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Private Provision of a Public Good in a General Equilibrium Model*

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Abstract

We analyze a general equilibrium model of a completely decentralized pure public good economy. Competitive firms using private goods as inputs produce the public good, which is privately provided by households. Previous studies on private provision of public goods typically use one private good, one public good models in which the public good is produced through a constant returns to scale technology. Two distinguishing features of our model are the presence of *several* private goods and *non-linear*, in fact strictly concave, production technology for the public good.

In this more general framework we revisit the question of neutrality - or non-effectiveness - of government interventions on private provision equilibrium outcomes. We confirm the well-known neutrality results when all households are contributing to the provision of the public goods and the non-neutrality results when there are some non-contributing households. We also show that relative price effects, which are absent with a single private good and under constant returns to scale technology for public good production, come to play an important role and generate new non-neutrality results. Specifically,

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if the number of private goods is greater than one, typically there exists a redistribution of endowments of the private good *among non-contributors* which increases the total supply of public good. More importantly, if a condition involving the number of households and private goods holds, typically there exists a choice of taxes on firms that Pareto improves upon the equilibrium outcome. *Therefore, a general non-neutrality result (in terms of utilities) holds even if all households are contributors.*

1 Introduction

In this paper, we analyze a general equilibrium model of a completely decentralized pure public good economy. Competitive rms using private goods as inputs produce the public good, which is nanced , or privately provided , or voluntarily contributed , by households. Previous studies on private provision of public goods typically use one private good, one public good models in which the public good is produced through a constant returns to scale technology. Two distinguishing features of our model are the presence of *several* private goods and *non-linear*, in fact strictly concave, production technology for the public good. In this more general framework we revisit the question of neutrality of government interventions on private provision equilibrium outcomes. We show that relative price effects, which are absent with a single private good and under constant returns to scale technology for public good production, come to play an important role in our more general framework. Relative price effects provide a powerful channel through which government interventions can bring about redistributive wealth effects, which, in turn, will change equilibrium outcomes.

The interest in a general equilibrium model with private provision of public goods lies in the fact that it serves as a benchmark extension of an analysis of completely decentralized private good economies to public good economies. Moreover, there are some relevant situations in which public goods are in fact privately provided: e.g., private donations to charity at a national and international level, campaign funds for political parties or special interests groups, and certain economic activities inside a family.

Warr (1983) provides the rst statement of the fact that in a private provision model of voluntary public good supply, small income redistributions among contributors to a public good are neutralized by changes in amounts contributed in equilibrium. Consumption of the private good and the total supply of the public good remain exactly the same as before redistribution. Warr (1982) also observes that small government contributions to a public good, paid for by lump sum taxes on contributors, will be offset completely by reductions in private contributions. Warr s results are derived in a partial equilibrium framework. Bernheim (1986) and Andreoni (1988) extend Warr s result to show that distortionary taxes and subsidies may also be neutralized by changes in private contributions.

Bergstrom, Blume and Varian (1986) - from now on quoted as BBV - discuss Warr s results in a simple general equilibrium model with one private and one public good and constant returns to scale in the production of the public good. They allow non-in nitesimal redistributions and possibility of zero contribution by some of the households.[1] They show that redistribution is not, in general, neutral if the amount of income distributed away from any household is more than his private contribution to the public good in the original equilibrium. Andreoni and Bergstrom (1996) argue that the neutrality results obtained in previous models all involve redistribution schemes in which only small changes in incomes, namely changes that do not exceed anyone s original equilibrium private contribution, are allowed. We show in this paper that it is not the smallness of redistributions allowed but the absence of relative price effects, through which further income effects arise, that leads to the neutrality results.[2]

With only one private good, assuming constant returns to scale, and therefore linearity of the production function, implies that there is no loss of generality in normalizing prices of both the private and the public good to one. These assumptions also allow taking pro ts of rms equal to zero, with the implication that the presence of rms basically plays no role in the model. With non-constant returns to scale in production allowed, such normalization of prices is not possible in our framework, even in the case of only one private good. For that reason, taxes and subsidies will have different effects in our framework than that of BBV s. In BBV s model, taxes on a subset of individuals change the distribution of wealth among those individuals, but not among individuals outside that subset. In our model, taxing any subset of households leads to changes in relative prices and therefore may affect the wealth of *any* household. This simple observation explains why some of our results will be different from those of BBV. Relative price effects, which are absent in BBV s setup, play a redistributive role in our framework.

With more than one private good and non-constant returns to scale, modeling of how the public good is produced becomes an issue. If a pro t-maximizing (private) rm is assumed to produce the public good, then how the (non-zero) pro ts of the rm are apportioned among its shareholders will have an impact on equilibrium outcomes. Alternatively, one can consider the production of the public good as being carried out by a non-pro t (public)

[1] See also the related papers Bergstrom, Blume and Varian (1992) and Fraser (1992).

[2] In a more recent contribution on the effects of taxation in an economy with private provision of public goods, Brunner and Falkinger (1999) consider a model with two private goods and public good, but the prices of the goods are taken as given in their model.

firm subject to a balanced budget constraint. In that case the contributions in monetary amounts collected from households would finance the cost of producing the public good. The amount of public good to be produced by the non-profit firm can be taken as the maximum amount that can be produced with the amount collected. Such a model assumes the presence of either the government itself or a non-profit firm in the production of the public good, which raises a host of issues related to what the objective functions of such entities should be.[3]

In the present paper, we study the alternative that we believe is the one most consistent with a decentralized framework, namely that profit-maximizing firms produce the public good in a competitive market.[4] Thus, the government is not involved in the production of the public good and only has the role of enforcing taxes and subsidies, if there are any, on households and firms. In analyzing crowding-out effects of government provision of public goods, it will be assumed that government makes purchases from the firms at market prices. We take this set of assumptions as describing a completely free market oriented policy benchmark applicable in principle to provision of any type of public good. The model can also be seen as a descriptive one covering cases in which a public institution purchases from private producers goods that will be consumed by households involved as public goods. Examples include fluoride purchased by a public agency to fluoridate a public water supply, pesticides purchased by government, packages of medicine bought by an international charitable organization for use in an underdeveloped country to control an epidemic disease, and so on.

As for the behavior of households, each household starts with endowments in private good only. Households also hold shares in the firms that produce the public good. There is no public good initially. Taking as given the prices of private goods, the price of the public good, the profits of the firms they hold shares in, as well as the amount of public good provided by others, each household determines their private good consumption bundle and the amount of public good they will privately provide. Private provision can take the form of each household actually purchasing the good in the amount they desire directly from a producer, or donating to a public or a voluntary organization a monetary amount with which the same amount can

[3] We analyze such a model in a forthcoming paper.

[4] The presence of more than one public good can be incorporated into our model, leaving the basic results unchanged - see Section 6.1.

be purchased by this agency at the equilibrium price for the public good. Note that in making their decisions on how much to contribute to the public good, the households will take into consideration relative prices. If the prices of medicines, doctor services, lab tests doubles, an household may choose to contribute more to fund raising for an hospital. When there are taxes or subsidies imposed by the government, households take these as given in their budget constraints. We assume that households fully understand and take into account the government s budget constraint.

Observe that household behavior described above amounts to assuming that the prices of private goods as well as the public good are taken as given by households in their maximization problem, while there is strategic interaction among them regarding the quantities chosen as private contribution to the public good. This type of behavior is plausible when the set of prospective consumers of the public good are small with respect to the economy in which they are embedded. Thus the households choices of the public good affects neither the prices of inputs that goes into production of the public good nor the price of the public good produced for the economy at large, while it does have an impact on others choices locally involved in the consumption of the good. For example, consider donations to a large agency that is involved in projects to eradicate poverty in Africa. Eradication of poverty in Africa is the public good for those who care about it in this case, and their cash or in kind contributions to the agency are not going to affect the prices of goods that go into the activities of the agency.[5]

Since the modeling of production technology for the public good is a key feature that distinguishes our model from that of BBV s, we further discuss it below.

With many inputs linearity of a production function implies constant returns to scale, but the converse will not hold (except for the single input case). Therefore, BBV s results extend to the case of many inputs only if the production function is linear.

When the production function is linear, and rms produce a strictly positive quantity of output, prices are completely determined by the production coefficients. Therefore, equilibrium prices are xed by the technology , i.e., they change only if technology changes, and they are not affected by changes

[5] The above discussion points out that goods may be (locally) public or private according to their use and not on the basis of their physical characteristics. Analysis of such a local public good model that explicitly takes into account the possibility that goods can be private or public according to their use is going to be the subject of another paper.

in endowments or preferences. Outside the case of linear production function, equilibrium prices are only partially fixed by the technology if returns to scale are constant, or they are not fixed at all by the technology if the production function is strictly concave.[6] Since price effects are the crucial factor in our analysis, we conjecture that our results apply to all situations in which prices are not fixed by the technology. That is, not only does it apply, to the case of strictly concave technology, as we show here, but it should also apply to the case of concave production functions with and without constant returns to scale.[7] Therefore, we can claim that, within the space of convex production technologies, our framework covers all but the rather extreme linear case.

Below are some of the results for the one good, linear technology case that are relevant to our analysis. All of the results except the last one, which is due to Cornes and Sandler (2000), are due to BBV. In all results, the distinction between *contributors*, i.e. the households that provide a strictly positive amount of public good, and *non-contributors* plays a crucial role. The tax schemes referred to involve small perturbations of the initial endowment distribution; hence named *local* tax schemes:

1. An equilibrium exists (BBV, Theorem 2, page 33) and is unique under a demand normality condition for the public good.

All the following results refer to an equilibrium situation.

2.a. Any local tax scheme applied to *contributors only* has no effect on the total supply of the public good (BBV, Theorem 1, page 29).

2.b. Any local tax scheme that redistributes wealth *from non-contributors to contributors* increases the total supply of the public good (BBV, Theorem 4.ii, page 36).

3. Suppose that the government supplies some amount of the public good, which it pays for through a local tax scheme among households. Then,

3a. If local positive taxes are imposed on *contributors*, the total supply of public good does not change: there is complete crowding-out (BBV, Theorem 6.i, page 42).

3b. If local positive taxes are imposed on *non-contributors*, then the total supply of public good increases (BBV, Theorem 6.ii, page 42).

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[6] See, for example, the discussion on properties of equilibria with constant returns to scale in production by Mas-Colell et al. (1995), Section 17.F, pages 606-616.

[7] The investigation of this conjecture is in fact the content of a forthcoming paper of ours.

4. Cornes and Sandler (2000) investigate the possibility for a government to increase all households welfare via an increase in the total supply of the public good. They observe that the possibility of such Pareto improvements is positively related to the number of non-contributors, marginal evaluation of the public good by noncontributors, and the change in the private provision of public good resulting from an increase in contributors total wealth.

Our approach is based on differential techniques, which amount to computing the derivative of the goal function - the total amount of provided public good or household welfare levels - with respect to some policy tools - taxes and/or government s direct provision of the public good - on the equilibrium manifold. Therefore, all our arguments are local in their nature. We also note that, since price effects may in principle go in any direction, all our non-neutrality results hold only *typically* in the relevant space of economies. First we observe that

1*. An equilibrium exists.

Existence of equilibrium, together with some regularity properties of equilibria that we use in the present paper, are proved in another paper by Villanacci and Zenginobuz (2001). Note that while BBV present their analysis in the case of unique equilibria, we allow for multiple equilibria.

The results of the present paper can be summarized as follows:

2a*. A neutrality result of the type described in 2a. holds and, in fact, it can be slightly generalized.

2b*. Local redistribution of wealth from non-contributors to contributors has the same effect as in the linear case.

2c*. If the number of private goods is greater than one, typically in the subset of economies for which there exist at least two non-contributors, there exists a redistribution of endowments of the private good *among non-contributors* which increases the total supply of public good.

3*. Results about crowding-out hold also in our case; more precisely, in our framework, results 3a. of BBV holds typically.

4*. If the number of households is smaller than the number of private goods, typically in the set of economies, there exists a choice of taxes on rms inputs and outputs that Pareto improves or impairs upon the equilibrium outcome. *Therefore, a general non neutrality result (in terms of utlities) holds even if all households are contributors.*

In relation to our result 4*, observe that other types of Pareto improving interventions could be studied applying the same general technique used in the paper.

