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SKEWNESS PREMIUM WITH LÉVY PROCESSES

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Skewness Premium with Lévy Processes

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Abstract

We study the skewness premium (SK) introduced by Bates (1991) in a general context using Lévy Processes. We obtain sufficient and necessary conditions for Bate's $x\%$ rule to hold. Then, We derive sufficient conditions for SK to be positive, in terms of the characteristic triplet of the Lévy Process under the risk neutral measure.

Keywords: Skewnes Premium; Lévy Process.
JEL Classification: C52; G10

1 Introduction

The option prices have been largely studied by many authors, an important fact from option prices is that relative prices of out-of-the-money calls and puts can be used as a measure of symmetry or skewness of the risk neutral distribution. Bates (1991), called this diagnosis "skewness premium", henceforth SK. He analyzed the behaviour of SK using three classes of stochastic processes: Constant Elasticity of Variance (CEV), Stochastic Volatility and Jump-diffusion. He found conditions on the parameters for the SP be positive or negative.

But, as many models in the literature have shown, the behaviour of the assets underlying options is very complex, the structure of jumps observed

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is more complex than Poisson jumps. They have higher intensity, see for example Aït-Sahalia (2004). For that reason diffusion models cannot consider the discontinuous sudden movements observed on asset prices. In that sense, the use of more general process as Lévy processes have shown to provide a better fit with real data, as was reported in Carr and Wu (2004) and Eberlein, Keller, and Prause (1998). On the other hand, the mathematical tools behind these processes are very well established and known.

In this paper we establish a theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process. In this way we establish a simply diagnostic for judging which distributions are consistent with observed option prices. Then we pass to study the SK and we obtain sufficient conditions for the SK be positive or negative.

The paper is organized as follows: in Section 2 we introduce the Lévy processes and we present the duality results. In Section 3 we present the Bate's rule. In Section 4 we analyze symmetry. In Section 5 we present the estimated parameters using real data and in Section 6 we study the skewness premium. Section 7 concludes.

2 Lévy processes and Duality

Consider a real valued stochastic process $X = \{X_t\}_{t\geq 0}$, defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, being càdlàg, adapted, satisfying $X_0 = 0$, and such that for $0 \leq s < t$ the random variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$, with a distribution that only depends on the difference $t - s$. Assume also that the stochastic basis $\mathcal{B}$ satisfies the usual conditions (see Jacod and Shiryaev (1987)). The process X is a Lévy process, and is also called a process with stationary independent increments (PIIS). For general reference on Lévy processes see Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996), Sato (1999). For Lévy process in Finance see Boyarchenko and Levendorskiĭ (2002), Schoutens (2003) and Cont and Tankov (2004).

In order to characterize the law of X under $\mathbb{Q}$, consider, for $q \in I\!R$ the

Lévy-Khinchine formula, that states

$$\mathbf{E}\, e^{iqX_t} = \exp\Big\{t\Big[iaq - \frac{1}{2}\sigma^2 q^2 + \int_{I\!R} \big(e^{iqy} - 1 - iqh(y)\big)\Pi(dy)\Big]\Big\}, \tag{1}$$

with

$$h(y) = y\mathbf{1}_{\{|y|<1\}}$$

a fixed truncation function, a and $\sigma \geq 0$ real constants, and Π a positive measure on $I\!R \setminus \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < +\infty$, called the *Lévy measure*. The triplet (a, σ^2, Π) is the *characteristic triplet* of the process, and completely determines its law.

Consider the set

$$\mathbb{C}_0 = \Big\{z = p + iq \in \mathbb{C}\colon \int_{\{|y|>1\}} e^{py}\Pi(dy) < \infty\Big\}. \tag{2}$$

The set $\mathbb{C}_0$ is a vertical strip in the complex plane, contains the line $z = iq$ ($q \in I\!R$), and consists of all complex numbers $z = p + iq$ such that $\mathbf{E}\, e^{pX_t} < \infty$ for some $t > 0$. Furthermore, if $z \in \mathbb{C}_0$, we can define the *characteristic exponent* of the process X, by

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{I\!R} \big(e^{zy} - 1 - zh(y)\big)\Pi(dy) \tag{3}$$

this function ψ is also called the *cumulant* of X, having $\mathbf{E}\,|e^{zX_t}| < \infty$ for all $t \geq 0$, and $\mathbf{E}\, e^{zX_t} = e^{t\psi(z)}$. The finiteness of this expectations follows from Theorem 21.3 in Sato (1999). Formula (3) reduces to formula (1) when $\mathrm{Re}(z) = 0$.

