Brownian motions under $\tilde{P}$, with correlation $\rho(V_t)$. The interest rate r and the dividend yield δ are constants. The coefficient functions $\tilde{b}(V)$ and $a(V)$ may be differentiated any number of times on $0 < V < \infty$.

Under Assumption 1, we can rewrite the PDE (**1**.4.10) for generalized European-style claims with price $F(S_t, V_t, t)$ and expiration T. That equation, defined in the region $0 < (S,V) < \infty$, $t < T$, becomes

$$-\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}F, \tag{1.2}$$

where
$$\tilde{\mathcal{A}}F = (r-\delta)S\frac{\partial F}{\partial S} + \tfrac{1}{2}VS^2\frac{\partial^2 F}{\partial S^2} + \tilde{b}(V)\frac{\partial F}{\partial V} + \tfrac{1}{2}a^2(V)\frac{\partial^2 F}{\partial V^2} + \rho(V)a(V)V^{1/2}S\frac{\partial^2 F}{\partial S\partial V}.$$

We almost always assume the payoff function is independent of volatility[1]. Then, European-style option prices are solutions to (1.2) with terminal condition $F(S,V,t=T) = g(S)$. As we will see below, sometimes there are multiple solutions to (1.2) with the same payoff function; briefly, this occurs because of volatility explosions. When that happens, we have to determine which solution is the "fair-value". Note that the first line defining the operator

---

[1] Our approach also accommodates very naturally a pure volatility-dependent payoff, such as a volatility future. The demands of traders for hedging and replication strategies under stochastic volatility would make such securities quite useful, although there are many real-world design issues

$\tilde{\mathcal{A}}$ is the linear operator of the B-S theory and the second line contains the stochastic volatility corrections.

2 The Transform-based Solution

In this section, we reduce (1.2) from two "space" variables to one. There are fundamental solutions to the reduced equation that provides a representation for the price of every (volatility independent) payoff function. As we will show, those fundamental solutions have a number of special properties.

This reduction to 1D is not the proverbial free lunch because the one variable PDE is then dependent upon a continuous transform parameter. Nevertheless, the reduction is extremely useful and it provides the basis for much of our subsequent development.

The first step is simply a change of variable from S to $x = \ln S$ in (1.2), letting $F(S,V,t) = f(x,V,t)$. Then f must solve, using subscripts for derivatives

$$\text{(2.1)} \qquad -f_t = -rf + (r - \delta - \tfrac{1}{2}V)\, f_x + \tfrac{1}{2}V\, f_{xx} + \tilde{b}\, f_V + \tfrac{1}{2}a^2\, f_{VV} + \rho a V^{1/2}\, f_{xV} \, .$$

Now consider the Fourier transform of $f(x,V,t)$ with respect to x:

$$\text{(2.2)} \qquad \hat{f}(k,V,t) = \int_{-\infty}^{\infty} e^{ikx}\, f(x,V,t)\, dx \, ,$$

where $i = \sqrt{-1}$ and k is the transform variable. The first issue is to determine under what conditions (2.2) exists for typical option solutions. The simplest case is $t \to T$ (expiration), where we know the functional form $f(x,V,T)$

For example, call option solutions are given at expiration by $C(S,V,T) = \text{Max}[S - K, 0] = (S-K)^+$, where K is the strike price. Hence, $f(x,V,T) = (e^x - K)^+$ and by a simple integration in (2.2),

$$\text{(2.3)} \qquad \hat{f}(k,V,T) = \left(\frac{\exp[(ik+1)x]}{ik+1} - K\,\frac{\exp(ikx)}{ik} \right)\Bigg|_{x=\ln K}^{x=\infty}$$

The upper limit $x = \infty$ in (2.3) does not exist unless $\text{Im}\, k > 1$, where Im means Imaginary part. Assuming this restriction holds, then (2.3) is well-defined, giving the payoff transform

$$\hat{f}(k,V,T) = -\frac{K^{1+ik}}{k^2 - ik}. \tag{2.4}$$

So the key to the existence of (2.2) is that the Fourier transform variable k has to have an imaginary part—making $k = k_r + ik_i$ a complex number[2]. Because k has been generalized to complex values, (2.2) is called a *generalized* Fourier transform[3]. In general, (2.2) exists for typical option payoffs only when $\mathrm{Im}\, k$ is restricted to a strip $\alpha < \mathrm{Im}\, k < \beta$. The reason that strips occur as a general feature of the theory is explained in Sec. 4. Given the transform $\hat{f}(k,V,t)$, the inversion formula is

$$f(x,V,t) = \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikx}\,\hat{f}(k,V,t)\,dk. \tag{2.5}$$

This is an integral along a straight line in the complex k-plane parallel to the real axis. In the case of the call option at expiration, this line can lie anywhere in the region $\mathrm{Im}\ k > 1$: say along $k_i = 3/2$ for example. Actually selecting a contour for computations is discussed further below. We can go through the same exercise for various standard payoff functions and see what restrictions are necessary for their Fourier transforms to exist. The results are summarized in Table 2.1 below.

Table 2.1 Generalized Fourier Transforms for Various Financial Claims

Financial Claim	Payoff Function	Payoff Transform	*k*-plane Restrictions
Call option	$\max[S_T - K, 0]$	$-\frac{K^{ik+1}}{k^2 - ik}$	$\mathrm{Im}\, k > 1$
Put option	$\max[K - S_T, 0]$	$-\frac{K^{ik+1}}{k^2 - ik}$	$\mathrm{Im}\, k < 0$
Covered call or cash-secured put	$\min[S_T, K]$	$\frac{K^{ik+1}}{k^2 - ik}$	$0 < \mathrm{Im}\, k < 1$
Delta function	$\delta(\ln S_T - \ln K)$	K^{ik}	none
Money market	1	$2\pi\delta(k)$	none

[2] If $z = x + iy = \mathrm{Re}\, z + i\,\mathrm{Im}\, z$ is any complex number, we write $|z|$ for the modulus or absolute value of z, and $z^* = x - iy$ for the complex conjugate.

[3] Sometimes the term *complex* Fourier transform is used. A comprehensive reference is Titchmarsh (1975).

The delta function. Two of the entries in the table use the Dirac delta function $\delta(x-y)$, which can be thought of as the limit of a function of *x* that is sharply peaked at $x=y$. In the limit, the function is zero everywhere else, while maintaining unit area under its "graph". More rigorously, the delta function is really a linear "functional" because it transforms well-behaved functions into numbers via $\int_{-\infty}^{\infty}\delta(x-y)f(x)dx = f(y)$. This function occurs naturally in the theory; for example, to prove the inversion formula, you insert (2.2) into (2.5) and rely upon this last equation and $\int_{ik_i-\infty}^{ik_i+\infty}\exp[-ik(x-y)]dk = 2\pi\delta(x-y)$.

Continuing with the development, we next translate (2.1) into a PDE for $\hat{f}(k,V,t)$. That's done by taking the time derivative of both sides of (2.2), and inside the integral replacing f_t by the (negative of the) right-hand-side of (2.1). Then, after parts integrations, the net effect is that *x*-derivatives of f in (2.1) become multiplications of $\hat{f}$ by $(-ik)$.