An interesting feature of the model analyzed in the paper is that it allows an (partial) answer to a more general problem. Several imperfections can be considered in a standard general equilibrium model. Is government intervention more or less effective when more than one imperfection is present? Does including one more imperfection in the presence of an initial one make government intervention more or less effective? Results 2a and 3a in BBV apply to general equilibrium models with incomplete markets or restricted participation besides public goods. Government intervention would have no effect in those cases if the economy also involves public goods that are privately provided. It is well known that, under certain assumptions a well chosen local redistribution among all households in a model with incomplete markets leads to a Pareto superior equilibrium (see Geanakoplos and Polemarchakis (1986) and Citanna et al. (1998)). The above observations suggest that a government intervention that would be effective against a single imperfection may turn out to be ineffective when the existence of other imperfections is taken into account. On the other hand, the analysis of Pareto improvement possibilities described above strongly supports the following result: In the case of the presence of two imperfections, there exists a *well-chosen* intervention which can reach the same goals as in the case of one imperfection.

The plan of our paper is as follows. In section 2, we present the set up of the model and the existence and regularity results proved in Villanacci and Zenginobuz (2001).

In sections 3, 4 and 5, following a strategy described, for example, in Citanna, Kajii and Villanacci (1998), we prove our main results on the possibility of a government intervention to in uence the total amount of public good and household welfare.[8]

2 Setup of the Model

We consider a general equilibrium exchange model with private provision of a public good.

There are C, $C \geq 1$, private commodities, labelled by $c = 1, 2, ..., C$. There are H households, $H > 1$, labelled by $h = 1, 2, ..., H$. Let $\mathcal{H} = \{1, ..., H\}$ denote the set of households. Let x_h^c denote consumption of private

[8] A more detailed version of the paper, containing even elementary proofs is available from the authors.

commodity c by household h; e_h^c embodies similar notation for the endowment in private goods.

The following standard notation is also used:

- $x_h \equiv (x_h^c)_{c=1}^C$, $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- $e_h \equiv (e_h^c)_{c=1}^C$, $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- p^c is the price of private good c, with $p \equiv (p^c)_{c=1}^C$; and p^g is the price of the public good. Let $\widehat{p} \equiv (p, p^g)$.
- $g_h \in R_+$ is the amount of public good that consumer h provides. Let $g \equiv (g_h)_{h=1}^H$, $G \equiv \sum_{h=1}^H g_h$, and $G_{\backslash h} \equiv G - g_h$.

Household h s preferences over the private goods and the public good are represented by a utility function $u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++}^1 \to \mathbb{R}$. Note that households preferences are de ned over the total amount of the public good, i.e. we have $u_h : (x_h, G) \mapsto u_h(x_h, G)$.

Assumption 1 *$u_h(x_h, G)$ is a smooth, differentiably strictly increasing (i.e., for every $(x_h, G) \in R_{++}^{C+1}$, $Du_h(x_h, G) \gg 0)$[9], differentiably strictly concave function (i.e., for every $(x_h, G) \in R_{++}^{C+1}$, $D^2 u_h(x_h, G)$ is negative de nite), and the closure (in the topology of R^{C+1}) of the indifference surfaces is contained in R_{++}^{C+1}.*

There are F rms, indexed by subscript f, that use a production technology represented here by a transformation function $t_f : \mathbb{R}^{C+1} \to \mathbb{R}$, where $t_f : \left(y_f, y_f^g\right) \mapsto t_f\left(y_f, y_f^g\right)$.

Assumption 2 *$t_f\left(y_f, y_f^g\right)$ is a C^2, differentiably strictly decreasing (i.e., $Df\left(y_f, y_f^g\right) \ll 0$), and differentiably strictly concave (i.e., $D^2 f\left(y_f, y_f^g\right)$ is negative de nite) function, with $f(0) = 0$.*

De ne $\widehat{y}_f \equiv \left(y_f, y_f^g\right)$ and $\widehat{y} \equiv (\widehat{y}_f)_{f=1}^F$. Using the convention that input components of the vector $\left(y_f, y_f^g\right)$ are negative and output components are

[9] For vectors y, z, $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the correponding element of z; $y > z$ means that $y \geq z$ but $y \neq z$.

positive, the pro t maximization problem for rm f is: For given $p \in \mathbb{R}^C_{++}$, $p^g \in \mathbb{R}_{++}$,

$$\begin{array}{ll} \underset{(y_f, y^g) \in \mathbb{R}^{C+1}}{\text{Max}} & p y_f + p^g y^g_f \\ & \text{s.t. } t_f\left(y_f, y^g_f\right) \geq 0 \quad (\alpha_f) \end{array}, \tag{1}$$

where the term in parenthesis next to the constraint is the associated Kuhn-Tucker multiplier.[10]

Remark 1 *From Assumption 2, it follows that if the above problem (1) has a solution, it is a unique one characterized by the Kuhn-Tucker (in fact, Lagrange) conditions.*

Let s_h be the share of the rm pro ts owned by household h, and de ne

$$s \equiv (s_h)_{h=1}^H \in \mathcal{S}^{H-1} \equiv \left\{ s' \equiv (s'_h)_{h=1}^H \in \mathbb{R}^H_+ : \sum_{h=1}^H s'_h = 1 \right\},$$

the $(H-1)$ dimensional simplex.

Household s maximization problem is then the following: For given $p \in \mathbb{R}^C_{++}$, $p^g \in \mathbb{R}_{++}$, $s_h \in [0,1]$, $e_h \in \mathbb{R}^C_{++}$, $G_{\backslash h} \in \mathbb{R}_+$, $(\widehat{y}) \in \mathbb{R}^{C+1}$,

$$\begin{array}{lll} \underset{(x_h, g_h) \in \mathbb{R}^{C+1}_{++}}{\text{Max}} & u_h\left(x_h, g_h + G_{\backslash h}\right) & \\ \text{s.t.} & -p\left(x_h - e_h\right) - p^g g_h + \widehat{p} \sum_{f=1}^F \widehat{y}_f \geq 0 & (\lambda_h) \\ & g_h \geq 0 & (\mu_h) \end{array} \tag{2}$$

Remark 2 *From Assumption 1, it follows that the above problem (2) has a unique solution characterized by Kuhn-Tucker conditions.*

Remark 3 *By de nition of u_h, observe that we must have $\sum_h g_h > 0$ and, therefore,*

1. *since $g_h \geq 0$ for all h, there exists h' such that $g_{h'} > 0$; and*
2. $\sum_{f=1}^F y^g_f > 0$.

[10] We will follow this convention of writing the Kuhn-Tucker multipliers next to the associated constraints throughout the paper.

Market clearing conditions are:

$$\begin{array}{ll} -\sum_{h=1}^{H} x_h + \sum_{h=1}^{H} e_h + \sum_{f=1}^{F} y_f & = 0 \\ -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g & = 0. \end{array}$$

The set of all utility functions of household h that satisfy Assumption 1 is denoted by $\mathcal{U}_h$; the set of all transformation functions of rm f that satisfy Assumption 2 is denoted by $\mathcal{T}_f$. Moreover de ne $\mathcal{U}' \equiv \times_{h=1}^{H} \mathcal{U}_h$ and $\mathcal{T} \equiv \times_{f=1}^{F} \mathcal{T}_f$.

Assumption 3 $\mathcal{U}'$ and $\mathcal{T}$ are endowed with the subspace topology of the C^2 uniform convergence topology on compact sets[11].

De nition 1 *An economy is a vector* $\pi \equiv (e, s, u, t) \in \Pi'$, *where* $\Pi' \equiv \mathbb{R}_{++}^{CH} \times \mathcal{S}^{H-1} \times \mathcal{U}' \times \mathcal{T}$.

Summing up consumers budget constraints, and observing that $\sum_{h=1}^{H} s_h = 1$, we get

$$-p\left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H} e_h + y\right) - p^g\left(\sum_{h=1}^{H} g_h + y^g\right) = 0,$$

i.e., the Walras law. Therefore, the market clearing condition for good C, for example, is redundant. Moreover, we can normalize the price of private good C without affecting the budget constraints of any household. With little abuse of notation, we denote the normalized private and public good prices with $p \equiv \left(p^{\setminus}, 1\right)$ and p^g, respectively.

Using Remarks 1-3, we can give the following de nition:

De nition 2 *A vector* $(x, g, p^{\setminus}, p^g, \widehat{y})$ *is an equilibrium for an economy* $\pi \in \Pi'$ *iff:*

1. *the rm maximizes, i.e.,* $(\widehat{y})$ *solves problem (1) at* $p \in \mathbb{R}_{++}^{C}$, $p^g \in \mathbb{R}_{++}$;
2. *households maximize, i.e., for* $h = 1, ..., H$, (x_h, g_h) *solves problem (2) at* $p^{\setminus} \in \mathbb{R}_{++}^{C-1}$, $p^g \in \mathbb{R}_{++}$, $e_h \in \mathbb{R}_{++}^{C}$, $G_{\setminus h} \in \mathbb{R}_{+}$, $s_h \in (0, 1)$, $(\widehat{y}) \in \mathbb{R}^{C+1}$; *and*

[11] A sequence of functions f^n with domain an open set A of $\mathbb{R}^m$ converges to f if and only if f^n, $Df^n$and $D^2 f^n$ uniformly converge to f, Df and $D^2 f$, respectively, on any compact subset of A.

3. markets clear , i.e., $(x, g, \widehat{y})$ *solves*

$$\begin{array}{ll} -\sum_{h=1}^{H} x_h + \sum_{h=1}^{H} e_h + \sum_{f=1}^{F} y_f & = 0 \\ -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g & = 0. \end{array} \tag{3}$$

The system of First Order Conditions characterizing the solutions to problems (1) and (2), and conditions (3) is displayed below - see Remarks 1 and 2.

$$\begin{array}{ll} \widehat{p} + \alpha_f D t_f (\widehat{y}_f) & = 0 \\ t_f (\widehat{y}_f) & = 0 \\ \ldots & \\ D_{x_h} u_h \left(x_h, g_h + G_{\backslash h}\right) - \lambda_h p & = 0 \\ D_{g_h} u_h u_h \left(x_h, g_h + G_{\backslash h}\right) - \lambda_h p^g + \mu_h & = 0 \\ \min \{g_h, \mu_h\} & = 0 \\ -p (x_h - e_h) - p^g g_h + \sum_{f=1}^{F} s_{fh} \widehat{p y_f} & = 0 \\ \ldots & \\ -\sum_{h=1}^{H} x_h^{\backslash} + \sum_{h=1}^{H} e_h^{\backslash} + \sum_{f=1}^{F} y_f^{\backslash} & = 0 \\ -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g & = 0 \end{array} \tag{4}$$

Observe that the number of equations is equal to the number of endogenous variables. De ne

$$\xi \equiv \left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g\right) \in \Xi$$

where $\Xi \equiv \mathbb{R}^{(C+1)F} \times \left(-\mathbb{R}^F_{++}\right) \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^H \times \mathbb{R}^H \times \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++} \times \mathbb{R}_{++}$, and

$$F : \Xi \times \Pi \to \mathbb{R}^{\dim \Xi}, \quad F : (\xi, \pi) \mapsto \text{Left Hand Side of (4).}$$

Observe that $\left(\widehat{y}, x, g, p^{\backslash}, p^g\right)$ is an equilibrium associated with an economy π iff there exists (α, μ, λ) such that $\widehat{\widehat{F}} \left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g, \pi\right) = 0$. With innocuous abuse of terminology, we will call $\xi \equiv \left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g\right)$ an equilibrium.

Using a homotopy argument applied to system (4), Villanacci and Zenginobuz (2001) prove the existence of equilibria.

Theorem 3 *For every economy* $(e, s, u, f) \in \Pi'$*, an equilibrium exists.*

After restricting the set of utility functions to a large and reasonable subset $\mathcal{U}$ of $\mathcal{U}'$ - see Assumption 4 below - Villanacci and Zenginobuz (2001) also show that there exists an open and dense subset Π^* of $\Pi \equiv \mathbb{R}^{CH}_{++} \times \mathcal{S}^{H-1} \times \mathcal{U} \times \mathcal{T}$ such that

$$\forall\, \pi^* \in \Pi^*, \;\; \forall \xi^* \in F^{-1}_{\pi^*}(0)\,, \;\; rank\; D_\xi F_{\pi^*}(\xi^*) \;\text{ is full.}$$

The assumption on $\mathcal{U}'$ needed for this regularity result is the following:

Assumption 4 For all h, $x_h \in \mathbb{R}^C_{++}$ and $G \in \mathbb{R}_{++}$, it is the case that

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, G) & \left[D_{x_h} u_h(x_h, G)\right]^T \\ D_{G x_h} u_h(x_h, G) & D_G u_h(x_h, G) \end{bmatrix} \neq 0. \qquad (5)$$

The economic meaning of Assumption 4 is given in the following Lemma.

