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EQUIVALENT MARTINGALE MEASURES AND LÉVY PROCESSES

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Equivalent Martingale Measures and Lévy Processes

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Abstract

In this paper we compute equivalent martingale measures when the asset price return is modelled by a Lévy process. We follow the approach introduced by Gerber and Shiu (1994).

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1 Introduction

The asset returns behavior have been studied by many authors, many models have been suggested. Some of them have captured a reasonable part of this behavior, such as fat tails, asymmetry, autocorrelation, etc. For a survey about stylized facts see Rydberg (1997).

The importance of the correct specification of asset returns is very well understood, due to the implications on derivative pricing and Value at Risk calculations. In that sense a class of processes called Lévy processes have shown to be a suitable context for the modelling of these asset returns, since a Lévy process is a simple Markov model with jumps that allow us to capture a huge class of asset returns without the necessity of introducing extreme parameter values. But the most important fact to consider discontinuous processes, as Lévy processes, is the fact that diffusion models can not consider the discontinuous sudden movements observed on asset prices, for that reason Lévy processes have shown to provide a good fit with real data, as we can see in Carr and Wu (2004) and Eberlein, Keller and Prause (1998). In the other hand, the mathematical tools behind these processes are very well established and known.

After defining the process that better captures the asset return behavior, we can construct the set of equivalent martingale measures (hereafter EMM), under the absence of arbitrage assumption. This set is very important, because knowing this set we know the set of derivative arbitrage free prices.

In this paper we show how to compute EMM when the asset return is modelled by a Lévy process, using the approach introduced by Gerber and Shiu (1994).

The paper is organized as follows: In Section 2, we describe the Lévy processes and give some examples, in Section 3, we introduce the stock price model. In Section 4, we discuss the characterization of EMM, in Section 5, we show how to compute an EMM. In Section 6 compute the EMM in two cases: when we have a diffusion with jumps and when we have a pure jump processes. In the last sections we have the conclusions and an appendix.

2 Lévy processes

In this section we introduce the Lévy processes, this name is due to the fact that was Paul Lévy, a French mathematician, who studied deeply processes with independent and stationary increments, obtaining the most important results and properties concerning these processes.

Definition 1 *We say that* $\{Y(t)\}_{t\geq 0}$ *is a Lévy process if*

- Y *has right continuous paths and left limits.*
- $Y(0) = 0$*, and given* $0 < t_1 < t_2 < ... < t_n$*, the random variables*

$$Y(t_1), Y(t_2) - Y(t_1), \cdots, Y(t_n) - Y(t_{n-1})$$

are independent.

- *The distribution of the increment $Y(t)-Y(s)$ is time-homogenous, that is, depends only on $t-s$.*

Observe that the first condition implies that the sample paths can present discontinuities at random times.

A key result in this context is the Lévy-Khintchine formula, that allow us to obtain the characteristic function of any Lévy process $\{Y(t)\}$:

$$\varphi_{Y(t)}(z) = E(e^{izY(t)}) = e^{t\psi(z)},$$

the function ψ is called *characteristic exponent* , and is given by:

$$\psi(z) = iaz - \frac{1}{2}c^2z^2 + \int_{\mathbb{R}} (e^{izy} - 1 - izy\mathbf{1}_{\{|y|<1\}})\Pi(dy),$$

where a and $c \geq 0$ are real constants, and Π is a positive measure in $\mathbb{R} - \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < \infty$, that is called *Lévy measure* and describes the jumps of the process.

An important consequence of this result is that the triplet (a, c, Π) completely characterizes the distribution of the Lévy processes $\{Y(t)\}$. In other words we must know if the process has tendency $(a \neq 0)$, diffusion component $(c \neq 0)$ and jumps $(\Pi \neq 0)$.

2.1 Examples

Now we present some examples of Lévy processes:

- Let $\{B(t)\}_{t\geq 0}$ be a Brownian Motion, that is the increments $B(t)-B(s)$ are independent and stationary with normal distribution of 0 mean and variance $t-s$. The characteristic function is given by:

  $$\varphi_{B(t)}(z) = e^{-tz^2/2},$$

  that is $\psi(z) = -\frac{z^2}{2}$, from here the triplet that characterize the Lévy process is $(0, 1, 0)$, that is, we have just a diffusion.