An important point is that we assumed that the boundary terms associated with the parts integrations can be neglected. This is similar to the issue that we discovered at (2.3) and led to our introduction of the generalized transform. Typically, there exists a strip $\alpha < \operatorname{Im} k < \beta$ such that the boundary terms vanish. This is proved in the subsection "Neglected boundary terms" below. It's also typical that α and β depend upon the parameters of the problem as well, such as the time to expiration. We also show examples of $\alpha(\tau)$ and $\beta(\tau)$ below. With $\operatorname{Im} k$ appropriately restricted, the PDE satisfied by $\hat{f}(k,V,t)$ is

$$-\hat{f}_t = [-r - ik(r-\delta)]\,\hat{f} - \tfrac{1}{2}V(k^2 - ik)\,\hat{f} + (\tilde{b} - ik\rho a V^{1/2})\,\hat{f}_V + \tfrac{1}{2}a^2\,\hat{f}_{VV}$$

We remove the dependence on *r* and δ, using $\tau = T - t$, and letting

(2.6) $$\hat{f}(k,V,t) = \exp\{[-r - ik(r-\delta)]\tau\}\,\hat{h}(k,V,\tau)\ .$$

Also, introducing $c(k) = (k^2 - ik)/2$, we see that $\hat{h}(k,V,\tau)$ satisfies the initial-value problem

(2.7) $$\frac{\partial \hat{h}}{\partial \tau} = \tfrac{1}{2}a^2(V)\frac{\partial^2 \hat{h}}{\partial V^2} + \left[\tilde{b}(V) - ik\rho(V)a(V)V^{1/2}\right]\frac{\partial \hat{h}}{\partial V} - c(k)V\,\hat{h}$$

The initial condition is that $\hat{h}(k,V,\tau=0)$ is given by the Fourier transform of the payoff function—the entries in Table 2.1.

The fundamental transform. Notice that the entries in Table 2.1 do not depend upon *V*. They don't because we have restricted our theory to volatility independent payoffs. Because of this assumption, it suffices to take the special case $\hat{h}(k,V,\tau=0)=1$. To obtain the solution to (2.7) for any other payoff of this type, multiply the solution for the special case by the "Payoff Transform" entry in Table 2.1 . This deserves a formal definition and some distinguishing notation:

Definition. A solution $\hat{H}(k,V,\tau)$ to (2.7) at a (complex-valued) point k, which satisfies the initial condition $\hat{H}(k,V,\tau=0)=1$, is called a *fundamental transform.*

Given the fundamental transform, to obtain a (not necessarily unique) solution $F(S,V,t)$ for a particular payoff, here are the steps:

- multiply the fundamental transform by the expiration payoff transform;
- further multiply by the factor that we removed in (2.6);
- invert the result with the k-plane integration (2.5), keeping $\operatorname{Im} k$ in an appropriate strip; this gives a solution $f(x,V,t)$ to (2.1);
- in terms of S, the solution is $F(S,V,t)=f(\ln S,V,t)$

For this procedure to work, we need a strip for which a fundamental solution to (2.7) exists; then we can carry out the inversion along any line contained within. Let's define a class of problems where this procedure is especially well-defined:

Definition. We call the initial-value problem (2.7) *regular*[4] if there exists a fundamental solution to (2.7) which is regular as a function of k within a strip $\alpha<\operatorname{Im} k<\beta$, where α and β are real numbers. We call this strip the *fundamental strip of regularity.* In typical examples, $\alpha<0$ and $\beta>1$.

Given the fundamental transform, the steps above are quite straightforward. For an example using Mathematica, see Appendix 2 to this chapter. For closed-form examples of the fundamental transform, see Sec. 3 below.

Call option Solution I. The call option payoff transform is given in Table 2.1 and it exists for $\operatorname{Im} k>1$. The call option solution in this subsection exists only under the following assumption: the initial-value problem (2.7) is regular in a

[4] A function $f(k)$ is *analytic* at a complex-valued point k if it has a derivative there. If it's both analytic and single-valued in a region, it's called *regular*

strip $\alpha < \text{Im}\, k < \beta$ *and* $\beta > 1$. In other words, we are assuming that the strip associated with the payoff transform and the fundamental strip intersect. If they don't, then this particular solution formula does not exist (but see below—there will always be an alternative formula that does exist). See Example II below for an example where there is such an intersection and further examples in Sec. 3. Carrying out the prescription above yields the solution representation

$$C_I(S,V,\tau) = -\frac{e^{-r\tau}}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S} e^{-ik(r-\delta)\tau} \frac{K^{ik+1}}{k^2 - ik} \hat{H}(k,V,\tau)\, dk \,, \; 1 < \text{Im}\, k < \beta \,.$$

We continue to employ $\tau = T - t$. This equation can be simplified by introducing the dimensionless variable

$$X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right].$$

Then, in terms of *X*, we have Solution I:

$$(2.8) \qquad C_I(S,V,\tau) = -Ke^{-r\tau} \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX} \frac{\hat{H}(k,V,\tau)}{k^2 - ik} dk \,, \qquad 1 < \text{Im}\, k < \beta \,.$$

Frequently, $\hat{H}(k,V,\tau)$ is the Fourier transform of a norm-preserving transition density for the risk-adjusted process. This is discussed further below. For now, we simply note that when $\hat{H}$ is norm-preserving, then one can show, by Fourier inversion, that

$$C_I(S,V,\tau) = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right],$$

which is the martingale-style solution. As we will see, sometimes there are other solutions and sometimes the martingale-style solution is not the arbitrage-free fair value.

Homogeneity. One immediate property of (2.8) is that the call option price is homogeneous of degree 1 in the stock price and the strike. That is, $C(S,V,\tau) = K\,c(S/K)$. If we multiply both the stock price and the strike by the same constant: $K \to \lambda K$ and $S \to \lambda S$, then $C \to \lambda C$. This is a well-known consequence of starting, as we did at Assumption (1.1), with a proportional stock price process. That is, the (risk-adjusted) stock price *return* distribution, although dependent upon the initial volatility, is independent of the level of *S*.[5]

[5] See Theorem 8.9 of Merton (1973).

Call option Solution II. In practice, we often do the k-plane integrations in $0 < \mathrm{Im}\, k < 1$: usually along $k_i = 1/2$. In this strip, $\hat{H}$ is often free of singularities—see Example II below and the discussion in Sec. 4. The reason that this strip is the "regular" one is that solutions to (2.7) are usually quite well-behaved as long as $\mathrm{Re}\, c(k) \geq 0$, which is true when $0 \leq \mathrm{Im}\, k \leq 1$. This strip is especially important both in the asymptotic $\tau \to \infty$ behavior of the theory, which is explained in Chapter 6, and when the martingale-style solution is not the fair value, which is explained in Chapter 9.