Lemma 4 *1. At the solution of the consumer problem, if* $\mu_h = 0$, *Assumption 4 is equivalent to*

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, G) & -p^T \\ D_{G x_h} u_h(x_h, G) & -p^g \end{bmatrix} \neq 0.$$

2. If $w \in \mathbb{R}_{++}$ *is the consumer s wealth, Assumption 3 is equivalent to* $D_w g_h(p, p^g, w) \neq 0$, *where* $g_h(p, p^g, w)$ *is the demand function of household s* h *for the public good when* $g_h > 0$ *and therefore* $\mu_h = 0$.

Lemma 4 follows directly from the First Order Conditions of the household s problem and from an application of the Implicit Function Theorem to those conditions.

Call $\mathcal{U}$ the subset of $\mathcal{U}'$ whose elements satisfy Assumption 4. Observe that $\mathcal{U}$ is an open subset of $\mathcal{U}'$. The main regularity results proven in Villanacci and Zenginobuz (2001) are now given.

Proposition 5 *1.* $\forall\; (s, u, f)\,,\; \exists$ *a full measure subset* $E^{**}_{(s,u,f)}$ *of* $\mathbb{R}^{CH}_{++}$, *such that* $\forall e \in E^{**}_{(s,u,f)}, \forall \xi^{**} \in F^{-1}_{(e,s,u,f)}(0)\,,$

$$\textit{for } h = 1, ..., H, \textit{ either } g_h > 0 \textit{ or } \mu_h > 0.$$

2. There exists an open and dense subset Π^{**} *of* Π *such that* $\forall \pi^{**} \in \Pi^{**}, \forall \xi^* \in F^{-1}_{\pi^{**}}(0)\,,$

$$\textit{for } h = 1, ..., H, \textit{ either } g_h > 0 \textit{ or } \mu_h > 0.$$

Proposition 6 *1.* $\forall\ (s,u,f)$, $\exists$ *a full measure subset* $E^*_{(s,u,f)}$ *of* $\mathbb{R}^{CH}_{++}$, *such that* $\forall e \in E^*_{(s,u,f)}, \forall \xi^* \in F^{-1}_{(e,s,u,f)}(0)$,

$$rank\ D_\xi F_{\pi^*}(\xi^*) = \dim \Xi.$$

2. There exists and open and dense subset Π^* *contained in* Π^{**} *such that for all* $\pi^* \in \Pi^*$, *and for all* $\xi^* \in F^{-1}_{\pi^*}(0)$, *we have that*

$$rank\ D_\xi F_{\pi^*}(\xi^*) = \dim \Xi.$$

The following result is needed in the next Sections.

Lemma 7 *In a an open and dense subset* S^{**} *of the endowment space, in equilibrium, rms are active.*

Proof. It is a simple consequence of Proposition 6 and of the Parametric Transversality Theorem. ■

3 Redistribution of Wealth and Quantity of Public Good

In this Section, we show the following results.

1. For all economies, local redistributions [12] of endowments of a private good *among contributors* do not change the set of equilibria.

2. For a generic subset of the economies for which there exists at least one non-contributor, there exists a redistribution of endowments of a private good *between contributors and non-contributors* which increases the level of provided public good;

3. For a generic subset of economies for which there exist at least two non contributors, there exists a redistribution of endowments of a private good *among non-contributors* which increases the level of provided public good.

The proof of statement 1. above is due to BBV. To prove statements 2, 3 as well as the main Theorems in the next Sections, we use a general methodology described, for example, in Citanna, Kajii and Villanacci (1998). We summarize this methodology, which is presented in the Appendix (Differential Analysis on the Equilibrium Manifold), below:

[12] By local redistribution we mean redistribution in a small enough neighborhood of a given endowment.

a. We de ne an equilibrium function F_1 taking into account the effects of the planner s intervention on agents behaviors via some policy tools ρ.

b. We introduce a function F_2 describing the constraints on planner s behavior and we consider a function $\widetilde{F} \equiv (F_1, F_2)$ whose zeros are equilibria with planner s intervention.

c. We observe that there are values of ρ at which equilibria with and without planner s intervention coincide.

d. We de ne a goal function G.

e. We study the local effect of a change in (a subvector of) ρ on G when endogenous variables move along the equilibrium manifold.

3.1 Redistributions among Contributors

The following Theorem is due to BBV.

Theorem 8 *Consider an equilibrium associated with an arbitrary economy and a redistribution of the private numeraire good among contributing households such that no household loses more wealth than her original contribution. All the equilibria after the redistribution are such that the consumption of private goods and the total amount of consumed public good are the same as before the redistribution.*

As a simple Corollary to Theorem 8, we get the following:

Proposition 9 *The set of equilibria after a local redistribution from an arbitrary set of non-contributors to one contributor is equal to the set of equilibria after a local redistribution from that same arbitrary set of non-contributors to an arbitrary set of contributors.*

The above result also explains why in all of the different types of planner interventions considered, we tax only one contributor.[13]

Remark 4 *Theorem 8 applies to all equilibrium models in which the constraint set for each household s problem is convex and utility functions are strictly quasi concave and continuous. Therefore, it holds true also for the*

[13] As we will see, for example in Lemma 12, showing non-neutrality amounts to showing that a well chosen Jacobian matrix has full rank. In fact, taxing more than one contributor would imply that matrix has two linearly dependent columns.

cases of exchange economies with a public good and incomplete markets or restricted participation[14]*, and a firm producing the public good, whatever objective functions it may have.*

The case of more than one public good is analyzed in BBV and it could easily be analyzed in our framework as well. All of our non-neutrality results will hold a fortiori in their case as well.

3.2 Redistributions between Contributors and Non-Contributors

We consider the case in which the planner redistributes endowments of one private good between two households, say $h = 1, 2$.

Before proceeding in the analysis, we first show that the set of economies for which there exists at least one "strict" non-contributor[15] is open (non-emptiness follows from a Cobb-Douglas utility function exercise).

Define as Π^{op} as the set economies with at least one strict non-contributors, i.e.,

$$\begin{array}{rl} \Pi^{op} \equiv \{ & \pi \in \Pi : \text{there exists } \xi \in F_{\pi}^{-1}(0) \text{ for which} \\ & g_h = 0 \text{ and } \mu_h > 0 \text{ for } h \in \mathcal{H}', \text{ and} \\ & g_h > 0 \text{ and } \mu_h = 0 \text{ for } h \in \mathcal{H} \backslash \mathcal{H}', \\ & \text{with } \mathcal{H}' \subseteq \mathcal{H} \text{ and } \# \mathcal{H}' \geq 1 \quad \} \end{array} .$$

Then consider

$$\begin{array}{rl} B \equiv \{ & (\xi, \pi) \in F^{-1}(0) : \\ & g_h \geq 0 \text{ and } \mu_h = 0 \text{ for } h \in \mathcal{H}', \text{ and} \\ & g_h = 0 \text{ and } \mu_h \geq 0 \text{ for } h \in \mathcal{H} \backslash \mathcal{H}' \\ & \text{with } \mathcal{H}' \subseteq \mathcal{H} \text{ and } \# \mathcal{H}' \geq 1 \quad \} \end{array} .$$

Define

$$\Phi : F^{-1}(0) \to \Pi, \qquad \Phi : (\xi, \pi) \mapsto \pi.$$

The set $B \subseteq F^{-1}(0)$ is closed, and $\Pi^{op} = \Pi \backslash \Phi(B)$. Since Φ is proper, Π^{op} is open[16].

[14] For the definition of a general equilibrium model with restricted participation on financial markets, see Balasko, Cass and Siconolfi (1990).

[15] Household h is a "strict" non-contributor if $g_h = 0$ and $\mu_h > 0$.

[16] Analogous results holds true for the set of economies with at least two strict non-contributors, which is studied in next section.

The equilibrium system taking into consideration the planner intervention is the following:

$$
\begin{array}{llll}
C+1 & \widehat{y}_f & \widehat{p}+\alpha_f Dt_f\left(\widehat{y}_f\right) & =0 \\
1 & \alpha_f & t_f\left(\widehat{y}_f\right) & =0 \\
 & & \ldots & \\
C & x_h & D_{x_h}u_h\left(x_h, g_h+G_{\backslash h}\right)-\lambda_h p & =0 \\
1 & g_h & D_{g_h}u_h u_h\left(x_h, g_h+G_{\backslash h}\right)-\lambda_h p^g+\mu_h & =0 \\
1 & \mu_h & \min\left\{g_h, \mu_h\right\} & =0 \\
1 & \lambda_h & -p\left(x_h-e_h+\rho_h\right)-p^g g_h+\sum_{f=1}^F s_{fh}\widehat{p}\widehat{y}_f & =0 \\
 & & \ldots & \\
C-1 & p^{\backslash} & -\sum_{h=1}^H x_h^{\backslash}+\sum_{h=1}^H e_h^{\backslash}+\sum_{f=1}^F y_f^{\backslash} & =0 \\
1 & p^g & -\sum_{h=1}^H g_h+\sum_{f=1}^F y_f^g & =0
\end{array}, \qquad (6)
$$

where

$$\rho_h \equiv 0 \text{ iff } h \neq 1,2.$$

In the above system, the rst column indicates the number equations in the corresponding rows and the second column indicates the vectors of endogenous variables with the same number of components as the corresponding equations. Consider $(\rho_1, \rho_h) \equiv \rho \in \mathbb{R}^2$ and

$$F_1 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}^{\dim \Xi}, \qquad F_1 : (\xi, \rho, \pi) \mapsto (\text{Left Hand Side of 6}),$$

$$F_2 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, \qquad F_2 : (\xi, \rho, \pi) \mapsto \rho_1 + \rho_{h^0},$$

$$\widetilde{F} : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}^{\dim \Xi + 1}, \qquad \widetilde{F} : (\xi, \rho, \pi) \mapsto \left(F_1(\xi, \rho, \pi), F_2(\xi, \rho, \pi)\right)$$

$$G_a : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, \qquad G_a : (\xi, \rho, \pi) \mapsto \sum_{h=1}^H g_h.$$

Since we are going to de ne different goal functions "G", we are going to distinguish among them using a subscript $i = a, b, c$. In sections 7 and 8, we are going to introduce the functions G_b and G_c. For ease of notation and with little abuse of notation, we do not add the subscript i to the functions F_1, F_2 and $\widetilde{F}$.

Observe that ξ is an equilibrium iff $\widetilde{F}(\xi, \rho = 0, \pi) = 0$.

We now show an important preliminary result.

Lemma 10 *There exists an open and dense subset* $\Pi_a \subseteq \Pi$ *such that for every* $\pi' \in \Pi_a$ *and for every* ξ' *such that* $\widetilde{F}(\xi', \rho = 0, \pi') = 0$,

$$D_{(\xi,\rho_2)}\widetilde{F}(\xi', 0, \pi') \text{ has full row rank (equal to } \dim \Xi + 1).$$

Proof. $D_{(\xi,\rho_2)}\widetilde{F}(\xi', 0, \pi', u)$ is

$$\begin{bmatrix} D_\xi F_1 & * \\ 0 & 1 \end{bmatrix}$$

and the desired result follows from the fact that in equilibrium $D_\xi F_1 = D_\xi F$and from Proposition 6. ■

>From the above result and the Implicit Function Theorem, it follows that there exist an open set $V \subseteq \mathbb{R}$ containing $\rho_1 = 0$ and a unique smooth function $h : V \to \mathbb{R}^{\dim \Xi + 1}$ such that h is C^1, $h(0) = (\xi', \rho_2 = 0)$, and

$$\text{for every } \tau_1 \in V, \quad \widetilde{F}(h(\rho_1), \rho_1, \pi') = 0\,, \text{ i.e., } \quad (h(\rho_1), \rho_1) \in \widetilde{F}_{\pi'}^{-1}(0)\,.$$

In words, the function h describes the effects of local changes of ρ_1 around 0 on the *equilibrium values* of ξ and ρ_2.