2.1 Lévy market

By a *Lévy market* we mean a model of a financial market with two assets: a deterministic savings account $B = \{B_t\}_{t\geq 0}$, with

$$B_t = e^{rt}, \qquad r \geq 0,$$

where we take $B_0 = 1$ for simplicity, and a stock $S = \{S_t\}_{t\geq 0}$, with random evolution modelled by

$$S_t = S_0 e^{X_t}, \qquad S_0 = e^x > 0, \tag{4}$$

where $X = \{X_t\}_{t\geq 0}$ is a Lévy process.

In this model we assume that the stock pays dividends, with constant rate $\delta \geq 0$, and that the given probability measure $\mathbb{Q}$ is the choosen equivalent martingale measure. In other words, prices are computed as expectations with respect to $\mathbb{Q}$, and the discounted and reinvested process $\{e^{-(r-\delta)t}S_t\}$ is a $\mathbb{Q}$-martingale.

In terms of the characteristic exponent of the process this means that

$$\psi(1) = r - \delta, \tag{5}$$

based on the fact, that $\mathbf{E}\, e^{-(r-\delta)t+X_t} = e^{-t(r-\delta+\psi(1))} = 1$, and condition (5) can also be formulated in terms of the characteristic triplet of the process X as

$$a = r - \delta - \sigma^2/2 - \int_{I\!R} \big(e^y - 1 - h(y)\big)\Pi(dy). \tag{6}$$

In the case, when

$$X_t = \sigma W_t + at \quad (t \geq 0), \tag{7}$$

where $W = \{W_t\}_{t\geq 0}$ is a Wiener process, we obtain the Black–Scholes–Merton (1973) model (see Black and Scholes (1973),Merton (1973)).

In the market model considered we introduce some derivative assets. More precisely, we consider call and put options, of both European and American types. Denote by $\mathcal{M}_T$ the class of stopping times up to a fixed constant time T, i.e:

$$\mathcal{M}_T = \{\tau :\ 0 \leq \tau \leq T,\ \tau \text{ stopping time w.r.t } \mathbf{F}\}$$

for the finite horizon case and for the perpetual case we take $T = \infty$ and denote by $\overline{\mathcal{M}}$ the resulting stopping times set. Then, for each stopping time $\tau \in \mathcal{M}_T$ we introduce

$$c(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+, \tag{8}$$

$$p(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+. \tag{9}$$

In our analysis (8) and (9) are auxiliary quantities, anyhow, they are interesting by themselves as random maturity options, as considered, for instance, in Schroder (1999) and Detemple (2001). If $\tau = T$, formulas (8) and (9) give the price of the European call and put options respectively. For the American finite case, prices and optimal stopping rules τ_c^* and τ_p^* are defined, respectively, by:

$$C(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+ = \mathbf{E}\, e^{-r\tau_c^*}(S_{\tau_c^*} - K)^+ \tag{10}$$

$$P(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+ = \mathbf{E}\, e^{-r\tau_p^*}(K - S_{\tau_p^*})^+, \tag{11}$$

and, for the American perpetual case, prices and optimal stopping rules are determined by

$$\overline{C}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \overline{\mathcal{M}}} \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+ \mathbf{1}_{\{\tau<\infty\}} = \mathbf{E}\, e^{-r\tau_c^*}(S_{\tau_c^*} - K)^+ \mathbf{1}_{\{\tau<\infty\}}, \tag{12}$$

$$\overline{P}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \overline{\mathcal{M}}} \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+ \mathbf{1}_{\{\tau<\infty\}} = \mathbf{E}\, e^{-r\tau_p^*}(K - S_{\tau_p^*})^+ \mathbf{1}_{\{\tau<\infty\}}. \tag{13}$$

2.2 Put Call duality and dual markets

Lemma 2.1 (Duality). *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$, defined in* (3)*, on the set $\mathbb{C}_0$ in* (2)*. Then, for the expectations introduced in* (8) *and* (9) *we have*

$$c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{14}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{I\!R} \left(e^{zy} - 1 - zh(y)\right)\tilde{\Pi}(dy) \tag{15}$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\tilde{\psi}(z) = \psi(1-z) - \psi(1), \quad \text{for } 1 - z \in \mathbb{C}_0,$$

and in consequence,

$$\begin{cases} \tilde{a} & = \delta - r - \sigma^2/2 - \int_{I\!R} \left(e^y - 1 - h(y)\right)\tilde{\Pi}(dy), \\ \tilde{\sigma} & = \sigma, \\ \tilde{\Pi}(dy) & = e^{-y}\Pi(-dy). \end{cases} \tag{16}$$

Proof. See Fajardo and Mordecki (2006). □

Remark 2.1. *The presented Lemma is very similar to Proposition 1 in Schroder (1999) and the results obtained in Eberlein and Papapantoleon (2005) and Fajardo and Mordecki (2005). The main difference is that the particular structure of the underlying process (Lévy process are a particular case of the models considered in Schroder (1999)) allows to completely characterize the distribution of the dual process $\tilde{X}$ under the dual martingale measure $\tilde{\mathbb{Q}}$, and to give a simpler proof. Considering Additive processes similar result, in the case of European plain vanilla options, were obtained by Eberlein and Papapantoleon (2005), see Corollary 4.2.*

If we take $\tau = T$ in the Duality Lemma we obtain the following put call relation.