- Let $\{N(t)\}$ be a Poisson process with parameter λ, for each $t > 0$ the random variable $N(t)$ has a Poisson distribution with parameter λt, that is:
$$P(N(t) = n) = e^{-\lambda t}(\lambda t)^n/n!, \quad (n = 0, 1, \cdots),$$
from here
$$Ee^{zN(t)} = \sum_{n=0}^{\infty} e^{zn} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(e^z \lambda t)^n}{n!} = e^{\lambda t(e^z - 1)}. \tag{1}$$
The characteristic exponent is given by:
$$\psi(z) = \lambda t(e^z - 1) = \int_{I\!R} (e^{zy} - 1)\Pi(dy), \text{ where } \Pi(dy) = \lambda \delta_1(dy),$$
here δ_1 is the Dirac delta measure, all the mass is concentrated in the point 1. Then, the triplet of this process is $(0, 0, \lambda\delta_1)$, we have a process with a finite number of jumps in a finite time interval (finite activity).

- Diffusion with Jumps: Let $\{X_t\}$ be a process defined by:
$$X_t = at + cB(t) + \sum_{k=1}^{N(t)} Z_k, \quad t > 0.$$
where a e c are constants, $\{B_t\}$ is a Brownian Motion, $\{N(t)\}$ is a Poisson process with parameter λ and $\{Z_n\}$ is a sequence of independents and identically distributed random variables, with distribution $F(x)$. Moreover, W, N, Z are mutually independent. Now, we find ψ:
$$Ee^{zX_t} = e^{azt} Ee^{zcW(t)} Ee^{z \sum_{k=1}^{N(t)} Z_k} = e^{t(az + c^2 \frac{z^2}{2} + \lambda \int_{I\!R} (e^{zy} - 1)F(dy))},$$
then $\psi(z) = az + c^2 \frac{z^2}{2} + \int_{I\!R} (e^{zy} - 1)\Pi(dy)$, where $\Pi(dy) = \lambda F(dy)$. Our process has the triplet $(a, c, \lambda F)$, that is we have trend, continuous part and a finite number of jumps in a finite time interval.

In these examples was relatively easy to find the characteristic exponent, due to the simple structure of jumps, that is we consider just finite activity processes, but when we consider infinite activity processes (infinite number of jumps in a finite time interval) the calculation can be hardly, due to the fact that we need to make an analytic integration. Fortunately, for the huge class of Lévy processes considered in the literature, the characteristic exponent have been already computed. Now we can used this class of processes to model asset prices.

3 Stock Price Model

We consider a risky asset called *stock*. We denote by $S(t)$ the stock price at each time $t \in [0, T]$, $T < \infty$. The evolution of this price is modelled by the following equation:

$$dS(t) = S(t^-)[\rho_t dt + \sigma_t dY(t)], \quad S(0) \in (0, \infty). \tag{2}$$

In this model the sources of risk are modelled by a Lévy process $Y(t), 0 \leq t \leq T$, and this process is defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and denote by $\mathbf{F} = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the $\mathbf{P}-$ augmentation[1] of the natural filtration generated by Y:

$$\mathcal{F}_Y(t) = \sigma(Y(s), 0 \leq s \leq t), \;\; 0 \leq t \leq T.$$

The positiveness of the stock price will be analyzed in the next section. The *interest rate* $\{r(t) : 0 \leq t \leq T\}$, is assumed finite, the *appreciation rate* $\{\rho(t), 0 \leq t \leq T\}$, and the *volatilities* $\{\sigma(t), 0 \leq t \leq T\}$ are deterministic continuous functions.