We can obtain a formula for the call option with this restriction by using the put/call parity relation

$$C(S,V,\tau) = S\exp(-\delta\tau) - [K\exp(-r\tau) - P(S,V,\tau)], \tag{2.9}$$

where $P(S,V,\tau)$ is the put option value. The expression in brackets in (2.9) is the cash-secured put entry in Table 2.1. As you can see from the table, the payoff function for the cash-secured put has (i) the same Fourier transform as the call option, except for a minus sign, and (ii) the different restriction $0 < \mathrm{Im}\, k < 1$. Now we assume that $\hat{H}$ is regular in a fundamental strip which intersects $0 < \mathrm{Im}\, k < 1$. With that assumption, we have solution II:

$$\boxed{\begin{gathered} C_{II}(S,V,\tau) = Se^{-\delta\tau} - Ke^{-r\tau} \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX} \frac{\hat{H}(k,V,\tau)}{k^2 - ik} dk\,, \\ \max[0,\alpha] < \mathrm{Im}\, k < \min[1,\beta] \end{gathered}} \tag{2.10}$$

In the same way, we define P_I to be the put option solution in its natural domain of definition, using Table 2.1:

$$P_I(S,V,\tau) = -Ke^{-r\tau} \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX} \frac{\hat{H}(k,V,\tau)}{k^2 - ik} dk\,, \qquad \alpha < \mathrm{Im}\, k < 0\,.$$

Again, when $\hat{H}(k,V,\tau)$ is the Fourier transform of a norm-preserving transition density, then

$$P_I(S,V,\tau) = e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right],$$

And, using (2.9) and (2.10), we also have the second put option solution in the same strip as C_{II}

$$P_{II}(S,V,\tau) = Ke^{-r\tau}\left[1 - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2 - ik}\,dk\right]$$

$$\max[0,\alpha] < \operatorname{Im} k < \min[1,\beta]$$

Relationships between the solutions. There is a very simple relationship between the Solution I and Solution II formulas under the assumption that the fundamental strip of regularity for $\hat{H}$ extends at least slightly above $\operatorname{Im} k = 1$ and at least slightly below $\operatorname{Im} k = 0$. In that case, one can apply the Residue Theorem (see Appendix 2.1) to show that

$$C_{II} = C_I + Se^{-\delta\tau}\left[1 - \hat{H}(k = i,V,\tau)\right]$$
$$P_{II} = P_I + Ke^{-r\tau}\left[1 - \hat{H}(k = 0,V,\tau)\right]$$

The meaning of these relationships is discussed further below and extensively in Chapter 9. For now, we simply note that in many situations, the fundamental transform is the transform of a norm-preserving transition density that is also *martingale-preserving*. These properties are defined below; when they hold, then

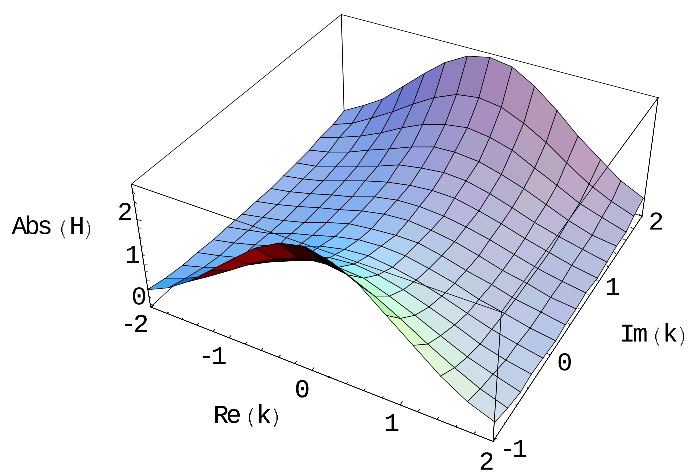
$$\hat{H}(k = 0,V,\tau) = \hat{H}(k = i,V,\tau) = 1 \quad \text{and so} \quad C_{II} = C_I \quad \text{and} \quad P_{II} = P_I \,.$$

Example I. *Constant or deterministic volatility*. In the case of constant volatility, the volatility process is $dV_t = 0$ and the fundamental transform satisfies $\hat{H}_\tau = -c(k)V\hat{H}$. Applying the initial condition, it's elementary to find $\hat{H}(k,V,\tau) = \exp[-c(k)V\tau]$. This is an entire function of *k*; i.e., analytic in the entire *k*-plane. So the only singularities of the integrands in both (2.8) and (2.10) are simple poles at $k = 0$ and $k = i$. In this case, (2.8) holds for the entire strip $1 < \operatorname{Im} k < \infty$ and (2.10) holds for the strip $0 < \operatorname{Im} k < 1$ and $C_{II} = C_I$. Of course, we should recover the B-S formula from both (2.8) or (2.10). This is shown in the Appendix 2.1 to this chapter.

In the case of deterministic volatility, the volatility process is $dV_t = b(V_t)dt$. The fundamental transform satisfies $\hat{H}_\tau = b(V)\hat{H}_V - c(k)V\hat{H}$. The solution to this equation is obtained by first finding $Y(u,V)$, which is defined as the solution to $dY/du = b(Y)$, $Y(0) = V$. Then, the fundamental transform is given by $H(k,V,\tau) = \exp[-c(k)U(V,\tau)]$, where $U(V,\tau) = \int_0^\tau Y(u,V)\,du$. So the k-plane behavior is identical to the case of constant volatility. Again the B-S formula is recovered, but the volatility V that appears in the formula is replaced by $v(V,\tau) = U(V,\tau)/\tau$. Again, see Appendix 2.1

Fig. 2.1 shows a plot of the modulus $|\hat{H}(k,1,1)|$, for the constant volatility case. Notice the saddle shape. Also the modulus is symmetrical about the $\text{Im}(k)$ axis; we show below that this *reflection symmetry* is a general feature of the fundamental transform:

Fig. 2.1 $|\hat{H}|$ **for the Constant Volatility Case**



Example II. *The square root model*. In the simplest case of this model, the volatility process is $dV_t = \xi\sqrt{V_t}\, dW_t$. Initially, we will assume that the volatility process is uncorrelated with the stock price process, but then subsequently relax that assumption.

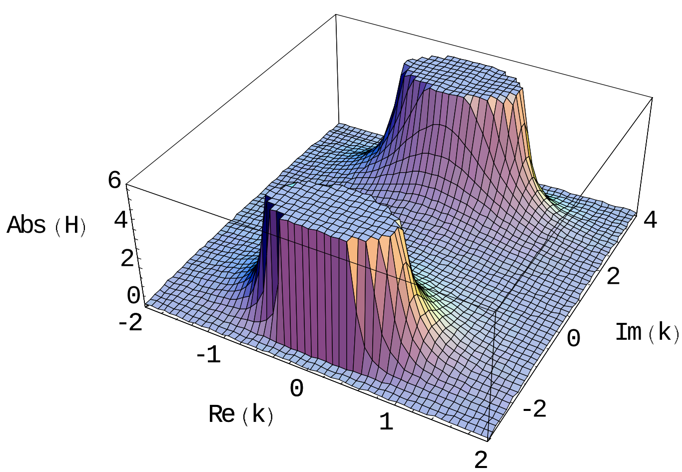
When $\rho = 0$, $\hat{H}$ satisfies $\hat{H}_\tau = (1/2)\xi^2 V \hat{H}_{VV} - c(k) V \hat{H}$. Applying the initial condition, the solution is

(2.11) $$\hat{H}(k,V,\tau) = \exp\left\{-\frac{V}{\xi}\sqrt{2c(k)}\tanh\left[\sqrt{\frac{c(k)}{2}}\,\xi\,\tau\right]\right\}.$$

The Taylor series for $\tanh z$ (the hyperbolic tangent) about $z = 0$ contains only odd powers of z and converges for $|z| < \pi/2$. This implies that $\hat{H}$ is analytic in c near $c = 0$. Because $c = 0$ at $k = 0$ and $k = i$, $\hat{H}$ is regular near those two points. Note that $\hat{H}(k = 0, V, \tau) = \hat{H}(k = i, V, \tau) = 1$.