Corollary 2.1 (European Options). *For the expectations introduced in* (8) *and* (9) *we have*

$$c(S_0, K, r, \delta, T, \psi) = p(K, S_0, \delta, r, T, \tilde{\psi}), \tag{17}$$

with ψ and $\tilde{\psi}$ as in the Duality Lemma.

To formulate the duality result for American Options, we observe that the optimal stopping rules for the American Call and Put options have, respectively, the form

$$\tau_c^* = \inf\{t \geq 0\colon S_t \geq B_c(t)\} \wedge T,$$
$$\tau_p^* = \inf\{t \geq 0\colon S_t \leq B_p(t)\} \wedge T.$$

where the curves B_c and B_p are the boundaries of the continuation region. (See 12.1.3 in Cont and Tankov (2004), or Theorem 6.1 in Boyarchenko and Levendorskiĭ (2002).)

Corollary 2.2 (American Options). *For the value functions in* (10) *and* (11) *we have*

$$C(S_0, K, r, \delta, T, \psi) = P(K, S_0, \delta, r, T, \tilde{\psi}), \tag{18}$$

with ψ and $\tilde{\psi}$ as in the Duality Lemma. Furthermore, when $\delta > 0$, for the optimal stopping boundaries, we obtain that

$$B_c(t)B_p(t) = S_0 K. \tag{19}$$

In case $\delta = 0$ we have $\tau_c^* = \tau_p^* = T$.

Remark 2.2. *The relation between the stopping boundaries is analogous to the one for Itô processes obtained by Detemple (2001) (see equation (30)).*

In what respects Perpetual Call and Put American Options, the optimal stopping rules have, respectively, the form

$$\begin{aligned}\tau_c^* &= \inf\{t \geq 0\colon S_t \geq S_c^*\},\\ \tau_p^* &= \inf\{t \geq 0\colon S_t \leq S_p^*\}.\end{aligned}$$

where the constants S_c^* and S_p^* are the critical prices. (See Theorem 1 and 2 in Mordecki (2002).)

Corollary 2.3 (Perpetual Options). *For prices of Perpetual Call and Put options in* (12) *and* (13) *we have*

$$\overline{C}(S_0, K, r, \delta, \psi) = \overline{P}(K, S_0, \delta, r, \tilde{\psi}), \tag{20}$$

with ψ and $\tilde{\psi}$ as in the Duality Lemma. Furthermore, when $\delta > 0$, for the optimal stopping levels, we obtain the relation

$$S_c^* S_p^* = S_0 K. \tag{21}$$

2.3 Dual Markets

The Duality Lemma motivates us to introduce the following market model. Given a Lévy market with driving process characterized by ψ in (3), consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \qquad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modelled by

$$\tilde{S}_t = K e^{\tilde{X}_t}, \qquad \tilde{S}_0 = K > 0,$$

where $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ is a Lévy process with characteristic exponent under $\tilde{\mathbb{Q}}$ given by $\tilde{\psi}$ in (15). The process $\tilde{S}_t$ represents the price of KS_0 dollars measured in units of stock S. This market is the *auxiliary market* in Detemple (2001), and we call it *dual market*; accordingly, we call *Put–Call duality* the relation (14). It must be noticed that Peskir and Shiryaev (2001) propose the same denomination for a different relation. Finally observe, that in the dual market (i.e. with respect to $\tilde{\mathbb{Q}}$), the process $\{e^{-(\delta-r)t}\tilde{S}_t\}$ is a martingale. As a consequence, we obtain the Put–Call symmetry in the Black–Scholes–Merton model: In this case $\Pi = 0$, we have no jumps, and the characteristic exponents are

$$\psi(z) = (r - \delta - \sigma^2/2)z + \sigma^2 z^2/2,$$
$$\tilde{\psi}(z) = (\delta - r - \sigma^2/2)z + \sigma^2 z^2/2.$$

and relation (14) is the result known as put–call symmetry. In the presence of jumps like the jump-diffusion model of Merton (1976), if the jump returns of S under $\mathbb{Q}$ and $\tilde{S}$ under $\tilde{\mathbb{Q}}$ have the same distribution, the Duality Lemma, implies that by exchanging the roles of δ by r and K by S_0 in (14) and (16), we can obtain an American call price formula from the American put price formula. Motivated by this analysis we introduce the definition of symmetric markets in the following section.