4 Equivalent Martingale Measures

An EMM is an absolutely continuous probability measure with respect to $\mathbf{P}$ that makes the discounted price process a martingale. Under absence of arbitrage the existence of EMM have been extensively studied, the most general result is due to Delbaen and Schachermayer (1994), they studied the implications of absence of arbitrage when the price process is a semimartingale, and Lévy processes are semimartingales. In this paper we will not discuss the existence of EMM, we will suppose that they exist. But, under minor assumptions, it is easy to verify that the set of EMM in not empty, moreover, since most of the Lévy processes present random jumps there can be more than one EMM.

Assuming that there a no arbitrage we describe the set of EMM, to do that we use some properties of Lévy processes. From the Lévy-Ito decomposition we know that all Lévy processes must be a linear combination of a standard

[1] The augmented filtration $\mathbf{F}$ is defined by $\mathcal{F}(t) = \sigma(\mathcal{F}_Y(t) \cup \mathcal{N})$, where $\mathcal{N} = \{E \subset \Omega : \exists G \in \mathcal{F} \text{ with } E \subseteq G, \mathbf{P}(G) = 0\}$ denotes the set of $\mathbf{P}$-null events.

Brownian Motion ($\{B(t)\}$) and a quadratic pure jump process[2] $\{N(t)\}$ which is independent of the Brownian Motion $B(t)$, then

$$Y(t) = cB(t) + N(t),$$

Now assume that[3]

$$E\left[exp(-bY(1))\right] < \infty, \quad \forall b \in (-b_1, b_2)$$

and

$$\int_{[|x|\geq 1]} e^{-bx} d\Pi(x) < \infty, \quad \forall b \in (-b_1, b_2)$$

Where $0 < b_1, b_2 \leq \infty$. The first assumption said that $Y(t)$ has all moments finite and the second is technical and will let us separate integrands. With this in mind we can return to the jumps and transform $N(t)$ into:

$$N(t) = M(t) + at,$$

where $\{M(t)\}$ is a discontinuous martingale and $a = EN(1)$, as a consequence the original process can be written as

$$Y(t) = M(t) + cB(t) + at. \tag{3}$$

Now we can use the Generalized Ito's Lemma[4] to obtain the solution of equation (2):

$$dS(t) = S(t^-)[\rho_t dt + \sigma_t dY(t)] = (a\sigma_t + \rho_t)S(t^-)dt + \sigma_t S(t^-)(cdB(t) + dM(t))$$

When the coefficients ρ_t and σ_t are deterministic continuous function the solution of this equation is given by the Doléans-Dade exponential[5]:

$$S(t) = S(0)exp\left\{ \int_0^t \sigma_s dY(s) + \int_0^t \left(\rho_s - \frac{c^2\sigma_s^2}{2}\right) ds \right\} \prod_{0<s\leq t} (1+\sigma_s \Delta Y(s))e^{-\sigma_s \Delta Y(s)},$$

[2]A process X is said to be a quadratic pure jump process if $\langle N\rangle^c \equiv 0$, where $\langle N\rangle^c$ is the continuous part of its quadratic variation $\langle N\rangle$. Remember that $\langle N\rangle$ is the process such that $(N(t))^2 - \langle N\rangle_t$ is a martingale.

[3]$E(\cdot)$ denote the expectation with respect to $\mathbf{P}$

[4]See appendix.

[5]See Jacod and Shiryaev (1987)

with (3) we obtain:

$$S(t) = S(0)exp\left\{ \int_0^t c\sigma_s dB(s) + \int_0^t c\sigma_s dM(s) + \int_0^t \left(a\sigma_s + \rho_s - \frac{c^2\sigma_s^2}{2} \right) ds \right\}$$

$$\cdot \prod_{0<s\leq t} (1 + \sigma_s \Delta M(s)) e^{-\sigma_s \Delta M(s)}, \tag{4}$$

to ensure that $S_t \geq 0,\ a.s.\ \forall t \in [0, T]$, we need that

$$1 + \sigma_t \Delta M(t) \geq 0, \quad \forall t \in [0, T]$$

If we assume the convention '$\sigma > 0$', we only need that the jumps of $M(t)$ be bounded from below, i.e. $\Delta M(t) \geq -\frac{1}{\sigma_t}$, it means that we consider only "semi-fat tailed" distributions as Poisson, Gamma, Hyperbolic and Normal Inverse Gaussian and we eliminate processes with heavy tails, it is worth noting that the stable distributions (without including the Gaussian case) were eliminated when we supposed that $Y(t)$ has all moments finite.