For every economy π, and every $\xi' \in F_\pi^{-1}(0)$, we can then

$$\widehat{g}_a : V \to \mathbb{R}, \quad \widehat{g}_a : \rho_1 \mapsto G(h(\rho_1), \rho_1, \pi) \tag{7}$$

such that $h(0) = (\xi', \rho_2 = 0)$.[17]

We are now ready to state the main result of this section.

Theorem 11 *For an open and dense subset* S_a^* *of the set of the economies for which there exists at least one non-contributor, at any equilibrium* ξ', *the function* $\widehat{g}_a$ *is locally onto around* 0. *That is, there exists a redistribution of the endowments of private good* C *between one contributor and one non-contributor which increases (or decreases) the level of provided public good.*

Remark 5 *Observe that we have imposed no restrictions on the number of private goods. Therefore, our analysis applies also to the case of only one private good, as analyzed by Bergstrom, Blume and Varian (1986) and (1992).*

[17] Observe that a heavier and more rigorous notation would have required that we write $\widehat{g}_{a|\pi}$.

Remark 6 *Because of Lemma 4, the local changes in the endowments mentioned in the statement of Theorem 11 can be made small enough to leave unchanged the set of households for which* $g_h > 0$*, i.e., the set of contributors.*

As explained for example in the Appendix, to prove Theorem 11 it suffices to show the following:

Lemma 12 *There exists an open and dense subset* $S_a^* \subseteq \Pi$ *such that for every* $\pi' \in S_a^*$ *and for every* $\xi' \in \widetilde{F}_{\pi'}^{-1}(0)$,

$$D_{(\xi,\rho)}\widetilde{F}(\xi', 0, \pi') \text{ has full rank.}$$

Lemma 12 is equivalent to the following one:

Lemma 13 *In an open and dense set of* Π*, the following system computed at* $\widetilde{F}(\xi, 0, \pi) = 0$ *has no solutions in the unknowns* $\widehat{d}$

$$\begin{array}{ll} \widehat{d}^T \begin{bmatrix} D\tilde{F} \\ DG \end{bmatrix} & = 0 \\ \widehat{d}^T \widehat{d} - 1 & = 0 \end{array} \tag{8}$$

Proof.

Openness: Define the projection

$$\phi : \widetilde{F}^{-1}(0) \to \Pi, \qquad \phi : (\xi, \pi) \mapsto \pi$$

and

$$M \equiv \{ (\xi, \rho, \pi) \in \Xi \times \mathbb{R}^2 \times \Pi : \widetilde{F}(\xi, \rho, \pi) = 0 \ , \ \rho = 0 \text{ and } rank \left[D_{\xi,\rho}\left(\widetilde{F}, G\right)\right] < n+2 \}.$$

Observe that $\phi(M) = \mathcal{E}^*$. Consider all the square submatrices of $\left[D_{\xi,\rho}\left(\widetilde{F}, G\right)\right]$ of dimension smaller than $(n+2)$. The rank condition in the definition of M requires that the determinants of all those submatrices are zero. Since the function determinant is a continuous function, M is a closed subset of the closed set $F^{-1}(0)$.

If the function ϕ is proper, then $\phi(M) = \mathcal{E}^*$ is closed, as desired. The proof of properness of ϕ is (almost) identical to the proof of Lemma 8 by Villanacci and Zenginobuz (2001).

Density: The proof of density goes through some steps. We rst compute $\left(D\widetilde{F}, DG\right)$. We perform some elementary row and column operations. We write system (8) at ξ such that $\widetilde{F}(\xi, 0, \pi) = 0$. Finally, distinguishing two cases, we prove the desired result.

For simplicity of notation, we take $h = 1$ as the household who is a contributor and whose corresponding columns are used to clean up columns of the matrix under analysis; $h = 2$ as a non-contributor who is taxed; $h = 3$ as a contributor who is not taxed; and $h = 4$ as a non-contributor who is not taxed.

We display below the matrix $\left(D\widetilde{F}, DG\right)$ after we performed the elementary row and column operations mentioned above:

$C{+}1$ d_{11}	α_1 D^2t_1	Dt_2^T												I 0 0	0 0 1
1 d_{12}	Dt_1														
$C{+}1$ d_{F1}			α_F D^2t_F	Dt_F^T										I 0 0	0 0 1
1 d_{F2}			Dt_F												
C c_{11}					D^1_{xx}	$-p^T$								$-\lambda_1$ $\widehat{I}$	
1 c_{12}					D^1_{Gx}	$-p^g$									$-\lambda_1$
1 c_{13}					$-p$					p^g				$-\widetilde{z}_1^{\backslash}$	$-\widetilde{g}_1$
C c_{21}							D^2_{xx}	$-p^T$						$-\lambda_2$ $\widehat{I}$	
1 c_{23}							$-p$							$-\widetilde{z}_2^{\backslash}$	$-\widetilde{g}_2$
C c_{31}									D^3_{xx}		$-p^T$			$-\lambda_3$ $\widehat{I}$	
1 c_{32}									D^3_{Gx}		$-p^g$				$-\lambda_3$
1 c_{33}									$-p$	$-p^g$				$-\tilde{z}_3^{\backslash}$	$-\tilde{g}_3$
1 c_{41}												D^4_{xx}	$-p^T$	$-\lambda_4$ $\widehat{I}$	
1 c_{43}												$-p^T$		$-\tilde{z}_4^{\backslash}$	$-\tilde{g}4$
1 c_{m1}	$I00$		$I00$		$-I0$		$-I0$		$-I0$			$-I0$			
1 c_{m2}	$\mathbf{0}01$		$\mathbf{0}01$												
1 c_{m3}															

where

$$\widehat{I} \equiv \begin{bmatrix} I_{C-1} \\ 1_{2\times(C-1)} \end{bmatrix}, \quad \widehat{\widehat{I}} \equiv \begin{bmatrix} I_{C-1} \\ 0_{2\times(C-1)} \end{bmatrix}, \quad \widetilde{I} \equiv \begin{bmatrix} 0_{(C+1)\times 1} \\ 0_{1\times 1} \end{bmatrix},$$

and for each h

$$\widetilde{z}_h \equiv x_h - e_h - \sum_f s_{fh} y_f \qquad \text{and} \qquad \widetilde{g}_h \equiv g_h + \sum_f s_{fh} y_f^g.$$

Next to each super row in the above matrix, we wrote the subvector of $\widehat{d} \equiv (b, d)$ that multiplies the elements in that super row[18]; and the number above each subvector is the number of its components.[19]

System (8), modi ed consistently with the above procedure, is displayed below.

$$\begin{array}{lll}
(f.1.1) & b_{11}\alpha_1 D^2 t_1 + b_{12} D t_1 + d_{m1}\left[I00\right] + d_{m2}\left[001\right] & = 0 \\
(f.1.2) & b_{11}\left[D t_1\right]^T & = 0 \\
(f.F.1) & b_{F1}\alpha_F D^2 t_F + b_{F2} D t_F + d_{m1}\left[I00\right] + d_{m2}\left[001\right] & = 0 \\
\left(\dot{f}F.2\right) & b_{F1}\left[D t_F\right]^T & = 0 \\
(1.1) & d_{11} D^1_{xx} + d_{12} D^1_{Gx} - d_{13} p - d_{m1}\left[I0\right] & = 0 \\
(1.3) & -d_{11} p^T - d_{12} p^g & = 0 \\
(2.1) & d_{21} D^2_{xx} - d_{23} p^T - d_{m1}\left[I0\right] & = 0 \\
(2.3) & -d_{21} p^T & = 0 \\
(3.1) & d_{31} D^3_{xx} + d_{32} D^3_{Gx} - d_{33} p - d_{m1}\left[I0\right] & = 0 \\
(3.2) & d_{13} p^g + d_{33} p^g & = 0 \\
(3.3) & -d_{31} p^T - d_{32} p^g & = 0 \\
(4.1) & d_{41} D^4_{xx} - d_{43} p^T - d_{m1}\left[I0\right] & = 0 \\
(4.1) & -d_{41} p^T & = 0 \\
(M.1) & \begin{array}{l} \sum_f b_{f1}\widehat{\widehat{I}} - d_{11}\lambda_1\widehat{I} - d_{13}\widetilde{z}_1^{\backslash} - d_{21}\lambda_2\widehat{I} + \\ -d_{23}\widetilde{z}_2^{\backslash} - d_{31}\lambda_3\widehat{I} - d_{33}\widetilde{z}_3^{\backslash}\widehat{I} - d_{41}\lambda_4\widehat{I} - d_{43}\widetilde{z}_4^{\backslash} \end{array} & = 0 \\
(M.2) & \begin{array}{l} \sum_f b_{f1}\widetilde{I} - d_{12}\lambda_1 - d_{13}\widetilde{g}_1 - d_{23}\widetilde{g}_2 + \\ -d_{32}\lambda_3 - d_{33}\widetilde{g}_3 - d_{43}\widetilde{g}_4 \end{array} & = 0 \\
(M.3) & -d_{13} + d_{m3} & = 0 \\
(M.4) & -d_{23} + d_{m3} & = 0 \\
(M.5) & d^T d - 1 & = 0
\end{array} \quad . \qquad (9)$$

---

[18] We used the letter b for rm related entries of the matrix.

[19] Note that blank entries stand for zero entries in the this and all other matrices.

The perturbation of the transformation function we are going to use has the following form

$$\overline{t_f}\left(y_f, y_f^G\right) = t_f\left(y_f, y_f^G\right) + \left(\left(y_f, y_f^G\right) - \left(\left(y_f^*, y_f^{G*}\right)\right)\right)^T \cdot A_f \cdot \left(\left(y_f, y_f^G\right) - \left(\left(y_f^*, y_f^{G*}\right)\right)\right),$$

where $\left(y_f^*, y_f^{G*}\right)$ are equilibrium values, and A_f is a symmetric negative de - nite matrix. Observe that the above proposed perturbation of the transformation function can be done because, generically, in equilibrium $y_f^G \neq 0$ - see Lemma 7. Moreover, the derivative of $d_{f1} \cdot A_f$ with respect to the elements of A_f has full row rank iff $d_{f1} \neq 0$.

The perturbation of the utility function we are going to use has the following form:

$$\overline{u}_h\left(x_h, g_h\right) = u\left(x_h, g_h\right) + \left(\left(x_h, g_h\right) - \left(x_h^*, g_h^*\right)\right)^T \begin{bmatrix} A_{xx}^h & 0 \\ 0 & a_{gg}^h \end{bmatrix} \left(\left(x_h, g_h\right) - \left(x_h^*, g_h^*\right)\right),$$

where $\left(x_h^*, g_h^*\right)$ are equilibrium values, A_{xx}^h is a symmetric negative de nite matrix and a_{gg}^h is a strictly negative number. Using the utility function perturbation proposed above, we can perturb independently equations $(h.1)$ and $(h.2)$ as long as $d_{h1} \neq 0$ and $d_{h2} \neq 0$. In fact, the derivative of

$$\begin{bmatrix} d_{h1} & d_{h2} \end{bmatrix} \begin{bmatrix} A_{xx}^h & 0 \\ 0 & a_{gg}^h \end{bmatrix}$$

with respect to the elements of A_{xx}^h and a_{gg}^h is the following:

$$\begin{bmatrix} a^{11} & a^{21} & & a^{C1} & \dots & a^{22} & & a^{C2} & & a^{CC} & a^{gg} & \\ d_{h1}^1 & d_{h1}^2 & & d_{h1}^C & \cdots & & & & & & & \\ & d_{h1}^1 & & & \cdots & d_{h1}^2 & & d_{h1}^C & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & d_{h1}^1 & \cdots & & & d_{h1}^2 & & d_{h1}^C & & \\ & & & & \cdots & & & & & & d_{h2} & \end{bmatrix}. \qquad (10)$$

We denote the utility function perturbation via the Hessian perturbation relative to private goods and public good for a generic household h by $\Delta\left(d_{h1}\right)$ and $\Delta\left(d_{h2}\right)$. In fact, in this Section, we are going to use only $\Delta\left(d_{h1}\right)$.[20] We denote the transformation function perturbation by $\Delta\left(d_{f1}\right)$.

[20] For a detailed account of the use of this methodology of proof, see Citanna, Kajii and Villanacci (1998) and also Villanacci et al. (forthcoming).