3 Market Symmetry

It is interesting to note that in a market with no jumps (i.e. in the Black-Scholes model), the distribution of the discounted and reinvested stock both in the given risk neutral and in the dual Lévy market, taking equal initial values, coincide. It is then natural to define a Lévy market to be *symmetric* when this relation hold, i.e. when

$$\mathcal{L}\big(e^{-(r-\delta)t+X_t} \mid \mathbb{Q}\big) = \mathcal{L}\big(e^{-(\delta-r)t-X_t} \mid \tilde{\mathbb{Q}}\big), \tag{22}$$

meaning equality in law. In view of (16), and due to the fact that the characteristic triplet determines the law of a Lévy processes, we obtain that a necessary and sufficient condition for (22) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy). \tag{23}$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows $a - (r - \delta) = \tilde{a} - (\delta - r)$, giving (22), as we always have $\tilde{\sigma} = \sigma$. Condition (23) answers a question raised Carr and Chesney (1996). Let us illustrate our result in an example.

4 Bates' Rule

Corollary 4.1. *Take $r = \delta$ and assume (23) holds, we have*

$$c(F_0, K_c, r, \tau, \psi) = (1 + x)\, p(F_0, K_p, r, \tau, \psi), \tag{24}$$

where $K_c = (1 + x)F_0$ and $K_p = F_0/(1 + x)$, with $x > 0$.

Proof. Follows directly from Proposition 1. Since $r = \delta$ and $\psi = \tilde{\psi}$. □

From here calls and puts at-the-money ($x = 1$) should have the same price. As we mention this $x\%-$rule, in the context of Merton's model was obtained by Bates (1997). That is, if the call and put options have strike prices $x\%$ out-of-the money relative to the forward price, then the call should be priced $x\%$ higher than the put.

5 Asymmetry in Lévy markets

Our intention is to review several concrete models proposed in the literature. We restrict ourselves to Lévy markets with jump measure of the form

$$\Pi(dy) = e^{\beta y}\Pi_0(dy), \tag{25}$$

where $\Pi_0(dy)$ is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$, everything with respect to the risk neutral measure $\mathbb{Q}$.

As a consequence of (23), we obtain that the market is symmetric if and only if $\beta = -1/2$. In view of this, we propose to measure the *asymmetry* in the market through the parameter $\beta + 1/2$. When $\beta + 1/2 = 0$ we have a symmetric market.

As we have seen when the market is symmetric, the skewness premium is obtained using the $x\%-$rule. The idea is to describe numerically the departure from the symmetry, the main difference with Bates (1997) is that the parameter β is a property of the market, independent of the derivative asset considered.

Although from the theoretical point of view the assumption (25) is a real restriction, most models in practice share this property, and furthermore, they have a jump measure that has a Radon-Nikodym density. In this case, we have

$$\Pi(dy) = e^{\beta y} p(y) dy, \tag{26}$$

where $p(y) = p(-y)$, i.e. the function $p(y)$ is even. See the examples below, in 5.1.1 - 5.1.3.

More precisely, all parametric models that we found in the literature, in what concerns Lévy markets, including diffusions with jumps, can be reparametrized in the form (26). As we will see, empirical risk-neutral markets are not symmetric, and in view of (26), we propose to model the asymmetry of the market through the parameter $\beta + 1/2$. Before considering concrete examples, we review the Esscher transform.

5.1 Esscher transform and asymmetry

As in the presented concrete examples we departure from historical data, we now review some notation and useful facts. All the developments up to now where with respect to the risk neutral martingale measure $\mathbb{Q}$. Now we consider that there is an historical probability measure $\mathbb{P}$ and that $\mathbb{Q}$ is the consequence of an Esscher transform. It is clear that this is one between several possibilities, and we refer to Chan (1999) and Shiryaev (1999) for a discussion on this topic. In consequence, when necessary, we add a subscript $\mathbb{P}$ to refer to parameters under the historical probability measure $\mathbb{P}$, i.e. $\psi_{\mathbb{P}}$, $\Pi_{\mathbb{P}}$, and even sometimes we use the subscript $\mathbb{Q}$ to distinguish risk neutral parameters. As we said, the link between $\mathbb{P}$ and $\mathbb{Q}$ is given by the Esscher transform, and this is stated through the change of measure

$$d\mathbb{Q}_t = e^{\theta X_t - t\psi_{\mathbb{P}}(\theta)} d\mathbb{P}_t, \tag{27}$$

where θ is a parameter to be determined. From (27) follows that

$$\psi(z) = \psi_{\mathbb{Q}}(z) = \psi_{\mathbb{P}}(z + \theta) - \psi_{\mathbb{P}}(z), \tag{28}$$