Fig. 2.2 again plots $|\hat{H}(k,1,1)|$ with $\xi = 1$; we still have reflection symmetry about the $\mathrm{Im}(k)$ axis, but now singularities on the $\mathrm{Im}\, k$ axis are suggested:

Fig. 2.2 $|\hat{H}|$ **for the Square Root Model (** $\rho = 0$ **)**



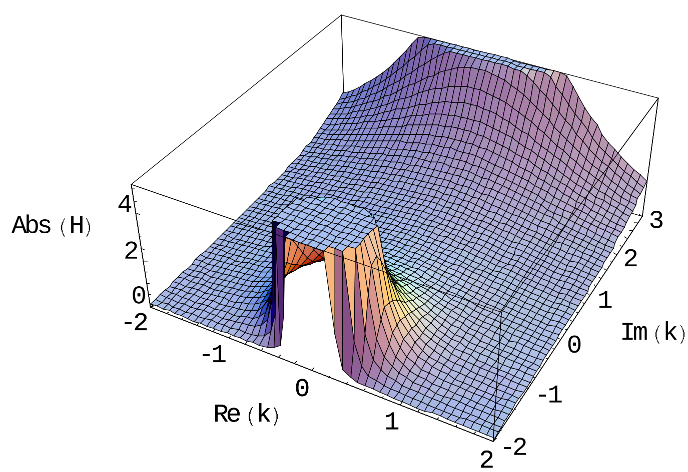
Along the pure imaginary axis, let $k = iy$ so that $c(k) = (y - y^2)/2$. This last expression becomes negative for $y < 0$ or $y > 1$, which means that the argument of the hyperbolic tangent, $(c/2)^{1/2}\xi\tau$, will be purely imaginary. So write $(c/2)^{1/2}\xi\tau = i\varphi$, where φ is a real number. But $\tanh(i\varphi) = i\tan\varphi$ which will of course diverge whenever $\varphi = (2n+1)\pi/2$, for $n = 0, \pm 1, \pm 2, \cdots$. Let k_n be the locations of the k-plane singularities of $\hat{H}$. The singularities in the figure correspond to the case $n = 0$. Setting $\varphi = \pi/2$, we find

$$k_0 = i y_\pm, \text{ where } y_\pm(\tau) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\pi^2}{\xi^2 \tau^2}} \cong \begin{cases} 3.68113 \\ -2.68113 \end{cases} \quad (\xi^2 = \tau = 1)$$

In the limits where $\xi^2 \to 0$ or $\tau \to 0$, we recover our previous results (an entire function) because the singularities move off to infinity. In the opposite limit where $\xi^2 \to \infty$, the singularities move to $y_\pm = 0,1$. So as long as ξ^2 is finite, we see that for this model, the integrand $\hat{H}(k,V,\tau)/(k^2 - ik)$ is free of singularities for the strips (i) $\alpha < \operatorname{Im} k < 0$ (ii) $0 < \operatorname{Im} k < 1$, and (iii) $1 < \operatorname{Im} k < \beta$, where $\alpha = y_-(\tau)$ and $\beta = y_+(\tau)$. This is typical.

In Fig 2.2, the line $\operatorname{Im} k = 1/2$ is symmetrically located between the two singularities. This occurs whenever $\rho = 0$. The square root model can also be solved when $\rho \neq 0$ (see Sec. 3 for formulas). Fig 2.3 shows the same model with the same parameters except that now $\rho = -1/2$; the reflection symmetry about $\operatorname{Re} k = 0$ is still present but now the symmetry about $\operatorname{Im} k = 1/2$ is lost.

Fig. 2.3 $|\hat{H}|$ **for the Square Root Model (**$\rho = -1/2$**)**



A Green function. Consider the entry in Table 2.1 for the delta function claim $\delta(\ln S_T - \ln K)$, but with $K = 1$. From the table, the transform of the payoff function is 1. So the fundamental transform is a solution to the problem with a delta function payoff and it's not too surprising that general claims can be developed in terms of this special one.

A closely related payoff function is $\delta(S_T - K)$, which has a fair value which is sometimes called a Green function or Arrow-Debreu security price. To get from one delta function to the other, apply the formula

$$\delta(f(x)) = \frac{\delta(x - x_0)}{| f'(x_0) |}, \quad \text{where } f(x_0) = 0\,.$$

Applying this in our case tells us that

$$\delta(S_T - K) = \frac{1}{K}\delta(\ln S_T - \ln K)\,.$$

That is, $\delta(S_T - K)$ has the payoff transform K^{ik-1} where k is any complex number. But, for times prior to expiration, we may still have a finite strip where the transform exists. So, a solution to the PDE (1.2) for this payoff, which we denote by $G(S,V,K,\tau)$ for Green function, is given by

$$G(S,V,K,\tau) = \frac{e^{-r\tau}}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S} e^{-ik(r-\delta)\tau} K^{ik-1} \hat{H}(k,V,\tau)\,dk$$

$$= \frac{e^{-r\tau}}{2\pi K} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\, \hat{H}(k,V,\tau)\,dk\,, \quad \alpha < \operatorname{Im} k < \beta$$

Interpretation of the fundamental transform. The last equation can be interpreted as follows. Associated with the martingale pricing process (1.1) is a risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$. Specifically $\tilde{p}\,dS_T$ is the probability that the stock price S with instantaneous variance V will, after the elapse of time-τ, reach the interval $(S_T, S_T + dS_T)$ with *any* variance. Since the stock price must end up *somewhere*, $\tilde{p}(S,V,S_T,\tau)$ is norm-preserving with respect to S_T. That is, $\int_0^\infty \tilde{p}(S,V,S_T,\tau)\,dS_T = 1$. Also, we have the initial value $\tilde{p}(S,V,S_T,0) = \delta(S - S_T)$. From the above, we know that both $G(S,V,S_T,\tau)$ and $\tilde{p}(S,V,S_T,\tau)$ satisfy the same PDE, (1.1), with the same initial condition. Are these two functions equal? The answer is yes, *if* $G(S,V,S_T,\tau)$ is norm-preserving. As we now show, there is a very simple test to determine when $G(S,V,S_T,\tau)$ is norm-preserving.

We can simply relabel $K \to S_T$ and re-write the last equation, using $\tilde{X} = \ln\left[\frac{S}{S_T}\right] + (r-\delta)\tau$, as

(2.12) $$G(S,V,S_T,\tau) = \frac{e^{-r\tau}}{2\pi S_T} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\tilde{X}} \hat{H}(k,V_t,\tau)\, dk \,,$$

Inversion. Multiply both sides of (2.12) by $\exp(ik'\tilde{X})$ and integrate with respect to S_T from $S_T = 0$ to $S_T = \infty$. On the right-hand-side this is accomplished by changing variables to $y = \ln S_T$ and using the delta function formula given above. The result is

$$\hat{H}(k,V,\tau) = \int_0^\infty e^{ik\tilde{X}} G(S,V,S_T,\tau)\, dS_T \,.$$

This last formula shows that $\hat{H}(k=0,V,\tau) = \int_0^\infty G(S,V,S_T,\tau)\, dS_T$; hence $G(S,V,S_T,\tau)$ is norm-preserving if, and only if, $\hat{H}(k=0,V,\tau)=1$. That is, we can identify the fundamental transform as the Fourier transform of the norm-preserving transition density in S_T if and only if $\hat{H}(k=0,V,\tau)=1$. In addition, the last formula shows that the martingale property for the stock price:

$$Se^{-\delta\tau} = e^{-r\tau} \int_0^\infty S_T G(S,V,S_T,\tau)\, dS_T \,,$$

is preserved by G, if and only if $\hat{H}(k=i,V,\tau)=1$. These results prompt the following definitions:

Definitions. A fundamental transform $\hat{H}(k,V,\tau)$ is called *norm-preserving* if it has the property $\hat{H}(k=0,V,\tau)=1$. If a fundamental transform is not norm-preserving, it's called *norm-defective.* A fundamental transform is called *martingale-preserving* if it has the property $\hat{H}(k=i,V,\tau)=1$; otherwise it's called *martingale-defective.*

Examples. The fundamental transform solution for the square root model is both norm-preserving and martingale-preserving. The fundamental transform solutions for the 3/2 model and the GARCH diffusion solution (see Sec. 3 below and Ch. 11) are sometimes norm-defective or martingale-defective.