For each h and f it is crucial whether d_{h1} and/or b_{f1} are or are not equal to *zero*.[21] In principle, we should therefore distinguish among all the possible $2^{4+2} = 64$ cases. Therefore, we preliminarily study the effect of each vector d_{h1} and b_{f1} being *zero* on the other vector. Such an analysis reduces dramatically the number of cases to be studied. The remaining cases are then discussed in detail. More precisely, we go through proving the following preliminary facts:

Step 1: If $d_{h'1} = 0$ for some h', then $d_{h1} = 0$ for each h.

Step 2: $b_{f1} = 0 \Rightarrow \widehat{d} = 0$. Therefore, there are only two cases to be considered in this step:

Case 1: $d_{h1} \neq 0$ for each h and $b_{f1} \neq 0$ for each f.

Case 2: $d_{h1} = 0$ for each h and $b_{f1} \neq 0$ for each f.

We now analyze each of these situations separately.

Step 1: We just show that $d_{11} = 0 \Rightarrow \widehat{d} = 0$.The other cases of the form $d_{h1} = 0 \Rightarrow \widehat{d} = 0$,for all h, are proved in a similar way.

$>$From (1.3) , $d_{12} = 0$.

$>$From (1.1) , $d_{13} = 0$ and $d_{m1} = 0$.

For $h = 2, 4$, from $(h.1)$ and $(h.3)$, $d_{h1} = 0$ and $d_{h3} = 0$.

$>$From (3.2) , $d_{33} = 0$.

$>$From (3.1) and (3.3) , $d_{31} = 0$ and $d_{32} = 0$.

$>$From $(M.3)$, $d_{m3} = 0$.

Step 2: $b_{f1} = 0$ for some $f \Rightarrow \widehat{d} = 0$.

$>$From equation $(f.1.1)$, $b_{f2} = 0$, $d_{m1} = 0$ and $d_{m2} = 0$.

$>$From $(fF1)$ and $(fF2)$, $b_{F1} = 0$ and $b_{F2} = 0$.

$>$From (2.1) and (2.3) , $d_{21} = 0$ and $d_{23} = 0$.

$>$From $(M.4)$, $d_{m3} = 0$.

$>$From $(M.3)$, $d_{13} = 0$.

$>$From (1.1) and (1.3) , $d_{11} = 0$,and then following the same steps as in Case 1, we can show that $d = 0$.

Case 1:

[21] Observe that we do not use d_{h2} to perturb the utility function.

The required perturbations are shown below:

$(f1.1)$	$\Delta(b_{11})$
$(f1.2)$	b_{11}^{C}
$(fF.1)$	$\Delta(b_{F1})$
$(fF,2):$	b_{F1}^{C}
$(1.1):$	$\Delta(d_{11})$
$(1.3):$	d_{11}^{C}
$(2.1):$	$\Delta(d_{21})$
$(2.3):$	d_{21}^{C}
$(3.1):$	$\Delta(d_{31})$
$(3.2):$	d_{33}
$(3.3):$	d_{31}^{C}
(4.1)	$\Delta(d_{41})$
(4.3)	d_{41}^{C}
$(M.1):$	$d_{11}^{\backslash C,G}$
$(M.2):$	d_{f1}^{G}
$(M.3):$	d_{m3}
$(M.4):$	d_{23}
$(M.5):$	d_{m2}

. (11)

Case 2:
>From Step 1, we are left with the following system:

		C^{+}	1	C^{+}	1	1	1
(11)	C^{+}	$b_{11}\cdot$ $\alpha_1 D^2 t_1$	b_{12} Dt_1			$d_{m2}\cdot$ $[\mathbf{0}01]$	
(12)	1	$b_{11}\cdot$ $[Dt_1]^T$					
$(F1)$	C^{+}			$b_{F1}\cdot$ $\alpha_F D^2 t_F$	b_{F2} Dt_F	$d_{m2}\cdot$ $[\mathbf{0}01]$	
$(F2)$	1			$b_{F1}\cdot$ $[Dt_F]^T$			
$(M.1)$	C^{-}	I 0 $b_{11}0$		I 0 $b_{F_1}0$			
$(M.2)$	1	0 0 b_{11} 0		0 0 $b_{F_1}0$			
$(M.5)$							

. (12)

Observe that in the above system, the number of extra equations is $(C-1)+1 \geq 1$. To perturb the remaining equations proceed as follows:

$(f1.1)$	$\Delta(b_{11})$
$(f1.2)$	b_{11}^{C}
$(fF.1)$	$\Delta(b_{F1})$
$(fF,2):$	b_{F1}^{C}
$(M.1):$	$b_{11}^{\setminus C,G}$
$(M.2):$	b_{f1}^{G}
$(M.5):$	d_{m2}

.

■

3.3 Redistributions between Non-Contributors

The only difference in the analysis of this case with respect to the case of the previous Section is that in system (6) we have

$$\rho_h = 0 \quad for \quad h \neq 2,4.$$

De ne $\widehat{g}_{a'} : \rho_2 \mapsto G(h(\rho_2), \rho_2, \pi)$, with meaning of h similar to that one introduced in relationship with the de nition of $\widehat{g}_a$ in (7).

Theorem 14 *If $H > 2$ and $C > 1$, for an open and dense subset $S_{a'}^{*}$ of the set Π of the economies for which there exists at least two non-contributors, at any equilibrium ξ', the function $\widehat{g}_{a'}$ is locally onto around 0. That is, there exists a redistribution of the endowments of private good C between two non-contributors which increases (or decreases) the level of provided public good.*

Remark 7 *The Assumption $C \geq 2$ is used below in Case 2 of the proof of the Theorem. The intuition behind this requirement is the following one. Consider the case of one public and one private good. Redistributing the private good among non-contributors does not change the demand of the public good: contributors are not affected by this intervention and non contributors do not become contributors (because we are not on the border line cases and taxes are small). Therefore, not even the demand of private goods changes. The total effect is just some changes on the demand of private goods by some non contributors.*

Proof. The only difference between this system and system (6) above is the fact that d_{43} and not d_{13} appears in equation $(M.3)$. As in Subsection 3.2, we preliminary study the effect of each vector d_{h1} and b_{f1} being *zero* on the other vectors, and then we analyze the relevant cases. More precisely, we go through the following steps:

Step 1: $[d_{11} = 0 \text{ or } d_{31} = 0] \Rightarrow \widehat{d} = 0$;

Step 2: $d_{21} = 0 \Leftrightarrow d_{41} = 0$;

Step 3: $b_{f'1} = 0 \Rightarrow \left[(b_{f1})_{f=1}^{F} = 0 \text{ and } d_{21} = d_{41} = 0\right]$. Therefore, the following cases are possible:

Case 1: $\forall\, f,\, b_{f1} \neq 0$ and $\forall h,\, d_{h1} \neq 0$.

Case 2: $\forall\, f,\, b_{f1} \neq 0,\, d_{21} = d_{41} = 0$ and $d_{11} \neq 0,\, d_{31} \neq 0$.

Case 3: $\forall\, f,\, b_{f1} \neq 0$ and $\forall h,\, d_{h1} = 0$.

Case 4: $\forall\, f,\, b_{f1} = 0,\, d_{21} = d_{41} = 0$ and $d_{11} \neq 0,\, d_{31} \neq 0$.

The argument in Steps 1, 2, 3 and in Cases 1 and 3 of Step 3 is very similar to that the analogous ones in Subsection 3.2. Therefore we are left with analyzing Cases 2 and 4.

Case 2: $\forall\, f,\, b_{f1} \neq 0,\, d_{21} = d_{41} = 0$ and $d_{11} \neq 0,\, d_{31} \neq 0$.

Lost unknowns	#		Lost equations	#		# Eqns. we can erase
d_{21}	C		(2.1)	C		
d_{23}	1		(2.3)	1		
d_{41}	C		(4.1)	C		
d_{43}	1		(4.3)	1		
d_{m1}	$C-1$					$C-1$
d_{m3}	1		$(M.3)$	1		
			$(M.4)$	1		-1
						−−−−−−− $(C-1)-1$ in total

In the above table, by lost unknowns we mean unknowns that are equal to zero, and by lost equations we mean equations which, in the case under analysis, take the form of identity $0 = 0$.

Observe from the table above that we have more equations than unknowns iff $C \geq 2$. Since we assumed so, we can then proceed with the perturbation used in Case 1. Observe that we lost unknowns we were using to perturb equations we lost as well: for example, in Case 1 to perturb equation $(f.1.1)$ we use b_{11}. Here we lose both those equations and unknowns.

Case 4: $\forall f$, $b_{f1} = 0$, $d_{21} = d_{41} = 0$ and $d_{11} \neq 0$, $d_{31} \neq 0$.

Lost unknowns	#		Lost equations	#		# Eqns. we can erase
b_{11}	$C+1$		(11)	$C+1$		
b_{12}	1		(12)	1		
b_{F1}	$C+1$		$(F.1)$	$C+1$		
b_{F2}	1		$(F.2)$	1		
d_{11}	C		(1.1)	C		
d_{23}	1		(2.3)	1		
d_{41}	C		(4.1)	C		
d_{43}	1		(4.3)	1		
d_{m1}	$C-1$					$C-1$
d_{m3}	1		$(M.3)$	1		
d_{m2}			$(M.4)$	1		
						−−−−−−− $(C-1)$ in total

Since $C - 1 \geq 1$, we can erase equation $(M.5)$ and perturb the other equation as we did in Case 1. ■

4 Crowding-out Effects

We will show that a planner can increase G if her intervention is as described below:

1. She taxes all non contributors and one contributor by an amount ρ_h of the numeraire good;

2. She uses those taxes to nance the purchase of an amount θ^g of the public good. This purchase occurs on the market at market equilibrium prices.

Equilibrium with planner intervention:

1. For each h, the amount of consumed public good is $\sum_{h=1}^{H} g_h + \theta^g$;

2. The budget set has to take into account the tax ρ_h for $h = 2$;

3. The purchase of θ^g has to be nanced with the revenue from tax collection, i.e.,

$$\rho_2 - p^g \theta^g = 0.$$

4. The market clearing condition becomes

$$-\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g - \theta^g = 0.$$

The equilibrium system with planner intervention in this case is the following:

$C+1$	$\widehat{y}_f$	$\widehat{p} + \alpha_f Dt_f \left(\widehat{y}_f\right)$	$= 0$
1	α_f	$t_f \left(\widehat{y}_f\right)$	$= 0$
		$\ldots$	
C	x_h	$D_{x_h} u_h \left(x_h, g_h + G_{\setminus h}\right) - \lambda_h p$	$= 0$
1	g_h	$D_{g_h} u_h u_h \left(x_h, g_h + G_{\setminus h}\right) - \lambda_h p^g + \mu_h$	$= 0$
1	μ_h	$\min \{g_h, \mu_h\}$	$= 0$
1	λ_h	$-p\left(x_h - e_h + \rho_h\right) - p^g g_h + \sum_{f=1}^{F} s_{fh} \widehat{p} \widehat{y}_f$	$= 0$
		$\ldots$	
$C-1$	$p^{\setminus}$	$-\sum_{h=1}^{H} x_h^{\setminus} + \sum_{h=1}^{H} e_h^{\setminus} + \sum_{f=1}^{F} y_f^{\setminus}$	$= 0$
1	p^g	$-\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g - \theta^g$	$= 0$

where

$$\rho_h \neq 0 \text{ iff } h = 2$$

$$F_2\left(\xi, \rho, \pi\right) = \left(\rho_2 - p^g \theta^g\right)$$

$$G_b\left(\xi, \rho, \theta, \pi\right) = \sum_{h=1}^{H} g_h + \theta^g.$$

Note that # goals =1, # constraints = 1, tools: ρ_2, θ^g and thus # tools = 2. Therefore, condition (24) in the Appendix is satis ed simply because $2 \geq 2$.

Observe that ξ is an equilibrium iff $\widetilde{F}\left(\xi, \rho = 0, \theta = 0, \pi\right) = 0$.

Observe that Walras law holds in the above case. Summing up consumers budget constraints, and observing that $\sum_{h=1}^{H} s_h = 1$, we get

$$-p\left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H} e_h - \sum_{f=1}^{F} y_f\right) - \rho_2 - p^g \left(\sum_{h=1}^{H} g_h - \sum_{f=1}^{F} y_f^g\right) = 0.$$

Since $\rho_2 - p^g \theta^g = 0$, we also have

$$-p\left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H} e_h - \sum_{f=1}^{F} y_f\right) - p^g \left(\sum_{h=1}^{H} g_h - \sum_{f=1}^{F} y_f^g + \theta^g\right) = 0,$$

i.e., the Walras law.