As we require that the discounted and reinvested stock is a martingale under $\mathbb{Q}$, i.e. $\{e^{-(r-\delta)t} S_t\}$ is a $\mathbb{Q}$-martingale, we obtain that

$$\psi(1) = \psi_{\mathbb{P}}(1 + \theta) - \psi_{\mathbb{P}}(1) = r - \delta,$$

and this determines θ. It is relevant for us, that from (28) follows that

$$\Pi_{\mathbb{Q}}(dy) = e^{\theta y}\Pi_{\mathbb{P}}(dy).$$

(See Theorem VII.3.2 in Shiryaev (1999).) If we combine this result with our model assumption (25) we conclude that

$$e^{\beta y}\Pi_0(dy) = e^{\theta y}\Pi_{\mathbb{P}}(dy),$$

meaning that the form of the jump measure under $\mathbb{P}$ is

$$\Pi_{\mathbb{P}}(dy) = e^{(\beta-\theta)y}\Pi_0(dy) = e^{\beta_{\mathbb{P}} y}\Pi_0(dy), \tag{29}$$

that is, the same form with a translated parameter. We conclude, that under the Esscher transform, our model assumption (25) is equivalent to the assumption (29), and that the relation between the symmetry parameters is

$$\beta = \beta_{\mathbb{Q}} = \beta_{\mathbb{P}} + \theta. \tag{30}$$

5.1.1 The Generalized Hyperbolic Model

This model has been proposed by Eberlein and Prause (2000) as they allow for a more realistic description of asset returns (see Eberlein and Prause (2000) and Eberlein, Keller, and Prause (1998)). This model, under $\mathbb{P}$, has $\sigma = 0$, and a Lévy measure given by (29), with

$$p(y) = \frac{1}{|y|}\Big(\int_0^\infty \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z\big(J_\lambda^2(\delta\sqrt{2z}) + Y_\lambda^2(\delta\sqrt{2z})\big)}dz + \mathbf{1}_{\{\lambda\geq 0\}}\lambda e^{-\alpha|y|}\Big),$$

where $\alpha, \beta_{\mathbb{P}}, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta_{\mathbb{P}}| < \alpha$, and $\delta > 0$; and J_λ, Y_λ are the Bessel functions of the first and second kind. Particular cases are the hyperbolic distribution, obtained when $\lambda = 1$; and the normal inverse gaussian (NIG) when $\lambda = -1/2$.

Using the daily returns from Brazilian Index Ibovespa for the period 07/01/1994 to 12/13/2001, Fajardo and Farias (2004), estimate the parameter $\beta_{\mathbb{P}} = -0.0035$ for the NIG distribution and the estimate $\beta_{\mathbb{Q}} = 80.65$ for the risk neutral distribution, given by (30). They also estimate the parameters for various Brazilian assets finding $\beta_{\mathbb{Q}} \neq -1/2$. This indicates absence of symmetry.

Index	$\hat{a}$	$\hat{b}$	θ	$\beta_{\mathbb{Q}} + 1/2$
Nikkei 225	0.02982825	0.12716244	0.42190524	5.18506
DAX	0.02612297	-0.50801886	-4.46513538	-23.4123
FTSE-100	0.01502403	-0.014336370	-4.34746821	-4.8017
SMI	0.02954687	-0.33888717	-3.97213216	-14.9416
Nasdaq Comp.	0.03346698	-0.49356259	-5.95888693	-20.2066
CAC-40.	0.02539854	-0.23804755	-5.77928595	-14.6518

Table 1: Estimates of the Meixner Distribution

5.1.2 The Meixner Model

The Meixner process was proposed to model financial data by Grigelionis (1999) and by Schoutens (2001). The Lévy process derived from this distribution has, under $\mathbb{P}$, the following Lévy measure:

$$\Pi(dy) = c\frac{e^{\frac{b}{a}y}}{y\sinh(\pi y/a)}dy,$$

where a, b and c are parameter of the Meixner density, such that $a > 0$, $-\pi < b < \pi$ and $c > 0$. The Lévy measure also corresponds to the form in (29), if we take $\beta_{\mathbb{P}} = b/a$, and

$$p(y) = \frac{c}{y\sinh(\pi y/a)}.$$

Using daily returns from various index Schoutens in Schoutens (2002), found parameters estimates $\hat{a}$ and $\hat{b}$ for the period 1/1/1997 to 31/12/1999. We resume this results and the corresponding parameter $\beta_{\mathbb{Q}} = \hat{b}/\hat{a} + \theta$ in Table 1.