With these definitions, we can assert that, when a fundamental transform is norm-preserving, then it's the Fourier transform of the risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$; i.e.,

(2.13)

$$\boxed{\begin{array}{c} \hat{H}(k,V,\tau) = \int_0^{\infty} e^{ik\tilde{X}}\, \tilde{p}(S,V,S_T,\tau)\, dS_T\,, \\ \text{where } \tilde{X} = \ln\left[\dfrac{S}{S_T}\right] + (r-\delta)\tau\,, \\ \text{if and only if } \hat{H}(k=0,V,\tau) = 1 \end{array}}$$

Failure of the martingale pricing formula. We shall find that it's possible for a fundamental transform, in very typical models, to be norm-preserving, but martingale-defective. Since it's norm-preserving, it's the Fourier transform of the risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$. In that case, as we noted earlier, we can interpret call option Solution I as an expectation

$$C_I(S,V,\tau) = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right].$$

The expectation is taken with respect to the norm-preserving density of the risk-adjusted process: $\tilde{p}(S,V,S_T,\tau)$. But, as we showed earlier, because the fundamental transform is martingale-defective, we have a *second* PDE solution $C_{II} \neq C_I$. Moreover, we show in Chapter 9 that the arbitrage-free fair value is given by C_{II}. In other words, the usual martingale pricing formula $e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right]$, while always a solution to the valuation PDE, does *not* always give the fair value of an option. Sometimes, option prices are not martingales, but only strictly local martingales.

Relationship to volatility explosions. When a fundamental transform is norm-preserving but martingale-defective, we also show in Chapter 9 that $1 - \hat{H}(k=i,V,\tau) = \hat{P}_{\exp}(V,\tau)$, where the right-hand-side is an *explosion* probability. Specifically, $\hat{P}_{\exp}(V,\tau)$ is the probability that a particular volatility process, the *auxiliary* volatility process, reaches $V = +\infty$ prior to time τ. Very briefly, to get a sense of what it going on in these cases, take $k = i$ in (2.7) and consider solutions to (2.7) $\hat{P}_{\exp}(V,\tau)$ with vanishing initial condition and with $\hat{P}_{\exp}(V=\infty,\tau) = 1$. If you can find such solutions, the auxiliary process can explode. Similarly, if the risk-adjusted volatility process can explode, then there exists a norm-defective fundamental transform such that $1 - \hat{H}(k=0,V,\tau) = P_{\exp}(V,\tau)$, where the right-hand-side is the explosion probability for the risk-adjusted process. In this case, take $k = 0$ in (2.7) Again, see Chapter 9 for a detailed discussion.

Reflection symmetry. Note that we always have the property, because the fundamental transform is the transform of a real-valued function, $\hat{H}^*(k,V,\tau) = \hat{H}(-k^*,V,\tau)$. This always holds, whether or not the transform is defective.

With the exception of the results for the 3/2 model given in Sec. 3 below, we generally assume without further comment that for the remaining development in this chapter, the fundamental transform is both norm- and martingale-preserving.

Power law behavior and scaling. Since $\tilde{X}$ is a function of the ratio S/S_T, then (2.13) shows that the transition density satisfies the scaling behavior

$$\tilde{p}(S,V,S_T,\tau) = \frac{1}{S_T}\varphi(u), \quad \text{where } u = S/S_T, \tag{2.14}$$

and $\varphi(u)$ is some *scaling* function. So if we know the behavior of $\tilde{p}(S,V,S_T,\tau)$ for $S_T \to \infty$, $(S_T \to 0)$ then we also know the behavior as $S \to 0$, $(S \to \infty)$ respectively. In fact, if the problem is regular, then we can deduce a lot about that behavior. For example, (2.13) exists for $k = iy$, where $y = \beta - \varepsilon$ for every $\varepsilon > 0$. That is, for any $S > 0$

$$\int_0^\infty S_T^{\beta-\varepsilon}\,\tilde{p}(S,V,S_T,\tau)\,dS_T < \infty .$$

This implies that $\tilde{p}(S,V,S_T,\tau) = O(S_T^{-1-\beta-\varepsilon})$ as $S_T \to \infty$, for every $\varepsilon > 0$. Similarly, taking $y = \alpha + \varepsilon$ implies that $\tilde{p}(S,V,S_T,\tau) = O(S_T^{-1+\alpha+\varepsilon})$ as $S_T \to 0$[6]. In turn, these relations may be restated in terms of the scaling function:

$$\varphi(u) = \begin{cases} O(u^{\beta+\varepsilon}) & \text{as } u \to 0 \\ O(u^{-\alpha+\varepsilon}) & \text{as } u \to \infty \end{cases} \quad \text{for every } \varepsilon > 0 . \tag{2.15}$$

Neglected boundary terms. One application of (2.14) is to show that the neglected boundary terms associated with the call option solution (2.8) can indeed be neglected. Writing $f(x) = C(e^x,V,t)$, where $x = \ln S$, the two neglected boundary terms from parts integrations were

---

[6] The notation, as $x \to x_0$, $f(x) = O(g(x))$, means that $f(x)/g(x)$ is bounded as $x \to x_0$. For a more rigorous discussion of the power law order behavior for $\tilde{p}$, see Fourier's theorem for analytic functions (Titchmarsh 1975, Theorem 26, p.44)

$$e^{ikx} f(x)\Big|_{x=-\infty}^{x=+\infty} \quad \text{and} \quad e^{ikx}\frac{\partial f(x)}{\partial x}\Big|_{x=-\infty}^{x=+\infty} \quad \text{where } 1 < \text{Im}\, k < \beta$$

In Appendix 2.3 to this chapter, we show the general arbitrage bounds $C \le S = e^x$, and $C_S \le 1$, which implies that both $f \le e^x$ and $\partial f / \partial x \le e^x$ for large enough *x*. So, since $\text{Im}\, k > 1$, both boundary terms vanish at the upper limit $x = +\infty$.