We now show an important preliminary result.

Lemma 15 *There exists an open and dense subset* $\Pi' \subseteq \Pi$ *such that for every* $\pi' \in \Pi'$ *and for every* ξ' *such that* $\widetilde{F}(\xi', \rho = 0, \pi') = 0$,

$$D_{(\xi,\rho_2,\theta^G)}\widetilde{F}(\xi', 0, \pi') \text{ has full row rank (equal to } \dim \Xi + 1).$$

Proof. $D_{(\xi,\rho_2,\theta^g)}\widetilde{F}(\xi', 0, \pi', u)$ is

$$\begin{bmatrix} D_\xi F_1 & * & * \\ 0 & 1 & * \end{bmatrix},$$

and the desired result follows. ■

>From the above result and the Implicit Function Theorem it follows that there exist an open set $V \subseteq \mathbb{R}$ containing $\rho_1 = 0$ and a unique smooth function $h : V \to \mathbb{R}^{\dim \Xi + 1}$ such that h is C^1, $h(0) = (\xi', \rho_2 = 0)$, and

$$\text{for every } \tau_1 \in V, \quad \widetilde{F}(h(\theta^g), \rho_1, \pi', u) = 0, \text{ i.e., } (h(\theta^g), \theta^g) \in \widetilde{F}^{-1}_{(\pi',u)}(0) \tag{13}$$

In words, the function h describes the effects of local changes of θ^g around 0 on the *equilibrium values* of ξ and ρ_2.

For every economy π, and every $\xi' \in F_\pi^{-1}(0)$, we can then de ne

$$\widehat{g}_b : V \to \mathbb{R}, \quad \widehat{g}_b : \rho_1 \mapsto G_b(h(\theta^g), \theta^g, \pi)$$

such that $h(0) = (\xi', \rho_2 = 0)$.

We are now ready to state the main result of this section.

Theorem 16 *For an open and dense subset* S_b^* *of the set of the economies for which there exists at least one non contributor, at any equilibrium* ξ', *the function* $\widehat{g}_b$ *is locally onto around* 0. *That is, there exists a tax on the endowments of private good* C *of one non-contributor and a choice of public good production which increases the total amount of produced public good.*

Proof. The proof of the Theorem goes through the usual steps. In fact, up to elementary row and column operations, the matrix to be analyzed to prove denseness is the same as in the analogous proof in Subsection 3.2. ■

5 Pareto Improving Interventions

Our techniques of proof require the number of independent tools to be not smaller than the number of goals. That observations implies that a simple redistribution of the numeraire good among all households would not work. That redistribution has to satisfy the constraint $\sum_{h=1}^{H} \rho_h = 0$, and, therefore, the number of independent tools is only $H - 1$, while the number of goals - the utility levels of all households - is H.

On the other hand, if the planner can tax not only households but also rms, more tools become available. In fact, we show that a planner can Pareto improve upon the market outcome if her intervention is as described below:

The planner imposes taxes the use of inputs and the production of output in a proportion τ^c for each good $c = 1, ..., C, G$.

De ne $\tau = (\tau^c)_{c=1}^{C,G}$. Observe that the rm f s problem becomes:

$$\max_{\left(y_f, y_f^g\right)} \sum_{c=1}^{C,G} (1 - \tau^c) \, p^c y_f^c \quad s.t. \qquad t_f\left(y_f, y_f^g\right) = 0.$$

Therefore to describe equilibria with planner intervention we have to change the equilibrium system as follows:

1. The tax collection has to balance:

$$\sum_{f=1}^{F} \sum_{c=1}^{C,G} \tau^c p^c y_f^c = 0.$$

2. The First Order Conditions for Firm f becomes

$$\begin{array}{ll} (1-\tau) \square (p, p^g) + \alpha_f D t_f \left(y_f, y_f^g\right) & = 0 \\ t_f\left(y_f, y_f^g\right) & = 0 \end{array},$$

where, for given $x, y \in \mathbb{R}^n$, $x \square y \equiv (x_i y_i)_{i=1}^n$.

Remark 8 *Here we choose to tax each rm in the same proportion with respect to each good. Another possibility would be to impose taxes (and subsidies) which depend on the type of goods and identity of rms. That would allow substituting for the requirement $C \geq H$ of the main Theorem of the Section a much weaker requirement of the form $FC \geq H$.*

We could also let the government have her own demand θ^g for the public good: that would increase by one the number of tools at her disposal.

Finally, we could let the government impose taxes on households; it turns out that this does not have any signi cant effect.

Another tool which may be effective would be a subsidy/tax on the price of the public good for the households. Observe that taxing only one good does not create any kink in the budget set, but it only rotates the budget plane.

Remark 9 *If we use production functions ($y_f^g = t_f(y_f)$) instead of transformation functions, there are two different ways to show the above mentioned generic result:*

1. Impose a condition of the following form

$$\lim_{y_f^c \to 0} D_{y_f^c} t_f(y_f) = -\infty.$$

2. Write down the problem as

$$\begin{array}{llll} \max_{y_f} p^g y_f^g - p y_f & s.t. & y_f^g = t_f(y_f) & \gamma_f \\ & & y_f \geq 0 & \beta_f \end{array}$$

Then, it is enough to show that generically it cannot be the case that

$$y_f = 0 \quad and \quad \beta_f = 0.$$

In this case, we can partition rms in the group of strictly active and strictly inactive ones, and proceed taxing and subsidizing the active ones only.

The function F_1 de ning the equilibrium with planner intervention in this case is the left hand side of the following system:

$$\begin{array}{ll}
(1-\tau)\,\square\,(p, p^g) + \alpha_f D t_f\left(y_f, y_f^g\right) & = 0 \\
t_f\left(y_f, y_f^g\right) & = 0 \\
\ldots & \\
D_{x_h} u_h\left(x_h, g_h + G_{\backslash h}\right) - \lambda_h p & = 0 \\
D_{g_h} u_h u_h\left(x_h, g_h + G_{\backslash h}\right) - \lambda_h p^g + \mu_h & = 0 \\
\min\{g_h, \mu_h\} & = 0 \\
-p(x_h - e_h) - p^g g_h + \sum_f s_{hf} \sum_{c=1}^{C,G} (1-\tau^c) p^c y_f^c & = 0 \\
\ldots & \\
-\sum_{h=1}^H x_h^\backslash + \sum_{h=1}^H e_h^\backslash + \sum_f y_f^\backslash & = 0 \\
-\sum_{h=1}^H g_h + \sum_f y_f^g & = 0
\end{array}$$

$$\rho_h \neq 0 \text{ iff } h \in \mathcal{H}^0 \cup \{1\}.$$

$$F_2 : (\xi, \rho, \pi) \mapsto \sum_{f=1}^{F} \sum_{c=1}^{C,G} \tau^c p^c y^c + \sum_{h \in \mathcal{H}^0 \cup \{1\}} \rho_h,$$

$$G : (\xi, \rho, \pi) \mapsto (u_h (x_h))_{h=1}^{H}.$$

Note that # goals $=H$, # constraints $= 1$, tools: $\left(\tau_f^c\right)_{c=1}^{C,G}$ and thus # tools $= C + 1$. Therefore, to be consistent with condition (24), we must have

$$C + 1 \geq H + 1, \text{ or, } \quad C \geq H. \tag{14}$$

Remark 10 *In fact, as shown in the proof of the main Theorem of this section, see Case 18 and Remark 12, we have to impose $C \geq H^+$. Moreover, since H^+ is an endogenous variable which at most can be equal to H, we are going to require $C \geq H$.*

Denote a H^+-dimensional subvector of τ by $\tau^* \in \mathbb{R}^{H^+}$.

Observe that ξ is an equilibrium iff $\widetilde{F}(\xi, (\rho, \tau) = 0, \pi) = 0$.

Observe that the Walras law holds in the above case. Summing up consumers budget constraints, and observing that $\sum_{h=1}^{H} s_h = 1$, we get

$$-p \left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H} e_h - \sum_{f=1}^{F} y_f \right) - \left(\sum_{h=1}^{H} \sum_{f=1}^{F} s_{hf} \sum_{c=1}^{C,G} \tau^c p^c y_f^c \right) - p^g \left(\sum_{h=1}^{H} g_h - \sum_{f=1}^{F} y_f^g \right) = 0.$$

Since $\sum_{h=1}^{H} \sum_{f=1}^{F} s_{hf} \sum_{c=1}^{C,G} \tau^c p^c y_f^c = 0$, we also have

$$-p \left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H} e_h - \sum_{f=1}^{F} y_f \right) - p^g \left(\sum_{h=1}^{H} g_h - \sum_{f=1}^{F} y_f^g \right) = 0,$$

i.e., the Walras law.

We now show an important preliminary result.

Lemma 17 *There exists an open and dense subset* $\Pi' \subseteq \Pi$ *such that for every* $\pi' \in \Pi'$ *and for every* ξ' *such that* $\widetilde{F}(\xi', (\tau, \rho) = 0, \pi', u) = 0$,

$$D_{(\xi,\tau,\rho)}\widetilde{F}(\xi', 0, \pi') \text{ has full row rank (equal to } \dim \Xi + 1).$$

Proof. $D_{(\xi,\rho_1)}\widetilde{F}(\xi', 0, \pi', u)$ is

$$\begin{bmatrix} D_\xi F_1 & * \\ 0 & \sum_{f=1}^{F} y_f^g > 0 \end{bmatrix},$$

and the desired result follows. Observe that in the above matrix, $D_\xi F_2 = 0$ because it is computed at $\tau = 0$. ■

As usual, from the above result, for every economy π, and every $\xi' \in F_\pi^{-1}(0)$, we can then de ne

$$\widehat{g}_c : V \to \mathbb{R}, \quad \widehat{g}_c : (\tau^*) \mapsto G_c(h(\tau^*), \tau^*, \pi)$$

such that $h(0) = (\xi', \tau^* = 0)$.

Theorem 18 *Assume that* $C \geq H$. *For an open and dense subset* S_c^* *of the set of the economies, at any equilibrium* ξ', *the function* $\widehat{g}_c$ *is locally onto around* 0. *That is, there exists a choice of taxes on the rm s inputs and outputs which Pareto improves or impairs upon the equilibrium* ξ'. *Moreover, in the subset of the economies for which there are* H^+ *contributors, it is enough to assume that* $C \geq H^+$.

Proof. For simplicity of notation, we take $h = 1$ as the household whose associated columns are used to clean up the columns of the matrix under analysis; $h = 2$ as a non-contributor, and $h = 3$ as a contributor.