Now we can characterize all the absolutely continuous measures with respect to $\mathbf{P}$, and then we can find necessary and sufficient conditions for these measures to be EMM. This is a very technical part and we discuss it in the appendix.

Our main concern now is how to compute in a very fast way one of these EMM. In the next section we present one way to do that.

5 Gerber and Shiu Approach

In this section we present the approach introduced by Gerber and Shiu (1994). Using a parameter $\theta \in I\!R$ we define a new probability by:

$$\frac{d\mathbf{P}_t^\theta}{d\mathbf{P}_t} = \mathcal{Z}^\theta(t) = e^{\{\theta Y_t - t \log \varphi(\theta)\}}. \tag{5}$$

Where $\phi(\theta) = Ee^{\theta Y(1)}$. When the stock price process has constant coefficients, Gerber and Shiu (1994) prove that for a given constant r it is possible to find a solution θ of the following equation:

$$r = \log\left(\frac{\phi(\theta + 1)}{\phi(\theta)} \right). \tag{6}$$

Then we can verify that the process $\hat{S}(t) = e^{-rt}S(t)$ is a martingale under $\mathbf{P}^\theta$,i.e. $\mathbf{P}^\theta$ is an EMM. Moreover, the original process is still a Lévy process under this new probability and is called the Esscher transform of the original process. In our model we consider time dependent functions, then we consider the generalized Esscher Transform:

$$\frac{d\mathbf{P}_t^\theta}{d\mathbf{P}_t} = \mathcal{Z}^\theta(t) = e^{\{\int_0^t \theta_s dY(s) - \int_0^t \log\phi(\theta_s)ds\}},$$

we can prove that this new probability is an EMM for some θ_s. Since, we can verify that equation (7) has an unique solution for which $\phi(\theta_s) < \infty$ and $\theta_s \in (-b_1, b_2)\ \forall s$,

$$-c^2\sigma_s\theta_s + a\sigma_s + \rho_s - r_s + \sigma_s \int_{I\!R} x(e^{-\theta_s x} - 1)\Pi(dx) = 0, \tag{7}$$

we can see that in fact the measure obtained from this solution is an EMM by taking $K(s,x) = exp(-\theta_s x)$, $k(s,x) = -\theta_s x$ and $R_s = -c\theta_s$ in equation (15) in section 8.2.

Although, this choose can be arbitrary, we can said that this measure minimize relative entropy[6] with respect to $\mathbf{P}$, i.e. this EMM is the EMM closest to $\mathbf{P}$ in terms of its information contents, since $\mathbf{P}$ contains information about the behavior of the market, but of course another criteria to choose EMM can be used.

6 Examples

In this section we compute the EMM for two cases: A diffusion with jumps and a Pure jump process, we make this choice, since the pure diffusion case is very well understood, remember that Black and Scholes used the following EMM for the geometric Brownian motion case:

$$\frac{dQ}{dP} = e^{\left(\frac{r-\mu}{\sigma}B_t - \frac{(\mu-r)^2}{2\sigma^2}t\right)}.$$

The presence of jumps in the stock price model will affect the density of the EMM as we will see in the examples below.

[6] See appendix.