When option prices are martingales, they are given by the pricing formula

$$C(S,V,\tau) = e^{-r\tau}\int_0^{\infty} \tilde{p}(S,V,S_T,\tau)\max[S_T - K,0]dS_T$$
$$= e^{-r\tau} S\int_0^{S/K} \varphi(u)\Big[1 - \frac{K}{S}u\Big]\frac{du}{u^2},$$

where we substituted from (2.14). Letting $S \to 0$, we have from (2.15) that $\varphi(u) = O(u^{\beta+\varepsilon})$ as $u \to 0$. Substituting this expression into the above integral implies that $C(S) = O(S^{\beta+\varepsilon})$ as $S \to 0$ for every $\varepsilon > 0$. Or, in other words both $f = O(e^{x(\beta+\varepsilon)})$ and $\partial f / \partial x = O(e^{x(\beta+\varepsilon)})$ as $x \to -\infty$. Since $\text{Im}\, k < \beta$, both boundary terms also vanish at the lower limit $x = -\infty$. ■

The fundamental transform as a characteristic function. By a characteristic function, we mean any function that has the form

(2.16) $$\hat{H}(k) = \int_{-\infty}^{\infty} e^{ikx} dG(x) = \int_{-\infty}^{\infty} e^{ikx} g(x)dx,$$

where $G(x)$ is a *cumulative distribution function* and $g(x) = dG/dx$ is its probability density. For our purposes in this chapter, a cumulative distribution function is function of a real variable *x* that is (i) non-decreasing, and (ii) satisfies $G(-\infty) = 0$, $G(+\infty) = 1$. Of course, for this to occur, then $g(x)$ must be non-negative and integrable.

To show that $\hat{H}$ is a characteristic function, change integration variables in (2.13) from S_T to $\tilde{X} = \ln(S/S_T) + (r-\delta)\tau$ and define a new function $g(\tilde{X}; S,V,\tau)$ by

$$\tilde{p}(S,V,S_T,\tau)S\, e^{(r-\delta)\tau-\tilde{X}} = g(\tilde{X};S,V,\tau).$$

Or, suppressing arguments again,

$$dG(\tilde{X}) = g(\tilde{X})\, d\tilde{X} = \tilde{p}\Big(S,V,Se^{(r-\delta)\tau-\tilde{X}},\tau\Big)S\, e^{(r-\delta)\tau-\tilde{X}} d\tilde{X}\ .$$

This shows that $H(\tilde{X})$ is non-negative and now (2.13) reads

$$\hat{H}(k) = \int_{-\infty}^{\infty} e^{ik\tilde{X}} dG(\tilde{X}) \,,$$

where
$$G(\tilde{X}) = \int_{-\infty}^{\tilde{X}} \tilde{p}\left(S, V, S e^{(r-\delta)\tau - x}, \tau\right) S\, e^{(r-\delta)\tau - x} dx$$

$$= \int_{S \exp[(r-\delta)\tau - \tilde{X}]}^{\infty} \tilde{p}(S, V, S_T, \tau) dS_T \,.$$

This last equation shows that $G(\tilde{X})$ is indeed non-decreasing and satisfies $G(-\infty) = 0$, $G(+\infty) = 1$. And, since $\hat{H}(k)$ is of the form (2.16), with $x = \tilde{X}$, this shows that $\hat{H}(k)$ is a characteristic function. In fact, the examples show that $\hat{H}(k)$ can typically be further characterized as an *analytic characteristic function*. This important topic is discussed in Sec. 4.

The martingale pricing density. We can also consider the probability density $p(S_t, V_t, S_T, \tau)$ that the *actual* volatility process, starting from (S_t, V_t) reaches S_T with any variance. The ratio of the two probabilities

$$M_t = M(S_t, V_t, S_T, \tau) = \frac{\tilde{p}(S_t, V_t, S_T, \tau)}{p(S_t, V_t, S_T, \tau)}$$

also values arbitrary payoffs. That is, we have two general pricing formulas that work for any volatility-independent claim price, when it's a martingale:

(2.17)
$$F(S_t, V_t, \tau) = e^{-r\tau} \int_0^{\infty} \tilde{p}(S_t, V_t, S_T, \tau) g(S_T) dS_T$$

$$= e^{-r\tau} \int_0^{\infty} M(S_t, V_t, S_T, \tau) p(S_t, V_t, S_T, \tau) g(S_T) dS_T \,.$$

These are explicit integral kernel versions of the martingale pricing formulas presented in Chapter 1. As a general rule, (2.17) is the long way around, however, from the Solution I and II formulas based upon a direct *k*-plane integration, since it forces you to do an extra integration. So we don't recommend (2.17) for most computations—but we have seen already that it was useful in considering the $S \to 0$ and $S \to \infty$ limits of the theory.

Forward contracts and options on forwards. The formulas are easily modified to handle forwards. For example, the forward stock price F_t is defined to be the fair value at time t for delivery of one share of the stock at time T. As usual, this price is determined by arbitrage to be $F_t = e^{(r-\delta)\tau} S_t$, where $\tau = T - t$. Hence by Ito's formula, the martingale pricing process $\tilde{P}$ of (1.1) becomes $dF_t = \sigma_t F_t d\tilde{B}_t$, with the same volatility evolution. Under $\tilde{P}$, the forward price behaves like a stock with a dividend yield of *r*. Using this idea, a call option on the forward, say solution II at $\text{Im}\,k = 1/2$, becomes

$$C_{II}(F,V,\tau) = e^{-r\tau}\left[F - K\frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2 - ik}\,dk\right],$$

where $X = \ln(F/K)$.

Summary. If the initial-value problem in the box below is regular in a strip $\alpha < \text{Im}\,k < \beta$ in the complex *k*-plane, then the solution can be used to determine option prices by a *k*-plane integration:

$$\text{(2.19)}\qquad \frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}a^2(V)\frac{\partial^2 \hat{H}}{\partial V^2} + \left[\tilde{b}(V) - ik\rho(V)a(V)V^{1/2}\right]\frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H}$$

where $c(k) = (k^2 - ik)/2$. In addition to $\hat{H}(k,V,\tau=0) = 1$, the fundamental solution has the following properties:

(2.20)

(i) $\hat{H}^*(k,V,\tau) = \hat{H}(-k^*,V,\tau)$

(ii) $\hat{H}(k=0,V,\tau) = 1 - P_{\text{exp}}(V,\tau)$

(iii) $\hat{H}(k=i,V,\tau) = 1 - \hat{P}_{\text{exp}}(V,\tau)$,

where $\hat{P}_{\text{exp}}$ and P_{exp} are the probabilities that the auxiliary volatility process and risk-adjusted volatility process can explode to $+\infty$

3 Some Models with Closed-form Solutions

In general, even with the assumption of a simple process for the *actual* volatility, the simplest risk-adjustments (via utility theory) can produce complex results for the martingale pricing process. Risk-adjustment is discussed in detail in Chapter 7. To obtain a model that can be solved in closed-form generally requires two assumptions: (i) a relatively simple process for the actual volatility, and (ii) a relatively simple preference model, such as the representative agent model with power utility.

Making both of these assumptions, here is a short list of models that can be solved in closed-form. Each volatility process has constant correlation ρ with the stock price process. All other parameters are also constants. The agent is assumed to be a pure investor (no consumption until a final date) with a distant planning horizon. The parameter γ is the representative's risk-aversion parameter. It's restricted to $\gamma \le 1$ plus some additional restrictions that are shown. The risk-aversion adjustments are derived in Chapter 7.