The proof of the Theorem goes through the usual steps. Properness follows in the same way as in the other cases. Denseness is proved in detail below. We have

$$
\begin{array}{lll}
(f.1.1) & b_{11}\alpha_1 D^2 t_1 + b_{12} D t_1 + d_{m1}\left[I00\right] + d_{m2}\left[001\right] & = 0 \\
(f,1.2) & b_{11}\left[D t_1\right]^T & = 0 \\
(f.F.1) & b_{F1}\alpha_F D^2 t_F + b_{F2} D t_F + d_{m1}\left[I00\right] + d_{m2}\left[001\right] & = 0 \\
(f.F.2) & b_{F1}\left[D t_F\right]^T & = 0 \\
(1.1) & d_{11} D^1_{xx} + d_{12} D^1_{Gx} - d_{13} p - d_{m1}\left[I0\right] + d_{u1} D^1_x & = 0 \\
(1.2) & \begin{array}{l} d_{11} D^1_{xG} + d_{12} D^1_{GG} - d_{13} p^g - d_{21} D^2_{xG} + \\ d_{31} D^3_{xG} + d_{32} D^3_{GG} + d_{u1} D^1_G + d_{u2} D^2_G + d_{u3} D^3_G \end{array} & = 0 \\
(1.3) & -d_{11} p^T - d_{12} p^g & = 0 \\
(2.1) & d_{21} D^2_{xx} - d_{23} p^T - d_{m1}\left[I0\right] + d_{u2} D^2_x & = 0 \\
(2.3) & -d_{21} p^T & = 0 \\
(3.1) & d_{31} D^3_{xx} + d_{32} D^3_{Gx} - d_{33} p - d_{m1}\left[I0\right] + d_{u3} D^3_x & = 0 \\
(3.2) & d_{13} p^g + d_{33} p^g & = 0 \\
(3.3) & -d_{31} p^T - d_{32} p^g & = 0 \\
(M.1) & \begin{array}{l} \sum_f \widehat{\widehat{I}} d_{f1} - d_{11}\lambda_1 \widehat{I} - d_{13}\widetilde{z}^{\backslash}_1 + \\ -d_{21}\lambda_2 \widehat{I} - d_{23}\widetilde{z}^{\backslash}_2 - d_{31}\lambda_3 \widehat{I} - d_{33}\widetilde{z}^{\backslash}_3 \widehat{I} \end{array} & = 0 \\
(M.2) & \begin{array}{l} \sum_f \widetilde{I} d_{f1} - d_{12}\lambda_1 - d_{13}\widetilde{g}_1 + \\ -d_{23}\widetilde{g}_2 - d_{32}\lambda_3 - d_{33}\widetilde{g}_3 \end{array} & = 0 \\
(M.3) & \begin{array}{l} \sum_f b_{f1} \cdot dg^*\left(\widehat{p}^*\right) - c_{33}\sum_f s_{3f}\left(\widehat{p}\Box\widehat{y}_f\right)^* + \\ +d_{m3}\sum_f\left(\widehat{p}\Box\widehat{y}\right)^* \end{array} & = 0 \\
(M.4) & d^T d - 1 & = 0
\end{array} ,
$$

where

$$
dg^*\left(\widehat{p}^*\right) \equiv \begin{bmatrix} \left[dg\left(\widehat{p}^*\right)\right]_{H^+\times H^+} \\ 0_{\left[(C+1)-H^+\right]\times H^+} \end{bmatrix},
$$

and

$$
\left(\widehat{p}\Box\widehat{y}\right)^* \equiv \left(p^*\Box y^*\right),
$$

(i.e. the variables with *) are H dimensional vectors of the same type as τ^*. Observe that $[b_{f'1} = 0$ for some $f']$ implies that ($b_{f'2} = 0$, $d_{m1} = 0$, $d_{m2} = 0$, and therefore) $[b_{f1} = 0$ for each $f]$.

Remark 11 *Moreover, observe that we cannot perturb equations* $(f1)$ *without using the perturbation of the transformation function:* b_{f1} *is going to be used to perturb equations* $(M.3)$. *For that reason, we have to consider the two cases* $b_{f1} \neq 0$ *and* $b_{f1} = 0$. *We rst consider the cases in which* $b_{f1} \neq 0$. *As pointed out in Remark 12, the cases under which* $b_{f1} = 0$ *reduces to only one case (Case 19 below).*

Assuming that $b_{f1} \neq 0$, we summarize the relevant cases in the following table:

Case			
1	$d_{11} \neq 0, c_{12} \neq 0$	$c_{21} \neq 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
2			$d_{31} \neq 0,\ d_{32} = 0$
3			$d_{31} = 0,\ d_{32} = 0$
4		$c_{21} = 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
5			$d_{31} \neq 0,\ d_{32} = 0$
6			$d_{31} = 0,\ d_{32} = 0$
7	$d_{11} \neq 0, c_{12} = 0$	$d_{21} \neq 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
8			$d_{31} \neq 0,\ d_{32} = 0$
9			$d_{31} = 0,\ d_{32} = 0$
10		$d_{21} = 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
11			$d_{31} \neq 0,\ d_{32} = 0$
12			$d_{31} = 0,\ d_{32} = 0$
13	$c_{11} = 0, c_{12} = 0$	$d_{21} \neq 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
14			$d_{31} \neq 0,\ d_{32} = 0$
15			$d_{31} = 0,\ d_{32} = 0$
16		$d_{21} = 0$	$d_{31} \neq 0,\ d_{32} \neq 0$
17			$d_{31} \neq 0,\ d_{32} = 0$
18			$d_{31} = 0,\ d_{32} = 0$

After that we should consider the same case when $d_{f1} = 0$. As pointed out in Remark 12 , those other Cases reduce to only one (Case 19).

Since the strategy of proof is similar, we analyze the most interesting cases.

Case 1: $b_{f1} \neq 0$, $for\ h \in H\ \ d_{h1} \neq 0$, $for\ h \in \mathcal{H}^{+}\ \ d_{2h} \neq 0$.[22]

(15)

$(f.1)$	$\Delta(b_{f1})$
$(f.2)$	b_{f1}^{C}
(1.1)	$\Delta(d_{11})$
(1.2)	$\Delta(d_{12})$
(1.3)	d_{11}^{C}
(2.1)	$\Delta(d_{21})$
(2.3)	d_{21}^{C}
(3.1)	$\Delta(d_{31})$
(3.3)	d_{31}^{C}
$(M.1)$	$d_{h1}^{\backslash}$
$(M.2)$	d_{h2}, $h \in \mathcal{H}^{+}$
$(M.3)$	$b_{f1}^{\backslash C}$
$(M.4)$	d_{uh} as long as $d_{12} \neq 0$ and $d_{h1} \neq 0$

We now show in more detail the perturbation of equation $(M.4)$:

$$\begin{array}{lllll} (M.4) & \leftarrow d_{uh} & \rightsquigarrow (1.2) & \leftarrow \Delta(d_{12}) \\ & & \rightsquigarrow (h.1) & \leftarrow \Delta(d_{h1}) \end{array} . \tag{16}$$

Observe that we could perturb equation $(M.4)$ also using d_{m2} - as long as $d_{12} \neq 0$ and $d_{fi} \neq 0$.[23]

Case 18. $d_{11} = 0$, $d_{12} = 0$, $d_{21} = 0$, $d_{31} = 0$, $d_{32} = 0$.

Lost unkns.	#	Lost eqns.	#	# Eqns. we can erase	Eqns. we erase	#
d_{11}	C			C	$(M.3)$	H
d_{12}	1	(1.3)	1		$(M.4)$	1
d_{21}	C	(2.3)	1	$C-1$	rst $(C-1)$ in (2.1)	$C-1$
d_{31}	C			C	rst $(C-1)$ in (3.1)	$C-1$
d_{32}	1	(3.3)				
				$3C-1$ in total		$2C+H$

(17)

[22] If $c_{12} = 0$, we have to erase (1.2). To perturb $(f.2)$, we need to use a d_{f1}^{c} with $c \neq 1, ..., H$. We take d_{f1}^{C} to perturb $(f.2)$, because d_{f1}^{C} does not appear in $(M.3)$.

[23] See Remark 12 for the reason for which we didn t choose c_{m2} as the perturbing variable.

Then, proceed as illustrated in the following table:

$(f.1)$	$\Delta(b_{f1})$
$(f.2)$	b_{f1}^{C}
(1.1)	d_{m1}, d_{u1}
(1.2)	d_{m2}
(1.3)	cancelled
(2.1)	d_{u2}
(2.3)	cancelled
(3.1)	d_{u3}
(3.3)	cancelled
$(M.1)$	$d_{f1}^{\setminus C,G}$
$(M.2)$	$d_{f1}^{\setminus C,G}$
$(M.3)$	cancelled
$(M.4)$	cancelled

. (18)

■

Remark 12 *We should now analyze the other remaining 18 Cases in which $b_{f1} = 0$ for each f. But it is enough to observe what follows.*

1. If $b_{f1} = 0$, then we have the following situation:

Lost unkns.	#	*Lost eqns.*	#	# *Eqns. we can erase*	*Eqns. we erase*	#
b_{f1}	$C+1$	$(f.1)$	$C+1$		$(H.7)$	H
b_{f2}	1	$(f.2)$	1			
d_{m1}	$C-1$			$C-1$		
d_{m2}	1			1		
				$\overline{C}$ *in total*		$\overline{H}$

.

In all rst 17 cases, we never used d_{m2} and we used b_{f1} and b_{f2} to perturb equations $(f.1)$, $(f.2)$ and $(M.3)$ which we erased.

2. The case in which all the perturbing variables are equal to zero is analyzed below.

Proof. Case 19. $b_{f1} = 0$ for each f; $d_{h1} = 0$ for $h \in H$; and $d_{2h} = 0$ for $h \in \mathcal{H}^{+}$.

>From $(f.1)$, $b_{f2} = 0$, $d_{m1} = 0$, and $d_{m2} = 0$. Therefore, the system reduces to the following:

(1.1)	C	$-d_{13}$ p				d_{u1} D^1_x		
12	1	$-d_{13}$ p^g				d_{u1} D^1_G	d_{u2} D^2_G	d_{u3} D^3_G
(2.1)	C		$-d_{23}$ p				d_{u2} D^2_x	
(3.1)	C			$-d_{33}$ p				d_{u3} D^3_x
(3.2)	1	d_{13} p^g		$-d_{33}$ p^g				
$(M.1)$	C^-	$-d_{13}$ $\widetilde{z}_1^{\backslash}$	$-d_{23}$ $\widetilde{z}_2^{\backslash}$	$-d_{33}$ $\widetilde{z}_3^{\backslash}$				
$(M.2)$	1	$-d_{13}$ $\widetilde{g}_1$	$-d_{23}$ $\widetilde{g}_2$	$-d_{33}$ $\widetilde{g}_3$				
$(M.3)$	H^+	$-d_{13}$ $s_1(\widehat{p}\square\widehat{y})^*$	$-d_{23}$ $s_2(\widehat{p}\square\widehat{y})^*$	$-d_{33}$ $s_3(\widehat{p}\square\widehat{y})^*$	d_{m3} $(\widehat{p}\square\widehat{y})^*$			
$(M.4)$	1							

. (19)

Using the rst order conditions of households problems, we can rewrite the rst part of the system as follows:

$(1.1C)$	1	$-c_{13}$			$c_{u1}\lambda_1$		
(1.2)	1	$-c_{13}$			$c_{u1}\lambda_1$	$c_{u2}\lambda_2$	$c_{u3}\lambda_3$
$(2.1C)$	1		$-c_{23}$			$c_{u2}\lambda_2$	
$(3.1C)$	1			$-c_{33}$			$c_{u3}\lambda_3$
(3.2)	1	c_{13}		$-c_{33}$			

.

Using $(1.1C)$ in (1.2), we get

$(1.1C)$	1	$-d_{13}$			$d_{u1}\lambda_1$		
(1.2)	1					$d_{u2}\lambda_2$	$d_{u3}\lambda_3$
$(2.1C)$	1		$-d_{23}$			$d_{u2}\lambda_2$	
$(3.1C)$	1			$-d_{33}$			$d_{u3}\lambda_3$
(3.2)	1	d_{13}		$-d_{33}$			

. (20)

>From (3.2), $d_{13} = d_{33}$.

>From $(3.1C)$, $d_{33}=d_{u3}\lambda_3$.
>From $(2.1C)$, $d_{23}=d_{u2}\lambda_2$.
>From (1.2), $d_{u2}\lambda_2=-d_{u3}\lambda_3$.
Therefore, $d_{13}=d_{33}=-d_{23}$, or more generally,

$$d_{13}=d_{h^{+\prime}3}=d_{h^+1}=-d_{h^{0\prime}3}=-d_{h^03}\quad \text{for } h^+,h^{+'}\in\mathcal{H}^+ \text{ and } h^0,h^{0'}\in\mathcal{H}^0.$$

Substituting in $(M.1)$, $(M.2)$ and $(M.3)$, and using some obvious notation, we get

$(M.1)$:

$$\begin{aligned}
&d_{13}\left(\sum_{h\in\mathcal{H}^+}(x_h^c-e_h^c)-\sum_{h\in\mathcal{H}^0}(x_h^c-e_h^c)-y^c\left(\sum_{h\in\mathcal{H}^+}s_h-\sum_{h\in\mathcal{H}^0}s_h\right)\right)\\
&=d_{13}\left(x_+^c-e_+^c-x_0^c+e_0^c-y^c(s_+-s_0)\right)\\
&=d_{13}\left(x_+^c-e_+^c-x_0^c+e_0^c-y^c(s_+-s_0)+(x_0^c-x_0^c-e_0^c+e_0^c-s_0y^c+s_0y^c)\right)\\
&=^{24}\ d_{13}\left(-x_0^c+e_0^c+s_0y^c-x_0^c+e_0^c+s_0y^c\right)\\
&=-2d_{13}\left(x_0^c-e_0^c-s_0y^c\right).
\end{aligned}$$

$(M.2)$:

$$\begin{aligned}
&d_{13}\left(\sum_{h\in\mathcal{H}^+}\widetilde{g}_h-\sum_{h\in\mathcal{H}^0}\widetilde{g}_h\right)=d_{13}\left(\sum_{h\in\mathcal{H}^+}g_h-y^c\left(\sum_{h\in\mathcal{H}^+}s_h-\sum_{h\in\mathcal{H}^0}s_h\right)\right)\\
&=^{25}\ d_{13}y^g(1-s_0+s_+)=2d_{13}s_0y^g.
\end{aligned}$$

$(M.3)$:

$$\begin{aligned}
&d_{13}p^cy^c\left(\sum_{h\in\mathcal{H}^+}s_h-\sum_{h\in\mathcal{H}^0}s_h\right)+d_{m3}p^cy^c\\
&=\left[d_{13}(s_+-s_0)+d_{m3}\right]p^cy^c.
\end{aligned}$$

To summarize, we have

$$\begin{array}{lll}
(M.1) & d_{13}\left(x_0^c-e_0^c-s_0y^c\right) & =0\\
(M.2) & d_{13}s_0y^g & =0\ .\\
(M.3) & \left[d_{13}(s_+-s_0)+d_{m3}\right]p^cy^c & =0
\end{array}$$

Therefore, if $d_{13}=0$, then, using $(M.3)$, and recalling that $p^gy^g\neq 0$, $d_{m3}=0$ and also $d=0$, a contradiction.