6.1 Brownian Motion and Two Jumps process

We consider the following parameter $\sigma = 36\%$ and $\rho = 0$ in (2), and $r = 16\%$. Now assume that $N(t) = \frac{N_1(t)-N_2(t)}{2}$, where N_i is a Poisson process with rate 1, so:

- $\Pi = \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}}$
- $a = EN(t) = 0$

Finally take $c = 1$. Then, the triplet of this process is $(0, 1, \Pi)$. Now we apply the approach presented in the last section to find an EMM, we obtain the parameter θ that satisfy equation (7):

$$-0.36\theta - 0.16 + 0.36 \int x(e^{-\theta x} - 1)(\delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}})(dx) = 0,$$

reducing this expression, we have:

$$-\theta - 4/9 + \frac{e^{-\frac{\theta}{2}} - e^{\frac{\theta}{2}}}{2} = 0,$$

the solution of this equation is $\theta^* \approx -0.2959$. In equation (5):

$$\mathcal{Z}^{\theta *}(t) = e^{\{\theta^* Y_t - t \log \phi(\theta^*)\}},$$

we have also $Y(t) = B(t) + N(t)$ and

$$\begin{aligned} \phi(\theta) &= Ee^{\theta B(1) + \theta N(1)} \\ &= Ee^{\theta B(1)} Ee^{\theta N(1)} \\ &= Ee^{\theta B(1)} Ee^{\theta N_1(1)/2} Ee^{-\theta N_2(1)/2}, \end{aligned}$$

where the first equality is due to the independence of $B(t)$ and $N(t)$ and the second is due to the fact that $N_1(t)$ and $N_2(t)$ are independent. Now, we use the expected value of a Log-normal random variable and equation (1) with $\lambda = 1$ and $t = 1$, to obtain:

$$\begin{aligned} \phi(\theta) &= e^{\frac{\theta^2}{2}} e^{(e^{\theta/2}-1)} e^{(e^{-\theta/2}-1)} \\ &= e^{\left(\frac{\theta^2}{2} + e^{\theta/2} + e^{-\theta/2} - 2\right)}, \end{aligned}$$

then, $\log(\phi(\theta^*)) = 0.0657$. From here we have

$$\mathcal{Z}^{\theta *}(t) = e^{\{-0.2959B(t) - 0.2959N(t) - 0,0657t\}}, \tag{8}$$

This is the density of the EMM.

6.2 Normal Inverse Gaussian Process

We consider the parameters $\sigma = 1$ and $\rho = 0$ in (2) and a Normal Inverse Gaussian distribution for the jump component[7] that is $N(1)$ has a $NIG(\alpha, \beta, \mu, \delta)$ distribution, which has the following density:

$$nig(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta}{\pi} \exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}},$$

$$\alpha,\ \delta \geq 0,\ |\beta| \leq \alpha,\ \mu \in I\!R$$

where K_1 is the modified Bessel function of the third kind. The Lévy measure is given by:

$$\Pi(x) = \frac{\delta\alpha}{\pi|x|} \exp\{\beta x\} K_1(\alpha|x|),$$

and we consider a Lévy process with the following triplet $(0, 0, \Pi)$, that is $Y(t) = N(t)$, this type of process is called a *Pure jump process.* From here we have

$$\phi(\theta) = Ee^{\theta N(1)} = \exp(\mu\theta + \delta[(\alpha^2 - \beta^2)^{1/2} - (\alpha^2 - (\beta + \theta)^2)^{1/2}]. \tag{9}$$

Then, in equation (6), we have

$$r = \mu + \delta[(\alpha^2 - (\beta + \theta)^2)^{1/2} - (\alpha^2 - (\beta + \theta + 1)^2)^{1/2}], \tag{10}$$

to obtain the parameter θ, we use the parameters obtained by Fajardo and Farias (2002) for Brazilian index Ibovespa. The parameters are,

$$(\alpha, \beta, \mu, \delta) = (31.9096, -0.0035, 0.0233, 0.0012)$$

Assuming $r = 13\%$, the solution of equation (10) is $\theta^* \approx 80.65$. Replacing this values in equation (9), we obtain:

$$\begin{aligned} \phi(\theta^*) &= e^{\{0.0233*80.65+0.0012*[(31.9096^2-0.0035^2)^{1/2}-(31.9096^2-(-0.0035+80.65)^2)^{1/2}]\}} \\ &= 324.27 \end{aligned}$$

[7] We make this choice, since Eberlein et all. (1998) and Barndorff-Nielsen, O.E. (1998), showed that this process has a good fit with real data.

