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SYMMETRY AND TIME CHANGED BROWNIAN MOTIONS

José Fajardo
Ernesto Mordecki

Symmetry and Time Changed Brownian Motions*

José Fajardo[†] and Ernesto Mordecki[‡]

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Abstract

In this paper we examine which Brownian Subordination with drift exhibits the symmetry property introduced by Fajardo and Mordecki (2006b). We obtain that when the subordination results in a Lévy process, a necessary and sufficient condition for the symmetry to hold is that drift must be equal to -1/2.

Key Words: Time Changed, Subordination, Symmetry.

1 Introduction

Asset returns have been studied during many years, one of the main findings is the presence of many small jumps in a finite time interval. To deal with that fact more realistic jump structures have been suggested, as for example the Generalized Hyperbolic (GH) model of Eberlein, Keller, and Prause (1998), the Variance-Gamma (VG) model of Madan, Carr, and Chang (1998) and the CGMY model of Carr, Geman, Madan, and Yor (2002). All that models are included in a huge family called Lévy processes.

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[†]IBMEC Business School, Rio de Janeiro - Brazil, e-mail: pepe@ibmecrj.br

[‡]Centro de Matemática, Facultad de Ciencias, Universidad de la República, Montevideo. Uruguay. e-mail: mordecki@cmat.edu.uy.

By a result due to Monroe (1978), we know that every semimartingale can be written as a time-changed Brownian motion. As a consequence many Lévy processes can be represented as Time-changed Brownian Motion. This fact is very useful for the pricing of multiasset derivatives, since we can correlate assets by correlating the Brownian motions.

On the other hand, it is not easy to express explicitly the time-change used. As for example in the case of CGMY process introduced by Carr, Geman, Madan, and Yor (2002) and the Meixner process, introduced by Grigelionis (1999) and Schoutens (2002). The time-changes used for these processes have been obtained recently by Madan and Yor (2006).

In this paper, based on this explicit time-changes, we study how does the symmetry concept, introduced by Fajardo and Mordecki (2006b), works in the CGMY and Meixner models and also in other Lévy processes obtained by Subordinating Brownian motion with drift.

The paper is organized as follows: in Section 2, we introduce Time-changed Brownian Motion. In Section 3, we describe the market model. In Section 4, we describe symmetry and obtain our main result. In last sections we have the conclusions and an appendix.

2 Time-Changed Brownian Motion

Let $X = (X^1, \ldots, X^d)$ be a d-dimensional Lévy process respect to the complete filtration $\mathbf{F} = \{\mathcal{F}_t, t \geq 0\}$, this process is defined on the probability space $(\Omega, \mathbf{F}, P)$, in other words X is a càdlàg process with independent and stationary increments.

We know by the Lévy-Khintchine formula that the characteristic function of X_t, $\phi_{X_t}(z) \equiv \mathsf{E}e^{izX_t} = \exp(t\psi(z))$ where the characteristic exponent ψ is given by:

$$\psi(z) = i(b, z) - \frac{1}{2}(z, \Sigma z) + \int_{I\!R^d} \Big(e^{i(z,y)} - 1 - i(z, y)\mathbf{1}_{\{|y| \leq 1\}}\Big)\Pi(dy), \qquad (1)$$

where $b = (b_1, \ldots, b_d)$ is a vector in $I\!R^d$, Π is a positive measure defined on $I\!R^d \setminus \{0\}$ such that $\int_{I\!R^d} (|y|^2 \wedge 1)\Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A.

Now let $t \mapsto \mathcal{T}_t, \ \ , t \geq 0$, be an increasing cádlág process, such that for each fixed t, $\mathcal{T}_t$ is a stopping time with respect to $\mathbf{F}$. Furthermore, suppose $\mathcal{T}_t$ is finite and positive $P - a.s.$, $\forall t \geq 0$ and $\mathcal{T}_t \to \infty$ as $t \to \infty$. Then $\{\mathcal{T}_t\}$ defines a random change on time, we can also impose $E\mathcal{T}_t = t$.

Now let $X_t = W_t$ be a Brownian motion. Then, consider the process Y_t defined by:

$$Y_t \equiv X_{\mathcal{T}_t}, \ \ t \geq 0,$$

this process is called Time-changed Brownian Motion. Using different time changes $\mathcal{T}_t$, we can obtain a good candidate for the underlying asset return process. We know that if $\mathcal{T}_t$ is a Lévy process we have that Y would be another Lévy process[1]. A more general situation is when $\mathcal{T}_t$ is modelled by a non-decreasing semimartingale:

$$\mathcal{T}_t = a_t + \int_0^t \int_0^\infty y\mu(dy, ds) \tag{2}$$

where a is a drift and μ is the counting measure of jumps of the time change. Now we can obtain the characteristic function of Y_t:

$$\phi_{Y_t}(z) = \mathsf{E}(e^{iz'X_{\mathcal{T}_t}}) = \mathsf{E}\left(\mathsf{E}\left(e^{iz'X_u} \mid T_t = u\right)\right)$$

If $\mathcal{T}_t$ and X_t are independent, then:

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)), \tag{3}$$

where $\mathcal{L}_{\mathcal{T}_t}$ is the Laplace transform of $\mathcal{T}_t$. So if the Laplace transform of $\mathcal{T}$ and the characteristic exponent of X have closed forms, we can obtain a closed form for ϕ_{Y_t}, as we show in the next examples.