To get $d_{13}=0$, it must be that either

1. for all $\mathcal{H}^0\subseteq\mathcal{H}$ it is the case that $\sum_{h\in\mathcal{H}'}(x_h^c-e_h^c-s_hy^c)\neq 0$, or
2. for all $\mathcal{H}^0\subseteq\mathcal{H}$ it is the case that $\sum_{h\in\mathcal{H}'}s_h\neq 0$.

[24] We used tha fact, from market clearing, $x_+^c-e_+^c+x_0^c-e_0^c-y^c=0$.
[25] We used tha fact, from market clearing, $g-y^g=0$.

The rst condition can be proved to hold true and the second condition is clearly true, *generically in the space of economies and as long as* $\mathcal{H}^0 \neq \emptyset$. Observe that it cannot be that $\mathcal{H}^0 = \mathcal{H}$.

The case in which $\mathcal{H}^0 = \emptyset$ *is analyzed below:*

If $\mathcal{H}^0 = \emptyset$, then equations (2.1) and all terms related to household 2 in the system disappear, and we get the following system.

(1.1)	C	$-d_{13}$ p			d_{u1} D_x^1	
12	1	$-d_{13}$ p^g			d_{u1} D_G^1	d_{u3} D_G^3
(3.1)	C		$-d_{33}$ p			d_{u3} D_x^3
(3.2)	1	d_{13} p^g	$-d_{33}$ p^g			
$(M.1)$	C^-	$-d_{13}$ $\tilde{z}_1^\backslash$	$-d_{33}$ $\tilde{z}_3^\backslash$			
$(M.2)$	1	$-d_{13}$ $\tilde{g}_1$	$-d_{33}$ $\tilde{g}_3$			
$M7$	H^+	$-d_{13}$ $s_1(\hat{p}\square\hat{y})^*$	$-d_{33}$ $s_3(\hat{p}\square\hat{y})^*$	d_{m3} $(\hat{p}\square\hat{y})^*$		
$M8$	1					

.

Using the rst order conditions of households problems, we can rewrite the rst part of this system as:

$(1.1C)$	1	$-d_{13}$		$d_{u1}\lambda_1$	
(1.2)	1	$-d_{13}$		$d_{u1}\lambda_1$	$d_{u3}\lambda_3$
$(3.1C)$	1		$-d_{33}$		$d_{u3}\lambda_3$
(3.2)	1	d_{13}	$-d_{33}$		

. (21)

Using $(1.1C)$ in (1.2), we get

$(1.1C)$	1	$-d_{13}$		$d_{u3}\lambda_1$	
(1.2)	1				$d_{u3}\lambda_3$
$(3.1C)$	1		$-d_{33}$		$d_{u3}\lambda_3$
(3.2)	1	d_{13}	$-d_{33}$		

. (22)

From(1.1), $d_{13} = d_{u1}\lambda_1$.

>From (3.2), $d_{13} = d_{33}$.
>From $(3.1C)$, $d_{33} = d_{u3}\lambda_3$.
>From (1.2), $d_{u3} = 0$.
Therefore, $d_{13} = d_{33} = d_{u1} = d_{u2}$.
Finally with $(M.3)$, and recalling that $p^g y^g \neq 0$, $d_{m3} = 0$ and also $d = 0$, we get a contradiction. ■

6 Appendix. Differential Analysis on the Equilibrium Manifold

The starting point of the analysis is a function whose zeros describe equilibria:

$$F : \Xi^{n_1} \times \Theta \times \mathcal{U} \to \mathbb{R}^{n_1},\ F : (\xi, \theta, u) \mapsto F(\xi, \theta, u),$$

where Ξ^{n_1} is an open subset of $\mathbb{R}^{n_1}$, the set of endogenous variables, Θ is the set of the exogenous variables, and $\mathcal{U}$ is the utility function space.

Then some new variables $\tau \in T \equiv \mathbb{R}^m$ are added. T is the set of the planner s tools. The function

$$F_1 : \Xi^{n_1} \times T \times \Theta \times \mathcal{U} \to \mathbb{R}^{n_1},\quad F_1 : (\xi, \tau, \theta, u) \mapsto F_1(\xi, \tau, \theta, u)$$

describes the equilibrium with planner intervention , and the function

$$F_2 : \Xi^{n_1} \times T \to \mathbb{R}^p,\quad F_2 : (\xi, \tau) \mapsto F_2(\xi, \tau)$$

describes the constraints on the planner intervention. De ne

$$\widetilde{F} : \Xi^{n_1} \times T \times \Theta \times \mathcal{U} \to \mathbb{R}^{n_1+p\equiv n},\quad \widetilde{F} : (\xi, \tau, \theta, u) \mapsto (F_1(\xi, \tau, \theta, u), F_2(\xi, \tau)) .$$

The set T of tools can be written as $T = T_1 \times T_2 = \mathbb{R}^{m-p} \times \mathbb{R}^p$, with $(\tau_1, \tau_2) \in T$, and where τ_1 can be interpreted as the vector of independent tools and τ_2 as the vector of dependent tools.

Step 1. There exists $\overline{\tau}_1$ such that for each $(\theta, u) \in \Theta \times \mathcal{U}$

$$\begin{aligned} &\{\xi \in \Xi^{n_1} : F(\xi, \theta, u) = 0\} \\ &= \left\{\xi \in \Xi^{n_1} : \exists!\ \overline{\tau}_2 \text{ such that } \widetilde{F}(\xi, \overline{\tau}_1, \overline{\tau}_2, \theta, u) = 0\right\} \end{aligned} \tag{23}$$

That is, we have An equilibrium without planner intervention is an equilibrium with planner intervention when the planner decides not to intervene .

In fact, we want to study the effects of changes in τ_1 around $\overline{\tau}_1$.

Finally, G describes the goals of the planner:

$$G : \Xi^{n_1} \times T \times \Theta \times \mathcal{U} \to \mathbb{R}^k, \quad G : (\xi, \tau, \theta, u) \mapsto G(\xi, \tau, \theta, u) \ .$$

Step 2. For every $u \in \mathcal{U}$, there exists an open and full measure subset Θ_u of Θ such that for every $\theta' \in \Theta_u$ and for every $\xi' \in \Xi$ such that $\widetilde{F}(\xi', \overline{\tau}_1, \overline{\tau}_2, \theta, u) = 0$,

$$rank\ D_{(\xi,\tau_2)} \widetilde{F}(\xi', \overline{\tau}_1, \overline{\tau}_2, \theta, u) \text{ is full.}$$

Usually, the above result follows from the fact that $D_{(\xi,\tau_2)} \widetilde{F}(\xi', \overline{\tau}_1, \overline{\tau}_2, \theta, u)$ is

$$\begin{bmatrix} D_\xi F_1 & D_{\tau_2} F_1 \\ D_\xi F_2 & D_{\tau_2} F_2 \end{bmatrix},$$

and, from a Regularity Lemma, $D_\xi F_1$ has full row rank in an open and full measure subset of Θ_u, the fact that $D_\xi F_2 = 0$, and that $D_{\tau_2} F_2$ has full row rank.

De ning

$$\widehat{g} : T_1 \to \mathbb{R}^k, \quad \widehat{g} : \tau_1 \mapsto G(\xi(\tau_1), \tau_2(\tau_1), \tau_1) \ ,$$

we want to show that in an open and dense set of economies $d\widehat{g}_{\overline{\tau}_1}$ is onto and, therefore, $\widehat{g}$ is locally onto around $\overline{\tau}_1$. As explained in Citanna, Kajii and Villanacci (1998), that condition is implied by the following one.

There exists an open and dense subset $S^* \subseteq \Theta \times \mathcal{U}$ such that for every $(\theta', u') \in \Theta \times \mathcal{U}$ and for every $\xi' \in \Xi^{n_1}$ such that $\widetilde{F}(\xi', \overline{\tau}_1, \overline{\tau}_2, \theta, u) = 0$

$$rank\ \left[D_{(\xi,\tau)} \widetilde{F}(\xi', \overline{\tau}_1, \overline{\tau}_2, \theta, u) \right]_{(n_1+p+k)\times(n_1+m)} = n_1 + p + k.$$

The above condition implies that it must be

$$m \geq p + k,$$

i.e.,

$$(number\ of\ tools \ \geq \ (number\ of\ constraints\ on\ planner\ intervention) \ + \ (number\ of\ goals) \tag{24}$$

or

$$m - p \geq k,$$

i.e., (number of independent tools)$\geq$ (number of goals).

The above statement is equivalent to showing that for $(\theta, u) \in S^*$ the following system has no solutions $(\xi, c) \in \Xi^{n_1} \times \mathbb{R}^{n_1+p+k}$:

$$\begin{cases} \widetilde{F}\left(\xi, \overline{\tau}_1, \overline{\tau}_2, \theta, u\right) & = & 0 \\ c^T \left[D_{(\xi,\tau)} \left(\widetilde{F}, G \right) (\xi, \tau, \theta, u) \right] & = & 0 \\ c^T c - 1 & = & 0, \end{cases},$$

or, using condition (23), that the following system has no solutions:

$$\begin{cases} F\left(\xi, \theta, u\right) & = & 0 \\ c^T \left[D_{(\xi,\tau)} \left(\widetilde{F}, G \right) (\xi, \tau, \theta, u) \right] & = & 0 \\ c^T c - 1 & = & 0. \end{cases}. \tag{25}$$

Step 3. Openness of S^*.
Since

$$M \equiv \{(\xi, \theta, u) \in \Xi \times \Theta \times \mathcal{U} : \text{ system (25) has a solution at } \tau = 0\}$$

is closed, it is sufficient to show that the following function is proper:

$$pr : F^{-1}(0) \to \Theta \times \mathcal{U}, \;\; pr : (\xi, \theta, u) \mapsto (\theta, u).$$

Step 4. Density of S^*.

We apply the Parametric Transversality Theorem to the function de ned by the left hand side of system (25). That amounts to show that the following matrix has full row rank:

$$\begin{array}{lll} & \xi & c\,,\ \alpha_u \\ & & \\ F\left(\xi, \theta, u\right) & D_\xi F\left(\xi, c, \theta, u_a\right) & B\left(\xi, c, \theta, u_a\right) \\ c^T \left[D_{(\xi,\tau)} \left(\widetilde{F}, G \right) (\xi, \tau, \theta, u) \right] & * & A\left(\xi, \tau, c, \theta, u_a\right) \\ c^T c - 1 & & \end{array},$$

where α_u is an element of an Euclidean space which is a nite dimensional local parametrization of the utility function space.

Again from a Regularity result, $D_\xi F$ has full row rank in an open and dense subset of $\Theta \times \mathcal{U}$. Moreover, it is crucial to have $B(\xi, c, \theta, u_a) = 0$. If that is the case, to get the result in Step 4, it is enough to show that the following matrix has full row rank

$$A(\xi, \tau, \theta, u) = \begin{bmatrix} \left[D_{(\xi,\tau)}\left(\widetilde{F}, G\right)(\xi, \tau, \theta, u)\right]^T & N(\alpha_u) \\ c & 0 \end{bmatrix}.$$

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