Now in (5), we have the density of the EMM:

$$\mathcal{Z}^{\theta*}(t) = e^{\{80.65Y(t)-324.27t\}}.$$

From the fact that NIG distributions are closed under convolutions, we have $Y(t) \sim nig(\alpha, \beta, t\mu, t\delta)$ and from the fact that under the change of measure $(\mathcal{Z}^{\theta*})$ the process N is still a Lévy process and $N(1)$ still has a NIG distribution, with the same α, δ and μ, but $\beta^* = \beta + \theta^*$, that is the new density is $nig(31.91, 80.65, 0.0233, 0.0012)$. We use the convolution property to obtain the distribution of $Y(T)$, that is, it has a density $nig(31.9096, 80.65, 0.0233T, 0.0012T)$, from here, depending on the maturity T, we can compute expectations under the EMM in order to obtain derivative prices.

In comparison with the EMM obtained in the Black and Scholes model we can see that jumps are also present in the density of the EMM. In the case of pure jump process we have just jumps in the density.

As we said there can be many EMM, the complete abstract characterization is given in the appendix.

7 Conclusions

In this paper we have used a Lévy process to model asset price returns, which allow us to capture more stylized facts from real data. Then, we have shown how to compute EMM using the approach introduced by Gerber and Shiu (1994). We compute the EMM in two cases, of course many other examples can be done. For a discussion about which type of jumps can be observed in asset returns see Aït-Sahalia (2004) and Huang and Wu (2004).

Interesting processes do not considered in this paper are processes with dependent increments and non-time-homogenous processes, with these processes we can model the autocorrelation observed in the square and absolute returns of the stocks and consider more flexible structures for implied volatilities on option prices, facts that can not be modelled with Lévy processes.

8 Appendix

8.1 Generalized Ito formula

For any measurable function $f(t,x)$ we have

$$\sum_{0<s\leq t} f(s,\Delta N_s) = \int_0^t \int_{I\!R} f(s,x)L(ds,dx), \tag{11}$$

and for any C^2 function f, we have *the Generalized Itô's formula* for càdlàg semimartingales $X^1,...,X^n$:

$$df(X_t^1,..,X_t^n) = \sum_i f_i(X_{t^-}^1,..,X_{t^-}^n)dX_t^i + \sum_{i,j}\frac{1}{2}f_{ij}(X_{t^-}^1,..,X_{t^-}^n)d[X^i,X^j]_t^c$$
$$+f(X_t^1,..,X_t^n) - f(X_{t^-}^1,..,X_{t^-}^n) - \sum_i f_i(X_{t^-}^1,..,X_{t^-}^n)\Delta X_t^i.$$

With $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i x_j}$ and $[X^i,X^j]^c$ the continuous part of the mutual variation[8] of X^i and X^j.

8.2 Equivalent measures

The process $N(t)$ has a Lévy decomposition: Let $L(dt,dx)$ be a Poisson measure on $I\!R^+ \times I\!R \setminus \{0\}$ with expectation (or compensator) measure $dt\times\Pi$ [9], then:

$$N(t) = \int_{[|x|<1]} x(L((0,t],dx) - t\Pi(dx)) + \int_{[|x|\geq 1]} xL((0,t],dx) \tag{12}$$
$$+tE\left[N_1 - \int_{|x|\geq 1} x\Pi(dx)\right].$$

The following step consists in characterizing all the measures that are absolutely continuous with respect to $\mathbf{P}$, to this end let:

$$\mathcal{M}(dt,dx) = L(dt,dx) - dt\Pi(dx),$$

[8]For more details see Shiryaev (1999), Ch. III, 5C.