Some solvable models and their volatility processes

Square root model

$$P: \; dV = (\omega - \theta V)\,dt + \xi\sqrt{V}dW$$

$$\tilde{P}: \; dV = \left\{\omega - \tilde{\theta} V\right\}dt + \xi\sqrt{V}dW \,,$$

where $\tilde{\theta} = (1-\gamma)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}$

Conditions: $\gamma(1-\gamma) \le \theta^2$

3/2 model

$$P: \; dV = (\omega V - \theta V^2)\,dt + \xi V^{3/2}dW$$

$$\tilde{P}: \; dV = \left\{\omega V - \tilde{\theta} V^2\right\}dt + \xi V^{3/2}dW \,,$$

where $\tilde{\theta} = -\frac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \frac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}$

Conditions: $\gamma(1-\gamma)\xi^2 \le (\theta + \frac{1}{2}\xi^2)^2$

Geometric Brownian motion

$$P: \; dV = -\theta V\,dt + \xi V\,dW$$

$$\tilde{P}: \; dV = \left\{-(1-\gamma)\rho\,\xi V^{3/2} + \tfrac{1}{2}\xi^2 V\left[1 + y\frac{K'_\mu(y)}{K_\mu(y)}\right]\right\}dt + \xi V\,dW \,,$$

where $y = \frac{2}{\xi}\sqrt{-\gamma(1-\gamma)V}$ and $\mu = 1 + \frac{2\theta}{\xi^2}$.

Conditions: ($-2\theta < \xi^2$ and $\gamma \le 0$) or $\gamma = 1$

The solution for the fundamental transform under geometric Brownian motion is quite complex and difficult to work with when the correlation is non-zero. In contrast, both the square root model and the 3/2 model have short solutions that we now show. Both models use the reduced variables

$$t = \tfrac{1}{2}\xi^2\tau\,, \qquad \tilde{\omega} = \frac{2}{\xi^2}\omega\,, \qquad \tilde{c} = \frac{2}{\xi^2}c(k)\,. \tag{3.1}$$

In terms of these variables, the fundamental transforms are given below. The results for all three models are derived in Chapter 11.

The square root model[7] [$\gamma \le 1$ and $\gamma(1-\gamma)\xi^2 \le \theta^2$]

$$\hat{H}(k,V,\tau) = \exp[\, f_1(t) + f_2(t)V\,]\,, \quad \text{using} \tag{3.2}$$

$$f_1(t) = \tilde{\omega}\,\left[t\,g - \ln\left(\frac{1-h\exp(d\;t)}{1-h}\right)\right], \qquad f_2(t) = g\,\left(\frac{1-\exp(d\;t)}{1-h\exp(d\;t)}\right)$$

$$d = [\hat{\theta}^2 + 4\tilde{c}]^{1/2}\,, \qquad g = \tfrac{1}{2}(\hat{\theta} + d)\,, \qquad h = \frac{\hat{\theta}+d}{\hat{\theta}-d}\,,$$

where $$\hat{\theta}(k) = \frac{2}{\xi^2}\left[(1-\gamma+ik)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\,\right]$$

The 3/2 model[8] [$\gamma \le 1$ and $\gamma(1-\gamma)\xi^2 \le (\theta + \frac{1}{2}\xi^2)^2$]

$$\hat{H}(k,V,\tau) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}\left[X\left(\frac{\tilde{\omega}}{V},\omega\,\tau\right)\right]^{\alpha} M\left[\alpha,\beta,-X\left(\frac{\tilde{\omega}}{V},\omega\,\tau\right)\right], \tag{3.3}$$

using $$X(x,t) = \frac{x}{e^t-1}\,, \quad \mu = \tfrac{1}{2}(1+\hat{\theta})\,, \quad \delta = \left[\mu^2 + \tilde{c}\right]^{1/2},$$

$$\alpha = -\mu + \delta\,, \quad \beta = 1 + 2\delta\,,$$

where $$\hat{\theta}(k) = -1 + \frac{2}{\xi^2}\left[\sqrt{(\theta+\tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2} + (1-\gamma+ik)\rho\xi\right].$$

---

[7] Heston's (1993) call option solution is also achieved with a transform-based approach: an ordinary Fourier transform with respect to the log-strike price. In Heston's approach, there are two transforms instead of the one here.

[8] Caution: this fundamental transform is sometimes either norm-defective or martingale-defective. Using risk-neutral preferences only, the 3/2 model has been independently developed by Heston (1997), using an approach similar to his 1993 paper.

In (3.3), $\Gamma(z)$ is the Gamma function and $M(\alpha,\beta,z)$ is a confluent hypergeometric function[9] . Also, note that the second argument for $X(\bullet,\bullet)$ in (3.3) uses $\omega\,\tau = \tilde{\omega}\,t$.

Determining the fundamental strip. Once you have $\hat{H}$ for a model, then you can analyze it to determine the fundamental strip of regularity: $\alpha < \operatorname{Im} k < \beta$ and whether it's norm- and/or martingale-preserving. Once you know that, you know the regions of validity for all of the option formulas presented previously. As an example, consider the square root model above. Rather than a complete analysis, let's just establish that the strip $0 < \operatorname{Im} k < 1$ is free from singularities—this places the boundaries of the fundamental strip outside this region.

The singularities occur where $1 = he^{dt}$, which causes divergences in both $f_1(t)$ and $f_2(t)$. We know the singularities occur along the imaginary axis, so consider $k = iy$, where y is real. We see from (3.1) that $\tilde{\theta}$ is real along that axis. Moreover, for $0 < \operatorname{Im} k < 1$, then $\tilde{c} > 0$ (and real). Hence d is real and satisfies $d > |\tilde{\theta}|$, which implies that $\tilde{\theta} + d > 0$ and $\tilde{\theta} - d < 0$. In other words, $h < 0$. Since d is real and $h < 0$, there can be no solutions to $h = e^{-dt}$ inside the strip $0 < \operatorname{Im} k < 1$. Hence $0 < \operatorname{Im} k < 1$ is free from singularities. ■

Integrating. Once you know where you can legally integrate, then you're a *k*-plane integration away from the call option price. For these remaining steps, see Appendix 2.2 to this chapter. When you obtain those prices, you'll find that both models exhibit the typical qualitative behavior that we discuss in subsequent chapters: implied volatility smile patterns (see Chapter 5) and an implied volatility term structure that flattens to a constant as $\tau \to \infty$ (see Chapter 6). For the derivation of the formulas (3.2) and (3.3) see Chapter 11.

4 Analytic Characteristic Functions

We have seen from examples that $\hat{H}(k,V,\tau)$ is often an analytic function of k in some neighborhood. In general, a characteristic function $\hat{f}(k)$ is *any* function which has the representation

---

[9] See Abramowitz and Stegun (1970) for properties of these and other special functions.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx \,,$$

where $p(x)$ is a probability density for some cumulative distribution function. Lukacs (1970, Chapter 7) proves two theorems that are relevant to our application. To achieve a more symmetrical notation, we write $\hat{H}(k)$ for our fundamental transform and $\hat{f}(k)$ for a generic characteristic function

If $\hat{f}(k)$ is regular in a neighborhood of $k = iy$, where y is real, then we call $\hat{f}(k)$ an analytic characteristic function (Lukacs takes $y = 0$). We have shown in a number of examples that the regions of regularity for $\hat{H}(k)$ are typically strips in the complex k-plane. And, we have suggested strips as regions of regularity in general. The rationale for the general case lies in the following theorem, quoted without proof:

THEOREM 2.1 (Lukacs Theorem 7.1.1): *If a characteristic function* $\hat{f}(k)$ *is regular in the neighborhood of* $k = 0$, *then it is also regular in a horizontal strip and can be represented in this strip by a Fourier integral. This strip is either the whole plane, or it has one or two horizontal boundary lines. The purely imaginary points on the boundary of the strip of regularity* (*if this strip is not the whole plane*) *are singular points of* $\hat{f}(k)$.