5.1.3 The CGMY model

This Lévy market model, proposed by Carr, Geman, Madan, and Yor (2002) is characterized by $\sigma = 0$ and Lévy measure given by (29), where the function $p(y)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}}e^{-\alpha|y|}.$$

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where C, G, M, Y are the parameters used in Carr, Geman, Madan, and Yor (2002).

For studying the presence of a pure diffusion component in the model, condition $\sigma = 0$ is relaxed, and risk neutral distribution is estimated in a five parameters model. Values of $\beta = (G - M)/2$ are given for different assets in Table 3 in Carr, Geman, Madan, and Yor (2002), and in the general situation, the parameter β is negative, and less than $-1/2$. The condition needed in this case for the market to be symmetric is $G = M - 1$.

In the particular case (25), when $\beta \neq -1/2$, equation (24) does not hold, in that case we need to analyze each market model to know for which parameters the skewness premium is positive or negative and compare with the available empirical data.

6 Skewness Premium

In order to study the sign of SK, lets analyze the following data on S&P500 American options in 08/31/2006 that matures in 09/15/2006 with Future price $F = 1303.82$. To verify if the Bates' rule holds we need to interpolate some non-observed option prices for some strike prices. To this end we use a cubic spline, as we can see in Fig. 1.

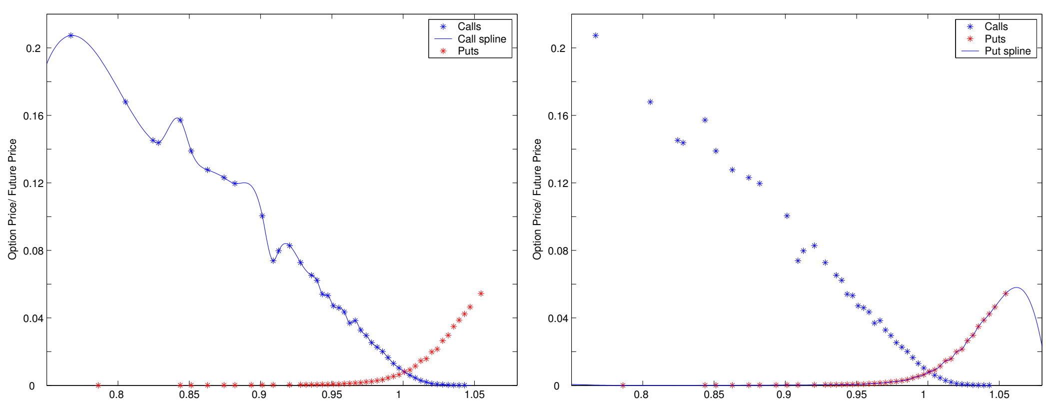

Figure 1: Observed Call and Put Prices on S&P500 08/31/2006.

The $x\%$ Skewness Premium is defined as the percentage deviation of $x\%$ OTM call prices from $x\%$ OTM put prices. The interpolating calls and put prices for the non-observed strikes are presented in Iables 2 and 3. We can see in both tables that this rule does not hold. Moreover, for OTM options usually $x_{obs} < x$, what implies $\frac{c}{p} - 1 < x$ and for ITM options, $x_{obs} > x$, implying $\frac{c}{p} - 1 > x$.

Then we want to know for what distributional parameter values we can capture the observed vies in these option price ratios. To this end we use the following definition introduced by Bates (1991).

$$SK(x) = \frac{c(S,T;X_c)}{p(S,T;X_p)} - 1, \text{ for European Options,} \tag{31}$$

$$SK(x) = \frac{C(S,T;X_c)}{P(S,T;X_p)} - 1, \text{ for American Options,}$$

where $X_p = \frac{F}{(1+x)} < F < F(1+x), \ \ x > 0$.

The SK was addressed for the following stochastic processes: Constant Elasticity of Variance (CEV), include arithmetic and geometric Brownian motion. Stochastic Volatility processes, the benchmark model being those for which volatility evolves independently of the asset price. And the Jump-diffusion processes, the benchmark model is the Merton's (1976) model. For that classes Bates (1991) and Bates (1996) obtained the following result.

Proposition 1. *For European options in general and for American options on futures, the SK has the following properties for the above distributions.*

i) $SK(x) \lesseqgtr x$ *for CEV processes with* $\rho \lesseqgtr 1$.

ii) $SK(x) \lesseqgtr x$ *for jump-diffusions with log-normal jumps depending on whether* $2\mu + \delta^2 \lesseqgtr 0$.

iii) $SK(x) \lesseqgtr x$ *for Stochastic Volatility processes depending on whether* $\rho_{S\sigma} \lesseqgtr 0$.

Now in equation (31) consider

$$X_p = F(1-x) < F < F(1+x), \ \ x > 0.$$

Then,

iv) $SK(x) < 0$ for CEV processes only if $\rho < 0$.

v) $SK(x) \geq 0$ for CEV processes only if $\rho \geq 0$.