[9]$\forall B \in I\!R^+ \times I\!R \setminus \{0\}$, $L(B)$ has Poisson distribution with parameter $(dt\times\Pi)(B)$

then

$$M_t = \int_0^t \int_{I\!R} x\mathcal{M}(ds, dx).$$

Now two useful results:

Lemma 1 *Let R_t and $K(t,x)$ be a previsible and a Borel previsible processes*[10] *respectively. Suppose that*

$$E(\int_0^t R_s^2 ds) < \infty,$$

and $K \geq 0,\ K(t,0) = 1\ \ \forall t \in I\!R^+$. Let $k(t,x)$ be another Borel previsible process such that

$$\int_{I\!R} [K(t,x) - 1 - k(t,x)]\,\Pi(dx) < \infty,$$

Define a process $\mathcal{Z}_t$ by

$$\mathcal{Z}_t = exp\left\{\int_0^t R_s dB_s - \frac{1}{2}\int_0^t R_s^2 ds + \int_0^t \int_{I\!R} k(s,x)\mathcal{M}(ds, dx)\right.$$

$$\left. - \int_{[0,t)\times I\!R} [K(s,x) - 1 - k(s,x)]\Pi(dx)ds\right\} \prod_{0<s\leq t} K(s, \Delta N_s)e^{-k(s,\Delta N_s)}.$$

Then $\mathcal{Z}$ is a local martingale with $\mathcal{Z}_0 = 1$ and $\mathcal{Z}$ is positive if and only if $K > 0$.

Proof See Chan (1999).□ The following Theorem is a Girsanov's type Theorem, that tell us how the triplet of the process change when we make a change of measure

[10] a Process $K_\omega(t,x)$ is said to be a Borel previsible function or process if the process $t \mapsto K_\omega(t,x)$ is a previsible function for fixed x and the function $x \mapsto K_\omega(t,x)$ is Borel-measurable for fixed t.

Theorem 1 *Let* $\mathbf{Q}$ *be a measure which is absolutely continuous with respect to* $\mathbf{P}$ *on* $\{\mathcal{F}_T\}$*. Then*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T} = \mathcal{Z}_T,$$

where $\mathcal{Z}$ *is as in the lemma 1, for some* R, K *and* k *for which* $E\mathcal{Z}_T = 1$*. Moreover under* $\mathbf{Q}$*, the process*

$$\hat{B}_t = B_t - \int_0^t R_s ds, \tag{13}$$

is a Brownian Motion and the process $N(t)$ *is a quadratic pure jump process with compensator measure given by* $dt\hat{\Pi}_t(dx)$ *with*

$$\hat{\Pi}_t(dx) = K(t,x)\Pi(dx),$$

and constant part given by

$$\hat{a}_t = E^{\mathbf{Q}} N(t) = at + \int_0^t \int_{I\!R} x(K(s,t) - 1)\Pi(dx)ds.$$

Proof: See Chan (1999).□ An implication of this results is that under $\mathbf{Q}$ the process $N(t)$ can be represented as

$$N(t) = \hat{M}_t + at + \int_0^t \int_{I\!R} x(K(s,t) - 1)\Pi(dx)ds,$$

with

$$\hat{M}_t = M_t - \int_0^t \int_{I\!R} x(K(s,t) - 1)\Pi(dx)ds. \tag{14}$$

This process is a $\mathbf{Q}-$martingale and it is easy to see that $\Delta\hat{M}_t = \Delta M_t$. Now let

$$\hat{S}_t = exp\left(-\int_0^t r_s ds\right) S_t,$$

be the discounted price process. Replacing the processes B_t and M_t in the equation (4) by their respective $\mathbf{Q}-$versions, we obtain:

$$\hat{S}_t = S_0 exp\left\{\int_0^t c\sigma_s d\hat{B}_s + \int_0^t c\sigma_s d\hat{M}_s + \int_0^t \left(a\sigma_s + c\sigma_s R_s + \rho_s - r_s - \frac{c^2\sigma_s^2}{2}\right) ds\right.$$

$$\left. + \int_0^t \sigma_s \int_{I\!R} x(K(s,x)-1)\Pi(dx)ds\right\} \prod_{0<s\leq t} (1+\sigma_s \Delta\hat{M}_s)e^{-\sigma_s\Delta\hat{M}_s}.$$