In this way we can obtain the distribution of Y_t for every t and in this way we can price some derivatives.

[1] See (Cont and Tankov (2004, Th. 4.2 pag. 108))

2.1 Subordinators

We say that $\mathcal{T}_t$ is a *Subordinator* if it is a Lévy Processes with non decreasing trajectories. As a consequence, trajectories take positive values almost sure. These properties are necessary in order to make a time change, a desired fact for a Time-changed and the choose of a Lévy process will allow us to obtain as a result a very good candidate to model asset returns.

2.1.1 Stable subortination

Now let $\mathcal{T}_t$ be a $\alpha-$Stable with zero drift and $\alpha \in (0,1)$, that is a Lévy process with Lévy measure given by:

$$\rho(x) = \frac{A}{x^{1+\alpha}}, \quad x > 0,$$

we can compute the Laplace transform of $\mathcal{T}_t$:

$$\mathcal{L}_{\mathcal{T}_t}(z) = A\int_0^{\infty} \frac{e^{zx}-1}{x^{\alpha+1}}dx = -\frac{A\Gamma(1-\alpha)}{\alpha}(-z)^{\alpha},$$

Let X_t be a symmetric $\beta-$stable process, that is using eq. (1) we have

$$\psi(z) = -B|z|^{\beta},$$

where A and B are positive constants. Then Using eq. (3), we have that $Y_t = X_{\mathcal{T}_t}$ has characteristic exponent given by

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = -C|z|^{\beta\alpha}$$

where $C = \frac{AB^{\alpha}\Gamma(1-\alpha)}{\alpha}$. That is Y_t is a $\beta\alpha-$Stable symmetric process. If X_t be a Brownian Motion, i.e. $\beta = 2$, then Y_t would be a $2\alpha-$Stable symmetric process. As $\alpha < 1$, we have that Y_t will be a process with heavy tails, which is an stylized fact of the majority of the observed asset returns.

2.1.2 Tempered subordination

Assume that $\mathcal{T}_t$ has Lévy measure given by

$$\rho(x) = \frac{Ce^{-\lambda x}}{x^{1+\alpha}}1_{x>0},$$

where $C, \lambda > 0$ and $\alpha \in (0, 1)$. Then, we have

$$\mathcal{L}_{\mathcal{T}_t}(z) = C\Gamma(-\alpha)\left[(\lambda - z)^\alpha - \lambda^\alpha\right]$$

Now let X_t be a Time-Changed Brownian motion with drift μ, i.e. $X_t = \mathcal{T}_t\mu + \sigma W(\mathcal{T}_t)$. Then, using eq. (1), we have:

$$\psi(z) = -\frac{z^2}{2}\sigma^2 + i\mu z,$$

and using eq. (3), we have that $Y_t = X_{\mathcal{T}_t}$ has characteristic exponent given by

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = C\Gamma(-\alpha)\left[(\lambda + \frac{z^2}{2}\sigma^2 - i\mu z)^\alpha - \lambda^\alpha\right]$$

3 Market Model

Consider a Time-changed Brownian market where we have a riskless asset, with price process denoted by $B = \{B_t\}_{t\geq 0}$, with

$$B_t = e^{rt}, \qquad r \geq 0,$$

where we take $B_0 = 1$ for simplicity, and a risky asset, with price process denoted by $S = \{S_t\}_{t\geq 0}$,

$$S_t = S_0 e^{Y_t}, \qquad S_0 = e^y > 0. \tag{4}$$

Where Y_t is a time-changed Brownian Motion with a drift, where the time change is an independent Subordinator. Denote by (b, σ, ν) the characteristics of the time-changed Brownian with drift process[2]. Also, we assume that the stock pays dividends, with constant rate $\delta \geq 0$, and we assume that the probability measure P is the chosen equivalent martingale measure. In other words, prices are computed as expectations with respect to P, and the discounted and reinvested process $\{e^{-(r-\delta)t}S_t\}$ is a P–martingale.

In order to this condition be satisfied, we need that

$$E\left[e^{-(r-\delta)t}S_t\right] = S_0, \forall t$$

[2]Here we assume conditions to guarantee that this process is a Lévy process, see Appendix.