When x is small, the two SK measures will be approx. equal. For in-the-money options ($x < 0$), the propositions are reversed. Calls $x\%$ in-the-money should cost $0\% - x\%$ less than puts $x\%$ in-the-money.

Proof. See Bates (1991). □

Then, we have the main SK sign results

Theorem 6.1. *Take $r = \delta$ and assume $\Pi(dy) = \lambda F(dy)$, if F is such that $\int e^y F(dy) \gtreqless 1$, then*

$$C(F_0, K_c, r, \tau, \psi) \gtreqless (1 + x)\ P(F_0, K_p, r, \tau, \psi), \tag{32}$$

where $K_c = (1 + x)F_0$ and $K_p = F_0/(1 + x)$, with $x > 0$.

Proof. The idea is to exploit the monotonicity property of option prices on jump intensity and jump size, as Ekström and Tysk (2005) have shown in the unidimensional case we can still preserve this monotonicity. Then, first assume that $\Pi(dy) = \lambda F(dy), \quad \lambda > 0$. Then $\widetilde{\Pi}(dy) = e^{-y}\lambda F(-dy)$.

Let $\widetilde{\lambda} = \lambda \int e^{-y} F(-dy)$ and $\widetilde{F}(dy) = \frac{e^{-y}F(-dy)}{\int e^{-y}F(-dy)}$. Then, $\widetilde{\Pi} = \widetilde{\lambda}\widetilde{F}(dy)$ and as F is such that $\int e^y F(dy) \gtreqless 1$, then $\widetilde{\lambda} \gtreqless \lambda$. Using Ekström and Tysk (2005) result, option prices are monotonic on jump intensity:

$$\begin{aligned} C(F_0, K_c, r, \tau, a, \sigma, \widetilde{\lambda}\widetilde{F}) &\gtreqless C(F_0, K_c, r, \tau, a, \sigma, \lambda F) \\ &= (1 + x)P(F_0, K_p, r, \tau, a, \sigma, \widetilde{\lambda}\widetilde{F}) \end{aligned}$$

where the last equality is obtained from duality.

□

Now lets obtain a sufficient condition on the symmetry parameter.

Theorem 6.2. *Take $r = \delta$ and assume that in the particular case (25), If $\beta \gtreqless -1/2$, then*

$$C(F_0, K_c, r, \tau, \psi) \gtreqless (1 + x)\ P(F_0, K_p, r, \tau, \psi), \tag{33}$$

where $K_c = (1 + x)F_0$ and $K_p = F_0/(1 + x)$, with $x > 0$.

Proof. We need monotonicity of call prices on the parameter β. We have that $\beta \gtrless -1/2 \Longleftrightarrow \beta \gtrless \widetilde{\beta} := -\beta - 1$. Then, $\Pi(dy) = e^{\beta y}\Pi_0(dy)$ has $\beta \gtrless \widetilde{\beta}$ of $\widetilde{\Pi} = e^{-(1+\beta)y}\Pi_0(dy)$. By monotonicity

$$\begin{aligned} C(F_0, K_c, r, \delta, \tau, a, \sigma, \Pi) &\gtrless C(F_0, K_c, r, \tau, a, \sigma, \widetilde{\Pi}) \\ &= (1+x)P(F_0, K_c, r, \tau, a, \sigma, \Pi), \end{aligned}$$

where the last equality is obtained from duality and the fact that $\widetilde{\widetilde{\Pi}} = \Pi$.

Now the same can be obtained if put were decreasing on β: $\beta \gtrless -1/2$ implies

$$\begin{aligned} (1+x)P(F_0, K_c, r, \tau, a, \sigma, \Pi) &\lessgtr (1+x)P(F_0, K_c, r, \tau, a, \sigma, \widetilde{\Pi}) \\ &= C(F_0, K_c, r, \tau, a, \sigma, \Pi), \ \forall x > 0. \end{aligned}$$

□

6.1 Diffusions with jumps

Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy,$$

and is direct to verify that condition (23) holds if and only if $2\mu + \delta^2 = 0$. This result was obtained by Bates (1997) for future options, that result is obtained as a particular case.

Note that in that model $\beta = \frac{\mu}{\delta^2}$, so we obtain that sufficient conditions in the above theorems are equivalent. That is,

$$\int e^{-y} F(-dy) = e^{\mu + \delta^2/2} \gtrless 1 \Longleftrightarrow \beta \gtrless -1/2.$$

But, in general that conditions are not equivalent neither one imply the other.