A necessary and sufficient condition for $\hat{S}_t$ be a $\mathbf{Q}-$martingale is the existence of R and $K>0$ $a.s.$ for which :

$$cR_s + a + \frac{\rho_s - r_s}{\sigma_s} + \int_{I\!R} x(K(s,x)-1)\Pi(dx) = 0 \ \ \forall s, \tag{15}$$

and $E\mathcal{Z}_t = 1$, $\forall t>0$. Since the process

$$exp\left\{\int_0^t c\sigma_s d\hat{B}_s + \int_0^t \sigma_s d\hat{M}_s - \int_0^t \frac{c^2\sigma_s^2}{2} ds\right\} \prod_{0<s\leq t} (1+\sigma_s\Delta\hat{M}_s)e^{-\sigma_s\Delta\hat{M}_s},$$

is a $\mathbf{Q}-$martingale.

8.3 Minimizing Relative Entropy

As we said the new measure obtained using the Esscher transform minimize entropy, to see this remember the definition of entropy:

$$I_{\mathbf{P}}(\mathbf{Q}) = E^{\mathbf{Q}}\left[\log\frac{d\mathbf{Q}}{d\mathbf{P}}\right],$$

where $\mathbf{Q}$ is any absolutely continuous measure with respect to $\mathbf{P}$, with Lemma 1 we have

$$I_{\mathbf{P}}(\mathbf{Q}) = E^{\mathbf{Q}}\left[\frac{1}{2}\int_0^T R_s^2 ds + \int_0^T \int_{I\!R} [K(s,x)(\log K(s,x)-1)+1]\Pi(dx)ds\right],$$

where **Q** depends on the choice of K and R, and these functions have to satisfy equation (15). We can show [11] that this minimum is obtained when $K = exp(-x\sigma\lambda)$ and $R = -c\sigma\lambda$, where λ is the lagrange multiplier associated to the constraint (15), this can justify the choice of the measure associated to $\theta = \sigma\lambda$.

References

[1] Aït-Sahalia (2004), " Disentangling Diffusion from Jumps". *Forthcoming Journal of Financial Economics.*

[2] Barndorff-Nielsen, O. (1998), "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, **2**, 41-68.

[3] Carr, P. and Wu, L. (2004), " Time-changed Lévy Processes and option pricing". *Journal of Financial Economics* **71**, 113–141.

[4] Chan, Terence. (1999), " Pricing Contingent Claims on Stocks Driven by Lévy Processes". *Annals of Applied Probability* **Vol. 9, no. 2**, 504–528.

[5] Delbaen, F., and W. Schachermayer (1994), " A General Version of the Fundamental Theorem of Asset Pricing". *Mathematische Annalen*, **Vol. 300**, 463–520

[6] Eberlein, E and Keller, U. (1995), "Hyperbolic distributions in finance", *Bernoulli* **1**,281–299.

[7] Eberlein, E, Keller, U. and Prause, K. (1998), "New insights into smiles, mispricing and Value at Risk: The Hyperbolic Model", *Journal of Business* **71**,371–405.

[8] Fajardo, J. and Farias, A. (2002), "Generalized Hyperbolic Distributions and Brazilian Data", Working Paper No. 52. Central Bank of Brazil.

[9] Gerber, H. and Shiu, E. (1994). "Option Pricing by Esscher-transform", *Transactions of the Society of Actuaries* , **vol. 46**, 99–191

[11] See Chan (1999)

[10] Huang, J. and Wu, L., (2004). "Specification Analysis of Option Pricing Models Based on Time-Changed Lévy Processes", *Journal of Finance*, **vol. LIX, no. 3**, 1405–1440.

[11] Jacod, J. and A.N. Shiryaev (1987), *Limit Theorems for Stochastic Processes.* Springer-Verlag, New York.

[12] Rydberg, T. (1997), "Why Financial Data are Interesting to Statisticians", Centre for Analytical Finance, Aarhus University Working Paper 5.

[13] Sato, Ken-iti. (1999), *Lévy processes and infinitely divisible distributions.* Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge.

[14] Shiryaev, A.N. (1999), *Essential of Stochastic Finance: Facts, Models and Theory.* World Scientific Pub. Co.