In other words, $E(e^{Y_t}) = e^{(r-\delta)t}$. That means that the characteristic exponent of Y must satisfy:

$$\psi_Y(1) = (r - \delta).$$

So to avoid arbitrage opportunities we have to restrict our attention to time-changed Brownian process such that the exponential process $e^{Y_t-(r-\delta)t}$ be a P–martingale.

4 Symmetry

Consider a Time-changed Brownian market described above with driving process characterized by (b, σ, ν). Now, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \qquad r \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modelled by

$$\tilde{S}_t = Ke^{\tilde{Y}_t}, \qquad S_0 = e^x > 0,$$

where $\tilde{Y} = \{\tilde{Y}_t\}_{t\geq 0}$ is a Lévy processes with characteristics under $\tilde{P}$ given by $(\tilde{b}, \tilde{\sigma}, \tilde{\nu})$. This market is the *dual market* in Fajardo and Mordecki (2006b). Observe, that in the dual market (i.e. with respect to $\tilde{P}$), the process $\{e^{-(\delta-r)t}\tilde{S}_t\}$ is a martingale.

It is interesting to notice, that in a market with no jumps the distribution (or laws) of the discounted (and reinvested) stocks in both the given and dual markets coincide. It is then natural to define a market to be *symmetric* when this relation hold, i.e. when

$$\mathcal{L}\big(e^{-(r-\delta)t+Y_t} \mid P\big) = \mathcal{L}\big(e^{-(\delta-r)t-Y_t} \mid \tilde{P}\big), \tag{5}$$

meaning equality in law. Fajardo and Mordecki (2006b) derived the characteristics of the *dual process* $\widetilde{Y}_t$, In particular they obtained that a necessary and sufficient condition for (5) to hold is

$$\nu(dx) = e^{-x}\nu(-dx). \tag{6}$$

This ensures $\tilde{\nu} = \nu$, and from this follows $b - (r - \delta) = \tilde{b} - (\delta - r)$, giving (5), as we always have $\tilde{\sigma} = \sigma$.

Now as we have assumed that $Y_t = \mathcal{T}_t\mu + W(\mathcal{T}_t)$. If we denote the Lévy measure of $\mathcal{T}_t$ by $\rho(dy)$. Then, we know by Sato (1999)[Th. 30.1] that Lévy measures are related by

$$\nu(dx) = \left[\int_0^\infty \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\mu y)^2}{2y}} \rho(dy)\right] dx,$$

we can express this relationship as

$$\nu(dx) = e^{\mu x} f(x) dx,$$

where

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2y}(x^2+\mu^2 y^2)} \rho(dy),$$

this density is even, i.e. $f(x) = f(-x)$.

Proposition 1. *A Time-Changed Brownian Market is symmetric if and only if the drift is equal -1/2.*

Proof. By condition (6) we have that a market is symmetric iff

$$e^{\mu x} f(x) dx = e^{-x}[e^{-\mu x} f(-x)],$$

from here $\mu = -1/2$. □

As an application of this Proposition we have the following

Corollary 4.1. *a) The CGMY Market model will be symmetric if and only if*

$$G - M = -1.$$

b) The Meixner Market Model will be symmetric if and only if

$$2b + a = 0$$

Proof. Since, Madan and Yor (2006) have obtained the explicitly representations of CGMY model and Meixner Model as a Time-changed Brownian motion with drift $\frac{G-M}{2}$ and $\frac{b}{a}$, respectively. The result follows. □

Moreover, in Carr, Geman, Madan, and Yor (2002) the drift is estimated under the market risk neutral measure and in the majority of cases the drift is negative and less that -0.5. Also, Schoutens (2001) estimates the values of a and b and obtain similar evidence. It give us evidence against market symmetry.

5 Conclusions

In a Time-changed Brownian market we have shown that market will be symmetric if and only if the drift is equal to -1/2.

Since, Time-Changed Brownian motion allow us to model correlations by correlating the Brownian motions. Another important application that can be address with duality techniques is the pricing of bidimensional derivatives in a Time-Changed Brownian context as is done by Fajardo and Mordecki (2006a) for the case of Lévy processes.

6 Appendix

The following Theorem is taken from Cont and Tankov (2004)[Th. 4.3, Pag. 113] and it gives conditions for a Time-Changed Brownian motion with drift to be a Lévy process.

Theorem 6.1. *Let ν be a Levy measure and $\mu \in I\!R$. There exists a Lévy process Y_t with Lévy measure ν such that $Y_t = W(Z_t) + \mu Z_t$ for some subordinator Z_t and some Brownian motion W_t independent from Z, if and only if the following conditions are satisfied:*

1. *ν is absolutely continuous with density $\nu(x)$*
2. *$\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$*
3. *$\nu(\sqrt(x))e^{-\mu\sqrt(x)}$ is a completely monotonic function on $(0, 1)$*

A function $f : [a, b] \to I\!R$ is called completely monotonic if all derivatives exist and $(-1)k\frac{d^k f(u)}{du^k} > 0, \ \forall k \geq 1$.

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