Discussion. In our application, we have often found that the fundamental transform $\hat{H}(k)$ is regular in the horizontal strip $\alpha < \operatorname{Im} k < \beta$, where $\alpha < 0$ and $\beta > 1$. We have already pointed out that the PDE (2.19) is especially well-behaved when $\operatorname{Re} c(k) > 0$, which occurs when $0 < \operatorname{Im} k < 1$. In this subsection, we try to understand a little better why the strip $0 < \operatorname{Im} k < 1$ is often free of singularities of $\hat{H}(k)$. We know from (2.13) that $\hat{H}(k)$ has the representation

$$\hat{H}(k,V,\tau) = \int_0^{\infty} e^{ik\tilde{X}(S_T)}\, \tilde{p}(S,V,S_T,\tau)\, dS_T \,,$$

where $\tilde{X}(S_T) = \ln\left[\dfrac{S}{S_T}\right] + (r-\delta)\tau$. Therefore

$$\hat{H}^{(m)}(k) = \frac{d^m}{dk^m}\hat{H}(k) = i^m \int_0^{\infty} \left[\tilde{X}(S_T)\right]^m e^{ik\tilde{X}}\, \tilde{p}(S,V,S_T,\tau)\, dS_T \,.$$

Let $k = k_r + iy$, where k_r and y are real. Then,

$$\hat{H}^{(m)}(k_r + iy) = i^m e^{-y(r-\delta)\tau} \int_0^{\infty} \left[\tilde{X}(S_T)\right]^m e^{ik_r\tilde{X}} \left(\frac{S_T}{S}\right)^y \tilde{p}(S,V,S_T,\tau)\, dS_T$$

Along the purely imaginary axis, we have

$$\hat{H}^{(m)}(iy) = i^m e^{-y(r-\delta)\tau} \int_0^\infty [\,\tilde{X}(S_T)\,]^m \left(\frac{S_T}{S}\right)^y \tilde{p}(S,V,S_T,\tau)\,dS_T\,. \tag{4.1}$$

And in particular for the fundamental transform itself, we have

$$\hat{H}(iy) = i^m e^{-y(r-\delta)\tau} \int_0^\infty \left(\frac{S_T}{S}\right)^y \tilde{p}(S,V,S_T,\tau)\,dS_T\,. \tag{4.2}$$

Now it's known from complex variable theory that if a function is analytic in a region *R*, then it has derivatives of all orders and a Taylor series in *R*. Consequently, if $\hat{H}(k)$ is regular near the point $k = iy$, then the series

$$\hat{H}(k) = \sum_{m=0}^{\infty} \frac{\hat{H}^{(m)}(iy)}{m!}(k-iy)^m$$

is convergent. This means that $\hat{H}(k)$ is an analytic characteristic function near $k = iy$ if and only if the following two conditions are satisfied:

(4.3) (i) $\hat{H}^{(m)}(iy)$ exists for all $m = 0,1,2,\ldots$

(4.4) (ii) $\lim_{m\to\infty}\left[\frac{|\,\hat{H}^{(m)}(iy)\,|}{m!}\right]^{1/m} = \frac{1}{\Delta}$ is finite.

Then if these conditions hold, $\hat{H}(k)$ is regular in the strip $(y-\Delta) < \operatorname{Im} k < (y+\Delta)$.

Now recall the normalization and martingale identity:

(a) $\int_0^\infty \tilde{p}(S,V,S_T,\tau)\,dS_T = 1$ and (b) $Se^{-\delta\tau} = e^{-r\tau}\int_0^\infty S_T\,\tilde{p}(S,V,S_T,\tau)\,dS_T$.

These two relations strongly restrict the possible behavior of $\tilde{p}(S_T)$ near $S_T = 0$ and $S_T = \infty$, where we suppress the other arguments in $\tilde{p}(S,V,S_T,\tau)$. Because of (a), it must be true that $\tilde{p}(S_T) = O(S_T^{-1+\varepsilon})$ for every $\varepsilon > 0$ as $S_T \to 0$. In other words $\tilde{p}(S_T)$, if it diverges at all as $S_T \to 0$, diverges no faster than $S_T^{-1+\varepsilon}$. Similarly, because of (b), it must be true that $\tilde{p}(S_T) = O(S_T^{-2-\varepsilon})$ for every $\varepsilon > 0$, as $S_T \to \infty$. Because of these two end-point behaviors, if you keep *y* in (4.2) in the range $0 < y < 1$, then you will have a convergent integral. Similarly, with the same restriction, (4.1) should exist for any *m* because, (I) as $x \to \infty$, $x^y\,|\ln(1/x)|^m = O(x)$ for any $y < 1$ and (II) as $x \to 0$, $x^y\,|\ln(1/x)|^m = O(1)$ for any $y > 0$

Unfortunately, this argument establishes (4.3) but not (4.4). Nevertheless, it provides some additional insight into why $0 < \mathrm{Im}\, k < 1$ is the "natural" strip for the financial claim problem.

Stationary points. In Chapter 6, "The Term Structure of Implied Volatility", we examine the asymptotic $\tau \to \infty$ behavior of the theory. It turns out that the asymptotic implied volatility is determined by an eigenvalue of a differential operator. This eigenvalue is also a stationary or saddle point of $\hat{H}(k)$ in the k-plane (recall the saddle shapes from the figures). We discover, in particular models, that these stationary points always lie along the purely imaginary axis. The general reason for this behavior lies in the following theorem:

THEOREM 2.3 (Lukacs Theorem 7.1.2): *Let $\hat{f}(k)$ be an analytic characteristic function. Then $|\hat{f}(k)|$ attains its maximum along any horizontal line contained in the interior of its strip of regularity on the imaginary axis. The derivatives $d^{2j}\hat{f}/dk^{2j}$ of even order of $\hat{f}$ have the same property.*

PROOF: We know that $\hat{f}(k)$ has the representation

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx\,, \qquad \alpha < \mathrm{Im}\, k < \beta\,.$$

Therefore $$\hat{f}^{(m)}(k) = \frac{d^m}{dk^m}\hat{f}(k) = i^m \int_{-\infty}^{\infty} x^m e^{ikx} p(x)\, dx\,.$$

Let $k = k_r + iy$, where k_r and y are real and where $\alpha < y < \beta$. Then,

$$|\hat{f}^{(m)}(k_r + iy)| \;\le\; \int_{-\infty}^{\infty} |x|^m \, e^{-yx} p(x)\, dx\,.$$

If $m = 2j \;\; (j = 0,1,2,\ldots)$ is an even integer, then this becomes

$$|\hat{f}^{(2j)}(k_r + iy)| \;\le\; \int_{-\infty}^{\infty} x^{2j} e^{-yx} p(x)\, dx \;=\; |\hat{f}^{(2j)}(iy)|,$$

so that $$\max_{-\infty < k_r < \infty} |\hat{f}^{(2j)}(k_r + iy)| \;=\; |\hat{f}^{(2j)}(iy)|\,.$$

The ridge property. The relation

$$|\hat{f}(k_r + iy)| \;\le\; |\hat{f}(iy)|$$

is very important in the theory of analytic characteristic functions, and is called the "ridge property". It plays an important role in our application in the asymptotic $\tau \to \infty$ theory. So we have learned that if the fundamental transform $\hat{H}(k)$ is an analytic characteristic function, then it is also a "ridge function".