7 Conclusions

Departing from *duality*, a relation between call and put prices, obtained through a change of numeraire, and corresponding to a change of probability measure in a Lévy market model under a given risk neutral probability measure, the main contribution of this paper is the characterization of *symmetry* in these market models, a notion that is also introduced.

This characterization allows to introduce a parameter in the risk neutral model that, in certain sense, measures the *asymmetry* of a Lévy market model. We also find the expression of this asymmetry parameter in the historical market model, assuming that we rely in the Esscher transform to obtain the given risk neutral measure. We analyze popular models used in the literature, concluding that, in general, markets are not symmetric. Then, we verify that when a market is symmetric the Bates's $x\%-$rule holds.

Finally, we analyze the sign of the Skewness premium. Using data from S&P500 we observe that this rule does not hold. In that case We derive sufficient conditions for SK to be positive, in terms of the characteristic triplet of the Lévy Process under the risk neutral measure. In particular on the symmetry parameter.

In this way we obtain a simple diagnostic to observe what Lévy model deals with the behaviour of the underlying and with the sign of the SK.

Implications of this result is that we need to correct the martingale measure using the symmetry parameter and that the volatility smiles depends on this parameter.

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K_c	$K_p = F^2/K_c$	$x = K_c/F - 1$	$x_{obs} = c_{obs}/p_{int} - 1$	$x - x_{obs}$
1230	1382.07	-0.05662	0.050681	-0.1073
1235	1376.475	-0.05278	0,13642	-0.1892
1240	1370.925	-0.04895	0.115006	-0.16395
1245	1365.419	-0.04511	0.197696	-0.24281
1250	1359.957	-0.04128	0.277944	-0.31922
1255	1354.539	-0.03744	0.280729	-0.31817
1260	1349.164	-0.03361	0.536286	-0.5699
1265	1343.831	-0.02977	0.574983	-0.60476
1270	1338.541	-0.02594	0.606719	-0.63266
1275	1333.291	-0.0221	0.675372	-0.69748
1280	1328.083	-0.01827	0.691325	-0.70959
1285	1322.916	-0.01443	0.966306	-0.98074
1290	1317.788	-0.0106	0.904839	-0.91544
1295	1312.7	-0.00676	0.794059	-0.80082
1300	1307.651	-0.00293	0.78018	-0.78311
1305	1302.641	0.000905	0.614561	-0.61366
1310	1297.669	0.00474	0.532798	-0.52806
1315	1292.735	0.008575	0.427299	-0.41872
1320	1287.838	0.01241	0.108911	-0.0965
1325	1282.979	0.016245	-0.11658	0.132826
1330	1278.155	0.020079	-0.45097	0.471053
1335	1273.368	0.023914	-0.50378	0.527697
1340	1268.617	0.027749	-0.61306	0.640807
1345	1263.901	0.031584	-0.73872	0.770305
1350	1259.22	0.035419	-0.81448	0.849896
1355	1254.573	0.039254	-0.80297	0.842224
1360	1249.961	0.043089	-0.82437	0.867454

Table 2: Options prices Interpolating Put prices

K_p	$K_c = F^2/K_p$	$x = F/K_p - 1$	$x_{obs} = c_{int}/p_{obs} - 1$	$x - x_{obs}$
1250	1359.957	0.043056	-0.88837	0.931421
1255	1354.539	0.0389	-0.86897	0.907873
1260	1349.164	0.034778	-0.85655	0.891331
1265	1343.831	0.030688	-0.78107	0.81176
1270	1338.541	0.02663	-0.70531	0.731941
1275	1333.291	0.022604	-0.63926	0.661869
1280	1328.083	0.018609	-0.51726	0.535865
1285	1322.916	0.014646	-0.31216	0.326801
1290	1317.788	0.010713	-0.20329	0.214005
1295	1312.7	0.006811	-0.03659	0.043397
1300	1307.651	0.002938	0.090739	-0.0878
1305	1302.641	-0.0009	0.130843	-0.13175
1310	1297.669	-0.00472	0.252541	-0.25726
1315	1292.735	-0.0085	0.261905	-0.27041
1320	1287.838	-0.01226	0.242817	-0.25507
1325	1282.979	-0.01598	0.346419	-0.3624
1330	1278.155	-0.01968	0.183207	-0.20289
1335	1273.368	-0.02336	0.237999	-0.26135
1340	1268.617	-0.027	0.145858	-0.17286
1345	1263.901	-0.03062	0.152637	-0.18325
1350	1259.22	-0.03421	0.101211	-0.13542
1355	1254.573	-0.03777	-0.03964	0.001869
1360	1249.961	-0.04131	0.028337	-0.06965
1365	1245.382	-0.04482	-0.0101	-0.03472
1375	1236.325	-0.05177	-0.0451	-0.00667

Table 3: Options prices Interpolating Call prices